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An explicitly solvable Heston model with stochastic interest rate

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ABSTRACT

This paper deals with a variation of the Heston hybrid model with stochastic interest rate illustrated in Grzelak and Oosterlee (2011). This variation leads to a multi-factor Heston model where one factor is the stochastic interest rate. Specifically, the dynamics of the asset price is described through two stochastic factors: one related to the stochastic volatility and the other to the stochastic interest rate. The proposed model has the advantage of being analytically tractable while preserving the good features of the Heston hybrid model in Grzelak and Oosterlee (2011) and of the multi-factor Heston model in Christoffersen et al. (2009). The analytical treatment is based on an appropriate parametrization of the probability density function which allows us to compute explicitly relevant integrals which define option pricing and moment formulas. The moments and mixed moments of the asset price and log-price variables are given by elementary formulas which do not involve integrals. A procedure to estimate the model parameters is proposed and validated using three different data-sets: the prices of call and put options on the U.S. S&P 500 index, the values of the Credit Agricole index linked policy, Azione Più Capitale Garantito Em.64., and the U.S. three-month, two and ten year government bond yields. The empirical analysis shows that the stochastic interest rate plays a crucial role as a volatility factor and provides a multi-factor model that outperforms the Heston model in pricing options. This model can also provide insights into the relationship between short and long term bond yields.

1. Introduction

The pricing of derivative products is probably one of the most challenging topics in modern financial theory. The modern fixed income market includes not only bonds but also derivative securities sensitive to interest rates. In this paper we consider a modified version of the hybrid model illustrated in Grzelak and Oosterlee (2011). This model is described by a system of stochastic differential equations (SDEs) which combines different models for equity, interest rate and volatility in order to efficiently price European vanilla call and put options with short and long maturities.

Specifically, we focus on the model which combines the Heston model 1993 for equity and its volatility and the Cox, Ingersoll, and Ross (1985) (CIR) model for interest rate. Roughly speaking, the proposed model can be interpreted as the Heston multi-factor model introduced by Christoffersen, Heston, and Jacob (2009) where one volatility factor is the stochastic interest rate. Indeed, a multi-factor Heston-like stochastic volatility model capable of explaining some stylized facts on interest rate volatility and of forecasting bond yields with different maturities has been introduced by Trolle and Schwartz (2009). This model describes the yield using a stochastic differential equation where the drift and the coefficients of the stochastic variances are dependent on time and maturity date.

Furthermore, Cieslak and Povala (2014) propose the joint use of a short term yield with another stochastic factor as an efficient tool to explain the volatility of the yields. Motivated by the results of these papers, the hybrid model proposed here has been applied to option pricing (see Sections 4.2 and 4.3) and, at a very preliminary stage, to bond yield analysis (see Section 4.4).

1.1. Literature review

Recent literature motivates the use of hybrid SDE models due to the empirical evidence that the asset volatility and the interest rate are not constant over time. Indeed, the relaxation of the constant volatility assumption in time continuous stochastic volatility models goes back to the end of 1980s with (Ball & Roma, 1994; Heston, 1993; Hull & White, 1988; Stein & Stein, 1991).

The Heston model is one of the most celebrated models because it allows for closed-form formulas for option pricing. In fact, this model accurately describes the asset price behavior when the

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assumption of constant interest rate is realistic and the volatility is not affected by abrupt oscillations. Improvements of this model can be found in the recent literature with the models of Christoffersen et al. (2009), Fatone, Mariani, Recchioni, and Zirilli (2009), Fatone, Mariani, Recchioni, and Zirilli (2013), Wong and Lo (2009), Date and Islyaev (2015), Islyaev and Date (2015), Pun, Chung, and Wong (2015). These models are modified versions of the Heston model which are able to better capture the price volatility dynamics in order to efficiently solve option pricing problems.

The relaxation of the constant interest rate assumption can be found in the literature of the last decade. Far from being exhaustive, we cite the papers of Chiarella and Kwon (2003), Trolle and Schwartz (2009), Andersen and Benzoni (2010), Christensen, Diebold, and Rudebush (2011), Moreno and Platania (2015) which show that stochastic interest rates should be used in order to capture the bond yield behavior.

In line with the attempt to deal with stochastic interest rates and stochastic volatility(ies) several hybrid SDE models have been introduced since 2000. In fact, Zhu (2000) introduces a model capable of generating a skew pattern for the equity using a stochastic interest rate not correlated to the equity. Later, Andreasen (2007) generalizes the Zhu model using the Heston stochastic volatility model and an indirect correlation between the equity and the interest rate process. Ahlip (2008) studies a model for the spot FX (Foreign Exchange) rate with stochastic volatility and stochastic domestic as well as foreign rates. Specifically, these rates are modeled by the Ornstein–Uhlenbbeck process and the volatility is modeled by a mean-reverting Ornstein–Uhlenbeck process correlated with the spot FX rate. Ahlip also derives an analytical formula for the price of European call options on the spot FX rate. Grzelak, Oosterlee, and Van Weeren (2012) propose the so called Schöbel–Zhu–Hull–White hybrid model. This is an affine model whose analytical treatment is shown by Grzelak et al. (2012) following the approach proposed by Duffie, Pan, and Singleton (2000). However, as highlighted in Grzelak and Oosterlee (2011), the model allows for negative volatility and interest rate. In order to overcome this problem they propose the use of a Cox–Ingersoll–Ross (CIR) process to describe the variance and interest rate processes. Local volatility models have also been extended to deal with stochastic interest rates. For example, Deelstra and Rayee (2012) propose a three factor pricing model with local volatility and domestic and foreign interest rates modeled by the Hull and White (HW) model 1993. In line with the latter, Benhamou, Gobet, and Miri (2012) provide analytical formulas for European option prices when the underlying asset is described by a local volatility model with stochastic rates.

1.2. Description of the results

As previously mentioned, this paper focuses on a modified version of the hybrid Heston-CIR model illustrated by Grzelak and Oosterlee (2011). However, the analytical treatment proposed here extends to the hybrid Heston-Hull and White model. Our contribution is threefold.

Firstly, we modify the hybrid SDE model of Grzelak and Oosterlee (2011) in order to preserve the affine structure and to permit a “direct” correlation between the equity and the interest rate. As highlighted by Grzelak and Oosterlee (2011) this correlation plays a fundamental role in getting a good match between the observed and the theoretical option prices.

Secondly, we present the analytical treatment of the model. Specifically, we derive a parametrization of the probability density function of the stochastic process by solving the backward Kolmogorov equation using some ideas illustrated in Fatone et al. (2009, 2013).

This parametrization allows us to express the prices of European call and put options as one dimensional integrals and to derive elementary formulas for the moments and mixed moments of the asset price variable as well as the moments of the log-price variable. These elementary formulas which do not involve integrals show that the existence of bounded moments of the price variable depends on the value of the correlation coefficients. This finding coheres with the results obtained by Lions and Musiela (2007) as well as Andersen and Piterbarg (2007) on the explosion of moments of some well known probability distributions. As a byproduct of this analytical treatment we obtain an efficient approximation of the stochastic integral appearing in the discount factor (i.e. an efficient approximation of the zero-coupon bond). This approximation is suggested by the explicit formula of the zero-coupon bond in the CIR model (see Eq. (29)) and permits us to approximate the option prices as one dimensional integrals of explicitly known elementary functions.

Thirdly, we calibrate the model in order to measure its performance in interpreting real data and forecasting European call and put option prices (see the empirical analysis on the U.S. S&P 500 index options in Section 4.2 and on the Credit Agricole pure endowment policy in Section 4.3) and in predicting the trend of the two and ten-year U.S. government bond yields (see Section 4.4).

The calibration procedure of the empirical analysis of Sections 4.2 and 4.3 is based on the solution of a nonlinear constrained optimization problem whose objective function measures the relative squared difference between the observed and theoretical call and/or put option prices while the calibration of Section 4.4 is based on the maximum likelihood approach. Numerical simulations show that the approximate formulas for the discount factor and option prices work satisfactorily for any maturities (see Section 4.1). Moreover, the empirical study conducted using real data shows that the model is capable of fitting and predicting satisfactorily call and put option prices using only one set of model parameters obtained from the calibration procedure (empirical analysis of call and put options on U.S. S&P 500 index). In addition, it is capable of forecasting values of long-term products (Credit Agricole pure endowment policy) and of interpreting the relationship between short and long term bond yields.

The empirical analysis regarding the pure endowment policy and the two/ten-year U.S. government bonds show the crucial role played by the stochastic interest rate factor when long maturities are considered. However, we show that the interest rate is a significant factor even in the case of European options on the U.S. S&P 500 index. To do this we use a two-stage calibration. In the first stage we estimate the parameters of the CIR model by using the U.S. three month government bond yields and then we estimate the remaining parameters by using option prices. The hybrid model calibrated in this way still outperforms the Heston model highlighting that the stochastic interest rate is a crucial factor (see Section 4.2).

1.3. Outline of the paper

The paper is organized as follows. In Section 2 we describe the hybrid SDE model and illustrate the main relevant formulas. In Section 3 we propose formulas to approximate the European vanilla call and put option prices as one-dimensional integrals of explicitly known functions. In Section 4 we illustrate some experiments involving the moments of the price variable and the zero coupon bond formula (Section 4.1). Furthermore, we estimate the model parameters implied by the option prices by solving constrained optimization problems. We use the daily prices of European vanilla call and put options on the U.S. S&P 500 index from April 2, 2012 to July 2, 2012 in Section 4.2, the weekly values of the Credit Agricole pure endowment policy from April 4, 2012 to April 13, 2015 in Section 4.3. In Section 4.4 the daily data of the three-month, two(ten)-year U.S.

government bond yields in the period December 31, 2002 to April 3, 2014 are analyzed by estimating the model parameters using the maximum likelihood approach. Finally, conclusions are drawn in Section 5 and in the Appendix we derive the formula for the probability density function and explicit formulas of the moments of the asset price.

2. The Heston hybrid SDE model

In this section we present the Heston hybrid model and some relevant formulas (i.e. transition probability density function, explicit moments of the price variable, the first order moment of the log-price variable, etc.) that are necessary for estimating the model parameters and deducing option pricing formulas.

Hereafter, we denote with S_t, v_t and r_t the equity price, its volatility and the interest rate at time $t > 0$ respectively. The hybrid model proposed is inspired to the Grzelak and Oosterlee (2011) model. More precisely, the model describes the dynamics of the process (S_t, v_t, r_t), $t > 0$, through the following system of stochastic differential equations:

$$dS_t = S_t r_t dt + S_t \sqrt{v_t} dW_t^{p,v} + S_t \Delta \sqrt{v_t} dW_t^v + S_t \Omega_t \sqrt{r_t} dW_t^{p,r}, \quad t > 0, \tag{1}$$

$$dv_t = \chi (v^* - v_t) dt + \gamma \sqrt{v_t} dW_t^v, \qquad t > 0, \tag{2}$$

$$dr_t = \lambda(\theta - r_t) dt + \eta \sqrt{r_t} dW_t^r, \quad t > 0 \tag{3}$$

where Δ is a positive constant, Ω_t is a positive function, $\chi, v^*, \gamma, \lambda, \theta$, η are positive constants and $W_t^{p,v}$, $W_t^{p,r}$, W_t^v, W_t^r are standard Wiener processes. We assume the following correlation structure:

$$E(dW_t^{p,v} dW_t^v) = \rho_{p,v}\, dt, \;\; E(dW_t^{p,v} dW_t^r) = 0, \quad t > 0, \tag{4}$$

$$E(dW_t^{p,r} dW_t^r) = \rho_{p,r}\, dt, \;\; E(dW_t^{p,r} dW_t^v) = 0, \;\; E(dW_t^r dW_t^v) = 0, \quad t > 0, \tag{5}$$

where $E(\cdot)$ denotes the expected value of $\cdot$ and $\rho_{p,v}$, $\rho_{p,r}$ are constant quantities known as correlation coefficients, $\rho_{p,v}, \rho_{p,r} \in (-1, 1)$.

Roughly speaking, this model generalizes the Heston model within a framework of stochastic interest rates. This stochastic rate is described by the Cox Ingersoll Ross (CIR for short) model resulting in a hybrid Heston model named HCIR model.

Indeed, this model is closely related to the multi-factor Heston model proposed by Christoffersen et al. (2009). In fact, the latter describes the dynamics of the asset price through several stochastic volatilities while assuming a constant risk free interest rate. We, on the other hand, have made the interest rate a factor driving the price. However, the proposed analytical treatment which allows us to deduce explicit formulas for the conditional moments and mixed moments can also be applied to the Heston multi-factor model in Christoffersen et al. (2009).

The system of Eqs. (1)–(3) is equipped with the following initial conditions:

$$S_0 = S_0^*, \;\; v_0 = v_0^*, \;\; r_0 = r_0^*, \tag{6}$$

where S_0^* and v_0^*, r_0^* are random variables concentrated in a point with probability one. For simplicity, we identify these random variables with the points where they are concentrated and we choose S_0^*, $v_0^* > 0$. In the following we continue to use S_0, v_0, r_0 to denote the random variables S_0^*, v_0^*, r_0^*. The quantities χ, v^*, γ, λ, θ, η describe two mean reverting processes: one for the price variance and the other for the stochastic interest rate. More precisely, the quantity χ is the speed of mean reversion of the variance process and v^* is its long term meanwhile γ is the so called volatility of variance process (vol of vol for short).

It is worth noting that the variance v_t remains positive for any $t > 0$ with probability one given that $2\chi\, v^*/\gamma^2 > 1$ and $v_0 = v_0^* > 0$ (see Heston, 1993). As a consequence, the equity price S_t remains positive for any $t > 0$ with probability one given that $S_0^* > 0$ with probability one.

The parameters λ, θ and η of the CIR model which describe the interest rate, r_t, have the same interpretation as the corresponding parameters χ, v^* and ϵ of the variance process, v_t. Hence, with regard to the variance process, v_t, the interest rate, r_t, described by the HCIR model remains positive with probability one for any $t > 0$ given that $2\lambda\theta/\eta^2 > 1$ and $r_0 = r_0^* > 0$ with probability one (see Heston, 1991).

As explained in Grzelak and Oosterlee (2011), the good performance of the HCIR model in interpreting real data is due to the terms $S_t \Delta \sqrt{v_t} dW_t^v$ and $S_t \Omega_t \sqrt{r_t} dW_t^{p,r}$ that correspond to the terms $S_t \Delta \sqrt{v_t} dW_t^v$ and $S_t \Omega_t \sqrt{r_t} dW_t^r$ in the Grzelak and Oosterlee model. Our model replaces the term $S_t \Omega_t \sqrt{r_t} dW_t^r$ with $S_t \Omega_t \sqrt{r_t} dW_t^{p,r}$. We modify this term to allow for a direct correlation between equity and interest rate. In the Grzelak and Oosterlee model the term $S_t \Omega_t \sqrt{r_t} dW_t^r$ implies a covariance between equity and interest rate equal to $S_t^2 \Omega_t^2\, r_t\, dt$ while in the proposed method the term $S_t \Omega_t \sqrt{r_t} dW_t^{p,r}$ implies a covariance equal to $S_t^2 \Omega_t^2\, \rho_{p,r}\, r_t\, dt$.

Furthermore, as in the Grzelak and Oosterlee model, the decomposition of the Wiener processes appearing in (1) allows us to show how the conditional expected value of the log-price depends on $\rho_{p,v}$ as well as on Δ and Ω (see Eq. (24)). This is possible thanks to the explicit formulas of the moments of the (log)price deduced by the analytical treatment of the model illustrated later in this section.

For simplicity, in order to illustrate the analytical treatment, we assume $\Omega_t = \Omega$, with Ω being a positive constant. More general choices of Ω_t can be considered while preserving the analytical treatment of the model.

Let us now give the main formulas derived in this paper and used in the simulation and empirical analysis. To this aim, we rewrite the model (1)–(3) in term of the log-price, $x_t = \ln(S_t/S_0)$, $t > 0$. Using Ito's lemma and Eq. (1) we obtain that the process (x_t, v_t, r_t) satisfies the following dynamics:

$$dx_t = \left[r_t - \frac{1}{2}\left(\tilde{\psi} v_t + \Omega^2 r_t\right)\right] dt + \sqrt{v_t} dW_t^{p,v} + \Delta \sqrt{v_t} dW_t^v + \Omega \sqrt{r_t} dW_t^{p,r}, \;\; t > 0, \tag{7}$$

$$dv_t = \chi (v^* - v_t) dt + \gamma \sqrt{v_t} dW_t^v, \quad t > 0, \tag{8}$$

$$dr_t = \lambda(\theta - r_t) dt + \eta \sqrt{r_t} dW_t^r, t > 0, \tag{9}$$

where $\tilde{\psi}$ is the quantity defined by:

$$\tilde{\psi} := 1 + \Delta^2 + 2\rho_{p,v}\,\Delta. \tag{10}$$

The system of stochastic differential Eqs. (7)–(9) is equipped with the initial conditions:

$$x_0 = x_0^* = 0, \quad v_0 = v_0^*, \quad r_0 = r_0^*. \tag{11}$$

In Eq. (11), as already specified, x_0^* is a random variable that we assume to be concentrated in a point with probability one.

