Unit 11: Applications – Econometrics

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Contents

1 Applications: Econometrics

In this unit, we study a few more advanced examples of how to use Python to perform common econometric tasks. We will go beyond just calling existing functions that someone implemented, but we instead will implement these ourselves to understand the underlying concepts.

1.1 Preliminaries: Drawing bivariate samples

In most of the exercises below, we'll need to draw a random sample that serves as an input. We therefore first define a routine which returns a sample drawn from a bivariate normal distribution.

In line with what we learned in unit 10, we check arguments and raise an exception if a an invalid value is encountered.

[1]:
```
import numpy as np
from numpy.random import default_rng

def draw_bivariate_sample(mean, std, rho, n, seed=123):
    """
    Draw a bivariate normal random sample.

    Parameters
    ----------
    mean : array_like
        Length-2 array of means
    std : array_like
        Length-2 array of standard deviations
    rho : float
        Correlation parameter
    n : int
        Sample size
    """

    if not -1 <= rho <= 1:
        raise ValueError(f'Invalid correlation parameter: {rho}')

    if np.any(np.array(std) <= 0):
        raise ValueError(f'Invalid standard deviation: {std}')
```

```python
    if n <= 0:
        raise ValueError(f'Invalid sample size: {n}')

    # initialize default RNG with given seed
    rng = default_rng(seed)

    # Unpack standard deviations for each dimension
    std1, std2 = std

    # Compute covariance
    cov = rho * std1 * std2

    # Create variance-covariance matrix
    vcv = np.array([[std1**2.0, cov],
                    [cov, std2**2.0]])

    # Draw MVN random numbers:
    # each row represents one sample draw.
    X = rng.multivariate_normal(mean=mean, cov=vcv, size=n)

    return X
```

1.2 Singular value decomposition (SVD) and principal components

Singular value decomposition is a matrix factorisation that is commonly used in econometrics and statistics. For example, we can use it to implement principal component analysis (PCA), principal component regression, OLS or Ridge regression.

Let $\mathbf{X} \in \mathbb{R}^{m \times n}$ be a matrix. For our purposes, we will assume that $m \geq n$ since $\mathbf{X}$ will be the matrix containing the data with observations in rows and variables in column. The (compact) SVD of $\mathbf{X}$ is given by

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}'$$

where $\mathbf{U} \in \mathbb{R}^{m \times n}$ and $\mathbf{V} \in \mathbb{R}^{n \times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

The elements σ_i are called singular values of $\mathbf{X}$, and Σ is arranged such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Since $\mathbf{U}$ is not necessarily square, it's not truly orthogonal, but its columns are still orthogonal to each other.

These matrices satisfy the following useful properties:

$$\begin{aligned} \mathbf{U}'\mathbf{U} &= \mathbf{I}_n \\ \mathbf{V}'\mathbf{V} &= \mathbf{V}\mathbf{V}' = \mathbf{I}_n \\ \mathbf{V}' &= \mathbf{V}^{-1} \end{aligned}$$

In Python, we compute the SVD using the `svd()` function from `numpy.linalg`.

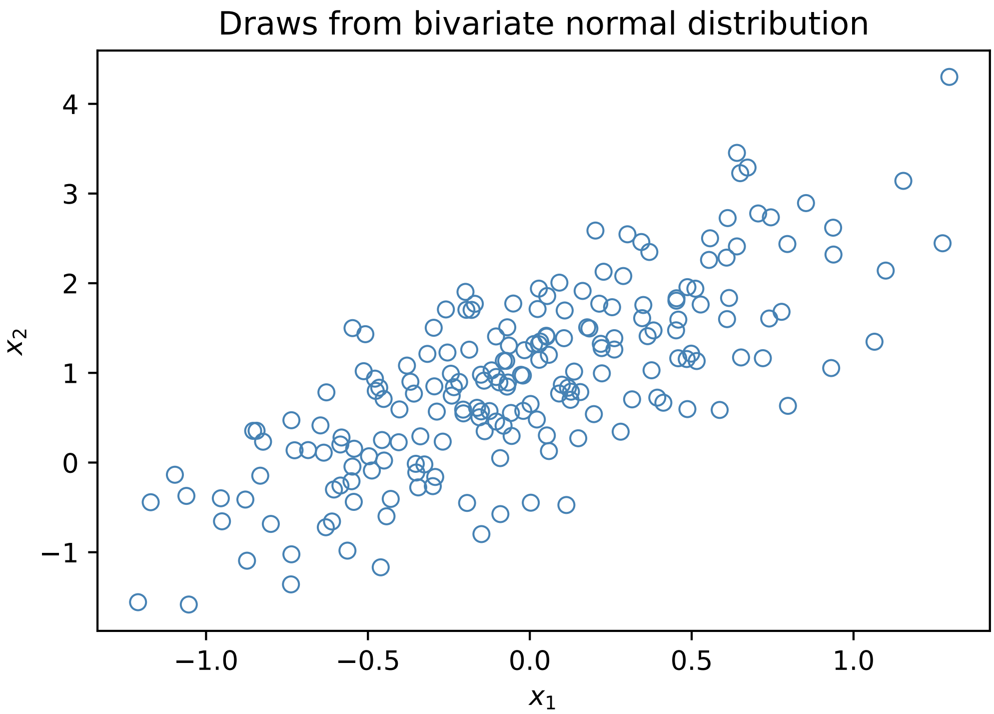
1.2.1 Example: Bivariate normal

Imagine we construct X as 200 random draws from a bivariate normal:

