Stochastic Volatility With an Ornstein-Uhlenbeck Process: An Extension

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Abstract

In this paper, we reexamine and extend the stochastic volatility model of Stein and Stein (1991) where volatility follows a mean-reversion Ornstein-Uhlenbeck process. Using Fourier inversion techniques we are able to allow for correlation between instantaneous volatilities and the underlying stock returns. A closed-form pricing solution for European options is derived and some numerical examples are given.

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1 Introduction

Modelling stochastic volatility is crucial to capture stock return variability. In order to describe the empirical leptokurtic distributions of stock returns, a number of different specifications of stochastic volatility have been suggested. Wiggins (1987) considers a case where the log-volatility $\ln v(t)$ follows a mean-reversion Ornstein-Uhlenbeck (O-U) process. Obviously, under this specification it is not guaranteed that volatilities are stationary. Hull and White (1987) solve a special option pricing problem with stochastic volatility by using a Taylor expansion technique. All models before 1990 including Scott (1987) do not present closed-form solutions and require an extensive use of numerical techniques. Stein and Stein (1991) (S&S) assume that the volatility follows a mean reversion O-U process and develop the analytic density function of stock returns to evaluate option prices. But since volatilities are uncorrelated with stock returns, their solution is not complete. Heston (1993) provides a new approach to derive a closed-form solution for options where the squared volatility (variance rate) is specified as a square-root process. The idea is that while the probability that the stock price is greater (less) than the strike price can not be expressed analytically, the corresponding characteristic function can indeed be described analytically in many cases, at least in the case of a square-root process for the variance rate. The probability function is then obtained via inverse Fourier transformation. Recently, using this approach, Bakshi, Cao and Chen (1997) develop and test a comprehensive closed-form option pricing formula including jump components of the stock price process, stochastic interest rates, and a square-root based stochastic volatility. Stochastic volatility option pricing models with closed-form solutions include also Bates (1994), Bakshi and Chen (1997) and Scott (1997). The first two are on currency options. The method used by Scott is somewhat different from the other papers. Instead of deriving the characteristic functions from a fundamental partial differential equation (PDE), he calculates them directly using martingale methods.

In this paper, we apply the method inspired by Scott (1997) to extend the model of S&S (1991) to the correlation case. That is, we assume that the stochastic volatility follows a mean-reversion O-U process and is possibly correlated with stock returns. Closed-form solutions for option prices and price sensitivities are then obtained via inverse Fourier transformation.

The remainder of the paper is organized as follows: Section 1 examines how to construct characteristic functions (CF) in a European-style option pricing formula and then develops a closed-form solution for our extension of S&S. Section 2 compares our model with S&S as well as with Heston's model. Section 3 concludes.

2 The Model

We consider that volatility follows a mean-reverting O-U process. As assumed by S&S (1991), the stochastic processes of the log of the stock price $x(t) = \ln S(t)$ and its instantaneous volatility $v(t)$ can be described by

$$dx(t) = \left(r - \tfrac{1}{2}v^2(t)\right) dt + v(t)dw_s(t), \tag{1}$$

and

$$dv(t) = \kappa\left(\theta - v(t)\right) dt + \sigma dw_v(t), \tag{2}$$

respectively. Here we suggest that dw_s and dw_v are possibly correlated, $dw_s dw_v = \rho dt$ which extends the S&S model. Both processes are interpreted under the risk-neutralized probability measure.[1]

In order to get a closed-form solution using inverse Fourier transformation, we have two methods available. The pure PDE approach used by Heston (1993), who ends up with a system of ODEs, and the more elaborated approach used by Scott (1997) who applies stochastic calculus to compute the characteristic functions (CF) directly. Given the above processes (1) and (2), we find that the PDE-approach is too cumbersome to get closed-form solutions for the CFs while the second method turns out to be much more elegant and straightforward.[2] It has the advantage that there is no need to guess a suitable form of the CFs. Instead these transformations can be perceived easily in following the calculations.

We start from a fairly general representation of the pricing formula for European-style (call) options with stock price S and strike price K. The value of a call option is the expected terminal value of the option relative to the money market account

$$\begin{aligned} C(S,t,T) &= \mathbb{E}^Q\left[e^{-r(T-t)}(S(T)-K)\cdot \mathbf{1}_{\{S(T)>K\}}\right] \\ &= \mathbb{E}^Q\left[e^{-r(T-t)}S(T)\cdot \mathbf{1}_{\{x(T)>\ln K\}}\right] - e^{-r(T-t)}K\mathbb{E}^Q\left[\mathbf{1}_{\{x(T)>\ln K\}}\right] \end{aligned} \tag{3}$$

where Q denotes the risk-neutralized martingal measure. According to Geman, El Karoui and Rochet (1995), Björk (1996) and others, in order to simplify calculations, we change numeraires. For the first term we choose the stock price S as numeraire and switch from measure Q to Q_1. For the second term we use the T-forward measure to switch from Q to Q_2. The Radon-Nikodym derivatives are then given by

$$\frac{dQ_1}{dQ} = g_1(t,T) = \exp\left\{-\int_t^T r(s)ds\right\}\frac{S(T)}{S(t)} \tag{4}$$

and

$$\frac{dQ_2}{dQ} = g_2(t,T) = \exp\left\{-\int_t^T r(s)ds\right\}\frac{1}{B(t,T)}, \tag{5}$$

respectively, where Q_1 and Q_2 are again two martingale measures. $B(t,T)$ is the price of a zero-bond maturing at time T. Since the short rate of interest is constant here, we have immediately

$$g_1(t,T) = \frac{e^{-r(T-t)}S(T)}{S(t)} \qquad \text{and} \qquad g_2(t,T) = 1. \tag{6}$$