Now, let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}^+$ the set of the positive real numbers and $\mathbf{R}^n$ the n-dimensional Euclidean vector space. Let $\underline{\Theta}_v$ and $\underline{\Theta}_r$ denote the vectors $\underline{\Theta}_v = (\gamma, \chi, v^*, \rho_{p,v}, \Delta) \in \mathbf{R}^5$ and $\underline{\Theta}_r = (\eta, \lambda, \theta, \rho_{p,r}, \Omega) \in \mathbf{R}^5$ containing the parameters of the volatility and interest rate processes respectively. Let $p_f(x, v, r, t, x', v', r', t')$, (x, v, r), $(x', v', r') \in \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+, t$, $t' \geq 0$, $t - t' > 0$, be the transition probability density function associated with the stochastic differential system (7)–(9), that is, the probability density function of having $x_{t'} = x'$, $v_{t'} = v'$, $r_{t'} = r'$ given that $x_t = x$, $v_t = v$, $r_t = r$, when $t' - t > 0$. This transition probability density function $p_f(x, v, r, t, x', v', r', t')$ as a function of the “past” variables (x, v, r, t) satisfies the following backward Kolmogorov equation:

$$-\frac{\partial p_f}{\partial t} = \frac{1}{2}\big[(1+\Delta^2+2\Delta\rho_{p,v})v+\Omega^2 r\big]\frac{\partial^2 p_f}{\partial x^2}+\frac{1}{2}\gamma^2 v\frac{\partial^2 p_f}{\partial v^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 p_f}{\partial r^2}+\gamma(\rho_{p,v}+\Delta)v\frac{\partial^2 p_f}{\partial x\partial v}+\eta\rho_{p,r}\Omega r\frac{\partial^2 p_f}{\partial x\partial r} + \chi(v^*-v)\frac{\partial p_f}{\partial v}+\lambda(\theta-r)\frac{\partial p_f}{\partial r} + \left(r-\frac{1}{2}\big[(1+2\rho_{p,v}\Delta+\Delta^2)v+\Omega^2 r\big]\right)\frac{\partial p_f}{\partial x}, \quad (x,v,r)\in\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^+,\ 0\le t<t', \tag{12}$$

with final condition:

$$p_f(x,v,r,t,x',v',r',t') = \delta(x'-x)\delta(v'-v)\delta(r'-r), \quad (x,v,r),(x',v',r')\in\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^+,\ t\ge 0, \tag{13}$$

and appropriate boundary conditions.

In Appendix A, using a suitable parametrization of the transition probability density function we prove that the following formula holds:

$$p_f(x,v,r,t,x',v',r',t') = e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\, e^{\imath k(x'-x)} \times L_{v,q}(t'-t,v,v',k;\underline{\Theta}_v)\,L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r),\ (x,v,r), (x',v',r')\in\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^+,\ t,t'\ge 0,\ q\in\mathbf{R},\ t'-t>0, \tag{14}$$

where $\imath$ is the imaginary unit, $L_{v,q}$ and $L_{r,q}$ are explicitly known functions given in Eqs. (78)–(80). The functions $L_{v,q}$ and $L_{r,q}$ depend on the modified Bessel functions, $I_{(2\chi v^*/\gamma^2)-1}$, $I_{(\lambda\theta/\eta^2)-1}$ (see, for example, Abramowitz & Stegun, 1970) where $(2\chi v^*/\gamma^2)-1$ and $(\lambda\theta/\eta^2)-1$ are real indices. The indices of these Bessel functions are positive under the conditions $2\chi v^*/\gamma^2>1$ and $2\lambda\theta/\eta^2>1$. Positive indices imply that the modified Bessel functions are bounded at zero and this guarantees that the function p_f given in (14) is a probability density function with respect to the future variables.

On the other hand, as already mentioned, these conditions are those that guarantee positive values of the variance and interest rate processes for any time (with probability one) given that the initial stochastic conditions v_0, r_0 are positive (with probability one).

Moreover, it is worth noting that formula (14) can be interpreted as the inverse Fourier transform of the convolution of the probability density functions associated with the stochastic processes described by Eqs. (7)–(9) when one of the two factors, v_t, r_t, is dropped. This specific form of the transition probability density function is a consequence of the correlation structure (4) and (5).

A further good feature of the functions $L_{v,q}$ and $L_{r,q}$ is that the integrals of $L_{v,q}$ and $L_{r,q}$ with respect the future variables v' and r' are given by elementary functions, $W^0_{v,q}$ and $W^0_{r,q}$ respectively (see formulas (83) and (84)) as well as their products for integer powers of the future variables v', r' (see formulas (85) and (86)). That is, let us define $W^m_{v,q}$ and $W^m_{r,q}$, $m=0,1,\ldots$, as follows:

$$W^m_{v,q}(t'-t,v,k;\underline{\Theta}_v) = \int_0^{+\infty}dv'\,(v')^m L_{v,q}(t'-t,v,v',k;\underline{\Theta}_v), \tag{15}$$

$$W^m_{r,q}(t'-t,r,k;\underline{\Theta}_r) = \int_0^{+\infty}dr'\,(r')^m L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \tag{16}$$

we can use these functions and the function $L_{r,q}$ to get an integral representation formula for the marginal probability density function, $D_{v,q}(x,v,r,t,x',r',t')$, of the future variables (x',r')

$$D_{v,q}(x,v,r,t,x',r',t') = \int_0^{+\infty}dv'\,p_f(x,v,r,t,x',v',r',t') = e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk e^{\imath k(x'-x)}W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v) \times L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \tag{17}$$

and for the marginal probability density function, $D_{v,r,q}(x,v,r,t,x',t')$, of the price variable x':

$$D_{v,r,q}(x,v,r,t,x',t') = \int_0^{+\infty}dr'\int_0^{+\infty}dv'\,p_f(x,v,r,t,x',v',r',t') = e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\,e^{\imath k(x'-x)} \times W^0_{v,q}(t'-t,v,k;\underline{\Theta}_v)W^0_{r,q}(t'-t,r,k;\underline{\Theta}_r)\,. \tag{18}$$

We highlight that in formulas (17) and (18) the variables x, v, r are the initial values of the log-price, the stochastic variance and the stochastic interest rate respectively. These two last variables are not observable in the financial market and should be estimated using an appropriate calibration procedure. The estimation of the initial stochastic volatility is a common practice as suggested by Bühler (2002).

In addition, formulas (17) and (18) are useful in estimating model parameters using a maximum likelihood approach and deriving the explicit formulas for the moments of the price variable and the mixed moments. In fact, the formula for the mth moment of the price $S_{t'}=S_0 e^{x'}$ conditioned to the observation at time $t=0$ is obtained computing the following integral:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0\int_{-\infty}^{+\infty}dx' e^{mx'}D_{v,r,q}(0,v_0,r_0,0,x',t') = S^m_0\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\left(\int_{-\infty}^{+\infty}dx' e^{mx'}e^{-qx'}e^{\imath kx'}\right)W^0_{v,q}(t',v_0,k;\underline{\Theta}_v) \times W^0_{r,q}(t',r_0,k;\underline{\Theta}_r)(S_0,v_0,r_0)\in\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^+,\ q\in\mathbf{R},\ t'>0. \tag{19}$$

When we choose $m=q$ the integral in the bracket gives us a Dirac's delta function of the conjugate variable k which allows us to have the following explicit formula for the moments:

$$\mathcal{M}_m = E(S^m_{t'}) = S^m_0 W^0_{v,m}(t',v_0,0;\underline{\Theta}_v)\,W^0_{r,m}(t',r_0,0;\underline{\Theta}_r), \quad (S_0,v_0,r_0)\in\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^+,\ t'>0. \tag{20}$$

Similarly, using the explicit formulas for the integrals (15) and (16) we obtain the following explicit formulas for the mixed moments:

$$E(S^{m_1}_{t'}r^{m_2}_{t'}) = S^{m_1}_0 W^0_{v,m_1}(t',v_0,0;\underline{\Theta}_v)\,W^{m_2}_{r,m_1}(t',r_0,0;\underline{\Theta}_r), \quad (S_0,v_0,r_0)\in\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^+,\ t'>0, \tag{21}$$

and

$$E(S^{m_1}_{t'}v^{m_2}_{t'}) = S^{m_1}_0 W^{m_2}_{v,m_1}(t',v_0,0;\underline{\Theta}_v)\,W^0_{r,m_1}(t',r_0,0;\underline{\Theta}_r), \quad (S_0,v_0,r_0)\in\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{R}^+,\ t'>0, \tag{22}$$

where the functions $W^0_{v,q}$, $W^0_{r,q}$, $W^m_{v,q}$, $W^m_{r,q}$, $m=1,2,\ldots$ are elementary functions given by (83), (84), (15) and (16) respectively. Moreover, using this approach we obtain the following expressions for the moments of the log-return variable used in Section 4:

$$E(x^m_{t'}) = \imath^m\frac{d^m}{dk^m}\big[W^0_{v,0}(t',v_0,k;\underline{\Theta}_v)\,W^0_{r,0}(t',r_0,k;\underline{\Theta}_r)\big]_{k=0}, \quad (v_0,r_0)\in\mathbf{R}^+\times\mathbf{R}^+,\ t'>0, \tag{23}$$

that when $m=1$ gives:

$$E(x_{t'}) = \left(1-\frac{\Omega^2}{2}\right)\left(\theta t'+(r_0-\theta)\frac{1-e^{-\lambda t'}}{\lambda}\right) - \frac{1}{2}(1+\Delta^2+2\rho_{p,v}\Delta)\left(v^* t'+(v_0-v^*)\frac{1-e^{-\chi t'}}{\chi}\right). \tag{24}$$

We highlight that formulas (20), (22), (21), (23), (24) are elementary formulas that do not involve integrals. The derivation of these formulas is possible thanks to Eq. (14). This equation reduces the computation of the transition probability density function, which depends on a "regularization" parameter, q, to a one dimensional integral whose integrand function is the product of smooth functions of the future variables. This smoothness is a result of the specific way in which the formula is deduced. This together with an appropriate choice for q permits us to derive elementary formulas for the marginal probability density functions and the moments illustrated above. These formulas are used to estimate the model parameters with great savings of computational time.

In fact, the marginal probability density function, $D_{\nu,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, we use $D_{\nu,q}$ to deduce formulas for European call and put vanilla options.

As stressed in Christoffersen et al. (2009) a multi-factor model is more flexible for conditional kurtosis and skewness. Thanks to formula (20) we can easily compute these indicators. Finally, a useful byproduct of these formulas is the moments and the mixed moments associated with the Heston model.

3. Integral formulas for European vanilla call and put options

In the framework of the model (7)–(9) we derive integral formulas to approximate the prices of European vanilla call and put options with strike price E and maturity time T. This is done using the no arbitrage pricing theory. As illustrated in Grzelak and Oosterlee (2011) we compute the option price as the expected value of the discounted payoff with respect to an equivalent martingale measure known as a risk-neutral measure (see, for example, Duffie, 2001; Schoutens, 2003; Wong & Heyde, 2006). Denoting the spot price at time zero by S_0 and the initial stochastic interest rate and variance by r_0 and ν_0 respectively, we can compute the prices of these European vanilla call and put options as follows:

$$C(S_0,T,E,r_0,\nu_0)=E^{Q}\left(\frac{(S_0e^{x_T}-E)_+}{e^{\int_0^T r_t\,dt}}\right),\quad S_0,\ T,\ r_0,\ \nu_0>0, \tag{25}$$

$$P(S_0,T,E,r_0,\nu_0)=E^{Q}\left(\frac{(E-S_0e^{x_T})_+}{e^{\int_0^T r_t\,dt}}\right),\ S_0,\ T,\ E,\ r_0,\ \nu_0>0, \tag{26}$$

where $(\,\cdot\,)_+=\max\{\cdot,0\}$, and the expectation is taken under the risk-neutral measure Q. In the numerical experiments we use only option prices to calibrate the model. That is, we use only the risk-neutral measure and not the physical one. As a consequence, it is not necessary to introduce the risk premia parameters.

Note that ν_0 is not observable in the market and that an appropriate choice for r_0 is unclear. For these reasons we consider ν_0 and r_0 as model parameters that must be estimated (see Section 4).

However, the model parameter estimation requires the numerical evaluation of formulas (25), (26) which is very time consuming. We overcome this difficulty using formulas introduced in the previous section while approximating the stochastic integral that defines the discount factor as follows:

$$e^{-\int_0^T r_t dt}\approx e^{-r_0\frac{T}{(1+e^{\lambda T})}-r_T\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}. \tag{27}$$

Roughly speaking, formula (27) has been obtained approximating r_t as a suitably weighted sum of the short rate r_t evaluated at $t=0$ and $t=T$. In order to measure its performance we consider the following approximation:

$$E^{Q}\left(e^{\int_0^T r_t\,dt}\right)\approx E^{Q}\left(e^{-r_0\frac{T}{(1+e^{\lambda T})}-r_T\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right). \tag{28}$$

The expected values appearing in formula (28) are given by explicit formulas. Specifically, the expected value on the left hand side of Eq. (28) is the exact formula to price zero-coupon bonds in the CIR model (see Cox et al., 1985) which is given by:

$$B(r_0,T)=\left(\frac{2\,he^{(\lambda-h)T/2}}{2h+(\lambda-h)(1-e^{-hT})}\right)^{\frac{2\lambda\theta}{\eta^2}}\times e^{-\left(\frac{2\,he^{(\lambda-h)T/2}}{2h+(\lambda-h)(1-e^{-hT})}\right)r_0e^{-(h+\lambda)T/2}(e^{hT}-1)/h},\ T>0,\ r_0>0, \tag{29}$$

where $h=\sqrt{\lambda^2+2\eta^2}$. The expected value on the right hand side of Eq. (28) is computed using formula (17) and the following formula (see Erdelyi, Magnus, Oberhettinger, and Tricomi, 1954, Vol. I, p. 197, formula (18)):

$$\int_0^{+\infty}dr'\left(r'\right)^{\nu_r/2}e^{-(M_{q,r}+b)r'}I_{\nu_r}(2M_{q,r}(r_qr')^{1/2})=[(M_{q,r})^2\tilde r_q]^{\frac{\nu_r}{2}}(M_{q,r}+b)^{-\nu_r-1}e^{\frac{(M_{q,r})^2\tilde r_q}{(M_{q,r}+b)}}, \tag{30}$$

where $M_{q,\,r}$ is given in (81). That is we have:

$$B_A(r_0,T)=e^{-\frac{r_0T}{(1+e^{\lambda T})}}\left(\frac{2\lambda}{2\lambda+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\eta^2(1-e^{-\lambda T})}\right)^{\frac{2\lambda\theta}{\eta^2}}\times e^{-\left(\frac{2\lambda}{2\lambda+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\eta^2(1-e^{-\lambda T})}\right)\left(\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\right)r_0e^{-\lambda T}},\ T>0,\quad r_0>0. \tag{31}$$

Note that in formulas (29) and (31) we have denoted the expected values appearing in the left and right hand sides of Eq. (28) with $B(r_0,\ T)$ and $B_A(r_0,\ T)$ respectively to highlight the dependence of these formulas on r_0 and T. It is worthy to note that formula (31) differs from that presented in Choi and Wirjanto (2007) in that the latter is based on a suitable approximation of the stochastic differential equation that defines the interest rate while our formula is obtained using the integral representation formula of the transition probability density and a specific approximation of the stochastic interest rate process.

The simulation study, illustrated in Section 4, shows that formula (31) satisfactorily approximates zero-coupon bonds with maturities up to twenty years. In fact, comparing the zero-coupon bond values obtained using Eqs. (31) and (29) we show that the approximation (31) guarantees at least five correct significant digits for maturity up to 20 years (see Table 1). This good performance of formula (31) guarantees a good approximation when used to price options.

