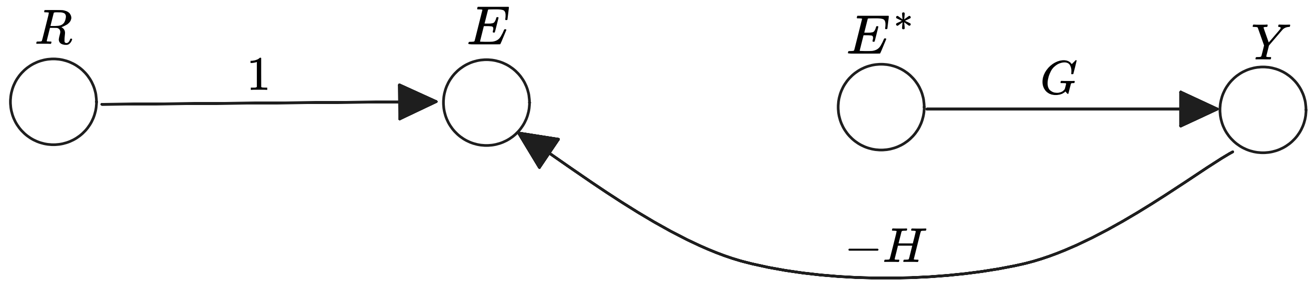
2019440102_이진우_HW02

1.a.

우선 original signal flow graph를 그립니다. ($G = \frac{1-e^{-Ts}}{s}G_p$)



이때 샘플러의 input은 E, output은 E^*입니다.

E와 Y를 나타냅니다.

$$E = R - HGE^*$$

$$Y = GE^*$$

위 두 식의 양 변에 모두 starred transform을 취합니다.

$$\begin{aligned} E^* &= R^* - \overline{HG}^* E^* \\ E^* &= \frac{R^*}{1+\overline{HG}^*} \end{aligned}$$

$$Y^* = G^*E^*$$

이 두 식을 정리하여 $\frac{Y^*}{R^*}$전달함수를 만듭니다.

$$\frac{Y^*}{R^*} = \frac{G^*}{1+\overline{HG}^*} \tag{1}$$

이 식을 z-transform하기 위해 각 $G, \overline{HG}$를 z-transform해줍니다.

$$\begin{aligned} G(s) &= \frac{1-e^{-Ts}}{s} \cdot \frac{1}{s} \\ G^d(z) &= (1-z^{-1})\mathcal{Z}\left[\frac{1}{s^2}\right] \\ &= \frac{z-1}{z} \cdot \frac{0.2z}{(z-1)^2} \\ &= \frac{0.2}{z-1} \end{aligned} \tag{2}$$

$$\begin{aligned} HG(s) &= \frac{1}{\tau s+1} \cdot \frac{1-e^{-Ts}}{s} \cdot \frac{1}{s} \\ \overline{HG}^d(z) &= (1-z^{-1})\mathcal{Z}\left[\frac{1/\tau}{s^2(s+1/\tau)}\right] \\ &= \frac{z-1}{z}\,\frac{z\left[(0.2/\tau - 1 + e^{-0.2/\tau})z + (1 - e^{-0.2/\tau} - 0.2/\tau e^{-0.2/\tau})\right]}{(1/\tau)(z-1)^2(z-e^{-0.2/\tau})} \\ &= \frac{(0.2/\tau - 1 + e^{-0.2/\tau})z + (1 - e^{-0.2/\tau} - (0.2/\tau)e^{-0.2/\tau})}{(1/\tau)(z-1)(z-e^{-0.2/\tau})} \end{aligned} \tag{3}$$

식 (1), (2), (3)에 따라 전달함수의 z-transform은 다음과 같습니다.

