Inference in Bayesian Nets

We last discussed in class how we can use Bayes Nets to model a complex set of variables over which we have to perform inference.

The kind of questions we wish to ask are of the type:

- What is the most likely value of variables A, B, ... X (for an arbitrary subset of variables in our problem)
- If I have observed values for certain variables, how does that change the most likely value for the rest?
- If I have to make a decision based on my current observations, what is the optimal decision I can make?

To answer these questions, we go back to the structure of the Bayes Net, and perform inference on it given any observations we have, and the structure of the network telling us how variables relate to each other.

The simplest process for performing inference on a given Bayes Net is to do ***sampling.*** In particular, we will random sample possible values for all the ***unobserved*** variables in the network using the associated ***conditional probability tables*** at each node in the net.

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Bayes Net Sampling:

   For N samples (with N suitably large)
       - Randomly draw values for each of the unobserved variables in the network
       - Determine the weight of each sample (more on this below)

   For each variable
       - Create a histogram of possible values (by counting how many times each
         possible value was observed in our random samples, multiplied by the
         corresponding weight for each sample).
       - Normalize the histogram to obtain a probability distribution of values
         for that variable
       - Determine the most likely value from the histogram
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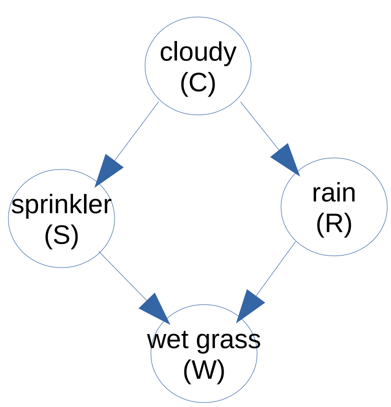
Example:

Consider the Bayes Net shown below



This network could be used to answer questions such as:

- If the grass is wet, what's the probability the sprinklers were on?
- If it's cloudy, what's the probability it will rain?
- If the grass is wet, what's the probability it's cloudy?

And so on. Let's assume that each of these four variables is Boolean, it can be TRUE, or FALSE. To fully determine the possible outcomes for variables in the network, we need four probability tables:

P(C) - Which depends on nothing else on the network
P(S|C) - Sprinkler being on depends on whether it's cloudy or not
P(R|C) - Whether it is rainy or not depends on whether it's cloudy or not
P(W|S,R) - Whether the grass is wet will depend on both rain and sprinklers

Let's put values in these tables (it's just an example, I didn't get stats from anywhere)

P(C)	
C=0	.7
C=1	.3

P(S|C)	C=0	C=1
S=0	.8	.5
S=1	.2	.5

P(R|C)	C=0	C=1
R=0	.99	.2
R=1	.01	.8

P(W|S,R)	S=0,R=0	S=0,R=1	S=1,R=0	S=1,R=1
W=0	.99	.1	.05	.01
W=1	.1	.9	.95	.99

How do we ***get a random sample*** from these tables and the structure of the network?

* Start with variables that do not depend on anything (in this case, C)
- We get a random number ***r*** , if ***r<=.7, C=0,*** otherwise ***C=1***
  e.g. r=.46271, C=0

Our sample at this point looks like so:

C=0 S=? R=? W=?

* Now sample variables that depend on only ***one*** other variable (in this case, S and R which depend on C).

- For sampling ***S***, get a random number ***r***. Then look in the CPT for P(S|C) and find the column for ***C=0*** which is the value we got from our random sample for C.
  There we find that if ***r<=.8, S=0, otherwise S=1***.
  e.g. r=.9231, so S=1

Our sample at this point looks like so:

C=0 S=1 R=? W=?

- For sampling ***R***, get a random number ***r***. Then look in the CPT for P(R|C) and find the column for ***C=0*** which is the value we got from our random sample for C. There we find that if ***r<=.99, R=0, otherwise R=1***. e.g. r=.1231, so R=0

Our sample at this point looks like so:

C=0 S=1 R=0 W=?

