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ELSEVIER Stochastic Processes and their Applications 115 (2005) 639–659

stochastic processes and their applications

www.elsevier.com/locate/spa

Time-inhomogeneous affine processes

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Received 19 July 2003; received in revised form 8 April 2004; accepted 19 November 2004
Available online 14 December 2004

Abstract

Affine processes are distinguished by their rich structural properties, which makes them favorite when it comes to computations in financial applications of all kind. This fact has been explored and illustrated for the time-homogeneous case in a recent paper by Duffie, Filipović and Schachermayer. However, there are many situations which require time-dependent parameters, such as when it comes to model calibration. This paper provides a rigorous treatment and complete characterization of time-inhomogeneous affine processes.

Keywords: Affine processes; Time-inhomogeneity

1. Introduction

Affine processes are distinguished by their rich structural properties, which makes them favorite when it comes to computations in financial applications of all kind. This fact has been explored and illustrated in [3] for the time-homogeneous case, which covers many of the relevant applications. However, there are many situations where time-inhomogeneity (that is, the explicit time dependence of some model parameters) is indispensable, such as for short rate models that perfectly fit the initial yield curve (see e.g. [4,6]), or for other calibration purposes.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2004.11.006

The defining feature of an affine process is the exponential-affine form of the characteristic function of its transition probabilities. This paper provides a rigorous treatment and complete characterization of time-inhomogeneous affine processes. It extends the main results in [3] (Theorems 2.13 and 2.14) to the time-inhomogeneous case, which can be roughly summarized as follows: a Markov process is affine if and only if the coefficients of its generator are affine functions of the state.

We face nontrivial difficulties, which were not present in the time-homogeneous case and which mainly arise from the presence of jumps (Example 2.12). For clarity and simplicity we restrict our considerations to the class of "strongly regular affine" processes (Definition 2.9), for which all the parameters depend continuously on time. This is a slight restriction, but covers essentially all applications, and it makes the theory of Feller processes available (Remark 2.11).

The structure of the paper is much like parts of [3], we often refer to passages therein. In Section 2 we provide the definitions and main results. The proof of the main results is divided into Sections 3–7, which are of interest on their own.

2. Definitions and main results

2.1. Markovian setup

For the stochastic terminology we refer to [5,8]. Let $(p_{t,T}(x, \mathrm{d}\xi))_{0 \leqslant t \leqslant T}$ be the transition function of a (possibly nonconservative) time-inhomogeneous Markov process with state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$, where $m \geqslant 0$, $n \geqslant 0$ and $d = m + n \geqslant 1$. We write $P_{t,T} f(x) = \int_D f(\xi) p_{t,T}(x, \mathrm{d}\xi)$ for $f \in bD$ (the $\mathbb{C}$-valued bounded measurable functions on D). Throughout we assume that

$$p_{t,T}(x, A) \text{ is jointly measurable in } (t, T, x), \text{ for all Borel sets } A \text{ in } D. \tag{2.1}$$

Then we can consider the associated time-homogeneous *space-time process* with state space $\mathbb{R}_+ \times D$ and transition semigroup $(\overline{P}_t)_{t \geqslant 0}$ acting on $b(\mathbb{R}_+ \times D)$ by

$$\overline{P}_t f(r, x) := P_{r,r+t} f(r + t, x) = \int_D f(r + t, \xi) p_{r,r+t}(x, \mathrm{d}\xi). \tag{2.2}$$

Indeed, we let $(\Theta, X) = (\Theta, Y, Z)$ denote the realization of $(\overline{P}_t)$ on the canonical filtered space $(\Omega, \mathscr{F}^0, (\mathscr{F}_t^0))$ consisting of paths $\omega : \mathbb{R}_+ \to (\mathbb{R}_+ \times D)_\Delta = (\mathbb{R}_+ \times D) \cup \{\Delta\}$ (the one-point compactification of $\mathbb{R}_+ \times D$) and equipped with the family of probability measures $(\mathbb{P}_{(r,x)})_{(r,x) \in \mathbb{R}_+ \times D}$ (see [8, Section I.3]). So that

$$\Theta_t = r + t \quad \text{and} \quad X_0 = x, \quad \mathbb{P}_{(r,x)}\text{-a.s.}$$

To avoid unnecessary repetitions we shall usually refer to the transition operators $(P_{t,T})$ when we mean any of the above notions related to the original time-inhomogeneous Markov process X.

2.2. Weakly regular affine processes

We follow the notation in [3] and define $f_u \in C(D)$ by

$$f_u(x) := \mathrm{e}^{\langle u,x\rangle},$$

for $u \in \mathbb{C}^d$ and

$$\mathscr{U} := \mathbb{C}^m_- \times \mathrm{i}\mathbb{R}^n, \quad \partial\mathscr{U} := \mathrm{i}\mathbb{R}^d, \quad \mathscr{U}^0 := \mathscr{U}\backslash\partial\mathscr{U} = \mathbb{C}^m_{--} \times \mathrm{i}\mathbb{R}^n,$$

such that $f_u \in C_b(D)$ if and only if $u \in \mathscr{U}$.

Definition 2.1. We call $(P_{t,T})$ *affine* if for every $0 \leqslant t \leqslant T$ and $u \in \partial\mathscr{U}$ there exists $\phi(t,T,u) \in \mathbb{C}$ and $\psi(t,T,u) = (\psi^{\mathscr{Y}}(t,T,u), \psi^{\mathscr{Z}}(t,T,u)) \in \mathbb{C}^m \times \mathbb{C}^n$ such that

$$P_{t,T} f_u(x) = \mathrm{e}^{\phi(t,T,u)+\langle\psi(t,T,u),x\rangle}, \quad \forall x \in D. \tag{2.3}$$

If $P_{t,T} = P_{T-t}$ is time-homogeneous then we are back to [3] with $\phi(t,T,u)$ and $\psi(t,T,u)$ replaced by $\phi(T-t,u)$ and $\psi(T-t,u)$, respectively. We follow the convention made in [3, Remark 2.3] and let $\phi(t,T,\cdot)$ in (2.3) denote the *unique* continuous function on $\mathrm{i}\mathbb{R}^n$ with $\phi(t,T,0)=0$. As noted in [3, Remark 2.2] we necessarily have $\phi(t,T,u) \in \mathbb{C}_-$ and $\psi(t,T,u) \in \mathscr{U}$, since $P_{t,T} f_u \in bD$, for all $u \in \partial\mathscr{U}$.

Definition 2.2. We call $(P_{t,T})$ *stochastically continuous* if $p_{s,S}(x,\cdot) \to p_{t,T}(x,\cdot)$ weakly on D for $(s,S) \to (t,T)$, for every $0 \leqslant t \leqslant T$ and $x \in D$.

Hence $(P_{t,T})$ is stochastically continuous if and only if $P_{t,T} f(x)$ is jointly continuous in (t,T), for all $x \in D$ and $f \in C_b(D)$. As in [3] we need further regularity assumptions. For technical reasons, as will be made clear in Example 2.12, we have to distinguish between a "weak" and a "strong" regularity hypothesis.

Definition 2.3. We call $(P_{t,T})$ *weakly regular* if it is stochastically continuous and the left-hand derivative

$$\tilde{\mathscr{A}}(t) f_u(x) := -\partial_s^- P_{s,t} f_u(x)|_{s=t} \tag{2.4}$$

exists for all $(t,x,u) \in \mathbb{R}_{++} \times D \times \mathscr{U}$ and is continuous at $u=0$ for all $(t,x) \in \mathbb{R}_{++} \times D$.

If $(P_{t,T})$ is weakly regular and affine we call it simply *weakly regular affine*.

Note that (2.1) implies joint measurability of $\tilde{\mathscr{A}}(t) f_u(x)$ in $(t,u) \in \mathbb{R}_{++} \times \mathscr{U}$, but the t-dependence can be arbitrarily irregular, even for weakly regular affine processes. For illustration we consider a simple deterministic situation.

Example 2.4. Let $f : \mathbb{R}_+ \to D$ be a measurable function such that $f(T) - f(t) \in D$ for all $0 \leqslant t \leqslant T$. Then

$$p_{t,T}(x, \mathrm{d}\xi) := \delta_{x+f(T)-f(t)}(\mathrm{d}\xi)$$

is an affine transition function with

$$\phi(t,T,u) = \langle u, f(T) - f(t)\rangle, \quad \psi(t,T,u) = u.$$

The dependence of $\phi(t,T,u)$ on (t,T) is implicit by f and can be very irregular.

2.3. Some notation

For the convenience of the reader we recall here the notional conventions from [3]. For $\alpha,\beta \in \mathbb{C}^k$ we write $\langle\alpha,\beta\rangle := \alpha_1\beta_1 + \cdots + \alpha_k\beta_k$ (notice that this is not the scalar product on $\mathbb{C}^k$). We let Sem^k be the convex cone of symmetric positive semidefinite $k\times k$ matrices. Denote by $\{e_1,\ldots,e_d\}$ the standard basis in $\mathbb{R}^d$, and write $\mathscr{I} := \{1,\ldots,m\}$ and $\mathscr{J} := \{m+1,\ldots,d\}$. Let $\alpha = (\alpha_{ij})$ be a $d\times d$-matrix, $\beta = (\beta_1,\ldots,\beta_d)$ a d-tuple and $I,J \subset \{1,\ldots,d\}$. Then we write α^T for the transpose of α, and $\alpha_{IJ} := (\alpha_{ij})_{i\in I, j\in J}$ and $\beta_I := (\beta_i)_{i\in I}$. Examples are $\chi_I(\xi) = (\chi_k(\xi))_{k\in I}$ or $\nabla_I := (\partial_{x_k})_{k\in I}$. We write $\mathbf{1} := (1,\ldots,1)$ without specifying the dimension whenever there is no ambiguity. For $i\in\mathscr{I}$ we define $\mathscr{I}(i) := \mathscr{I}\backslash\{i\}$ and $\mathscr{J}(i) := \{i\}\cup\mathscr{J}$. The Kronecker Delta is denoted by δ_{kl}, which equals 1 if $k=l$ and 0 otherwise.

