Overview

In this note, we derive the semidefinte programming (SDP) and alternating direction method of multipliers (ADMM) gain design strategies used for the distributed motion planner of ACLSwarm [1]. The original SDP formulation follows Fathian et al. [2, 3]. The SDP formulation derived here is cast in standard form, which is particularly amenable to solving using ADMM methods as outlined in [4]. For completeness, we first present the distributed motion planning problem [2].

Swarm Motion Planning

Our goal is to describe the formation flying strategy that brings a swarm of n agents into a desired formation. A desired formation is defined by a graph $\mathcal{G}$ with vertices located at 3D points $p_1, \ldots, p_n$ and edges connecting the vertices. We assume that $\mathcal{G}$ is undirected, connected, and universally rigid [5]. Before the motion planning step, each agent in the swarm is assigned a unique formation point in $\mathcal{G}$ through task assignment. Here, we assume an identity assignment map for clarity; thus, agent 1 is assigned to p_1 and so on.

For motion planning, we model the i^{th} agent with single-integrator dynamics

$$\dot{q}_i = u_i, \tag{1}$$

where $q_i \in \mathbb{R}^3$ is position in a common global coordinate frame (unknown to the agent) and $u_i \in \mathbb{R}^3$ is the velocity control law. To bring the swarm into the desired formation, the control law can be computed as

$$u_i := \sum_{j \in \mathcal{N}_i} A_{ij} \left(q_j - q_i \right), \tag{2}$$

where $\mathcal{N}_i$ is the set of neighbors to agent i as defined by $\mathcal{G}$ and $A_{ij} \in \mathbb{A}$ is a constant gain matrix. These matrices lie in the space defined by

$$\mathbb{A} \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}. \tag{3}$$

By stacking each agent's position vector into $q = \begin{bmatrix} q_1^\top & \cdots & q_n^\top \end{bmatrix}^\top \in \mathbb{R}^{3n}$, the closed-loop dynamics under the control law (2) can be expressed as

$$\dot{q} = Aq, \tag{4}$$

$$A \stackrel{\text{def}}{=} \begin{bmatrix} -\sum_j A_{1j} & A_{12} & \cdots & A_{1n} \\ A_{21} & -\sum_j A_{2j} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & -\sum_j A_{nj} \end{bmatrix} \in \mathbb{S}^{3n}, \tag{5}$$

where for $j \notin \mathcal{N}_i$ the A_{ij} block is defined as a zero matrix and $\mathbb{S}^m$ is the space of symmetric $m \times m$ matrices.

Given desired formation points $p_1, \ldots, p_n$, we would like the swarm to be invariant to scale and formation heading (i.e., orientation about each agent's common z-axis). For $p_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^\top$, let $\bar{p}_i \stackrel{\text{def}}{=} \begin{bmatrix} -y_i & x_i & z_i \end{bmatrix}^\top$, $\bar{\bar{p}}_i \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & z_i \end{bmatrix}^\top$. Let $e_x \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$, $e_y \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$, $e_z \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top$. Using the matrix

$$N \stackrel{\text{def}}{=} \begin{bmatrix} p_1 & \bar{p}_1 & \bar{\bar{p}}_1 & e_x & e_y & e_z \\ p_2 & \bar{p}_2 & \bar{\bar{p}}_2 & e_x & e_y & e_z \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_n & \bar{p}_n & \bar{\bar{p}}_n & e_x & e_y & e_z \end{bmatrix} \in \mathbb{R}^{3n \times 6}. \tag{6}$$

we can leverage the following theorem for swarm invariance.

Theorem 1 (See [3, 6])**.** *Consider a swarm of agents with closed-loop dynamics (4). Assume blocks A_{ij} in (5) are chosen such that*

(i) the columns of N form a basis for $\ker(A)$*,*

(ii) all nonzero eigenvalues of A have negative real parts.

Then the swarm globally converges to the desired formation up to a translation, a rotation about the common z-axis, a scaling along the z-direction, and a scaling along the x-y directions of the common coordinate frame.

