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John Hull and Alan White*

Abstract

This paper compares different approaches to developing arbitrage-free models of the term structure. It presents a numerical procedure that can be used to construct a wide range of one-factor models of the short rate that are both Markov and consistent with the initial term structure of interest rates.

I. Introduction

During the last 15 years, there have been many attempts to describe yield curve movements using a one-factor model. The traditional approach has been to propose a plausible model for the short-term interest rate and deduce from the model the current yield curve and the way it can evolve. The parameters of the model are then chosen so that it reflects market data as closely as possible. Examples of this approach are provided by the work of Vasicek (1977), Dothan (1978), Courtadon (1982), and Cox, Ingersoll, and Ross (1985). Recently some researchers have adopted a different approach. They have taken market data, such as the current term structure of interest rates, as given, and have developed a no-arbitrage yield curve model so that it is perfectly consistent with the data. The main purpose of this paper is to provide some general procedures that can be used when this second approach is adopted.

Ho and Lee (1986) were pioneers in the development of no-arbitrage yield curve models. Their model, which was presented in the form of a binomial tree for discount bond prices, provides an exact fit to the current term structure of interest rates. An alternative to the Ho and Lee model was proposed by Black, Derman, and Toy (1990), who use a binomial tree to construct a one-factor model of the short rate that fits the current volatilities of all discount bond yields as well as the

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current term structure of interest rates. Hull and White (1990b) suggest two one-factor models of the short rate that are also capable of fitting both current discount bond yield volatilities and the current term structure of interest rates. They show how the parameters of the process followed by the short-term interest rate in the models can be determined from the market data. The continuous time version of the Ho and Lee model is a particular case of one of the models considered by Hull and White. Heath, Jarrow, and Morton (1990), (1992) consider the process followed by instantaneous forward rates and provide general results that must hold for all arbitrage-free yield curve models.

This brief review of the literature reveals that there have been three main approaches to constructing arbitrage-free models of the term structure. These involve modeling discount bond prices, modeling instantaneous forward rates, and modeling the short rate. This paper starts by explaining the relationship among the three approaches. Attention is then restricted to one-factor models where the short rate is Markov. A general numerical procedure is presented involving the use of trinomial trees for constructing these models so that they are consistent with initial market data.

The procedure proposed in this paper is robust and efficient. It provides a convenient way of implementing models that have already been suggested in the literature, such as the extended-Vasicek and extended-CIR models in Hull and White (1990b) and the lognormal interest rate model in Black and Karasinski (1991). It also enables many other new models to be developed and implemented. The procedure provides an easy way for academics and practitioners to test the effect of a wide range of different assumptions about the interest-rate process on the prices of interest-rate derivatives. It is important to be able to do this because there is no general agreement on which set of assumptions is best.

The rest of this paper is organized as follows. Section II compares alternative approaches to constructing arbitrage-free yield curve models and explains the approach used in this paper. Section III presents a procedure for fitting a one-factor model of the short rate to the initial yield curve using a trinomial interest-rate tree. Section IV shows how the procedure can be extended so that the model is fitted to both the initial yield curve and the initial volatilities of all discount bond yields. Section V compares alternative models that can be implemented using the procedure. Section VI explains how the length of the time step on the tree can be changed. Conclusions are in Section VII.

II. Alternative Approaches to Modeling the Term Structure

There are three broad approaches to constructing arbitrage-free models of the term structure. The first approach, used by Ho and Lee (1986) and Hull and White (1993), involves specifying the process followed by all discount bond prices at all times. The second approach, used by Heath, Jarrow, and Morton (1992), involves specifying the process followed by all instantaneous forward rates at all future times. The third approach used by Black, Derman, and Toy (1990), Hull and White (1990b), and Black and Karasinski (1991) involves specifying the process for the short rate. This section explores the relationship among the approaches.

It is assumed that a single factor drives the whole term structure, but the analysis can be extended to the situation where there are several factors.

A. The Processes for Discount Bond Prices and Forward Rates

The following notation is adopted for this paper:

$P(t, T)$: price at time t of a discount bond maturing at time T,

$v(t, T)$: volatility of $P(t, T)$,

$F(t, T)$: instantaneous forward rate as seen at time t for a contract maturing at time T,

$r(t)$: short-term risk-free interest rate at time t,

$f(t, T_1, T_2)$: forward rate as seen at time t for the period between time T_1 and time T_2, and

$dz(t)$: Wiener process driving term structure movements.

The process that would be followed by $P(t, T)$ in a risk-neutral world is

$$dP(t,T) = r(t)P(t,T)dt + v(t,T)P(t,T)dz(t). \tag{1}$$

The volatility, $v(t, T)$, in the most general form of the model can be any well-behaved function of past and present Ps. However, since a bond's price volatility declines to zero at maturity,[1]

$$v(t,t) = 0.$$

The forward rate $f(t, T_1, T_2)$ can be related to discount bond prices as follows

$$f(t,T_1,T_2) = \frac{\mathrm{Log}[P(t,T_1)] - \mathrm{Log}[P(t,T_2)]}{T_2 - T_1}. \tag{2}$$

From (1),

$$d\,\mathrm{Log}[P(t,T_1)] = \left[r(t) - \frac{v(t,T_1)^2}{2}\right] dt + v(t,T_1)dz(t),$$

and

$$d\,\mathrm{Log}[P(t,T_2)] = \left[r(t) - \frac{v(t,T_2)^2}{2}\right] dt + v(t,T_2)dz(t),$$

so that

$$df(t,T_1,T_2) = \frac{v(t,T_2)^2 - v(t,T_1)^2}{2(T_2 - T_1)}\,dt + \frac{v(t,T_1) - v(t,T_2)}{T_2 - T_1}\,dz(t). \tag{3}$$

Equation (3) shows that the risk-neutral process for f depends only on the vs. It depends on r and the Ps only to the extent that the vs themselves depend on these variables.

When $T_1 = T$ and $T_2 = T + \Delta T$ are substituted in (4) and then limits are taken as ΔT tends to zero, $f(t, T_1, T_2)$ becomes $F(t, T)$, the coefficient of $dz(t)$ becomes

[1] $v(t, t) = 0$ is equivalent to the assumption that all discount bonds have finite drifts at all times. This is because, if the volatility of the bond does not decline to zero at its maturity, an infinite drift may be necessary to ensure that the bond's price equals its face value at maturity.