Under the new measures Q_1 and Q_2,[3] the option price (3) can be restated as

$$\begin{aligned} C(S,t,T) &= S(t)\mathbb{E}^{Q_1}\left[\mathbf{1}_{\{x(T)>\ln K\}}\right] - e^{-r(T-t)}K\mathbb{E}^{Q_2}\left[\mathbf{1}_{\{x(T)>\ln K\}}\right] \\ &= S(t)F^{Q_1}\left(S(T)>K\right) - e^{-r(T-t)}KF^{Q_2}\left(S(T)>K\right). \end{aligned} \tag{7}$$

A powerful way to get closed-form solutions for the probabilities F^{Q_1} and F^{Q_2} is to derive their corresponding CFs, which are defined by

$$f_j(\phi) \equiv \mathbb{E}^{Q_j}\left[\exp\left\{i\phi x(T)\right\}\right] \quad j = 1,2. \tag{8}$$

Using the above two Radon-Nikodym derivatives, we obtain new expressions for the CFs $f_j(\phi)$ under the original martingale measure Q:

$$f_1(\phi) \equiv \mathbb{E}^{Q_1}\left[\exp\left\{i\phi x(T)\right\}\right] = \mathbb{E}^{Q}\left[\frac{e^{-r(T-t)}S(T)}{S(t)}\exp\left\{i\phi x(T)\right\}\right], \tag{9}$$

and

$$f_2(\phi) \equiv \mathbb{E}^{Q_2}\left[\exp\left\{i\phi x(T)\right\}\right] = \mathbb{E}^{Q}\left[\exp\left\{i\phi x(T)\right\}\right], \tag{10}$$

where the interest rate here is specified as constant for simplicity.[4] With these new expressions we start calculating the CFs $f_j(\phi)$ under the martingale measure Q. As shown in the Appendix, we obtain $f_1(\phi)$ which has the following form [5]

$$
\begin{aligned}
f_1(\phi) &= \mathbb{E}^Q\left[\exp\left\{-r\left(T-t\right)-x(t)+(1+i\phi)x(T)\right\}\right] \\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)-\tfrac{1}{2}\rho(1+i\phi)\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\}\times \\
&\qquad\qquad \times\mathbb{E}^Q\left[\exp\left\{-s_1\int_t^T v^2(u)du-s_2\int_t^T v(u)du+s_3v^2(T)\right\}\right] \\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)-\tfrac{1}{2}\rho(1+i\phi)\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\}\times \\
&\qquad \times\exp\left\{\tfrac{1}{2}D(t,T;s_1,s_3)v^2(t)+B(t,T;s_1,s_2,s_3)v(t)+C(t,T;s_1,s_2,s_3)\right\}
\end{aligned} \tag{11}
$$

with

$$
\begin{aligned}
s_1 &= -\tfrac{1}{2}\left(1+i\phi\right)^2\left(1-\rho^2\right)+\tfrac{1}{2}\left(1+i\phi\right)\left(1-2\kappa\rho\sigma^{-1}\right), \\
s_2 &= (1+i\phi)\kappa\theta\rho\sigma^{-1}, \\
s_3 &= \tfrac{1}{2}(1+i\phi)\rho\sigma^{-1}.
\end{aligned}
$$

The functions $D(t,T)$, $B(t,T)$ and $C(t,T)$ are also derived in the Appendix. Similarily, $f_2(\phi)$ is given by

$$
\begin{aligned}
f_2(\phi) &= \mathbb{E}^Q\left[\exp\left\{i\phi x(T)\right\}\right] \\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)-\tfrac{1}{2}i\phi\rho\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\}\times \\
&\qquad\qquad \times\mathbb{E}^Q\left[\exp\left\{-\hat{s}_1\int_t^T v^2(u)du-\hat{s}_2\int_t^T v(u)du+\hat{s}_3v^2(T)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
f_2(\phi) &= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)-\tfrac{1}{2}i\phi\rho\left[\sigma^{-1}v^2(t)+\sigma\left(T-t\right)\right]\right\}\times \\
&\qquad \times\exp\left\{\tfrac{1}{2}D(t,T;\hat{s}_1,\hat{s}_3)v^2(t)+B(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)v(t)+C(t,T;\hat{s}_1,\hat{s}_2,\hat{s}_3)\right\}
\end{aligned} \tag{12}
$$

with

$$\widehat{s}_1 = \tfrac{1}{2}\phi^2\left(1-\rho^2\right) + \tfrac{1}{2}i\phi\left(1-2\kappa\rho\sigma^{-1}\right),$$

$$\widehat{s}_2 = i\phi\kappa\theta\rho\sigma^{-1},$$

$$\widehat{s}_3 = \tfrac{1}{2}i\phi\rho\sigma^{-1}.$$

Given the CFs, the probability distribution functions F_1 and F_2 can be calculated using the Fourier inversion formula:

$$F_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(f_j(\phi)\frac{\exp\{-i\phi\ln K\}}{i\phi}\right)d\phi, \quad j=1,2. \tag{13}$$

