ECE6703J

Computer-Aided Design of Integrated Circuits

Multi-level Logic Synthesis:
Factor Operation

Outline

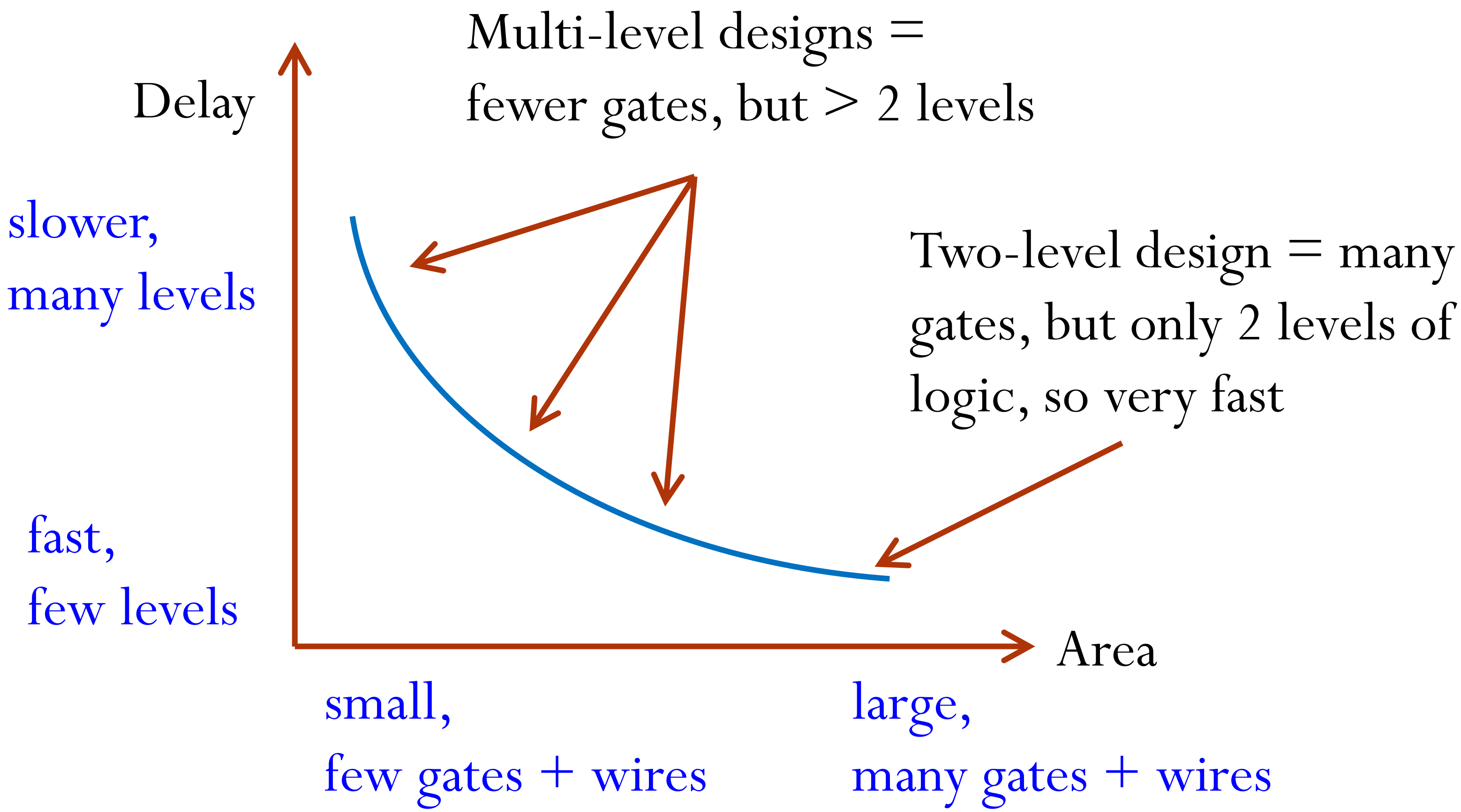
- Introduction to Multi-Level Logic Synthesis

- Factor operation
  - Algebraic Division
  - Kernels
  - Obtaining Kernels

Why Multi-level Logic?

- Two-level forms are too **restrictive**
- It has small delay but large area
  - **Area** = gates + literals (wires), i.e., things that take space on a chip
  - **Delay** = maximum levels of logic gates

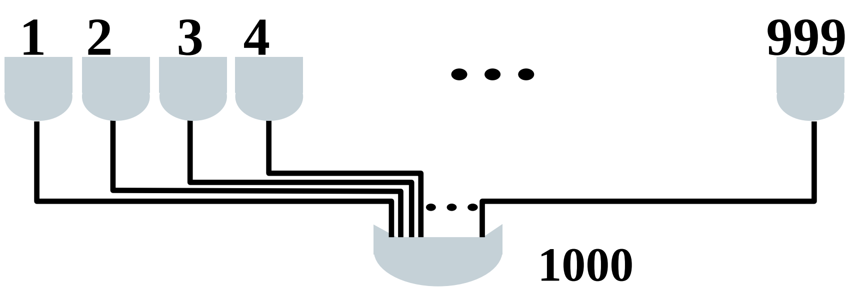
Area versus Delay Tradeoff



Why Multi-level Logic?

- **<u>Rarely</u>** see 2-level designs for really big things…
  - We use 2-level logic mostly for **components** of bigger things.
  - Even small things routinely done as **multi-level**.