As shown in Section 4 this approximation works well also for long maturity. Furthermore, the use of formula (27) allows us to reduce the computation of the option prices to the evaluation of a one dimensional integral. In fact, let $p_f(x,\nu,t,x',\nu',t')$, $(x,\nu,t),(x',\nu',t')\in\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^+,t,t'\ge0,\tau=t'-t>0$, be the transition probability density function of the stochastic process described by Eqs. (7)–(9), using formula (14) for p_f and (27) we obtain:

$$C_A(S_0,T,E,r_0,\nu_0)=e^{-r_0\frac{T}{(1+e^{\lambda T})}}\int_{\ln(E/S_0)}^{+\infty}dx'\,(S_0e^{x'}-E)\times\int_0^{+\infty}dr'e^{-r'\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}D_{\nu,q}(0,\nu_0,r_0,0,x',r',T),\quad S_0,\ T,\ E,\ r_0,\ \nu_0>0,\ q>1, \tag{32}$$

$$P_A(S_0,T,E,r_0,\nu_0)=e^{-r_0\frac{T}{(1+e^{\lambda T})}}\int_{-\infty}^{\ln(E/S_0)}dx'\,(E-S_0e^{x'})\times\int_0^{+\infty}dr'e^{-r'\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}D_{\nu,q}(0,\nu_0,r_0,0,x',r',T),\quad S_0,\ T,\ E,\ r_0,\ \nu_0>0,\ q<-1, \tag{33}$$

where $D_{\nu,q}$ is given in (17). Using formulas (17) and (30) with $q=2$ in (32) we obtain the following approximation of the call option

Table 1
Performance of formula (31) to approximate the zero-coupon bond value.

τ	True bond value	Approximate bond value	Relative error
Set A: $\theta = 0.02$, $\eta = 0.01$, $\lambda = 0.01$, $\rho_{p,r} = -0.23$, $\Omega = 1$, $r_0 = 0.02$ $\Delta = 0.01$, $\rho_{p,\nu} = -0.3$, $\chi = 0.3$, $\gamma = 0.6$, $\nu^* = 0.05$, $\nu_0 = 0.05$, $S_0 = 100$			
0.25	9.950125e−1	9.950125e−1	1.292331e−9
0.50	9.900499e−1	9.900499e−1	1.026083e−8
0.75	9.851121e−1	9.851120e−1	3.436840e−8
1.00	9.801990e−1	9.801989e−1	8.084669e−8
1.50	9.704466e−1	9.704464e−1	2.686953e−7
2.00	9.607920e−1	9.607914e−1	6.270951e−7
3.00	9.417728e−1	9.417709e−1	2.050759e−6
5.00	9.048737e−1	9.048657e−1	8.896235e−6
10.0	8.189837e−1	8.189348e−1	5.969022e−5
20.0	6.718534e−1	6.716448e−1	3.105461e−4
Set B: $\theta = 0.00044$, $\eta = 0.0098$, $\lambda = 3.62$, $\rho_{p,r} = -0.81$, $\Omega = 2.51$, $r_0 = 0.00022$ $\Delta = 1.98$, $\rho_{p,\nu} = -0.97$, $\chi = 0.65$, $\gamma = 0.018$, $\nu^* = 0.0345$, $\nu_0 = 0.089$, $S_0 = 12.456$			
0.25	9.999262e−1	9.999250e−1	1.206098e−6
0.50	9.998308e−1	9.998245e−1	6.328031e−6
0.75	9.997268e−1	9.997125e−1	1.429458e−5
1.00	9.996192e−1	9.995961e−1	2.314066e−5
1.50	9.994007e−1	9.993621e−1	3.865490e−5
2.00	9.991811e−1	9.991322e−1	4.893588e−5
3.00	9.987416e−1	9.986838e−1	5.785582e−5
5.00	9.978631e−1	9.978026e−1	6.057676e−5
10.0	9.956702e−1	9.956100e−1	6.049504e−5
20.0	9.912989e−1	9.912398e−1	5.963593e−5

price C:

$$C_A(S_0,T,E,r_0,\nu_0) = e^{-r_0\frac{T}{(1+e^{\lambda T})}}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty} dk\,\frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)}}{-k^2-3\imath k+2}\cdot$$
$$W^0_{\nu,q}(T,\nu_0,k;\underline{\Theta}_\nu)W^0_{r,q}(T,r_0,k;\underline{\Theta}_r)\left(\frac{M_{q,r}}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right)^{\nu_r+1}$$
$$\times\, e^{-\left(\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\right)\left(\frac{M_{q,r}\bar{r}_q}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right)},\ S_0,\ T,\ E,\ r_0,\ \nu_0,\quad q=2, \qquad (34)$$

Proceeding in a similar way, we obtain the following approximation, P_A, of the put option price P:

$$P_A(S_0,T,E,r_0,\nu_0) = e^{-r_0\frac{T}{(1+e^{\lambda T})}}\frac{S_0}{2\pi}\int_{-\infty}^{+\infty} dk\,\frac{\left(\frac{S_0}{E}\right)^{-(3+\imath k)}}{-k^2+5\imath k+6}\cdot$$
$$W^0_{\nu,q}(T,\nu_0,k;\underline{\Theta}_\nu)W^0_{r,q}(T,r_0,k;\underline{\Theta}_r)\left(\frac{M_{q,r}}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right)^{\nu_r+1}$$
$$\times\, e^{-\left(\frac{Te^{\lambda T}}{(1+e^{\lambda T})}\right)\left(\frac{M_{q,r}\bar{r}_q}{M_{q,r}+\frac{Te^{\lambda T}}{(1+e^{\lambda T})}}\right)},\ S_0,\ T,\ E,\ r_0,\ \nu_0,\quad q=-2. \qquad (35)$$

Taking the limit $\Omega \to 0^+$, $\lambda \to 0^+$, $\eta \to 0^+$ in Eqs. (34) and (35) we derive the following exact formulas for the price of the European call and put options under the Heston model:

$$C_H(S_0,T,E,r_0,\nu_0) = e^{-r_0T}S_0e^{2r_0T}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\,\frac{\left(\frac{S_0}{E}\right)^{(1-\imath k)}e^{-\imath k r_0 T}}{-k^2-3\imath k+2}$$
$$\times\, W^0_{\nu,q}(T,\nu_0,k;\underline{\Theta}_\nu),\ S_0,\ T,\ E,\ r_0,\ \nu_0,\quad q=2, \qquad (36)$$

$$P_H(S_0,T,E,r_0,\nu_0) = e^{-r_0T}S_0e^{-2r_0T}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\,\frac{\left(\frac{S_0}{E}\right)^{-(3+\imath k)}e^{-\imath k r_0 T}}{-k^2+5\imath k+6}$$
$$\times\, W^0_{\nu,q}(T,\nu_0,k;\underline{\Theta}_\nu),\ S_0,\ T,\ E,\ r_0,\ \nu_0,\quad q=-2. \qquad (37)$$

We use these formulas in the empirical analysis to compare the performance of the Heston model and the stochastic model proposed here in interpreting real data. It is worth noting that the integrand functions appearing in formulas (34)–(37) are smooth functions whose integration does not require a specific care. This regularity is due to the specific approach used to derive them.

4. Simulation and empirical studies

In this section we illustrate some experiments on simulated data as well as an empirical analysis relative to the U.S. S&P 500 index and its European options and some insurance policy known as pure endowment policy. This analysis uses a suitable calibration of the model parameters.

4.1. Simulation study

In this subsection we propose two experiments to validate the formula to approximate the stochastic integral (see Eqs. (31) and (27)) and the formulas of the moments (see Eq. (20)).

The first experiment analyzes the performance of formula (31) in approximating the bond price.

We consider two sets of parameter values.

The first one, *Set A*, is as follows: $\theta = 0.02$, $\eta = 0.01$, $\lambda = 0.01$, $\rho_{p,r} = -0.23$, $\Omega = 1$, $r_0 = 0.02$, $\Delta = 0.01$, $\rho_{p,\nu} = -0.3$, $\chi = 0.3$, $\gamma = 0.6$, $\nu^* = 0.05$, $\nu_0 = 0.05$, $S_0 = 100$.

The parameters of the volatility process are those employed by Grzelak and Oosterlee (2011) in Table 1.

The second one, *Set B*, is as follows: $\theta = 0.00044$, $\eta = 0.0098$, $\lambda = 3.62$, $\rho_{p,r} = -0.81$, $\Omega = 2.51$, $r_0 = 0.00022$, $\Delta = 1.98$, $\rho_{p,\nu} = -0.97$, $\chi = 0.65$, $\gamma = 0.018$, $\nu^* = 0.0345$, $\nu_0 = 0.089$, $S_0 = 12.456$.

Set B are the parameter values estimated in the empirical analysis illustrated in the following subsection.

Table 1 shows, from left to right, the time to maturity τ, the true bond value computed via formula (29) the approximate formula (31) and the relative error given by the ratio of the absolute error (i.e. the absolute value of the difference between the true and approximate values) to the true value. The notation *x.xxx e−n* is equivalent to $x.xxx \times 10^{-n}$.

The relative errors in Table 1 show that the approximate formula guarantees at least five correct significant digits for maturity up to 20 years. This is a satisfactory result that supports the approximation of the discount factor given in formula (27).

In the second experiment we compare the first two theoretical moments given in Eq. (20) (i.e. $m = 1$, $m = 2$) with those attained using the Monte Carlo method. The latter is implemented integrating numerically the stochastic differential equations with the explicit Euler method with variable step-size. The largest value of the Euler step-size is 10^{-5} and the number of the Monte Carlo simulations is 10,000.

The upper panels of Fig. 1 show the first and second moments of the asset price variable as a function of time computed using Eq. (20) (solid line) and the Monte Carlo method (dotted line). The lower panel shows the corresponding relative errors. The values of the model parameters are those of *Set A*. The moments obtained using the Monte Carlo method approximate quite well those obtained with the theoretical formulas. However, the theoretical formulas lead to significant savings in computing time. In fact, the Monte Carlo method requires about ten minutes while formula (20) requires only a few milliseconds when their Matlab code is run on a laptop with Intel core i3 processor and 8 gigabyte of RAM.

4.2. An empirical analysis of index options

In this subsection, we conduct an empirical analysis calibrating the model parameters against real data. The calibration procedure is based on the solution of an appropriate nonlinear constrained least

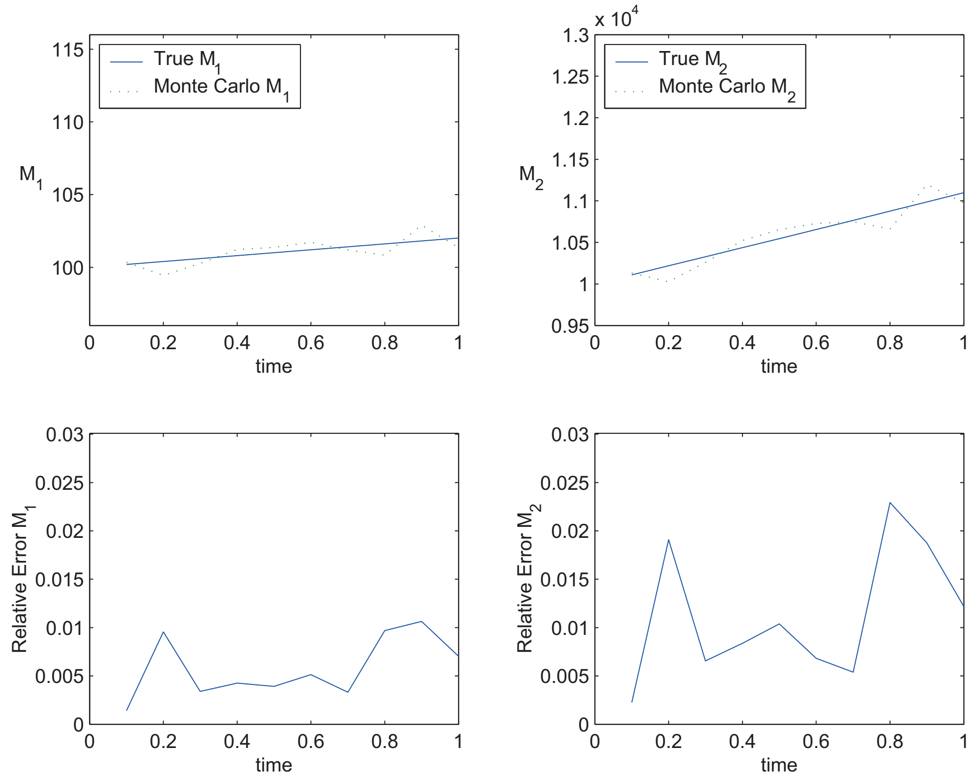

Fig. 1. Graphs of $\mathcal{M}_1$ (upper left panel) and $\mathcal{M}_2$ (upper right panel) versus time computed using formula (20) (solid line) and Monte Carlo method (dotted line). The lower panels show the corresponding relative errors. The values of the model parameters are those of *Set A*.

squares problem, whose set of feasible vectors, $\mathcal{V}$, is as follows:

$$\mathcal{V} = \Big\{ \underline{\Theta} = (\Delta, \gamma, \upsilon^*, \chi, \rho_{p,\upsilon}, \upsilon_0, \eta, \lambda, \theta, \rho_{p,r}, r_0, \Omega) \in \mathbf{R}^{12} \mid \Delta, \gamma, \upsilon^*, \chi, \upsilon_0, \eta, \lambda, \theta > 0, \frac{2\chi\upsilon^*}{\gamma^2} > 1, \quad -1 < \rho_{p,\upsilon}, \rho_{p,r} < 1 \Big\}, \quad (38)$$

where $\mathbf{R}^{12}$ denotes the 12-dimensional Euclidean real space.

We highlight that the set $\mathcal{V}$ also includes the initial stochastic volatility and interest rate, υ_0, r_0, of the stochastic model (7)–(9). That is, the initial values υ_0, r_0 are considered parameters to be estimated via the calibration procedure.

This choice is motivated by the fact that υ_0 cannot be observed in the market while r_0 refers to a risk free interest rate associated with the risk neutral measure and, subsequently, its value is not clearly identified by the financial market. The estimation of the initial stochastic volatility is a common practice since 2002 (see, for example, Bühler, 2002), while the interest rate has been estimated only recently (see, for example, Fatone et al., 2009, 2013; Grzelak & Oosterlee, 2011).

In order to formulate the least square problem, we have to define the objective function. Let n_D be a positive integer, $\tilde{t} \geq 0$ be the observation time and $\tilde{S}_{\tilde{t}}$ be the asset price observed at time $t = \tilde{t}$. In addition, let $C^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$, $C^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$, $i = 1, 2, \ldots, n_D$ be the observed price and the model price (34) at time $t = \tilde{t}$ of the European call option having maturity times T_i and strike prices E_i, $i = 1, 2, \ldots, n_D$. Similarly, we denote with $P^{\tilde{t}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$ and $P^{\tilde{t},\underline{\Theta}}(\tilde{S}_{\tilde{t}}, T_i, E_i)$ the observed price and the model price (35) at time $t = \tilde{t}$ of the European put option having maturity times T_i and strike prices E_i where $i = 1, 2, \ldots, n_D$.

Let n_T be a positive integer, the objective function, F_{n_T}, of our constrained optimization problem is as follows:

$$F_{n_T}(\underline{\Theta}) = \frac{1}{n_T}\sum_{j=1}^{n_T}\frac{1}{n_D}\sum_{i=1}^{n_D}\left[\frac{C^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i) - C^{\tilde{t}_j,\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}{C^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}\right]^2 + \frac{1}{n_T}\sum_{j=1}^{n_T}\frac{1}{n_D}\sum_{i=1}^{n_D}\left[\frac{P^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i) - P^{\tilde{t}_j,\underline{\Theta}}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}{P^{\tilde{t}_j}(\tilde{S}_{\tilde{t}_j}, T_i, E_i)}\right]^2, \quad \underline{\Theta} \in \mathcal{V}, \quad (39)$$

where $\tilde{t}_j$, $j = 1, 2, \ldots, n_T$, are the observation times.

The optimization problem used to estimate the model parameters can be stated as follows:

$$\min_{\underline{\Theta}\in\mathcal{V}} F_{n_T}(\underline{\Theta}). \quad (40)$$

We also calibrate the Heston model (Heston, 1993) by solving problem (40) with the appropriate adjustments. That is, we remove the parameters of the stochastic interest rate model with the exception of the parameter r_0.

We use formulas (34)–(37) to evaluate option prices. The one-dimensional integrals appearing in these formulas are computed using the midpoint quadrature rule with 2^{14} nodes.

