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The Pricing of Options on Assets with Stochastic Volatilities

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ABSTRACT

One option-pricing problem that has hitherto been unsolved is the pricing of a European call on an asset that has a stochastic volatility. This paper examines this problem. The option price is determined in series form for the case in which the stochastic volatility is independent of the stock price. Numerical solutions are also produced for the case in which the volatility is correlated with the stock price. It is found that the Black-Scholes price frequently overprices options and that the degree of overpricing increases with the time to maturity.

ONE OPTION-PRICING PROBLEM that has hitherto remained unsolved is the pricing of a European call on a stock that has a stochastic volatility. From the work of Merton [12], Garman [6], and Cox, Ingersoll, and Ross [3], the differential equation that the option must satisfy is known. The solution of this differential equation is independent of risk preferences if (a) the volatility is a traded asset or (b) the volatility is uncorrelated with aggregate consumption. If either of these conditions holds, the risk-neutral valuation arguments of Cox and Ross [4] can be used in a straightfoward way.

This paper produces a solution in series form for the situation in which the stock price is instantaneously uncorrelated with the volatility. We do not assume that the volatility is a traded asset. Also, a constant correlation between the instantaneous rate of change of the volatility and the rate of change of aggregate consumption can be accommodated. The option price is lower than the Black-Scholes (B-S) [1] price when the option is close to being at the money and higher when it is deep in or deep out of the money. The exercise prices for which overpricing by B-S takes place are within about ten percent of the security price. This is the range of exercise prices over which most option trading takes place, so we may, in general, expect the B-S price to overprice options. This effect is exaggerated as the time to maturity increases. One of the most surprising implications of this is that, if the B-S equation is used to determine the implied volatility of a near-the-money option, the longer the time to maturity the lower the implied volatility. Numerical solutions for the case in which the volatility is correlated with the stock price are also examined.

The stochastic volatility problem has been examined by Merton [13], Geske [7], Johnson [10], Johnson and Shanno [11], Eisenberg [5], Wiggins [16], and

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Scott [15]. The Merton and Geske papers provide the solution to special types of stochastic volatility problems. Geske examines the case in which the volatility of the firm value is constant so that the volatility of the stock price changes in a systematic way as the stock price rises and falls. Merton examines the case in which the price follows a mixed jump-diffusion process. Johnson [10] studies the general case in which the instantaneous variance of the stock price follows some stochastic process. However, in order to derive the differential equation that the option price must satisfy, he assumes the existence of an asset with a price that is instantaneously perfectly correlated with the stochastic variance. The existence of such an asset is sufficient to derive the differential equation, but Johnson was unable to solve it to determine the option price. Johnson and Shanno [11] obtain some numerical results using simulation and produce an argument aimed at explaining the biases observed by Rubinstein [14]. Eisenberg [5] examines how options should be priced relative to each other using pure arbitrage arguments. Numerical solutions are attempted by Wiggins [16] and Scott [15].

Section I of this paper provides a solution to the stochastic volatility option-pricing problem in series form. Section II discusses the numerical methods that can be used to examine pricing biases when the conditions necessary for the series solution are not satisfied. Section III investigates the biases that arise when the volatility is stochastic but when a constant volatility is assumed in determining option prices. Conclusions are in Section IV.

I. The Stochastic Volatility Problem

Consider a derivative asset f with a price that depends upon some security price, S, and its instantaneous variance, $V = \sigma^2$, which are assumed to obey the following stochastic processes:

$$dS = \phi S\,dt + \sigma S\,dw \tag{1}$$

$$dV = \mu V\,dt + \xi V\,dz. \tag{2}$$

The variable ϕ is a parameter that may depend on S, σ, and t. The variables μ and ξ may depend on σ and t, but it is assumed, for the present, that they do not depend on S. The Wiener processes dz and dw have correlation ρ. The actual process that a stochastic variance follows is probably fairly complex. It cannot take on negative values, so the instantaneous standard deviation must approach zero as σ^2 approaches zero. Since S and σ^2 are the only state variables affecting the price of the derivative security, f, the risk-free rate, which will be denoted by r, must be constant or at least deterministic.