[2]:
```python
import numpy as np
import matplotlib.pyplot as plt
from numpy.random import default_rng
```

```
# Draw a bivariate normal sample using the function we defined above
mu = [0.0, 1.0]           # Vector of means
sigma = [0.5, 1.0]        # Vector of standard deviations
rho = 0.75                # Correlation coefficient
Nobs = 200                # Sample size
X = draw_bivariate_sample(mu, sigma, rho, Nobs)
x1, x2 = X.T

# Scatter plot of sample
plt.scatter(x1, x2, linewidths=0.75, c='none', edgecolors='steelblue')
plt.xlabel(r'$x_1$')
plt.ylabel(r'$x_2$')
plt.title('Draws from bivariate normal distribution')
```

[2]:
```
Text(0.5, 1.0, 'Draws from bivariate normal distribution')
```



We can now perform the SVD as follows:

[3]:
```
from numpy.linalg import svd

# svd() returns transposed V!
# We use full_matrices=False to get the compact factorisation, otherwise
# U is 200 x 200.
U, S, Vt = svd(X, full_matrices=False)
```

[4]:
```
# Check that U'U is a 2x2 identity matrix
U.T @ U          # or np.dot(U.T, U)
```

[4]:
```
array([[ 1.00000000e+00, -6.12878704e-17],
       [-6.12878704e-17,  1.00000000e+00]])
```

[5]:
```
# Check that V'V = VV' is a 2x2 identity matrix
Vt.T @ Vt        # or np.dot(Vt.T, Vt)
```

```
[5]: array([[ 1.00000000e+00, -2.26167254e-18],
            [-2.26167254e-18,  1.00000000e+00]])
```

```
[6]: # svd() does not return S as a matrix but only its diagonal!
     S
```

```
[6]: array([19.73152572,  5.99498933])
```

```
[7]: # We can convert it to a diagonal matrix using np.diag()
     np.diag(S)
```

```
[7]: array([[19.73152572,  0.        ],
            [ 0.        ,  5.99498933]])
```

Finally, we can multiply the output of `svd()` to verify that the result is equal to $\mathbf{X}$:

```
[8]: X_svd = U * S @ Vt

     # Compute the max. absolute difference
     diff = np.amax(np.abs(X - X_svd) )
     print(f"Max. absolute difference between X and USV': {diff:.2e}")
```

```
Max. absolute difference between X and USV': 2.89e-15
```

1.2.2 Example: Principal components

We use principal component analysis (PCA) as a dimension reduction technique, which allows us to identify an alternate set of axes along which the data in $\mathbf{X}$ varies the most. In machine learning, PCA is one of the most basic unsupervised learning techniques.

To perform the PCA, it is recommended to first demean the data:

```
[9]: X = draw_bivariate_sample(mu, sigma, rho, Nobs)

     # Demean variables in X
     Xmean = np.mean(X, axis=0)

     # Matrix Z stores the demeaned variables
     Z = X - Xmean[None]
```

We can now use the SVD factorisation to compute the principal components. Once we have computed the matrices $\mathbf{U}$, Σ and $\mathbf{V}$, the matrix of principal components (one in each column) is given by

$$PC = \mathbf{U}\Sigma$$

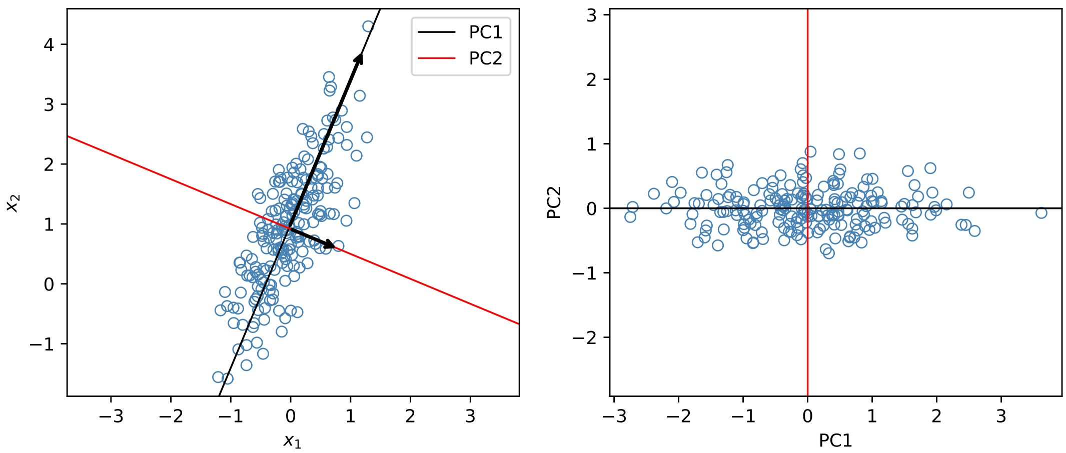
```
[10]: # Apply SVD to standardised values
      U, S, Vt = svd(Z, full_matrices=False)

      # Compute principal components
      PC = U * S          # same as U @ np.diag(S)

      # Variance is highest for first component
      var_PC = np.var(PC, axis=0, ddof=1)
      print(f'Principal component variances: {var_PC}')
```

```
Principal component variances: [1.17607859 0.09444617]
```

We can plot the principal component axes in the original data space (left columns). Moreover, the right column shows the data rotated and rescaled so that each axes corresponds to a principal component. Most of the variation clearly occurs along the first axis!

