2023 SNU COURSE WORK - DECISION MAKING

HW #2

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1 Problem 2

$$\dot{x}_1 = ax_1 - x_1^3 + x_2, \tag{1}$$
$$\dot{x}_2 = u. \tag{2}$$

$x_{1_d} : [0, \infty) \to \mathbb{R}$, desired trajectory of x_1, is given. $a \in \mathbb{R}$ is assumed to be a constant scalar.

$$z_1(t) := x_1(t) - x_{1_d}(t) \tag{3}$$
$$z_2(t) := x_2(t) - x_{2_d}(t) \tag{4}$$

Here, $x_{2_d}(t)$ is the desired x_2 by regarding x_2 in (1) as a virtual input, which will be designed in each section.

1.1 (a) Backstepping control (a=1 is known)

x_{1_d} is assumed to be twice differentiable.

Lyapunov function for the outer-loop control:

$$V_1 := \frac{1}{2} z_1^2. \tag{5}$$

$$\dot{V}_1 := z_1 \dot{z}_1 \tag{6}$$
$$= z_1(x_2 + ax_1 - x_1^3 - \dot{x}_{1_d}). \tag{7}$$

Let $x_{2_d} := -k_1 z_1 - (ax_1 - x_1^3 - \dot{x}_{1_d})$ with a positive constant $k_1 \in \mathbb{R}_{>0}$. Then,

$$\dot{V}_1 = -k_1 z_1^2 + z_1 z_2, \tag{8}$$

and if $z_2 = 0$ (exact tracking of x_2 to x_{2_d}), the virtual input x_{2_d} makes $z_1 = 0$ a globally asymptotically stable equilibrium point of the outer-loop control system.

Note that the time derivative of x_{2_d} is given as follows,

$$\dot{x}_{2_d} = -k_1(\dot{x}_1 - \dot{x}_{1_d}) - (a\dot{x}_1 - 3x_1^2 \dot{x}_1 - \ddot{x}_{1_d}), \tag{9}$$

and this quantity requires the knowledge of a.

Now, define the Lyapunov function for the whole control system:

$$V_2 := V_1 + \frac{1}{2} z_2^2. \tag{10}$$

$$\dot{V}_2 := -k_1 z_1^2 + z_1 z_2 + z_2 \dot{z}_2 \tag{11}$$
$$= -k_1 z_1^2 + z_1 z_2 + z_2(u - \dot{x}_{2_d}) \tag{12}$$

Letting $u = \dot{x}_{2_d} - z_1 - k_2 z_2$ implies

$$\dot{V}_2 = -k_1 z_1^2 - k_2 z_2^2 \prec 0, \tag{13}$$

which makes $(z_1, z_2) = (0, 0)$ a globally asymptotically stable equilibrium point, and thus $x_1(t) \to x_{1_d}(t)$ as $t \to \infty$.

1.2 (b) Adaptive backstepping control (a=1 is unknown)

Assumption 1.1. $x_{1_d}, \dot{x}_{1_d}, \ddot{x}_{1_d}$ are all bounded.

We will introduce a new state $\hat{a}(t) \in \mathbb{R}$ for adaptive control. Since a is unknown, we modify the virtual input as $\hat{x}_{2_d} = x_{2_d}|_{a=\hat{a}} = -k_1 z_1 - (\hat{a}x_1 - x_1^3 - \dot{x}_{1_d})$ and corresponding error $\hat{z}_2 := x_2 - \hat{x}_{2_d}$. Define the estimation error $\tilde{a} := \hat{a} - a$. Also, note for $\hat{\dot{x}}_1 := \dot{x}_1|_{a=\hat{a}}$ that $\hat{\dot{x}}_1 - \dot{x}_1 = \tilde{a}x_1$ Use the following Lyapunov function candidate:

$$V := \frac{1}{2}z_1^2 + \frac{1}{2}\hat{z}_2^2 + \frac{1}{2}\gamma^{-1}\tilde{a}^2. \tag{14}$$

$$\dot{V} = z_1 \dot{z}_1 + \hat{z}_2 \dot{\hat{z}}_2 + \gamma^{-1}\tilde{a}\dot{\hat{a}} \tag{15}$$

$$\begin{aligned}
z_1 \dot{z}_1 &= z_1(\hat{\dot{x}}_1 - (\hat{\dot{x}}_1 - \dot{x}_1) - \dot{x}_{1_d}) && (16)\\
&= z_1((\hat{a}x_1 - x_1^3 + x_2) - \tilde{a}x_1 - \dot{x}_{1_d}) && (17)\\
&= z_1((\hat{a}x_1 - x_1^3 + \hat{x}_{2_d}) - \tilde{a}x_1 - \dot{x}_{1_d}) + z_1\hat{z}_2 && (18)\\
&= -k_1 z_1^2 + z_1\hat{z}_2 - z_1\tilde{a}x_1. && (19)
\end{aligned}$$