One reason why this problem has not previously been solved is that there is no asset that is clearly instantaneously perfectly correlated with the state variable σ^2. Thus, it does not seem possible to form a hedge portfolio that eliminates all the risk. However, as was shown by Garman [6], a security f with a price that depends on state variables θ_i must satisfy the differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i,j} \rho_{ij}\,\sigma_i\sigma_j\,\frac{\partial^2 f}{\partial\theta_i\partial\theta_j} - rf = \sum_i \theta_i\,\frac{\partial f}{\partial\theta_i}\quad[-\mu_i + \beta_i(\mu^* - \mathrm{r})], \tag{3}$$

where σ_i is the instantaneous standard deviation of θ_i, ρ_{ij} is the instantaneous correlation between θ_i and θ_j, μ_i is the drift rate of θ_i, β_i is the vector of multiple-regression betas for the regression of the state-variable "returns" $(d\theta/\theta)$ on the market portfolio and the portfolios most closely correlated with the state variables, μ^* is the vector of instantaneous expected returns on the market portfolio and the portfolios most closely correlated with the state variables, and r is the vector with elements that are the risk-free rate r. When variable i is traded, it satisfies the $(N + 1)$-factor CAPM, and the ith element of the right-hand side of (3) is $-r\theta_i \partial f/\partial\theta_i$.

In the problem under consideration, there are two state variables, S and V, of which S is traded. The differential equation (3) thus becomes

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left[\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\sigma^3\xi S \frac{\partial^2 f}{\partial S \partial V} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2}\right] - rf = -rS\frac{\partial f}{\partial S} - [\mu - \beta_V(\mu^* - \mathrm{r})]\sigma^2 \frac{\partial f}{\partial V}, \tag{4}$$

where ρ is the instantaneous correlation between S and V. The variable β_V is the vector of multiple-regression betas for the regression of the variance "returns" (dV/V) on the market portfolio and the portfolios most closely correlated with the state variables, and μ^* is as defined above. Note that, since these expected returns depend on investor risk preferences, this means that, in general, the option price will depend on investor risk preferences. We shall assume that $\beta_V(\mu^* - \mathrm{r})$ is zero or that the volatility is uncorrelated with aggregate consumption. This is not an unreasonable assumption and means that the volatility has zero systematic risk.[1] The derivative asset must then satisfy:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left[\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\sigma^3\xi S \frac{\partial^2 f}{\partial S \partial V} + \xi^2 V^2 \frac{\partial^2 f}{\partial V^2}\right] - rf = -rS\frac{\partial f}{\partial S} - \mu\sigma^2 \frac{\partial f}{\partial V}. \tag{5}$$

It will also be assumed that $\rho = 0$, i.e., that the volatility is uncorrelated with the stock price. As the work of Geske [7] shows, this is equivalent to assuming no leverage and a constant volatility of firm value.

An analytic solution to (5) for a European call option may be derived by using the risk-neutral valuation procedure. Since neither (5) nor the option boundary conditions depend upon risk preferences, we may assume in calculating the option value that risk neutrality prevails. Thus, $f(S, \sigma^2, t)$ must be the present value of the expected terminal value of f discounted at the risk-free rate. The price of the option is therefore

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(S_T, \sigma_T^2, T) p(S_T \mid S_t, \sigma_t^2)\, dS_T, \tag{6}$$

[1] This assumption can be relaxed to: $\beta_V(\mu^* - \mathrm{r})$ is constant. The solution is then the same as the solution when $\beta_V(\mu^* - \mathrm{r}) = 0$ except that μ is replaced by $\hat{\mu}$, where $\hat{\mu} = \mu - \beta_V(\mu^* - \mathrm{r})$. We are grateful to George Athanassakos, a Ph.D. student at York University, for pointing this out to us.

where

$$T = \text{time at which the option matures;}$$

$$S_t = \text{security price at time } t;$$

$$\sigma_t = \text{instantaneous standard deviation at time } t;$$

$$p(S_T \mid S_t, \sigma_t^2) = \text{the conditional distribution of } S_T \text{ given the security price and variance at time } t;$$

$$\mathrm{E}(S_\tau \mid S_t) = S_t e^{r(\tau - t)};$$

and $f(S_T, \sigma_T^2, T)$ is max[0, $S - X$]. The condition imposed on $\mathrm{E}(S_\tau \mid S_t)$ is given to make it clear that, in a risk-neutral world, the expected rate of return on S is the risk-free rate.

