Northeastern University
CS5100 – Foundations of Artificial Intelligence
Fall 2022 Semester, Richard Hoshino

Problem Set #5

This Problem Set is due at **6PM on Wednesday November 16**.

This Problem Set will be marked out of 40, and is worth 8% of the final course grade. There are five problems, each worth eight marks.

Problems #1 and #2 and #3 are to be completed individually, while Problems #4 and #5 are to be completed in your assigned small groups. For the two group problems, you are strongly encouraged, though not required, to have a different lead author for each of parts (a), (b), and (c).

Please type (or neatly handwrite) your solutions on standard 8.5×11 paper, with your name(s) at the top of each solution. Ensure that you submit five *separate* PDF files on Canvas, one for each problem. Make sure you label your Problem Set #5 submissions appropriately - e.g. richard5-1.pdf, richard5-2.pdf, richard5-3.pdf, richard5-4.pdf, richard5-5.pdf.

Given that the last two problems are done in a group, your final two PDF files will be identical to some of your classmates. (For example, richard5-4.pdf might be identical to yvonne5-4.pdf and bethany5-4.pdf). This is completely fine, and enables you to have a record of all of your submitted work in this course.

While a solution must be absolutely perfect to receive full marks, I will be generous in awarding partial marks for incomplete solutions that demonstrate progress.

So that there is no ambiguity, there are two non-negotiable rules. A violation of either rule constitutes plagiarism and will result in you receiving an F for this course.

(a) If you meet with a classmate to discuss one of the three Individual Problems, the articulation of your thought process (i.e., what you submit to me), must be an individual activity, done in your own words, away from others. Please remember that the solution-writing process is where so much of your learning will occur in this course: much more than anything we do in class, and even more than the time you spend on solving the problems. Do not be surprised if it takes you 3 to 5 times as long to write up a solution than it takes you to actually solve the problem. (For me, as an academic researcher writing formal proofs for publications, my ratio is significantly higher!)

(b) This Problem Set has been designed to be challenging, because struggling through problems is how we learn best. Your educational experience is cheapened by going online and finding the solution to a problem; even using the Internet to look for a "small hint" is unacceptable. In return, I will be readily available during our optional problem-solving workshops on Monday morning, and upon request, I will post hints to any questions you have on our class Canvas Page.

Problem #1 – INDIVIDUAL

Bob and Chris are major rivals, and they play the following "Stackelberg game" to decide who is superior.

There are two dice: Dice X and Dice Y. Both dice are fair, so each of the six faces is equally likely to turn up.

Dice X has the following numbers on its six faces: {7, 3, 3, 3, 3, 0}.
Dice Y has the following numbers on its six faces: {7, 7, 7, 0, 0, 0}.

Bob plays first. On his turn, he selects either of the two dice, rolls it, and scores the number of points that are rolled. Chris plays next. On his turn, he selects either of the two dice, rolls it, and scores the number of points that are rolled.

Whoever has the higher score wins the game.

If the game is tied after one round of rolls, then Bob and Chris repeat this game until there is a winner, with Bob always being the first to play in each round. Note that each player can select either Dice X or Dice Y on each of their turns.

(a) Suppose that in each round, b is the probability that Bob wins, c is the probability that Chris wins, and t is the probability of a tie. The game continues until one of the two players has won.

Let W_B be the probability that Bob wins the game, and let W_C be the probability that Chris wins the game. Prove that $W_B = \frac{b}{b+c}$ and $W_C = \frac{c}{b+c}$.

Hint: there are many ways to solve this problem, including using an infinite geometric series: $a + ar + ar^2 + ar^3 + \ldots$.

(b) Suppose that Bob always chooses Dice Y. If Bob rolls 7, explain why Chris will always respond by choosing Dice Y, and if Bob rolls 0, explain why Chris will always respond by choosing Dice X.

If Bob always chooses Dice Y in each round, prove that $b = \frac{1}{4}$, $c = \frac{5}{12}$, and $t = \frac{1}{3}$.

Use this information to determine the winning probabilities W_B and W_C in the case that Bob always chooses Dice Y.