$v_T(t,T)$, and the coefficient of dt becomes $v(t,T)v_T(t,T)$, where subscripts denote partial derivatives. It follows that

$$dF(t,T) = v(t,T)v_T(t,T)\,dt - v_T(t,T)\,dz(t),$$

or, since without loss of generality, the sign of $dz(t)$ may be changed,

$$dF(t,T) = v(t,T)v_T(t,T)\,dt + v_T(t,T)\,dz(t). \tag{4}$$

Once $v(t,T)$ has been specified for all t and T, the risk-neutral processes for the $F(t,T)$s are known. The $v(t,T)$s are therefore sufficient to fully define a one-factor interest-rate model.

Integrating $v_T(t,\tau)$ between $\tau = t$ and $\tau = T$ yields

$$v(t,T) - v(t,t) = \int_t^T v_T(t,\tau)\,d\tau.$$

Since $v(t,t) = 0$, this becomes

$$v(t,T) = \int_t^T v_T(t,\tau)\,d\tau.$$

If $m(t,T)$ and $s(t,T)$ are the instantaneous drift and standard deviation of $F(t,T)$, it follows from (4) that

$$m(t,T) = s(t,T)\int_t^T s(t,\tau)d\tau. \tag{5}$$

This is a key result in Heath, Jarrow, and Morton (1992).

B. The Process for the Short Rate

In this section, the process for $r(t)$ is derived from bond price volatilities and the initial term structure. Since

$$F(t,t) = F(0,t) + \int_0^t dF(\tau,t)$$

and $r(t) = F(t,t)$, it follows from (4) that

$$r(t) = F(0,t) + \int_0^t v(\tau,t)v_t(\tau,t)d\tau + \int_0^t v_t(\tau,t)dz(\tau). \tag{6}$$

Differentiating with respect to t, and using the result that $v(t,t) = 0$,

$$\begin{aligned} dr(t) = {} & F_t(0,t)dt + \left\{\int_0^t \left[v(\tau,t)v_{tt}(\tau,t) + v_t(\tau,t)^2\right]\,d\tau\right\}dt \\ & + \left\{\int_0^t v_{tt}(\tau,t)\,dz(\tau)\right\}dt + \left[v_t(\tau,t)|_{\tau=t}\right]dz(t). \end{aligned} \tag{7}$$

This is the risk-neutral process for r at time t. It is the process for r that is consistent with the risk-neutral process for bond prices in (1). For the purposes of derivative security pricing, this paper needs only to be concerned with the risk-neutral process for r. This is because derivative securities can be priced by assuming that r follows its risk-neutral process and by using the risk-free interest rate for discounting. The procedures that will be described in this paper lead directly to a tree representing the risk-neutral process for r. No assumptions are required about the market price of risk.

It is interesting to examine the terms on the right-hand side of (7). The first and fourth terms are straightforward. The first term shows that one component of the drift in r is time dependent and equal to the slope of the initial forward rate curve. The fourth term shows that the instantaneous standard deviation of r is $v_t(\tau, t)|_{\tau=t}$. The second and third terms are more complicated, particularly when v is stochastic. The second term depends on the history of v because it involves $v(\tau, t)$ when $\tau < t$. The third term depends on the history of both v and dz. The two terms are therefore liable to cause the process for r to be non-Markov.

Non-Markov models of r are, in general, less tractable than Markov models. It is computationally feasible to use a non-Markov model when European options are being valued.[2] However, when American options are valued, it is highly desirable that r be Markov. This is because a Markov process can always be represented by a recombining tree where the number of nodes considered at time $i\Delta t$ grows linearly with i (see Nelson and Ramaswamy (1990) and Hull and White (1990a)). For a non-Markov process, the trees that are constructed are, in general, not recombining and the number of nodes at time $i\Delta t$ grows exponentially with i so that accurate pricing is computationally extremely time consuming.

One special case when $r(t)$ is Markov is $v(t,T) = (T - t)\sigma$, where σ is a constant. Equation (7) then reduces to

$$dr = \left[F_t(0,t) + \sigma^2 t\right] dt + \sigma\, dz(t). \tag{8}$$

This is the continuous time version of the Ho and Lee (1986) model. More generally, Hull and White (1993) show that, when v is nonstochastic, $r(t)$ is Markov if and only if $v(t,T)$ has the functional form

$$v(t,T) = x(t)[y(T) - y(t)]. \tag{9}$$

The process for r then has the general form

$$dr = [\theta(t) - \phi(t)r]\, dt + \sigma(t)\, dz(t).$$

This is the extended-Vasicek model considered by Hull and White (1990b).

This paper provides a procedure that can be used to construct a wide range of Markov one-factor arbitrage-free models for r. The models are more general than the extended-Vasicek model. For example, the standard deviation of r can be a function of r and the drift of r need not be linear in r.

[2] For example, the time considered can be divided into intervals of length Δt and the Wiener process, $\Delta z(t)$ can be simulated with the values of the $r(t)$, $F(t,T)$, and $P(t,T)$ being calculated as necessary from the discrete versions of Equations (1), (4), and (7).

C. The Approach in This Paper

When v is allowed to be stochastic, it does not seem to be possible to derive a condition similar to (9) for a Markov r. It is even difficult to find particular v functions that lead to a Markov r. To expand the range of Markov arbitrage-free models of the short rate available to researchers, an alternative approach is taken. A Markov risk-neutral process is specified for r in terms of an unknown function of time, $\theta(t)$, and a procedure is developed for choosing this function of time so that the model is consistent with the initial term structure of interest rates. As an extension to the procedure, it is shown that, if two unknown functions of time are included in the expression assumed for the drift of r, the model can be fitted to both initial term structure and initial volatility data.

Using one factor and allowing the drift of r to be a function of time provides a relatively simple model that captures the information contained in the initial term structure on expected future trends in r. An alternative approach is to expand the number of factors. This can be expected to reduce the extent to which the drift must be dependent on time in order to fit the initial term structure, but it does not eliminate the need for time dependence completely. Unless the initial term structure is constrained in some way, the process for r is time dependent in all arbitrage-free models of the term structure that involve a finite number of factors.[3]

This paper's procedure for determining the functions of time involves using a trinomial tree. As shown by Hull and White (1990a), a trinomial tree is a useful representation of a one-factor model of the short rate. It is capable of duplicating at each node both the expected drift of the short rate and its instantaneous standard deviation. Derivative security prices calculated using the tree converge to the solution of the underlying differential equation for the security price as the length of the time step approaches zero. In a binomial tree with constant time steps, it is not, in general, possible to match both the expected drift and instantaneous standard deviation at each node without the number of nodes increasing exponentially with the number of time steps.[4]

This paper uses the trinomial tree in a different way from Hull and White (1990a). Whereas Hull and White (1990a) assume that the short-term interest rate process is known and build a tree to represent that process, this paper assumes that the short-term interest rate process has been specified in terms of unknown functions of time and uses the trinomial tree for the additional purpose of determining these functions.