Finally, the call option pricing formula is given by

$$C(S,v,t,T) = S(t)F_1 - e^{-r(T-t)}KF_2. \tag{14}$$

The formula for European puts can easily be obtained by using put-call parity. Formula (14) is more general than S&S since here the volatilities are correlated with the stock prices. Furthermore, our closed-form solution has a clear structure and gives the two probabilities F_1 and F_2 explicitly. Hence, the hedge ratio Δ and other Greeks can be given analytically. This is certainly a positive feature for the application of the model. Some popular hedge ratios are:

$$\Delta_S(S,v,t,T) = \frac{\partial C}{\partial S} = F_1, \tag{15}$$

$$\Delta_v(S,v,t,T) = \frac{\partial C}{\partial v} = S(t)\frac{\partial F_1}{\partial v} - e^{-r(T-t)}K\frac{\partial F_2}{\partial v}, \tag{16}$$

$$\Gamma_S(S,v,t,T) = \frac{\partial^2 C}{\partial S^2} = \frac{\partial F_1}{\partial S}, \tag{17}$$

$$\Gamma_v(S,v,t,T) = \frac{\partial^2 C}{\partial v^2} = S(t)\frac{\partial^2 F_1}{\partial v^2} - e^{-r(T-t)}K\frac{\partial^2 F_2}{\partial v^2}, \tag{18}$$

where for $h=S,v$ and $j=1,2$

$$\frac{\partial F_j}{\partial h} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\partial f_j(\phi)}{\partial h}\frac{\exp(-i\phi\ln K)}{i\phi}\right)d\phi, \quad j=1,2.$$

and

$$\frac{\partial^2 F_j}{\partial h^2} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{\partial^2 f_j(\phi)}{\partial h^2}\frac{\exp(-i\phi \ln K)}{i\phi}\right) d\phi, \quad j = 1, 2.$$

Our results on hedge ratios are similar to those in Bakshi, Cao and Chen (1997). While Heston (1993), and Bakshi, Cao and Chen (1997) got $f_j(\phi)$ via two-dimensional partial differential equations, as shown in the Appendix, we only need to solve a one dimensional PDE using the Feynman-Kač theorem after reducing the explicit form of $f_j(\phi)$ to a one dimensional problem. In the next section, we will compare our model with the S&S model and with Heston's solution.

3 Comparison with Other Stochastic Volatility Models

Heston shows that if the volatility in his model follows an O-U process with mean-reversion level equal to zero [Equation (2) in Heston (1993)], that is

$$dv(t) = -\beta v(t)dt + \delta dw, \tag{19}$$

then, from Itô's Lemma, the squared volatility $y(t) = v(t)^2$, namely the variance of instantaneous stock returns, follows a square-root process

$$dy(t) = \kappa_h(\theta_h - y(t))dt + \sigma_h\sqrt{y(t)}dw, \tag{20}$$

with

$$\beta = \frac{\kappa_h}{2}, \quad \delta = \frac{\sigma_h}{2}, \quad \theta_h = \frac{\delta^2}{\kappa_h}. \tag{21}$$

Hence, the only difference between equation (2) and equation (19) is the mean-reversion parameter θ. While θ in (2) generally differs from zero, in (19) it is always nil. Since θ gives the level of volatility in the long run, process (19) seems not very reasonable. Therefore, this restricted Heston model can be considered as a special case of our model in the sense of processes (19) and (20). Our model is reduced to Heston's model by setting the following parameters:

$$\kappa = \frac{\kappa_h}{2}, \quad \sigma = \frac{\sigma_h}{2}, \quad \theta = 0, \quad \theta_h = \frac{\sigma^2}{\kappa_h}. \tag{22}$$

However, note that the parameters in (20) are overdetermined by (21). This means that for a wide range of values for κ_h, σ_h and θ_h, the volatility process (20) can not be derived from (19). Therefore the two processes (19) and (20) are not mutually consistent for many parameter values. Hence, in this sense the Heston model is not a special case of our model. A favorite property of our model is that both volatility and the squared volatility perform mean-reversion, whereas in the Heston model, only the squared volatility exhibits mean-reversion.[6]

Here we present some option prices based on formula (14). For comparison, we calculate the Black-Scholes (BS) option prices according to the spot volatility v with κ, ρ and σ nil. The calculated BS prices are identical to these prices by using the BS formula. In Table 1, we choose the parameters suggested by S&S. In order to show the impact of correlation on the option prices, let ρ range from -1.0 to 1.0. Some observations are in order. First, options with different moneyness have different sensitivity to the correlation ρ. Overall the values of at-the-money (ATM) options do not change remarkably. However, the sensitivity of out-of-the-money (OTM) options to ρ is more conspicuous than of in-the-money (ITM) options. For example, in Panel C the relative changes of the OTM option prices due to the correlation ρ for $K = 120$ is about $\pm 14\%$ of the S&S value which is here 2.64.