* Now sample any variables that depend on ***two other variables***, in this case, ***W***.

- Get a random number ***r***. Then look in the CPT for P(W|S,R) and find the column for ***S=1, R=0*** which are the current values for these variables in our sample. We see that if ***r<=.05, W=0, otherwise W=1***. e.g. r=.31415, so W=1

And we have a complete sample! It has values

C=0 S=1 R=0 W=1

The weight for this sample is 1.0 because ***all variables were randomly sampled from the corresponding tables.***

We repeat the process above until we have obtained *N* random samples for possible assignments to variables in our network, we'll have a table that looks like so:

C	S	R	W	Weight
0	1	0	1	1.0
0	0	0	0	1.0
1	0	1	1	1.0
.	.	.	.	.
.	.	.	.	.
1	1	0	0	1.0
0	1	0	0	1.0

Suppose we have a few hundred thousand samples, or a couple million. Then we can approximate very closely the actual probability of any arbitrary setting for any subset of variables in the network from this table.

For example, say we want to know p(R=1), this is just:

$$P(R=1) = \frac{\#\ of\ rows\ in\ the\ table\ with\ R{=}1}{\#\ of\ rows}$$

Say we want to find out P(C=1, W=0), this is just:

$$P(C=1, W=0) = \frac{\#\ of\ rows\ in\ the\ table\ with\ C{=}1{,}W{=}0}{\#\ of\ rows}$$

It becomes a simple matter of counting how often the outcome we're interested in actually happens in our random sample!

Of course this is just an approximation, and how good it is depends on how much time we want to spend doing random sampling. But with current computers, drawing millions of samples for even a large network is not unreasonable.

Still, there may be outcomes which are unlikely, and if we don't do enough sampling we may never actually observe them in our random sample set. So for some networks, and in some cases, it may require a fairly large amount of sampling to get accurate probabilities for low-probability events.

Note: We can also ask questions about conditional probabilities – by using the identity P(A|B)=P(A,B)/P(B). We can also use Bayes rule, together with the conditional probability definition, to infer posterior probabilities as needed.

What about observed variables?

If you pay attention to the samples above, you'll see that we have a weight column with all weights equal to 1.0. This is because in all our random samples, all variables were randomly sampled from their corresponding probability tables.

This is wasteful when we have observations about any of the variables in our network. For instance, say we have ***observed R=1.*** In other words, we actually know it's raining.

We can follow the random sampling process above, now, suppose we want to find out what's the probability of ***C=0***.

Before observing that it is rainy, we would have counted every row in which C=0, and divided by the total number of rows.

$$P(C=1) = \frac{\#\ of\ rows\ in\ the\ table\ with\ C{=}1}{\#\ of\ rows}$$

After observing that R=1, we have

$$P(C=1|R=1) = \frac{\#\ of\ rows\ in\ the\ table\ with\ C{=}1{,}R{=}1}{\#\ of\ rows with R{=}1}$$

This follows from the conditional probability definition, since we are now given the knowledge that R=1.

Inference still works exactly the same as we did before, and we ***could continue to use our regular sampling process, with equal weights for all samples, just like before***.

However, notice that a large proportion of the samples in our table will now go unused – they contribute nothing to our probability estimate because they happen to have R=0, the value of R we know not to be correct.

This can get pretty bad if the ***observed values for variables given to us*** have low probability of occurring. In this case, most of the entries in our sample table could be useless – they do not correspond to the observed values we have.

So, either we sample a whole lot, or we get smart about sampling!

Importance sampling for networks with observed variables

Let's do things smartly – we will modify (slightly) our sampling process to ***not generate any samples that correspond to the wrong values of observed variables***. Simply put, we will ***not sample*** the observed variables, ***using instead their observed value***.