The conditional distribution of S_T depends on both the process driving S and the process driving σ^2. Making use of the fact that, for any three related random variables x, y, and z the conditional density functions are related by

$$p(x \mid y) = \int g(x \mid z) h(z \mid y)\, dz,$$

equation (6) may be greatly simplified. Define $\bar{V}$ as the mean variance over the life of the derivative security defined by the stochastic integral

$$\bar{V} = \frac{1}{T - t} \int_t^T \sigma_\tau^2\, d\tau.$$

Using this, the distribution of S_T may be written as

$$p(S_T \mid \sigma_t^2) = \int g(S_T \mid \bar{V}) h(\bar{V} \mid \sigma_t^2)\, d\bar{V},$$

where the dependence upon S_t is suppressed to simplify the notation. Substituting this into (6) yields

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int\int f(S_T) g(S_T \mid \bar{V}) h(\bar{V} \mid \sigma_t^2)\, dS_T\, d\bar{V},$$

which can then be written as

$$f(S_t, \sigma_t^2, t) = \int \left[e^{-r(T-t)} \int f(S_T) g(S_T \mid \bar{V})\, dS_T \right] h(\bar{V} \mid \sigma_t^2)\, d\bar{V}. \tag{7}$$

Under the prevailing assumptions ($\rho = 0$, μ and ξ independent of S), the inner term in (7) is the Black-Scholes price for a call option on a security with a mean variance $\bar{V}$, which will be denoted $C(\bar{V})$. To see this we need the following lemma:

LEMMA: *Suppose that, in a risk-neutral world, a stock price S and its instantaneous variance σ^2 follow the stochastic processes*

$$dS = rS\, dt + \sigma S\, d\tilde{z} \tag{a}$$

$$d\sigma^2 = \alpha\sigma^2\, dt + \xi\sigma^2\, d\tilde{w} \tag{b}$$

where r, the risk-free rate, is assumed constant, α and ξ are independent of S, and $d\tilde{z}$ and $d\tilde{w}$ are independent Wiener processes. Let $\bar{V}$ be the mean variance over some time interval $[0, T]$ *defined by*

$$\bar{V} = \frac{1}{T}\int_0^T \sigma^2(t)\, dt. \tag{c}$$

Given (a), (b), and (c), the distribution of $\log\{S(T)/S(0)\}$ *conditional upon $\bar{V}$ is normal with mean $rT - \bar{V}T/2$ and variance $\bar{V}T$.*

It is important to distinguish between the distributions of $\{S(T)/S(0) \mid \bar{V}\}$, $\{S(T)/S(0)\}$, and $\bar{V}$. The first is lognormal; the last two are not.

To see that the lemma is true, first let us suppose that σ^2 is deterministic but not constant. In this case, the terminal distribution of $\log\{S(T)/S(0)\}$ is normal with mean $rT - \bar{V}T/2$ and variance $\bar{V}T$. Note that the parameters of the lognormal distribution depend only on the risk-free rate, the initial stock price, the time elapsed, and the mean variance over the period. Thus, any path that σ^2 may follow and that has the same mean variance $\bar{V}$ will produce the same lognormal distribution. If σ^2 is stochastic, there are an infinite number of paths that give the same mean variance $\bar{V}$, but all of these paths produce the same terminal distribution of stock price. From this we may conclude that, even if σ^2 is stochastic, the terminal distribution of the stock price given the mean variance $\bar{V}$ is lognormal.