Second, a comparison of Panels A, B and C shows that the mean-reversion level θ is important for the pricing of options. Keeping other parameters unchanged, the differential $\theta - v$ (mean-reversion level minus current volatility) has a great impact on the option values. From Panel A to C, $\theta - v$ is 0, -0.1 and 0.1 respectively, and the differences in option prices across these panels are mostly between 0.60$ and 1.50$. Since the expectation of the future spot price volatility approaches θ, the prices of options, especially the options with a long-term maturity, should be mainly affected by θ. It is not surprising that the price differences between BS and our model are the smallest for Panel A where $\theta = v$.

Finally, ITM options and OTM options react to correlation just oppositely. Whereas ITM options ($K = 90, 95$) decrease in value with increasing correlation ρ, OTM option prices ($K = 105, 110, 115, 120$) go up. This finding is consistent with Hull's (1997, pages 492-500) excellent intuitive explanation of how correlation affects option prices. It is also empirically evident that stock returns are inversely correlated with the underlying volatilities. Panels A to C show that a negative correlation ρ leads to ITM (OTM) option prices in our model that are greater (less) than the corresponding BS option prices. MacBeth and Merville (1979) found that the BS option prices undervalue (overvalue) on average market prices for ITM (OTM) options. Hence, this indicates that option prices in our model should be closer to the actual market prices. S&S report an overall overvaluation due to stochastic volatility. Our numerical examples from Panel A to Panel C show that this upward pricing bias is caused by the zero correlation assumption between volatility and its underlying asset returns. For $\rho = 0$ the direction of the movement of $S(t)$ is not affected by stochastic volatility, and any stochastic volatility raises only the additional uncertainty of $S(t)$. Consequently, S&S overvalues option prices relative to BS. Therefore, this result of S&S is no longer surprising. Ball and Roma (1994) found that the S&S misdiagnosis of stochastic volatility effects is due to using an "inappropriate" value for variance in the BS price. In context of our model, these arguments are not confirmed. The volatility smile may be reasonably explained only in presence of correlation. If correlation is different from zero, the direction of the movement of $S(t)$ is influenced by $v(t)$. The correlation between volatility and spot returns is thus necessary to create skewness and kurtosis in the distribution of spot returns. Not surprisingly, all of these discussions are similarily true for the Heston model with and without zero correlation. Bakshi, Cao and Chen (1997) reported that taking stochastic volatility is of first-order importance in eliminating the volatility smile, but only in presence of correlation. Their results should be also valid for our model.

(Insert Table 1)

In Table 2, we examine how the option prices vary with the mean-reversion level θ. The finding that option prices are very sensible to θ is confirmed. Since θ indicates the long-run level of volatility, this sensitivity can be considered as the sensitivity of option prices to their volatilities in the long run. Furthermore, it seems to be that θ is more important than the spot volatility $v(t)$ for the pricing of options in the framework of a mean-reversion process. If the true process of volatility performs mean-reversion, and option prices are evaluated using the spot volatility from the BS formula, a significant pricing bias will occur. All prices in Panel E correspond to the case of S&S. The numbers in italics in Panels D, E, and F are option prices under the restricted Heston model in the sense of equations (19) and (20). The implied zero level of the mean-reversion leads to an overall undervaluation of options compared with BS.

(Insert Table 2)

Table 3 demonstrates the impact of ρ on Delta Δ_S which is of first-order importance for hedging purposes whenever stochastic volatility models are used. First, for the given parameters almost all of the Deltas are a decreasing with correlation ρ except a few deep-ITM and deep-OTM options across the three Panels G, H and I. Second, the changes of values of near-ATM options relative to the correlation ρ are more sensitive than these of deep-ITM and deep-OTM options. The differences between Δ_S in our model and the BS model for near-ATM options should not be neglected. For a negative correlation, using Δ_S of the BS model seems to cause a severe underhedging for near-ATM options. Furthermore, the long-term level of volatility θ is also important for hedging. Keeping other parameters unchanged, the greater θ is, the smaller (greater) Δ_S will be for ITM (OTM) options. The sensitivity of Δ_S to θ is remarkable and can be studied more detailed by the second derivative $\Delta_{S\theta}$. We conclude that an unbiased estimate of θ is crucial for Delta-hedging.