This quadrature rule gives satisfactory approximations since the integrand functions appearing in Eqs. (34)–(37) are smooth functions whose numerical integration does not require special care.

Moreover, problem (40) is solved using a steepest descent algorithm with variable metric (see, for example, Fatone et al., 2013; Recchioni & Scoccia, 2000). The real data analyzed are the daily closing values of the U.S. S&P 500 index and the daily closing prices of the European call and put options on this index. These options have an expiry date of March 16, 2013 with strike prices $E_i = 1075 + 25(i - 1)$, $i = 1, 2, \ldots, 4$, $E_5 = 1170$.

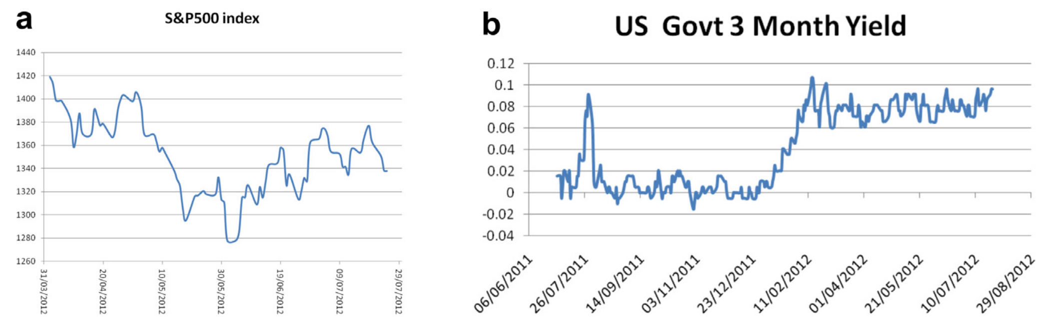

Fig. 2. The U.S. S&P S500 index (a) and U.S. three-month government yield (b) versus time.

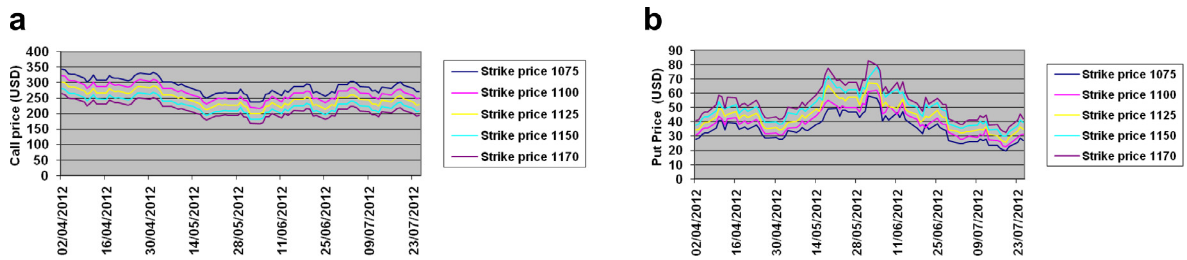

Fig. 3. Prices of the call (a) and put (b) options on the U.S. S&P 500 index with strike prices $E_i = 1075 + 25(i-1)$, $i = 1, 2, \ldots, 4$ and $E_5 = 1170$, and with expiry date $T =$ March 16, 2013 versus time.

Fig. 2 (a) shows the U.S. S&P 500 index while Fig. 3(a) and (b) shows the corresponding call and put option prices as a function of time (April 2, 2012, July 27, 2012). Fig. 2(b) shows the U.S. three month government yields (in percent) as a function of time. Usually, the short-dated government bonds are used as proxy of the risk free interest rate and we expect that the initial stochastic interest rate r_0 and the long-term mean θ have values similar to those in Fig. 2(b) from April 2 to July 27, 2012.

In the empirical analysis we consider a rolling window of six consecutive trading day data (i.e. $n_T = 6$). We move this window by one day along the historical series. The time window covers the period April 2 to July 2, 2012. In this way we solve $66 - n_T$ calibration problems (40). That is, one problem for each six-day window, j, where $j = 1, 2, \ldots, 66 - n_T$. As a consequence, in each window sixty option values are used to calibrate the twelve parameters of the model (i.e. $n_D = 5$ put option price and $n_D = 5$ call option prices for $n_T = 6$ days).

Shifting the rolling window by one day along the time series and the choice of $n_T = 6$ have a twofold effect. First, we have a sufficient number of data to validate the model (sixty option values). Second, we obtain a “daily” time series of the estimated parameters. The values of the parameters obtained in the jth window are representative of the last day of the jth window.

It is worth noting that when the time series of the parameter values are constant the model (7)–(9) will correctly interpret the asset price dynamics. In fact, when the values of the estimated parameters are constant in time, the model is able to reproduce the asset price dynamics in the analyzed period by using only one set of model parameters. Figs. 4 and 5 show the parameter values as a function of the index j, $j = 1, 2, \ldots, 66 - n_T$.

We can observe that these values are relatively constant as a function of time.

We observe a significant difference between the initial interest rate r_0 of the hybrid Heston model (i.e. $r_0 \approx 0.00022$) and the Heston model (i.e. $r_0 \approx 0.04$). The two models also differ in the value of the vol of vol γ and the correlation coefficient $\rho_{p,v}$. In fact, as already stressed by Grzelak and Oosterlee (2011), the hybrid Heston model shows lower values of the parameter γ and a more negative correlation coefficient $\rho_{p,v}$ with respect to the Heston model. The lower value of γ may be due to the additional volatility coming from the interest rate process, while the more negative correlation may be due to an increase of the leverage effect caused by the previously mentioned additional volatility.

The values of the initial stochastic interest rate r_0 and of θ are about 0.02 percent and 0.04 percent, that is values comparable with those shown in Fig. 2(b). Moreover, Fig. 2(b) shows an abrupt change of the yield trend in February. This abrupt change may explain the fluctuations of the initial stochastic rate shown in Fig. 5.

Figs. 6 and 7 show the in-sample values of the European call and put option prices obtained using the Heston (dashed line) and the hybrid Heston (dotted line) models with the parameters estimated in the period April 2, 2012 to July 2, 2012.

These figures show that the theoretical option prices of the hybrid Heston model provide satisfactory approximations of observed put prices for all values of the strike prices and time to maturity. These values outperform those obtained with the Heston model. Both models overestimate the observed values of the call options, but their relative errors are reduced by over one half using the hybrid Heston model. In fact, the sample mean of the relative errors of the call and put options are 9.6 percent and 6.9 percent for the hybrid Heston model while 21.2 percent and 11.2 percent for the Heston model.

Figs. 4–7 show that the hybrid Heston model is capable of matching with sufficient accuracy both call and put option prices for several strike prices and expiry dates using only one set of parameters. This results from the use of a stochastic interest rate.

We use the value of the model parameters estimated in the last window, June 25, 2012–July 2, 2012, to evaluate the out-of-sample European call and put option prices. The out-of-sample period is July 3 to July 27, 2012. The time to maturity for this period is 176–160 days. We measure the performance of the stochastic model proposed and its parameter estimation procedure with an “a posteriori” validation. That is, we compare the observed out-of-sample option prices with those obtained using formulas (36), (37), (34), (35) which use estimated parameters and observed spot prices.

Figs. 8 and 9 show the out-of-sample option prices for the hybrid model (dotted line) and for the Heston model (dashed line).

The out-of-sample put option prices of the Heston model are very accurate while the call option prices are not. The hybrid Heston model provides accurate approximations of put option prices and

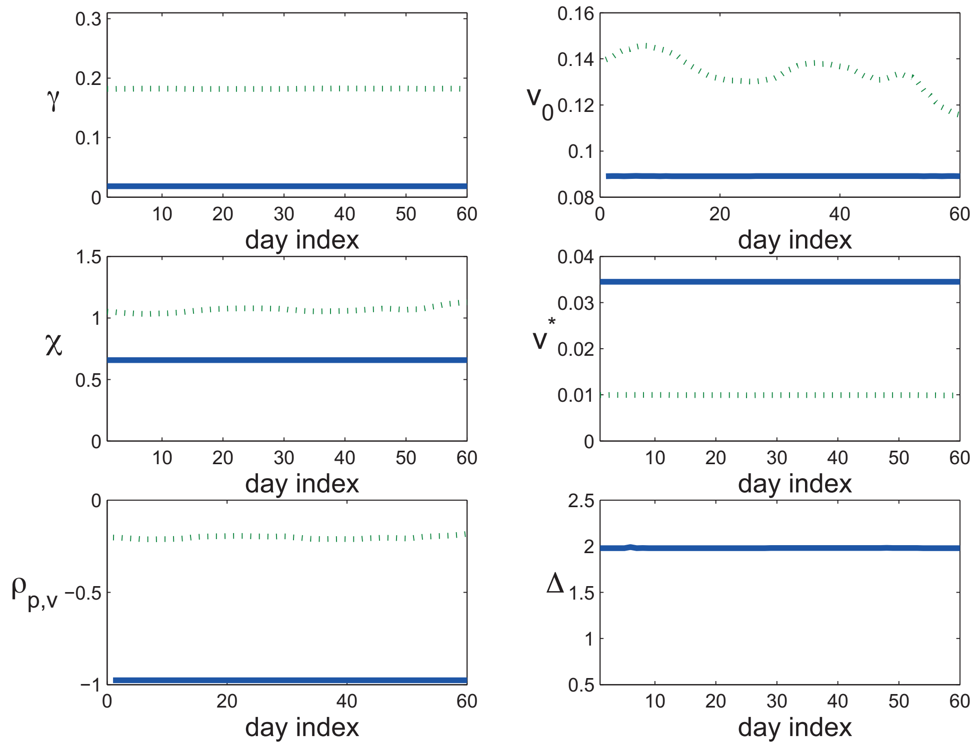

Fig. 4. Estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line) and the Heston model (dotted line).

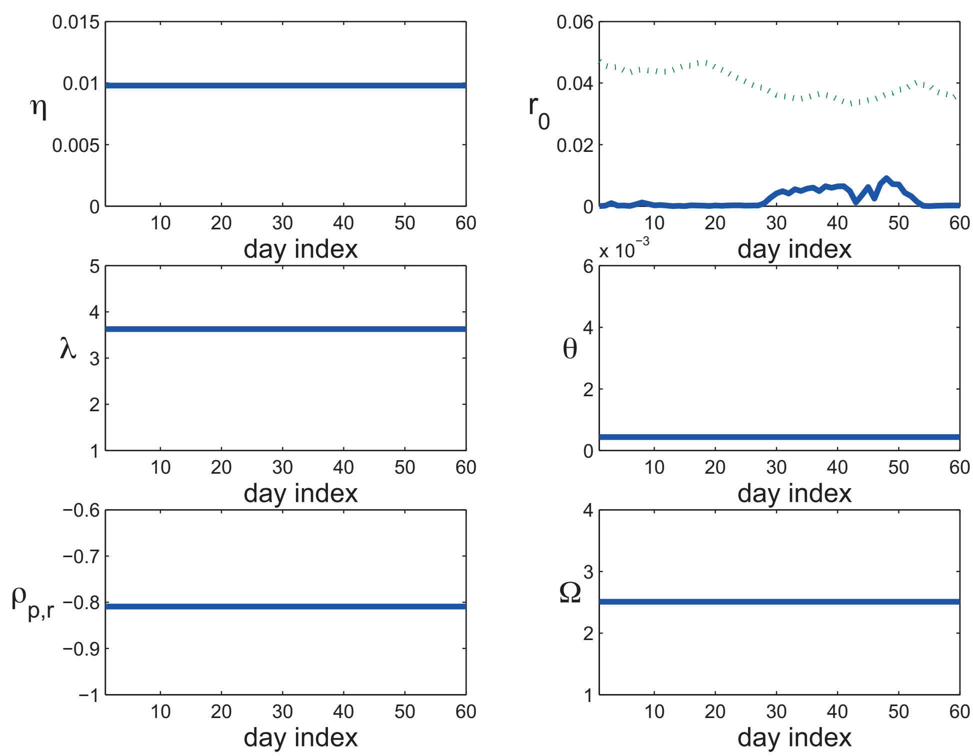

Fig. 5. Estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω versus window index (six-day window) resulting from the calibration of the hybrid Heston model (solid line). Note that the dotted line in the upper right panel shows the estimated values of the parameter r_0 resulting from the calibration of the Heston model.

outperforms the Heston model in approximating the call options. In fact, the sample mean of the relative errors on the put and call options obtained using the hybrid Heston model are 9.5 percent and 7.8 percent and using the Heston model are 9.6 percent and 17.9 percent.

In conclusion, the empirical analysis shows that the hybrid model interprets satisfactorily the real data considered in the period April 2 to July 27 2012 using only one set of model parameters. Moreover, the values of the initial stochastic rate, r_0, could be considered a proxy of the short-dated government bond yield. The results illustrated here

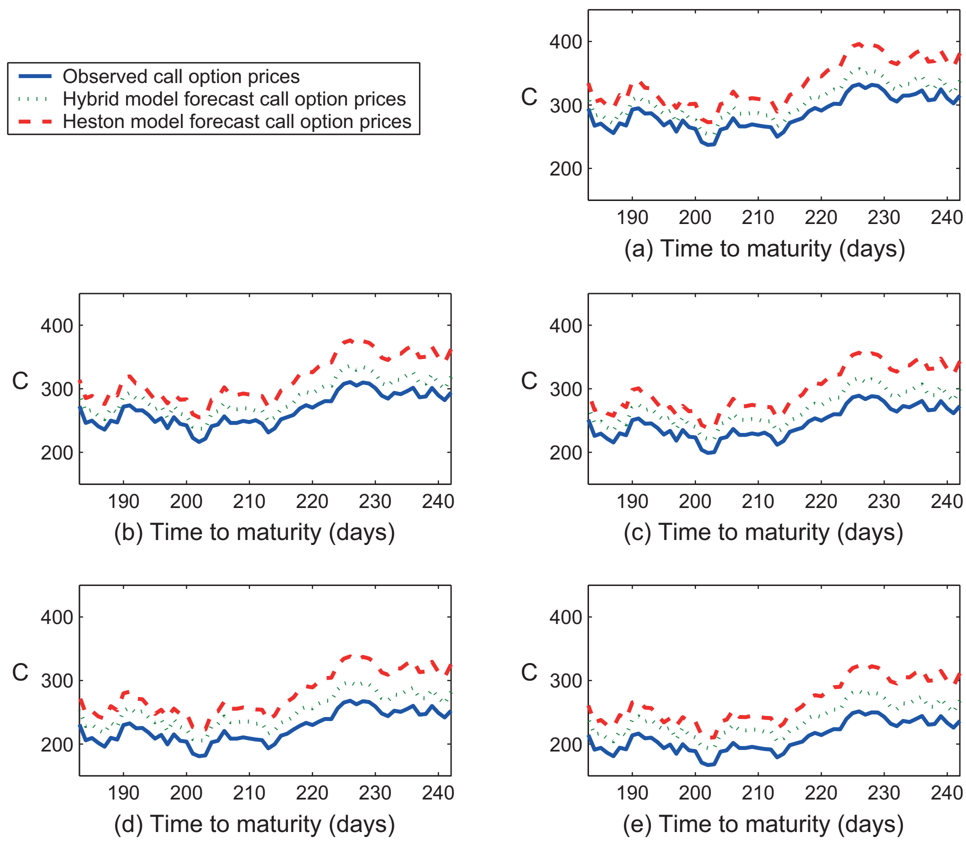

Fig. 6. Observed (solid line) and in-sample call option prices (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