[11]:
```
# Plot principal components

# Scatter plot of sample
fig, axes = plt.subplots(1, 2, figsize=(10,4))
axes[0].scatter(X[:, 0], X[:, 1], linewidths=0.75, c='none',␣
↪edgecolors='steelblue')
axes[0].axis('equal')
axes[0].set_xlabel(r'$x_1$')
axes[0].set_ylabel(r'$x_2$')
axes[0].axline(Xmean, Xmean + Vt[0], label='PC1', lw=1.0, c='black', zorder=1)
axes[0].axline(Xmean, Xmean + Vt[1], label='PC2', lw=1.0, c='red', zorder=1)

PC_arrows = Vt * np.sqrt(var_PC[:, None])
for v in PC_arrows:
    # Scale up arrows by 3 so that they are visible!
    axes[0].annotate('', Xmean + v*3, Xmean, arrowprops=dict(arrowstyle='->',␣
↪linewidth=2))

axes[0].legend()

# Plot in principal component coordinate system
axes[1].scatter(PC[:, 0], PC[:, 1], linewidths=0.75, c='none',␣
↪edgecolors='steelblue')
axes[1].set_xlabel('PC1')
axes[1].set_ylabel('PC2')
axes[1].axis('equal')
axes[1].axvline(0.0, lw=1.0, c='red')
axes[1].axhline(0.0, lw=1.0, c='black')
```

[11]:
```
<matplotlib.lines.Line2D at 0x7ff5ec0f4430>
```



Of course, in real applications we don't need to manually compute the principal components, but can use a library such as scikit-learn to do it for us:

[12]:
```
from sklearn.decomposition import PCA

# Draw the same sample as before
X = draw_bivariate_sample(mu, sigma, rho, Nobs)

# Create PCA with 2 components (which is the max, since we have only two
# variables)
pca = PCA(n_components=2)
```

```
# Perform PCA on input data
pca.fit(X)

# The attribute components_ can be used to retrieve the V' matrix
print("Principal components (matrix V'):")
print(pca.components_)

# The attribute explained_variance_ stores the variances of all PCs
print(f'Variance of each PC: {pca.explained_variance_}')

# Fraction of variance explained by each component:
print(f'Fraction of variance of each PC: {pca.explained_variance_ratio_}')
```

```
Principal components (matrix V'):
[[ 0.38420018  0.92324981]
 [ 0.92324981 -0.38420018]]
Variance of each PC: [1.17607859 0.09444617]
Fraction of variance of each PC: [0.92566365 0.07433635]
```

1.3 Ordinary least squares (OLS)

Consider the regression

$$y_i = \mathbf{x}_i' \beta + u_i$$

where $\mathbf{x}_i$ is a vector of regressors (explanatory variables) that is assumed to include a constant. Recall that the OLS estimator $\widehat{\beta}$ is given by

$$\widehat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{y}$$

where $\mathbf{X}$ is the regressor matrix that contains all stacked $\mathbf{x}_i'$, and $\mathbf{y}$ contains all observations of the dependent variable.

1.3.1 Example 1: Bivariate data

We first demonstrate how to run OLS using bivariate normal data. With only one regressor, the regression simplifies to

$$y_i = \alpha + \beta x_i + u_i$$

where α is the intercept and β is the slope coefficient. In this special case, the population coefficient β can be computed using the formula

$$\beta = \frac{E[(Y - \overline{Y})(X - \overline{X})]}{E[(X - \overline{X})]} = \frac{Cov(Y, X)}{Var(X)}$$

where the numerator contains the covariance of the random variables Y and X, and the denominator contains the variance of X. Given a sample of values, the estimator $\widehat{\beta}$ is computed using the corresponding sample moments:

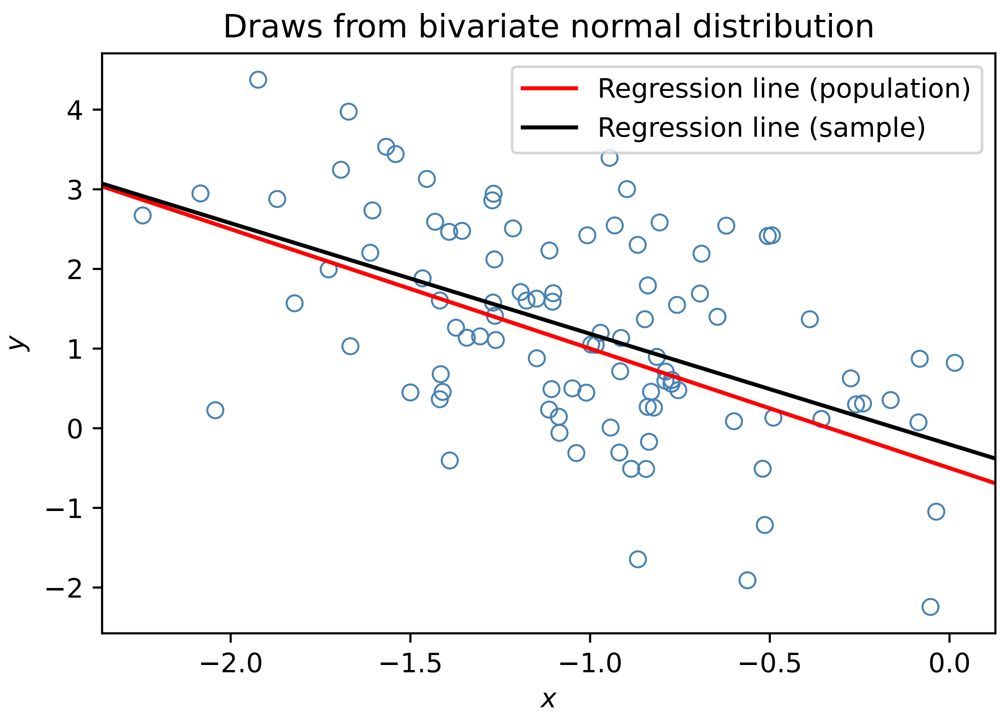
$$\widehat{\beta} = \frac{\widehat{Cov}(y, x)}{\widehat{Var}(x)}$$