(Insert Table 3)

4 Conclusions

Stochastic volatility option pricing models provide us with new insights into derivative security markets. Generally stochastic volatilities have been specified at least by two

classes of stochastic processes. The first specification is the mean-reversion square-root process in the line with the famous interest rate process of Cox, Ingersoll and Ross (1985b). The closed-form pricing formula for options with squared volatility (variance) following such a process is given by Heston (1993). The advantage of square-root process might be obvious: Squared volatilities never become negative.
The second specification is a mean-reversion O-U process. In this paper, we have derived a closed-form pricing formula for the general case where volatility is allowed to display arbitrary correlation with the underlying stock price. Since in a diffusion context negative volatilities only mean that upward moves of the driving Brownian motion become downward moves of the stock price and vice versa, we believe that this is not a severe theoretical restriction and suggest this new closed-form pricing formula is an alternative to Heston's solution: Not surprisingly, squared volatilities never become negative here either.
Certainly it is interesting to study the empirical evidence of this second specification and compare its performance with the Heston model and its generalizations. This is left for future research.

Notes

[1] Because the volatility v is not a traded asset, this risk-adjusted martingale measure is not unique but depends on the market price of volatility risk λ which is (implicitly) determined by the market participants. A common way to specify λ is to assume $\lambda dt = \gamma Cov[dv, dC/C]$ where γ and C are the relative-risk aversion parameter and consumption respectively. From the Cox, Ingersoll and Ross (1985a) equilibrium model, one can get a consumption process [also see equation (8) in Heston (1993)]:

$$dC = \mu_c v(t)^2 C dt + \sigma_c v(t) C dw_c(t),$$

where the investor is assumed to have log-utility, i.e. $\gamma = 1$. Consequently, the risk premium is proportional to $v, \lambda(v) = \lambda v$ with λ a constant.
[2] Heston (1993) wrote in the appendix of his paper: " *Although Stein and Stein (1991) assume the volatility process is uncorrelated with the spot asset, one can generalize this to allow* $z_1(t)$ *and* $z_2(t)$ *to have constant correlation.* $\cdots\cdots$". Following this suggestion, we found that this leads to a rather cumbersome procedure. His method does not lead to a decomposition of the PDE into several ordinary differential equations which can be solved successively.
[3] Our choice of probability measure to calculate the CFs corresponds with Scott (1997). Depending on the two numeraires "stock price" and "default-free discount-bond price" the two functions g_1 in (4) and g_2 in (5) define two likelihood processes which are martingales themselves. Therefore we can switch from one measure to the other without violating the no arbitrage condition. For a constant interest rate, g_2 has a value of one. Thus in this case, the measure Q^2 is identical to the original measure Q. See Geman, El Karoui and Rochet (1995) for details.
[4] In fact, we can embody stochastic interest rates into this option pricing model by asssuming that stochastic interest rate and stochastic volatility are mutually in-

dependent. Interest rates can be specified either as a mean-reverting O-U process (Vasicek, 1977) or as a mean-reverting square-root process (Cox, Ingersoll and Ross, 1985b). The derivation of the corresponding CFs follows the same lines as shown in the Appendix.

[5] In these calculations, two facts from stochastic calculus are employed. One is the decomposition of a standard Brownian motion. If two standard Brownian motions dw_1 and dw_2 are correlated with $dw_1 dw_2 = \rho dt$, so dw_1 can be expressed as $dw_1 = \rho dw_2 + \sqrt{1-\rho^2}dw$ where $dwdw_2 = 0$ and $dwdw_1 = \sqrt{1-\rho^2}dt$. The second is the so-called Itô isometry which says $var[\int_0^t v(u)dw(u)] = \mathbb{E}[\int_0^t v^2(u)du]$ for any Itô process $v(t)$.

[6] Applying Itô's Lemma once again, we obtain the process of $v(t) = \sqrt{y(t)}$:

$$dv(t) = [\tfrac{1}{2}(\kappa_h\theta_h - \tfrac{1}{4}\sigma_h^2)v(t)^{-1} - \tfrac{1}{2}\kappa_h v(t)]dt + \tfrac{1}{2}\sigma_h dw(t).$$

Obviously, if (21) is not satisfied, $\kappa_h\theta_h - \frac{1}{4}\sigma_h^2$ will not be zero. Hence the term $v(t)^{-1}$ will appear in the volatility process. As a consequence, the specification of $y(t) = v(t)^2$ such as (20) should also be examined carefully. If the volatility follows a mean-reversion O-U process as (2), the process for the squared volatility is

$$dy(t) = [\sigma^2 + 2\kappa\theta\sqrt{y(t)} - 2\kappa y(t)]dt + 2\sigma\sqrt{y(t)}dw_v(t).$$

This is also a mean reversion square-root process with an additional term $2\kappa\theta\sqrt{y(t)}$.

