APM466 A2

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April 2, 2020

Questions - 100 points

1. *(40 points)* Suppose that company X has four states of solvency: good, bad, crisis, and default. Suppose also that the one year transition (between solvency states) probability matrix is given by:

$$P = \begin{array}{c|cccc} \text{state} & \text{good} & \text{bad} & \text{crisis} & \text{default} \\ \hline \text{good} & 8/10 & 1/10 & 1/10 & 0 \\ \text{bad} & 1/10 & 5/10 & 2/10 & 2/10 \\ \text{crisis} & 1/10 & 3/10 & 3/10 & 3/10 \\ \text{default} & 0 & 0 & 0 & 1 \end{array}$$

For the following questions, feel free to use a computer to aid your calculations. For part a)&b), you must state your final answer with a small explanation (explicit calculations discouraged in your report). For parts c)& d), a formal proof is not needed, just a 1 or 2 sentence explanation.

(a) *(10 points)* What is the two year transition probability matrix?

Answer: The two year transition probability matrix is:

$$P^2 = \begin{pmatrix} 0.66 & 0.16 & 0.13 & 0.15 \\ 0.15 & 0.32 & 0.17 & 0.36 \\ 0.14 & 0.25 & 0.16 & 0.45 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

because: Using python to compute the result, code in appendix.

(b) *(10 points)* What is the probability that if company X is currently in a "crisis" solvency state, they will default within the next month?

Answer: If company X is currently in a "crisis" solvency state, the probability that they will default within the next month is 0.0376.

because: By using Python to compute $P^{\frac{1}{12}} = \begin{pmatrix} 0.9802 & 0.0088 & 0.0142 & 0 \\ 0.0106 & 0.9287 & 0.0444 & 0.0163 \\ 0.0124 & 0.0675 & 0.8826 & 0.0376 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we see wanted is 0.0376.

(c) *(10 points)* What is $\lim_{t\to\infty} P^t$?

Answer:

$\lim_{t\to\infty} P^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ because: Since we the state of solvency can only be worse, and default state has 100% to stay in the same state. We can expected after infinite time, the company start with any state will go to default.

(d) *(10 points)* If $t \in \mathbb{N}$, $(t < \infty)$, given that the company X has not yet defaulted, is it guaranteed (/with probability 1) that company X will default within t years?
(Hint: Either use induction or show that $\exists t < \infty$ for which $P^t_{ij} = 0 \; \forall j \neq 4, P^t_{ij} = 1$ if $j = 4$.)

Answer: No, because: The question is asking given the company is not in "default" state, the probability of it move to "non-default" rate after finite time is 0.
This is not correct, since the probability of "non-default" state to another "non-default" state is not 0 in Markov Transition Matrix in each period. That is, with in finite n years, the probability of a "non-default" state remains "non-default" state in 0-yr to 1 yr, 1yr to 2 yr ... n-1 yr to n yr, is always greater than 0.
ie, with finite t, we have $P^t \neq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. *(40 points)* Assume that Germany's bonds are risk-free and Italy's bonds are risk-prone, and that each country issues zero coupon bonds with a face value of 1. We denote a German bond with an outstanding term of i years simply by its current price P_i^G, and an Italian bond with outstanding term of i years also simply by P_i^I. Finally, assume everything henceforth is priced using continuous discounting, and zero recovery under default.

(a) *(10 points)* Given $\{P_1^G, \ldots, P_n^G\}$ and $\{P_1^I, \ldots, P_n^I\}$, derive a closed form formula for the credit spread, h_i, at time $i \in \{1, \ldots, n\}$ for Italy in terms of i, P_i^G, and P_i^I.

Answer: Since Both Bonds are zero coupon with face value of 1, then we have
$P_i^G = 1 \times e^{-r_i^G \cdot i}$, $P_i^I = 1 \times e^{-r_i^I \cdot i}$
taking log on both side, we got $r_i^G = \frac{-lnP_i^G}{i}; r_i^I = \frac{-lnP_i^I}{i}$
Since Italy's Bond is risky-prone, then we have $P_k^I = 1 \times e^{-r_i^I} \cdot i = 1 \times e^{-(r_i^G + h_i) \cdot i)}$
then, $h_i = r_i^I - r_i^G = \frac{lnP_i^G}{i} - \frac{lnP_i^I}{i}$
Thus,

$$h_i = \frac{ln(\frac{P_i^G}{P_i^I})}{i}$$

(b) *(10 points)* Under a two state markov chain model (solvency and default), write Italy's ith-year probability transition matrix, P^i, in terms of just i and h_i.

Answer: Let q_i be the probability of Italy is solvent at ith year.
Then we have $P_i^I = 1 \times e^{-r_i^G \cdot i} \cdot q_i$
By 2a, we know $P_i^I = 1 \times e^{-(r_i^G + h_i) \cdot i} = 1 \cdot e^{-r_i^G \cdot i} \cdot e^{-h_i \cdot i}$, where r_i^G is the risk-free yield for i years.
Then we can see that $q_i = e^{-h_i \cdot i}$, and $1 - q_i = 1 - e^{-h_i \cdot i}$
Therefore we have:

$$P^i = \begin{pmatrix} e^{-h_i \cdot i} & 1 - e^{-h_i \cdot i} \\ 0 & 1 \end{pmatrix}$$

(c) *(10 points)* If the Italian government issues a one-off asset, A, that pays $C_i, i = 1, \ldots, n$, at time i, find the price of this asset in terms of $\{1, \ldots, n\}$, $\{h_1, \ldots, h_n\}$, $\{P_1^G, \ldots, P_n^G\}$, and $\{C_1, \ldots, C_n\}$.

Answer: Let A be the price of the asset. r_i^I be the risky yield of i years of Italy's bond. then, by discounting each payment using r_i^I, we have A $= \sum_{i=1}^{n} C_i e^{-r_i^I \cdot i}$
We also have $h_i = r_i^I - r_i^G$, where $r_i^G = -\frac{lnP_i^G}{i}$
then $r_i^I = r_i^G + h_i = -\frac{lnP_i^G}{i} + h_i$
get this back to the summation formula of A we see that:

$$A = \sum_{i=1}^{n} C_i e^{-r_i^I \cdot i} = \sum_{i=1}^{n} C_i P_i^G e^{-h_i \cdot i}$$

(d) *(10 points)* First find $\partial_{h_i} A$, then use this to say what would happen to the price of A given Italy's probability of default (by any time $i \geq 1$) increases.

Answer: Taking the derivative in h_i, we see that:

$$\partial_{h_i} A = -i \cdot C_i P_i^G e^{-h_i \cdot i}.$$

And thus since exponential function is always positive, $p_i^G, C_i \geq 0$ as well, then we see that $\partial_{h_i} A < 0$. We see that if Italy's probability of default increases, A's price will decrease.

3. *(20 points)* List 4 simplifications (I.e., assumptions that might not be true in real life) that are made under Merton's Credit Risk Model.

Max 1 sentence per assumption.

(a) *Assumption 1: No Taxes or Transaction costs.*

(b) *Assumption 2: Risk-free is constant.*

(c) *Assumption 3: The value of a firm's assets A_t follows a geometric Brownian motion, such that $dA_t =_t dt + \sigma A_t dW_t$*

(d) *Assumption 4:Volatility is constant*

Appendix

Supplementary codes linked in GitHub.
https://github.com/JinhanM/Credit-Risk-APM466-Assignment-2