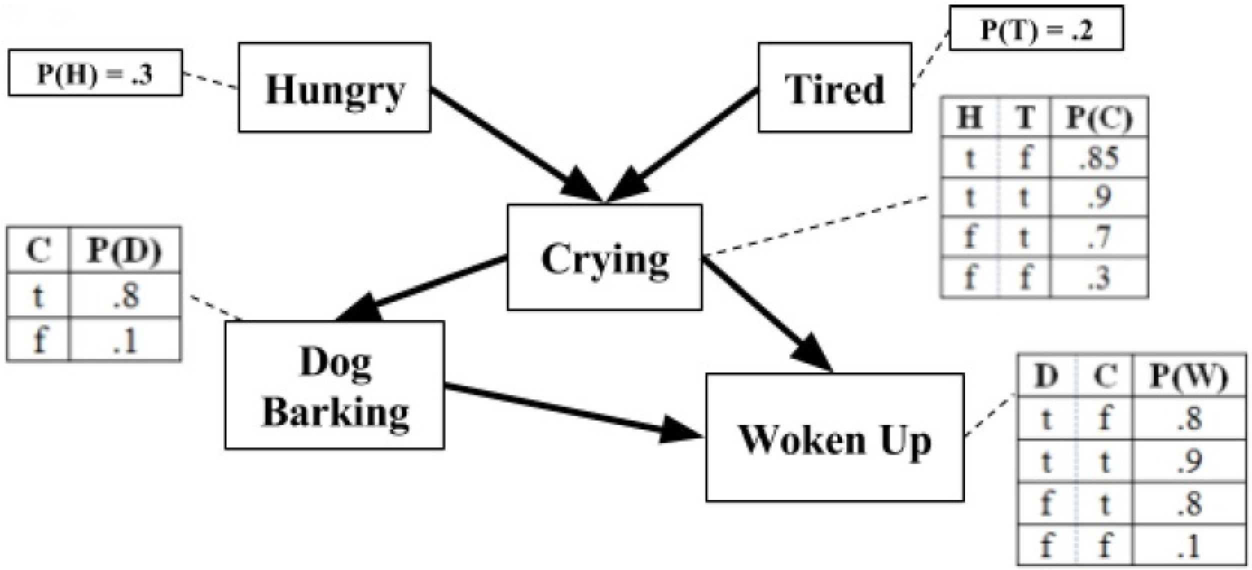
(c) Both Bob and Chris play this game optimally.

Determine W_B and W_C. Clearly justify your answer.

p.s. For those of you into sports, you may have noticed that this Stackelberg game models overtime in football. Incidentally, Bob Stoops and Chris Petersen were the head football coaches of the two universities (Oklahoma and Boise State) who played arguably the most famous college football game in history.

Problem #2 – INDIVIDUAL

Consider the following Bayesian Network.



In this Bayesian Network, there are five states: **H** (baby is hungry), **T** (baby is tired), **C** (baby is crying), **D** (dog is barking), and **W** (parent is woken up).

The above probability tables provide all of the information in this Bayesian Network.

For example, $P(+W \mid +D, -C) = 0.8$ is the probability that the parent is woken up given that the dog is barking and the baby is not crying. And $P(+W \mid +D, +C) = 0.9$ is the probability that the parent is woken up given that the dog is barking and the baby is crying.

Answer the following questions. Clearly and carefully justify each answer, showing all of your steps.

(a) Determine $P(+C)$, $P(+D)$, and $P(+W)$.

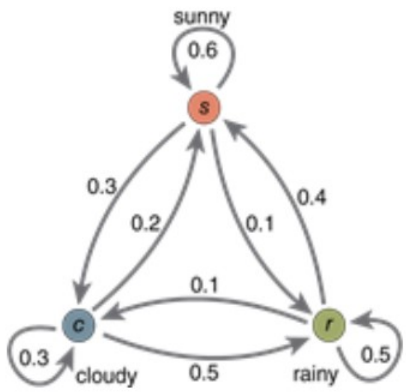
These three probabilities are respectively the probability that the baby is crying, the probability that the dog is barking, and the probability that the parent is woken up.