An alternative way to consider this problem is to assume that the variance changes at only n equally spaced times in the interval from 0 to T. Define S_i as the stock price at the end of the ith period and V_{i-1} as the volatility during the ith period. Thus, $\log(S_i/S_{i-1})$ has a normal distribution with mean

$$\frac{rT}{n} - \frac{V_{i-1}T}{2n}$$

and variance

$$\frac{V_{i-1}T}{n}.$$

If S and V are instantaneously uncorrelated, this is also the probability distribution of $\log(S_i/S_{i-1})$ conditional on V_i. The probability distribution of $\log(S_T/S_0)$ conditional on the path followed by V is therefore normal with mean $rT - \bar{V}T/2$ and variance $\bar{V}T$. This distribution depends only on $\bar{V}$. By letting $n \to \infty$, the lemma is seen to be true.

It is important to realize that the lemma does not hold when S and V are instantaneously correlated. In this case, $\log(S_i/S_{i-1})$ and $\log(V_i/V_{i-1})$ are normal distributions that in the limit have correlation ρ. The density function of $\log(V_i/V_{i-1})$ is normal with mean

$$\frac{\mu T}{n} - \frac{\xi^2 T}{2n}$$

and variance

$$\frac{\xi^2 T}{n},$$

so that $\log(S_i/S_{i-1})$ conditional on V_i is normal with mean

$$\frac{rT}{n} - \frac{V_{i-1}T}{2n} + \rho\,\frac{\sqrt{V_{i-1}}}{\xi}\left[\log(V_i/V_{i-1}) - \frac{\mu T}{n} + \frac{\xi^2 T}{2n}\right]$$

and variance

$$\frac{V_{i-1}T}{n}(1 - \rho^2).$$

Thus, $\log(S_T/S_0)$ conditional on the path followed by V has a normal distribution with mean

$$rT - \frac{\bar{V}T}{2} + \textstyle\sum_i \displaystyle\frac{\rho\sqrt{V_{i-1}}}{\xi}\left[\log\left(\frac{V_i}{V_{i-1}}\right) - \frac{\mu T}{n} + \frac{\xi^2 T}{2n}\right]$$

and variance

$$\bar{V}T(1 - \rho^2).$$

This distribution clearly depends on attributes of the path followed by V other than $\bar{V}$.

It is also interesting to note that the lemma does not carry over to a world in which investors are risk averse. In such a world, the drift rate of the stock price depends on σ^2 through the impact of σ^2 on the stock's β. This means that the mean of the terminal stock price distribution depends on the path that a nonconstant σ^2 follows. Different paths for σ^2 that have the same mean variance produce distributions for the log of the terminal stock price that have the same variance but different means. In this case, it is not true that the terminal distribution of the stock price given the mean variance $\bar{V}$ is lognormal.

Since $\log(S_T/S_0)$ conditional on $\bar{V}$ is normally distributed with variance $\bar{V}T$ when S and V are instantaneously uncorrelated, the inner integral in equation (7) produces the Black-Scholes price $C(\bar{V})$, which is

$$C(\bar{V}) = S_t\mathrm{N}(d_1) - Xe^{-r(T-t)}\mathrm{N}(d_2),$$

where

$$d_1 = \frac{\log(S_t/X) + (r + \bar{V}/2)(T - t)}{\sqrt{\bar{V}(T - t)}}$$

$$d_2 = d_1 - \sqrt{\bar{V}(T - t)}.$$

Thus, the option value is given by

$$f(S_t, \sigma_t^2) = \int C(\bar{V})h(\bar{V} \mid \sigma_t^2)\, d\bar{V}. \tag{8}$$