III. Fitting a Model to the Term Structure

This section considers models for r where the drift has been specified in terms of a single unknown function of time. It shows how to choose this function so that the model provides an exact fit to the initial term structure of interest rates.

[3]The drift of r in the multifactor version of Equation (7) always has $F_t(0,t)$ as its leading term. The other terms in the drift are functions of time and the factors. For an arbitrary $F(0,t)$, the drift is a function of time.

[4]Nelson and Ramaswamy (1990) show that, if the expected drift and the standard deviation at each node is required to be correct only in the limit as the length of the time step goes to zero, it is possible to construct a binomial tree where the number of nodes increases linearly with the number of time steps.

A. The Model $dr = \mu[\theta(t), r, t]\,dt + \sigma\,dz(t)$

First, a model is considered where the instantaneous standard deviation of r is constant. It is assumed

$$dr = \mu[\theta(t), r, t]\,dt + \sigma\,dz(t),$$

where σ is a known constant, the functional form for μ is known, and $\theta(t)$ is the unknown function of time. A particular case of the model that is of interest because of its analytic tractability is[5]

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz(t). \tag{10}$$

This is a version of the extended-Vasicek model discussed by Hull and White (1990b). It has the property that

$$v(t,T) = \frac{\sigma}{a}\left[1 - e^{-a(T-t)}\right]. \tag{11}$$

A tree is constructed whose geometry is similar to that in Hull and White (1990a). The short rate r is defined as the continuously compounded yield on a discount bond maturing in time Δt. The values of r on the tree are equally spaced and have the form $r_0 + j\Delta r$ for some Δr, where r_0 is the current value of r, and j is a positive or negative integer. The time values considered by the tree are also equally spaced, having the form $i\Delta t$ for some Δt, where i is a nonnegative integer. The variables Δr and Δt must be chosen so that Δr is between $\sigma\sqrt{3\Delta t}/2$ and $2\sigma\sqrt{\Delta t}$. As pointed out by Hull and White (1990a), there are some theoretical advantages to choosing $\Delta r = \sigma\sqrt{3\Delta t}$.

For convenience, the node on the tree where $t = i\Delta t$ and $r = r_0 + j\Delta r$ $(i \geq 2)$ will be referred to as the (i,j) node. The following notation is used:

$R(i)$: yield at time zero on a discount bond maturing at time $i\Delta t$,

r_j: $r_0 + j\Delta r$,

$\mu_{i,j}$: the drift rate of r at node (i,j), and

$p_1(i,j)$, $p_2(i,j)$, $p_3(i,j)$: probabilities associated with the upper, middle, and lower branches emanating from node (i,j).

Suppose that the tree has already been constructed up to time $n\Delta t$ $(n \geq 0)$ so that it is consistent with the $R(i)$ and consider how it can be extended one step further. Since the interest rate, r, at time $i\Delta t$ is assumed to apply to the time period between $i\Delta t$ and $(i+1)\Delta t$, a tree constructed up to time $n\Delta t$ reflects the values of $R(i)$ for $i \leq n+1$. In constructing the branches comprising the tree between times $n\Delta t$ and $(n+1)\Delta t$, a value of $\theta(n\Delta t)$ must be chosen so that the tree is consistent with

[5]This particular case of the model has the property that the function $\theta(t)$ can be determined analytically from the term structure. This makes it possible to use the approach in Hull and White (1990a) that is based on a known process for r. The authors of this current paper prefer not to use this approach. Errors are introduced by assuming that the analytic value of θ at time t is correct for the whole time period between t and $t+\Delta t$ and attempts to correct this error by, for example, integrating $\theta(t)$ between t and $t+\Delta t$ are not totally satisfactory. Building a tree in the way that will be described here automatically chooses a value for θ between t and $t+\Delta t$ that matches the current forward bond price. The approach in Hull and White (1990a) is best suited to models where the parameters are non-time-dependent.

$R(n+2)$. The procedure for doing this is explained in Appendix A. Note that for the purposes of constructing the tree, θ and μ are assumed to be constant within each of the time steps of length Δt.

Once $\theta(n\Delta t)$ has been determined, the drift rates $\mu_{n,j}$ for r at the nodes at time $n\Delta t$ are calculated using

$$\mu_{n,j} = \mu[\theta(n\Delta t), r_0 + j\Delta r, n\Delta t].$$

The branches emanating from the nodes at time $n\Delta t$ and their associated probabilities are then chosen to be consistent with the $\mu_{n,j}$s and with σ. The three nodes that can be reached by the branches emanating from node (n,j) are

$$(n+1, k+1), \qquad (n+1, k), \qquad \text{and} \qquad (n+1, k-1),$$

with the value of k being chosen so that r_k (the value of r reached by the middle branch) is as close as possible to $r_j + \mu_{n,j}\Delta t$ (the expected value of r). The probabilities are given by

$$p_1(n,j) = \frac{\sigma^2 \Delta t}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} + \frac{\eta}{2\Delta r},$$

$$p_2(n,j) = 1 - \frac{\sigma^2 \Delta t}{\Delta r^2} - \frac{\eta^2}{\Delta r^2},$$

$$p_3(n,j) = \frac{\sigma^2 \Delta t}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} - \frac{\eta}{2\Delta r},$$

where[6]

$$\eta = \mu_{n,j}\Delta t + (j-k)\Delta r.$$

Providing that Δr is chosen within the range $\sigma\sqrt{3\Delta t}/2$ to $2\sigma\sqrt{\Delta t}$ mentioned above, the probabilities are always between 0 and 1.

Figure 1 illustrates the procedure by showing the tree that is constructed for the model in (10) when $a = 0.1$, $\sigma = 0.014$, and $\Delta t = 1$. The term structure is assumed to be upward sloping with the yields on one-, two-, three-, four-, and five-year discount bonds being 10 percent, 10.5 percent, 11.0 percent, 11.25 percent, and 11.5 percent, respectively. Table 1 shows the results of using the same model with progressively smaller values of Δt to calculate the prices of one-year European call options on five-year discount bonds. Since these option prices are known analytically, the results provide a test of the speed of convergence of the procedure. The table illustrates that convergence is reasonably fast.