[13]:
```
import numpy as np
import matplotlib.pyplot as plt

mu = [-1.0, 1.0]            # Mean of X and Y
std = [0.5, 1.5]            # Std. dev. of X and Y
rho = -0.5                  # Correlation coefficient
Nobs = 100                  # Sample size

# We transpose the return value and unpack individual rows into X and Y
x, y = draw_bivariate_sample(mu, std, rho, Nobs).T

# Compute beta (slope coefficient) from distribution moments.
```

```
# This is the true underlying relationship given our data generating process.
cov = rho * np.prod(std)
beta = cov / std[0]**2.0
print(f'Slope of population regression line: {beta}')

# Compute beta from sample moments
# Sample variance-covariance matrix (ddof=1 returns the unbiased estimate)
cov_hat = np.cov(x, y, ddof=1)[0, -1]
var_x_hat = np.var(x, ddof=1)
beta_hat = cov_hat / var_x_hat
# Sample intercept
alpha_hat = np.mean(y) - beta_hat * np.mean(x)

print(f'Slope of sample regression line: {beta_hat}')

# Scatter plot of sample
plt.scatter(x, y, linewidths=0.75, c='none', edgecolors='steelblue')
plt.xlabel(r'$x$')
plt.ylabel(r'$y$')
plt.title('Draws from bivariate normal distribution')
plt.axline(mu, slope=beta, color='red', label='Regression line (population)')
plt.axline((0, alpha_hat), slope=beta_hat, color='black', label='Regression line␣
↪(sample)')
plt.legend()
```

```
Slope of population regression line: -1.5
Slope of sample regression line: -1.3889613032802288
```

[13]:
```
<matplotlib.legend.Legend at 0x7ff5c7733fd0>
```



1.3.2 Example 2: OLS using matrix algebra

With more than one regressor, we need to use matrix algebra to perform the OLS estimation. For demonstration purposes, we continue using the bivariate data generated above, but now we write the OLS regression as

$$y_i = \mathbf{x}_i'\gamma + u_i$$

where $\gamma = (\alpha, \beta)$, and the regressors now contain a constant, $\mathbf{x_i} = (1, x_i)'$. As stated above, the OLS estimator is given by

$$\widehat{\gamma} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

Naive solution You might be tempted to solve the above equation system by explicitly computing the inverse of $\mathbf{X}'\mathbf{X}$ using NumPy's `inv()` like this:

[14]:
```
from numpy.linalg import inv

# We transpose the return value and unpack individual rows into X and Y
x, y = draw_bivariate_sample(mu, std, rho, Nobs).T

# Create vector of ones (required to estimate the intercept)
ones = np.ones((len(x), 1))
# Prepend constant to vector of regressors to create regressor matrix X
X = np.hstack((ones, x[:, None]))

# Compute inverse of X'X
XXinv = inv(X.T @ X)

print("Explicitly computed (X'X)^(-1):")
print(XXinv)

# Compute naive estimate of gamma
gamma_naive = XXinv @ X.T @ y
print(f'Naive estimate of gamma: {gamma_naive}')
```

```
Explicitly computed (X'X)^(-1):
[[0.05633363 0.04521468]
 [0.04521468 0.04412275]]
Naive estimate of gamma: [-0.20352351 -1.3889613 ]
```

This might seems like a straightforward way to implement OLS, but in practice you should *never* do this. Explicitly taking the inverse of a matrix to solve an equation system is rarely a good idea and numerically unstable, even though in this particular case it yields the same result!

Solving as a linear equation system One numerically acceptable way to run OLS is to view it as a linear equation system. Recall that a linear equation system can be written in matrix notation as

$$\mathbf{A}\mathbf{z} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{k \times k}$ is a coefficient matrix of full rank, $\mathbf{b} \in \mathbb{R}^k$ is a vector, and $\mathbf{z} \in \mathbb{R}^k$ is a vector of k unknows we want to solve for. The OLS estimator can be written in this form if we set

$$\mathbf{A} = \mathbf{X}'\mathbf{X}$$
$$\mathbf{b} = \mathbf{X}'\mathbf{y}$$
$$\mathbf{z} = \widehat{\gamma}$$

so that we have

$$(\mathbf{X}'\mathbf{X})\widehat{\gamma} = \mathbf{X}'\mathbf{y}$$

We can use NumPy's `solve()` to find $\widehat{\gamma}$:

[15]:
```
from numpy.linalg import solve

# Compute X'X
A = X.T @ X
# Compute X'y
b = X.T @ y

# Solve for coefficient vector
gamma_solve = solve(A, b)
print(f'Estimate of gamma using solve(): {gamma_solve}')
```

```
Estimate of gamma using solve(): [-0.20352351 -1.3889613 ]
```

Of course, running OLS (or equivalently: solving an overdetermined linear equation system) is a common task, so NumPy has the function `lstsq()` which allows you do to it without explicitly computing $\mathbf{X}'\mathbf{X}$ or $\mathbf{X}'\mathbf{y}$:

[16]:
```
from numpy.linalg import lstsq

# Estimate using lstsq(). Pass rcond=None to suppress a warning.
gamma_lstsq, *rest = lstsq(X, y, rcond=None)

print(f'Estimate of gamma using lstsq(): {gamma_lstsq}')
```

```
Estimate of gamma using lstsq(): [-0.20352351 -1.3889613 ]
```

1.3.3 Example 3: Implementing OLS yourself

NumPy's `lstsq()` uses SVD to compute the solution. Since we covered SVD in a previous exercise, we already have the tools to build our own implementation.