Equation (8) is always true in a risk-neutral world when the stock price and volatility are instantaneously uncorrelated. If, in addition, the volatility is uncorrelated with aggregate consumption, we have shown that the option price is independent of risk preferences and that the equation is true in a risky world as well. Equation (8) states that the option price is the B-S price integrated over the distribution of the mean volatility. It does not seem to be possible to obtain an analytic form for the distribution of $\bar{V}$ for any reasonable set of assumptions about the process driving V. It is, however, possible to calculate all the moments of $\bar{V}$ when μ and ξ are constant. For example, when $\mu \neq 0$,

$$\mathrm{E}(\bar{V}) = \frac{e^{\mu T} - 1}{\mu T} V_0$$

$$\mathrm{E}(\bar{V}^2) = \left[\frac{2e^{(2\mu+\xi^2)T}}{(\mu+\xi^2)(2\mu+\xi^2)T^2} + \frac{2}{\mu T^2}\left(\frac{1}{2\mu+\xi^2} - \frac{e^{\mu T}}{\mu+\xi^2}\right)\right] V_0^2,$$

and, when $\mu = 0$,

$$\mathrm{E}(\bar{V}) = V_0$$

$$\mathrm{E}(\bar{V}^2) = \frac{2(e^{\xi^2 T} - \xi^2 T - 1)}{\xi^4 T^2} V_0^2$$

$$\mathrm{E}(\bar{V}^3) = \frac{e^{3\xi^2 T} - 9e^{\xi^2 T} + 6\xi^2 T + 8}{3\xi^6 T^3} V_0^3.$$

The proofs of these results are available from the authors on request; they have been produced independently by Boyle and Emanuel [2].

Expanding $C(\bar{V})$ in a Taylor series about its expected value, $\bar{\bar{V}}$, yields

$$f(S_t, \sigma_t^2) = C(\bar{\bar{V}}) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{\bar{V}}} \int (\bar{V} - \bar{\bar{V}})^2 h(\bar{V})\, d\bar{V} + \cdots$$

$$= C(\bar{\bar{V}}) + \frac{1}{2}\frac{\partial^2 C}{\partial \bar{V}^2}\bigg|_{\bar{\bar{V}}} \mathrm{Var}(\bar{V}) + \frac{1}{6}\frac{\partial^3 C}{\partial \bar{V}^3}\bigg|_{\bar{\bar{V}}} \mathrm{Skew}(\bar{V}) + \cdots,$$

where $\mathrm{Var}(\bar{V})$ and $\mathrm{Skew}(\bar{V})$ are the second and third central moments of $\bar{V}$. For sufficiently small values of $\xi^2(T - t)$, this series converges very quickly. Using the moments for the distribution of $\bar{V}$ given above this series becomes when $\mu =$ 0:

$$\begin{aligned} f(S, \sigma^2) = C(\sigma^2) & \\ & + \frac{1}{2}\frac{S\sqrt{T-t}\,\mathrm{N}'(d_1)(d_1 d_2 - 1)}{4\sigma^3} \times \left[\frac{2\sigma^4(e^k - k - 1)}{k^2} - \sigma^4\right] \\ & + \frac{1}{6}\frac{S\sqrt{T-t}\,\mathrm{N}'(d_1)[(d_1 d_2 - 3)(d_1 d_2 - 1) - (d_1^2 + d_2^2)]}{8\sigma^5} \\ & \times \sigma^6\left[\frac{e^{3k} - (9 + 18k)e^k + (8 + 24k + 18k^2 + 6k^3)}{3k^3}\right] + \cdots, \end{aligned} \tag{9}$$

where

$$k = \xi^2(T - t)$$

and the t subscript has been dropped to simplify the notation. The choice of $\mu = 0$ is justified on the grounds that, for any nonzero μ, options of different maturities would exhibit markedly different implied volatilities. Since this is never observed empirically, we must conclude that μ is at least close to zero.