[6]Later, this branching process is used when the values of r at time $(n+1)\Delta t$ have the form $r_0 + j\Delta r$, but those at time $n\Delta t$ do not. Suppose that the value of an r at time $n\Delta t$ is r^* and its drift is μ^*. The value of k is chosen so that r_k is as close as possible to $r^* + \mu^*\Delta t$ and the expressions for the probabilities are still correct with

$$\eta = r^* + \mu^*\Delta t - (r_0 + k\Delta r).$$

FIGURE 1

Tree constructed for the model,

$$dr = [\theta(t) - ar]\, dt + \sigma\, dz(t),$$

when σ = 0.014, a = 0.1, and Δt = 1 year. The zero coupon interest rates for maturities of one, two, three, four, and five years are 10 percent, 10.5 percent, 11 percent, 11.25 percent, and 11.5 percent. The values calculated for θ are: $\theta(0)$ = 0.0201, $\theta(1)$ = 0.0213 $\theta(2)$ = 0.0124, $\theta(3)$ = 0.0175.

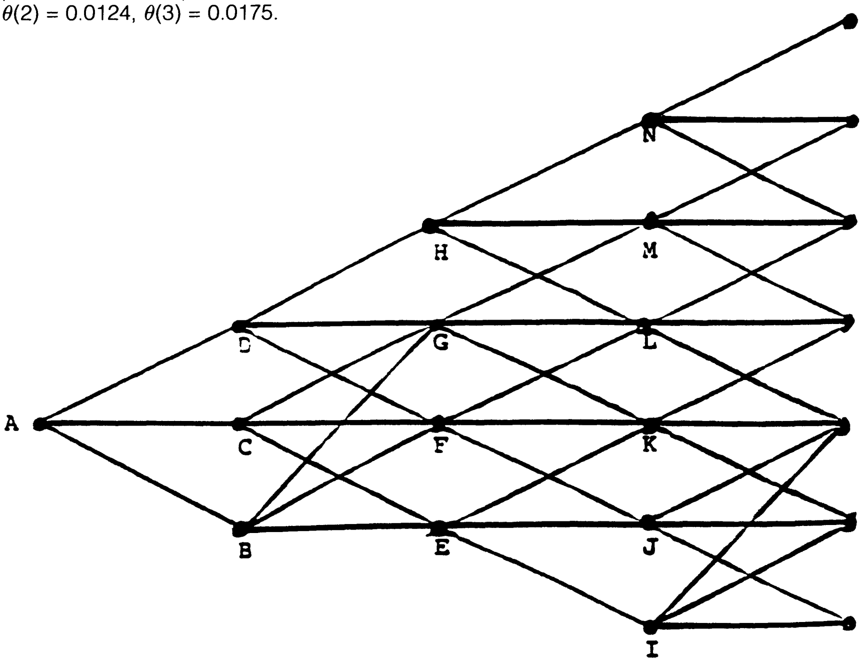


Table of Rates and Probabilities

Node	A	B	C	D	E	F	G
Rate	10.00	7.58	10.00	12.42	7.58	10.00	12.42
p_1	0.462	0.044	0.507	0.415	0.286	0.221	0.166
p_2	0.493	0.477	0.451	0.534	0.627	0.657	0.667
p_3	0.045	0.479	0.042	0.051	0.087	0.122	0.167
Node	H	I	J	K	L	M	N
Rate	14.85	5.15	7.58	10.00	12.42	14.85	17.27
p_1	0.121	0.042	0.455	0.370	0.293	0.228	0.171
p_2	0.657	0.426	0.499	0.570	0.623	0.654	0.667
p_3	0.222	0.532	0.046	0.060	0.084	0.118	0.162

B. Other Models

The approach just considered can be extended to the general class of models,

$$dr = \mu(\theta(t), r, t)\, dt + \sigma(r, t)\, dz(t). \tag{12}$$

TABLE 1

Convergence of the proposed procedure for a one-year call option on a five-year discount bond when the model

$$dr = [\theta(t) - ar]\,dt + \sigma\,dz(t)$$

is used with $a = 0.1$ and $\sigma = 0.014$. The term structure increases linearly from 9.5 percent to 11 percent over the first three years and then increases linearly from 11 percent to 11.5 percent over the next two years.

Total Number of Time Steps	Exercise Price[a] 0.96	0.98	1.00	1.02	1.04
5	2.30	1.31	1.00	0.69	0.37
25	2.47	1.68	0.95	0.58	0.25
50	2.48	1.64	1.00	0.55	0.26
100	2.48	1.64	0.99	0.54	0.26
Analytic Value	2.48	1.64	0.99	0.53	0.26

[a]The exercise price is expressed as a proportion of the forward bond price.

Here, the instantaneous standard deviation of r is a general function of r and t. One particular case of (12) that is of interest is

$$dr = [\theta(t) - ar]\,dt + \sigma\sqrt{r}\,dz(t). \tag{13}$$

This is a version of the extended CIR that is considered by Cox, Ingersoll, and Ross (1985) and Hull and White (1990b). It has the property that

$$v(t,T) = \sigma\sqrt{r}\frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma},$$

where

$$\gamma = \sqrt{a^2 + 2\sigma^2}.$$

A more general family of models corresponding to (12) is

$$dr = [\theta(t) - ar]\,dt + \sigma r^{\beta}\,dz(t), \tag{14}$$

where β is a constant. When $\beta = 0$, (14) reduces to the model in (10); when $\beta = 0.5$, it reduces to the model in (13). When $\beta = 0$, the model is capable of fitting any initial term structure. When $\beta > 0$, r must be nonnegative for the standard deviation of r to be well defined. This means that as r tends to zero, the drift of r must be nonnegative. One consequence of this is that the condition $\theta(t) \geq 0$ must be satisfied. It can be shown that it is impossible for a $\beta > 0$ model to satisfy this condition and to fit all initial term structures. In particular, term structures where $F(0,t)$ is positive, but $F_t(0,t)$ is highly negative for some values of t, cannot be fitted.[7]

[7]A lognormal model that does not have this problem is a version of Black and Karasinski (1991),

$$d\,\mathrm{Log}\,r = [\theta(t) - a\,\mathrm{Log}\,r]\,dt + \sigma dz.$$

An alternative to the extended-CIR model ($\beta = 0.5$) that does not have the problem is

$$dr = r[\theta(t) - ar]\,dt + \sigma\sqrt{r}\,dz.$$

It appears that both of these models can be fitted to any initial term structure where $F(0,t) > 0$. They can be implemented using the procedure described in this paper.