Throughout, we fix a continuous truncation function $\chi : \mathbb{R}^d \to [-1,1]^d$ such that $\chi(\xi) = \xi$ on some neighborhood of 0 (in [3] this function was unnecessarily defined in an explicit form).

Important convention: we tacitly write $x = (y,z)$ or $\xi = (\eta,\zeta)$ for a point in $D = \mathbb{R}^m_+ \times R^n$ and $u = (v,w)$ for an element in $\mathbb{C}^d = \mathbb{C}^m \times \mathbb{C}^n$. Also, we have that

$$\psi^{\mathscr{Y}}(t,T,u) = \psi_{\mathscr{I}}(t,T,u) \quad \text{and} \quad \psi^{\mathscr{Z}}(t,T,u) = \psi_{\mathscr{J}}(t,T,u)$$

(since these mappings play a distinguished role we introduced a "coordinate-free" notation).

2.4. Strongly regular affine processes

This section contains the main theorems of the paper. The proofs are postponed to Section 7. First, here is the extension of [3, Definition 2.6].

Definition 2.5. The t-dependent parameters

$$(a,\alpha,b,\beta,c,\gamma,m,\mu) = (a(t),\alpha(t),b(t),\beta(t),c(t),\gamma(t),m(t),\mu(t)), \quad t\in\mathbb{R}_+,$$

are called *weakly admissible* if for each fixed $t\in\mathbb{R}_+$ they are admissible in the sense of [3, Definition 2.6], that is,

- $a(t)\in\mathrm{Sem}^d$ with $a_{\mathscr{I}\mathscr{I}}(t) = 0$ (hence $a_{\mathscr{I}\mathscr{J}}(t) = 0$ and $a_{\mathscr{J}\mathscr{I}}(t) = 0$), (2.5)
- $\alpha(t) = (\alpha_1(t),\ldots,\alpha_m(t))$ with $\alpha_i(t)\in\mathrm{Sem}^d$ and $\alpha_{i;\mathscr{I}(i)\mathscr{I}(i)}(t) = 0$, (hence $\alpha_{i;kl}(t) = \alpha_{i;ii}(t)\delta_{ik}\delta_{kl}$ for all $k,l\in\mathscr{I}$), (2.6)

- $b(t) \in D$, (2.7)
- $\beta(t) \in \mathbb{R}^{d\times d}$ such that $\beta_{\mathscr{J}\mathscr{J}}(t) = 0$ and $\beta_{i,\mathscr{I}(i)}(t) \in \mathbb{R}^{m-1}_{+}$ for all $i \in \mathscr{I}$, (hence $\beta_{\mathscr{I}\mathscr{I}}(t)$ has nonnegative off-diagonal elements), (2.8)
- $c(t) \in \mathbb{R}_+$, (2.9)
- $\gamma(t) \in \mathbb{R}^m_+$, (2.10)
- $m(t)$ is a Borel measure on $D\backslash\{0\}$ satisfying $M(t, D\backslash\{0\}) < \infty$ with
$$M(t, \mathrm{d}\xi) := (\langle \chi_{\mathscr{I}}(\xi), \mathbf{1}\rangle + \|\chi_{\mathscr{J}}(\xi)\|^2) m(t, \mathrm{d}\xi), \tag{2.11}$$
- $\mu(t) = (\mu_1(t), \ldots, \mu_m(t))$ where $\mu_i(t)$ is a Borel measure on $D\backslash\{0\}$ satisfying $\mathscr{M}_i(t, D\backslash\{0\}) < \infty$ with
$$\mathscr{M}_i(t, \mathrm{d}\xi) := (\langle \chi_{\mathscr{I}(i)}(\xi), \mathbf{1}\rangle + \|\chi_{\mathscr{J}(i)}(\xi)\|^2) \mu_i(t, \mathrm{d}\xi). \tag{2.12}$$

They are called *strongly admissible* if in addition they satisfy the following continuity conditions:

- $(a(t), \alpha(t), b(t), \beta(t), c(t), \gamma(t))$ are continuous in $t \in \mathbb{R}_+$, (2.13)
- $M(t, \mathrm{d}\xi)$ and $\mathscr{M}_i(t, \mathrm{d}\xi)$ are weakly continuous on $D\backslash\{0\}$ in $t \in \mathbb{R}_+$. (2.14)

Example 2.6. Let us illustrate the above definition for the case $(m, n) = (2, 1)$. Conditions (2.5), (2.6) and (2.8) then say that

$$a(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}, \quad \alpha_1(t) = \begin{pmatrix} + & 0 & * \\ 0 & 0 & 0 \\ * & 0 & + \end{pmatrix}, \quad \alpha_2(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & + & * \\ 0 & * & + \end{pmatrix}$$

and

$$\beta(t) = \begin{pmatrix} * & + & 0 \\ + & * & 0 \\ * & * & * \end{pmatrix},$$

where $*$ and $+$ stand for real and nonnegative real numbers, respectively.

We first state a representation result for weakly regular affine processes.

Theorem 2.7. *Suppose $(P_{t,T})$ is weakly regular affine. Then there exist some weakly admissible parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ such that, for all $t > 0$, $u = (v, w) \in \mathscr{U}$, $x = (y, z) \in D$,*

$$\tilde{\mathscr{A}}(t) f_u(x) = (F(t, u) + \langle R^{\mathscr{Y}}(t, u), y\rangle + \langle R^{\mathscr{Z}}(t, u), z\rangle) f_u(x), \tag{2.15}$$

with

$$F(t, u) = \langle a(t)u, u\rangle + \langle b(t), u\rangle - c(t) + \int_{D\backslash\{0\}} (\mathrm{e}^{\langle u, \xi\rangle} - 1 - \langle u_{\mathscr{J}}, \chi_{\mathscr{J}}(\xi)\rangle) m(t, \mathrm{d}\xi), \tag{2.16}$$

$$R_i^{\mathscr{Y}}(t,u) = \langle \alpha_i(t)u, u\rangle + \langle \beta_i^{\mathscr{Y}}(t), u\rangle - \gamma_i(t) + \int_{D\setminus\{0\}} (\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u_{\mathscr{J}(i)}, \chi_{\mathscr{J}(i)}(\xi)\rangle)\mu_i(t,\mathrm{d}\xi), \tag{2.17}$$

$$R^{\mathscr{Z}}(t,u) = \beta^{\mathscr{Z}}(t)w, \tag{2.18}$$

and

$$\beta_i^{\mathscr{Y}}(t) := (\beta^{\mathrm{T}}(t))_{i\{1,\dots,d\}} \in \mathbb{R}^d, \tag{2.19}$$

$$\beta^{\mathscr{Z}}(t) := (\beta^{\mathrm{T}}(t))_{\mathscr{J}\mathscr{J}} \in \mathbb{R}^{n\times n}, \tag{2.20}$$

for $i \in \mathscr{I}$. *Representations* (2.16)–(2.18) *of the functions* $F(t,\cdot)$, $R^{\mathscr{Y}}(t,\cdot)$ *and* $R^{\mathscr{Z}}(t,\cdot)$ *on* $\mathscr{U}$ *by* $a(t), \alpha(t), b(t), \beta(t), c(t), \gamma(t), m(t), \mu(t)$ *are unique.*

If $\tilde{\mathscr{A}}(t)f_u(x)$ *has a continuous extension in* t *on* $\mathbb{R}_+$ *and* (2.14) *holds, then* $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ *are strongly admissible and* (2.15) *also holds for* $t = 0$.

Remark 2.8. Relation (2.19)–(2.20) between $\beta(t)$ and $(\beta^{\mathscr{Y}}(t), \beta^{\mathscr{Z}}(t))$ is made clearer by considering

$$\langle \beta(t)x, u\rangle = \langle x, \beta^{\mathrm{T}}(t)u\rangle = \sum_{i=1}^{m} \langle \beta_i^{\mathscr{Y}}(t), u\rangle y_i + \langle \beta^{\mathscr{Z}}(t)w, z\rangle, \tag{2.21}$$

which implies $\tilde{\mathscr{A}}(t)f_u(x) = \mathscr{A}(t)f_u(x)$, see (2.23).

As in [3, Definition 5.1] we shall call the parameters $(a, \alpha, b, \beta^{\mathscr{Y}}, \beta^{\mathscr{Z}}, c, \gamma, m, \mu)$ *weakly* (*strongly*) *admissible* if $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ are weakly (strongly) admissible where $\beta(t) \in \mathbb{R}^{d\times d}$ is given by $\beta_{\mathscr{J}\mathscr{I}}(t) := 0$ and (2.19)–(2.20).

Definition 2.9. We call $(P_{t,T})$ *strongly regular affine* if it is weakly regular affine and the parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ from Theorem 2.7 are strongly admissible.

Remark 2.10. In the time-homogeneous case [3] the distinction between weakly and strongly regular affine becomes redundant: every "regular affine" process (in the notation of [3]) is strongly regular affine.

Remark 2.11. The strong regularity (continuity of the parameters) is assumed to make the theory of Feller processes available (Theorem 2.13). The continuity is used for e.g. the existence of classical solutions of the ODEs (2.24)–(2.26), and for the technical points (4.11) and (5.5) (here we explicitly use (2.14)).

Piecewise continuous parameters (regime switches) can be approximated by continuous parameters. Hence for applications it seems to be more than enough to have the characterization, existence and uniqueness results for time-inhomogeneous affine Markov processes under the strong regularity hypothesis. Yet, we conjecture that similar results can be derived on the level of semimartingales (see Theorem 2.14), and leave this open for future research.