Proof. Note that since (4) is a linear time-invariant system, trajectories will converge to $\ker(A)$. As N is a basis of $\ker(A)$, the aggregate position vector q can converge to any linear combination of the columns of N. For a complete proof, see [6, Theorem 3, p. 29]. □

SDP Gain Design

To satisfy the conditions of Theorem 1, we formulate an SDP. We begin with the following naïve transcription of the swarm motion planning problem

$$\begin{aligned} \underset{A\in\mathcal{S}^{dn}_{-}}{\text{minimize}} \quad & \lambda_{\max}\left(A\right) \\ \text{subject to} \quad & AN = 0 \\ & A_{ij} \in \mathbb{A} && \forall_{i,j} \\ & A_{ij} = 0 && \forall_i \; \forall_{j\notin\mathcal{N}_i} \\ & \operatorname{tr}(A) = \text{constant}, \end{aligned} \tag{7}$$

where d is the dimension of the n formation points (i.e., 2D or 3D) and the trace constraint on A prevents the problem from becoming unbounded.

Note however that the objective in (7) is not very effective: the constraint that N be in the kernel of A requires that $\dim(N)$ of the eigenvalues of A be zero. Hence, the objective to minimize the maximum eigenvalue of A and this constraint are at odds. Using the orthogonal complement of N (i.e., $N^\perp$), we can use restrict the optimization objective to only the non-zero eigenvalues of A. A matrix that spans the orthogonal complement of N is found using the singular value decomposition (SVD) (see, e.g., Beard [7]).

Let $N = USV^\top$ be the SVD of $N \in \mathbb{R}^{dn\times r}$, with $\operatorname{rank}(N) = r$ (for the 2D problem, $r = 4$). The matrix $U \in \mathbb{R}^{dn\times dn}$ is decomposed into $U = (U_1 \;\; U_2)$, where $\mathcal{R}(N) = \operatorname{span}(U_1)$, $\mathcal{N}(N^\top) = \operatorname{span}(U_2)$, and $U_1 \in \mathbb{R}^{dn\times r} \oplus U_2 \in \mathbb{R}^{dn\times(dn-r)} = \mathbb{R}^{dn\times dn}$ (i.e., are orthogonal complements). Notice that U_1 also forms a basis of $\ker(A)$ since N forms a basis of $\ker(A)$ and $\operatorname{span}(U_1) = \mathcal{R}(N)$. Thus, the columns of U_2 do *not* null A—instead, they can be used to restrict A onto the orthogonal complement of $\mathcal{R}(N)$, removing the zero eigenvalues of A. Note that this restriction yields a $dn-r\times dn-r$ matrix.

Using $Q := U_2$, we can then write the following more effective optimization problem

$$\begin{aligned} \underset{A\in\mathcal{S}^{dn}_{-}}{\text{minimize}} \quad & \lambda_{\max}\left(Q^\top A Q\right) \\ \text{subject to} \quad & AN = 0 \\ & A_{ij} \in \mathbb{A} && \forall_{i,j} \\ & A_{ij} = 0 && \forall_i \; \forall_{j\notin\mathcal{N}_i} \\ & \operatorname{tr}(A) = \text{constant}, \end{aligned} \tag{8}$$

which is found in equation (5) of [1]. Note that this SDP gain design, as found in [3], does not scale well to a large number of vehicles.

Decoupling the 3D Formation Problem

Here, we consider a decoupling of the 3D gain design that will allow us to utilize ADMM. Observe from (3) that $A_{ij} \in \mathbb{A}$ has a block diagonal structure which can be expressed as

$$A_{ij} = \begin{bmatrix} D_{ij} & 0 \\ 0 & c_{ij} \end{bmatrix} \in \mathbb{R}^{3\times 3}. \tag{9}$$

This structure allows us to conclude that vehicle trajectories along the x-y and z components are decoupled and rely only on D_ij and c_{ij}, respectively. Therefore, solving the 3D gain design problem (8) with $d = 3$ is the same as solving a 2D subproblem and a 1D subproblem and the appropriately combining each ij block as in (9).

ADMM Gain Design

In [1] the gain design is formulated using ADMM for SDPs [4], which shows superior scalability and efficiency over the SDP approach. We first recast (8) into the following standard form suitable for applying ADMM (see [4])

$$\begin{aligned} \underset{X\in\mathcal{S}^{p}_{+}}{\text{minimize}} \quad & \langle C, X\rangle \\ \text{subject to} \quad & \mathcal{A}(X) = b, \end{aligned} \tag{10}$$

where the linear map $\mathcal{A} : \mathcal{S}^p \to \mathbb{R}^l$ is defined as $\mathcal{A}(X) := \begin{bmatrix}\langle A^{(1)}, X\rangle & \dots & \langle A^{(l)}, X\rangle\end{bmatrix}^\top$ for l constraints. Note that the constraint $\mathcal{A}(X) = b$ is equivalent to $\mathbf{A}\operatorname{vec}(X) = b$ where

$$\mathbf{A} := \begin{bmatrix}\operatorname{vec}\left(A^{(1)}\right) & \dots & \operatorname{vec}\left(A^{(l)}\right)\end{bmatrix}^\top \in \mathbb{R}^{l\times p^2}. \tag{11}$$

In pursuit of this standard form, we consider the 2D subproblem alone ($d = 2$).