Analogously to Hull and White (1990a), (12) is dealt with by defining a function of r, $x(r)$, that has a constant instantaneous standard deviation. This function is

$$x(r) = \sigma(r_0, 0) \int \frac{dr}{\sigma(r,t)}.$$

It follows the process

$$dx = [\mu(\theta(t), r, t)u(r,t) + w(r,t)]\,dt + \sigma(r_0, 0)\,dz(t), \tag{15}$$

where

$$u(r,t) = \frac{\sigma(r_0, 0)}{\sigma(r,t)},$$

$$w(r,t) = -\frac{\sigma(r_0, 0)}{2}\frac{\partial \sigma(r,t)}{\partial r} + \frac{\partial x}{\partial t}.$$

A tree is constructed for x where the spacing between the x-values, Δx, is constant and equal to $\sigma(r_0, 0)\sqrt{3\Delta t}$. Assume that a tree has been constructed up to time $n\Delta t$. The value of $\theta(n\Delta t)$ is calculated as described in Appendix A with r_j now being defined as the value of r at the (i, j) node for x. The branching process for x between times $n\Delta t$ and $(n + 1)\Delta t$ is calculated to represent (15) using the same procedure as that described for r in Section III.A.

IV. Fitting a Model to Term Structure and Volatility Data

This section moves on to consider models for r that involve two functions of time, $\theta(t)$ and $\phi(t)$, and can be fitted to both the initial term structure of interest rates and initial volatility data. It is assumed that the volatility data consist of discount bond yield volatilities estimated from historical data.

It is important to emphasize that the models developed in this section match the volatilities of the yields on discount bonds only at time zero. There is no guarantee that the pattern of discount bond yield volatilities at later times will be similar to the pattern at time zero.[8] In practice, it is found that they are sometimes quite different. Models such as (14) in Section III are more robust. Although they do not match the volatilities at time zero exactly, they have the advantage that they give rise to stationary volatility structures.[9] They may be more appropriate than the models in this section for valuing long-lived options because the price of a long-lived option can be quite sensitive to the way bond yield volatilities evolve.

One model involving two functions of time is

$$dr = [\theta(t) - \phi(t)r]\,dt + \sigma r^{\beta}\,dz(t). \tag{16}$$

This has the same general flavor as (14). Another is

$$d\,\mathrm{Log}\, r = [\theta(t) - \phi(t)\,\mathrm{Log}\, r]\,dt + \sigma\,dz(t).$$

[8]In general, a non-Markov model is necessary if the volatilities of the yields on discount bonds are required to have some particular pattern at all times.

[9]When $\beta = 0$ or $\beta = 0.5$ in Equation (14), $v(t, T)$ is a known function of $T - t$ and r. In general, for models similar to those in Equation (14) and footnote 7, the volatility structure is stationary in the sense that $v(t, T)$ can depend only on $T - t$ and the term structure at time t.

This is considered by Black and Karasinski (1991).

For the sake of generality, it is assumed

$$dr \quad = \quad \mu(\theta(t), \phi(t), r, t)\, dt + \sigma(r, t)\, dz(t). \tag{17}$$

As in Section III.B, a transformed variable, x, whose instantaneous standard deviation is constant, is used. For computational convenience, one change is made to the geometry of the tree. The tree is made binomial during the first time step and trinomial thereafter. During the first time step, there is a probability of 0.5 of moving up a new node U and a probability 0.5 of moving down to a new node D. Define

r_u, r_d: values of r at nodes U and D, respectively;
x_u, x_d: values of x at nodes U and D, respectively;
$R_u(i)$, $R_d(i)$: yields at nodes U and D, respectively, on a discount bond maturing at time $i\Delta t$;
μ_u, μ_d: the drift rates of r at nodes U and D, respectively;
$V(i)$: volatility at time zero of the yield on a discount bond maturing at time $i\Delta t$.

Unlike the values of x considered in Section III.B, the values of x_u and x_d are not necessarily equal to $x(r_0) + j\Delta x$ for any integer j. However, the values of x considered at time $n\Delta t$ when $n > 1$ do have this form.

The first step in the construction of the tree is to determine $R_u(i)$ and $R_d(i)$ for all $i \geq 1$. These must be consistent with the known values of $R(i)$ so that

$$e^{-r_0\Delta t}\left[0.5e^{-(i-1)R_u(i)\Delta t} + 0.5e^{-(i-1)R_d(i)\Delta t}\right] \quad = \quad e^{-iR(i)\Delta t}. \tag{18}$$

They must also be consistent with the known values of $V(i)$. Since $V(i)\sqrt{\Delta t}$ is the standard deviation of the distribution of the natural logarithm of the yield on a discount bond maturing at time $i\Delta t$,

$$V(i)\sqrt{\Delta t} \quad = \quad 0.5\,\mathrm{Log}\,\frac{R_u(i)}{R_d(i)}. \tag{19}$$

Equations (18) and (19) can be solved for $R_u(i)$ and $R_d(i)$ using the Newton-Raphson procedure. Since $R_u(2)$ and $R_d(2)$ are r_u and r_d, respectively, the solution to (18) and (19) when $i = 2$ determines the two nodes at time Δt.

The tree is constructed from time Δt onward using an approach similar to that in Section III. There are two functions of time, $\theta(t)$ and $\phi(t)$. These are chosen to be consistent with $R_u(i)$s and $R_d(i)$s using the procedure explained in Appendix B. The branching process is determined as described in Section III. Footnote 6 describes a minor modification necessary for the segment of the tree between Δt and $2\Delta t$.

Figure 2 illustrates the procedure by showing the tree that is produced for the model,

$$dr \quad = \quad [\theta(t) - \phi(t)r]\, dt + \sigma r^{\beta}\, dz(t), \tag{20}$$

when $\sigma = 0.14$, $\beta = 1$, and $\Delta t = 1$ year. The term structure of interest rates is assumed to be flat at 10 percent per annum. The volatilities of one-year, two-year, three-year, four-year, and five-year rates are assumed to be 13 percent, 12 percent, 11 percent, 10 percent, and 9 percent per annum, respectively.

FIGURE 2

Tree constructed for the model $dr = [\theta(t) - \phi(t)r]\,dt + \sigma r\,dz(t)$ when $\sigma = 0.14$ and the time step is one year. The term structure of interest rates is flat at 10 percent per annum. The current volatilities of one-year, two-year, three-year, four-year, and five-year discount bond yields are 13 percent, 12 percent, 11 percent, 10 percent, and 9 percent per annum, respectively. The values calculated for θ and ϕ are: $\theta(1) = 0.0165$, $\theta(2) = 0.0193$, $\theta(3) = 0.0244$, $\phi(1) = 0.164$, $\phi(2) = 0.190$, $\phi(3) = 0.241$.