The following example shows that there are weakly regular affine processes that are not strongly regular affine, even though $F(t,u)$ and $R(t,u)$ are uniformly continuous in $(t,u) \in \mathbb{R}_+ \times \mathcal{U}$.

Example 2.12. Let $(m,n) = (0,1)$, set $R(t,u) \equiv 0$ and

$$F(t,u) = \int_{\mathbb{R}\setminus\{0\}} (\mathrm{e}^{uz} - 1 - uz)\,\frac{1}{z^2}\,\delta_{x(t)}(\mathrm{d}z) = \frac{\mathrm{e}^{ux(t)} - 1 - ux(t)}{x(t)^2},$$

where $x(t)$ is continuous with $x(0) = 0$ and $0 < x(t) \leqslant 1$ for $t > 0$. Thus $F(t,u)$ is continuous in t for all $u \in \mathbb{C}$. But $F(0,u) = \lim_{t\to 0} F(t,u) = u^2/2$, and thus $b(t) \equiv c(t) \equiv 0$,

$$a(t) = \begin{cases} 1/2 & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad m(t,\mathrm{d}z) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{1}{z^2}\,\delta_{x(t)}(\mathrm{d}z) & \text{otherwise,} \end{cases}$$

in the representation (2.16) do not satisfy the continuity conditions (2.13) and (2.14). A similar example can be constructed with a discontinuous $b(t)$.

Theorem 2.13. *Suppose $(P_{t,T})$ is strongly regular affine and $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ the corresponding strongly admissible parameters. Then (Θ, X) is a Feller process. Let $\overline{\mathcal{A}}$ be its infinitesimal generator. Then $C_c^\infty(\mathbb{R}_+ \times D)$ is a core of $\overline{\mathcal{A}}$, $C_c^{1,2}(\mathbb{R}_+ \times D) \subset \mathcal{D}(\overline{\mathcal{A}})$ and for $f \in C_c^{1,2}(\mathbb{R}_+ \times D)$ we have*

$$\overline{\mathcal{A}} f(t,x) = \partial_t f(t,x) + \mathcal{A}(t) f(t,x), \tag{2.22}$$

where $\mathcal{A}(t)$ acts on the function $f(t,\cdot)$ as follows:

$$\begin{aligned} &\mathcal{A}(t) f(t,x) \\ &\quad := \sum_{k,l=1}^{d} (a_{kl}(t) + \langle \alpha_{\mathcal{I},kl}(t), y\rangle)\frac{\partial^2 f(t,x)}{\partial x_k \partial x_l} + \langle b(t) + \beta(t)x, \nabla_x f(t,x)\rangle \\ &\qquad - (c(t) + \langle \gamma(t), y\rangle) f(t,x) \\ &\qquad + \int_{D\setminus\{0\}} (f(t,x+\xi) - f(t,x) - \langle \nabla_{\mathcal{J}} f(t,x), \chi_{\mathcal{J}}(\xi)\rangle) m(t,\mathrm{d}\xi) \\ &\qquad + \sum_{i=1}^{m} \int_{D\setminus\{0\}} (f(t,x+\xi) - f(t,x) - \langle \nabla_{\mathcal{J}(i)} f(t,x), \chi_{\mathcal{J}(i)}(\xi)\rangle) y_i \mu_i(t,\mathrm{d}\xi). \end{aligned} \tag{2.23}$$

Moreover, (2.3) *holds for all $0 \leqslant t \leqslant T$ and $u \in \mathcal{U}$ where $\phi(t,T,u)$ and $\psi(t,T,u)$ solve the generalized Riccati equations*

$$\phi(t,T,u) = \int_t^T F(s, \psi(s,T,u))\,\mathrm{d}s, \tag{2.24}$$

$$-\partial_t \psi^{\mathscr{Y}}(t,T,u) = R^{\mathscr{Y}}\left(t, \psi^{\mathscr{Y}}(t,T,u), \mathrm{e}^{\int_t^T \beta^{\mathscr{Z}}(s)\,\mathrm{d}s} w\right), \quad \psi^{\mathscr{Y}}(T,T,u) = v, \tag{2.25}$$

$$\psi^{\mathscr{Z}}(t,T,u) = \mathrm{e}^{\int_t^T \beta^{\mathscr{Z}}(s)\,\mathrm{d}s} w, \tag{2.26}$$

with F, $R^{\mathscr{Y}}$ and $\beta^{\mathscr{Z}}$ given by (2.16)–(2.20).

Conversely, let $(a,\alpha,b,\beta,c,\gamma,m,\mu)$ be strongly admissible parameters. Then there exists a unique, strongly regular affine Markov process $(P_{t,T})$ whose associated space-time process (Θ, X) has the infinitesimal generator (2.22), *and* (2.3) *holds for all $0 \leqslant t \leqslant T$ and $u \in \mathscr{U}$ where $\phi(t,T,u)$ and $\psi(t,T,u)$ are given by* (2.24)–(2.26).

We now give some conventions and a brief summary of facts about Feller processes, the proofs of which can be found in e.g. [8, Chapter III.2]. Let (Θ, X) be the Feller process from Theorem 2.13. Since we deal with an entire family of probability measures, $(\mathbb{P}_{(r,x)})_{(r,x)\in\mathbb{R}_+\times D}$, we make the convention that "a.s." means "$\mathbb{P}_{(r,x)}$-a.s. for all $(r,x) \in \mathbb{R}_+ \times D$". Then X admits a cadlag modification, and "X" will from now on always stand for such a cadlag version. Let

$$\tau_{(\Theta,X)} := \inf\{t \in \mathbb{R}_+ \mid (\Theta,X)_{t-} = \Delta \text{ or } (\Theta,X)_t = \Delta\}$$

(remember that $(\mathbb{R}_+ \times D) \cup \{\Delta\}$ is the one-point compactification of $\mathbb{R}_+ \times D$). Then we have $(\Theta,X)_\cdot = \Delta$ on $[\tau_X, \infty)$ a.s. Hence (Θ, X) is conservative if and only if $\tau_{(\Theta,X)} = \infty$ a.s. Write $\mathscr{F}^{(r,x)}$ for the completion of $\mathscr{F}^0$ with respect to $\mathbb{P}_{(r,x)}$ and $(\mathscr{F}_t^{(r,x)})$ for the filtration obtained by adding to each $\mathscr{F}_t^0$ all $\mathbb{P}_{(r,x)}$-nullsets in $\mathscr{F}^{(r,x)}$. Define

$$\mathscr{F}_t := \bigcap_{(r,x)\in\mathbb{R}_+\times D} \mathscr{F}_t^{(r,x)}, \qquad \mathscr{F} := \bigcap_{(r,x)\in\mathbb{R}_+\times D} \mathscr{F}^{(r,x)}.$$

Then the filtrations $(\mathscr{F}_t^{(r,x)})$ and $(\mathscr{F}_t)$ are right-continuous, and (Θ, X) is still a Markov process with respect to $(\mathscr{F}_t)$, for every $\mathbb{P}_{(r,x)}$.

By convention, we call X a *semimartingale* if $(X_t 1_{\{t<\tau_{(\Theta,X)}\}})$ is a semimartingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}_{(r,x)})$ for every $(r,x) \in \mathbb{R}_+ \times D$. For the definition of the characteristics of a semimartingale with respect to χ we refer to [5, Section II.2].

Theorem 2.14. *Let $(P_{t,T})$ be strongly regular affine and $(a,\alpha,b,\beta,c,\gamma,m,\mu)$ the corresponding strongly admissible parameters. Then X is a semimartingale. If $(P_{t,T})$ is conservative then X admits the $\mathbb{P}_{(r,x)}$-characteristics (B,C,ν),*

$$B_t = \int_0^t (\tilde{b}(r+s) + \tilde{\beta}(r+s)X_s)\,\mathrm{d}s, \tag{2.27}$$

$$C_t = 2\int_0^t \left(a(r+s) + \sum_{i=1}^m \alpha_i(r+s) Y_s^i\right) \mathrm{d}s, \tag{2.28}$$

$$\nu(\mathrm{d}t, \mathrm{d}\xi) = \left(m(r+t, \mathrm{d}\xi) + \sum_{i=1}^m Y_t^i \mu_i(r+t, \mathrm{d}\xi)\right) \mathrm{d}t, \tag{2.29}$$

for every $(r,x)\in\mathbb{R}_+\times D$, *where* $\tilde b(t)\in D$ *and* $\tilde\beta(t)\in\mathbb{R}^{d\times d}$ *are given by*

$$\tilde b(t) := b(t) + \int_{D\setminus\{0\}} (\chi_{\mathscr{I}}(\xi), 0) m(t, \mathrm{d}\xi), \tag{2.30}$$

$$\tilde\beta_{kl}(t) := \begin{cases} \beta_{kl}(t) + (1-\delta_{kl}) \int_{D\setminus\{0\}} \chi_k(\xi)\,\mu_l(t,\mathrm{d}\xi) & \text{if } l\in\mathscr{I}, \\ \beta_{kl}(t) & \text{if } l\in\mathscr{J}, \end{cases} \quad \text{for } 1\leqslant k\leqslant d. \tag{2.31}$$

Moreover, let $X'=(Y',Z')$ *be a* D*-valued semimartingale defined on some filtered probability space* $(\Omega',\mathscr{F}',(\mathscr{F}'_t),\mathbb{P}')$ *with* $\mathbb{P}'[X'_0=x]=1$. *Suppose* X' *admits the characteristics* (B',C',v'), *given by* (2.27)–(2.29) *where* X *is replaced by* X'. *Then* $\mathbb{P}'\circ X'^{-1}=\mathbb{P}_{(r,x)}$.