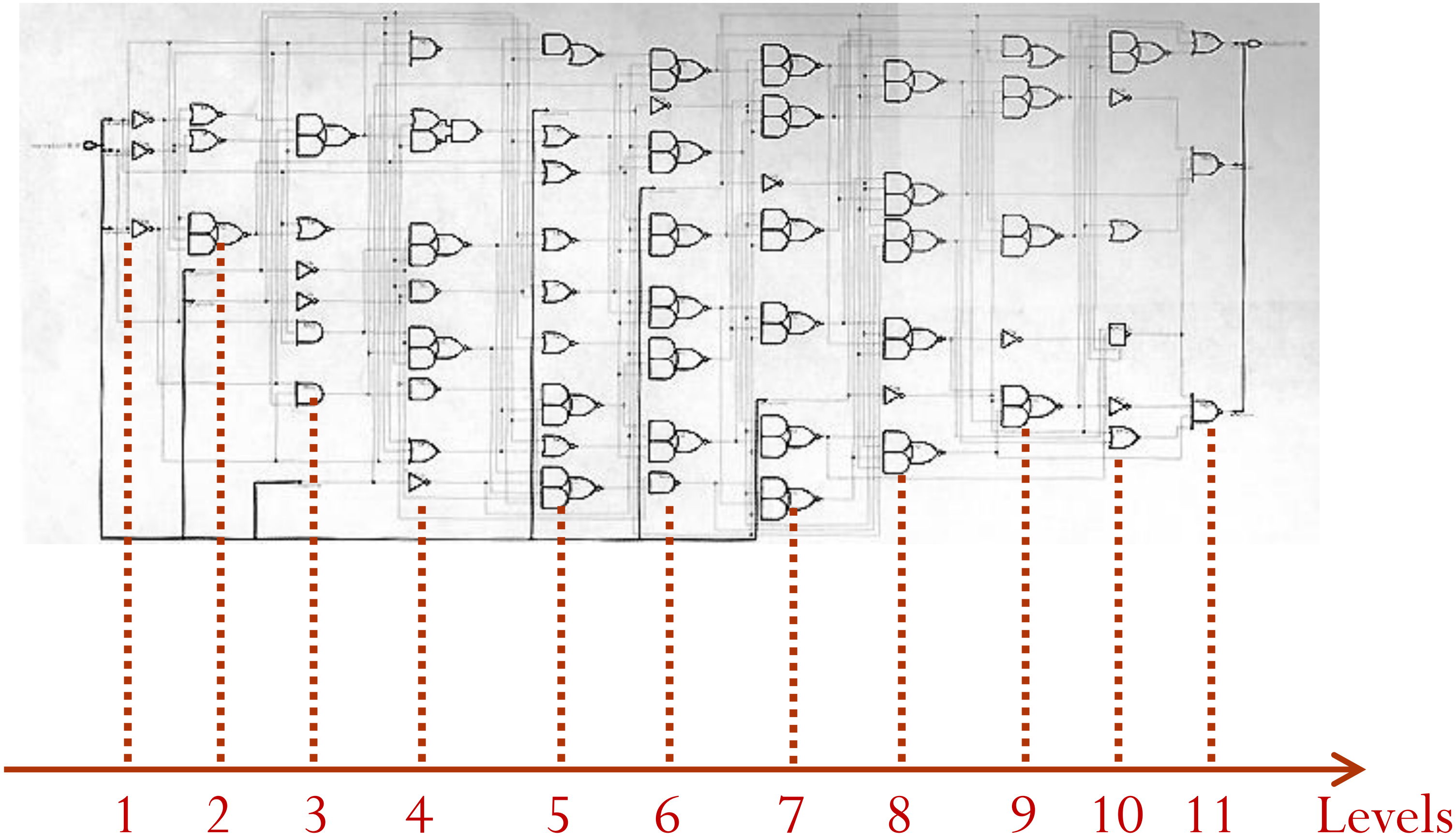
- What does a 2-level design with 1000 gates look like?



This is just **NOT** going to be the preferred logic network structure…

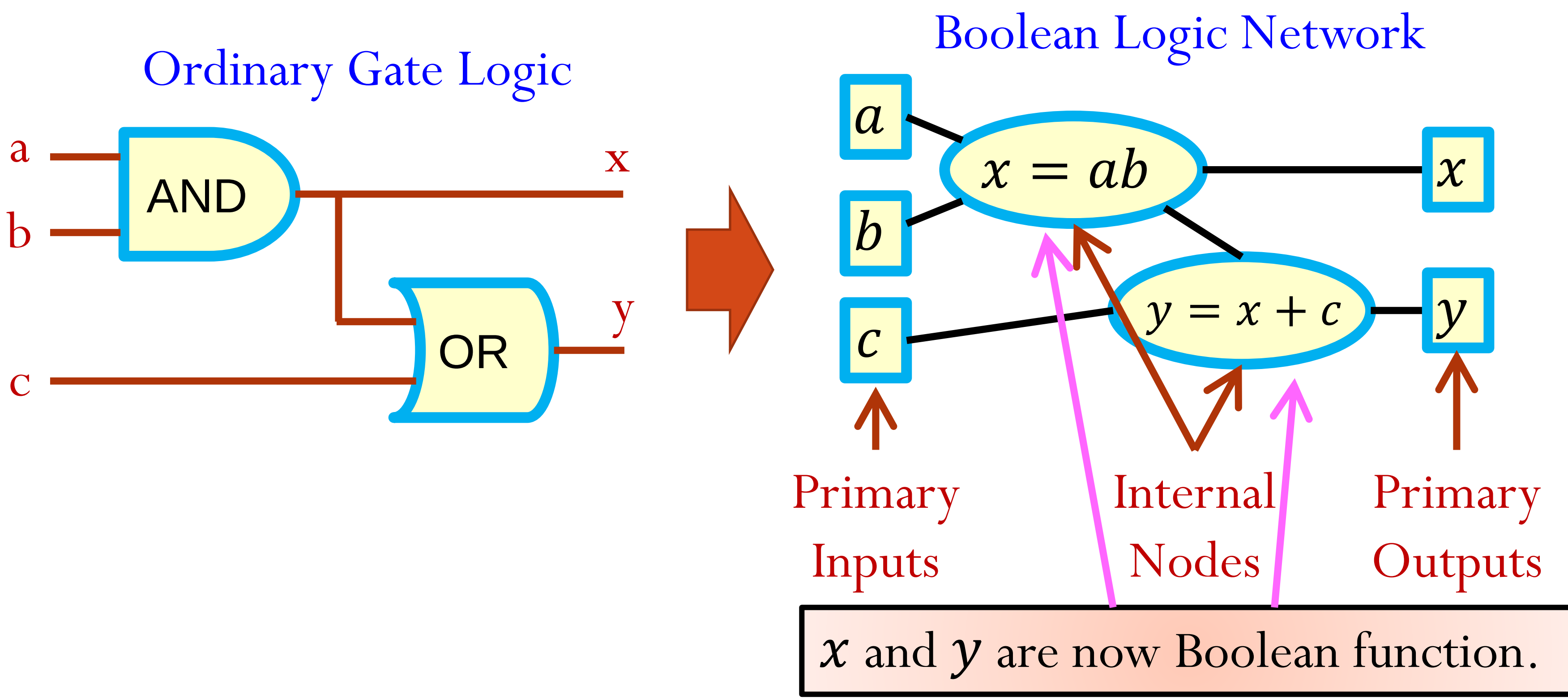
Real Multilevel Example

- A small design, done by commercial synthesis tool.