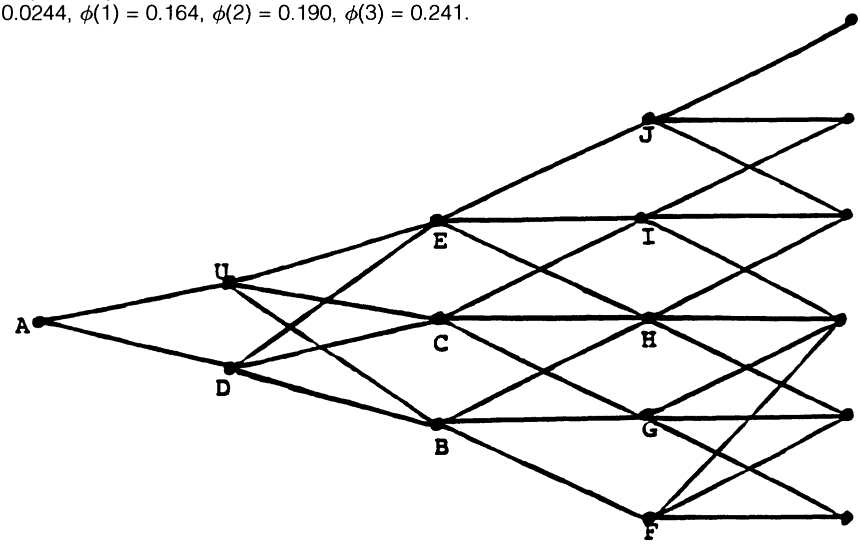


Table of Rates and Probabilities

Node	A	D	U	B	C	E	F	G	H	I	J
Rate	10.00	8.81	11.20	7.85	10.00	12.74	6.16	7.85	10.00	12.74	16.24
p_1	0.500	0.042	0.411	0.282	0.154	0.088	0.047	0.321	0.153	0.074	0.045
p_2	0.500	0.450	0.537	0.629	0.666	0.627	0.506	0.605	0.665	0.606	0.494
p_3		0.508	0.052	0.089	0.180	0.285	0.447	0.074	0.182	0.320	0.461

V. Comparison of Models

The procedures in Section IV allow two different models to be fitted to the same term structure of interest rates and volatility data. This makes it possible to assess the impact of different model assumptions on option prices. For bond options and bond futures options lasting less than nine months, there is very little difference between the option prices produced by different models. This means that most exchange-traded options can be expected to be relatively insensitive to the model chosen once the term structure of interest rates and discount bond yield volatilities have been determined.

Over-the-counter bond options and swap options frequently last considerably longer than exchange-traded options.[10] It is therefore important to investigate the impact of the model chosen on longer-dated bond options. As the life of the option is increased beyond nine months, there is very little difference between the

[10] A swap option is an option to exchange a fixed rate bond for a floating rate bond and can be regarded as a type of bond option.

prices produced by different models for at-the-money options.[11] But for deep-in and deep-out-of-the-money options, the absolute differences are greater. This is illustrated by Table 2, which shows the prices for one-, two-, three-, and four-year call options on five-year discount bonds using the model in (20) with β = 0, 0.5, and 1.0. The term structure and initial volatility data are the same as in Figure 2, and σ is chosen so that the initial instantaneous standard deviation of r is the same for all models and equal to 0.014. All results are based on 100 time steps. The pattern of results in Table 2 can be explained by the effect of β on the skewness of the distributions of future interest rates.[12] In proportional terms, the differences between the prices produced by the models are quite high for deep-out-of-the-money options.

TABLE 2

Prices of European call options on a five-year discount bond with a face value of $100 when the interest rate model

$$dr = [\theta(t) - \phi(t)r]dt + \sigma r^{\beta} dz(t)$$

is fitted to the term structure of interest rates and initial discount bond yield volatilities using a trinomial tree. The term structure of interest rates is flat at 10 percent per annum. The volatility of the yield on a t-year discount bond is assumed to be $(14 - t)$% per annum. The parameter, σ, is chosen so that σr^{β} equals 0.014 when $r = 0.1$.

Option Maturity (Years)		Exercise Price[a]				
	β	0.96	0.98	1.00	1.02	1.04
	0.0	2.55	1.58	0.84	0.38	0.14
1.0	0.5	2.57	1.60	0.84	0.37	0.12
	1.0	2.59	1.61	0.84	0.36	0.10
	0.0	2.56	1.59	0.85	0.39	0.15
2.0	0.5	2.58	1.61	0.86	0.37	0.13
	1.0	2.61	1.63	0.86	0.35	0.10
	0.0	2.48	1.44	0.67	0.24	0.06
3.0	0.5	2.50	1.46	0.67	0.22	0.04
	1.0	2.52	1.48	0.67	0.20	0.03
	0.0	2.43	1.26	0.37	0.05	0.00
4.0	0.5	2.43	1.27	0.37	0.03	0.00
	1.0	2.44	1.28	0.37	0.02	0.00

[a]The exercise price is expressed as a proportion of the forward bond price.

One particularly popular over-the-counter interest-rate option is a cap. This is designed to provide insurance against the rate of interest paid on a floating-rate loan rising above a predetermined level (the cap rate). As explained in Hull and White (1990b), a cap can be regarded as a portfolio of put options on bonds. Table 3 compares cap prices when β = 0, 0.5, and 1.0 and the models are fitted to the same yield curve and volatility data. The results are generally similar to those

[11]An at-the-money European bond option is defined as one where the strike price equals the forward bond price.

[12]As β increases, low interest rates (high bond prices) become less likely and high interest rates (low bond prices) become more likely. This means that the prices of out-of-the-money calls increase. The prices of out-of-the-money puts decrease and from put-call parity, the prices of in-the-money calls must therefore also decrease.

for bond options in Table 2. The model chosen makes very little difference for at-the-money caps, but can have a significant effect on the prices of out-of and in-the-money caps.[13]

TABLE 3

Prices, as a percent of principal, of instruments designed to cap an interest rate that is reset every three months. The term structure is flat at 10 percent per annum with quarterly compounding. The yield volatilities are those for the $\beta = 0$ model with $\sigma = 0.015$ and $a = 0.1$.