That means ***every sample*** in our table will now have the correct values for observed variables, and no samples are wasted – we will be able to obtain a much better estimate of any probabilities we're interested in for the remaining variables. ***But***, the samples are no longer ***fair***. We are not sampling certain variables, so we need to account for the probability of these ***observed variables*** having the value they have, given the rest of the variables in the network.

We will do this by computing a weight for each sample that accounts for how likely the observed variable values are given all the values that were ***randomly sampled***.

Let's see an example of how this would work in the case above. Suppose we observe R=1

P(C)	
C=0	.7
C=1	.3

P(S|C)	C=0	C=1
S=0	.8	.5
S=1	.2	.5

P(R|C)	C=0	C=1
R=0	.99	.2
R=1	.01	.8

P(W|S,R)	S=0,R=0	S=0,R=1	S=1,R=0	S=1,R=1
W=0	.99	.1	.05	.01
W=1	.1	.9	.95	.99

We use random sampling (as described above) to get random assignments for C, S, and W (note that for W we will use R=1). Suppose we obtain the following sample:

C=0 S=0 R=1 W=1

The value for R is shown in red because it was ***not sampled***, it's ***fixed for all our samples because it was observed***.

The sample above is a perfectly possible sample, however, we need to weight it by the probability that R=1 given the values for C, S, and W in that specific random sample.

R depends only on C, so we head over to the probability table for P(R|C), and we look at the column for C=0 (which is the value in this specific random sample), there we find that P(R=1 | C=0)=.01.

So the weight for this sample is:

weight = 1.0 * P(R=1 | C=0) = 1.0 * .01 = .01

The 1.0 corresponds to the weight of a standard sample, in which all variables were randomly sampled. Then we multiply that for the conditional probability of R=1 given the sample's other values. The final weight is .01 which reflects the fact that there's a very low chance of observing rain if it's not cloudy!

Let's do another sample:

C=1 S=0 R=1 W=1

From P(R|C) we see that P(R=1|C=1)=.99

So the weight for this sample would be:

weight = 1.0 * P(R=1 | C=1) = 1.0 * .99 = .99

This weight is telling us the sample above is a lot more likely as a possible assignment to the variables in our network.

We would repeat this process N times, obtain N total samples all of which has R=1, and each of which has an appropriate weight, and then we can infer probabilities for any specific assignment of variables in the network by slightly modifying what we had before so it accounts for the weights of samples. For example, say we want to know P(C=0):

$$P(C=0) = \frac{\sum_{rows\ with\ C=0} w_{row}}{\sum_{all\ rows} w_{row}}$$

That is, instead of simply counting and dividing by the total number of samples, now we accumulate the weights for all rows with C=0, and divide that by the sum of the weights of all the rows in the table.

Similarly, you can compute the probability of any other possible outcome by adding up the weights of the rows with that specific outcome, and dividing by the sum of the weights of all the rows.

Conditional probabilities and Bayes rule can be applied like before!

The advantage of this process is that we do not waste samples on assignments that do not correspond to the observed values of variables. This means we'll obtain an accurate estimate of the probabilities we care about with far less sampling. But, we should be able to obtain the same probability estimates using the regular sampling process (random sampling all variables) if we sample enough!

In summary: Inference in Bayes Nets can be carried out by random sampling and counting – it's a general procedure, works for any network, and requires us only to have enough computing power/time to obtain enough samples that our probability estimates are meaningful. We can also use importance sampling to obtain good estimates with less sampling by using the observed values of some variables, and weighting the samples accordingly.

Now go and try it out!

Problems:

1) Figure out what the sampling process would need to do for the examples above, if we observe W=1, R=0. What would be the weight of a sample C=0, S=1, R=0, W=1?

2) Write a little program to estimate the probability of specific events in the network above. It has to carry out the random sampling process, and provide you with a table from which you can compute the corresponding probabilities.

You can do this in your favorite language :) - I don't have enough time at this point to give you Matlab starter, so off you go to do it however you prefer – but do it! Then you'll fully understand the sampling process and the probability estimation process.