Notions (2.30) and (2.31) are not substantial and only introduced for notational compatibility with [5]. Indeed, we simply replaced $\langle\nabla_{\mathscr{J}}f(t,x),\chi_{\mathscr{J}}(\xi)\rangle$ and $\langle\nabla_{\mathscr{J}(i)}f(t,x),\chi_{\mathscr{J}(i)}(\xi)\rangle$ in (2.23) by $\langle\nabla_x f(t,x),\chi(\xi)\rangle$, which is compensated by replacing b and β by $\tilde b$ and $\tilde\beta$, respectively.

3. Preliminary results

This section corresponds to [3, Section 3]. We now have to distinguish between forward and backward equations. Note that the Chapman–Kolmogorov equation

$$P_{s,t}P_{t,T}=P_{s,T}, \quad 0\leqslant s\leqslant t\leqslant T,$$

is a composition rule which is backward in time. This is in contrast to the common "evolution systems", say $(U_{t,s})_{0\leqslant s\leqslant t}$, arising in the study of time-inhomogeneous Cauchy problems, where the composition rule is forward in time,

$$U_{T,t}U_{t,s}=U_{T,s}, \quad 0\leqslant s\leqslant t\leqslant T,$$

see [7]. We take this distinction into account by introducing the notion

$$B_{t,T} := P_{T-t,T}. \tag{3.1}$$

Throughout we fix $T>0$, and we frequently replace "P_t" in [3] with $B_{t,T}$. This is justified since, as for "P_t", there exists a (sub)stochastic kernel, $p_{T-t,T}(x,\mathrm{d}\xi)$, such that

$$B_{t,T}f(x)=\int_D f(\xi)\,p_{T-t,T}(x,\mathrm{d}\xi), \quad \forall f\in bD,$$

Moreover, the semigroup property of "(P_t)" was not used for the analysis in [3, Sections 4 and 5].

We suppose from now on that $(P_{t,T})$ is weakly regular affine. First, we want to extend the (t,u)-range of validity of (2.3), which a priori is $[0,T]\times\partial\mathscr{U}$ and $\{T\}\times\mathscr{U}$. That is, we fix $u\in\mathscr{U}$ and see how far we can go in the $-t$-direction. Therefore, we define

$$\mathcal{O}(T) := \{(t,u) \in [0,T] \times \mathcal{U} \mid P_{s,T} f_u(0) \neq 0,\ \forall s \in [t,T]\}, \tag{3.2}$$

which contains $\{T\} \times \mathcal{U}$. The following result can be proved as [3, Lemma 3.1].

Lemma 3.1. *$P_{t,T} f_u(x)$ is jointly continuous in (t,T,u), for every $x \in D$. The set $\mathcal{O}(T)$ is simply connected and open in $[0,T] \times \mathcal{U}$, and there exists a continuous extension of $\phi(\cdot,T,\cdot)$ and $\psi(\cdot,T,\cdot)$ on $\mathcal{O}(T)$ such that (2.3) holds for all $(t,u) \in \mathcal{O}(T)$.*

Later, it can be shown that $\mathcal{O}(T) = [0,T] \times \mathcal{U}$, see Proposition 4.3.

We now derive the ODEs for $\phi(\cdot,T,\cdot)$ and $\psi(\cdot,T,\cdot)$. First, we have

$$\phi(T,T,u) = 0, \quad \psi(T,T,u) = u, \quad \forall u \in \mathcal{U}. \tag{3.3}$$

Let $(t,u) \in \mathcal{O}(T)$ and $s \leqslant t$ such that $(s,u) \in \mathcal{O}(T)$ and $(s,\psi(t,T,u)) \in \mathcal{O}(t)$. Lemma 3.1 and the Chapman–Kolmogorov equation yield

$$\begin{aligned} \mathrm{e}^{\phi(s,T,u)+\langle\psi(s,T,u),x\rangle} &= \int_D p_{s,t}(x,\mathrm{d}\xi) \int_D p_{t,T}(\xi,\mathrm{d}\tilde{\xi}) f_u(\tilde{\xi}) \\ &= \mathrm{e}^{\phi(t,T,u)} \int_D p_{s,t}(x,\mathrm{d}\xi) \mathrm{e}^{\langle\psi(t,T,u),\xi\rangle} \\ &= \mathrm{e}^{\phi(t,T,u)+\phi(s,t,\psi(t,T,u))+\langle\psi(s,t,\psi(t,T,u)),x\rangle}, \quad \forall x \in D, \end{aligned} \tag{3.4}$$

hence

$$\phi(s,T,u) = \phi(t,T,u) + \phi(s,t,\psi(t,T,u)), \tag{3.5}$$

$$\psi(s,T,u) = \psi(s,t,\psi(t,T,u)). \tag{3.6}$$

According to Definition 2.3 and Lemma 3.1, the left-hand derivatives

$$F(t,u) := -\partial_s^- \phi(s,t,u)|_{s=t}, \tag{3.7}$$

$$R(t,u) = (R^{\mathcal{Y}}(t,u), R^{\mathcal{Z}}(t,u)) := -\partial_s^- \psi(s,t,u)|_{s=t} \tag{3.8}$$

exist and are measurable in $(t,u) \in \mathbb{R}_{++} \times \mathcal{U}$. From (3.5) and (3.6) we conclude that, for all $(t,u) \in \mathcal{O}(T)$ with $t > 0$,

$$-\partial_t^- \phi(t,T,u) = F(t,\psi(t,T,u)), \tag{3.9}$$

$$-\partial_t^- \psi(t,T,u) = R(t,\psi(t,T,u)). \tag{3.10}$$

As for the mappings F and R, we observe that we have from (2.4),

$$\tilde{\mathcal{A}}(t) f_u(x) = \partial_s^+ B_{s,t} f_u(x)|_{s=0} = (F(t,u) + \langle R(t,u),x\rangle) f_u(x), \tag{3.11}$$

for all $x \in D$, and hence

$$F(t,u) = \tilde{\mathcal{A}}(t) f_u(0), \tag{3.12}$$

$$R_i(t,u) = -F(t,u) + \frac{\tilde{\mathcal{A}}(t) f_u(e_i)}{f_u(e_i)}, \quad i = 1,\ldots,d, \tag{3.13}$$

for all $(t,u) \in \mathbb{R}_{++} \times \mathcal{U}$. Hence it is enough to find $\tilde{\mathcal{A}}(t)f_u(x)$ on the coordinate axes, that is, for $x = re_i$ and $x = se_j$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, $r \geqslant 0$, $s \in \mathbb{R}$, in order to determine F and R. This can be done exactly as in [3, Sections 4 and 5], see Section 7 below.

4. Generalized Riccati equations

Let $(a, \alpha, b, \beta^{\mathcal{Y}}, \beta^{\mathcal{Z}}, c, \gamma, m, \mu)$ be some strongly admissible parameters, and let $F(t,u)$ and $R(t,u) = (R^{\mathcal{Y}}(t,u), R^{\mathcal{Z}}(t,u))$ be given by (2.16)–(2.18). In this section we discuss the generalized Riccati equations

$$-\partial_t \Phi(t,T,u) = F(t, \Psi(t,T,u)), \quad \Phi(T,T,u) = 0, \tag{4.1}$$

$$-\partial_t \Psi(t,T,u) = R(t, \Psi(t,T,u)), \quad \Psi(T,T,u) = u, \quad 0 \leqslant t \leqslant T. \tag{4.2}$$

Observe that (4.1) is a trivial differential equation. A solution of Eqs. (4.1)–(4.2) is a pair of continuously differentiable mappings $\Phi(\cdot,T,u)$ and $\Psi(\cdot,T,u) = (\Psi^{\mathcal{Y}}(\cdot,T,u), \Psi^{\mathcal{Z}}(\cdot,T,u))$ from $[0,T]$ into $\mathbb{C}$ and $\mathbb{C}^m \times \mathbb{C}^n$, respectively, satisfying (4.1)–(4.2) or, equivalently,

$$\Phi(t,T,u) = \int_t^T F(s, \Psi(s,T,u))\,\mathrm{d}s, \tag{4.3}$$

$$-\partial_t \Psi^{\mathcal{Y}}(t,T,u) = R^{\mathcal{Y}}\left(t, \Psi^{\mathcal{Y}}(t,T,u), \mathrm{e}^{\int_t^T \beta^{\mathcal{Z}}(s)\,\mathrm{d}s} w\right), \quad \Psi^{\mathcal{Y}}(T,T,u) = v, \tag{4.4}$$

$$\Psi^{\mathcal{Z}}(t,T,u) = \mathrm{e}^{\int_t^T \beta^{\mathcal{Z}}(s)\,\mathrm{d}s} w, \tag{4.5}$$

for $0 \leqslant t \leqslant T$ and $u = (v,w) \in \mathbb{C}^m \times \mathbb{C}^n$. As shown in [3], $\mathbb{R}^{\mathcal{Y}}(t,\cdot)$ may fail to be Lipschitz continuous at $\partial\mathcal{U}$. Yet the following can be proved.

Proposition 4.1. *For every $T > 0$ and $u \in \mathcal{U}^0$ there exists a unique solution $\Phi(\cdot,T,u)$ and $\Psi(\cdot,T,u)$ of (4.1)–(4.2) with values in $\mathbb{C}_-$ and $\mathcal{U}^0$, respectively. Moreover, $\Phi(t,T,u)$ and $\Psi(t,T,u)$ are jointly continuous in (t,T,u), for $0 \leqslant t \leqslant T$ and $u \in \mathcal{U}^0$, and*

$$\partial_T^+ \Phi(t,T,u)|_{T=t} = F(t,u), \tag{4.6}$$

$$\partial_T^+ \Psi(t,T,u)|_{T=t} = R(t,u), \tag{4.7}$$

for all $(t,u) \in \mathbb{R}_+ \times \mathcal{U}^0$.