$$
\begin{aligned}
Y^d(z)/R^d(z) &= \frac{G^d(z)}{1+\overline{HG}^d(z)} \\
&= \frac{\frac{0.2}{z-1}}{1+\frac{(0.2/\tau-1+e^{-0.2/\tau})z+(1-e^{-0.2/\tau}-(0.2/\tau)e^{-0.2/\tau})}{(1/\tau)(z-1)(z-e^{-0.2/\tau})}} \\
&= \frac{0.2(1/\tau)(z-e^{-0.2/\tau})}{(1/\tau)(z-1)(z-e^{-0.2/\tau})+((0.2/\tau)-1+e^{-0.2/\tau})z+(1-e^{-0.2/\tau}-(0.2/\tau)e^{-0.2/\tau})} \\
&= \frac{0.2(z-e^{-0.2/\tau})}{(z-1)(z-e^{-0.2/\tau})+(0.2-\tau+\tau e^{-0.2/\tau})z+(\tau-\tau e^{-0.2/\tau}-0.2e^{-0.2/\tau})} \\
&\left(\epsilon = e^{-0.2/\tau}\right) \\
&= \frac{0.2(z-\epsilon)}{z^2+(-(0.8+\tau)+(\tau+1)\epsilon)z+(\tau+(0.8-\tau)\epsilon)}
\end{aligned}
$$

1.b.

전달함수의 분모와 분자에 $V^d(z)$를 곱하고 각 Y^d, R^d에 분자, 분모를 할당합니다. 또한, 그 식에 inverse z-transform을 취합니다.

$$
\begin{aligned}
Y^d(z) &= 0.2(z-\epsilon)V^d(z) \\
\rightarrow y^d(k) &= 0.2v^d(k) - 0.2\epsilon v^d(k+1)
\end{aligned}
$$

$$
\begin{aligned}
R^d(z) &= \{z^2+(-(0.8+\tau)+(\tau+1)\epsilon)z+(\tau+(0.8-\tau)\epsilon)\}V^d(z) \\
\rightarrow r^d(k) &= v^d(k+2)+(-(0.8+\tau)+(\tau+1)\epsilon)v^d(k+1)+(\tau+(0.8-\tau)\epsilon)v^d(k) \\
v^d(k+2) &= ((0.8+\tau)-(\tau+1)\epsilon)v^d(k+1)+(-\tau+(\tau-0.8)\epsilon)v^d(k)+r^d(k)
\end{aligned}
$$

위 식을 토대로 이산 시간 상태변수 모델을 만듭니다.

$$
\left(x^d(k) = \begin{bmatrix} v^d(k) \\ v^d(k+1) \end{bmatrix},\ x^d(k+1) = \begin{bmatrix} v^d(k+1) \\ v^d(k+2) \end{bmatrix}\right)
$$

$$
\begin{aligned}
x^d(k+1) &= \begin{bmatrix} 0 & 1 \\ -\tau+(\tau-0.8)\epsilon & (0.8+\tau)-(\tau+1)\epsilon \end{bmatrix} x^d(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r^d(k) \\
y^d(k) &= [0.2 \quad -0.2\epsilon] x^d(k)
\end{aligned}
$$

$$
\begin{aligned}
A^d &= \begin{bmatrix} 0 & 1 \\ -\tau+(\tau-0.8)e^{-0.2/\tau} & (0.8+\tau)-(\tau+1)e^{-0.2/\tau} \end{bmatrix} \\
B^d &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C^d &= \begin{bmatrix} 0.2 & -0.2e^{-0.2/\tau} \end{bmatrix}
\end{aligned}
$$

1.c.

A^d의 특성방정식을 구합니다.

$$
\begin{aligned}
0 &= \det(A^d - I\lambda) \\
&= -\lambda\left((0.8+\tau)-(\tau+1)e^{-0.2/\tau}-\lambda\right)+\tau+(0.8-\tau)e^{-0.2/\tau} \\
&= \lambda^2 - \left((0.8+\tau)-(\tau+1)e^{-0.2/\tau}\right)\lambda+\tau+(0.8-\tau)e^{-0.2/\tau}
\end{aligned}
$$

이때 이 시스템이 안정하려면 A^d의 모든 eigenvalue가 단위원에 들어와야 합니다. 따라서, 위 이차식의 근의 크기가 1보다 작아야 합니다. 이 조건을 식으로 나타내면 다음과 같습니다.