(b) Determine $P(+W|+H)$, the probability that the parent is woken up, given that the baby is hungry.

(c) Determine $P(+H|+W)$, the probability that the baby is hungry, given that the parent is woken up.

Problem #3 – INDIVIDUAL

In Chennai, India, each day's weather is one of three states: Sunny, Rainy, and Cloudy.



The diagram above provides the Markov Chain for how these three states are related. For example, if the weather is Sunny on day t, then on day $t+1$ the probability of Sunny is 0.6, the probability of Cloudy is 0.3, and the probability of Rainy is 0.1.

Let $P(X_t) = \begin{bmatrix} S_t \\ C_t \\ R_t \end{bmatrix}$ be a 3-dimensional vector representing the probability of the weather being Sunny, Cloudy, Rainy on day t. By definition, $S_t + C_t + R_t = 1$ for all days t.

Richard has no idea about the weather in Chennai, so his prior probability is $P(X_0) = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$. Thus,

$$P(X_1) = \begin{bmatrix} 0.6 \times 1/3 + 0.2 \times 1/3 + 0.4 \times 1/3 \\ 0.3 \times 1/3 + 0.3 \times 1/3 + 0.1 \times 1/3 \\ 0.1 \times 1/3 + 0.5 \times 1/3 + 0.5 \times 1/3 \end{bmatrix} = \begin{bmatrix} 0.400 \\ 0.233 \\ 0.367 \end{bmatrix}.$$

On each day, Niranjan's mood is either Happy or Angry. His mood is always influenced by that day's weather in his hometown of Chennai. Specifically,

> If it is Sunny, there is a 90% chance Niranjan is Happy and a 10% chance Niranjan is Angry.
>
> If it is Cloudy, there is a 80% chance Niranjan is Happy and a 20% chance Niranjan is Angry.
>
> If it is Rainy, there is a 70% chance Niranjan is Happy and a 30% chance Niranjan is Angry.

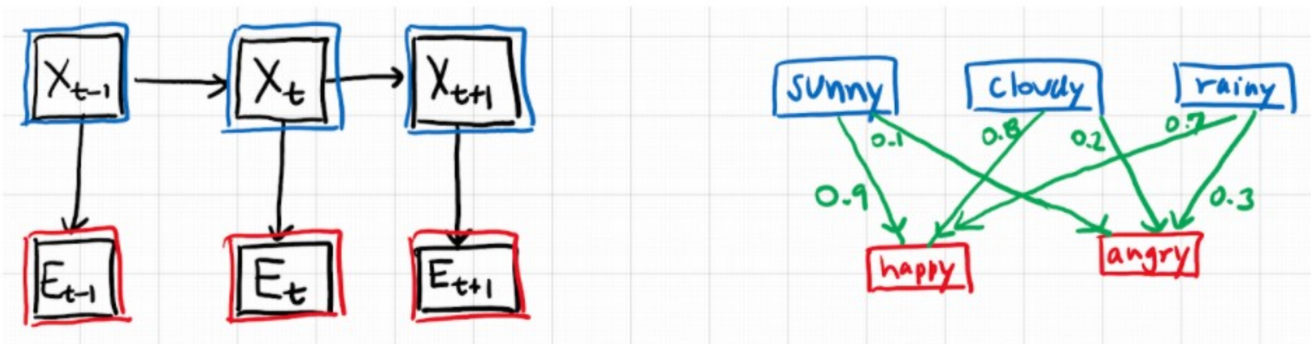
Richard and Niranjan meet on three consecutive days, and Richard notices that Niranjan is Happy on Day 1, Happy on Day 2, and Angry on Day 3.

Thus, $E_1 = Happy$, $E_2 = Happy$, $E_3 = Angry$.

From the above information, Richard can *update* his belief about Chennai's weather on Day 1, given that Niranjan is Happy on Day 1. Specifically, Richard can now calculate $P(X_1|E_1 = Happy)$.

On the previous page, we provided the transition probabilities for the (hidden) weather state, from X_t to X_{t+1}, as well as the emission (or sensor) probabilities from X_t to E_t.