Cap Maturity (Years)	β	Cap Rate %				
		9.0	9.5	10.0	10.5	11.0
	0.0	0.76	0.49	0.27	0.14	0.07
1.0	0.5	0.76	0.48	0.27	0.14	0.07
	1.0	0.75	0.48	0.27	0.15	0.08
	0.0	1.85	1.30	0.81	0.52	0.30
2.0	0.5	1.83	1.29	0.81	0.53	0.32
	1.0	1.82	1.28	0.81	0.54	0.34
	0.0	2.96	2.16	1.43	1.00	0.64
3.0	0.5	2.92	2.14	1.43	1.02	0.67
	1.0	2.89	2.12	1.44	1.04	0.71
	0.0	4.03	3.02	2.08	1.51	1.02
4.0	0.5	3.98	2.99	2.08	1.54	1.07
	1.0	3.93	2.97	2.08	1.57	1.12
	0.0	5.06	3.86	2.73	2.03	1.41
5.0	0.5	4.98	3.82	2.73	2.07	1.48
	1.0	4.91	3.78	2.73	2.10	1.54

The correct value for β is an issue that is difficult to resolve by analyzing interest rate data. Chan et al. (1992) have considered the model in (16) with the two functions of time being constants. Using monthly data on one-month Treasury bill yields between June 1964 and December 1989, they conclude that $\beta = 1.499$ provides the best fit. Empirical research carried out by the authors shows that maximum likelihood estimates of β are greatly influenced by the few observations in a sample where large interest-rate movements take place.

An alternative approach to choosing between the models in Equation (14) is to use the market prices of options. The test of a model's quality is how little the volatility parameters must be varied in order to price correctly a wide range of interest-rate options. This is similar to the approach suggested by MacBeth and Merville (1979) and Rubinstein (1985) for evaluating equity option pricing models. The difference is that there are two volatility parameters, a and σ, rather than one.

[13]Since caps pay off when interest rates are high (bond prices low), the impact of changing beta on the prices of in-the-money and out-of-the-money caps is the reverse of that for in-the-money and out-of-the-money call options on bonds (see footnote 12).

VI. Changing the Length of the Time Step

For many one-factor interest rate models, bond prices are not known analytically as a function of the short rate.[14] When bond options are valued, the interest rate tree must therefore have the same life as the bond. This presents a problem when a short-dated option on a long-dated bond is being valued. The Δt required during the life of the option is generally much smaller than that required for the period of time between the end of the life of the option and the end of the life of the bond. For example, when a three-month option on a 10-year bond is being valued, it might be appropriate to use 50 time steps each of length 0.005 years during the first three months and 39 steps each of length 0.25 years during the remaining 9.75 years. This section shows how this type of variation in Δt can be achieved.

Suppose that at time τ it is required to change the length of time steps from Δt_1 to Δt_2. The new time step length, Δt_2, is assumed to be an integral multiple of the old one, Δt_1. The tree is constructed using time steps of length Δt_1 until time $\tau + \Delta t_2$ as described in Sections III and IV. The tree is then used to calculate the value of the Δt_2-maturity interest rate at each of the nodes at time τ. This calculation is necessary because up to time τ, the short rate on the tree is the Δt_1-maturity rate. From time τ onward, it is the Δt_2-maturity rate. Once the calculation has been carried out, the tree constructed between time τ and time $\tau + \Delta t_2$ can be dispensed with.

From time τ onward, the tree is constructed in time steps of length Δt_2. The new Δx is chosen to be equal to the old Δx times $\sqrt{\Delta t_2/\Delta t_1}$. At time $\tau + \Delta t_2$, one of the short rates to be considered can be chosen arbitrarily. The rest are determined by the new Δx. Footnote 6 describes the modifications to the standard calculations necessary to define the branching process between times τ and $\tau+\Delta t_2$.

The geometry of a tree where the size of the time step increases by a factor of three after two steps is illustrated in Figure 3.

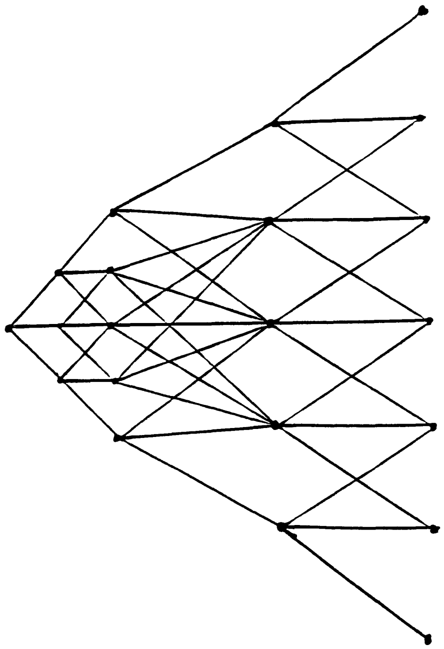
VII. Conclusions

From a computational perspective, there are compelling arguments in favor of using a yield-curve model that is Markov when valuing interest rate derivative securities. It is also desirable that the model fit the initial term structure so that it prices at least one well-known class of securities correctly. One approach to developing a Markov model of the short rate that fits the term structure is to specify a process for the short rate that has one or two unknown functions of time in the drift and then estimate these functions so that the process is consistent with the initial term structure and other market data. Examples of models that are developed using this approach are the extended-Vasicek and extended-CIR models in Hull and White (1990b), and the lognormal interest rate model in Black and Karasinski (1991).

[14]The extended-Vasicek model is the only model where bond prices are known analytically. For the extended-CIR model in (13), Cox, Ingersoll, and Ross (1985) provide an expression for bond prices in terms of an analytically intractable integral. It is found to be computationally more efficient to construct a trinomial tree for calculating bond prices in the extended-CIR model than to evaluate the integral numerically.

FIGURE 3

Geometry of Tree when Δt Increases by a Factor of 3 after the Second Step



This paper has developed a general procedure, involving the construction of a trinomial tree for the short rate, that implements the approach. The procedure is robust and numerically efficient. It provides an alternative to the model-specific procedures for constructing trees that are suggested by authors such as Ho and Lee (1986) and Black and Karasinski (1991). It provides a way in which the extended-Vasicek, the extended-CIR model, and a wide range of other one-factor models can be implemented.

Once the tree has been constructed, bond options and other non-path-dependent interest rate derivative securities can be valued in the usual way by working back through the tree from the end of the life of the security to time zero. Path-dependent derivative securities can be valued by using Monte Carlo simulation to randomly sample paths through the tree. If required, the length of the time step can be changed during the life of the tree.

Appendix A

In this appendix, it is assumed that the tree has been constructed up to time $n\Delta t$ and it is shown how $\theta(n\Delta t)$ is obtained. Define $Q(i,j)$ as the value of a security that pays off \$1 if node (i,j) is reached and zero otherwise. It is assumed that the $Q(i,j)$s are calculated as the tree is being constructed using the relationship

$$Q(i,j) = \sum_{j^*} Q(i-1,j^*)q(j^*,j)e^{-r_{j^*}\Delta t},$$

where $q(j^*,j)$ is the probability of moving from node $(i-1,j^*)$ to node (i,j).[15] (For any given j^*, this is zero for all except three of the js.) This means that when $\theta(n\Delta t)$ is being estimated, the $Q(i,j)$s are known for all $i \leq n$.

