COMP8650 Assignment 3. Longfei Zhao u5976992.

Shortest path.

a)

A path starts at node 1:

$$x_{i1} = 0, \text{ for } i = 1, \dots, n$$
$$\sum_{j=1}^{n} x_{1j} = 1$$

A path ends at node n:

$$x_{nj} = 0, \text{ for } j = 1, \dots, n$$
$$\sum_{i=1}^{n} x_{in} = 1$$

b)
$$\sum_{j=1}^{n} x_{ij} \le 1, \text{ for } i = 1, \dots, n$$
$$\sum_{i=1}^{n} x_{ij} \le 1, \text{ for } j = 1, \dots, n$$

c)
$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ji}, \text{ for } i = 2, \dots, n-1$$

d)
$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = tr(C^T X)$$

e)
$$\begin{aligned}
\text{minimize} \quad & tr(C^T X) \\
\text{s.t.} \quad & x_{i1} = 0, \text{ for } i = 1, \dots, n \\
& \sum_{j=1}^{n} x_{1j} = 1 \\
& x_{nj} = 0, \text{ for } j = 1, \dots, n \\
& \sum_{i=1}^{n} x_{in} = 1 \\
& \sum_{j=1}^{n} x_{ij} \le 1, \text{ for } i = 1, \dots, n \\
& \sum_{i=1}^{n} x_{ij} \le 1, \text{ for } j = 1, \dots, n \\
& \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ji}, \text{ for } i = 2, \dots, n-1 \\
& X \preccurlyeq \text{edge} \\
& x_{ij} \in \{0, 1\}, \text{ for } i = 1, \dots n, \ j = 1, \dots, n
\end{aligned}$$

f). Yes, it's convex.

4.3

$\nabla f_0(x) = \frac{1}{2}(P+P^T)x + q = Px + q$

$x^* = (1, \frac{1}{2}, -1)$

$\Rightarrow \nabla f_0(x^*) = Px^* + q = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$\therefore \nabla f_0(x^*)^T(y-x) = [-1\ 0\ 2]\begin{bmatrix} y_1 - 1 \\ y_2 - \frac{1}{2} \\ y_3 + 1 \end{bmatrix} = 3 - y_1 + 2y_3$

$\because -1 \le y_i \le 1,\ i = 1,2,3$

$\therefore \nabla f_0(x^*)(y-x) \geq 0$

Therefore, x^* is optimal for optimization problem.

4.8

a) ① if $b \notin R(A)$.

The problem is infeasible, the optimal value $p^* = +\infty$

② if $b \in R(A)$ ~~and~~

Notice that if there exists a μ that $c^T = \mu^T A$.

$\Rightarrow c^T x = \mu^T A x = \mu^T b$. That means A is not full rank.

if such μ do not exist. ~~That means that the rank of c~~

so assume $c^T = \mu^T A + v^T \ (Av^T = 0)$

$\Rightarrow c^T x = (\mu^T A + v^T) \cdot x = \mu^T b + v^T x$

$\Rightarrow x = tv + x_0 \ (Ax_0 = b),\ t \in \mathbb{R})$

$\therefore c^T x = \mu^T b + t v^T v \ (t \in \mathbb{R})$

Therefore, the problem is unbounded.

$\therefore p^* = \begin{cases} +\infty, & b \notin R(A) \\ \mu^T b, & \text{if } c^T = \mu^T A \text{ for some } \mu. \\ -\infty, & \text{otherwise.} \end{cases}$

由 扫描全能王 扫描创建

c) $l \preceq x \preceq u$

$\Rightarrow x_i - l_i \geq 0$, for $i = 1, \dots, n$

$x_i - u_i \leq 0$, for $i = 1, \dots, n$.

Notice that $c^T x = \sum_{i=1}^n c_i x_i$

$\therefore$ if $c_i > 0$, then $x_i^* = l_i$

if $c_i = 0$, then $x_i^* = \forall x \in [l_i, u_i]$

if $c_i < 0$, then $x_i^* = u_i$

$\therefore p^* = \sum_{i=1}^n \left(\max\{c_i, 0\} \cdot l_i + \min\{c_i, 0\} \cdot u_i \right)$.

e) ~~Assume~~

we denote by $c_{[i]}$ the ith smallest component of c, i.e.

$c_{[1]} \leq c_{[2]} \leq \cdots \leq c_{[n]}$

$\therefore$ ① α is an integer

The optimal is easy to be ~~obser~~ observed.

assign $c_{[1]} = c_{[2]} = \cdots = c_{[\alpha]} = 1$,

other component of c be 0.

$\therefore p^* = \sum_{i=1}^{\alpha} c_{[i]}$.

② α is not an integer

the different is that the last weight is decimal.

$\therefore p^* = \sum_{i=1}^{\lfloor \alpha \rfloor} c_{[i]} + (\alpha - \lfloor \alpha \rfloor) \cdot c_{[\lceil \alpha \rceil]}$.

③ . $1^T x \leq \alpha$

$\therefore$ if $c_{[\alpha]} \leq 0$, $p^* = \sum_{i=1}^{\alpha} c_{[i]}$

if $c_{[\alpha]} > 0$, $p^* = \sum c_{[i]}$ where $c_{[i]} \leq 0$.

4.9. assume $y = Ax$. Therefore, the original is equivalent to.

$$\begin{aligned} \text{minimize} \quad & c^T A^{-1} y. \\ \text{s.t.} \quad & y \preceq b. \end{aligned}$$

$\therefore$ If $A^{-T} c \preceq 0$, $y^* = b \Rightarrow p^* = c^T A^{-1} b$.
otherwise, the problem is unbounded.

5.12. assume $y_i = b_i - a_i^T x$ for $i = 1, \cdots, m$
the problem is equivalent to.

$$\begin{aligned} \text{minimize} \quad & -\sum_{i=1}^m \log y_i \\ \text{s.t} \quad & \cancel{y - b + a_i^T x = 0} \quad y - b + Ax = 0 \end{aligned}$$

$\therefore L(x, y, \nu) = -\sum_{i=1}^m \log y_i + \nu^T (y - b + Ax)$

$\therefore$ dual function is

$$g(\nu) = \inf_{x, y} \left(-\sum_{i=1}^m \log y_i + \nu^T (y - b + Ax) \right)$$

$\nu^T A x$ is a function of x, which is unbounded below if $A^T \nu \neq 0$.
$\nu^T y$ is a function of y, which is unbounded below if $\nu \not\succ 0$.
If $A^T \nu = 0$ and $\nu \succ 0$,

$$\nabla_y L(x, y, \nu) = -\frac{1}{y_i} + \nu_i = 0$$

$$\Rightarrow \quad y_i = \frac{1}{\nu_i}, \text{ for } i = 1, \dots, m.$$

$$\therefore g(\nu) = \begin{cases} \sum_{i=1}^m \log \nu_i + m - b^T \nu & A^T \nu = 0, \nu \succ 0 \\ -\infty & \text{otherwise} \end{cases}$$

$\therefore$ the dual problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^m \log \nu_i + m - b^T \nu \\ \text{s.t.} \quad & A^T \nu = 0 \end{aligned}$$

5.31.

$\because$ it's convex problem

$\therefore$ $f_i(x^*) + \nabla f_i(x^*)^T (x - x^*) \le f_i(x) \le 0$

$\because$ $\lambda_i^* \geq 0$

$\therefore$ $\lambda_i^* (f_i(x^*) + \nabla f_i(x^*)^T (x - x^*)) \le 0$

$\because$ $\lambda_i^* f_i(x^*) = 0$

$\therefore$ $\lambda_i^* \nabla f_i(x^*)^T (x - x^*) \le 0$

$\therefore$ $\sum_{i=1}^m \lambda_i^* \nabla f_i(x^*)^T (x - x^*) \le 0$

$\because$ $\nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) = 0$

$\therefore$ $-\nabla f_0(x^*) (x - x^*) \le 0$

$\therefore$ $\nabla f_0(x^*) (x - x^*) \geq 0.$