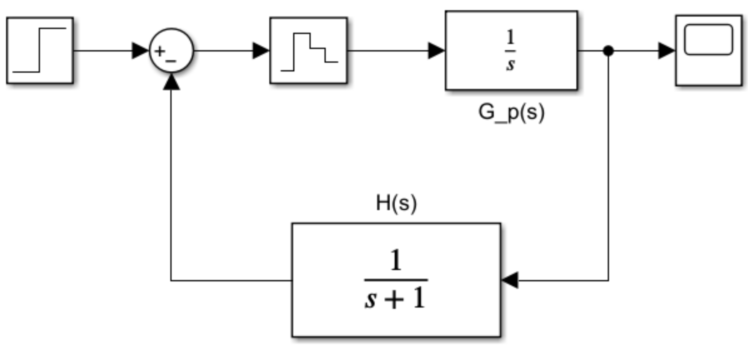
$$
\begin{aligned}
&\left(b = -(0.8+\tau)+(\tau+1)e^{-0.2/\tau}\right) \\
&\left(c = \tau+(0.8-\tau)e^{-0.2/\tau}\right) \\
&\lambda = -\frac{b}{2} \pm \sqrt{c}
\end{aligned}
$$

$$
\begin{aligned}
&c \geq 0 : \left|-\frac{b}{2} \pm \sqrt{c}\right| \leq 1 \\
&c < 0 : \frac{b^2}{4} + c^2 \leq 1
\end{aligned}
$$

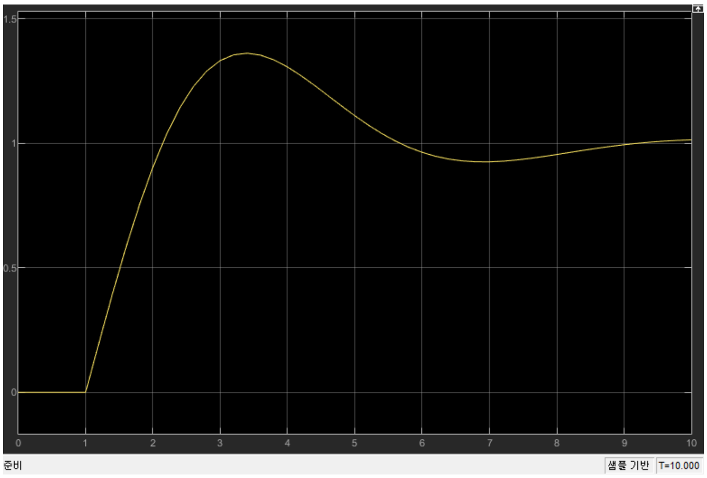
1.d.

$\tau = 1$ 일 때 $b = -0.16254,\ c = 0.83625$로 1.c의 조건을 만족합니다. 따라서 $H(s) = \frac{1}{s+1}$을 사용하였습니다.

폐루프 시스템 :



출력 파형 :



2.a.

matlab을 이용해 $G^d(z)$를 구하면 다음과 같습니다.