When the volatility is stochastic, the B-S price tends to overprice at-the-money options and underprice deep-in-the-money and deep-out-of-the-money options. (We define an at-the-money option as one for which $S = Xe^{-r(T-t)}$.) The easiest way to see this is to note that (8) is just the expected B-S price, the expectation being taken with respect to $\bar{V}$,

$$f = \mathrm{E}[C(\bar{V})].$$

When C is a concave function, $\mathrm{E}[C(\cdot)] < C(\mathrm{E}[\cdot])$, while, for a convex function, the reverse is true. The B-S option price $C(\bar{V})$ is convex for low values of $\bar{V}$ and concave for higher values. Thus, at least when ξ is small, we find that the B-S price tends to underprice for low values of $\bar{V}$ and overprice for high values of $\bar{V}$. It seems strange that a stochastic variance can lower the option price below the price it would have if the volatility were nonstochastic. However, this is consistent with the results Merton [13] derived for the mixed jump-diffusion process. There he showed that, if the option is priced by using the B-S results based on the expected variance (the expectation being formed over both jumps and continuous changes), then the price might be greater or less than the correct price.

To determine the circumstances under which the B-S price is too high or too low, examine the second derivative of $C(\bar{V})$.

$$C''(\bar{V}) = \frac{S\sqrt{T-t}}{4\bar{V}^{3/2}} \mathrm{N}'(d_1)(d_1 d_2 - 1),$$

where d_1 and d_2 are as defined above. The curvature of C is determined by the sign of C'', which depends on the sign of $d_1 d_2 - 1$. The point of inflection in $C(\bar{V})$ is given when $d_1 d_2 = 1$, that is, when

$$\bar{V} = \frac{2}{T-t}\left[\sqrt{1 + [\log(S/X) + r(T-t)]^2} - 1\right].$$

Denote this value of $\bar{V}$ by I. When $\bar{V} < I$, $C'' > 0$ and C is a convex function of $\bar{V}$. When $\bar{V} > I$, $C'' < 0$ and C is a concave function of $\bar{V}$. If $S = Xe^{-r(T-t)}$, then $I = 0$; this means that C is always a concave function of $\bar{V}$, and, regardless of the distribution of $\bar{V}$, the actual option price will always be lower than the B-S price. As $\log(S/X) \to \pm\infty$, I becomes arbitrarily large, and C is always convex so that the actual option price is always greater than the B-S price. Thus, we find that the B-S price always overprices at-the-money options but underprices options that are sufficiently deeply in or out of the money.

It is clear from this argument that $\partial f/\partial \xi$ may be positive or negative. The comparative statics with respect to the remaining six parameters, S, X, r, σ_t, $T - t$, and μ, are consistent with Merton's [12] distribution-free theorems. Since

μ and ξ are presumed independent of S, the distribution $h(\bar{V})$ is independent of S, X, and r. Thus, with respect to these three parameters, the comparative statics of $f(\cdot)$ are the same as the comparative statics of $C(\cdot)$. This follows since $C(\cdot)$ is monotonic in these three parameters, and h is everywhere non-negative. Thus, we find, as one might expect,

$$\frac{\partial f}{\partial S} = \mathrm{E}\left[\frac{\partial C(\bar{V})}{\partial S}\right] > 0$$

$$\frac{\partial f}{\partial X} = \mathrm{E}\left[\frac{\partial C(\bar{V})}{\partial X}\right] < 0$$

$$\frac{\partial f}{\partial r} = \mathrm{E}\left[\frac{\partial C(\bar{V})}{\partial r}\right] > 0.$$

The remaining three parameters $T - t$, μ, and σ_t affect both $C(\cdot)$ and $h(\cdot)$. The effect of increasing any of them is to increase the option price:

$$\frac{\partial f}{\partial \mu} > 0, \quad \frac{\partial f}{\partial \sigma_t^2} > 0, \quad \frac{\partial f}{\partial T} > 0.$$

To see this, note that $\partial f/\partial T$, $\partial f/\partial \mu$, and $\partial f/\partial \sigma_t^2$ are positive for every possible sample path of σ^2. Thus, they must also be positive when averaged across all possible sample paths.