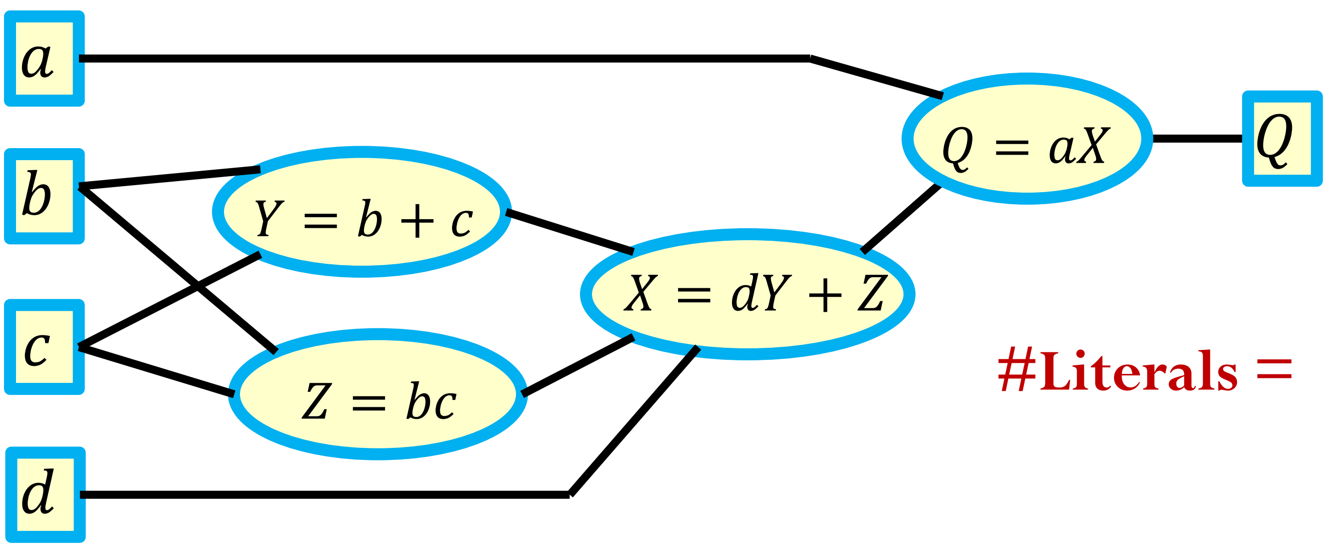
Boolean Logic Network Model

- Need more sophisticated model: **Boolean Logic Network**
  - Idea: it's a **graph** of connected blocks, like any logic diagram, but now individual component blocks are **2-level Boolean functions in SOP form**.



Multilevel Logic: What to Optimize?

- A simplistic but surprisingly useful metric:
  Total literal count
  - Count **every** appearance of **every** variable on **right hand side** of "=" in **every internal node**.
  - Delays also matter, but for our lecture, only focus on **area**.



#Literals = 9

Optimizing Multilevel Logic: Big Ideas

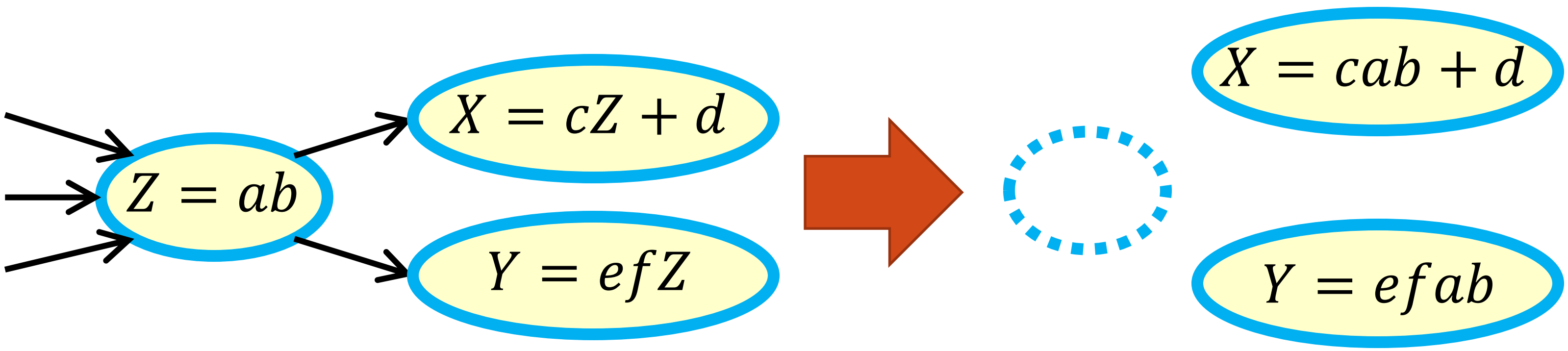
- **Boolean logic network** is a **data structure**. So, what **operators** do we need?
- 3 basic kinds of operators:
  - **Simplify** network nodes: **<u>no change</u>** in # of nodes, just simplify insides, which are **SOP form**.
  - **Remove** network nodes: take "too small" nodes, **substitute them back** into fanouts.
    - This is not too hard. This is mostly manipulating the graph, simple SOP edits.
  - **Add** new network nodes: this is **factoring**. Take big nodes, split into smaller nodes.
    - This is a **big deal**. This is new. This is what we will show you…

Simplifying a Node

- You already know this! This is **2-level synthesis**.
- Just run ESPRESSO on 2-level form **inside** the node, to reduce # literals.
- As structural changes happen across network, "**insides**" of nodes may present opportunity to simplify.

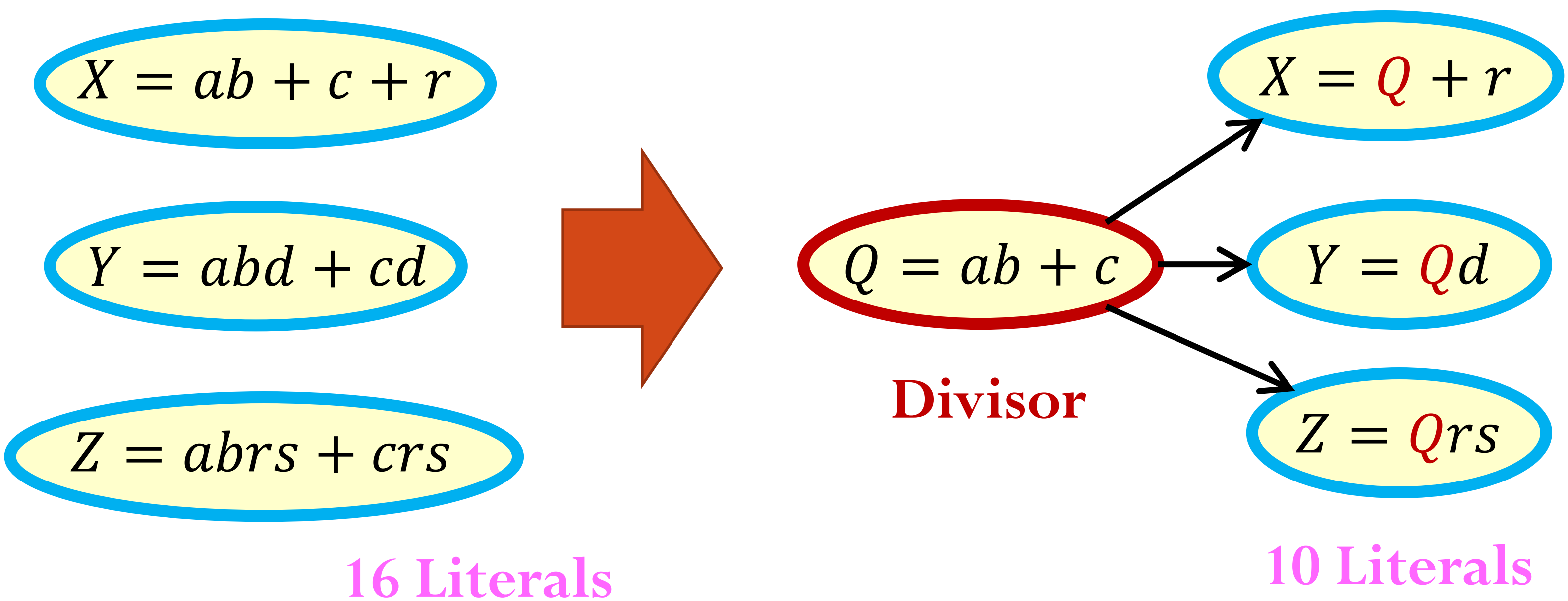
Removing a Node

- Typical case is you have a "**small**" factor which doesn't seem to be worth making it a **separate** node.
- "**Push**" it back into its fanouts, make those nodes bigger, and hope you can use 2-level simplification on them.



