Homework 2

- Collaboration: Homeworks are individual work. Visit here for more info on academic integrity.
- Formatting: You are supposed to upload two files (`hw2_writeup.pdf` and `hw2_code.py`). For the pdf file, You can produce the file however you like (e.g. LaTeX, Microsoft Word, scanner), as long as it is readable.

Create a Python file `hw2_code.py` and do the following two problems. Make sure you import the necessary python libraries. Create a pdf file `hw2_writeup.pdf`, include the answer in this pdf file whenever you are asked to do so. To include computational outputs (including matrices, eigenvectors and eigenvalues etc.), round the outputs to only two decimals in `hw2_writeup.pdf`. You can do this manually or using the NumPy `round()` function.

Problem 1 (Eigenvectors and Eigenvalues)

(a)[**2pt**] (a) Define a 3×3 real symmetric matrix `A` that is also positive definite (Ideally define `A` as a NumPy array). Please avoid making a diagonal matrix. Use `print` to print the matrix `A` and check it is symmetric. Include the matrix in your `hw2_writeup.pdf`.

Hint: What can we say about the diagonal entries of A

(b)[**4pt**] Use a Python library (e.g. `numpy.linalg`) to find the eigenvalues and eigenvectors of this matrix. Use `print` to print your eigenvalues and the corresponding eigenvectors. Include your eigenvalues and the corresponding eigenvectors in `hw2_writeup.pdf`. (For each eigenvector, you need to specify its corresponding eigenvalue)

(c)[**1pt**] Conclude from your answer to part (b) that the matrix you define is indeed positive definite. Do this part in `hw2_writeup.pdf` only.

Problem 2 (Cholesky Decomposition)

For the real, symmetric, positive-definite matrix `A` you defined in problem 1, we can find its Cholesky Decomposition $A = L \cdot L^T$, where L is lower triangular with positive diagonal entries.

(a)[**3pt**] Use a Python library (e.g. `numpy.linalg`) to compute the Cholesky Decomposition of `A`. Use `print` to print the matrix `L` . Include `L` in `hw2_writeup.pdf`.

Now suppose we have some uncorrelated sample data vector `u` and we treat the matrix `A` as the covariance matrix of `u`. Applying the martrix `L` to `u` produces a correlated vector Lu with the covariance matrix (approximately) `A`.

(b)[**2pt**] Sample $100,000$ points for each of 3 standard normal variables and name the data vector by `u`, i.e. `u` has NumPy shape $(3, 100000)$. Compute the sample covariance matrix cov_u of u and convince yourself that cov_u is approximately the identity matrix. Include cov_u in hw2_writeup.pdf.

(c)[**3pt**] Apply L to u to get the correlated data v, i.e. $v = Lu$. Comopute the sample covariance matrix cov_v of v and convince yourself cov_v is approxiamtely A. Include cov_v in hw2_writeup.pdf.

Problem 3 (Singular Value Decomposition)

Singular Value Decomposition is a generalization of Eigen Decomposition to arbitrary matrices of any size.

Definition: A **singular value decomposition** of a matrix $A_{m,n}$ is a factorization

$$A = U\Sigma V^T$$

where:

- Σ is a $(m \times n)$ diagonal matrix whose i^{th} entry is the i^{th} singular value of A for $i = 1, ..., r$.
- V is a $(n \times n)$ orthogonal matrix. The columns of V are orthonormal eigenvectors $v_1, ..., v_n$ of $A^T A$, where $A^T A v_i = \sigma_i^2 v_i$.
- U is an $(m \times m)$ orthogonal matrix. The i^{th} column of U is $\sigma_i^{-1} A v_i$. For $i > r$, the i^{th} column of U can be obtained by arbitrarily extending to an orthonormal basis for R^m.

(a)[**1pt**] Let $\mathbf{v} \in \mathbb{R}_n$ be a real-valued vector. Write expressions for the dot product $\mathbf{v}^T\mathbf{v}$ and the matrix product $\mathbf{v}\mathbf{v}^T$.

(b)[**1pt**] Give a one sentence answer for why the rank of $\mathbf{v}\mathbf{v}^T$ is 1.

(c)[**2pts**] Recall the SVD of A can be written as $U\Sigma V^T = \sum_{i=1}^r \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T + \sum_{i=r+1}^n 0\boldsymbol{u}\boldsymbol{v}^T$. Show that the rank of each $\boldsymbol{u}_i \boldsymbol{v}_i^T$ is 1.

(d)[**3pts**] Let $A_{(m,n)}$ be a matrix. Prove that $A^T A$ is positive semi-definite.

Hint: Consider $\mathbf{z}^T A^T A \mathbf{z}$ for any $\mathbf{z} \in \mathbb{R}^n$

(e)[**1pt**] (Hard) Let $A_{(m,n)}$ be a matrix. Prove that the rank of A is equal to the number of non-zero singular values of A. You may use, without proof, the fact that $A^T A$ and A have the same null space (kernel).

Hint: Consider using the Rank-nullity theorem.