Appendix

$f_1(\phi)$ can be calculated as follows (The expansion of $f_2(\phi)$ follows the same way.):

$$\begin{aligned}
f_1(\phi) &= \mathbb{E}^Q\left[\exp\left\{-r(T-t) - x(t) + (1+i\phi)x(T)\right\}\right] \\
&= \mathbb{E}^Q\left[\exp\left\{-r(T-t) - x(t) + (1+i\phi)\left(x(t) + \int_t^T r du - \tfrac{1}{2}\int_t^T v^2(u)du + \int_t^T v(u)dw_s(u)\right)\right\}\right] \\
&= \exp\left\{i\phi\left(r(T-t) + x(t)\right)\right\}\mathbb{E}^Q\left[\exp\left\{(1+i\phi)\left(-\tfrac{1}{2}\int_t^T v^2(u)du + \int_t^T v(u)dw_s(u)\right)\right\}\right]
\end{aligned}$$

$$\begin{aligned}
f_1(\phi) &= \exp\left\{i\phi\left(r(T-t) + x(t)\right)\right\} \times \\
&\times \mathbb{E}^Q\left[\exp\left\{(1+i\phi)\left(-\tfrac{1}{2}\int_t^T v^2(u)du + \rho\int_t^T v(u)dw_v(u) + \sqrt{1-\rho^2}\int_t^T v(u)dw(u)\right)\right\}\right]
\end{aligned}$$

Note that dw is uncorrelated with dw_v:

$$
\begin{aligned}
f_1(\phi) &= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)\right\}\times\\
&\times\mathbb{E}^Q\left[\exp\left\{(1+i\phi)\left(-\tfrac{1}{2}\int_t^T v^2(u)du+\rho\int_t^T v(u)dw_v(u)\right)+\tfrac{1}{2}(1+i\phi)^2(1-\rho^2)\int_t^T v^2(u)du\right\}\right]\\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)\right\}\times\\
&\times\mathbb{E}^Q\left[\exp\left\{\tfrac{1}{2}(1+i\phi)\left((1+i\phi)(1-\rho^2)-1\right)\int_t^T v^2(u)du+(1+i\phi)\rho\int_t^T v(u)dw_v(u)\right\}\right]\\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)\right\}\mathbb{E}^Q\left[\exp\left\{\tfrac{1}{2}(1+i\phi)\left((1+i\phi)(1-\rho^2)-1\right)\int_t^T v^2(u)du+\right.\right.\\
&\qquad\left.\left.+(1+i\phi)\frac{\rho}{2\sigma}\left(v^2(T)-v^2(t)-\sigma^2(T-t)-2\kappa\theta\int_t^T v(u)du+2\kappa\int_t^T v^2(u)du\right)\right\}\right]\\
&= \exp\left\{i\phi\left(r\left(T-t\right)+x(t)\right)\right\}\mathbb{E}^Q\left[\exp\left\{\tfrac{1}{2}(1+i\phi)\left((1+i\phi)(1-\rho^2)-1+\frac{2\kappa\rho}{\sigma}\right)\int_t^T v^2(u)du-\right.\right.\\
&\qquad\left.\left.-(1+i\phi)\frac{\rho}{2\sigma}\left(v^2(t)+\sigma^2(T-t)\right)+(1+i\phi)\frac{\rho}{2\sigma}v^2(T)-(1+i\phi)\frac{\kappa\theta\rho}{\sigma}\int_t^T v(u)du\right\}\right]\\
&= \exp\left\{i\phi\left[x(t)+r\left(T-t\right)\right]-(1+i\phi)\frac{\rho}{2\sigma}\left(v^2(t)+\sigma^2(T-t)\right)\right\}\times\\
&\qquad\qquad\times\mathbb{E}^Q\left[\exp\left\{-s_1\int_t^T v^2(u)du-s_2\int_t^T v(u)du+s_3v^2(T)\right\}\right]
\end{aligned}
$$

Now we have to calculate the expectation

$$
\begin{aligned}
y(v,t,T) &= \mathbb{E}^Q\left[\exp\left\{-s_1\int_t^T v^2(u)du-s_2\int_t^T v(u)du+s_3v^2(T)\right\}\right]\\
&= \mathbb{E}^Q\left[\exp\left(\int_t^T(-s_1v^2(u)-s_2v(u))du\right)\exp(s_3v^2(T))\right]
\end{aligned}
$$

for arbitrary complex numbers s_1, s_2 and s_3 and $-s_1v^2(u)-s_2v(u)$ is lower bounded. According to the Feynman-Kač formula, y satisfies the following differential equation [see Karlin and Taylor (1975) and Øksendal (1995)]

$$
\frac{1}{2}\sigma^2\frac{\partial^2 y}{\partial v^2}+\kappa(\theta-v)\frac{\partial y}{\partial v}-(s_1v^2+s_2v)y+\frac{\partial y}{\partial t} = 0
$$

with boundary condition

$$y(v,T,T) = \exp(s_3 v^2).$$

It can be shown that the above differential equation has a solution of the form

$$\begin{aligned} y(v,t,T) &= \exp\left\{\tfrac{1}{2}A(t,T)v^2(t) + B(t,T)v(t) + C(t,T) + s_3 v^2(t)\right\} \\ &= \exp\left\{\tfrac{1}{2}(A(t,T) + 2s_3)v^2(t) + B(t,T)v(t) + C(t,T)\right\} \\ &= \exp\left\{\tfrac{1}{2}(D(t,T)v^2(t) + B(t,T)v(t) + C(t,T)\right\} \end{aligned}$$

with $D(t,T) = A(t,T)+2s_3$. Substituting this into the differential equation, we obtain a system of three ordinary differential equations that determine $D(t,T)$, $B(t,T)$ and $C(t,T)$.