Note that

$$\begin{aligned}
\dot{\hat{x}}_{2_d} &= -k_1(\dot{x}_1 - \dot{x}_{1_d}) - (\dot{\hat{a}}x_1 + \hat{a}\dot{x}_1 - 3x_1^2\dot{x}_1 - \ddot{x}_{1_d}) && (20)\\
&= -k_1(\hat{\dot{x}}_1 - \dot{x}_{1_d}) - (\dot{\hat{a}}x_1 + \hat{a}\hat{\dot{x}}_1 - 3x_1^2\hat{\dot{x}}_1 - \ddot{x}_{1_d}) + (k_1 + \hat{a} - 3x_1^2)(\hat{\dot{x}}_1 - \dot{x}_1) && (21)\\
&= -k_1(\hat{\dot{x}}_1 - \dot{x}_{1_d}) - (\dot{\hat{a}}x_1 + \hat{a}\hat{\dot{x}}_1 - 3x_1^2\hat{\dot{x}}_1 - \ddot{x}_{1_d}) + (k_1 + \hat{a} - 3x_1^2)\tilde{a}x_1, && (22)
\end{aligned}$$

and define its estimate as

$$\hat{\dot{\hat{x}}}_{2_d} := \dot{\hat{x}}_{2_d}|_{a=\hat{a}} = -k_1(\hat{\dot{x}}_1 - \dot{x}_{1_d}) - (\dot{\hat{a}}x_1 + \hat{a}\hat{\dot{x}}_1 - 3x_1^2\hat{\dot{x}}_1 - \ddot{x}_{1_d}), \tag{23}$$

and the estimation error is $\hat{\dot{\hat{x}}}_{2_d} - \dot{\hat{x}}_{2_d} = -(k_1 + \hat{a} - 3x_1^2)\tilde{a}x_1$.

Then,

$$\begin{aligned}
\hat{z}_2\dot{\hat{z}}_2 &= \hat{z}_2(u - \dot{\hat{x}}_{2_d}) && (24)\\
&= \hat{z}_2(u - \hat{\dot{\hat{x}}}_{2_d} - (k_1 + \hat{a} - 3x_1^2)\tilde{a}x_1). && (25)
\end{aligned}$$

Letting $u = \hat{\dot{\hat{x}}}_{2_d} - z_1 - k_2\hat{z}_2$ implies

$$\hat{z}_2\dot{\hat{z}}_2 = -z_1\hat{z}_2 - k_2\hat{z}_2^2 - \hat{z}_2(k_1 + \hat{a} - 3x_1^2)\tilde{a}x_1. \tag{26}$$

Therefore, the following adaptive law,

$$\dot{\hat{a}} = \gamma(z_1 + \hat{z}_2(k_1 + \hat{a} - 3x_1^2)), \tag{27}$$

implies

$$\dot{V} = -k_1 z_1^2 - k_2 \hat{z}_2^2 - (z_1 + \hat{z}_2(k_1 + \hat{a} - 3x_1^2))\tilde{a}x_1 + \gamma^{-1}\tilde{a}\dot{\hat{a}} = -k_1 z_1^2 - k_2 \hat{z}_2^2 \leq 0. \tag{28}$$

Note that V is non-increasing, and therefore, z_1, $\hat{z}_2$, and $\tilde{a}$ is bounded. This also implies that $\hat{a}$ is bounded. By assumption, $x_1 = z_1 + x_{1_d}$ is also bounded. Now, observe this:

$$\ddot{V} = -2(k_1 z_1 \dot{z}_1 + k_2 \hat{z}_2 \dot{\hat{z}}_2). \tag{29}$$

From (16) and (24), $z_1 \dot{z}_1$ and $\hat{z}_2 \dot{\hat{z}}_2$ contains x_1, z_1, $\hat{z}_2$, $\hat{a}$, $\tilde{a}$ only, and therefore, $\ddot{V}$ is bounded.

By Barbalat's lemma, this implies that $\dot{V} \to 0$ as $t \to \infty$, which shows that $(z_1, \hat{z}_2) = (0, 0)$ is a globally asymptotically stable equilibrium point.

Theorem 1.1 (Barbalat's lemma, modified)**.** Given $f : [0, \infty) \to \mathbb{R}$, if $f(t)$ has a finite limit and $\ddot{f}$ is bounded, then $\dot{f} \to 0$ as $t \to \infty$.