Proposition 1. *Let $\bar{A} := -Q^\top A Q \in \mathbb{R}^{dm\times dm}$. Then $A = -Q\bar{A}Q^\top$ and Problem (8) is equivalent to*

$$\begin{aligned}
\underset{\bar{A}\in\mathcal{S}_+^{dm}}{\text{maximize}} \quad & \lambda_{\min}\left(\bar{A}\right) \\
\text{subject to} \quad & \bar{A}_{ij} \in \mathbb{A} && \forall_{i,j} \\
& \left[Q\bar{A}Q^\top\right]_{ij} = 0 && \forall_i \ \forall_{j\notin\mathcal{N}_i} \\
& \operatorname{tr}(\bar{X}) = \text{constant},
\end{aligned}$$

Proof. First, we note that Q is orthogonal, therefore $Q^{-1} = Q^\top$ and $A = -Q\bar{A}Q^\top$. For the optimization problems to be equivalent, this coordinate transform must preserve the structure of A so that $AN = 0$ and $A \in \mathbb{A}$. It is clear to see that $AN = -Q\bar{A}Q^\top N = 0$ because $Q^\top N = N^\top Q = 0$ by orthogonality ($\operatorname{span}(Q) = \mathcal{N}(N^\top)$). It remains to show that the linear transformation of $\bar{A}$ through Q preserves the structure of $\mathbb{A}$. Because Q is orthogonal, the structure of the 2×2 block is simply *rotated*. Consider the 2D complex representation of the formation problem for further insights. $\square$

To bring the result of Proposition 1 into standard form, we need to bring the constraints of the gain design problem into the form of the linear constraint $\mathcal{A}(\bar{A}) = \bar{b}$. First, we prove the following lemmas.

Lemma 1. *Suppose A satisfies Theorem 1. Then blocks $A_{ij} = 0_{3\times 3}$ if and only if $\operatorname{tr}(A_{ij}) = 0$.*

Proof. Sufficiency is trivial. To see necessity, note that blocks A_{ij} must have eigenvalues with strictly negative real parts (c.f. Theorem 1 (ii)). If $\operatorname{tr}(A_{ij}) = 0$, then $a = c = 0$ and $\lambda_{1,2} = \pm jb$ with $\operatorname{Re}\{\lambda_{1,2}\} = 0$, which is a contradiction. $\square$

Lemma 2. *Given $C = BAB^\top$, where $B^\top = \begin{bmatrix}\mathbf{b}_1 & \dots & \mathbf{b}_m\end{bmatrix} \in \mathbb{R}^{n\times m}$, the $ij-$th element is $C_{ij} = \operatorname{tr}(\mathbf{b}_j\mathbf{b}_i^\top A)$.*

Proof. By example. $\square$

Proposition 2. *The constraint that gain blocks $\left[Q\bar{A}Q^\top\right]_{ij} = 0$ for agents i,j that are not neighbors can be written in linear form as*

$$\operatorname{vec}(\mathbf{q}_s\mathbf{q}_r^\top)^\top\operatorname{vec}(\bar{A}) = 0 \qquad \text{and} \qquad \operatorname{vec}(\mathbf{q}_p\mathbf{q}_q^\top)^\top\operatorname{vec}(\bar{A}) = 0,$$

where $Q^\top = \begin{bmatrix}\mathbf{q}_1 & \dots & \mathbf{q}_{dn}\end{bmatrix} \in \mathbb{R}^{dm\times dn}$ and the indices are calculated as

$$\begin{aligned}
s &:= d(i-1)+1 & r &:= d(j-1)+1 \\
p &:= d(i-1)+1 & q &:= d(j-1)+2
\end{aligned}$$

where (s,r) corresponds to the $1,1$ entry of A_{ij} and (p,q) corresponds to the $1,2$ entry.

