Halfspace.

since $\|x\|_2 \geq 0$. for any x

$\therefore \|x - x_0\|_2 \leq \|x - x_1\|_2$

$\Rightarrow \|x - x_0\|_2^2 \leq \|x - x_1\|_2^2$

$\Rightarrow (x - x_0)^T (x - x_0) \leq (x - x_1)^T (x - x_1)$

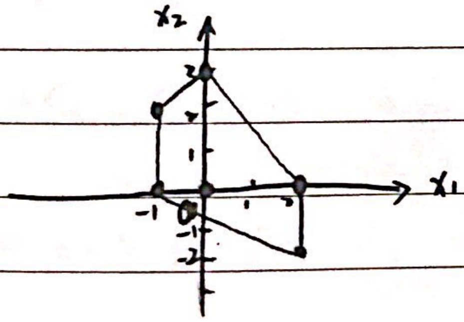
$\Rightarrow x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2x_1^T x + x_1^T x_1$

$\Rightarrow 2(x_1^T - x_0^T) x \leq (x_1^T x - x_0^T x)$

It satisfies the defination of halfspace. A non-strict linear inequality means a closed halfspace.

· Polyhedron

all points are shown in the figure.



It's easy to see that lines are the convex hull.

Therefore, the polyhedron can be represented by 5 halfspaces.

$\{x \in R^2 \mid x_1 \leq 2\}$, $\{x \in R^2 \mid 2x_1 + 3x_2 \geq -2\}$, $\{x \in R^2 \mid x_1 \geq -1\}$, $\{x \in R^2 \mid -x_1 + x_2 \leq 3\}$,

$\{x \in R^2 \mid 3x_1 + 2x_2 \leq 6\}$.

Therefore, the polyhedron is in the form $Ax \preceq b$. could be.

$$\begin{bmatrix} 1 & 0 \\ -2 & -3 \\ -1 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}$$

2.12.

a) it's a convex set since it's an intersection of 2 halfspaces.

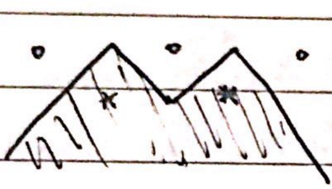
b) it's convex since it's an intersection of halfspaces.

c) it's convex since it's an intersection of 2 halfspaces

d) it's convex since for each y, $\{x \mid \|x-x_0\|_2 \le \|x-y\|_2\}$ is a halfspace. For all $y \in S$

it's an intersection of halfspaces

e) it's not convex. for example:



• for points in S

* for points in T

the shadow is not convex

f) it's convex.

Proof.

$x + S_2 \subseteq S_1 \iff$ for all $t \in S_2$, $x + t \in S_1$

$\therefore \{x \mid x + S_2 \subseteq S_1\} \iff \bigcap_{\forall t \in S_2} \{x \mid x + t \in S_1\}$

$\because$ S_1 is convex and t is a certain point

$\therefore$ $S_1 - t$ is also ~~convex~~ convex

$\therefore$ it's an intersection of convex sets

$\therefore$ it's convex

g) it's convex.

Proof. since $\|x\|_2 \ge 0$, for any x.

$\therefore \|x-a\|_2 \le \theta \|x-b\|_2$

$\Rightarrow \|x-a\|_2^2 \le \theta^2 \|x-b\|_2^2$

$\Rightarrow (1-\theta^2)x^Tx - 2(a-\theta^2 b)^T x + (a^Ta - \theta^2 b^Tb) \le 0$

∴ ~~θ~~ if $\theta=0$, the set is just one point, a, it's convex.

if $\theta=1$, ~~the~~ it's a halfspace. it's convex

if $0<\theta<1$, it can be represented, a sphere

$$(x-\frac{a-\theta^2 b}{1-\theta})^T(x-\frac{a-\theta^2 b}{1-\theta}) \le \frac{\theta^2(a-b)^2+(1-\theta^2)b^Tb)}{(1-\theta^2)^2}$$, which is larger than 0.

2.15.

a) $\alpha \le \sum_{i=1}^n p_i f(a_i) \le \beta$, which is an intersection of 2 halfspace. Therefore it's convex

b) It's convex since $\sum_{i=a}^n p_i \le \beta$ is a halfspace.

e) It's convex since $\sum_{i=1}^n p_i a_i^2 \ge \alpha$ is a halfspace

f) It's not convex.