Recall that SVD factorises a regressor matrix $\mathbf{X}$ into three matrices,

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}'$$

We can use the orthogonality properties of $\mathbf{U}$ and $\mathbf{V}$ described above to transform the OLS estimator. We will be using the fact that the transpose of $\mathbf{X}$ is

$$\mathbf{X}' = \mathbf{V}\Sigma'\mathbf{U}' = \mathbf{V}\Sigma\mathbf{U}'$$

which follows since Σ is a diagonal (and thus symmetric) matrix. The OLS estimator can then be expressed as follows:

$$\begin{aligned}
\widehat{\gamma} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\
&= \left(\mathbf{V}\Sigma\mathbf{U}'\mathbf{U}\Sigma\mathbf{V}'\right)^{-1}\mathbf{V}\Sigma\mathbf{U}'\mathbf{y} \\
&= \left(\mathbf{V}\Sigma\mathbf{I}_k\Sigma\mathbf{V}'\right)^{-1}\mathbf{V}\Sigma\mathbf{U}'\mathbf{y} \\
&= \left(\mathbf{V}\Sigma^2\mathbf{V}'\right)^{-1}\mathbf{V}\Sigma\mathbf{U}'\mathbf{y}
\end{aligned}$$

This follows since $\mathbf{U}'\mathbf{U} = \mathbf{I}_k$ is an identity matrix where $k = 2$ is the number of coefficients we are estimating. Next, we can compute the inverse using the orthogonality properties of $\mathbf{V}$,

$$\begin{aligned}
\mathbf{V}\mathbf{V}' &= \mathbf{V}'\mathbf{V} = \mathbf{I} \\
\mathbf{V}' &= \mathbf{V}^{-1}
\end{aligned}$$

Therefore,

$$\left(\mathbf{V}\Sigma^2\mathbf{V}'\right)^{-1} = (\mathbf{V}')^{-1}\Sigma^{-2}\mathbf{V}^{-1} = \mathbf{V}\Sigma^{-2}\mathbf{V}'$$

Plugging this into the expression for the OLS estimator, we see that

$$
\begin{aligned}
\widehat{\gamma} &= \left(\mathbf{V}\Sigma^2\mathbf{V}'\right)^{-1}\mathbf{V}\Sigma\mathbf{U}'\mathbf{y} \\
&= \mathbf{V}\Sigma^{-2}\mathbf{V}'\mathbf{V}\Sigma\mathbf{U}'\mathbf{y} \\
&= \mathbf{V}\Sigma^{-2}\mathbf{I}_k\Sigma\mathbf{U}'\mathbf{y} \\
&= \mathbf{V}\Sigma^{-1}\mathbf{U}'\mathbf{y}
\end{aligned}
$$

Why is this preferable to the original expression? Since Σ is a diagonal matrix, its inverse is trivially computed as the element-wise inverse of its diagonal elements!

[17]:
```
from numpy.linalg import svd

# Request "compact" SVD, we don't need the full matrix U.
U, S, Vt = svd(X, full_matrices=False)

# Note that S returned by svd() is a vector that contains the diagonal
# of the matrix Sigma.
gamma_svd = Vt.T * S**(-1) @ U.T @ y
print(f'Estimate of gamma using SVD: {gamma_svd}')
```

```
Estimate of gamma using SVD: [-0.20352351 -1.3889613 ]
```

1.3.4 Example 4: OLS standard errors

All of the above methods only computed the *point estimates*, i.e., the coefficient vector. Usually, we are interested in performing inference, i.e., testing some hypothesis, for example whether our estimate is significantly different from zero. To this end, we need to compute standard errors which reflect the sampling uncertainty of our estimates.

Under the assumption of homoskedastic errors, the variance-covariance matrix of the OLS estimator $\widehat{\gamma}$ is given by the expression

$$
\begin{aligned}
Var(\widehat{\gamma}) &= \widehat{\sigma}^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} \\
\widehat{\sigma}^2 &= \frac{1}{n-k}\sum_{i=1}^{n}\widehat{u}_i^2
\end{aligned}
$$

where $\widehat{\sigma}^2$ is the sample variance of the residuals (recall that we have included an intercept in the model, so the mean of $\widehat{u}_i$ is zero!). Note the degree-of-freedom correction in the denominator for a model with k parameters (including any intercept).

Luckily, we can directly use our insights from the previous section and instead of computing $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ directly (which is numerically undesirable), we can rewrite it using the SVD factorisation as follows:

$$
\begin{aligned}
\left(\mathbf{X}'\mathbf{X}\right)^{-1} &= \left(\mathbf{V}\Sigma\mathbf{U}'\mathbf{U}\Sigma\mathbf{V}'\right)^{-1} \\
&= \left(\mathbf{V}\Sigma\mathbf{I}_k\Sigma\mathbf{V}'\right)^{-1} \\
&= \left(\mathbf{V}\Sigma^2\mathbf{V}'\right)^{-1} \\
&= \mathbf{V}\Sigma^{-2}\mathbf{V}'
\end{aligned}
$$

Extending the code from above, we can now compute the point estimate and the standard errors:

[18]:
```
from numpy.linalg import svd

# Request "compact" SVD, we don't need the full matrix U.
U, S, Vt = svd(X, full_matrices=False)

# Compute point estimate as before
gamma = Vt.T * S**(-1) @ U.T @ y
```

```python
# Compute (X'X)^-1
XXinv = Vt.T * S**(-2) @ Vt

# Residuals are given as u = y - X*gamma
residuals = y - X @ gamma

# Variance of residuals
k = X.shape[1]
var_u = np.var(residuals, ddof=k)

# Variance-covariance matrix of estimates
var_gamma = var_u * XXinv

# Standard errors are square roots of diagonal elements of Var(gamma)
gamma_se = np.sqrt(np.diag(var_gamma))

print(f'Point estimate of gamma: {gamma}')
print(f'Standard errors of gamma: {gamma_se}')
```

```
Point estimate of gamma: [-0.20352351 -1.3889613 ]
Standard errors of gamma: [0.26527596 0.23477147]
```

1.3.5 Example 5: Complete OLS estimation routine

We can combine all our previous code and encapsulate it in a function called `ols`, which makes sure the input data are NumPy arrays and have the same number of observations. We also add the optional parameter `add_const` which allows callers to automatically include a constant in the model.