Proof. We only have to consider (4.4). We set

$$(\alpha(t), \beta^{\mathcal{Y}}(t), \beta^{\mathcal{Z}}(t), \gamma(t), \mu(t, \mathrm{d}\xi)) := (\alpha(0), \beta^{\mathcal{Y}}(0), \beta^{\mathcal{Z}}(0), \gamma(0), \mu(0, \mathrm{d}\xi)) \quad \text{for } t \leqslant 0.$$

Let $T > 0$, and consider the initial value problem

$$\partial_t g(t,T,u) = R^{\mathcal{Y}}\left(T - t, g(t,T,u), \mathrm{e}^{\int_{T-t}^T \beta^{\mathcal{Z}}(s)\,\mathrm{d}s} w\right), \quad g(0,T,u) = v. \tag{4.8}$$

Obviously, $g(t,T,u)$ satisfies (4.8) for $0\leqslant t\leqslant T$ if and only if

$$\Psi^{\mathscr{Y}}(\cdot,T,u)=g(T-\cdot,T,u)$$

is a solution of (4.4). It follows as in [3, Lemma 5.3] that the map

$$(t,v,w)\mapsto R^{\mathscr{Y}}\left(T-t,v,\mathrm{e}^{\int_{T-t}^{T}\beta^{\mathscr{Z}}(s)\,\mathrm{d}s}w\right):\mathbb{R}\times\mathscr{U}\to\mathbb{C}^m \tag{4.9}$$

is analytic in $v\in\mathbb{C}^m_{--}$ with, jointly in (t,v,w), continuous v-derivatives on $\mathbb{R}\times\mathscr{U}^0$. Hence (4.9) is locally Lipschitz continuous in $v\in\mathbb{C}^m_{--}$, uniformly in (t,w) on compact sets in $\mathbb{R}\times\mathrm{i}\mathbb{R}^n$. This implies that, for every $u=(v,w)\in\mathscr{U}^0$, there exists a unique $\mathbb{C}^m_{--}$-valued local solution $g(\cdot,T,u)$ to (4.8) with maximal lifetime in $\mathbb{C}^m_{--}$

$$\tau_{T,u}:=\lim_{n\to\infty}\inf\{t\in[0,T]\mid \|g(t,T,u)\|\geqslant n \text{ or } g(t,T,u)\in\mathrm{i}\mathbb{R}^m\}\leqslant T.$$

Since all coefficients in (4.8) depend continuously on t, it follows literally as in the proof of [3, Proposition 6.1] that $\tau_{T,u}=T$ and that $g(t,T,u)\in\mathbb{C}^m_{--}$ for all $(t,u)\in[0,T]\times\mathscr{U}^0$.

The continuity of $g(t,T,u)$ in (t,T,u) for $0\leqslant t\leqslant T$ and $u\in\mathscr{U}^0$ is a standard result and follows from the regularity properties of (4.9), see e.g. [1, Section II.8].

Now let $(t,u)\in\mathbb{R}_+\times\mathscr{U}^0$. Eq. (4.2) yields

$$\Psi(t,T,u)=u+\int_t^T R(s,\Psi(s,T,u))\,\mathrm{d}s,\quad T\geqslant t, \tag{4.10}$$

and hence

$$\frac{\Psi(t,t+h,u)-\Psi(t,t,u)}{h}=\frac{1}{h}\int_t^{t+h}R(s,\Psi(s,t+h,u))\,\mathrm{d}s\to R(t,u)$$
$$\text{for } h\downarrow 0, \tag{4.11}$$

by the joint continuity of $R(s,\Psi(s,t+h,u))$ in s and h. Whence (4.7), and similarly one shows (4.6). □

Remark 4.2. It is difficult to say more about the T-differentiability of $\Psi(t,T,u)$ for $T>t$ in general. In view of (4.10) we have

$$\begin{aligned}\frac{\Psi(t,T+h,u)-\Psi(t,T,u)}{h}&=\frac{1}{h}\int_T^{T+h}R(s,\Psi(s,T+h,u))\,\mathrm{d}s\\&\quad+\int_t^T\frac{R(s,\Psi(s,T+h,u))-R(s,\Psi(s,T,u))}{h}\,\mathrm{d}s.\end{aligned}$$

Since $R(s,v,w)$ is not differentiable in w in general ($\mu_i(t,\mathrm{d}\xi)$ in (2.17) can be any probability distribution on $\{0\}\times\mathbb{R}^n\subset D$ with infinite first moment), it is not clear what the limit for the second integrand should be as $h\to 0$. The candidate for $\partial_T\Psi(t,T,u)$ would be the solution to

$$\partial_T\Psi(t,T,u)=R(T,u)+\int_t^T D_uR(s,\Psi(s,T,u))\cdot\partial_T\Psi(s,T,u)\,\mathrm{d}s,$$

which is well-defined if $D_u R(s,u)$ exists and is continuous (which again is implied by a first moment condition on the measures μ_i, see [3, Lemmas 5.3 and 6.5]). The argument in (4.11) only works for $T = t$ in general.

Now let $(P_{t,T})$ be strongly regular affine and $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ the corresponding strongly admissible parameters. Recall the definition of $\mathcal{O}(T)$, see (3.2).

Proposition 4.3. *We have* $\mathcal{O}(T) = [0,T] \times \mathcal{U}$, *and* $\phi(t,T,u)$ *and* $\psi(t,T,u)$ *satisfy* (4.1)–(4.7), *for all* $0 \leqslant t \leqslant T$ *and* $u \in \mathcal{U}$.

Proof. Follows as in [3, Proposition 6.4] and by (4.11). □

5. $\mathscr{C} \times \mathscr{C}^{(m,n)}$-semiflows

As in [3, Section 7] we provide here the tools for proving the existence of weakly and strongly regular affine processes. We denote by $\mathscr{C}$ the convex cone of continuous functions $\phi : \mathcal{U} \to \mathbb{C}_+$ of the form

$$\phi(u) = \langle Aw, w \rangle + \langle B, u \rangle - C + \int_{D \setminus \{0\}} (\mathrm{e}^{\langle u, \xi \rangle} - 1 - \langle w, \chi_{\mathscr{J}}(\xi) \rangle) M(\mathrm{d}\xi), \tag{5.1}$$

for $u = (v, w) \in \mathcal{U}$, where $A \in \mathrm{Sem}^n$, $B \in D$, $C \in \mathbb{R}_+$ and $M(\mathrm{d}\xi)$ is a nonnegative Borel measure on $D \setminus \{0\}$ integrating $\langle \chi_{\mathscr{J}}(\xi), \mathbf{1} \rangle + \|\chi_{\mathscr{J}}(\xi)\|^2$. Moreover, we define the convex cone $\mathscr{C}^{(m,n)} \subset \mathscr{C}^m \times \mathscr{C}^n$ of mappings $\psi : \mathcal{U} \to \mathcal{U}$ by

$$\mathscr{C}^{(m,n)} := \{\psi = (\psi^{\mathscr{Y}}, \psi^{\mathscr{Z}}) \mid \psi^{\mathscr{Y}} \in \mathscr{C}^m \text{ and } \psi^{\mathscr{Z}}(v, w) = Bw, \text{ for some } B \in \mathbb{R}^{n \times n}\}.$$

We recall some basic facts about $\mathscr{C}$ and $\mathscr{C}^{(m,n)}$, see [3, Lemma 7.1 and Proposition 7.2].

Lemma 5.1. *There exists a unique and infinitely divisible sub-stochastic measure* μ *on* D *such that*

$$\int_D f_u \, \mathrm{d}\mu = \mathrm{e}^{\phi(u)}, \quad \forall u \in \mathcal{U}, \tag{5.2}$$

if and only if $\phi \in \mathscr{C}$. *Moreover, representation* (5.1) *of* $\phi(u)$ *by* A, B, C *and* M *is unique.*

Proposition 5.2. *Let* $\phi, \phi_k \in \mathscr{C}$ *and* $\psi, \psi_k \in \mathscr{C}^{(m,n)}$, $k \in \mathbb{N}$.

(i) *For every* $x \in D$ *there exists a unique and infinitely divisible sub-stochastic measure* $\mu(x, \mathrm{d}\xi)$ *on* D *such that*

$$\int_D f_u(\xi) \, \mu(x, \mathrm{d}\xi) = \mathrm{e}^{\langle \psi(u), x \rangle}, \quad \forall u \in \mathcal{U}.$$

(ii) *The composition* $\phi \circ \psi$ *is in* $\mathscr{C}$.
(ii) *The composition* $\psi_1 \circ \psi$ *is in* $\mathscr{C}^{(m,n)}$.
(iv) *If* ϕ_k *converges pointwise to a continuous function* ϕ^* *on* $\mathcal{U}^0$, *then* ϕ^* *has a continuous extension on* $\mathcal{U}$ *and* $\phi^* \in \mathscr{C}$.

(v) *If* ψ_k *converges pointwise to a continuous mapping* ψ^* *on* $\mathscr{U}^0$, *then* ψ^* *has a continuous extension on* $\mathscr{U}$ *and* $\psi^* \in \mathscr{C}^{(m,n)}$.

An extension of [3, Definition 7.3] is now straightforward.

Definition 5.3. A two-parameter family $\{(\phi_{t,T}, \psi_{t,T})\}_{0 \leqslant t \leqslant T}$ of elements in $\mathscr{C} \times \mathscr{C}^{(m,n)}$ is called a (time-inhomogeneous) $\mathscr{C} \times \mathscr{C}^{(m,n)}$-*semiflow* if

$$\phi_{s,T}(u) = \phi_{t,T}(u) + \phi_{s,t}(\psi_{t,T}(u)) \quad \text{and} \quad \phi_{T,T} = 0,$$

$$\psi_{s,T}(u) = \psi_{s,t}(\psi_{t,T}(u)) \quad \text{and} \quad \psi_{T,T}(u) = u,$$

for all $0 \leqslant s \leqslant t \leqslant T$ and $u \in \mathscr{U}$.

