MAT1856/APM466: Mathematical Finance **Winter 2020**

Assignment #2: Credit Risk

Professor: Luis Seco, TA: Jonathan Mostovoy

Due: April 5, 2020, at 10PM EST. Submissions accepted up to 3 days late at a 5% penalty a day. No submissions accepted after April 8, 2020.
Please bring any questions about this assignment to your TA's, Jonathan's, weekly (virtual) office hour on the Facebook group.

Expectations

1. Please have your final report typeset using LaTeX **and** using this template: `https://www.overleaf.com/read/wgswhrdwmycj`. Please also structure your answers in line with the mock answers provided.
2. You may, and are encouraged, to discuss how to do these questions with your peers. However, your write-up must be done individually, and the sharing of your write-up before April 13th is prohibited.

Additional Notes: Marks will be awarded for each question as either full-, half-, or zero-marks according to if the question was answered with a few small mistakes, substantial mistakes but fundamental idea still correct, or fundamental idea wrong / no answer respectively. -10 marks if not typeset in LaTeXusing the template provided as intended.

2 Questions- 100 points

1. *(40 points)* Suppose that company X has four states of solvency: good, bad, crisis, and default. Suppose also that the one year transition (between solvency states) probability matrix is given by:

$$P = \begin{array}{c|cccc} \text{state} & \text{good} & \text{bad} & \text{crisis} & \text{default} \\ \hline \text{good} & 8/10 & 1/10 & 1/10 & 0 \\ \text{bad} & 1/10 & 5/10 & 2/10 & 2/10 \\ \text{crisis} & 1/10 & 3/10 & 3/10 & 3/10 \\ \text{default} & 0 & 0 & 0 & 1 \end{array}$$

   For the following questions, feel free to use a computer to aid your calculations. For part a)&b), you must state your final answer with a small explanation (explicit calculations discouraged in your report). For parts c)& d), a formal proof is not needed, just a 1 or 2 sentence explanation.

   (a) *(10 points)* What is the two year transition probability matrix?

   (b) *(10 points)* What is the probability that if company X is currently in a "crisis" solvency state, they will default within the next month?

   (c) *(10 points)* What is $\lim_{t\to\infty} P^t$?

   (d) *(10 points)* If $t \in \mathbb{N}$, $(t < \infty)$, given that the company X has not yet defaulted, is it guaranteed (/with probability 1) that company X will default within t years?
   (Hint: Either use induction or show that $\exists t < \infty$ for which $P^t_{ij} = 0\ \forall j \neq 4, P^t_{ij} = 1$ if $j = 4$.)

2. *(40 points)* Assume that Germany's bonds are risk-free and Italy's bonds are risk-prone, and that each country issues zero coupon bonds with a face value of 1. We denote a German bond with an outstanding term of i years simply by its current price P^G_i, and an Italian bond with outstanding term of i years also simply by P^I_i. Finally, assume everything henceforth is priced using continuous discounting, and zero recovery under default.

(a) *(10 points)* Given $\{P_1^G, \ldots, P_n^G\}$ and $\{P_1^I, \ldots, P_n^I\}$, derive a closed form formula for the credit spread, h_i, at time $i \in \{1, \ldots, n\}$ for Italy in terms of i, P_i^G, and P_i^I.

(b) *(10 points)* Under a two state markov chain model (solvency and default), write Italy's ith-year probability transition matrix, P^i, in terms of just i and h_i.

(c) *(10 points)* If the Italian government issues a one-off asset, A, that pays $C_i, i = 1, \ldots, n$, at time i, find the price of this asset in terms of $\{1, \ldots, n\}$, $\{h_1, \ldots, h_n\}$, and $\{P_1^G, \ldots, P_n^G\}$.

(d) *(10 points)* First find $\partial_{h_i} A$, then use this to say what would happen to the price of A given Italy's probability of default increases.

3. *(20 points)* List 4 simplifications (I.e., assumptions that might not be true in real life) that are made under Merton's Credit Risk Model.

   Max 1 sentence per simplification.

	state	good	bad	crisis	default
	good	8/10	1/10	1/10	0
$P =$	bad	1/10	5/10	2/10	2/10
	crisis	1/10	3/10	3/10	3/10
	default	0	0	0	1

(a) *(10 points)* What is the two year transition probability matrix?