Adding new Nodes

- This is Factoring, this is new, and hard.
  - Look at existing nodes, identify **common divisors**, extract them, connect as **<u>fanins</u>**.
  - Tradeoff: **more** delay (another level of logic), but **fewer literals** (less gate area).



Multilevel Synthesis Scripts

- Multilevel synthesis like 2-level synthesis is **heuristic**.
- …but it's also more complex. Write **scripts of basic operators**.
  - Do **several passes** of different optimizations over the Boolean logic network.
  - Do some "cleanup" steps to get rid of "too small" nodes (**remove nodes**).
  - Look for "easy" **factors**: just look at existing nodes, and try to use them.
  - Look for "hard" **factors**: do some work to extract them, try them, and keep the good ones.
  - Do 2-level optimization of insides of each logic node in network (**simplify nodes** by ESPRESSO).
  - Lots of "art" in the engineering of these scripts.
- For us, the one big thing you don't know: **How to factor**…

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Another New Model: Algebraic Model

- Factoring: How do we really do it?
  - Develop another model for Boolean functions, cleverly designed to let us do this
  - Tradeoff: lose some "expressivity" – some aspects of Boolean behavior and some Boolean optimizations we just **cannot do**, but we **gain practical factoring**.

- **New model**: **Algebraic model**
  - Term "algebraic" comes from **pretending** that Boolean expressions behave like **polynomials of real numbers**, not like Boolean algebra.
  - **Big new Boolean operator**: **Algebraic Division** (also known as **"Weak" Division**).

Algebraic Model

- Idea: keep just those **rules** that work for **BOTH** polynomials of reals **AND** Boolean algebra, but **get rid of** the rest.

Real numbers

$$a \cdot b = b \cdot a \quad a + b = b + a$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
$$a + (b + c) = (a + b) + c$$
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$a \cdot 1 = a \quad a \cdot 0 = 0$$
$$a + 0 = a$$

Same

Boolean algebra

$$a \cdot b = b \cdot a \quad a + b = b + a$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
$$a + (b + c) = (a + b) + c$$
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$a \cdot 1 = a \quad a \cdot 0 = 0$$
$$a + 0 = a$$

$$a \cdot \bar{a} = 0 \quad a + \bar{a} = 1$$
$$a \cdot a = a \quad a + a = a$$
$$a + 1 = 1$$
$$(a + b)(a + c) = a + b \cdot c$$

Not Allowed

Algebraic Model

- If we only get to use algebra rules from real numbers…
  - **<u>Consequence</u>**: A variable and its complement must be treated as **totally unrelated**.
    - Since no expression like $a + \bar{a} = 1$ allowed.

$$F = ab + \bar{a}x + \bar{b}y$$

$$\downarrow \quad \boxed{\text{Let } R = \bar{a}, S = \bar{b}}$$

$$F = ab + Rx + Sy$$

- **<u>Aside</u>**: this is one of the losses of "expressive power" of Boolean algebra.

Algebraic Model: Representation

- Boolean functions manipulated in SOP form like **polynomials**.
- Each product term in such an expression is just **a set of variables**, e.g., $abRy$ is the set (a, b, R, y).
- SOP expression itself is just a **list of these products (cubes)**, e.g., $ab + Rx$ is the list $((a, b), (R, x))$.

Algebraic Division: Our Model for Factoring

- Given function F, we want to factor it as:

$$F = \underbrace{D}_{\text{divisor}} \cdot \underbrace{Q}_{\text{quotient}} + \underbrace{R}_{\text{remainder}}$$

  - If remainder $R = 0$, we call the divisor as a "**factor**".

Example: $F = ac + ad + bc + bd + e$

$$= \underbrace{(a + b)}_{\text{divisor}}\underbrace{(c + d)}_{\text{quotient}} + \underbrace{e}_{\text{remainder}}$$

Algebraic Division

- Example: $F = ac + ad + bc + bd + e$
  - **<u>Want</u>**: $F = D \cdot Q + R$.

Divisor is a **factor** if $R = 0$.

Divisor (D)	Quotient (Q)	Remainder (R)	Is D Factor?
$ac + ad + bc$ $+bd + e$	1	0	Yes
$a + b$	$c + d$	e	No
$c + d$	$a + b$	e	No
a	$c + d$	$bc + bd + e$	No
b	$c + d$	$ac + ad + e$	No
c	$a + b$	$ad + bd + e$	No
d	$a + b$	$ac + bc + e$	No
e	1	$ac + ad + bc + bd$	No

Algebraic Division: Very Nice Algorithm

- **<u>Inputs</u>**: A Boolean expression F and a divisor D, represented as lists of cubes (and each cube as a set of literals).
- **<u>Output</u>**
  - Quotient $Q = F/D$ as a cube list, or **empty** if $Q = 0$.
  - Remainder R as a cube list, or **empty** if D was a **factor**.

- **<u>Strategy</u>**
  - Cube-wise walk through cubes in divisor D, trying to divide each of them into F.
  - … intersect all the division results.

Algebraic Division Algorithm

```
AlgebraicDivision( F, D ) {  // divide D into F
  for ( each cube d in divisor D ) {
    let C = { cubes in F that contain this product term d };
    if ( C is empty )  return ( quotient = 0,  remainder = F );
    let C = cross out literals of cube d in each cube of C;
    if ( d is the first cube we have looked at in divisor D ) let Q = C;
    else  Q = Q ∩ C;
  }
  R = F − ( Q • D );
  return (quotient = Q,  remainder = R);
}
```

Example:
Cube **xyzw** contains product term **yz**

Example:
Suppose **C = xyz + yzw +pqyz** and **d = yz**. Then crossing out all the **yz** parts yields **x + w + pq**

Algebraic Division: Example

$$F = axc + axd + axe + bc + bd + de, D = ax + b$$

F cube	D cube: ax	D cube: b
axc	$axc \rightarrow c$	$-$
axd	$axd \rightarrow d$	$-$
axe	$axe \rightarrow e$	$-$
bc	$-$	$bc \rightarrow c$
bd	$-$	$bd \rightarrow d$
de	$-$	$-$

$$C = c + d + e \qquad C = c + d$$

$$Q = (c + d + e) \cap (c + d) = c + d$$

$$R = F - QD = axe + de$$

Algebraic Division: Warning

- Remember: No "**Boolean**" simplification, only "**algebraic**".
  - So what? Well, suppose you have this

$$F = a\bar{b}\bar{c} + ab + ac + bc, D = ab + \bar{c}$$

and you want F/D.

  - You must let $X = \bar{b}$ and $Y = \bar{c}$ and transform F and D to something like

$$F = aXY + ab + ac + bc, D = ab + Y$$

  - Because we must treat the true and complement forms of variables as **totally unrelated**.

One More Constraint: Redundant Cubes

- To do F/D, function F must have no **redundant** cubes
  - Technical definition is: **minimal** with respect to **single-cube containment**.
  - **Means**: no one cube is **completely** covered by **one of the other** cubes in SOP cover.
    - E.g., $abcd$ is completely covered by ab.

