Vectors

1. (10 points) Given the following vectors:

$$p = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix}, \qquad r = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Determine the following **by hand** (remember to show your work):

a. $p + 2q$

b. $p \cdot r$ and $r \cdot p$ where "$\cdot$" denotes the dot product

a. $p + 2q = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -4 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 15 \end{bmatrix}$

b. $p \cdot r = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = -5 + 2 + 3 = 0$

$r \cdot p = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = -5 + 2 + 3 = 0$

c. $q \times r$ and $r \times q$ where "$\times$" denotes the cross product

d. $||p||$, and $||q||$ where $|| \cdot ||$ denotes the Euclidean norm of a vector

e. distance between the tips of p and q

$q \times r = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -2 & 7 \\ -1 & -2 & 3 \end{bmatrix} = \vec{i} \cdot \left((-2)\cdot(3) - (7)\cdot(-2) \right)$

$- \vec{j} \cdot \left[(-4)(3) - (7)(-1) \right]$

$+ \vec{k} \cdot \left[(-4)\cdot(-2) - (-2)\cdot(-1) \right]$

$= \vec{i}(8) - \vec{j}(-5) + \vec{k}(6)$

$= \begin{bmatrix} 8 \\ 5 \\ 6 \end{bmatrix}$

$$r \times q = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 3 \\ -4 & -2 & 7 \end{bmatrix} = \vec{i} \cdot [(-1)(-2)-(-2)(-4)] - \vec{j}[(-1)(7)-(3)(-4)] + \vec{k}[(-1)(-2)-(-2)(-4)]$$

$$= \vec{i} \cdot (-8) - \vec{j} (5) + \vec{k} \cdot (-6) = \begin{bmatrix} -8 \\ -5 \\ -6 \end{bmatrix}$$

d. $$\|p\| = \sqrt{5^2+1^2+1^2} = 3\sqrt{3}$$

$$\|q\| = \sqrt{(-4)^2+(-2)^2+7^2} = \sqrt{69}$$

e: $$d = \sqrt{(5-(-4))^2 + (-1-(-2))^2 + (1-7)^2}$$

$$= \sqrt{9^2 + 1^2 + 6^2} = \sqrt{118}$$

2. (10 points) Find all k such that $p = \begin{bmatrix} -2 \\ 1 \\ -k \end{bmatrix}$ and $q = \begin{bmatrix} 2 \\ -3k \\ -k \end{bmatrix}$ are orthogonal **by hand**.

$$\vec{p} \cdot \vec{q} = 0 = -2 \cdot 2 + (-3k) \cdot 1 + (-k)(-k)$$

$$= k^2 - 3k - 4 = (k+1)(k-4)$$

so $k_1 = -1 \quad k_2 = 4$

$k_1 = -1$

$$\vec{p}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{q}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \vec{p}_1 \cdot \vec{q}_1 = -4+3+1 = 0$$

$k_2 = 4$

$$\vec{p}_2 = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} \quad \vec{q}_2 = \begin{bmatrix} 2 \\ -12 \\ -4 \end{bmatrix} \quad \vec{p}_2 \cdot \vec{q}_2 = -4-12+16 = 0$$

3. (5 points) Partition the following matrix into submatrices (i.e. find W, X, Y, and Z) **by hand**:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where $W \in \mathbb{R}^{2\times 1}$ and $Z \in \mathbb{R}^{2\times 3}$.

$W \in \mathbb{R}^{2\times 1}$ $\quad Z \in \mathbb{R}^{2\times 3}$

So: $\quad X \in \mathbb{R}^{2\times 3} \quad Y \in \mathbb{R}^{2\times 1}$

$$\Rightarrow W = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad X = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$$

$$Y = \begin{bmatrix} 9 \\ 13 \end{bmatrix} \quad Z = \begin{bmatrix} 10 & 11 & 12 \\ 14 & 15 & 16 \end{bmatrix}$$

4. (10 points) Perform the following matrix multiplication **by hand**:

$$\begin{bmatrix} 3 & -1 & -3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbb{R}^{2\times 3} \cdot \mathbb{R}^{3\times 3} \cdot \mathbb{R}^{3\times 1} = \mathbb{R}^{2\times 1}$$

$$= \begin{bmatrix} 3\times 0+(-1)\times(-1)+(-3)\times 0, & 3\times 1+(-1)\cdot 0+(-3)(-2), & 3\times 2+(-1)(-1)+(-3)(-1) \\ -1\cdot 0+0\cdot(-1)+2\cdot 0, & -1\cdot 1+0\cdot 0+2\cdot(-2), & -1\cdot 2+(0)(-1)+2\cdot(-1) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 & 10 \\ 0 & -5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2+9\times 1+10\times(-2) \\ 0\times 2+1\times(-5)+(-4)\times(-2) \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$$