[19]:
```python
def ols(X, y, add_const=False):
    """
    Run the OLS regression y = X * beta + u
    and return the estimated coefficients beta and their variance-covariance
    matrix.

    Parameters
    ----------
    X : array_like
        Matrix (or vector) of regressors
    y : array_like
        Vector of observations of dependent variable
    add_const : bool, optional
        If True, prepend a constant to regressor matrix X.
    """

    # Make sure we have a regressor matrix even if there is only a single
    # regressor
    X = np.atleast_2d(X)
    y = np.atleast_1d(y)

    # Check that arrays are of conformable dimensions, and raise an exception
    # if that is not the case
    Nobs = y.size
    if X.shape[0] != Nobs:
        raise ValueError('Non-conformable arrays X and y')

    # Check whether we need to prepend a constant
    if add_const:
        ones = np.ones((Nobs, 1))
        X = np.hstack((ones, X))

    # Request "compact" SVD, we don't need the full matrix U.
    U, S, Vt = svd(X, full_matrices=False)
```

```
    # Compute point estimate using SVD factorisation
    beta = Vt.T * S**(-1) @ U.T @ y

    # Compute (X'X)^-1 using SVD factorisation
    XXinv = Vt.T * S**(-2) @ Vt

    # Residuals are given as u = y - X*beta
    residuals = y - X @ beta

    # Number of model parameters
    k = X.shape[1]

    # Variance of residuals
    var_u = np.var(residuals, ddof=k)

    # Variance-covariance matrix of estimates
    var_beta = var_u * XXinv

    return beta, var_beta
```

1.4 OLS using housing data

We now proceed to run a more meaningful regression using the `ols()` function developed above. To this end, we use monthly observations from the file `HOUSING.csv` which contains various variables related to the US housing market. In particular, we will take the number of housing unit construction starts (variable `NHSTART`) in a given month and regress it on the average sales price of new homes (variable `ASPNHS`) lagged by 3, 6 and 12 months. We run the regression in logs, so the estimated coefficient should be interpreted as elasticities.

If you are familiar with Stata, the regression we are trying to run will look like this:

```
. regress log_nhstart L3.log_aspnhs L6.log_aspnhs L12.log_aspnhs if year >= 2000

      Source |       SS           df       MS      Number of obs   =       259
-------------+----------------------------------   F(3, 255)       =     19.27
       Model |  7.50833492         3  2.50277831   Prob > F        =    0.0000
    Residual |  33.1247614       255  .129901025   R-squared       =    0.1848
-------------+----------------------------------   Adj R-squared   =    0.1752
       Total |  40.6330963       258  .157492621   Root MSE        =    .36042

------------------------------------------------------------------------------
 log_nhstart |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  log_aspnhs |
         L3. |   2.293265   .5024764     4.56   0.000     1.303733    3.282797
         L6. |   .8001798   .5538507     1.44   0.150    -.2905242    1.890884
        L12. |   -1.94804   .4377757    -4.45   0.000    -2.810156   -1.085924
             |
       _cons |  -6.484041   2.783441    -2.33   0.021     -11.9655   -1.002581
------------------------------------------------------------------------------
```

1.4.1 Load and visually inspect the data

We first load and inspect the data using pandas's `read_csv()` function:

[20]:
```
import numpy as np
import pandas as pd

file = '../data/HOUSING.csv'
df = pd.read_csv(file)

# Inspect first and last rows of the DataFrame
df
```

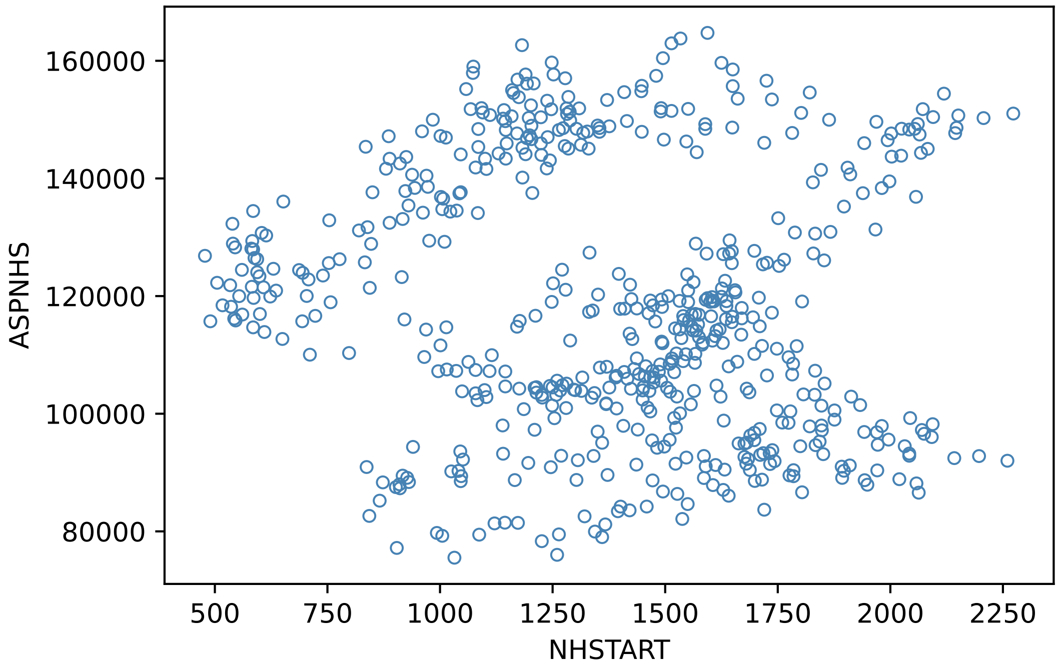
[20]:
```
     Year  Month  NHSTART  MORTGAGE30  CSHPRICE  HSN1F    ASPNHS    CPI  \
0    1975      1   1032.0         9.4       NaN  416.0   39500.0   52.3
1    1975      2    904.0         9.1       NaN  422.0   40600.0   52.6
2    1975      3    993.0         8.9       NaN  477.0   42100.0   52.8
3    1975      4   1005.0         8.8       NaN  543.0   42000.0   53.0
4    1975      5   1121.0         8.9       NaN  579.0   43200.0   53.1
..    ...    ...      ...         ...       ...    ...       ...    ...
554  2021      3   1725.0         3.1     245.5  873.0  414700.0  264.8
555  2021      4   1514.0         3.1     249.8  796.0  434800.0  266.8
556  2021      5   1594.0         3.0     254.4  720.0  442500.0  268.6
557  2021      6   1650.0         3.0     259.0  701.0  429600.0  271.0
558  2021      7   1534.0         2.9       NaN  708.0  446000.0  272.3

     HSUPPLY
0        9.9
1       10.4
2        8.9
3        7.2
4        6.8
..       ...
554      4.2
555      4.8
556      5.5
557      6.0
558      6.2

[559 rows x 9 columns]
```