- Why **no** redundant cubes?
  - **Consider**: $F = a + ab + bc$ and $D = a$.
    - **Note**: F has redundant cube ab.
  - What is F/D by our **algebraic division algorithm**?

$$\boxed{Q = F/D = 1 + b}$$

However, we don't have 1+stuff operation in algebraic model!

One More Constraint: Redundant Cubes

- … So, we should remove redundant cubes to make the SOP **minimal** with respect to **single-cube containment**.
  - Not hard.

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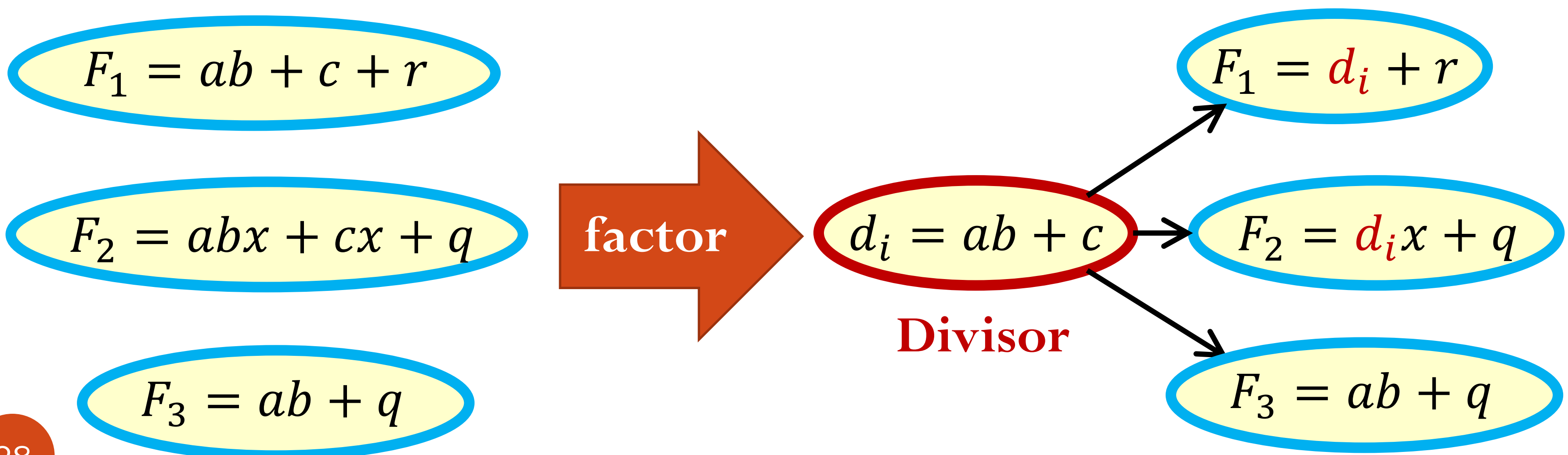
Multilevel Logic Synthesis: Where are We?

- For Boolean function F and D, can compute $F = Q \cdot D + R$ via **algebraic model**.
  - It is great, but still **not enough**: don't know how to **find** a good divisor D.
  - **Another problem**: given n functions $F_1, F_2, \ldots, F_n$, find **a set of good common divisors**.



Where To Look For Good Divisors?

- Surprisingly, the **algebraic model** has a beautiful answer.

- Where to look for divisors of function F?
  - In the set of **kernels** of F, denoted $K(F)$.
  - $K(F)$ is another set of 2-level SOP forms which are the special, foundational structure of any function F

- How to find a kernel $k \in K(F)$?
  - **Algebraically divide** F by one of its **co-kernels**, c.

Kernels and Co-Kernels of Function F

- **Kernel** of a Boolean expression F is:
  - A **cube-free** quotient k obtained by **algebraically dividing** F by a **single cube** c.
  - This single cube c also has a name: it is a **co-kernel** of function F.

quotient Q

divisor D | expression F

⋮

remainder R

$$F = D \cdot Q + R$$

kernel k (cube-free)

$c = 1$ cube | expression F

⋮

remainder R

$$F = c \cdot k + R$$

Kernels Are Cube-Free...

- **Cube-free** means…?
  - You cannot factor out a **<u>single</u>**-cube divisor that **leaves no remainder**.
  - Technically: has no **one cube** that is a **factor** of expression.
  - Pick a cube c. If you can "**cross out**" c in **<u>each</u>** product term of F, then F is **not** a **kernel**.
  - Note: cubes "1" and "a" are **NOT** cube-free

Expression F	$F = D \cdot Q + R$	F Cube-free?
a	$a \cdot 1 + 0$	No
$a + b$	--	Yes
$ab + ac$	$a \cdot (b + c) + 0$	No
$abc + abd$	$ab \cdot (c + d) + 0$	No
$ab + acd + bd$	--	Yes

Some Kernel Examples

- Suppose $F = abc + abd + bcd$

Divisor cube d	$Q = F/d$	Is Q a kernel of F?
1	$abc + abd + bcd$	No, has cube $= b$ as factor
a	$bc + bd$	No, has cube $= b$ as factor
b	$\boxed{ac + ad + cd}$	Yes! co-kernel $= b$
ab	$\boxed{c + d}$	Yes! co-kernel $= ab$

- Any Boolean function F can have many different kernels.
  - The **set** of kernels of F is denoted as $K(F)$.

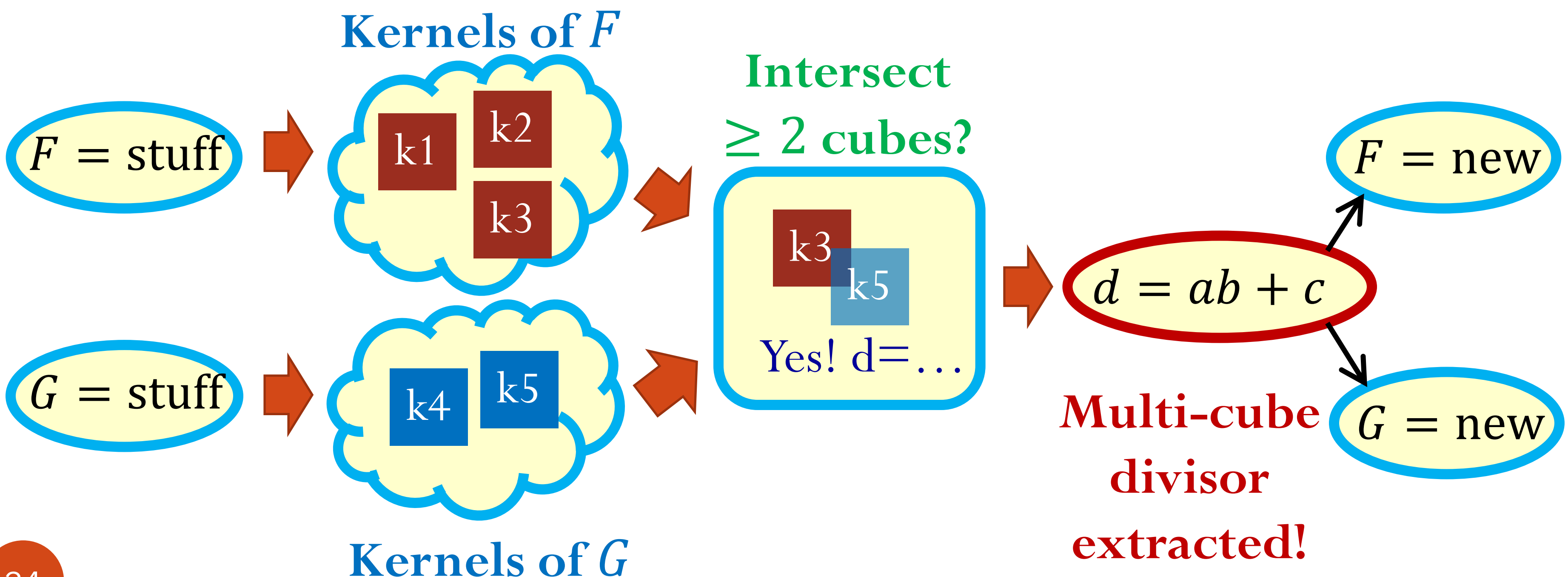
Kernels: Why Are They Important?

