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APM496 Assignment 2

Problem 1 (Eigenvectors and Eigenvalues)

(a)

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My PD Matrix is:
 [[ 2 -1  0]
 [-1  2 -1]
 [ 0 -1  2]]
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(b)

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The eigenvalue of the matrix is:  3.41 ,  2.0 ,  0.59

The correspoding eigenvoector is:  [-0.5  -0.71  0.5 ] ,  [0.71 0.   0.71] ,  [-0.5   0.71  0.5 ]
```

(c)

Since we see that all the eigenvalue of the PD Matrix is positive, 3.41, 2, 0.59 > 0, that is the Matrix is indeed positive definite.

Problem 2 (Cholesky Decomposition)

(a)

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The Cholesky Decomposition of matrix A is L =
 [[ 1.41  0.    0.  ]
 [-0.71  1.22  0.  ]
 [ 0.   -0.82  1.15]]

Lu = L*v*A*v^(-1)=
 [[ 2.82842712e+00  1.41421356e+00 -6.66133815e-16]
 [-1.89468691e-01  1.74238296e+00 -1.22474487e+00]
 [-8.16496581e-01 -2.78769370e+00  3.12589766e+00]]
```

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(b)

```
con_u is :
 [[ 0.99099442 -0.00247663  0.00252653]
 [-0.00247663  1.00720571 -0.00277093]
 [ 0.00252653 -0.00277093  1.00021405]]

Covariance matrix of 100,000 sampled data from normal distribution is close to Identity matrix:  True
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Since the diagonal of the covariance matrix is the variance of the sampled data, ie $\sigma_{nn} = var(x_n)$, and all the data are sampled from standard normal distribution, that is, all the variance of $x_{nn} = 1$, and the diagonal of the covariance matrix is 1. Besides, since $\sigma_{nm} = cov(x_n,x_m)$, and all three X are sampled independently, so when $m \neq n$, $cov(x_n,x_m) = 0$, that is, all the matrix besides the diagonal approaches 0 when the sample size goes large enough.
In conclusion, the covariance matrix is approximately the identity matrix.

(c)

```
con_v is :
 [[ 1.99833757e+00 -1.00360431e+00  3.29794531e-04]
 [-1.00360431e+00  2.01316036e+00 -1.00445005e+00]
 [ 3.29794531e-04 -1.00445005e+00  1.99137927e+00]]

Covariance matrix of Lu is close to A:  True
```

This is because that L is a Lower Triangle Matrix, that is, all the columns of L is linearly independent. Then, cov(Lu) = cov(u) = A.

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下午8:07 3月29日周日 54%

(a)[1pt] Let $\mathbf{v} \in \mathbb{R}_n$ be a real-valued vector. Write expressions for the dot product $\mathbf{v}^T\mathbf{v}$ and the matrix product $\mathbf{v}\mathbf{v}^T$.

let $V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$. then $V^TV = V_1^2 + V_2^2 + \cdots + V_n^2$

$$VV^T = \left[\begin{array}{c|c|c|c} V_1V & V_2V & V_3V & \cdots V_nV \end{array}\right]$$

(b)[1pt] Give a one sentence answer for why the rank of $\mathbf{v}\mathbf{v}^T$ is 1.

since rank is the number of linearly independent columns of matrix, we see from (a) that all columns of VV^T is linearly dependent, thus, rank of VV^T is 1

(c)[2pts] Recall the SVD of A can be written as $U\Sigma V^T = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T + \sum_{i=r+1}^{n} 0\boldsymbol{u}\boldsymbol{v}^T$. Show that the rank of each $\boldsymbol{u}_i \boldsymbol{v}_i^T$ is 1.

let $U_i = \begin{bmatrix} U_{i1} \\ U_{i2} \\ \vdots \\ U_{in} \end{bmatrix}$ $V_i = \begin{bmatrix} V_{i1} \\ V_{i2} \\ \vdots \\ V_{im} \end{bmatrix}$

$$U_iV_i^T = \left[\begin{array}{c|c|c|c} V_{i1}U_i & V_{i2}U_i & \cdots & V_{in}U_i \end{array}\right]$$

from (b), we see that $U_iV_i^T$ is rank 1. (Note that $U_i \neq \vec{0}$, then $\text{rank}(U_iV_i^T) \neq 0$, $\text{rank}(U_iV_i^T) = 1$)

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(d)[3pts] Let $A_{(m,n)}$ be a matrix. Prove that $A^T A$ is positive semi-definite.

let v be a n×1 vector. WTS: $v^T A^T A v \geq 0$

$v^T A^T A v = (Av)^T (Av)$

let $y = Av$

then $v^T A^T A v = y^T y \geq 0$

that is, $A^T A$ is PSD. ∎

(e)[1pt] (Hard) Let $A_{(m,n)}$ be a matrix. Prove that the rank of A is equal to the number of non-zero singular values of A. You may use, without proof, the fact that $A^T A$ and A have the same null space (kernel).

Hint: Consider using the Rank-nullity theorem.

Since $\text{rank}(A) + \dim(\text{Null}(A)) = n$.

$\text{rank}(A) + \dim(\text{Null}(A)) = n$

$\dim(\text{Null}(A)) = \dim(\text{Null}(A^T A))$

then we have $\text{rank}(A) = \text{rank}(A^T A) = n - \dim(\text{Null}(A^T A))$

$\dim(\text{Null}(A)) = \dim(\text{Null}(V \Sigma^T U^T U \Sigma V^T))$

$= \dim(\text{Null}(V \Lambda V^T))$ (where $\Lambda = \Sigma^T \Sigma$)

$=$ # of singular in A

thus $\text{rank}(A) =$ # of non-singular in A.

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