The data contains several variables which we won't be using in this analysis, such as the Case-Shiller house price index (`CSHPRICE`) which has missing values for some of the earlier dates (missing values are denoted as `NaN`).

Let's first plot the bivariate relationship between new house starts and the (concurrent) average sales price. The price is in current dollars, so we first need to deflate it (using the `CPI`) to make the values comparable across this 45-year period.

[21]:
```python
import matplotlib.pyplot as plt

# Convert average selling price to 1982-1984 dollars.
# The value of 100 corresponds to the average price level between 1982-1984.
df['ASPNHS'] /= df['CPI'] / 100.0

df.plot.scatter('NHSTART', 'ASPNHS', linewidths=0.75,␣
 ↪color='none',edgecolor='steelblue')
```

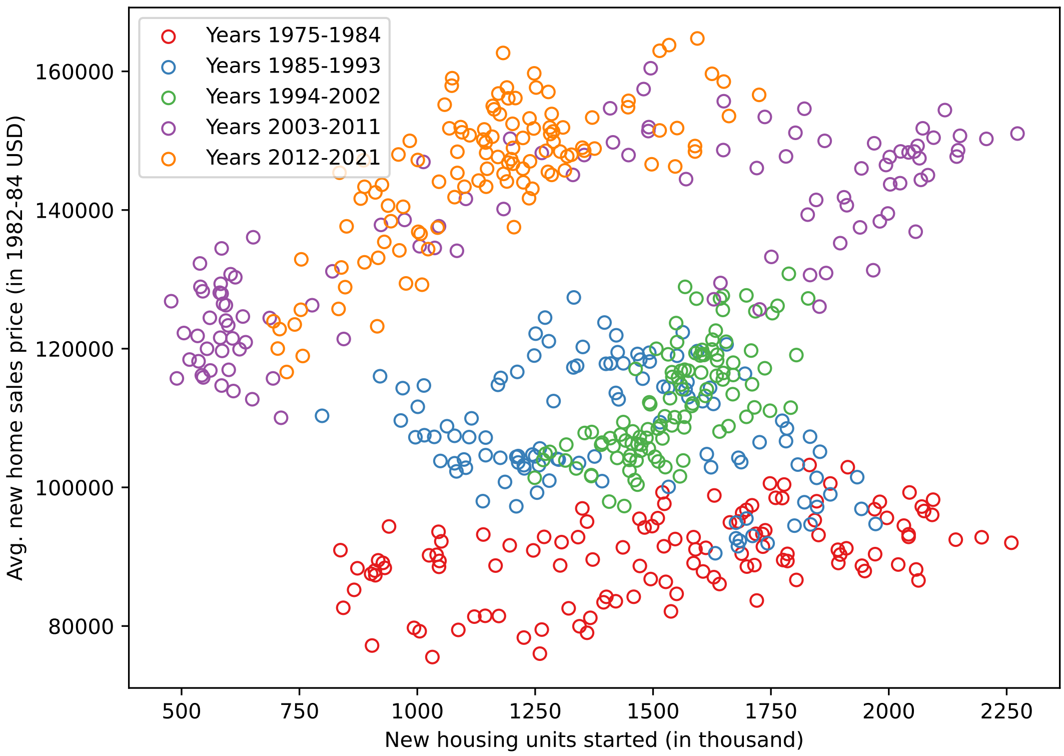
[21]:
```
<AxesSubplot:xlabel='NHSTART', ylabel='ASPNHS'>
```



This scatter plot looks somewhat unexpected as there seems to be no clear relationship between housing supply and house prices. This might be because the relationship has not remained stable over the decades covered by our data.

To see this more clearly, we bin the time periods into five blocks and recreate the plots using different colours:

[22]:
```
# Create 5 approximately equally-sized bins based on the calendar year
df['Year_bin'] = pd.cut(df['Year'], bins=5, labels=False)

# Plot each group of years using a different color
fig, ax = plt.subplots(1,1, figsize=(8,6))
colors = ['#e41a1c','#377eb8','#4daf4a','#984ea3','#ff7f00']

# Iterate over bins, plot each one separately
bins = df['Year_bin'].unique()
for bin in bins:
    # Restrict data set to relevant years
    df_i = df.loc[df['Year_bin'] == bin].copy()

    # Extract initial and terminal year of this block
    yfrom, yto = df_i['Year'].min(), df_i['Year'].max()

    ax.scatter(df_i['NHSTART'], df_i['ASPNHS'],
        label=f'Years {yfrom:.0f}-{yto:.0f}',
        edgecolors=colors[bin], color='none')

ax.set_xlabel('New housing units started (in thousand)')
ax.set_ylabel('Avg. new home sales price (in 1982-84 USD)')
ax.legend()

del df['Year_bin']
```



As you see, our suspicion was correct and there are clear changes across the sample of 45 years. At this point we could do something more elaborate, but for illustrative purposes we just restrict our analysis to the period after the year 2000, where we have an upwards-sloping relationship.

1.4.2 Prepare the data

Before we can call the function `ols()`, we need to pre-process the data so that we end up with NumPy arrays (the only type of data our function accepts).

