Hw5

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1

The probability that a node being active is P = 0.4

Then, the probability that a node is **not** being active is P = 1 - 0.4 = 0.6

1. P(I) = the probability that 0 node is active:

$$\begin{aligned} P(I) &= 0.6^6 \\ &= 0.0467 \end{aligned}$$

   P(S) = the probability that only 1 node is active, out of 6 nodes. That means it belongs to Binomial distribution

$$\begin{aligned} P(I) &= \binom{6}{1} \times 0.4^1 \times 0.6^5 \\ &= 6 \times 0.4^1 \times 0.6^5 \\ &= 0.187 \end{aligned}$$

2. Since *P(E) = 0.5*, *P(I|E) = 0.2*, *P(S|E) = 0.3*

   Thus, the probability the slot will be an collision slot given that the eavesdropper is listening:

$$\begin{aligned} P(C|E) &= 1 - P(I|E) - P(S|E) \\ &= \frac{P(C \cap E)}{P(E)} = 0.5 \\ \rightarrow P(C \cap E) &= 0.5 \times P(E) \\ &= 0.5 \times 0.5 \\ &= 0.25 \end{aligned}$$

   And, the probability that the time slot is collision is P(C):

$$\begin{aligned} P(C) &= 1 - P(I) - P(S) \\ &= 1 - 0.0467 - 0.187 \\ &= 0.766 \end{aligned}$$

Thus, the probability that the eavesdropper is listening given that the time slot is a collision:

$$\begin{aligned} P(E|C) &= \frac{P(E \cap C)}{P(C)} \\ &= \frac{P(C \cap E)}{P(C)} \\ &= \frac{0.25}{0.766} \\ &= 0.326 \end{aligned}$$

2

1. The CDF for k <= 10 is 0.

   Then, the CDF for k > 10 is:

$$\begin{aligned} F_X(k) &= P(X \leq k) \\ &= \int_{10}^{k} f(x)dx \\ &= \int_{10}^{k} \frac{10}{x^2}dx \\ &= 1 - \frac{10}{k} \end{aligned}$$

2. The probability that the device will fail within the first 15 hours, which mean $k = 15$

$$\begin{aligned} P(X \leq 15) &= 1 - \frac{10}{15} \\ &= 0.333 \end{aligned}$$

3. Now we know that the probability that the device will fail within the first 15 hours \$ P(fail)\$ $= \frac{1}{3}$

   Thus, the probability that the device will **not** fail within the first 15 hours \$ P(works) \$$= 1$ - $\frac{1}{3} = \frac{2}{3}$

   The probability that the $4th$ device will be the **first one** that **does not** fail within the first 15 hours, which belongs to Geometric distribution.

   So, k = 4.

$$\begin{aligned} P(X = k = 4) &= (\frac{1}{3})^{k-1} \times \frac{2}{3} \\ &= (\frac{1}{3})^3 \times \frac{2}{3} \\ &= \frac{2}{81} \\ &= 0.0247 \end{aligned}$$

3

According to the problem, $\lambda_A = \frac{1}{10}$, $\lambda_B = \frac{1}{5}$

The probability that have to wait more than 8 mins for a bus to arrive, t = 8:

$$\begin{aligned}
P(X > 8) &= P(X_A > 8 \cap X_B > 8) \\
&= (1 - P_A(X \leq 8)) \times (1 - P_B(X \leq 8)) \\
&= (1 - F_A(8)) \times (1 - F_B(8)) \\
&= (1 - (1 - e^{-\lambda_A t})) \times (1 - (1 - e^{-\lambda_B t})) \\
&= e^{-\frac{1}{10}8} \times e^{-\frac{1}{5}8} \\
&= 0.0907
\end{aligned}$$

4

1. In this problem, the mean $\mu = 71$ GBytes and the standard deviation $\sigma = 2.5$ GBytes.

Thus, map it to stander normal distribution: $N(0, 1)$, $Z = \frac{X-\mu}{\sigma}$.

$$\begin{aligned}
Z(72) &= \frac{X - \mu}{\sigma} \\
&= \frac{72 - 71}{2.5} \\
&= 0.4
\end{aligned}$$

By using the Z table, the probability that the flows are greater than 72 GBytes: $P(X > 72) = 1 - P(X \leq 72)$

$$\begin{aligned}
P(X > 72) &= 1 - P(X \leq 72) \\
&= 1 - P(Z \leq Z(72)) \\
&= 1 - \Phi(Z(72)) \\
&= 1 - \Phi(0.4) \\
&= 1 - 0.6554 \\
&= 0.3446 \\
&= 34.46\%
\end{aligned}$$

2. Given P(X < m) = 88.30%, find m. Mapping it to stander normal distribution: $N(0, 1)$, $Z = \frac{X-\mu}{\sigma}$.

$$\begin{aligned}
Z(m) &= \frac{X - \mu}{\sigma} \\
&= \frac{m - 71}{2.5}
\end{aligned}$$

Thus,

$$\begin{aligned}
P(X < m) &= P(Z < Z(m)) \\
&= P(Z < \frac{m - 71}{2.5}) \\
&= \Phi(\frac{m - 71}{2.5}) \\
&= 0.8830
\end{aligned}$$

According the Z-table, $\Phi(1.19) = 0.8830$. Thus,

$$\begin{aligned}
\frac{m - 71}{2.5} &= 1.19 \\
m &= 73.975
\end{aligned}$$

5

If use these percentages and take L = 1 to mean being a liar and F = 1 to mean failing the test. That means:

$$\text{Sensitivity } (\eta) = P(L = 1|F = 1) \text{ Liar Fail Test}$$

Since the machine pass 10 percent of the liars and fail 20 percent of the truth-tellers.

Thus,

$$\begin{aligned}
\text{Sensitivity } (\eta) &= P(L = 1|F = 1) \\
&= 1 - 10\% \\
&= 90\%
\end{aligned}$$

So, when L = 0 and F = 0 means being a truth teller pass the test.

$$\text{Specificity } (\theta) = P(T = 0|D = 0) \text{ Truth Teller Pass Test}$$

$$\begin{aligned}
\text{Specificity } (\theta) &= P(L = 0|F = 0) \\
&= 1 - 20\% \\
&= 80\%
\end{aligned}$$

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