$$\begin{aligned} D_t &= -\sigma^2 D^2 + 2\kappa D + 2s_1 \\ B_t &= \left(\kappa - \sigma^2 D\right) B - \kappa\theta D + s_2 \\ C_t &= -\tfrac{1}{2}\sigma^2 B^2 - \kappa\theta B - \tfrac{1}{2}\sigma^2 D \end{aligned}$$

where $D(T,T) = 2s_3$ and $B(T,T) = C(T,T) = 0$. Solving these equations is straightforward but tedious. We get

$$D(t,T) = \frac{1}{\sigma^2}\left(\kappa - \gamma_1 \frac{\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}\right)$$

$$B(t,T) = \frac{1}{\sigma^2\gamma_1}\left(\frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3) + \gamma_3\left(\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}\right)}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}} - \kappa\theta\gamma_1\right)$$

$$\begin{aligned} C(t,T) &= -\tfrac{1}{2}\ln\left(\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}\right) + \tfrac{1}{2}\kappa(T-t) + \\ &\quad + \frac{\left(\kappa^2\theta^2\gamma_1^2 - \gamma_3^2\right)}{2\sigma^2\gamma_1^3}\left(\frac{\sinh\{\gamma_1(T-t)\}}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}} - \gamma_1(T-t)\right) + \\ &\quad\quad + \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\,\gamma_3}{\sigma^2\gamma_1^3}\left(\frac{\cosh\{\gamma_1(T-t)\} - 1}{\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}\right) \end{aligned}$$

with

$$\gamma_1 = \sqrt{2\sigma^2 s_1 + \kappa^2}, \qquad \gamma_2 = \frac{1}{\gamma_1}\left(\kappa - 2\sigma^2 s_3\right), \qquad \gamma_3 = \kappa^2\theta - s_2\sigma^2.$$

Using the time dependent functions $D(t,T)$, $B(t,T)$ and $C(t,T)$, we obtain closed-form solutions for $f_j(\phi)$.

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Table 1. The impact of ρ on option prices.

ρ \ K	90	95	100	105	110	115	120
BS	15.12	11.34	8.14	5.58	3.66	2.29	1.38
-1.00	15.42	11.62	8.31	5.576	3.47	1.96	0.99
-0.75	15.35	11.56	8.28	5.58	3.53	2.06	1.11
-0.50	15.29	11.50	8.24	5.60	3.58	2.16	1.22
-0.25	15.22	11.44	8.21	5.61	3.64	2.25	1.32
0.00	*15.16*	*11.38*	*8.18*	*5.62*	*3.69*	*2.33*	*1.42*
0.25	15.08	11.31	8.14	5.63	3.75	2.42	1.51
0.50	15.00	11.24	8.11	5.64	3.80	2.50	1.61
0.75	14.92	11.17	8.07	5.65	3.86	2.58	1.70
1.00	14.83	11.09	8.03	5.66	3.91	2.65	1.77

A: $\theta = 0.2, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

ρ \ K	90	95	100	105	110	115	120
-1.00	14.73	10.60	6.98	4.06	1.98	0.74	0.18
-0.75	14.68	10.54	6.94	4.07	2.05	0.85	0.28
-0.50	14.63	10.48	6.89	4.075	2.12	0.96	0.37
-0.25	14.57	10.42	6.85	4.085	2.19	1.05	0.46
0.00	*14.52*	*10.35*	*6.80*	*4.093*	*2.25*	*1.14*	*0.54*
0.25	14.56	10.27	6.75	4.10	2.32	1.23	0.62
0.50	14.40	10.19	6.60	4.11	2.38	1.31	0.70
0.75	14.33	10.10	6.65	4.12	2.43	1.39	0.77
1.00	14.26	10.00	6,59	4.125	2.49	1.46	0.85

B: $\theta = 0.1, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

ρ \ K	90	95	100	105	110	115	120
-1.00	16.36	12.85	9.78	7.186	5.08	3.45	2.24
-0.75	16.30	12.80	9.75	7.198	5.13	3.53	2.34
-0.50	16.24	12.75	9.73	7.210	5.18	3.61	2.44
-0.25	16.17	12.70	9.71	7.223	5.23	3.69	2.54
0.00	*16.11*	*12.65*	*9.69*	*7.236*	*5.28*	*3.77*	*2.64*
0.25	16.04	12.60	9.66	7.251	5.33	3.84	2.73
0.50	15.96	12.54	9.64	7.265	5.38	3.92	2.82
0.75	15.85	12.49	9.62	7.280	5.43	3.99	2.91
1.00	15.81	12.43	9.60	7.296	5.47	4.07	2.99

C: $\theta = 0.3, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

The italic numbers correspond to the model of S&S.

Table 2. The impact of θ on option prices.