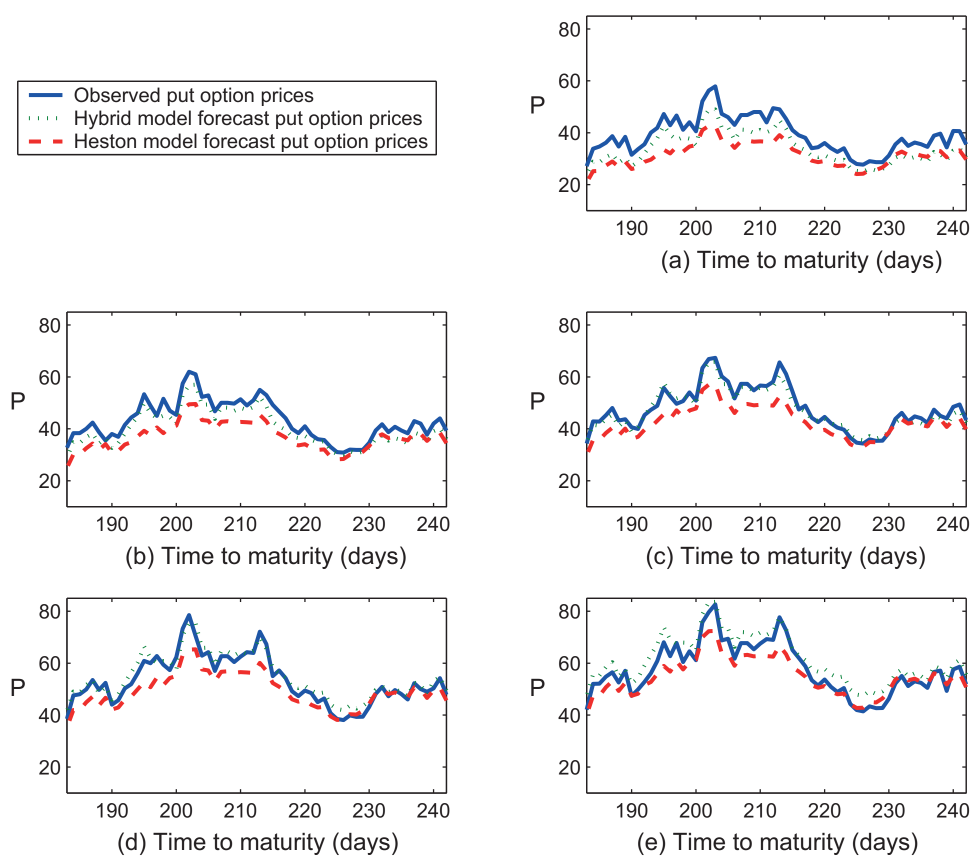

Fig. 7. Observed (solid line) and in-sample put option price (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

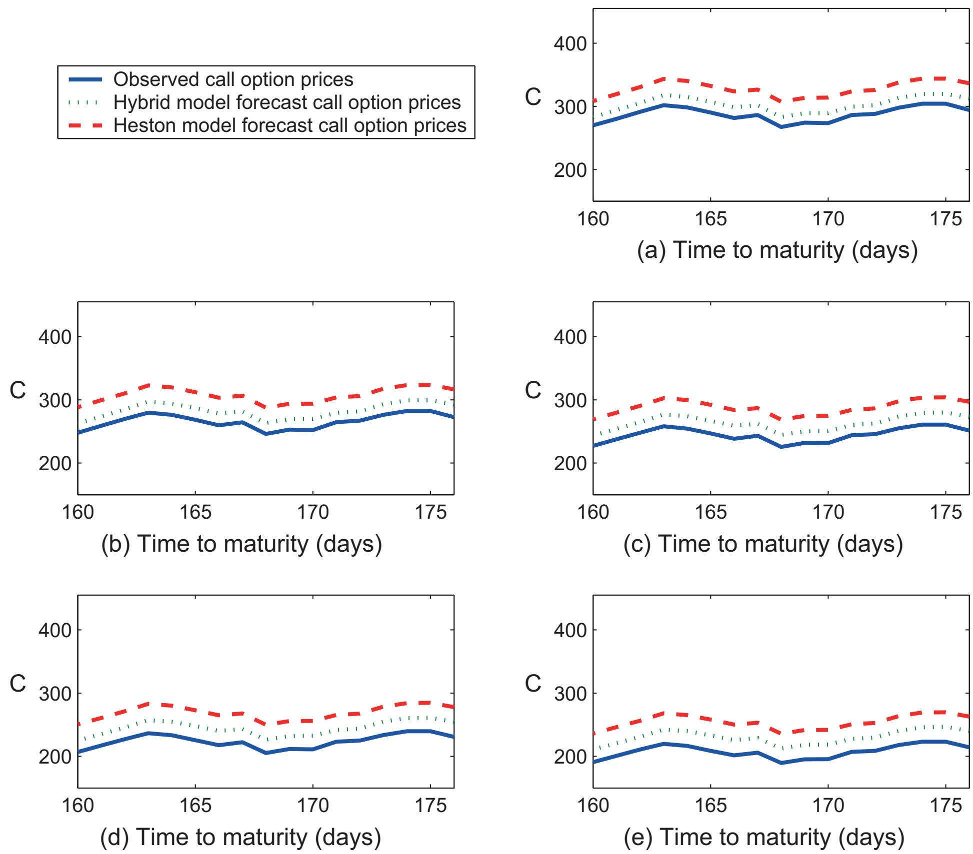

Fig. 8. Observed (solid line) and out-of-sample call option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

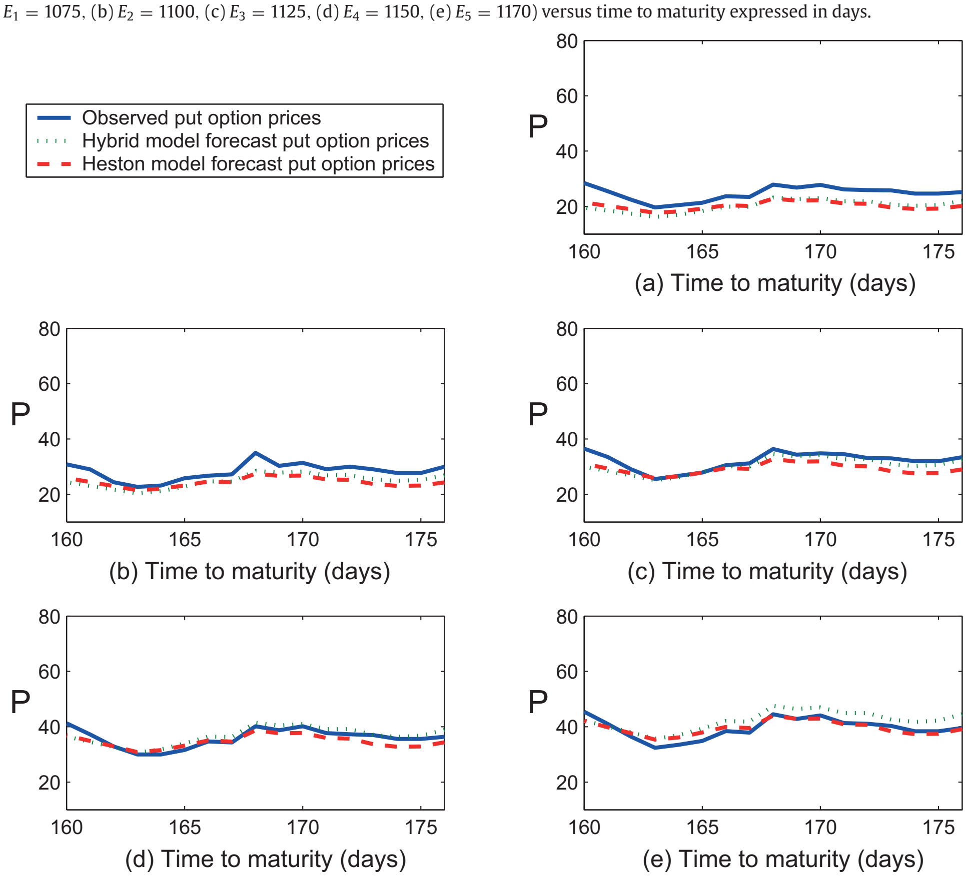

Fig. 9. Observed (solid line) and out-of-sample put option price forecast (in USD) obtained using the hybrid Heston (dotted line) and the Heston (dashed line) models for five different strike prices: ((a) $E_1 = 1075$, (b) $E_2 = 1100$, (c) $E_3 = 1125$, (d) $E_4 = 1150$, (e) $E_5 = 1170$) versus time to maturity expressed in days.

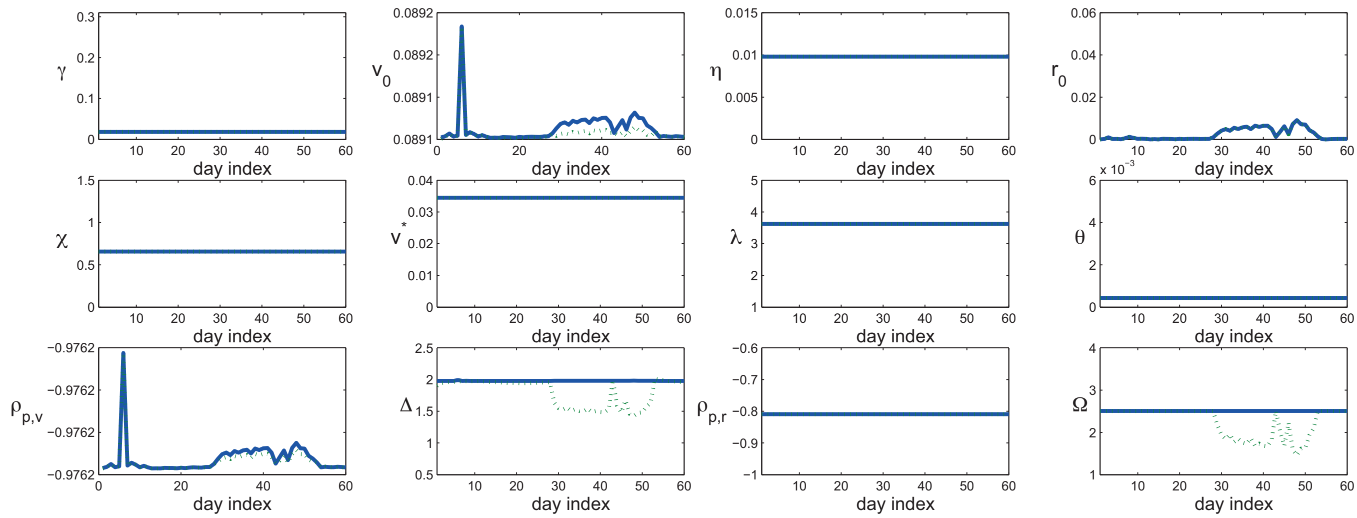

Fig. 10. Estimated parameters $\gamma, v_0, \chi, v^*, \rho_{p,v}, \Delta$ (left panel) and $\eta, r_0, \lambda, \theta, \rho_{p,r}$ and Ω (right panel) versus window index (six-day window) resulting from the calibration of the hybrid Heston model using only option prices (solid line) and the two stage procedure (dotted line).

are consistent with those obtained using the multi-factor stochastic volatility model of Christoffersen et al. (2009). Indeed, in the latter the authors use two stochastic factors but they are not specified. The results shown in this subsection suggest that the stochastic interest rate is one of these volatility factors.

To provide further evidence on the role of the stochastic interest rate, we conclude this subsection by repeating the previous analysis with an alternative calibration procedure which includes the U.S. three month government bond yields as data in the parameter estimation. Specifically, we use a two-stage calibration procedure to estimate the model parameters. Several two stage procedures applied to multi-factor stochastic volatility models can be found in the literature such as those illustrated in Christoffersen et al. (2009) and Islyaev and Date (2015).

The two stage calibration used here consists of a first stage where we estimate the parameters λ, θ, η and r_0 of the interest rate process using the U.S. three month government bond yields. This is done by minimizing the squared residuals of the observed and theoretical bond values employing the same rolling window of the experiment described immediately above. Then, in the second stage we calibrate the remaining parameters using option prices.

As in the previous experiment, we compare both in-sample and out-of-sample observed option prices with the option prices provided by the hybrid Heston model calibrated with this two stage procedure. The sample mean of the in-sample relative errors of the call and put options are 8.3 percent and 9.4 percent for the hybrid Heston model calibrated with the two stage procedure while, as already mentioned, they are 21.2 percent and 11.2 percent for the Heston model. The sample mean of the out-of-sample relative errors of the call and put options are 8.3 percent and 8.8 percent for the hybrid Heston model calibrated with the two stage procedure while they are 17.9 percent and 9.6 percent for the Heston model. Fig. 10 shows the parameters estimated with the two stage approach (dotted line) and those estimated using option prices (solid line) which have already been shown in Figs. 4 and 5. As we can see the parameter values estimated by using the two step procedure are basically the same obtained without using the two stage calibration except for Δ and Ω in the time windows indexed by 30 to 55 which corresponds to the period May 23, 2012 to June 25, 2012. Note that in this period the U.S. S&P 500 index shows abrupt oscillations (see Fig. 2(a)). We note that the y-scale of the graphs relative to the parameters $\rho_{p,v}$, v_0 of Fig. 10 differs from the y-scale of Fig. 4 in order to show the slight difference in the estimated values of these parameters.

The results obtained using the two stage calibration provides empirical evidence that the stochastic interest rate is a crucial volatility factor. In fact, the hybrid Heston model significantly outperforms the Heston one even when we use the two-stage calibration where the interest rate process parameters are estimated without using option prices.

4.3. An empirical analysis of a long-term endowment policy

In this subsection we show the ability of the hybrid model to price long-run products. We estimate the model parameters by using the time series of a pure endowment policy of the Crédit Agricole insurance company. Specifically, we use the weekly data of the index linked policy, Azione Più Capitale Garantito Em.64, freely available at the website http://www.previdoc.it/d/Ana/CREM64/_credit-agricole-vita-azione-piu-em64-01082017.

This policy is a single-premium index linked life insurance policy whose benefits are directly linked to the performance of the Dow Jones Eurostoxx 50 Index. The duration of the Azione Più Capitale Garantito Em.64 policy covers the period from October 29, 2010 (date of issue) to August 1, 2017 (expiration date). On the maturity date Action Capital Piú Guaranteed Em.64 policy guarantees the insured, should he be living, the payment of the premium plus a variable bonus obtained by multiplying the premium by 35 percent of the relative difference $(S_T - S_r)/S_r$ of the Dow Jones Eurostoxx 50 values between October 29, 2010 (i.e. $S_r = 2844.99$) and July 22, 2017, in the case of a positive difference. In the case of a negative difference the variable bonus will be equal to zero. The payoff of this policy is given by:

$$P_1\left(\frac{S_T}{S_r}, T\right) = N + 0.35N \max\left[\frac{S_T - S_r}{S_r}; 0\right] = N + 0.35\frac{N}{S_r}\max\left[S_T - S_r; 0\right] \quad (41)$$

where N is the nominal capital paid at the start of the contract. For this contract $N = 100$.

We model the risky asset specified in the contract under the risk-neutral measure (using the hybrid Heston model) and the mortality risk under the physical measure (using the mean reverting Gompertz model, see Milevsky and Promislow, 2001) with the assumption that these two measures are independent. As a consequence, the pricing of this policy consists in the evaluation of the following product:

$$C_P(S_0, T) = E(e^{-\int_0^T r_\tau\, d\tau} P_1(S_T))\, E(e^{-\int_0^T h_u du}), \quad (42)$$

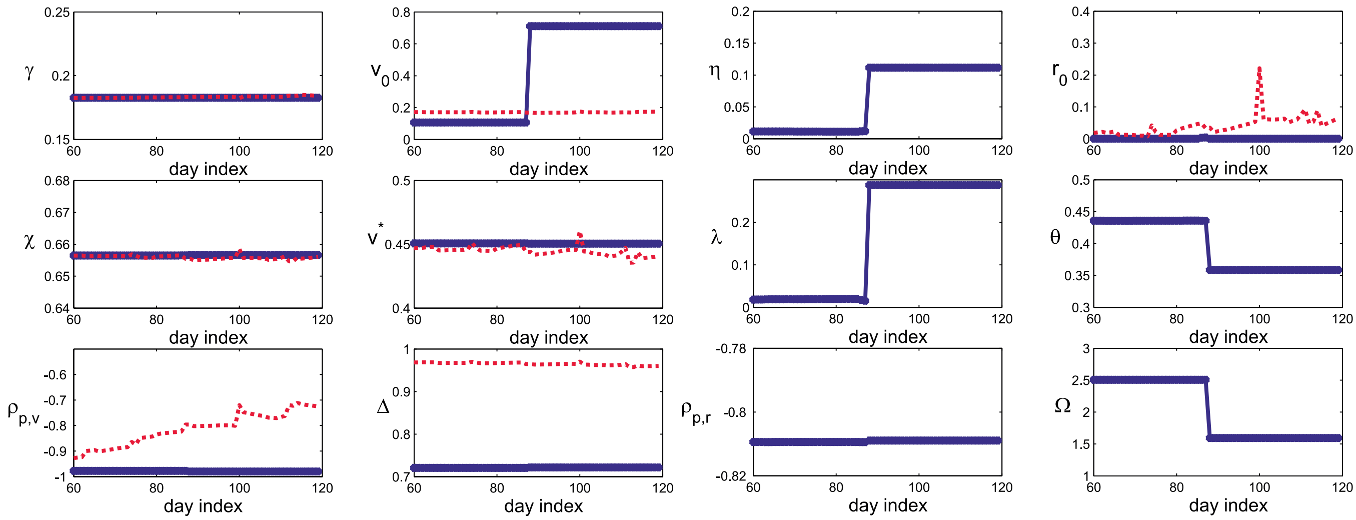

Fig. 11. Left panel: estimated parameters γ, v_0, χ, v^*, $\rho_{p,v}$ and Δ versus window index resulting from the calibration of the hybrid Heston model (solid line) and the Heston model (dotted line). Right panel: estimated parameters η, r_0, λ, θ, $\rho_{p,r}$ and Ω versus window index resulting from the calibration of the hybrid Heston model (solid line). Note that the dotted line in the upper right sub-panel shows the estimated values of the parameter r_0 resulting from the calibration of the Heston model. The data used in the calibration are the Credit Agricole pure endowment policy values.