In this section, it was shown that, if the stochastic volatility is independent of the stock price, the correct option price is the expected Black-Scholes price where the expectation is taken over the distribution of mean variances. This is given in equation (8). If the solution (8) is substituted into the differential equation (5), the equation is separable in h and C. The details of this substitution are available from the authors on request. The density function $h(\bar{V})$ is shown to satisfy the following differential equation:

$$\frac{\partial h}{\partial t} + \frac{\bar{V} - V_t}{T - t}\frac{\partial h}{\partial \bar{V}} + \frac{1}{2}\xi^2 \bar{V}_t^2 \frac{\partial h}{\partial V_t} + \mu V_t \frac{\partial h}{\partial V_t} = 0,$$

where $V_t = \sigma_t^2$. This can, in principle, be solved for the density function of the mean variance.

II. Other Numerical Procedures

We now consider efficient ways in which Monte Carlo simulation can be used to calculate the option price when some of the assumptions necessary for the series solution in (9) are relaxed. For our first result, we continue to assume that $\rho = 0$. However, we allow ξ and μ to depend on σ and t. This means that V can follow a mean-reverting process. One simple such process occurs when

$$\mu = a(\sigma^* - \sigma) \tag{10}$$

and ξ, a, and σ^* are constants.

The result in (8) still holds (i.e., the call price is the B-S price integrated over the distribution of $\bar{V}$). An efficient way of carrying out the Monte Carlo simulation involves dividing the time interval $T - t$ into n equal subintervals. Independent standard normal variates v_i $(1 \leq i \leq n)$ are sampled and are used to generate the variance V_i at time $t + i(T - t)/n$ using the formula:

$$V_i = V_{i-1}e^{[(\mu-\xi^2/2)\Delta t+v_i\xi\sqrt{\Delta t}]},$$

where $\Delta t = (T - t)/n$ and, if μ and ξ depend on σ, their values are based on $\sigma = \sqrt{V_{i-1}}$. The B-S option price, p_1, is calculated with the volatility set equal to the arithmetic mean of the V_i's $(0 \leq i \leq n)$. The procedure is then repeated using the antithetic standard normal variables, $-v_i$ $(0 \leq i \leq n)$, to give a price, p_2, and

$$y = \frac{p_1 + p_2}{2}$$

is calculated. The mean value of y over a large number of simulations gives an excellent estimate of the option price. This can be compared with the B-S price based on V_0 to give the bias.

Note that it is not necessary to simulate both V and S. Also, the antithetic variable technique that is described in Hammersley and Handscomb [8] considerably improves the efficiency of the procedure. In the mean-reverting model in (10) when $S = X = 1$, $r = 0$, $T = 90$ days, $\sigma_0 = 0.15$, $\xi = 1.0$, $a = 10$, $\sigma^* = 0.15$, and $n = 90$, 1000 simulations gave a value for the option of 0.029 with a standard error of 0.000014. The bias is -0.00038 (with the same standard error). The method can be used to deal with the situation where the conditions for the series solution in (9) hold but where ξ is too large for the series to converge quickly. Table I compares the values given by this Monte Carlo procedure with the values given by (9) for particular cases.

For our second result, we allow ρ to be nonzero and allow μ and ξ to depend on S as well as σ and t. We continue to assume that V is uncorrelated with aggregate consumption so that risk-neutral valuation can be used. In this case, it is necessary to simulate both S and V. The time interval is divided up as before, and two independent normal variates u_i and v_i $(1 \leq i \leq n)$ are sampled and used to generate the stock price S_i and variance V_i at time i in a risk-neutral world using the formulae:

$$S_i = S_{i-1}e^{[(r-V_{i-1}/2)\Delta t+u_i\sqrt{V_{i-1}\Delta t}]}$$

$$V_i = V_{i-1}e^{[(\mu-\xi^2/2)\Delta t+\rho u_i\xi\sqrt{\Delta t}+\sqrt{1-\rho^2}v_i\xi\sqrt{\Delta t}]}. \tag{11}$$