- Big result: Brayton & McMullen Theorem
  - **From**: R. Brayton and C. McMullen, "The decomposition and factorization of Boolean expressions." In *IEEE International Symposium on Circuits and Systems*, pages 49–54, 1982.

Expressions F and G have a **common multiple-cube divisor** d **if and only if**:
there are kernels $k1 \in K(F)$ and $k2 \in K(G)$ such that $d = k1 \cap k2$ and d is an expression with **at least 2** cubes in it (i.e., $k1$ and $k2$ have **common cubes**).

Multiple-Cube Divisors and Kernels

- Brayton & McMullen Theorem **in words**:
  - The **only** place to look for **multiple-cube divisors** is in the **intersection of kernels**!
    - Indeed, this intersection of kernels contains **all** divisors.



Brayton-McMullen: Informal Illustration

F = cube1 • **kernel1** + remainder1

G = cube2 • **kernel2** + remainder2

Assume:

kernel1 ∩ kernel2 = X + Y

F = cube1 • **[X + Y + stuff1]** + remainder1

G = cube2 • **[X + Y + stuff2]** + remainder2

F = cube1 • **[X + Y]** + [cube1•stuff1 + remainder1]

G = cube2 • **[X + Y]** + [cube2•stuff2 + remainder2]

d=**X+Y**

F = cube1•**d** + …

G = cube2•**d** + …

Kernels: Real Example

$F = ae + be + cde + ab$

$G = ad + ae + bd + be + bc$

Kernels	Co-kernel
$\boxed{a + b + cd}$	e
$b + e$	a
$a + e$	b
$ae + be + cde + ab$	1

Kernels	Co-kernel
$\boxed{a + b}$	d or e
$d + e$	a
$c + d + e$	b
$ab + ae + bd + be + bc$	1

Intersecting these 2 kernels: $(a + b + cd) \cap (a + b) = a + b$

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Kernels: Very Useful, But How To Find?

- Another **recursive** algorithm ("recursive" again!)
  - There are 2 more useful properties of kernels we need to see first…

- Start with a function F and a kernel $k1$ in $K(F)$

$$F = \text{cube1} \bullet \text{k1} + \text{remainder1}$$

- Then: a new, interesting question: what about $K(k1)$?
  - $k1$ is a perfectly nice Boolean expression, so it has got **its own** kernels.
  - Do $k1$'s kernels have anything interesting to say about $K(F)$?

How $K(k1)$ Relates to $K(F)$…

- We know this: F = cube1•k1 + remainder1
- Suppose k2 is a kernel in K(k1), then we know

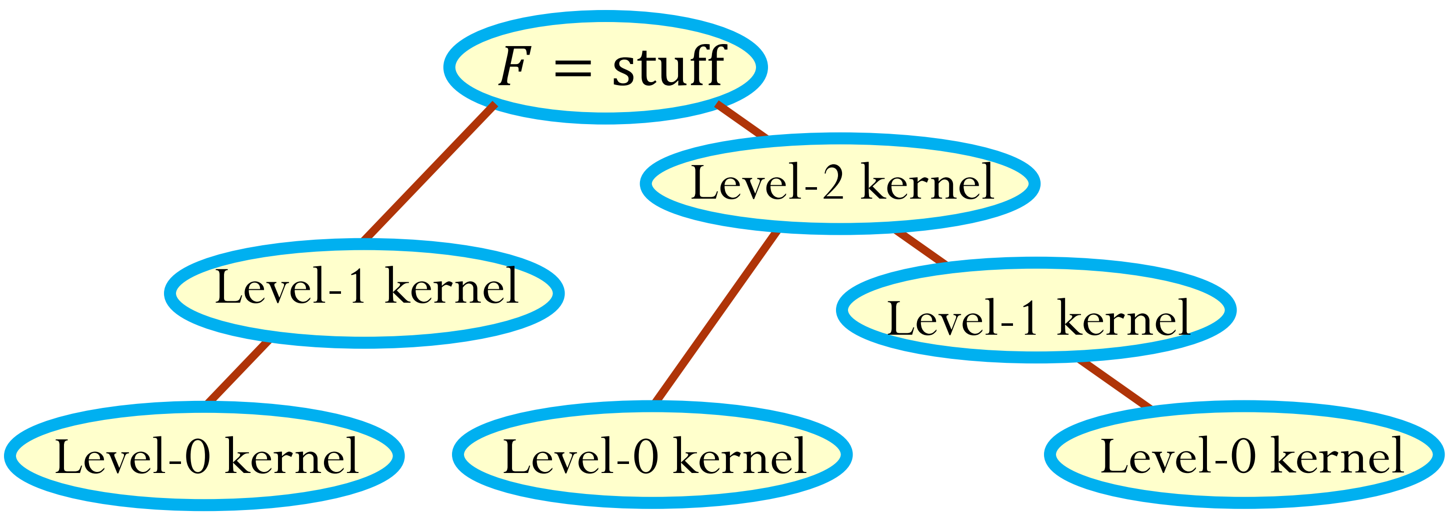
  k1 = cube2•k2 + remainder2
- Substitute this expression for k1 in original expression for F

  F = cube1•[cube2•k2 + remainder2] + remainder1
- Since cube1•cube2 is itself just another **single cube**, we have:

  F = (cube1•cube2)•[k2] + [cube1•remainder2 + remainder1]
- <u>**Conclusion**</u>: k2 is also a **kernel** of original F (with **co-kernel** cube1•cube2)

There is a Hierarchy of Kernels Inside F

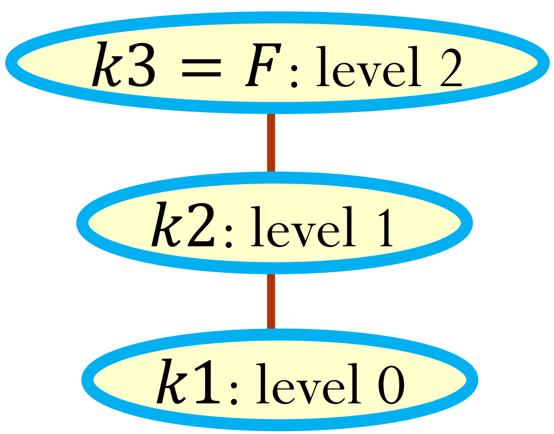
- **Definition**: $k \in K(F)$ is
  - A **level-0 kernel** if it contains no kernels inside it except itself.
    - In words: Only cube you can pull out to get a cube-free quotient is "1".
  - A **level-n kernel** if it contains **at least one** level-(n-1) kernel, and no other level-n kernels except itself.
    - In words: a level-1 kernel only has level-0 kernels inside it. A level-2 kernel only has level-1 and level-0 kernels in it, etc…



Kernel Hierarchy: Example

- $F = abe + ace + de + gh$ has three kernels:
  - $k1 = b + c$, with co-kernel ae.
  - $k2 = ab + ac + d$, with co-kernel e.
  - $k3 = F$ with co-kernel 1.