By python, we know $P^2 = \begin{bmatrix} 0.66 & 0.16 & 0.13 & 0.05 \\ 0.15 & 0.32 & 0.17 & 0.36 \\ 0.14 & 0.25 & 0.16 & 0.45 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

P^2 is wanted

(b) *(10 points)* What is the probability that if company X is currently in a "crisis" solvency state, they will default within the next month?

By python, we know that $P^{\frac{1}{12}} = \begin{bmatrix} 0.9802 & 0.0088 & 0.0142 & 0 \\ 0.0106 & 0.9287 & 0.0444 & 0.0163 \\ 0.0124 & 0.0675 & 0.8826 & \underline{0.0376} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

wanted $= 0.0376$

(c) *(10 points)* What is $\lim_{t\to\infty} P^t$?

By python, we see when $t = 1000$, we have $P^{1000} = \begin{bmatrix} 1\times10^{-52} & 8\times10^{-53} & 5\times10^{-53} & 1 \\ 7\times10^{-53} & 3\times10^{-53} & 2\times10^{-53} & 1 \\ 6\times10^{-53} & 3\times10^{-53} & 2\times10^{-53} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

then we can say $\lim_{t\to\infty} P^t = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) *(10 points)* If $t \in \mathbb{N}$, $(t < \infty)$, given that the company X has not yet defaulted, is it guaranteed (/with probability 1) that company X will default within t years?
(Hint: Either use induction or show that $\exists t < \infty$ for which $P_{ij}^t = 0\ \forall j \neq 4, P_{ij}^t = 1$ if $j = 4$.)

No. proof by induction.

let $P(t)$ be: " $t \in \mathbb{N}, t < \infty$. $P_{ij}^t \neq 0\ \forall j \neq 4 \wedge P_{ij}^t \neq 1$ if $j=4$ "

Base Case. $t=1$.

$P(t) = P$. we see in P, we have $\forall j \neq 4, P_{ij} \neq 0, \wedge\ j=4, P_{ij} = 1$.

thus, $P(1)$ holds.

Inductive step:

let $k \in \mathbb{N}, k < \infty$. Assume $P(k)$ holds, ... (IH)

WTS: $P(k+1)$ holds.

$$P^{k+1} = P^k \cdot P = \begin{bmatrix} P_{11}^k & P_{12}^k & P_{13}^k & P_{14}^k \\ P_{21}^k & P_{22}^k & P_{23}^k & P_{24}^k \\ P_{31}^k & P_{32}^k & P_{33}^k & P_{34}^k \\ P_{41}^k & P_{42}^k & P_{43}^k & P_{44}^k \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.8P_{11}+0.1P_{12}+0.1P_{13} & 0.1P_{11}+0.5P_{12}+0.3P_{13} & 0.1P_{11}+0.2P_{12}+0.3P_{13} & 0.2P_{12}+0.3P_{13}+P_{14} \\ 0.8P_{21}+0.1P_{22}+0.1P_{23} & 0.1P_{21}+0.5P_{22}+0.3P_{23} & 0.1P_{21}+0.2P_{22}+0.3P_{23} & 0.2P_{22}+0.3P_{23}+P_{24} \\ 0.8P_{31}+0.1P_{22}+0.1P_{23} & 0.1P_{31}+0.5P_{32}+0.3P_{33} & 0.1P_{31}+0.2P_{32}+0.3P_{33} & 0.2P_{32}+0.3P_{33}+P_{34} \\ 0.8P_{41}+0.1P_{42}+0.1P_{43} & 0.1P_{41}+0.5P_{42}+0.3P_{43} & 0.1P_{41}+0.2P_{42}+0.3P_{43} & 0.2P_{42}+0.3P_{43}+P_{44} \end{bmatrix}$$

since we know $\forall j \neq 4, P_{ij}^k \neq 0$. $P_{ij} \geq 0$, then $\forall j \neq 4, P_{ij}^{k+1} \neq 0$.

since P^{k+1}, P^k, P are probability matrix, then $\sum_{j=1}^{4} P_{ij}^{k+1} = 1$.

then since $\forall j \neq 4, P_{ij}^{k+1} \neq 0$. $P_{ij}^{k+1} \geq 0$; then $P_{i4}^{k+1} = 1 - \sum_{j=1}^{3} P_{ij} < 1$.

that is, $P(k+1)$ holds.

By principle of simple induction, we see that $\nexists t < \infty, t \in \mathbb{N}$, that $P_{ij}^t = 0, \forall j \neq 4$, and $P_{ij}^t = 1$. if $j=4$.