Proof. $var(x) = Ex^2-(Ex)^2$

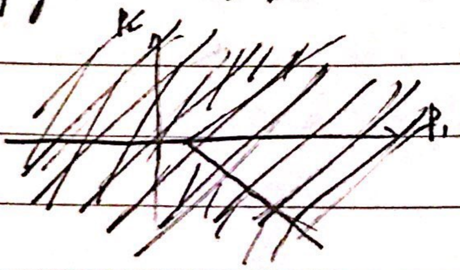
$= \sum_{i=1}^n p_i a_i^2 - (\sum_{i=1}^n p_i a_i)^2$

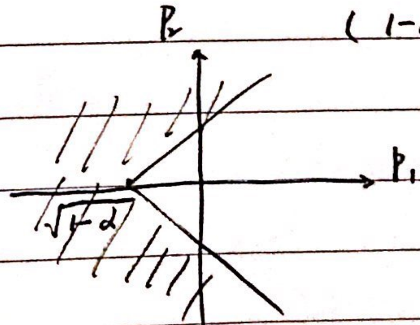
for example, let $a_1=-1$, $a_2=1$.

∴ $var(x) = \sum_{i=1}^{2} p_i - (p_2-p_1)^2 = -(p_2-p_1)^2+1$

∴ $var(x) \le \alpha \Rightarrow (p_2-p_1)^2 \ge 1-\alpha \Rightarrow p_2-p_1 \ge \sqrt{1-\alpha}$ or $p_2-p_1 \le -\sqrt{1-\alpha}$

the figure should be like, $(1-\alpha \ge 0)$





It's easy to see, the shadow is not convex. for ~~ex~~ example, $(0,\sqrt{1-\alpha})$, $(0,-\sqrt{1-\alpha})$ in the set, but $(0,0)$ is not in the set. which could be represented by $(0,\sqrt{1-\alpha})+(0,-\sqrt{1-\alpha})$

3.14

a) since $f(x,z)$ is a concave function of z, for each fixed x.

$\therefore \nabla^2_{zz} f(x,z) = \frac{\partial^2 f(x,z)}{\partial z_i z_j} \leq 0$

in a similar way, $\nabla^2_{xx} f(x,z) = \frac{\partial^2 f(x,z)}{\partial x_i x_j} \geq 0$.

$\therefore \nabla^2 f(x,z) = \begin{bmatrix} \nabla^2_{xx} f(x,z) & \nabla^2_{xz} f(x,z) \\ \nabla^2_{zx} f(x,z) & \nabla^2_{zz} f(x,z) \end{bmatrix}$

$\therefore$ For Hessian $\nabla^2 f(x,z)$, the top-left ~~matrix~~ $n \times n$ matrix $\succeq 0$

the bottom-right $m \times m$ matrix $\preceq 0$.

b) Fix $\tilde{x}$, $\nabla_z f(\tilde{x},\tilde{z}) = 0$. since it's concave, therefore $\nabla f(\tilde{x},z)$ is non-increasing

~~$\therefore (\tilde{x},\tilde{z})$ is the maximum for $f(\tilde{x},z)$~~

~~minimum point~~

~~$\therefore f(\tilde{x},z) \geq f(\tilde{x},\tilde{z})$~~

$\therefore$ for $z < \tilde{z}$, $\nabla_z f(\tilde{x},z) \geq 0$

for $z > \tilde{z}$, $\nabla_z f(\tilde{x},z) \leq 0$.

$\Rightarrow f(\tilde{x},\tilde{z})$ is the maximum for $f(\tilde{x},z)$

$\Rightarrow f(\tilde{x},\tilde{z}) \geq f(\tilde{x},z)$

in a similar way, $f(\tilde{x},\tilde{z}) \leq f(x,\tilde{z})$

$\Rightarrow f(\tilde{x},z) \leq f(\tilde{x},\tilde{z}) \leq f(x,\tilde{z})$

c). since $f(\tilde{x},z) \leq f(\tilde{x},\tilde{z}) \leq f(x,\tilde{z})$

$\therefore f(\tilde{x},\tilde{z})$ is the maximum for $f(\tilde{x},z)$

$\because \nabla f(\tilde{x},z)$ is non-increasing

$\therefore \nabla_z f(\tilde{x},\tilde{z}) = 0$

in a similar way, $\nabla_x f(\tilde{x},\tilde{z}) = 0$ $\Rightarrow \nabla f(\tilde{x},\tilde{z}) = 0$

3.16.

a) $f(x) = e^x - 1 \Rightarrow f''(x) = e^x > 0$

$\therefore$ it's convex ~~and~~, quasiconvex and quasiconcave. It's not concave.

b) $f(x) = x_1 x_2$ on R^2_{++}

$\therefore \nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore$ eigenvalues are $1, -1$

$\therefore$ it's not convex and it's not concave.

for $\{(x_1, x_2) \in R^2_{++} \mid x_1 x_2 \geq \alpha\}$, it's a convex set

$\therefore$ it's quasiconcave but it's not quasiconvex.