It is called a *weakly regular* $\mathscr{C} \times \mathscr{C}^{(m,n)}$-*semiflow* if $\phi_{t,T}(u)$ and $\psi_{t,T}(u)$ are jointly continuous in (t, T) and the left-hand derivatives

$$F_t(u) = \partial_s^- \phi_{s,t}(u)|_{s=t} \quad \text{and} \quad R_t(u) = \partial_s^- \psi_{s,t}(u)|_{s=t}$$

exist for all $(t, u) \in \mathbb{R}_{++} \times \mathscr{U}$ and are continuous at $u = 0$ for all $t > 0$.

Here is the link to weakly regular affine processes.

Proposition 5.4. *Suppose* $\{(\phi_{t,T}, \psi_{t,T})\}_{0 \leqslant t \leqslant T}$ *is a weakly regular* $\mathscr{C} \times \mathscr{C}^{(m,n)}$-*semiflow. Then there exists a unique weakly regular affine Markov process with state-space D and exponents* $\phi(t, T, u) = \phi_{t,T}(u)$ *and* $\psi(t, T, u) = \psi_{t,T}(u)$.

Proof. This follows as [3, Proposition 7.4]. $\square$

Definition 5.5. A weakly regular $\mathscr{C} \times \mathscr{C}^{(m,n)}$-semiflow is called *strongly regular* if the induced Markov process (Proposition 5.4) is strongly regular affine.

The counterpart to Proposition 5.4 is the following.

Proposition 5.6. *Let* $(a, \alpha, b, \beta^{\mathscr{Y}}, \beta^{\mathscr{Z}}, c, \gamma, m, \mu)$ *be strongly admissible parameters. Then the solution* Φ *and* Ψ *of* (4.1)–(4.2) *uniquely defines a strongly regular* $\mathscr{C} \times \mathscr{C}^{(m,n)}$-*semiflow* $\{(\phi_{t,T}, \psi_{t,T})\}_{0 \leqslant t \leqslant T}$ *by* $\phi_{t,T} = \Phi(t, T, \cdot)$ *and* $\psi_{t,T} = \Psi(t, T, \cdot)$.

Proof. We fix $T > 0$ and first suppose that

$$\int_{D \setminus \{0\}} \chi_i(\xi)\, \mu_i(t, \mathrm{d}\xi) < \infty, \tag{5.3}$$

$$\alpha_{i,ik}(t) = \alpha_{i,ki}(t) = 0, \quad \forall k \in \mathscr{J}(i), \tag{5.4}$$

for all $t \in [0, T]$ and $i \in \mathscr{I}$. Consequently, $R_i^{\mathscr{Y}}$ can be written in the form

$$R_i^{\mathscr{Y}}(t, u) = \tilde{R}_i^{\mathscr{Y}}(t, u) - c_i v_i \text{ with } \tilde{R}_i^{\mathscr{Y}}(t, \cdot) \in \mathscr{C} \quad \text{and} \quad c_i = \sup_{t \in [0,T]} |\beta_{ii}^{\mathscr{Y}}(t)|,$$

for $t \in [0,T]$ and $i \in \mathscr{I}$. Then Eq. (4.4) is equivalent to the following integral equations

$$\Psi_i^{\mathscr{Y}}(t,T,u) = \mathrm{e}^{-c_i(T-t)} v_i + \int_t^T \mathrm{e}^{-c_i(s-t)} \tilde{R}_i^{\mathscr{Y}}(s, \Psi(s,T,u))\,\mathrm{d}s, \quad i \in \mathscr{I}.$$

By a classical fixed point argument, the solution $\Psi_i^{\mathscr{Y}}(t,T,u)$ is the pointwise limit of the sequence $(\Psi_i^{\mathscr{Y},(k)}(t,T,u))_{k\in\mathbb{N}_0}$, for $(t,u) \in [0,T] \times \mathscr{U}^0$, obtained by the iteration

$$\Psi_i^{\mathscr{Y},(0)}(t,T,u) = v_i,$$

$$\Psi_i^{\mathscr{Y},(k+1)}(t,T,u) = \mathrm{e}^{-c_i(T-t)} v_i + \int_t^T \mathrm{e}^{-c_i(s-t)} \tilde{R}_i^{\mathscr{Y}}\left(s, \Psi^{\mathscr{Y},(k)}(s,T,u), \mathrm{e}^{\int_s^T \beta^{\mathscr{X}}(r)\,\mathrm{d}r} w\right) \mathrm{d}s.$$

Proposition 5.2(ii) and the convex cone property of $\mathscr{C}$ yield $\Psi_i^{\mathscr{Y},(k)}(t,T,\cdot) \in \mathscr{C}$, for all $k \in \mathbb{N}_0$. In view of Proposition 5.2(iv) there exists a unique continuous extension of $\Psi_i^{\mathscr{Y}}(\cdot,T,\cdot)$ on $[0,T] \times \mathscr{U}$, and $\Psi_i^{\mathscr{Y}}(t,T,\cdot) \in \mathscr{C}$. Hence $\Psi(t,T,\cdot) \in \mathscr{C}^{(m,n)}$. Since $F(t,\cdot) \in \mathscr{C}$, by Proposition 5.2(ii) also $\Phi(t,T,\cdot) = \int_t^T F(s,\Psi(s,T,\cdot))\,\mathrm{d}s \in \mathscr{C}$ and the proposition is proved if (5.3)–(5.4) hold. For the general case it is enough to notice that the solution of (4.4) depends continuously on the right-hand side of (4.4) with respect to uniform convergence on compacts. Now Lemma 5.7 below completes the proof. □

Lemma 5.7. *Let $i \in \mathscr{I}$ and $T > 0$. There exists a sequence of functions $(g_k)_{k\in\mathbb{N}}$ which converges uniformly on compacts in* $[0,T] \times \mathscr{U}$ *to $R_i^{\mathscr{Y}}$. Moreover, every g_k is of the form* (2.17), *satisfies the corresponding strong admissibility conditions* (2.13) *and* (2.14) *and also* (5.3), (5.4).

Proof. The proof is essentially the same as for [3, Lemma 7.5], we only have to clarify a few points concerning the t-dependence of the parameters. We consider a sequence of functions $\rho^{(k)} \in C_b(D)$ with $0 \leqslant \rho^{(k)} \leqslant \rho^{(k+1)} \leqslant 1$ and

$$\rho^{(k)}(\xi) = \begin{cases} 0 & \text{for } \|\xi\| \leqslant 1/k, \\ 1 & \text{for } \|\xi\| > 2/k. \end{cases}$$

Now introduce the finite measures on $D \backslash \{0\}$,

$$\mu_i^{(k)}(t,\mathrm{d}\xi) := \rho^{(k)}(\xi)\mu_i(t,\mathrm{d}\xi), \quad k \in \mathbb{N},$$

and denote by $\tilde{g}_k$ the corresponding map given by (2.17) with μ_i replaced by $\mu_i^{(k)}$. Note that $\tilde{g}_k$ satisfies (5.3) and the corresponding strong admissibility conditions (2.13) and (2.14), since $\rho^{(k)} \in C_b(D)$.

We now write

$$d(\xi) := \langle \chi_{\mathscr{I}(i)}(\xi), \mathbf{1} \rangle + \|\chi_{\mathscr{J}(i)}(\xi)\|^2.$$

Then, for fixed $t \in [0,T]$, the bounded measures

$$\overline{\mu}_i^{(k)}(t,\mathrm{d}\xi) := d(\xi)\mu_i^{(k)}(t,\mathrm{d}\xi), \quad k \in \mathbb{N},$$

converge weakly on $D\backslash\{0\}$ to

$$\overline{\mu}_i(t,\mathrm{d}\xi) := d(\xi)\mu_i(t,\mathrm{d}\xi).$$

The function h_u is defined by

$$h_u(\xi) = \frac{\mathrm{e}^{\langle u,\xi\rangle} - 1 - \langle u_{\mathscr{J}(i)}, \chi_{\mathscr{J}(i)}(\xi)\rangle}{d(\xi)}.$$

Let $K \subset \mathscr{U}$ be compact. Then there is a constant C such that

$$\begin{aligned}&\left|\int_{D\backslash\{0\}} h_u(\xi)\,\overline{\mu}_i^{(k)}(t,\mathrm{d}\xi) - \int_{D\backslash\{0\}} h_u(\xi)\,\overline{\mu}_i(t,\mathrm{d}\xi)\right| \\ &\quad = \left|\int_{D\backslash\{0\}} h_u(\xi)(\rho^{(k)}(\xi)-1)\overline{\mu}_i(t,\mathrm{d}\xi)\right| \leqslant C\int_{D\backslash\{0\}} |1-\rho^{(k)}(\xi)|\overline{\mu}_i(t,\mathrm{d}\xi),\end{aligned}$$

for all $u \in K$ and $k \in \mathbb{N}$. Observe that, by construction,

$$\int_{D\backslash\{0\}} |1-\rho^{(k)}(\xi)|\overline{\mu}_i(t,\mathrm{d}\xi) \downarrow 0 \quad \text{as } k \to \infty$$

monotonically for all $t \in [0,T]$. In view of (2.14), each $\int_{D\backslash\{0\}} |1-\rho^{(k)}(\xi)|\overline{\mu}_i(t,\mathrm{d}\xi)$ is continuous in $t \in [0,T]$. Hence

$$\sup_{t\in[0,T]} \int_{D\backslash\{0\}} |1-\rho^{(k)}(\xi)|\overline{\mu}_i(t,\mathrm{d}\xi) \to 0 \quad \text{as } k \to \infty, \tag{5.5}$$

by a theorem of Dini [2]. As a consequence we have that $\tilde{g}_k$ converge to $R_i^{\mathscr{Y}}$ uniformly on compacts in $[0,T]\times\mathscr{U}$.