Proof. Follows from Lemma 1 and Lemma 2. $\square$

Remark 1. *In the $d = 1$ case, only one constraint in Proposition 2 is added, using the (s,r) indices.*

Lemma 3. *Consider an $md\times md$ matrix A with $d\times d$ blocks A_{ij}. The r,c-th element inside the i,j-th block can be indexed as $A_{\bar{r},\bar{c}}$ where*

$$\begin{aligned}
\bar{r} &= d(i-1)+r \\
\bar{c} &= d(j-1)+c.
\end{aligned}$$

Lemma 4. *Consider a $p\times p$ matrix X. Its vectorized form is $x = \operatorname{vec}(X) \in \mathbb{R}^{p^2}$. Element $X_{ij} = x_s$ where*

$$s = p(j-1)+i.$$

Proposition 3. *The structure constraint $\bar{A}_{ij} \in \mathbb{A}$ (for $d = 2$) can be written in linear form as*

$$\begin{bmatrix} 0 & \dots & 1_{1\times s_1} & \dots & -1_{1\times s_2} & \dots & 0 \\ 0 & \dots & 1_{1\times s_3} & \dots & 1_{1\times s_4} & \dots & 0 \end{bmatrix} \operatorname{vec}(\bar{A}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where

$$\begin{aligned}
s_1 &:= dm([d(j-1)+1]-1)+d(i-1)+1 & s_2 &:= dm([d(j-1)+2]-1)+d(i-1)+2 \\
s_3 &:= dm([d(j-1)+2]-1)+d(i-1)+1 & s_4 &:= dm([d(j-1)+1]-1)+d(i-1)+2
\end{aligned}$$

Proof. Follows from Lemma 3 and Lemma 4. □

For transforming the minimum eigenvalue objective into standard form, we use the epigraph formulation to recast it as a linear objective. Momentarily ignoring the constraints of the gain design problem, we have the following equivalences

$$\underset{\bar{A}\in\mathcal{S}_+^{dm}}{\text{maximize}} \quad \lambda_{\min}\left(\bar{A}\right) \quad\iff\quad \begin{aligned} &\underset{\bar{A}\in\mathcal{S}_+^{dm},\, t\in\mathbb{R}}{\text{maximize}} && t \\ &\text{subject to} && \lambda_{\min}\left(\bar{A}\right)\geq t \end{aligned} \quad\iff\quad \begin{aligned} &\underset{\bar{A}\in\mathcal{S}_+^{dm},\, t\in\mathbb{R}}{\text{maximize}} && t \\ &\text{subject to} && \bar{A}\succeq tI \end{aligned}$$

This is further equivalent to the following minimization problem where the p.s.d constraint on $\bar{A}$ is stated explicitly

$$\begin{aligned} &\underset{\bar{A}\in\mathcal{S}_+^{dm},\, t\in\mathbb{R}}{\text{minimize}} && \bar{t} \\ &\text{subject to} && \bar{A}-\bar{t}^{-1}I\succeq 0 \\ & && \bar{A}\succeq 0. \end{aligned} \tag{12}$$

Definition 1. *Let X be a symmetric matrix given by $X=\begin{bmatrix}A & B\\ B^\top & C\end{bmatrix}$. Then the Schur complement of A in X can be written $X/A=C-B^\top A^{-1}B$.*

Remark 2. *Given an X defined as in Definition 1, if $A\succ 0$ then $X\succeq 0 \iff X/A\succeq 0$.*

Using Definition 1, (12) can be written in standard form as

$$\underset{X\in\mathcal{S}_+^{2dm}}{\text{minimize}} \quad \langle C, X\rangle \qquad \text{with} \qquad C:=\begin{bmatrix}I_{dm} & 0_{dm}\\ 0_{dm} & 0_{dm}\end{bmatrix},\ X:=\begin{bmatrix}tI_{dm} & I_{dm}\\ I_{dm} & \bar{A}\end{bmatrix}. \tag{13}$$

Note that the p.s.d constraints on X and $\bar{A}-\bar{t}^{-1}I$ are satisfied given Remark 2 since $tI_{dm}\succ 0$.

Remark 3. *Although X is a decision variable of (13), a specific structure is required of it. This structure is enforced by including constraints on blocks X_{11}, $X_{12}=X_{21}$ and $\bar{A}$ in the linear constraint set $\mathcal{A}(X)=b$.*

Proposition 4. *If the formation graph is fully-connected, then $A=-QQ^\top$.*

Proof. Is this guaranteed? It seems to work empirically. □

Flat Planar Formations

Given a matrix A that satisfies Theorem 1, a team of robots can achieve 3D formations. Alternatively, flat planar formations may also be achieved. In this case, note that the rank of N drops by one because $\exists\alpha,\ \alpha\bar{\bar{p}}_i=e_z\ \ \forall i$. It is important to detect and adapt to this rank deficiency in the design of the gains A.

References

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