θ \ K	90	95	100	105	110	115	120
BS	14.51	10.37	6.86	4.18	2.32	1.18	0.55
0.0	*14.19*	*9.46*	*5.14*	*2.17*	*0.76*	*0.24*	*0.075*
0.1	14.26	9.84	6.13	3.47	1.81	0.89	0.425
0.2	14.72	10.80	7.55	5.04	3.24	2.01	1.22
0.3	15.61	12.08	9.11	6.70	4.83	3.41	2.38

D: $\rho = 0.5, \kappa = 4, \sigma = 0.1, v = 0.15, T - t = 0.5, S = 100, r = 0.0953$

θ \ K	90	95	100	105	110	115	120
0.0	*14.20*	*9.53*	*5.27*	*2.17*	*0.65*	*0.15*	*0.030*
0.1	14.35	10.00	6.25	3.45	1.68	0.73	0.292
0.2	14.87	10.95	7.63	5.02	3.12	1.84	1.04
0.3	15.75	12.20	9.16	6.68	4.73	3.26	2.19

E: $\rho = 0.0, \kappa = 4, \sigma = 0.1, v = 0.15, T - t = 0.5, S = 100, r = 0.0953$

θ \ K	90	95	100	105	110	115	120
0.0	*14.22*	*9.60*	*5.37*	*2.15*	*0.50*	*0.06*	*0.004*
0.1	14.44	10.13	6.36	3.44	1.53	0.54	0.155
0.2	15.00	11.08	7.71	5.00	3.00	1.66	0.842
0.3	15.89	12.31	9.21	6.65	4.63	3.09	1.99

F: $\rho = -0.5, \kappa = 4, \sigma = 0.1, v = 0.15, T - t = 0.5, S = 100, r = 0.0953$

The italic numbers correspond to the restricted Heston model.

Table 3.The impact of ρ on Delta Δ_S.

ρ \ K	90	95	100	105	110	115	120
BS	0.8755	0.7795	0.6582	0.5249	0.3950	0.2807	0.1890
-1.00	0.8751	0.7949	0.6895	0.5644	0.4299	0.3002	0.1883
-0.75	0.8708	0.7916	0.6825	0.5547	0.4204	0.2941	0.1881
-0.50	0.8751	0.7881	0.6751	0.5449	0.4113	0.2889	0.1883
-0.25	0.8752	0.7844	0.6673	0.5350	0.4027	0.2843	0.1887
0.00	*0.8754*	*0.7802*	*0.6591*	*0.5251*	*0.3945*	*0.2802*	*0.1891*
0.25	0.8756	0.7757	0.6504	0.5153	0.3868	0.2765	0.1896
0.50	0.8759	0.7707	0.6413	0.5055	0.3795	0.2732	0.1899
0.75	0.8761	0.7649	0.6317	0.4957	0.3725	0.2701	0.1904
1.00	0.8761	0.7584	0.6216	0.4862	0.3659	0.2673	0.1907

G: $\theta = 0.2, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

ρ \ K	90	95	100	105	110	115	120
-1.00	0.9246	0.8490	0.7302	0.5684	0.3816	0.2048	0.0763
-0.75	0.9267	0.8479	0.7229	0.5554	0.3694	0.2025	0.0867
-0.50	0.9293	0.8467	0.7151	0.5424	0.3584	0.2012	0.0947
-0.25	0.9323	0.8456	0.7068	0.5293	0.3485	0.2003	0.1012
0.00	*0.9359*	*0.88445*	*0.6978*	*0.5163*	*0.3395*	*0.1996*	*0.1065*
0.25	0.9403	0.8433	0.6881	0.5033	0.3311	0.1989	0.1111
0.50	0.9458	0.8421	0.6774	0.4903	0.3234	0.1982	0.1149
0.75	0.9530	0.8404	0.6656	0.4773	0.3163	0.1975	0.1182
1.00	0.9629	0.8379	0.6526	0.4645	0.3095	0.1967	0.1211

H: $\theta = 0.1, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

ρ \ K	90	95	100	105	110	115	120
-1.00	0.8317	0.7552	0.6646	0.5644	0.4606	0.3599	0.2682
-0.75	0.8300	0.7512	0.6584	0.5568	0.4532	0.3541	0.2652
-0.50	0.8282	0.7469	0.6519	0.5492	0.4459	0.3487	0.2626
-0.25	0.8264	0.7424	0.6452	0.5416	0.4389	0.3437	0.2604
0.00	*0.8243*	*0.7376*	*0.6383*	*0.5339*	*0.4322*	*0.3390*	*0.2584*
0.25	0.8221	0.7324	0.6311	0.5263	0.4256	0.3347	0.2566
0.50	0.8196	0.7269	0.6237	0.5187	0.4193	0.3306	0.2550
0.75	0.8169	0.7210	0.6162	0.5112	0.4133	0.3267	0.2536
1.00	0.8137	0.7147	0.6084	0.5038	0.4074	0.3231	0.2522

I: $\theta = 0.3, \kappa = 4, \sigma = 0.1, v = 0.2, T - t = 0.5, S = 100, r = 0.0953$

The italic numbers correspond to the model of S&S.