Hw7

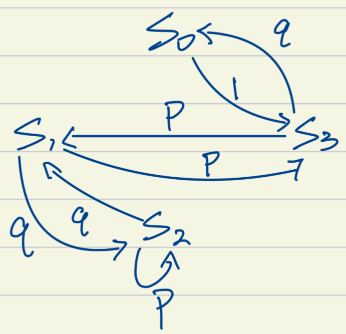
June 7, 2024

1

1. The transition matrix (in right-stochastic form) in terms of p and q:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p & q & 0 & 0 \\ 0 & p & q & 0 \\ 0 & 0 & p & q \end{bmatrix}$$

$S = \{0, 1, 2, 3\}$ is the number of umbrellas at current location.



$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & q & p \\ 2 & 0 & q & p & 0 \\ 3 & q & p & 0 & 0 \end{array}$$

2.

3. According to the π calculated by the R code below: $\bar{\lambda} = \pi = (0.1666667, 0.2777778, 0.2777778, 0.2777778)$

   So, The probability that the person will get wet = P(At state 0) $\times$ P(from 0 to 3 when it is rainy):

$$
\begin{aligned}
P(Wet) &= P(X_n = 0)P(X_{n+1} = 3|X_n = 0) \\
&= \lambda_0 \times P_{03} \\
&= 0.1666667 \times 0.4 \\
&= 0.06666667
\end{aligned}
$$

Thus, $P(Wet) = 0.06666667$

[1]:
```
m = matrix(0, nrow=4,ncol=4)  #define a vector
m[1,] = c(0,0,0,1)        #specify the row entries
m[2,] = c(0,0,0.6, 0.4)
m[3,] = c(0,0.6,0.4,0)
m[4,] = c(0.6, 0.4, 0, 0)
print(m)

e = eigen(t(m))                   #solve for the eigenvalues and eigenvectors of
 ↪the transpose matrix
print(e)                          #Note, the leading eigenvalue is 1, and all the
 ↪rest are smaller

pi = e$vectors[,1]/sum(e$vectors[,1])  #Extract the corresponding eigenvector
 ↪and normalize it
print(pi)
print(pi[1] * 0.4)
```

```
     [,1] [,2] [,3] [,4]
[1,]  0.0  0.0  0.0  1.0
[2,]  0.0  0.0  0.6  0.4
[3,]  0.0  0.6  0.4  0.0
[4,]  0.6  0.4  0.0  0.0
eigen() decomposition
$values
[1]  1.0000000 -0.9105598  0.6664830 -0.3559231

$vectors
           [,1]       [,2]       [,3]       [,4]
[1,] -0.3273268  0.4764328  0.4568312 -0.3605697
[2,] -0.5455447  0.4548329 -0.2965605  0.7111017
[3,] -0.5455447 -0.2082314 -0.6677210 -0.5644238
[4,] -0.5455447 -0.7230342  0.5074503  0.2138918

[1] 0.1666667 0.2777778 0.2777778 0.2777778
[1] 0.06666667
```

2

Given the expected value 16, $\mu = 16$.

Thus, according to Markov inequality:

$$\begin{aligned} P(X \geq 18) &= \frac{\mu}{18} \\ &= \frac{16}{18} \\ &= 0.8889 \end{aligned}$$

3

[12]:
```
set.seed(123)

K = c(100, 200, 400, 800, 1600, 3200, 6400)
mean = numeric(0)
var = numeric(0)
x = 1
for(i in K){
    sample = rpois(i,10)
    mean[x] = mean(sample)
    var[x] = var(sample)
    x = x + 1
}

print(mean)
print(var)
```

```
[1]  9.670000  9.930000 10.112500  9.951250  9.898125  9.975625 10.030625
[1]  7.718283 10.768945  9.318139 10.366832  9.763849  9.756205 10.225972
```