This information can be represented using the following Hidden Markov Model.



Answer the following questions. Clearly and carefully justify each answer, showing all of your steps.

(a) As $t \to \infty$, $P(X_t)$ converges to a *steady-state vector* $P(X^*)$. Determine $P(X^*)$.

(b) Determine the filtering probability $P(X_3 \mid E_1 = Happy, E_2 = Happy, E_3 = Angry)$.

(c) Determine the prediction probability $P(X_5 \mid E_1 = Happy, E_2 = Happy, E_3 = Angry)$.

(d) (**OPTIONAL BONUS**).
Determine the smoothing probability $P(X_1 \mid E_1 = Happy, E_2 = Happy, E_3 = Angry)$.

Problem #4 – GROUP

To prepare yourself for this problem, carefully read (and re-read) the Gorodnichy-Hoshino paper discussed in class, "Calibrated Confidence Scoring for Biometric Identification".

This paper can be found on Canvas by clicking on Notes From Class, and then selecting "Richard's Biometrics Paper" from Week #9.

On the table are four coins: a nickel, a dime, a quarter, and a loonie. For non-Canadian students, these four coins have denomination \$0.05, \$0.10, \$0.25, and \$1.00, respectively.

You learn that three of these coins are "fair" (i.e., on any toss, heads and tails are each likely to come up with probability $\frac{1}{2}$). However, the fourth coin is biased, so that heads lands $\frac{2}{3}$ of the time and tails lands $\frac{1}{3}$ of the time.

You do not have any information on which coin is biased, so assume that each coin is equally likely to be the biased coin.

(a) I flip each of these coins 10 times, and we note that the nickel lands heads 6 times, the dime lands heads 5 times, the quarter lands heads 4 times, and the loonie lands heads 4 times.

Given this result, determine the probability that each of these coins is the biased coin.

(b) I flip each of these coins 20 times, and we note that the nickel lands heads 12 times, the dime lands heads 10 times, the quarter lands heads 8 times, and the loonie lands heads 8 times.

Given this result, determine the probability that each of these coins is the biased coin.

Also clearly explain why these probabilities are different from the answers you got in part (a), even though the proportion of landed heads is the same in both questions for all four coins.

(c) Prove the general formula: There are k coins, with one of them biased (with heads landing $\frac{2}{3}$ of the time and tails landing $\frac{1}{3}$ of the time), and the other $k-1$ coins being fair.

Suppose that you flip each of the k coins n times. For each $1 \leq i \leq k$, suppose that the i^{th} coin lands heads exactly h_i times.

Find a formula for c_i, the calibrated confidence probability score that the i^{th} coin is the biased coin.

Does your formula depend on the value of n? Explain.

Problem #5 – GROUP

In this course, you have done a great deal of *problem-solving*, where you solve interesting and engaging problems created by me. As you can imagine, I (Richard) have gained a deep conceptual mastery of key concepts in the CS 5100 course through my experience of problem-posing.

In this question, you too will develop the skill of *problem-posing*, where you will create an interesting and engaging problem, and solve it yourself. According to one education researcher, "when students pose their own problems, they can enhance their mathematical (and computer science) knowledge, stimulate critical thinking, and improve computational skills by exploring their curiosity about specific concepts".

For each of these four questions, you will pose an original problem, and solve it using the method of your choice (e.g. writing a computer program, solving it by hand). You are permitted to use the Internet to generate ideas for your problem, but the problem itself must be original and not be a problem you find in a textbook or website.

Each member of your group must make a contribution, i.e., each of you will choose one of these four topics, and create an original problem.

(a) Create an original problem that relates to **Markov Decision Processes**, and solve it.

(b) Create an original problem that relates to **Reinforcement Learning**, and solve it.

(c) Create an original problem that relates to **Bayesian Networks**, and solve it.

(d) Create an original problem that relates to **Hidden Markov Models**, and solve it.

IMPORTANT NOTE: if you are in a 3-person group, please drop one of these four questions. You choose which three questions to keep.