The value as seen at node (n,j) of a bond maturing at time $(n+2)\Delta t$ is

$$e^{-r_j\Delta t}E\left[e^{-r(n+1)\Delta t}|r(n) = r_j\right],$$

where E is the risk-neutral expectations operator and $r(i)$ is the value of r at time $i\Delta t$. The value at time zero of a discount bond maturing at time $(n+2)\Delta t$ is therefore given by

$$\text{(A-1)} \qquad e^{-(n+2)R(n+2)\Delta t} = \sum_j Q(n,j)e^{-r_j\Delta t}E\left[e^{-r(n+1)\Delta t}|r(n) = r_j\right].$$

If $\epsilon(n,j)$ is defined as the value of $\{r(n+1) - r(n)|r(n) = r_j\}$,

$$\text{(A-2)} \qquad E\left[e^{-r(n+1)\Delta t}|r(n) = r_j\right] = e^{-r_j\Delta t}E\left[e^{-\epsilon(n,j)\Delta t}\right].$$

Expanding $e^{-\epsilon(n,j)\Delta t}$ as a Taylor series, taking expectations, and ignoring terms of higher order than Δt^2,

$$\text{(A-3)} \qquad E\left[e^{-r(n+1)\Delta t}|r(n) = r_j\right] = e^{-r_j\Delta t}\left[1 - \mu_{n,j}\Delta t^2\right].$$

Since the drift of the short rate, $\mu_{n,j}$, is a known function of $\theta(n\Delta t)$, $\theta(n\Delta t)$ can be determined from (A-1) and (A-2). For example, in the models in (10) and (14),

$$\mu_{n,j} = \theta(n\Delta t) - ar_j,$$

so that

$$\text{(A-4)} \qquad \theta(n\Delta t) = \frac{\sum_j Q(n,j)e^{-2r_j\Delta t}\left(1 + ar_j\Delta t^2\right) - e^{-(n+2)R(n+2)\Delta t}}{\sum_j Q(n,j)e^{-2r_j\Delta t}\Delta t^2}.$$

The estimates of $\theta(n\Delta t)$ given by this equation are found to be satisfactory for most purposes. They lead to a tree where discount bond prices calculated from the tree at time zero replicate those in the market to at least four significant figures. Any errors in the estimates tend to be self-correcting. For example, if the estimate for $\theta(n\Delta t)$ is slightly low, the estimate for $\theta[(n+1)\Delta t]$ tends to compensate for this by being slightly too high. If an even better fit to the initial yield curve is required, more terms in the Taylor series expansion can be used or an iterative procedure can be developed.

[15] A useful byproduct of storing the Qs is that all discount bond prices and European-style derivative securities can be valued immediately as $\sum_j Q(N,j)U(N,j)$, where $N\Delta t$ is the time when the security matures and $U(N,j)$ is the payoff at node (N,j).

In the case of the extended-Vasicek model in (10), the expectation in (A-2) is known analytically

$$E\left[e^{-r(n+1)\Delta t}|r(n)=r_j\right] \quad = \quad e^{-r_j\Delta t}e^{\left[-\theta(n\Delta t)+ar_j+\sigma^2\Delta t/2\right]\Delta t^2}.$$

Using (A-1), this leads to

$$\text{(A-5)}\quad \theta(n\Delta t) \quad = \quad \frac{1}{\Delta t}(n+2)R(n+2)+\frac{\sigma^2\Delta t}{2}+\frac{1}{\Delta t^2}\,\text{Log}\sum_j Q(n,j)e^{-2r_j\Delta t+ar_j\Delta t^2}.$$

This is an improvement over the estimate in (A-4) and leads to a tree that replicates discount bond prices at time zero with about eight significant figure accuracy.[16]

Appendix B

This appendix describes how $\theta(n\Delta t)$ and $\phi(n\Delta t)$ are calculated when a model is being fitted to both the term structure of interest rates and the current volatility structure. It is assumed that the tree has been constructed up to time $n\Delta t$. Define:

$Q_u(i,j)$: the value as seen at node U of a security that pays off \$1 if node (i,j) is reached and zero otherwise,

$Q_d(i,j)$: the value as seen at node D of a security that pays off \$1 if node (i,j) is reached and zero otherwise.

It is assumed that the $Q_u(i,j)$s and $Q_d(i,j)$s are known for $i \leq n$. As with the Qs in Appendix A, it is possible to calculate them as the tree is being constructed.

Analogously to (A-1), when $n \geq 2$, the values as seen at nodes U and D of bonds maturing at time $(n+2)\Delta t$ are given by

$$\text{(B-1)}\qquad e^{-(n+1)R_u(n+2)\Delta t} \quad = \quad \sum_j Q_u(n,j)e^{-r_j\Delta t}E\left[e^{-r(n+1)\Delta t}|r(n)=r_j\right],$$

and

$$\text{(B-2)}\qquad e^{-(n+1)R_d(n+2)\Delta t} \quad = \quad \sum_j Q_d(n,j)e^{-r_j\Delta t}E\left[e^{-r(n+1)\Delta t}|r(n)=r_j\right],$$

respectively.

Equation (A-3) is still true. In this case, $\mu_{n,j}$ is a known function of both $\theta(n\Delta t)$ and $\phi(n\Delta t)$. Using (A-3) in conjunction with (B-1) and (B-2), therefore, provides a pair of simultaneous equations for determining $\theta(n\Delta t)$ and $\phi(n\Delta t)$. In the case of (16), the equations are linear in these unknowns,

$$\theta(n\Delta t)\sum_j Q_u(n,j)e^{-2r_j\Delta t}\Delta t^2 - \phi(n\Delta t)\sum_j Q_u(n,j)e^{-2r_j\Delta t}r_j\Delta t^2$$

$$= \sum_j Q_u(n,j)e^{-2r_j\Delta t} - e^{-(n+1)R_u(n+2)\Delta t},$$

[16] The tree does not replicate the discount bond prices exactly even though Equation (A-5) is exact. This is because the trinomial distribution for interest rates in the tree is not a perfect representation of the normal distribution being assumed.

$$\theta(n\Delta t)\sum_j Q_d(n,j)e^{-2r_j\Delta t}\Delta t^2 - \phi(n\Delta t)\sum_j Q_d(n,j)e^{-2r_j\Delta t}r_j\Delta t^2$$
$$= \sum_j Q_d(n,j)e^{-2r_j\Delta t} - e^{-(n+1)R_d(n+2)\Delta t}.$$

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