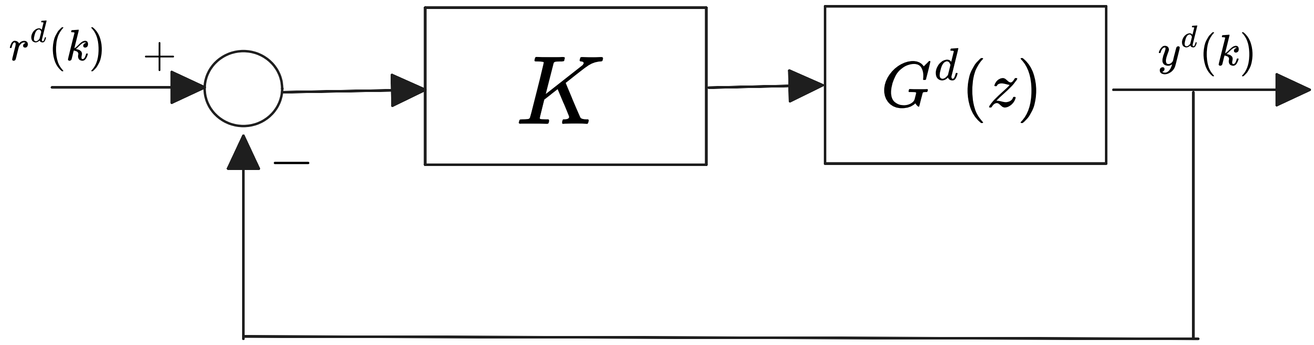
```
Gp_tf = tf(1,[1 1 0]);
T = 0.2;
Gd_tf = c2d(Gp_tf,T,'zoh')
```

```
Gd_tf =

   0.01873 z + 0.01752
  ----------------------
  z^2 - 1.819 z + 0.8187

샘플 시간: 0.2 seconds
이산시간 전달 함수입니다.
```

그러면 다음과 같은 폐루프 시스템과 같습니다



따라서 이것의 전달함수는 다음과 같습니다.

$$\begin{aligned}
Y^d(z)/R^d(z) &= \frac{KG^d(z)}{1+KG^d(z)} \\
&= \frac{K\frac{0.01873z+0.01752}{z^2-1.819z+0.8187}}{1+K\frac{0.01873z+0.01752}{z^2-1.819z+0.8187}} \\
&= \frac{K(0.01873z+0.01752)}{z^2-1.819z+0.8187+K(0.01873z+0.01752)} \\
&= \frac{K(0.01873z+0.01752)}{z^2+(K0.01873-1.819)z+(0.8187+0.01752K)}
\end{aligned}$$

2.b.

$K=1$일 때 final value theorem을 사용하여 정상상태 추종 오차를 구합니다.

$$\begin{aligned}
y^d(\infty) &= \lim_{z\to 1}(z-1)\frac{0.01873z+0.01752}{z^2-1.80027z+0.83622}\cdot\frac{z}{z-1} \\
&= \frac{0.03625}{0.03595} \\
&= 1.0083
\end{aligned}$$

$$e_{ss} = 1 - y^d(\infty) = -0.0083$$

$K=10$일 때 final value theorem을 사용하여 정상상태 추종 오차를 구합니다.

$$\begin{aligned}
y^d(\infty) &= \lim_{z\to 1}(z-1)\frac{0.1873z+0.1752}{z^2-1.6317z+0.9939}\cdot\frac{z}{z-1} \\
&= \frac{0.3625}{0.3622} \\
&= 1.00083
\end{aligned}$$

$$e_{ss} = 1 - y^d(\infty) = -0.00083$$

2.c.

final value theorem을 사용하여 정상상태 추종 오차를 구해봅니다.

$$\begin{aligned}
e_{ss} &= \lim_{k\to\infty}(r^d(k)-y^d(k)) \\
&= \lim_{z\to 1}(z-1)\left(1-Y^d(z)/R^d(z)\right)R^d(z) \\
&= \lim_{z\to 1}(z-1)\left[1-\frac{K(0.01873z+0.01752)}{z^2+(K0.01873-1.819)z+(0.8187+0.01752K)}\right]\cdot\frac{Tz}{(z-1)^2} \\
&= \lim_{z\to 1}\frac{z^2+(K0.01873-1.819)z+(0.8187+0.01752K)-K(0.01873z+0.01752)}{(z^2+(K0.01873-1.819)z+(0.8187+0.01752K))(z-1)}Tz \\
&= \lim_{z\to 1}\frac{(z^2-1.879z+0.8187)Tz}{(z^2+(K0.01873-1.819)z+(0.8187+0.01752K))(z-1)} \\
&= -\infty
\end{aligned}$$

추종 오차의 크기는 무한대로 발산하여 최소값은 없습니다.