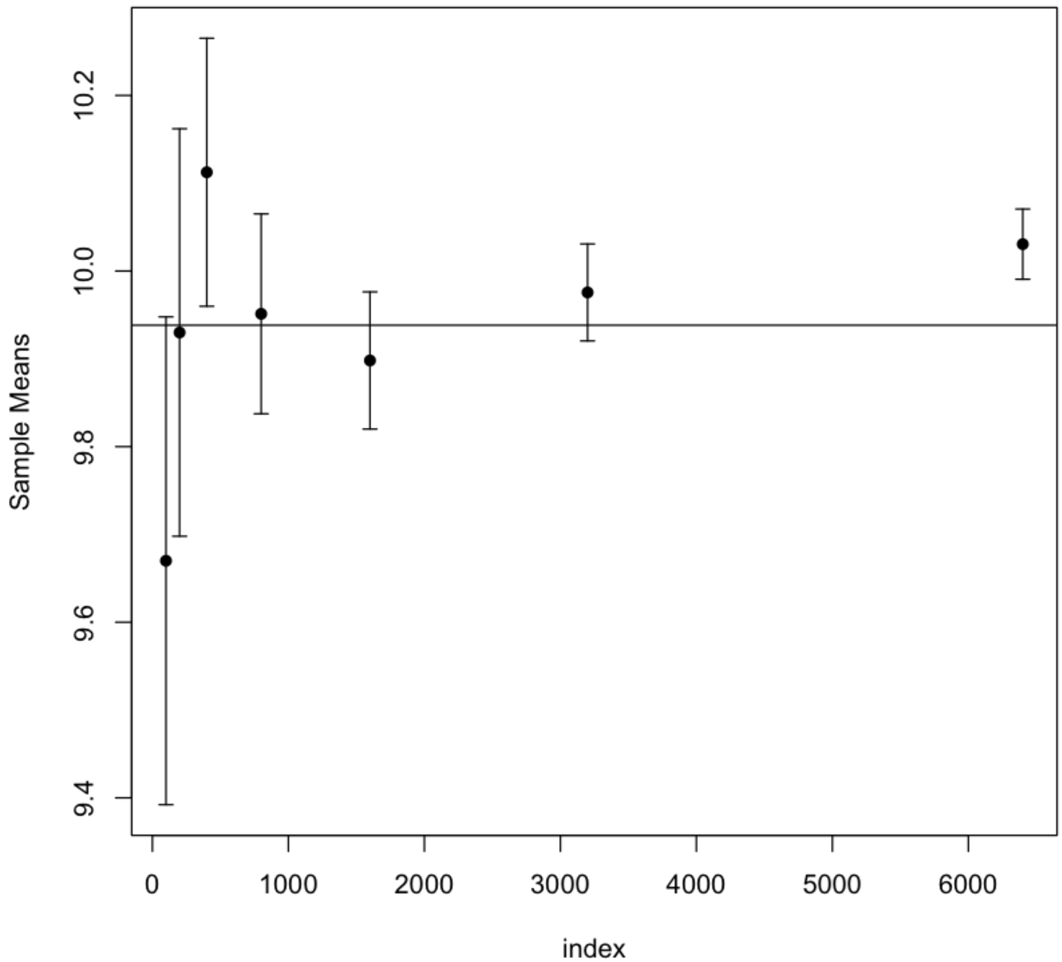
[3]:
```
library(Hmisc)
```

k	Sample mean	Sample variance
100	9.670000	7.718283
200	9.930000	10.768945
400	10.112500	9.318139
800	9.951250	10.366832
1600	9.898125	9.763849
3200	9.975625	9.756205
6400	10.030625	10.225972

1. The table:

2.

[4]:
```
stderr = sqrt(var/K)
errbar(K, mean, mean-stderr, mean+stderr, xlab = "index", ylab = "Sample Means")
abline(h=mean(mean))
```



3. According to the output of the R code below, the minimum number of observations = 61

[13]:
```
set.seed(123)
K = 0
P = 0
Sample = 0
Mean = 0
Var = 0
while (TRUE){
    K = K + 1                                  #number of samples
    Sample = rpois(K,10)
    Mean = mean(Sample)          #mean on K samples
    Var =  var(Sample)           # variance on K samples
```

```
    a = Mean-0.01*Mean                          #u-0.01*u
    b = Mean+0.01*Mean                          #u+0.01*u

    P = pnorm(b, Mean, Var/K) - pnorm(a, Mean, Var/K)  #P(  - 0.01  < X <   + 0.
↪01 ) = P(a < X < b)
    P = round(P, 2)

     if (!is.na(P) && abs(P - 0.97) < 0.01) {  # using a tolerance for␣
↪comparison
        break
    }
}
print(P)
print(K)
```

```
[1] 0.97
[1] 182
```

4

1. Sicne $X_1 = 1$, \$X_2 = 2 \$, $X_3 = 1 \sim$ Geom(P)

Thus,

$$\begin{aligned} L &= P \times (1-P)P \times P \\ &= P^3 - P^4 \end{aligned}$$

Set $\frac{dL}{dP} = 0$:

$$\frac{dL}{dP} = 3P^2 - 4P^3 = 0$$

$$P = \frac{3}{4}$$

Check if $\frac{d^2L}{dP^2} < 0$:

$$\begin{aligned} \frac{d^2L}{dP^2}(P = \frac{3}{4}) &= 6P - 12P^2 \\ &= -2.25 \end{aligned}$$

So, $P = \frac{3}{4}$ indeed maximizes L

2. Sicne $X_1 = 1$, \$X_2 = 2 \$, $X_3 = 2 \sim$ Geom(P) Thus,

$$\begin{aligned} L &= P \times (1-P)P \times (1-P)P \\ &= P^3 + P^5 - 2P^4 \end{aligned}$$

Set $\frac{dL}{dP} = 0$:

$$\frac{dL}{dP} = 3P^2 + 5P^4 - 8P^3 = 0$$

$$P_1 = 1, P_2 = 0.6$$

Check if $\frac{d^2L}{dP^2} < 0$ for $P_1 = 1, P_2 = 0.6$:

$$\begin{aligned}\frac{d^2L}{dP^2}(P=1) &= 6P + 20P^3 - 24P^2 \\ &= 2\end{aligned}$$

$$\begin{aligned}\frac{d^2L}{dP^2}(P=0.6) &= 6P + 20P^3 - 24P^2 \\ &= -0.72\end{aligned}$$

So, $P_2 = 0.6$ maximizes L

[ ]: