COMP4680/COMP8650: Advanced Topics in SML

Assignment #1: Background

Due: 5:00pm on Friday 3 August, 2018.
Submit as a single PDF file via Wattle.

1. **Determinant of a transpose (10 marks).** Prove that $\det A^T = \det A$.

2. **Identity matrix (10 marks).** Show that for any n, the $n \times n$ identity matrix has determinant equal to one.

3. **Triangular matrices (10 marks).** Show that for upper or lower triangular matrices the eigenvalues are equal to the diagonal elements.

4. **Non-negative matrices (10 marks).** A non-negative matrix A is a matrix whose elements are all greater than or equal to zero, i.e., $a_{ij} \geq 0$. Show that a non-negative matrix can have negative eigenvalues. That it, find a non-negative matrix A with at least one negative eigenvalue.

5. **Gradients (30 marks).** Compute the gradients of the following functions:

   (a) $f(x) = a^T x + b$ for $a \in \mathbb{R}^n, b \in \mathbb{R}$

   (b) $f(x) = \frac{1}{2} x^T P x + q^T x + r$ for $P \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n, r \in \mathbb{R}$

   (c) $f(x) = \frac{1}{2} x^T P x$ for $P = P^T \in \mathbb{R}^{n \times n}$

   (d) $f(x) = \dfrac{\exp(a^T x + b)}{1 + \exp(a^T x + b)}$ for $a \in \mathbb{R}^n, b \in \mathbb{R}$

   (e) $f(x) = \dfrac{1}{1 + \exp(-g(x))}$ in terms of $f(x)$ and $\nabla_x g(x)$

6. **Trace. (10 marks).** Show that the trace operator is a valid inner product for the vector space of real $m \times n$ matrices. That is, for $X, Y \in \mathbb{R}^{m \times n}$, show that $\langle X, Y \rangle = \mathbf{tr}\left(X^T Y\right)$ satisfies the properties of an inner product.

7. **Anti-symmetric matrix (10 marks).** Consider an arbitrary anti-symmetric matrix $A \in \mathbb{R}^{n \times n}$, i.e., $A^T = -A$. Prove that $x^T A x = 0$ for all $x \in \mathbb{R}^n$.

8. **Random variables (10 marks).** Let X and Y be two zero-mean random variables. Prove

$$(\mathbf{Cov}(X, Y))^2 \leq \mathbf{Var}(X)\mathbf{Var}(Y)$$

   where $\mathbf{Cov}(X, Y)$ is the covariance between X and Y and $\mathbf{Var}(X)$ is the variance of X.