Table 2
Parameter values of the demographic component of the Crédit Agricole index linked policies.

Cohort	h_0	a	b	g	σ^*
1977	0.0001175	0.0005	0.6315	0.0722	0.0311

where P_1 is the payoff function associated with the policy, r is the stochastic interest rate and h_t is the mortality rate. The first expected value in (42) is the value of a European option in the hybrid Heston model and the second expected value is the survival probability. We model the mortality rate h_t, $t > 0$, using the stochastic model:

$$dh_t = (g + \frac{1}{2}(\sigma^*)^2 + b\ln(\hat{h}_0) + bgt + b\ln(h_t))h_t\,dt + \sigma^* e^{at}\, h_t dQ_t,\ t > 0, \tag{43}$$

where the quantities g, b, h_0, a, σ^* are real constants and Q_t, $t > 0$, is a standard Wiener process. The survival probability is evaluated using the following approximation (for further details see Recchioni & Screpante, 2014):

$$E(e^{-\int_t^T h_\tau d\tau} \mid h_0 = \hat{h}_0) \approx e^{-\hat{h}_0 e^{gt}(e^{g(T-t)}-1)/g} \cdot \Bigg\{1 - \frac{\hat{h}_0(\sigma^*)^2}{4(a+b)} \times \Bigg[\Bigg(\frac{e^{(g+2a)t}(e^{(g+2a)(T-t)}-1)}{g+2a} - \frac{e^{(g-2b)t}(e^{(g-2b)(T-t)}-1)}{g-2b}\Bigg)\Bigg] + \frac{\hat{h}_0^2(\sigma^*)^2}{2}\Bigg[-\frac{1}{2(a+b)}\Bigg(\frac{e^{(g-b)t}(e^{(g-b)(T-t)}-1)}{g-b}\Bigg)^2 + \frac{1}{(a+b)}\,\frac{1}{(g+b+2a)}\cdot\Bigg[\Bigg(\frac{e^{2(g+a)t}(e^{2(g+a)(T-t)}-1)}{2(g+a)}\Bigg) - \Bigg(\frac{e^{(g-b)(T+t)+2(a+b)t} - e^{2(g+a)t}}{g-b}\Bigg)\Bigg]\Bigg]\Bigg\},\ 0 \le t < T. \tag{44}$$

The parameters appearing in formula (44) are chosen as shown in Table 2. These values are motivated by the analysis proposed in Recchioni and Screpante (2014) on a similar Crédit Agricole pure endowment policy.

We estimate the model parameters solving the calibration problem (40) after appropriate adjustments (i.e. removing the put option prices and replacing the call option prices with policy prices). We use formula (34) to evaluate the first expected value appearing in Eq. (42). As in Section 4.2 the integrals are computed by using the midpoint quadrature rule with 2^{14} nodes.

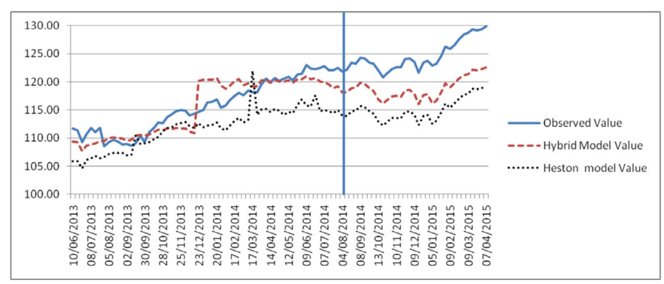

Fig. 12. Credit Agricole pure endowment policy. In-sample (on the left of the vertical bar) and out-of-sample (on the right of the vertical bar) approximations obtained using the hybrid model (dashed line) and the Heston model (dotted line) and the observed policy values (solid line).

We use the weekly prices covering the period April 4, 2012 to April 13, 2015 corresponding to 152 observations. In formula (39) we choose $n_T = 60$ (i.e. sixty weekly observations) and $n_D = 1$ (i.e. one strike price S_r). We consider a rolling window of size n_T and we move this window along the time series discarding the oldest observed price of the window and inserting the new observed one. In this way we solve sixty problems covering the period April 4, 2012 to August 4, 2014 and we obtain a time series of each parameter made of weekly observations. The parameter value estimated in a given time window is associated with the last date of the window. Fig. 11 shows the estimated values as a function of the window index. We can observe that the time series are approximately constant except for the time series of r_0 which shows some oscillations in the year 2014.

We use the model parameters estimated in the last window to evaluate the policy out-sample that is in the period August 11, 2014 to April 13, 2015. Fig. 12 shows the in-sample and out-sample observed policy values (solid line), the approximations obtained with the hybrid model (dashed line) and those obtained with the Heston model (dotted line). The observed policy values and their approximations before the vertical bar are in-sample while those after the vertical bar are out-of sample. We can observe that the hybrid model outperforms the Heston one in both the in-sample and out-of-sample. More specifically, the average relative error of the hybrid model

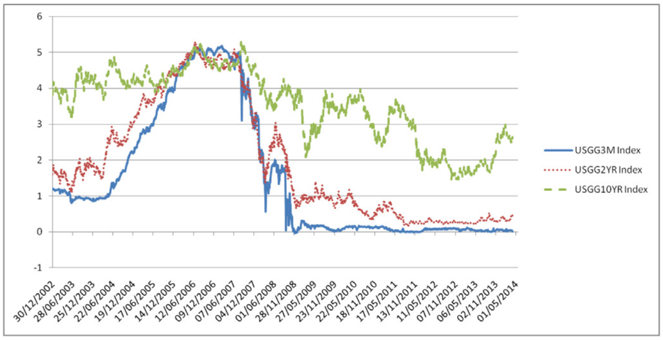

Fig. 13. Daily observations of U.S. three-month (solid line), two-year (dotted line) and ten-year (dashed line) government bond yields.

approximations is 2.5 percent and the maximum is 5.7 percent, while those of the Heston approximations is 4.9 percent and 8.4 percent respectively.

4.4. An empirical analysis of U.S. government bond yields

In this section we illustrate some preliminary results regarding the application of the hybrid model in interpreting bond yield term structure. Specifically, we show the model's ability to capture the relationship between short and long term bond yields and to forecast their upward/downward trend. This application is possible because the hybrid model proposed here is closely related to the multi-factor stochastic volatility model of Trolle and Schwartz (2009). In fact, roughly speaking, the hybrid model is similar to the Trolle and Schwartz model when only two stochastic factors are chosen: one is the short term yield and the other is the long-term yield volatility. Recently, Cieslak and Povala (2014) showed that the use of the short term rate as a volatility factor provides a good fitting of the bond yield term structure.

Specifically, we describe the generic U.S. two(ten)-year government bond yield using Eq. (7) where one factor is the volatility of the two(ten)-year bond yield and another factor is the generic U.S. three-month government bond yield. Fig. 13 shows the yield values in the period December 31, 2002 to April 3, 2014.

In this experiment we estimate the model parameters by maximizing the likelihood function using n_T (i.e. n_T = 22, 44, 66, 132) daily data for each bond (i.e. the daily data observed in one month $n_T = 22$, two months $n_T = 44$, three months $n_T = 66$ and six months $n_T = 132$). After having estimated the model parameters, we move this window along the time series discarding the n_T observations already used and inserting the new n_T observations. That is, we solve the following problem:

$$\max_{\Theta \in \mathcal{V}} L_{n_T}(\Theta). \tag{45}$$

where $\mathcal{V}$ is given in (38), n_T is the number of daily observations used and L_{n_T} is the objective function given by:

$$L_{n_t} = \sum_{j=1}^{n_T} D_{v,r,q}(x_j, v_0, r_{t_1}, t_j, x_{j+1}, t_{j+1}). \tag{46}$$

In Eq. (46) the function $D_{v,r,q}$ is given by (18) where q is equal to zero, x_j is the observation of the two(ten)-year bond yield at $t = t_j$, $j = 1, 2, \ldots, n_T$, and r_{t_1} is the observed value of the three month government bond on the first day of the time window (i.e. $t = t_1$). We solve 128 ($n_T = 22$), 64 ($n_T = 44$), 42 ($n_T = 66$) and 21 ($n_T = 132$) calibration problems. To each parameter value obtained solving problem (45) we associate the last date $t = t_{n_T}$ of the time window used in the estimation procedure. In this way we have a time series of monthly observations for $n_T = 22$, bi-monthly for $n_T = 44$ and so on. As mentioned above, in this experiment we choose the initial stochastic interest rate, r_0, to be the value of the three-month bond yield on the first date of the time window used in the calibration.

Fig. 14 shows the model parameters estimated with daily data of the three-month and two-year government bond yields using a time window of one month, $n_T = 22$ (dotted line) and a time window of three months, $n_T = 66$ (solid line). We can observe that the estimates obtained behave similarly except for the parameters v_0, Δ and Ω. In fact, the estimates of these parameters obtained using one-month time windows show oscillations which are more pronounced than those obtained using a three month window. This is especially true when the index value is in the interval [1900, 2200] which corresponds approximately to the period July 2010 to September 2011. These oscillations seem to precede the U.S. debt-ceiling crisis of 2011.

We then use formula (24) to forecast the log-return at $t = t_{n_T} + \delta t$. That is, we forecast $r_{t_{n_T}+\delta t} = x_{t_{n_T}+\delta t} - x_{t_{n_T}}$ where x_t is the logarithm of the two(ten)-year bond yield at time t. We compute the forecasts using different δt: $\delta t = 1$ month, $\delta t = 2$ months, $\delta t = 3$ months and $\delta t = 6$ months.

We compare these forecasts (hereafter, model forecasts) with those obtained using the following naive formula, $\hat{r}_{t_{n_T}+\delta t}$

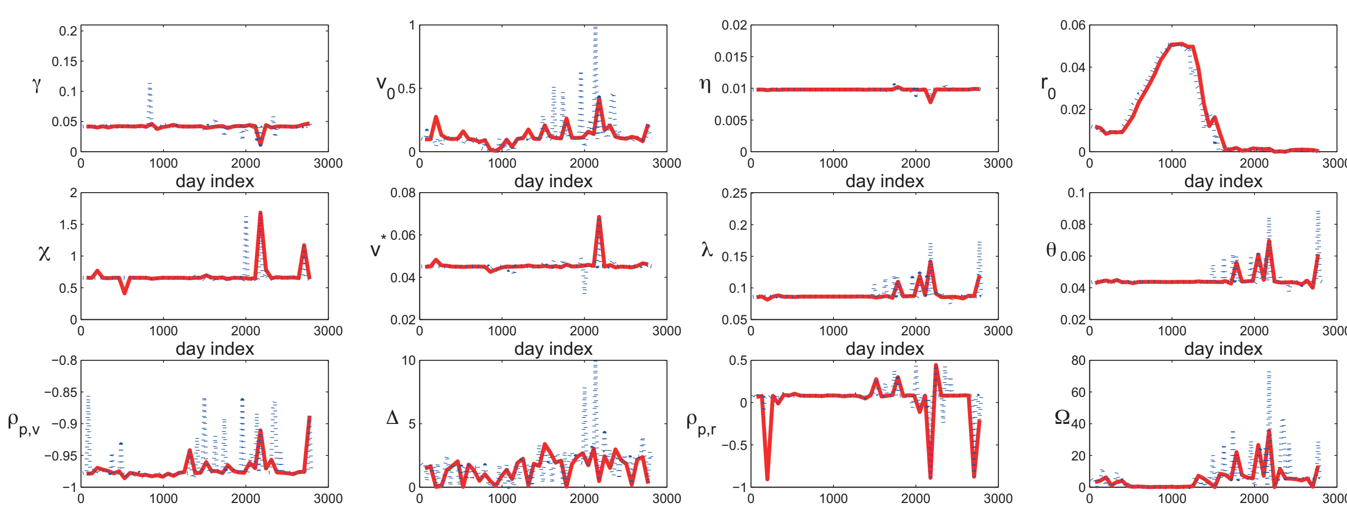

Fig. 14. Parameter estimated by using a time window of one month (dotted line) and a time window of three months (solid line). The data are daily data of the three-month and two-year government bond yields.

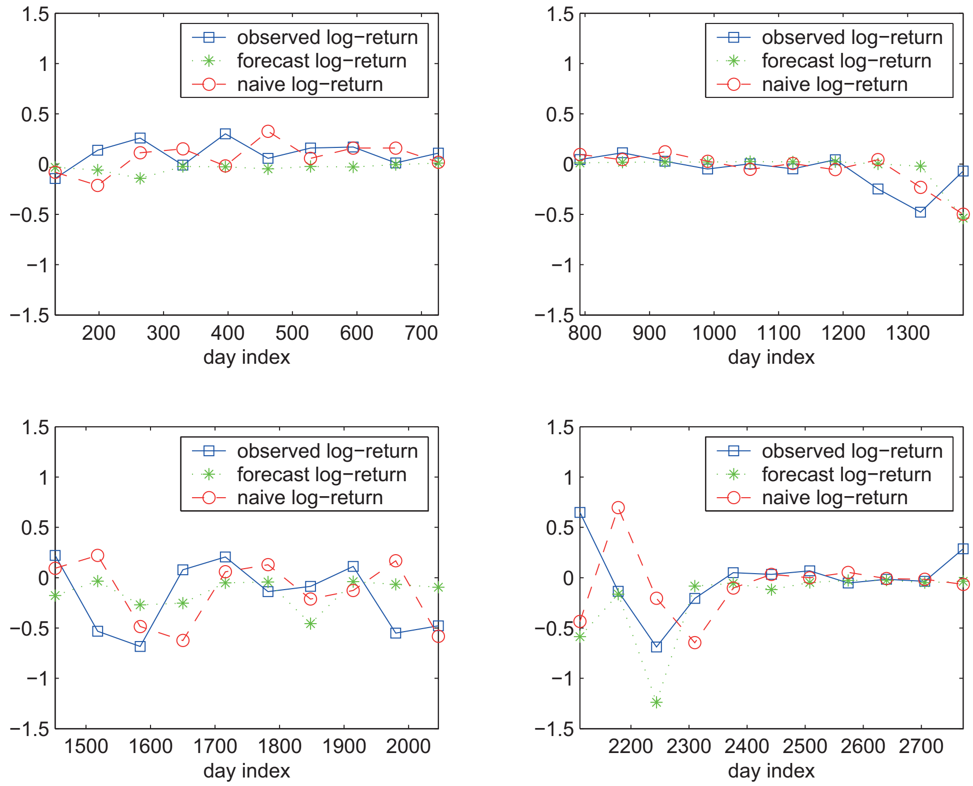

Fig. 15. Three month ahead 2-year bond yield forecasts. True log-returns (solid line), model log-return forecasts (dotted line) and naive log-return forecasts (dashed line). Upper left panel: January 2003–March 2006. Upper right panel: April 2006–July 2008. Lower left panel August 2008–April 2011. Lower right panel: May 2011–April 2014.