It remains to show that, for all $k \in \mathbb{N}$, there exists a sequence $(\tilde{g}_k^{(l)})_{l\in\mathbb{N}}$ of functions that are of the form (2.17) and satisfy (5.3)–(5.4), such that $\tilde{g}_k^{(l)} \to \tilde{g}_k$ uniformly on compacts in $[0,T]\times\mathscr{U}$. The lemma is then proved by choosing an appropriate subsequence $\tilde{g}_k^{(l_k)} =: g_k$, $k \in \mathbb{N}$.

To simplify the notation we suppress the index k in what follows and assume that μ_i already satisfies (5.3). We proceed literally as in the proof of [3, Lemma 7.5] and construct $\tilde{g}^{(l)}(t,u)$ that is of the form (2.17) and satisfies (5.3)–(5.4). Adapting the notation from [3] we then derive

$$\tilde{g}^{(l)}(t,u) - R_i^{\mathscr{Y}}(t,u) = \frac{2}{p^*(t)}(\tilde{h}_u(\lambda_l(t)) - \tilde{h}_u(\lambda_\infty(t))),$$

where $p^*(t)$, $\lambda_l(t)$ and $\lambda_\infty(t)$ are continuous in $t \in [0,T]$, and $\lambda_l \to \lambda_\infty$ uniformly in $t \in [0,T]$. It then follows that $\tilde{g}^{(l)}$ converges to $R_i^{\mathscr{Y}}$ uniformly on compacts in $[0,T]\times\mathscr{U}$, and the lemma is proved. □

6. Feller property

Let $(P_{t,T})$ be strongly regular affine and $(a,\alpha,b,\beta(\beta^{\mathscr{Y}},\beta^{\mathscr{Z}}),c,\gamma,m,\mu)$ the corresponding strongly admissible parameters, given by Theorem 2.7. In this section we show that $(\overline{P}_t)$ in (2.2) shares the Feller property, and we establish a connection between the strongly admissible parameters and the infinitesimal generator of $(\overline{P}_t)$.

First, we provide some preliminary results. For $f \in C^{1,2}(\mathbb{R}_+ \times D)$ and a closed sub-set $I \times U$ in $\mathbb{R}_+ \times D$ we write

$$[f]_{\mathscr{Y}}(t,x) := (1+\|y\|)\left(|f(t,x)| + \|\nabla_x f(t,x)\| + \sum_{k,l=1}^{d} \left| \frac{\partial^2 f(t,x)}{\partial x_k \partial x_l} \right| \right), \tag{6.1}$$

$$\begin{aligned} [f]_{\mathscr{Z}}(t,x) &:= |\langle z, \beta^{\mathscr{Z}}(t) \nabla_{\mathscr{J}} f(t,x)\rangle|, \\ \|f\|_{\sharp; I\times U} &:= \sup_{(t,x)\in I\times U} \{ |\partial_t f(t,x)| + [f]_{\mathscr{Y}}(t,x) + [f]_{\mathscr{Z}}(t,x)\}. \end{aligned} \tag{6.2}$$

Let $\mathscr{A}(t)$ be given by (2.23). The meaning of (6.1)–(6.2) becomes clear by the next lemma.

Lemma 6.1. *For any $f \in C^{1,2}(\mathbb{R}_+ \times D)$ we have*

$$|\mathscr{A}(t)f(t,x)| \leqslant C \times AP(t) \left(\sup_{\xi \in x+D} [f]_{\mathscr{Y}}(t,\xi) + [f]_{\mathscr{Z}}(t,x) \right), \tag{6.3}$$

for all $(t,x) \in \mathbb{R}_+ \times D$, with the continuous function

$$\begin{aligned} AP(t) := {} & \|a(t)\| + \|\alpha(t)\| + \|b(t)\| + \|\beta(t)\| + |c(t)| + \|\gamma(t)\| \\ & + M(t, D\backslash\{0\}) + \sum_{i\in\mathscr{I}} \mathscr{M}_i(t, D\backslash\{0\}), \end{aligned}$$

see (2.13) and (2.14), *and the constant C only depends on d.*

Proof. This follows as [3, Lemma 8.1]. □

We write

$$\overline{\mathscr{A}}^{\sharp} f(t,x) := \partial_t f(t,x) + \mathscr{A}(t) f(t,x),$$

for $f \in C^{1,2}(\mathbb{R}_+ \times D)$, and define the linear space

$$\mathscr{D}^{\sharp} := \{ f \in C^{1,2}(\mathbb{R}_+ \times D) \mid \partial_t f,\, [f]_{\mathscr{Y}},\, [f]_{\mathscr{Z}} \in C_0(\mathbb{R}_+ \times D)\}.$$

Let $f \in \mathscr{D}^{\sharp}$ and $(t,x) \in \mathbb{R}_+ \times D$. Then there exists $\xi_0 \in D$ such that the right-hand side of (6.3) equals

$$C \times AP(t)([f]_{\mathscr{Y}}(t, x+\xi_0) + [f]_{\mathscr{Z}}(t,x)).$$

But this tends to zero if $\|(t,x)\| \to \infty$, hence $\overline{\mathscr{A}}^{\sharp} f \in C_0(\mathbb{R}_+ \times D)$.

Lemma 6.2. *If $f \in C^{1,2}(\mathbb{R}_+ \times D)$ is such that*

$$\partial_t f,\, [f]_{\mathscr{Y}},\, \overline{\mathscr{A}}^{\sharp} f \in C_0(\mathbb{R}_+ \times D),$$

then also $[f]_{\mathscr{Z}} \in C_0(\mathbb{R}_+ \times D)$ and hence $f \in \mathscr{D}^{\sharp}$.

Proof. With the same arguments as for Lemma 6.1 [3, Lemma 8.1], it follows from (2.23) and (2.21) that

$$[f]_{\mathscr{X}}(t,x) \leqslant |\mathscr{A}(t)f(t,x)| + C \times AP(t)[f]_{\mathscr{Y}}(t, x+\xi_0),$$

for some $\xi_0 \in D$, for all $(t,x) \in \mathbb{R}_+ \times D$. This yields the claim. $\square$

Proposition 6.3. *The semigroup $(\overline{P}_t)$ is Feller. Let $\overline{\mathscr{A}}$ be its infinitesimal generator. Then $C_c^\infty(\mathbb{R}_+ \times D)$ is a core of $\overline{\mathscr{A}}$, $C_c^{1,2}(\mathbb{R}_+ \times D) \subset \mathscr{D}(\overline{\mathscr{A}})$ and* (2.22) *holds for $f \in C_c^{1,2}(\mathbb{R}_+ \times D)$.*

Proof. We define the sets of functions $\Theta_0 \subset \Theta \subset \mathscr{S}_d$ (= the space of rapidly decreasing C^∞-functions on R^d) and their complex linear hulls $\mathscr{L}(\Theta_0)$ and $\mathscr{L}(\Theta)$, respectively, as in the proof of [3, Proposition 8.2]. That is, any $h \in \Theta_0$ ($h \in \Theta$) is of the form

$$h(y,z) = \int_{\mathbb{R}^n} f_{(v,\mathrm{i}q)}(y,z) g(q)\,\mathrm{d}q,$$

for some $v \in \mathbb{C}^m_{--}$ and $g \in C_c^\infty(\mathbb{R}^n)$ ($g \in \mathscr{S}_n$). In addition we define

$$\overline{\Theta}_0 := \{\theta h \mid \theta \in C_c^\infty(\mathbb{R}_+),\, h \in \Theta_0\}, \quad \overline{\Theta} := \{\theta h \mid \theta \in C_c^\infty(\mathbb{R}_+),\, h \in \Theta\}.$$

Now let $f = \theta h \in \overline{\Theta}_0$. Proposition 4.3 and (4.5) imply

$$\begin{aligned}\overline{P}_s f(t,x) &= \theta(t+s) P_{t,t+s} h(x) \\ &= \theta(t+s) \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \exp(\int_t^{t+s} \beta^{\mathscr{Z}}(r)\,\mathrm{d}r) q, z\rangle} \mathrm{e}^{\phi(t,t+s,v,\mathrm{i}q) + \langle \psi^{\mathscr{Y}}(t,t+s,v,\mathrm{i}q), y\rangle} g(q)\,\mathrm{d}q, \end{aligned} \tag{6.4}$$

pointwise. By (4.6) and (4.7), Proposition 4.3 and dominated convergence we thus obtain

$$\begin{aligned}&\partial_s^+ \overline{P}_s f(t,x)|_{s=0} \\ &\quad = \partial_t \theta(t) h(x) + \theta(t) \int_{\mathbb{R}^n} (F(t,v,\mathrm{i}q) + \langle R(t,v,\mathrm{i}q), x\rangle) f_{(v,\mathrm{i}q)}(x) g(q)\,\mathrm{d}q \\ &\quad = \partial_t \theta(t) h(x) + \theta(t) \int_{\mathbb{R}^n} \mathscr{A}(t) f_{(v,\mathrm{i}q)}(x) g(q)\,\mathrm{d}q \\ &\quad = \overline{\mathscr{A}}^\sharp f(t,x), \end{aligned} \tag{6.5}$$

and in particular

$$\lim_{s \downarrow 0} \overline{P}_s f(t,x) = f(t,x),$$

pointwise for all $(t,x) \in \mathbb{R}_+ \times D$.

With the same arguments as in [3], and since the complex linear span $\mathscr{L}(\overline{\Theta}_0)$ of $\overline{\Theta}_0$ is dense in $C_0(\mathbb{R}_+ \times D)$, we conclude that $(\overline{P}_t)$ is Feller.