3.a.

$$F^d(z) = z^4 + Kz^3 - 0.4z^2 + 0.5K = 0$$

z^0	z^1	z^2	z^3	z^4
$0.5K$	0	-0.4	K	1
1	K	-0.4	0	$0.5K$
$\frac{1}{4}K^2 - 1$	$-K$	$-\frac{1}{5}K + \frac{2}{5}$	$\frac{1}{2}K^2$	
$\frac{1}{2}K^2$	$-\frac{1}{5}K + \frac{2}{5}$	$-K$	$\frac{1}{4}K^2 - 1$	
$-K^2 + 1$	$-\frac{3}{20}K^3 + K - \frac{1}{5}$	$\frac{19}{20}K^3 + \frac{1}{10}K^2 + \frac{1}{5}K - \frac{2}{5}$		

$$\begin{aligned} &|0.5K/1| < 1 \to |K| < 2 \\ &\to K < 2 \end{aligned}$$

$$\begin{aligned} &\left|\left(\frac{1}{4}K^2 - 1\right) \Big/ \left(\frac{1}{2}K^2\right)\right| < 1 \\ &\to \left|\frac{1}{2} - \frac{2}{K^2}\right| < 1 \\ &\to \frac{2}{\sqrt{3}} \approx 1.1547 < K \end{aligned}$$

$$\begin{aligned} &\left|\left(-K^2 + 1\right) / \left(\frac{19}{20}K^3 + \frac{1}{10}K^2 + \frac{1}{5}K - \frac{2}{5}\right)\right| < 1 \\ &\to K > 0.812957 \end{aligned}$$

$$\therefore \frac{2}{\sqrt{3}} < K < 2$$

3.b.

$$F^d(z) = z^4 + Kz^3 - 0.4z^2 + 0.5K = 0$$

bilinear transform하여 w에 대해 특성다항식을 정리합니다.

$$F^d\left(\frac{2 - 2Tw}{T + T^2w}\right) = 0$$

$$\left(\frac{2 - 2Tw}{T + T^2w}\right)^4 + K\left(\frac{2 - 2Tw}{T + T^2w}\right)^3 - 0.4\left(\frac{2 - 2Tw}{T + T^2w}\right)^2 + 0.5K = 0$$

$$(2 - 2Tw)^4 + K(2 - 2Tw)^3(T + T^2w) - 0.4(2 - 2Tw)^2(T + T^2w)^2 + 0.5K(T + T^2w)^4 = 0$$

$$\begin{aligned} &\left(16 - 1.6T^2 + (0.5T^4 + 2T)K\right) \\ &+ \left(-64T + 6.4T^3 + (2T^5 - 32T^2)K\right)w \\ &+ \left(96T^2 - 19.2T^4 + (3T^6 + 112T^3)K\right)w^2 \\ &+ \left(-64T^3 + 6.4T^6 + (2T^7 - 96T^4)K\right)w^3 \\ &+ \left(16T^4 - 1.6T^8 + 0.5T^8K\right)w^4 = 0 \end{aligned}$$

정리된 특성다항식에 Routh-Hurwitz 판별법을 사용합니다.

$$\begin{array}{c|ccc} w^4 & 16T^4 - 1.6T^8 + 0.5T^8K & 96T^2 - 19.2T^4 + (3T^6 + 112T^3)K & 16 - 1.6T^2 + (0.5T^4 + 2T)K \\ w^3 & -64T^3 + 6.4T^6 + (2T^7 - 96T^4)K & -64T + 6.4T^3 + (2T^5 - 32T^2)K & 0 \\ w^2 & & & \\ w^1 & & & \\ w^0 & & & \end{array}$$

3.c.

3.d.

3.e.