[23]:
```
# Keep only relevant variables, rest just clutters the DataFrame
varlist = ['Year', 'Month', 'ASPNHS', 'NHSTART']
df = df[varlist].copy()

# Create YYYY-MM date index
df['Date'] = pd.PeriodIndex(year=df['Year'], month=df['Month'], freq='M')
df = df.set_index('Date')

# Create 3-month, 6-month and 12-month lags of house prices
lags = 3, 6, 12
for lag in lags:
    df[f'L{lag}ASPNHS'] = df['ASPNHS'].shift(lag)

# Restrict data to year >= 2000
df = df.loc[df['Year'] >= 2000].copy()

# Drop year, month, these are no longer needed
df = df.drop(columns=['Year', 'Month'])

# Plot first 13 rows, which clearly shows the lagged values
df.head(13)
```

[23]:
```
              ASPNHS  NHSTART       L3ASPNHS       L6ASPNHS      L12ASPNHS
Date
2000-01  118310.691081   1636.0  119095.776324  113437.312537  111050.394657
2000-02  117176.470588   1737.0  125593.824228  115559.545183  116211.293260
2000-03  119824.561404   1604.0  119727.488152  116090.584029  114866.504854
2000-04  121299.005266   1626.0  118310.691081  119095.776324  115852.923448
2000-05  116822.429907   1575.0  117176.470588  125593.824228  113192.771084
2000-06  114808.362369   1559.0  119824.561404  119727.488152  116807.228916
2000-07  117081.644470   1463.0  121299.005266  118310.691081  113437.312537
2000-08  115923.566879   1541.0  116822.429907  117176.470588  115559.545183
2000-09  119988.479263   1507.0  114808.362369  119824.561404  116090.584029
2000-10  123691.776883   1549.0  117081.644470  121299.005266  119095.776324
2000-11  120952.927669   1551.0  115923.566879  116822.429907  125593.824228
2000-12  119186.712486   1532.0  119988.479263  114808.362369  119727.488152
2001-01  119020.501139   1600.0  123691.776883  117081.644470  118310.691081
```

Now that we have created all the lagged variables, we drop all rows with missing data and convert the relevant columns to NumPy arrays.

[24]:
```python
# List of variables to include in model
var_X = [f'L{lag}ASPNHS' for lag in lags]
var_y = 'NHSTART'

# Restrict to relevant variables
df = df[var_X + [var_y]].copy()

# drop all rows with missing observations
df = df.dropna()

# Extract raw data from data frame
X = df[var_X].to_numpy()
y = df[var_y].to_numpy()

# Estimate as elasticity in logs
log_X = np.log(X)
log_y = np.log(y)

# Print first 5 observations
log_X[:5]
```

[24]:
```
array([[11.68768329, 11.63900565, 11.61773938],
       [11.74080836, 11.65754122, 11.66316531],
       [11.69297351, 11.66212606, 11.65152591],
       [11.68106942, 11.68768329, 11.66007676],
       [11.67143637, 11.74080836, 11.63684758]])
```

1.4.3 Estimating the model

We are now ready to run the OLS regression.

[25]:
```python
# Run our own ols() function. This returns the coefficient vector and the
# variance-covariance matrix.
coefs, vcv = ols(log_X, log_y, add_const=True)

# Compute standard errors from the VCV matrix
se = np.sqrt(np.diag(vcv))

print(f'Estimated coefficients: {coefs}')
print(f'Standard errors: {se}')
print(f'Number of obs: {len(log_y)}')
```

```
Estimated coefficients: [-6.48405842  2.2932615   0.80018127 -1.94803646]
Standard errors: [2.78344134 0.50247618 0.55385034 0.43777545]
Number of obs: 259
```

1.4.4 Running OLS using statsmodels

As you can imagine, estimating an OLS regression is a common task so there are packages which already implement this functionality for you. One such package is `statsmodels`, which we will now use to verify our results.

[26]:
```python
import statsmodels.api as sm

# Explicitly augment the regressor matrix with a constant
log_X1 = sm.add_constant(log_X)

# Define the linear model
model = sm.OLS(log_y, log_X1)

# Estimate the model
result = model.fit()

# Print a summary of the results
result.summary()
```

[26]:
```
<class 'statsmodels.iolib.summary.Summary'>
"""
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.185
Model:                            OLS   Adj. R-squared:                  0.175
Method:                 Least Squares   F-statistic:                     19.27
Date:                Wed, 15 Sep 2021   Prob (F-statistic):           2.72e-11
Time:                        23:15:10   Log-Likelihood:                -101.18
No. Observations:                 259   AIC:                             210.4
Df Residuals:                     255   BIC:                             224.6
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const         -6.4841      2.783     -2.330      0.021     -11.966      -1.003
x1             2.2933      0.502      4.564      0.000       1.304       3.283
x2             0.8002      0.554      1.445      0.150      -0.291       1.891
x3            -1.9480      0.438     -4.450      0.000      -2.810      -1.086
==============================================================================
Omnibus:                       31.603   Durbin-Watson:                   0.184
Prob(Omnibus):                  0.000   Jarque-Bera (JB):                9.818
Skew:                          -0.144   Prob(JB):                      0.00738
Kurtosis:                       2.090   Cond. No.                     2.55e+03
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
specified.
[2] The condition number is large, 2.55e+03. This might indicate that there are
strong multicollinearity or other numerical problems.
"""
```

As you can see, the point estimates and standard errors are exactly the same as the ones we computed.

As for the interpretation, the regression says that a 1% increase in the average sales price is associated with a 2.3% increase in new house construction starts in three months time. The elasticity is only 0.8% if we consider a lag of 6 months (albeit not statistically significant), and even reverses its sign at a 12-month horizon. This interpretation of course assumes that prices are independent over time, which is not overly plausible.