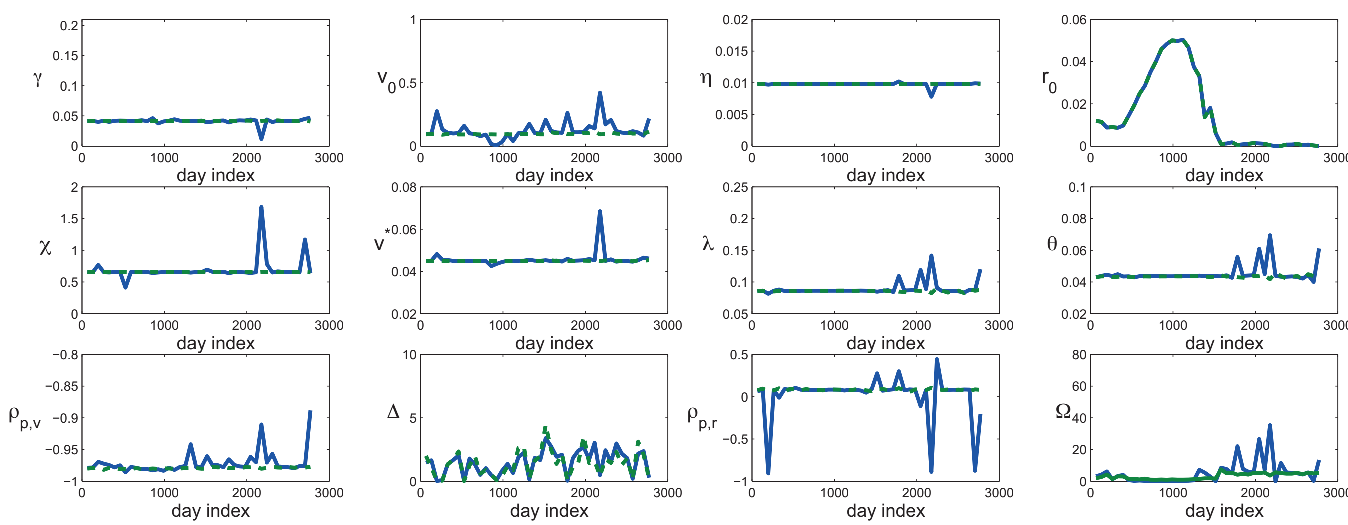

Fig. 16. Parameter estimated with a three month rolling time window using 3-month and 2-year bond yields (solid line) and using 3-month and 10-year bond yields (dashed line).

$= x_{t_{n_T}} - x_{t_{n_T}-\delta t}$. Fig. 15 shows the results obtained using the three-month bond yield as a volatility factor to describe the two year government bond yield. Specifically, Fig. 15 displays the true log-returns (solid line and squares), the forecast log-returns (dotted line and stars) and the naive forecast log-returns (dashed line and circles). We can observe that the forecast values obtained using the hybrid model are able to capture the trend of the log-return better than the naive approach in periods where the yields experienced strong fluctuations such as the crisis that followed the Lehman bankruptcy in the middle of September 2008 (lower left panel of Fig. 15) and the U.S. debt-ceiling crisis of 2011 (lower right panel of Fig. 15).

To better analyze the quality of the forecasts in predicting the trend, we compute how many times the forecast values match the upward/downward trend of the observed values. Note that the upward/downward trend is given by the sign of the difference between the value at $t = \tau$ and at $t = \tau + \delta t$. A negative (positive) sign signifies downward (upward) trend in the yield.

Table 3 illustrates the results of the trend forecasts obtained by first using the three-month and two-year government bond yields and then using the three-month and ten-year ones. Table 3 shows that the hybrid model forecasts outperform the naive ones in all cases considered except for the one-month ahead forecasts of the 10-year bond yield. This may be due to the fact that the maturity of the

Table 3
Percentage of correct trend forecasts in one, two, three and six month ahead bond yield forecasts.

2-year bond yield trend forecasts			10-year bond yield trend forecasts		
Forecast horizon	Model forecast (percent)	Naive forecast (percent)	Forecast horizon	Model forecast (percent)	Naive forecast (percent)
One-month	47.24	39.37	One-month	58.51	63.41
Two-month	69.35	37.10	Two-month	52.38	46.03
Three-month	57.50	32.50	Three-month	70.0	32.50
Six-month	52.63	31.58	Six-month	52.63	36.84

short-term bond yield used in the calibration is three months which may be too long for one-month ahead forecasts. Finally, Fig. 16 compares the estimates of the parameters obtained calibrating the hybrid model with a rolling time window of three months. The solid line indicates the values of the model parameters obtained by three-month and two year bond yields while the dashed line those obtained by three-month and ten-year yields. It is worth noting that the estimated values of the parameters obtained using the ten-year bond yields display less fluctuations than those obtained using two-year yields. This is particularly true for the parameters γ, χ, v_0, v^* and η, λ, θ which are related to the volatility of the stochastic processes which describe the long-term yield. This finding is coherent with the fact that bond yields with long-maturity usually are less volatile than those with short maturity. As previously mentioned, this is only a preliminary analysis on the use of this hybrid model to study the term structure of yields. However, the results seem to be encouraging.

5. Conclusions

We present a Heston model with a stochastic interest rate. This model can be seen as a modified version of the model in Grzelak and Oosterlee (2011) as well as of the multi-factor Heston model in Christoffersen et al. (2009) where a volatility factor is the interest rate. This hybrid model is analytically tractable. The analytical treatment is based on a simple "trick" which allows us to suitably parameterize the transition probability density function. That is, the transition probability density function is expressed as a one-dimensional integral of a smooth integrand function which depends on a real parameter q. Thanks to this formula and suitable choices for the parameter q, we deduce explicit elementary formulas for the moments and mixed moments of the price and log-price variables as well as efficient formulas to approximate the option prices.

We conduct an empirical analysis of three different data-sets. The empirical analysis of European call and put options on the U.S. S&P 500 index shows that the hybrid Heston model outperforms the Heston model in interpreting both call and put option prices. In fact, the use of a two stage calibration provides empirical evidence that the stochastic interest rate plays a significant role as a volatility factor in the option pricing. Moreover, the hybrid Heston model outperforms the Heston one even when the Credit Agricole policy values are analyzed. This result is not surprising since the assumption of constant interest rate is not realistic in the pricing of long-term products.

Finally, the preliminary results of the empirical analysis on U.S. government bond yields confirm the validity of the Heston-like multi-factor stochastic volatility models to interpret bond yield term structure. The use of short-term yields as a volatility factor is a recent approach of Cieslak and Povala (2014) that deserves further investigation.

In conclusion, the hybrid model seems to interpret satisfactorily call and put option prices and, consequently, the volatility implied by these prices. It also holds promise in analyzing the term structure of yields.

The use of this model to forecast implied volatility as well as the use of the moment formulas to calibrate the model following the approach of Date and Islyaev (2015) will be the object of future research.

Acknowledgments

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Appendix A. The analytical treatment of the hybrid stochastic model

In this section we derive an integral representation formula for the transition probability density function of the process described by Eqs. (7)–(9) and initial conditions (11).

In addition, we deduce the moments of the price variable and the mixed moments.

As mentioned in the previous section we assume that v_0^*, χ, λ, γ, η, v^*, θ are positive constant and that $\frac{2\chi v^*}{\gamma^2} > 1$ and $\frac{2\lambda\theta}{\eta^2} > 1$.

Note that since the coefficients of the Kolmogorov backward equation (12) and condition (13) are invariant by time translation and log-price translation, the function p_f can be expressed as follows:

$$p_f(x, v, r, t, x', v', r', t') = e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbf{R}} dk\, e^{\imath k(x'-x)} f(\tau, v, r, v', r', k),$$
$$(v, r), (v', r') \in \mathbf{R}^+ \times \mathbf{R}^+,\ k \in \mathbf{R},\ \tau > 0, q \in \mathbf{R}. \tag{47}$$

The representation formula (47) for the transition probability density function p_f depends on a "regularization" parameter, q, which plays a crucial role in deriving explicit formulas for option pricing and conditional mixed moments.

Using (47) into (12) we obtain that the function f is the solution of the following problem:

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= -\frac{k^2}{2}(\tilde{\psi} v + \Omega^2 r) f + \frac{1}{2}\gamma^2 v \frac{\partial^2 f}{\partial v^2} + \frac{1}{2}\eta^2 r \frac{\partial^2 f}{\partial r^2} \\ &\quad + \left[(-\imath k)\gamma\tilde{\rho}_{p,v} v + q\gamma\tilde{\rho}_{p,v} v\right]\frac{\partial f}{\partial v} \\ &\quad + \left[(-\imath k)\eta\rho_{p,r}\Omega r + q\eta\rho_{p,r}\Omega r\right]\frac{\partial f}{\partial r} + \chi(v^* - v)\frac{\partial f}{\partial v} \\ &\quad + \lambda(\theta - r)\frac{\partial f}{\partial r} + \left[\frac{v}{2}\left((q^2 - q)\tilde{\psi} - \imath k(2q-1)\tilde{\psi}\right)\right. \\ &\quad \left. + \frac{r}{2}\left((q^2 - q)\Omega^2 + 2q - \imath k((2q-1)\Omega^2 + 2)\right)\right] f \\ &\quad \times (k, v, r) \in \mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}^+,\ \tau > 0,\ q \in \mathbf{R}\end{aligned} \tag{48}$$

with the initial condition:

$$f(0, v, r, v', r', k) = \delta(v' - v)\delta(r' - r),\ (v, r),$$
$$(v', r') \in \mathbf{R}^+ \times \mathbf{R}^+,\ k \in \mathbf{R}, \tag{49}$$

where $\tilde{\psi}$ is given by Eq. (10) and $\tilde{\rho}_{p,v}$ is as follows:

$$\tilde{\rho}_{p,v} = \rho_{p,v} + \Delta\ . \tag{50}$$

Now let us represent f as the inverse Fourier transform of the future variables (v', r') whose conjugate variables are denoted by (l, ξ), that is:

$$f(\tau, v, r, v', r', k) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} dl\, e^{\imath l v'} \times \int_{-\infty}^{+\infty} d\xi\, e^{\imath \xi r'} g(\tau, v, r, k, l, \xi),$$
$$(v, r) \in \mathbf{R}^+ \times \mathbf{R}^+,\ (k, l, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R},\ \tau > 0. \tag{51}$$

It is easy to see that the function g satisfies Eq. (48) with the following initial condition:

$$g(0, v, r, k, l, \xi) = e^{-\imath \xi r} e^{-\imath l v},\ (v, r) \in \mathbf{R}^+ \times \mathbf{R}^+,\ (k, l, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}. \tag{52}$$

The coefficients of the partial differential operator appearing on the right hand side of (48) are first degree polynomials in v and r so that we seek a solution of problem (48), (49) in the form (see Lipton, 2001):

$$g(\tau, v, r, k, l, \xi) = e^{A(\tau,k,l,\xi)} e^{-v B_v(\tau,k,l)} e^{-r B_r(\tau,k,\xi)},\ (v, r) \in \mathbf{R}^+ \times \mathbf{R}^+,\ (k, l, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R},\ \tau > 0. \tag{53}$$

Substituting formula (53) into Eq. (48), we obtain that the functions A, B_v and B_r must satisfy the following ordinary differential equations for $(k, l, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}, \tau > 0$:

$$\frac{dA}{d\tau}(\tau, k, l, \xi) = -\lambda\,\theta\, B_r(\tau, k, \xi) - \chi\, v^* B_v(\tau, k, l), \tag{54}$$

$$\frac{dB_v}{d\tau}(\tau, k, l) = \varphi_q(k)\tilde{\psi} - (\chi + (\imath k - q)\,\gamma\, \tilde{\rho}_{p,v})B_v(\tau, k, l) - \frac{\gamma^2}{2} B_v^2(\tau, k, l), \tag{55}$$

$$\frac{dB_r}{d\tau}(\tau, k, \xi) = \varphi_q(k)\Omega^2 + \imath k - q - \frac{\eta^2}{2} B_r^2(\tau, k, \xi) - (\lambda + (\imath k - q)\Omega\, \rho_{p,r}\eta)B_r(\tau, k, \xi), \tag{56}$$

with initial conditions:

$$A(0, k, l, \xi) = 0, \quad B_v(0, k, l) = \imath l, \quad B_r(0, k, \xi) = \imath \xi, \tag{57}$$

where φ_q is the quantity given by:

$$\varphi_q(k) = \frac{k^2}{2} + \imath\frac{k}{2}(2q - 1) - \frac{1}{2}(q^2 - q),\ k \in \mathbf{R}, \tag{58}$$

and the quantities $\tilde{\psi}$, $\tilde{\rho}_{p,v}$ are given by (10) and (50) respectively. Eqs. (55) and (56) are Riccati equations that can be easily solved by substituting their solutions into Eq. (54) and integrating with respect to τ in order to obtain A.

Let us solve Eq. (55). The solution of Eq. (56) can be obtained analogously. We seek for the solution of Eq. (55) in the following form:

$$B_v(\tau, k, l) = \frac{2}{\gamma^2}\frac{\frac{d}{d\tau}C_v}{C_v}. \tag{59}$$

Replacing (59) into Eq. (55) we obtain C_v:

$$C_v(\tau, k, l) = e^{(\mu_{q,v}+\zeta_{q,v})\tau}\left[\frac{s_{q,v,b} + \imath l \frac{\gamma^2}{2} s_{q,v,g}}{2\zeta_{q,v}}\right], \tag{60}$$

where $\mu_{q,v}$, $\zeta_{q,v}$, $s_{q,v,g}$, $s_{q,v,b}$ are given by:

$$\mu_{q,v} = -\frac{1}{2}(\chi + (\imath k - q)\,\gamma\, \tilde{\rho}_{p,v}), \tag{61}$$

$$\zeta_{q,v} = \frac{1}{2}\left[4\mu_{q,v}^2 + 2\gamma^2\varphi_q(k)\tilde{\psi}\right]^{1/2}, \tag{62}$$

$$s_{q,v,g} = 1 - e^{-2\zeta_{q,v}\tau}, \tag{63}$$

$$s_{q,v,b} = (\zeta_{q,v} + \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} - \mu_{q,v}), \tag{64}$$

and the quantities φ_q, $\tilde{\rho}_{p,v}$ in (61) are given by (58) and (50) respectively. Using Eq. (60) into Eq. (59) we obtain:

$$B_v(\tau, k, l) = \frac{2}{\gamma^2}\frac{((\zeta_{q,v}^2 - \mu_{q,v}^2)s_{q,v,g} + \frac{\gamma^2}{2}\imath l\, s_{q,v,d})}{s_{q,v,b} + \imath l \frac{\gamma^2}{2} s_{q,v,g}}, \tag{65}$$

where $s_{q,v,d}$ are given by:

$$s_{q,v,d} = (\zeta_{q,v} - \mu_{q,v})e^{-2\zeta_{q,v}\tau} + (\zeta_{q,v} + \mu_{q,v}). \tag{66}$$

The solution of Eq. (56) is obtained analogously:

$$B_r(\tau, k, \xi) = \frac{2}{\eta^2}\frac{\frac{d}{d\tau}C_r}{C_r} = \frac{2}{\eta^2}\frac{((\zeta_{q,r}^2 - \mu_{q,r}^2)s_{q,r,g} + \frac{\eta^2}{2}\imath \xi\, s_{q,r,d})}{s_{q,r,b} + \imath \xi \frac{\eta^2}{2} s_{q,r,g}}, \tag{67}$$

where C_r is given by:

$$C_r(\tau, k, l) = e^{(\mu_{q,r}+\zeta_{q,r})\tau}\left[\frac{s_{q,r,b} + \imath \xi \frac{\eta^2}{2} s_{q,r,g}}{2\zeta_{q,r}}\right]. \tag{68}$$

and the quantities $\mu_{q,r}$, $\zeta_{q,r}$, $s_{q,r,g}$, $s_{q,r,b}$ are given by:

$$\mu_{q,r} = -\frac{1}{2}(\lambda + (\imath k - q)\,\eta\,\Omega\,\rho_{p,r}), \tag{69}$$

$$\zeta_{q,r} = \frac{1}{2}\left[4\mu_{q,r}^2 + 2\eta^2\left(\varphi_q(k)\Omega^2 - q + \imath k\right)\right]^{1/2}, \tag{70}$$

$$s_{q,r,g} = 1 - e^{-2\zeta_{q,r}\tau}, \tag{71}$$

$$s_{q,r,b} = (\zeta_{q,r} + \mu_{q,r})e^{-2\zeta_{q,r}\tau} + (\zeta_{q,r} - \mu_{q,r}), \tag{72}$$

where the quantity φ_q is given in Eq. (58) while $s_{q,r,d}$ is given by:

$$s_{q,r,d} = (\zeta_{q,r} - \mu_{q,r})e^{-2\zeta_{q,r}\tau} + (\zeta_{q,r} + \mu_{q,r}). \tag{73}$$

Let us carry out the final steps of the computation. From Eq. (54) we obtain:

$$A(\tau, k, l, \xi) = -\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau, k, l) - \frac{2\lambda\theta}{\eta^2}\ln C_r(\tau, k, \xi), \tag{74}$$

so that we obtain the following expression for the function g (see Eq. (53)):

$$g(\tau, v, r, k, l, \xi) = e^{-\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)} e^{-\frac{2v}{\gamma^2}\frac{dC_v}{d\tau}(\tau,k,l)/C_v} e^{-\frac{2\lambda\theta}{\eta^2}\ln C_r(\tau,k,\xi)} \times e^{-\frac{2r}{\eta^2}\frac{dC_r}{d\tau}(\tau,k,\xi)/C_r},\ (v, r) \in \mathbf{R}^+ \times \mathbf{R}^+,\ (k, l, \xi) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R},\ \tau > 0. \tag{75}$$