c) $f(x) = \frac{1}{x_1 x_2}$

$\therefore \nabla^2 f(x) = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$

eigenvalues are $(\frac{1}{x_1^2}+\frac{1}{x_2^2}) \pm \sqrt{(\frac{1}{x_1^2})^2+(\frac{1}{x_2^2})^2-\frac{3}{x_1^2 x_2^2}} > 0$

$\therefore$ it's convex and quasiconvex. but it's not concave or quasiconcave.

d) $f(x_1, x_2) = \frac{x_1}{x_2}$ on R^2_{++}

$\therefore \nabla^2 f(x) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$

let λ be eigenvalue

$\therefore \lambda(\frac{2x_1}{x_2^3} - \lambda) = -\frac{1}{x_2^4} < 0$

it's easy to see that $\lambda_1 < 0$ and $\lambda_2 > \frac{2x_1}{x_2^3} > 0$ (assume $\lambda_1 < \lambda_2$)

$\therefore$ it's not convex or concave.

$\{(x_1, x_2) \in R^2_{++} \mid \frac{x_1}{x_2} \geq \alpha\}$ is a halfspace.

$\therefore$ it's quasiconvex and quasiconcave.

e) $f(x_1, x_2) = \frac{x_1^2}{x_2}$ on $R \times R_{++}$

$\therefore \nabla^2 f(x_1, x_2) = \begin{bmatrix} 2x_1^{-1} & -2x_1 x_2^{-2} \\ -2x_1 x_2^{-2} & 2x_1^2 x_2^{-3} \end{bmatrix}$

$\therefore$ eigenvalues are $\frac{2x_1^2x_2^{-2}+1}{2} \pm \sqrt{\left(\frac{2x_1^2x_2^{-2}+1}{2}\right)^2 - x_1^2 x_2^{-2}} > 0$

$\therefore$ it's convex and quasiconvex but not ~~conv~~ concave or quasiconcave.

f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on R_{++}^2

$\nabla^2 f(x_1, x_2) = -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{bmatrix} x_1^{-2} & -x_1^{-1}x_2^{-1} \\ -x_1^{-1}x_2^{-1} & x_2^{-2} \end{bmatrix}$

~~$= -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha}$~~

let $t = \begin{bmatrix} x_1^{-1} \\ -x_2^{-1} \end{bmatrix}$

$\therefore \nabla^2 f(x_1, x_2) = -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \cdot t t^T$

$\because 0 \le \alpha \le 1$

$\therefore -\alpha(1-\alpha) \le 0$

$\therefore \nabla^2 f(x_1, x_2) \preceq 0$

$\therefore$ it's concave and quasiconcave but not convex or quasiconvex.

3.36

$$f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$$

a)

① if y has a negative value, let $y_k < 0$. so we could let a $x_k \to -\infty$. Therefore $y_k x_k \to \infty$. $f^*(y) \to \infty$

② $y \succeq 0$ and if $\sum_i y_i < 1$. so we could let $x_1 = x_2 = \cdots = x_n \to -\infty$. $\therefore f^*(y) \to \infty$

③ $y \succeq 0$ and if $\sum_i y_i > 1$, so we could let $x_1 = x_2 = \cdots = x_n \to +\infty$. $\therefore f^*(y) \to \infty$

② $y \succeq 0$ and $\sum_i y_i = 1$. $\therefore$ $y^T x \le y^T \cdot \mathrm{Max}\ x = \mathrm{Max}\ x$

$\therefore$ $y^T x - f(x) \le 0$.

$$\therefore f^*(y) = \begin{cases} 0, & \text{if } y \succeq 0,\ \sum_i y_i = 1 \\ \infty, & \text{otherwise} \end{cases}$$

b)

① same as (a), if there is $y_k < 0$, $f^*(y) = \infty$.

② $y \succeq 0$. assume. $x_1 = x_2 = x_3 = \cdots = x_n = t$.

$\therefore$ $y^T x - f(x) = (\sum_i y_i) \cdot t - t \cdot r$

$\therefore$ if $\sum_i y_i > r$ or $\sum_i y_i < r$. $f^*(y) = \infty$

$$\therefore f^*(y) = \begin{cases} 0. & \text{if } y \succeq 0,\ \sum_i y_i = r \\ \infty, & \text{otherwise} \end{cases}$$

e) ①

① if y has a positive value. let $y_k > 0$.

So assume $X = [1, 1, \ldots, \underset{x_k}{t}, 1, \ldots 1]^T$

$\therefore$ $t \to \infty \Rightarrow f^*(y) \to \infty$.

②

$$\therefore f^*(y) = \begin{cases} 0, & \text{if } y \preceq 0. \\ \infty, & \text{otherwise} \end{cases}$$

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