- <u>Note</u>: $k1$ is level 0, $k2$ is level 1, and $k3$ is level 2.



Kernels

- Second useful result [by Brayton et al.]:
  - **Co-kernels** of a Boolean expression in SOP form correspond to **intersections of 2 or more cubes** in this SOP form.
- **Note**: **Intersections** here means that we regard a cube as **a set of literals**, and look at common subsets of literals.
  - This is not like "AND" for products. This just extracts **common literals**.
  - Example: ace + bce + de + g

$$\text{ace} \cap \text{bce} = \text{ce} \quad \rightarrow \text{ce is a potential co-kernel}$$

$$\text{ace} \cap \text{bce} \cap \text{de} = \text{e} \quad \rightarrow \text{e is a potential co-kernel}$$

How to Find Kernels Using These 2 Results?

- Find the kernels **recursively**.
  - Whenever find one kernel, call **FindKernels()** on that kernel, to find (if any) lower level kernels inside.
- Use **algebraic division** to divide function by potential co-kernels, to drive recursion.
  - Use 2nd result – co-kernels are **intersections** of the cubes: If there're at least 2 cubes, then look at the intersection of those cubes, and use that intersected result as our potential co-kernel cube.

- One technical point: need to start with a **cube-free function** F to make things work right.
  - If not cube-free, just divide by biggest common cube to simplify F.

Kernel Algorithm

```
FindKernels( cube-free SOP expression F ) {
  K = empty;
  for ( each variable x in  F  ) {
     if ( there are at least 2 cubes in F that have variable x ) {
        let S = { cubes in F that have variable x in them };
        let co =  cube that results from intersection of all cubes in S,
                        this will be the product of those literals
                        that appear in each of these cubes in S;
        K = K ∪ FindKernels( F / co) ;
     }
  }
  K = K ∪ F ;
  return( K );
}
```

Cube-free **F** is always its own kernel, with trivial co-kernel = **1**

Kernelling Example

```
FindKernels( F ):
for (each var x in  F ) {
    if (x in ≥ 2 cubes in F) {
        co = intersection of cubes;
        K=K ∪ FindKernels(F/co) ;
    }
}
K = K ∪ F ;
return( K );
```

$$F = ace + bce + de + g$$

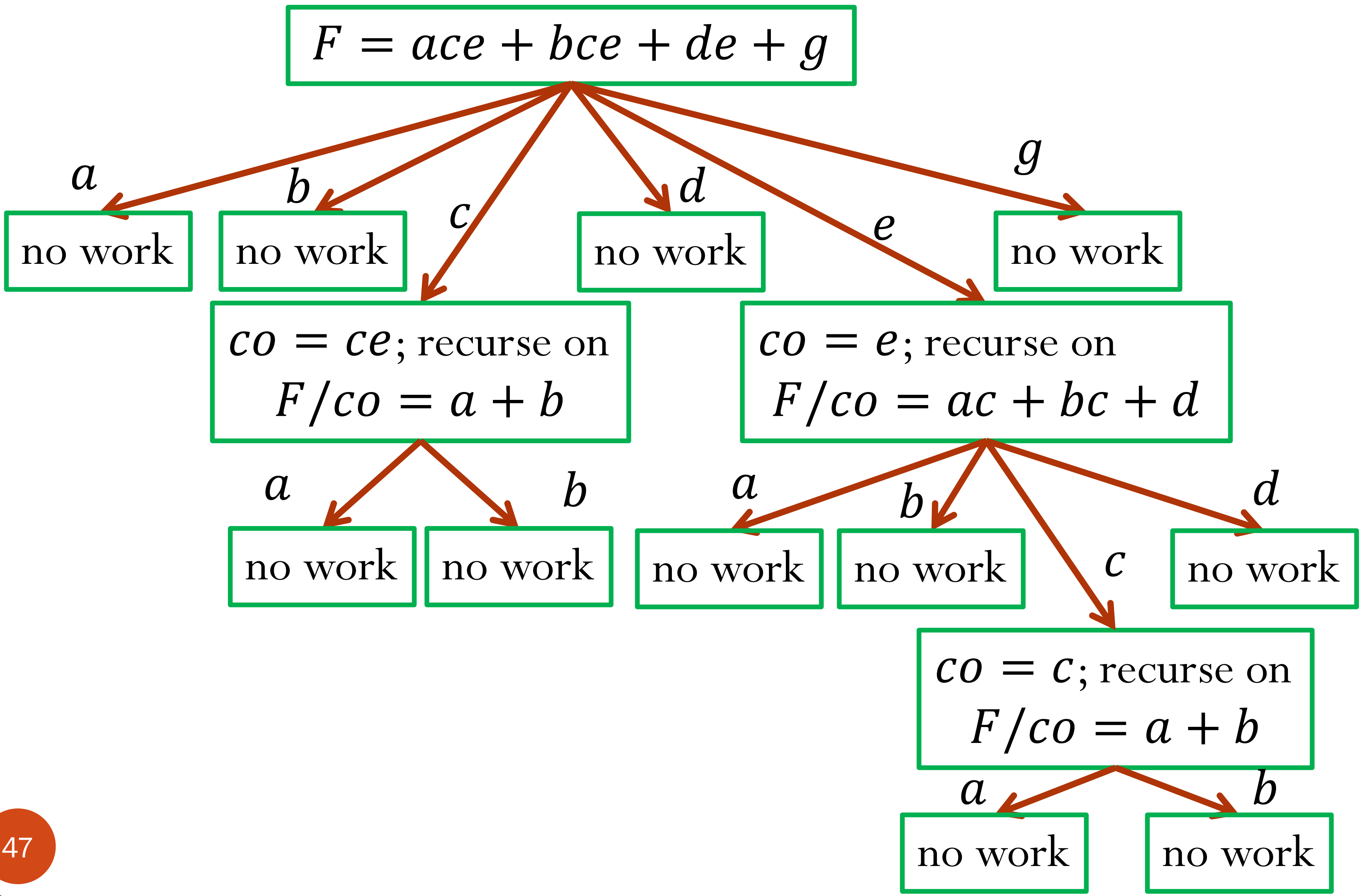
- a: only 1 cube with a, no work.
- b: only 1 cube with b, no work.
- c: two cubes ace and bce with c.
  - $co = ace \cap bce = ce$
  - $F/co = a + b$
  - **Recurse** on $a + b$
- d: only 1 cube with d, no work.
- e: three cubes ace, bce, and de with e.
  - $co = ace \cap bce \cap de = e$
  - $F/co = ac + bc + d$
  - **Recurse** on $ac + bc + d$
- g: only 1 cube with g, no work.