Finally, in order to get an explicit expression for f (see Eq. (51)) we have to compute the following integrals:

$$L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl\, e^{\imath l v'} e^{-\frac{2\chi v^*}{\gamma^2}\ln C_v(\tau,k,l)} e^{-\frac{2v}{\gamma^2}\frac{dC_v}{d\tau}(\tau,k,l)/C_v} \tag{76}$$

$$L_{r,q}(\tau, r, r', k; \underline{\Theta}_r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi\, e^{\imath \xi r'} e^{-\frac{2\lambda\theta}{\eta^2}\ln C_r(\tau,k,\xi)} e^{-\frac{2r}{\eta^2}\frac{dC_r}{d\tau}(\tau,k,\xi)/C_r}. \tag{77}$$

Let us show how to compute the integral in Eq. (76). The integral in Eq. (77) can be obtained in a similar fashion as Eq. (76). Using the explicit expression of C_v of Eq. (60) in Eq. (76) and the change of variable $l' = -s_{q,v,g}(\gamma^2/2)l/s_{q,v,b}$ and formula (34) on p. 156 in Oberhettinger (1973) we obtain the final expression of $L_{v,q}$:

$$L_{v,q}(\tau, v, v', k; \underline{\Theta}_v) = e^{-(2\chi v^*/\gamma^2)[\ln(s_{q,v,b}/(2\zeta_v)) + (\mu_{q,v}+\zeta_{q,v})\tau]} \times e^{-(2v/\gamma^2)(\zeta_{q,v}^2 - \mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}} \cdot M_{q,v}\left(\frac{v'}{\tilde{v}_q}\right)^{\nu_v/2} e^{-M_{q,v}\tilde{v}} e^{-M_{q,v}v'} \times I_{\nu_v}(2M_{q,v}(\tilde{v}_q v')^{1/2}),\ v, v' > 0,\ k \in \mathbf{R}, \tag{78}$$

where I_ν is the modified Bessel function of order ν (see, for example, Abramowitz & Stegun, 1970), $\nu_v = 2\chi v^*/\gamma^2 - 1$, and $M_{q,v}$, $\tilde{v}_q$ are given by:

$$M_{q,v} = \frac{2}{\gamma^2}\frac{s_{q,v,b}}{s_{q,v,g}}, \quad \tilde{v}_q = \frac{4\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,b}^2}, \quad M_{q,v}\tilde{v}_q = \frac{8}{\gamma^2}\frac{\zeta_{q,v}^2 v e^{-2\zeta_{q,v}\tau}}{s_{q,v,g}s_{q,v,b}}. \tag{79}$$

Arguing similarly, from Eq. (77) we obtain:

$$L_{r,q}(\tau, r, r', k; \underline{\Theta}_r) = e^{-(2\lambda\theta/\eta^2)[\ln(s_{q,r,b}/(2\zeta_{q,r}))+(\mu_{q,r}+\zeta_{q,r})\tau]}$$
$$\times e^{-(2r/\eta^2)(\zeta_{q,r}^2-\mu_{q,r}^2)s_{q,r,g}/s_{q,r,b}} \cdot M_{q,r}\left(\frac{r'}{\tilde{r}_q}\right)^{\nu_r/2} e^{-M_{q,r}\tilde{r}_q}e^{-M_{q,r}r'}$$
$$\times I_{\nu_r}(2M_{q,r}(\tilde{r}_q r')^{1/2}), \ r, r' > 0, \ k \in \mathbf{R}, \tag{80}$$

where $\nu_r = 2\lambda\theta/\eta^2 - 1$ and the quantities $M_{q,r}$, $\tilde{r}_q$ are given by:

$$M_{q,r} = \frac{2}{\eta^2}\frac{s_{q,r,b}}{s_{q,r,g}}, \quad \tilde{r}_q = \frac{4\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,b}^2}, \quad M_{q,r}\tilde{r}_q = \frac{8}{\eta^2}\frac{\zeta_{q,r}^2 r e^{-2\zeta_{q,r}\tau}}{s_{q,r,g}s_{q,r,b}}. \tag{81}$$

Finally, using (78) and (80) in formula (51) and then in (47) we obtain formula (14), that is:

$$p_f(x, v, r, t, x', v', r', t') = \frac{e^{q(x-x')}}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}$$
$$\times L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v)L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r), \ (x, v, r),$$
$$(x', v', r') \in \mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^+, \ t, t' \geq 0, \ q \in \mathbf{R}, \ t'-t > 0. \tag{82}$$

As explained in Section 2, the parameter q is a "regularization" parameter whose choice allows us to explicitly compute some integrals appearing in the formulas of the moments/mixed moments and option prices.

In fact, using formula (18) in Erdelyi et al. (1954) Vol. I, p. 197 we obtain:

$$W_{v,q}^0(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv'\, L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v)$$
$$= e^{-2\chi v^*\ln(s_{q,v,b}/(2\zeta_{q,v}))/\gamma^2}e^{-2\chi v^*(\zeta_{q,v}+\mu_{q,v})(t'-t)/\gamma^2}$$
$$\times e^{-(2v/\gamma^2)(\zeta_{q,v}^2-\mu_{q,v}^2)s_{q,v,g}/s_{q,v,b}}, \tag{83}$$

and $W_{r,q}^0$ is the function given by:

$$W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r) = \int_0^{+\infty} dr'\, L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r)$$
$$= e^{-2\lambda\theta\ln(s_{q,r,b}/(2\zeta_{q,r}))/\eta^2}e^{-2\lambda\theta(\zeta_{q,r}+\mu_{q,r})(t'-t)/\eta^2}$$
$$\times e^{-(2r/\eta^2)(\zeta_{q,r}^2-\mu_{q,r}^2)s_{q,r,g}/s_{q,r,b}}. \tag{84}$$

Furthermore, using Abramowitz and Stegun (1970, p. 375 formula (9.6.3), p. 486 formula (11.4.28), p. 505 formula (13.1.27), p. 509 formula (13.6.9)) we obtain:

$$W_{v,q}^m(t'-t, v, k; \underline{\Theta}_v) = \int_0^{+\infty} dv'\, (v')^m L_{v,q}(t'-t, v, v', k; \underline{\Theta}_v)$$
$$= W_{v,q}^0(t'-t, v, k; \underline{\Theta}_v)\frac{1}{(M_{v,q})^m}\frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)}\frac{m!}{(\nu_v+1)_m}L_m^{(\nu_v)}$$
$$\times(-M_{v,q}\tilde{v}_q), \tag{85}$$

$$W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r) = \int_0^{+\infty} dr'\, (r')^m L_{r,q}(t'-t, r, r', k; \underline{\Theta}_r)$$
$$= W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r)\frac{1}{(M_{r,q})^m}\frac{\Gamma(m+1+\nu_r)}{\Gamma(\nu_r+1)}\frac{m!}{(\nu_r+1)_m}L_m^{(\nu_r)}$$
$$\times(-M_{r,q}\tilde{r}_q), \tag{86}$$

where $L_m^{(\nu)}$ are the generalized Laguerre polynomials (see Abramowitz and Stegun, 1970, p. 775), $m!$ is the factorial of the integer m (i.e. $m! = \prod_{j=1}^m j$) and $(\nu)_m = \prod_{j=1}^m(\nu+j-1)$. The explicit expression of $L_m^{(\nu)}$ is given by (see Abramowitz and Stegun, 1970, p. 775):

$$L_m^{(\nu)}(y) = \sum_{j=0}^m(-1)^j\binom{m+\nu}{m-j}\frac{1}{j!}y^j, \ y \geq 0. \tag{87}$$

Using (87) into Eqs. (85) and (86) we deduce the following elementary formulae

$$W_{v,q}^m(t'-t, v, k; \underline{\Theta}_v) = W_{v,q}^0(t'-t, v, k; \underline{\Theta}_v)$$
$$\times\frac{\Gamma(m+1+\nu_v)}{\Gamma(\nu_v+1)}\frac{m!}{(\nu_v+1)_m}\sum_{j=1}^m\binom{m+\nu_v}{m-j}\frac{\tilde{v}_q^j}{j!}(M_{v,q})^{(j-m)}, \tag{88}$$

$$W_{r,q}^m(t'-t, r, k; \underline{\Theta}_r) = W_{r,q}^0(t'-t, r, k; \underline{\Theta}_r)$$
$$\times\frac{\Gamma(m+1+\nu_r)}{\Gamma(\nu_r+1)}\frac{m!}{(\nu_r+1)_m}\sum_{j=1}^m\binom{m+\nu_r}{m-j}\frac{\tilde{r}_q^j}{j!}(M_{r,q})^{(j-m)}. \tag{89}$$

Using Eqs. (83)–(86) and the definitions of the marginal distribution and of the moments with an easy computation we obtain formulas (17), (18), (20), (22), (21).

We conclude this section with some comments on the existence of the moments. Formula (20) shows that the existence of the moments is guaranteed when $\zeta_{m,v}(0)$, $\zeta_{m,r}(0)$ (see formulas (61) and (62) and (69) and (70)) are real numbers. It is easy to see that sufficient conditions for the existence of $E(S_{t'}^m)$, $m = 2, 3, \ldots$, are:

$$\Delta \leq 1, \quad \rho_{p,v} \leq \frac{\chi-\gamma\Delta}{m\gamma}$$
$$-\frac{1}{m\gamma}\sqrt{(m-1)\gamma(2\chi\Delta-\Delta^2\gamma+m\gamma)}, \tag{90}$$

$$\Omega \geq \sqrt{2}, \quad \rho_{p,r} \leq \frac{\lambda}{m\eta\Omega}$$
$$-\frac{1}{m\eta\Omega}\sqrt{m\eta^2((m-1)\Omega^2+2)}. \tag{91}$$

Finally, formula (20) provides the moments of the price variable of the Heston model in the limit $\Omega \to 0^+$, $\eta \to 0^+$ and $\lambda \to 0^+$, that is:

$$\mathcal{M}_m^H(S, v, r, t, t') = S^m e^{m\,r(t'-t)}W_{v,m}^0(t'-t, v, 0; \underline{\Theta}_v),$$
$$(x, v) \in \mathbf{R}\times\mathbf{R}^+, \ t, t' \geq 0, \ t'-t > 0. \tag{92}$$

References

Abramowitz, M., & Stegun, I. A. (1970). *Handbook of mathematical functions*. New York: Dover.

Ahlip, R. (2008). Foreign exchange options under stochastic volatility and stochastic interest rates. *International Journal of Theoretical and Applied Finance, 11*, 277–294.

Andersen, L. B. G., & Piterbarg, V. V. (2007). Moment explosions in stochastic volatility models. *Finance and Stochastics, 11*(1), 29–50.

Andersen, T. G., & Benzoni, L. (2010). Do bonds span volatility risk in the U.S. treasury market? A specification test for affine term structure models. *The Journal of Finance, 65*(2), 603–653.

Andreasen, J. (2007). Closed form pricing of FX options under stochastic rates and volatility. *International Journal of Theoretical and Applied Finance, 11*(3), 277–294.

Ball, C. A., & Roma, A. (1994). Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis, 29*, 589–607.

Benhamou, E., Gobet, E., & Miri, M. (2012). Analytical formulas for local volatility model with stochastic rates. *Quantitative Finance, 12*(2), 185–198.

Bühler, H. (2002). *Applying stochastic volatility models for pricing and hedging derivatives*. Global Quantitative Research Deutsche Bank AG Global Equities. Freely downloadable from the website: http://www.quantitative-research.de/dl/021118SV.eps

Chiarella, C., & Kwon, O. K. (2003). Finite dimensional affine realisations of HJM models in terms of forward rates and yields. *Review of Derivatives Research, 5*, 129–155.

Choi, Y., & Wirjanto, T. S. (2007). An analytic approximation formula for pricing zero-coupon bonds. *Finance Research Letters, 4*, 116–126.

Christensen, J. H. E., Diebold, F. X., & Rudebush, G. D. (2011). The affine arbitrage-free class of Nelson-Siegel term structure models. *Journal of Econometrics, 164*, 4–20.

Christoffersen, P., Heston, S., & Jacob, K. (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science, 55*, 1914–1932.

Cieslak, A., & Povala, P. (2014). Information in the term structure of yield curve volatility. *Journal of Finance, Forthcoming*. Available at SSRN: http://ssrn.com/abstract=1458006 or http://dx.doi.org/10.2139/ssrn.1458006

Cox, J. C., Ingersoll, J. E., & Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica, 53*, 385–407.

Date, P., & Islyaev, S. (2015). A fast calibrating volatility model for option pricing. *European Journal of Operational Research, 243*(2), 599–606.

Deelstra, G., & Rayee, G. (2012). Local volatility pricing models for long-dated FX derivatives. *Applied Mathematical Finance, 20*(4), 380–402.

Duffie, D. (2001). *Asset pricing theory*. Princeton, New Jersey, USA: Princeton University Press.

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump diffusions. *Econometrica, 68*, 1343–1376.

Erdelyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. (1954). *Tables of integral transforms*. New York, USA: McGraw-Hill Book Company.

Fatone, L., Mariani, F., Recchioni, M. C., & Zirilli, F. (2009). An explicitly solvable multiscale stochastic volatility model: option pricing and calibration. *Journal of Futures Markets, 29*(9), 862–893.

Fatone, L., Mariani, F., Recchioni, M. C., & Zirilli, F. (2013). The analysis of real data using a multiscale stochastic volatility model. *European Financial Management, 19*(1), 153–179.

Grzelak, L. A., & Oosterlee, C. W. (2011). On the Heston model with stochastic interest rates. *SIAM Journal on Financial Mathematics, 2*, 255–286.

Grzelak, L. A., Oosterlee, C. W., & Van Weeren, S. (2012). Extension of stochastic volatility equity models with Hull-White interest rate process. *Quantitative Finance, 12*, 89–105.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies, 6*, 327–343.

Hull, J., & White, H. (1988). The pricing of options on assets with stochastic volatilities. *Journal of Finance, 3*, 327–343.

Islyaev, S., & Date, P. (2015). Electricity futures price models: Calibration and forecasting. *European Journal of Operational Research, 247*, 144–154.

Lions, P. L., & Musiela, M. (2007). Correlation and bounds for stochastic volatility models. *Annales de l'Institut Henri Poincaré , AN, 24*, 1–16.

Lipton, A. (2001). *Mathematical methods for foreign exchange*. Singapore: World Scientific Publishing Co. Pte. Ltd.

Milevsky, M. A., & Promislow, S. D. (2001). Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics, 29*, 299–318.

Moreno, M., & Platania, F. (2015). A cyclical square-root model for term structure of interest rates. *European Journal of Operational Research, 247*, 144–154.

Oberhettinger, F. (1973). *Fourier transforms of distributions and their inverses. A collection of tables*. New York: Academic Press.

Pun, C. S., Chung, S. F., & Wong, H. Y. (2015). Variance swap with mean reversion, multifactor stochastic volatility and jumps. *European Journal of Operational Research, 245*, 571–580.

Recchioni, M. C., & Scoccia, A. (2000). A stochastic algorithm for constrained global optimization. *Journal of Global Optimization, 16*, 257–270.

Recchioni, M. C., & Screpante, F. (2014). A hybrid method to evaluate pure endowment policies: Crédit Agricole and Ergo Index linked policies. *Insurance: Mathematics and Economics, 57*, 114–124.

Schoutens, W. (2003). *Lévy processes in finance*. Chichester, UK: Wiley & Sons.

Stein, E. M., & Stein, J. C. (1991). Stock price distributions with stochastic volatility: an analytic approach. *The Review of Financial Studies, 4*, 727–752.

Trolle, A. B., & Schwartz, E. S. (2009). A general stochastic volatility model for the pricing of interest rate derivatives. *The Review of Financial Studies, 22*(5), 2007–2057.

Wong, B., & Heyde, C. C. (2006). On changes of measure in stochastic volatility models. *Journal of Applied Mathematics and Stochastic Analysis, 2006*, 1–13. ID 18130.

Wong, H. Y., & Lo, Y. W. (2009). Option pricing with mean reversion and stochastic volatility. *European Journal of Operational Research, 197*, 179–187.

Zhu, J. (2000). *Modular pricing of options*. Berlin: Springer Verlag.