Moreover, in view of [9, Lemma 31.7], (6.5) and the easy fact that $\overline{\Theta_0} \subset \mathscr{D}^\sharp$ (and hence $\overline{\mathscr{A}}^\sharp f \in C_0(\mathbb{R}_+ \times D)$), we derive that $\mathscr{L}(\overline{\Theta}_0) \subset \mathscr{D}(\overline{\mathscr{A}})$. Since $\mathscr{L}(\overline{\Theta_0})$ is $\|\cdot\|_{\sharp;\mathbb{R}_+ \times D}$-dense in $\mathscr{L}(\overline{\Theta})$, we easily infer from (6.3) and the closedness of $\overline{\mathscr{A}}$ that also $\mathscr{L}(\overline{\Theta}) \subset \mathscr{D}(\overline{\mathscr{A}})$ and (2.22) holds for all $f \in \mathscr{L}(\overline{\Theta})$.

With a Stone–Weierstrass argument (similar to [3, Lemma 8.1]) one can see that $\mathscr{L}(\overline{\Theta}_0)$ is dense in $C_c^{1,2}(\mathbb{R}_+ \times D)$ with respect to the norm

$$\sup_{(t,x)\in\mathbb{R}_+\times D}\left(|\partial_t f(t,x)| + |f(t,x)| + \|\nabla_x f(t,x)\| + \sum_{k,l=1}^{d}\left|\frac{\partial^2 f(t,x)}{\partial x_k \partial x_l}\right|\right).$$

Similarly, as in [3] one can then construct, for any given $h \in C_c^{1,2}(\mathbb{R}_+ \times D)$, a sequence (h_k) in $\mathscr{L}(\overline{\Theta})$ with $\|h - h_k\|_{\sharp;\mathbb{R}_+\times D} \to 0$ as $k \to \infty$. Also one can show that $C_c^{1,2}(\mathbb{R}_+ \times D)$ is $\|\cdot\|_{\sharp;\mathbb{R}_+\times D}$-dense in $\mathscr{D}^\sharp$. Again using (6.3) and the closedness of $\overline{\mathscr{A}}$, we conclude that $\mathscr{D}^\sharp \subset \overline{\mathscr{A}}$ and (2.22) holds for all $f \in \mathscr{D}^\sharp$.

It remains to consider cores. We show that

$$\overline{P}_s \mathscr{L}(\overline{\Theta}_0) \subset \mathscr{D}^\sharp, \quad \forall s \in \mathbb{R}_+. \tag{6.6}$$

Let $f = \theta h \in \overline{\Theta}_0$. Since $\psi^{\mathscr{Y}}(t,T,u) \in \mathbb{C}^m_{--}$ for all $u \in \mathscr{U}^0$ and $0 \leqslant t \leqslant T$ (Proposition 4.1), we see from (6.4) that $[\overline{P}_s f]_{\mathscr{Y}} \in C_0(\mathbb{R}_+ \times D)$ for all $s \in \mathbb{R}_+$. Let $s > 0$, then

$$\overline{P}_s f \in \overline{P}_s \overline{\Theta}_0 \subset \overline{P}_s \mathscr{D}(\mathscr{A}) \subset \mathscr{D}(\mathscr{A}),$$

and hence $\partial_s \overline{P}_s f = \overline{P}_s \mathscr{A} f$ exists and is in $C_0(\mathbb{R}_+ \times D)$. Eq. (6.4) thus implies that

$$(t,x) \mapsto \partial_T P_{t,T} h(x)|_{T=t+s} = \partial_s \overline{P}_s f(t,x) - \partial_t \theta(t+s) P_{t,t+s} h(x) \tag{6.7}$$

exists and is in $C_0(\mathbb{R}_+ \times D)$ (first one argues locally in (t,x) and chooses θ constant around t). Since $s > 0$ was arbitrary, and since $\overline{P}_s \mathscr{A} f(t,x)$ is jointly continuous in s,t,x, we conclude that $P_{t,T} h(x)$ is continuously differentiable in $T > t$. On the other hand, we see as in (6.5) that

$$\partial_t P_{t,T} h(x) = -\mathscr{A}(t) P_{t,T} h(x),$$

pointwise. We can now apply the chain rule and derive

$$\begin{aligned}
\overline{\mathscr{A}}^\sharp \overline{P}_s f(t,x) &= \partial_t \theta(t+s) P_{t,t+s} h(x) + \theta(t+s)(-\mathscr{A}(t) P_{t,t+s} h(x) \\
&\quad + \partial_T P_{t,T} h(x)|_{T=t+s}) + \theta(t+s) \mathscr{A}(t) P_{t,t+s} h(x) \\
&= \partial_t \theta(t+s) P_{t,t+s} h(x) + \theta(t+s) \partial_T P_{t,T} h(x)|_{T=t+s} \\
&= \partial_s \overline{P}_s f(t,x).
\end{aligned}$$

Hence $\overline{\mathscr{A}}^\sharp \overline{P}_s f \in C_0(\mathbb{R}_+ \times D)$ and Lemma 6.2 implies that $\overline{P}_s f \in \mathscr{D}^\sharp$, whence (6.6).

The rest of the proposition now follows as in [3]: property (6.6) together with the fact that $\mathscr{L}(\overline{\Theta}_0)$ and $\mathscr{D}^\sharp$ are dense in $C_0(\mathbb{R}_+ \times D)$ and $C_c^\infty(\mathbb{R}_+ \times D)$ is $\|\cdot\|_{\sharp;\mathbb{R}_+\times D}$-dense in $\mathscr{D}^\sharp$ yields the assertion. $\square$

Remark 6.4. What makes the proof of (6.6) a bit clumsy is the fact that we cannot and do not require any differentiability of $\phi(t,T,u)$ and $\psi(t,T,u)$ in T or u (see Remark 4.2). It is noteworthy that yet we could derive T-differentiability of $P_{t,T}h(x)$ (without computing it explicitly), see (6.7) and below.

To make this more clear we present here an alternative proof of (6.6) (without using Lemma 6.2) *assuming* that $\phi(t,T,u)$ and $\psi(t,T,u)$ are continuously differentiable in u. Let $f = \theta h \in \overline{\Theta}_0$. We have to show that

$$[\overline{P}_s f]_{\mathscr{Z}} \in C_0(\mathbb{R}_+ \times D). \tag{6.8}$$

Consider (6.4) for fixed t, s, v, y, z. The function $\tilde{g} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\tilde{g}\left(e^{\int_t^{t+s} \beta^{\mathscr{Z}}(r)\,dr} q\right)\left|\det\left(e^{\int_t^{t+s} \beta^{\mathscr{Z}}(r)\,dr}\right)\right|$$
$$:= \exp(\phi(t, t+s, v, iq) + \langle \psi^{\mathscr{Y}}(t, t+s, v, iq), y\rangle) g(q)$$

is in $C_c^1(\mathbb{R}^n)$, and we have

$$\overline{P}_s f(t, x) = \theta(t+s) \int_{\mathbb{R}^n} e^{i\langle q,z\rangle} \tilde{g}(q)\,dq.$$

Now

$$\langle z, \beta^{\mathscr{Z}}(t) \nabla_z e^{i\langle q,z\rangle}\rangle = \langle \beta^{\mathscr{Z}}(t) q, \nabla_q e^{i\langle q,z\rangle}\rangle.$$

Integration by parts yields

$$\int_{\mathbb{R}^n} \nabla_q e^{i\langle q,z\rangle} \tilde{g}(q)\,dq = -\int_{\mathbb{R}^n} e^{i\langle q,z\rangle} \nabla_q \tilde{g}(q)\,dq.$$

By the Riemann–Lebesgue theorem the right-hand side as a function of z is in $C_0(\mathbb{R}^n)$, whence (6.8) is proved.

7. Proof of the main results

7.1. Proof of Theorem 2.7

This is an extension of the arguments in [3, Sections 3–5]. Fix $t>0$. Replacing "P_s" by $B_{s,t}$ in [3, Sections 4 and 5] (see (3.1) and (3.11)) yields (2.16)–(2.18) for all $u \in \mathscr{U}$, such that (2.5)–(2.12) hold. The uniqueness of representations (2.16)–(2.18) is a classical result, see [9, Theorem 8.1]. Hence the first part of the theorem is proved.

Now suppose $\tilde{\mathscr{A}}(t) f_u(x)$ has a continuous extension in t on $\mathbb{R}_+$, for all $u \in \mathscr{U}$. Eq. (2.15) implies that $F(t, u)$, $R^{\mathscr{Y}}(t, u)$ and $\beta^{\mathscr{Z}}(t)$ have a continuous extension on $\mathbb{R}_+$. If (2.14) holds as well, then (2.16) and (2.17) yield that

$$\langle a(t)u, u\rangle + \langle b(t), u\rangle - c(t) \quad \text{and} \quad \langle \alpha_i(t)u, u\rangle + \langle \beta_i^{\mathscr{Y}}(t), u\rangle - \gamma_i(t)$$

have a continuous extension in t on $\mathbb{R}_+$, for all $u \in \mathscr{U}$. But this readily implies that $a(t), \alpha_i(t), b(t), \beta_i^{\mathscr{Y}}(t), c(t), \gamma_i(t)$ have a continuous extension in t on $\mathbb{R}_+$, whence the assertion. □

7.2. Proof of Theorem 2.13

The first part of the theorem is a summary of Theorem 2.7 and Propositions 4.3 and 6.3. The second part follows from Theorem 2.7 (see also (3.7) and (3.8)), Propositions 4.1, 5.4 and 5.6. □

7.3. Proof of Theorem 2.14

Utilizing again the continuity properties (2.13) and (2.14), this theorems follows by similar arguments as in the proof of [3, Theorem 2.12]. □

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