Kernelling Example (cont.)

- **Recurse** on $a + b$
  - No work for variables a and b, since one cube with a/b.

- **Recurse** on $ac + bc + d$
  - No work for variables a, b, d, since one cube with $a/b/d$.
  - c: two cubes ac and bc with c.
    - $co = ac \cap bc = c$
    - $F/co = a + b$
    - **Recurse** on $a + b$ (the same as above)

Kernelling Example (cont.)



Kernelling Example (cont.)

FindKernels(**F**):
for (each var **x** in **F**) {
  …
 K=**K** ∪ **FindKernels**(**F/co**) ;
}
K = **K** ∪ **F** ;
return(**K**);

$F = ace + bce + de + g$

Kernels $K = \{a+b,\ ac+bc+d,\ ace+bce+de+g\}$

return $K = \{a+b,\ ac+bc+d\}$

e

$co = e$; recurse on $F/co = ac + bc + d$

c

return $K = \{a+b\}$

$co = ce$; recurse on $F/co = a + b$

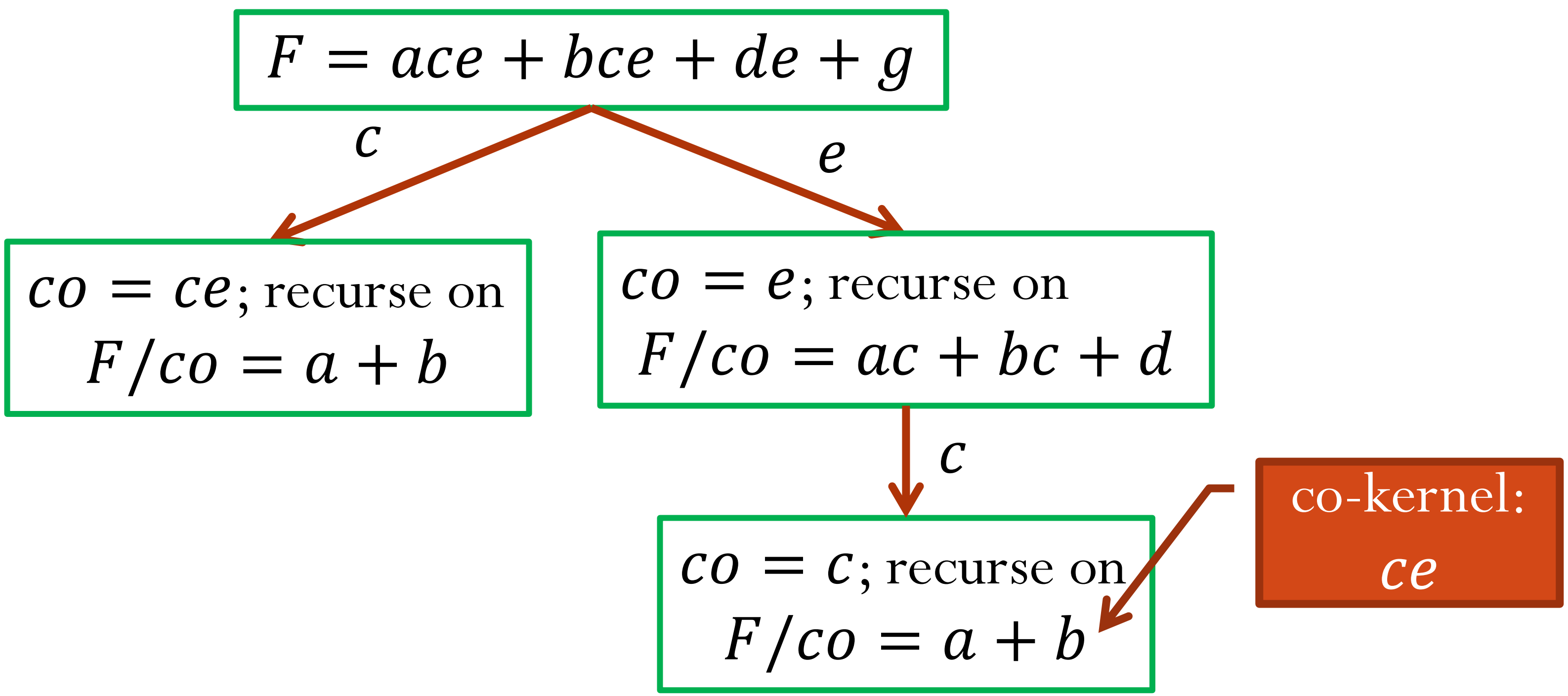
return $K = \{a+b\}$

c

$co = c$; recurse on $F/co = a + b$

Get Co-Kernels

- With this algorithm …
  - Can find **all** the kernels and **co-kernels** too.
  - Get co-kernels by **ANDing** the divisor co cubes **<u>up recursion tree</u>**.



One Tiny Problem

$$
\begin{tikzcd}
 & \boxed{F = ace + bce + de + g} \arrow[dl, "c"'] \arrow[dr, "e"] & \\
\boxed{F/co = a + b} & & \boxed{F/co = ac + bc + d} \arrow[d, "c"] \\
 & & \boxed{F/co = a + b}
\end{tikzcd}
$$

- The algorithm will revisit same (co-kernel, kernel) pair **multiple** times.
  - Why? Kernel you get for co-kernel abc is same as for cba, but current algorithm **doesn't know this** and will find same kernel for both co-kernels.
- Solution: remember which variables already tried in co-kernels. A little extra book keeping solves this.

Multilevel Synthesis Models: Summary

- **Boolean network model**
  - Like a gate network, but each node in network is an SOP form.
  - Supports many operations to add, reduce, simplify nodes in network.
- **Algebraic model** & **algebraic division**
  - Simplifies Boolean functions to behave like polynomials of reals.
  - Divides one Boolean function by another:

    F = (divisor D)•(quotient Q) + remainder R
- **Kernels** / **Co-kernels** of a function F
  - **Kernel** = **cube-free** quotient obtained by dividing by a single cube (**co-kernel**)
  - **Intersections** of kernels of two functions give all **multiple-cube common divisors** (Brayton & McMullen theorem).

Notes

- The **algebraic model** (and **division**) are not the only options.
  - There are also "**Boolean division**" models and algorithms that don't lose expressivity.
  - ..but they are more complex.
  - Rich universe of models & methods here.

Good References

- R.K. Brayton, R. Rudell, A. Sangiovanni-Vincentelli, A.R. Wang, "MIS: A Multiple-Level Logic Optimization System," *IEEE Transactions on CAD of ICs*, vol. CAD-6, no. 6, November 1987, pp. 1062-1081.
- Giovanni De Micheli, *Synthesis and Optimization of Digital Circuits*, McGraw-Hill, 1994.