Again, the values of μ and ξ are based on $\sigma^2 = V_{i-1}$ and $S = S_{i-1}$. The value of

$$e^{-r(T-t)}\max[S_n - X, 0]$$

is calculated to give one "sample value," p_1, of option price. A second price, p_2, is calculated by replacing u_i with $-u_i$ $(1 \leq i \leq n)$ and repeating the calculations; p_3 is calculated by replacing v_i with $-v_i$ $(1 \leq i \leq n)$ and repeating the calculations; p_4 is calculated by replacing u_i with $-u_i$ and v_i with $-v_i$ $(1 \leq i \leq n)$ and repeating the calculations. Finally, two sample values of the B-S price q_1 and q_2 are calculated by simulating S using $\{u_i\}$ and $\{-u_i\}$, respectively, with V kept constant

Table I

Comparison of Monte Carlo Procedure and Series Solution;
Option Parameters: $\sigma_0 = 10\%$, $\xi = 1$, $\mu = 0$, $T - t = 180$ Days

	Price		B-S Price Bias		
			Equation 9	Monte Carlo	
S/X	B-S	Equation 9	Percent Bias	Percent Bias	Standard Error
0.75	0.0000	0.0000	******	******	237.85
0.76	0.0000	0.0000	******	******	139.41
0.77	0.0000	0.0000	******	970.57	153.57
0.78	0.0000	0.0000	786.47	787.43	133.70
0.79	0.0000	0.0000	588.78	383.43	44.22
0.80	0.0000	0.0001	436.12	336.43	39.21
0.81	0.0000	0.0001	354.37	330.68	46.90
0.82	0.0000	0.0001	232.00	173.55	21.21
0.83	0.0001	0.0002	164.02	134.14	14.91
0.84	0.0001	0.0003	114.54	102.17	10.67
0.85	0.0002	0.0004	78.32	69.55	8.41
0.86	0.0004	0.0006	52.14	54.55	6.74
0.87	0.0006	0.0008	33.53	37.95	5.43
0.88	0.0009	0.0011	20.55	23.50	3.02
0.89	0.0013	0.0015	11.70	16.46	2.74
0.90	0.0019	0.0021	5.83	10.07	1.99
0.91	0.0027	0.0028	2.07	5.53	1.45
0.92	0.0039	0.0039	−0.23	2.49	1.09
0.93	0.0053	0.0052	−1.53	0.22	0.90
0.94	0.0071	0.0069	−2.17	−1.45	0.78
0.95	0.0094	0.0091	−2.40	−2.36	0.58
0.96	0.0119	0.0117	−2.38	−2.53	0.38
0.97	0.0151	0.0148	−2.22	−2.61	0.29
0.98	0.0188	0.0185	−1.98	−2.52	0.25
0.99	0.0231	0.0228	−1.72	−2.32	0.21
0.01	0.0281	0.0276	−1.45	−2.16	0.19
1.01	0.0334	0.0330	−1.20	−1.61	0.16
1.02	0.0394	0.0390	−0.97	−1.24	0.12
1.03	0.0461	0.0456	−0.76	−1.09	0.13
1.04	0.0529	0.0526	−0.58	−0.65	0.10
1.05	0.0603	0.0601	−0.41	−0.35	0.08
1.06	0.0682	0.0681	−0.28	−0.19	0.08
1.07	0.0765	0.0764	−0.16	−0.05	0.07
1.08	0.0850	0.0850	−0.06	0.06	0.06
1.09	0.0939	0.0939	0.01	0.13	0.05
1.10	0.1030	0.1030	0.07	0.17	0.05
1.11	0.1122	0.1124	0.11	0.20	0.04
1.12	0.1216	0.1218	0.13	0.19	0.03
1.13	0.1312	0.1314	0.15	0.19	0.03
1.14	0.1409	0.1411	0.15	0.19	0.03
1.15	0.1506	0.1509	0.15	0.13	0.02
1.16	0.1605	0.1607	0.14	0.14	0.02
1.17	0.1703	0.1706	0.13	0.10	0.01
1.18	0.1802	0.1804	0.11	0.10	0.01
1.19	0.1902	0.1904	0.10	0.08	0.01
1.20	0.2001	0.2003	0.08	0.08	0.01
1.21	0.2101	0.2102	0.07	0.05	0.01
1.22	0.2201	0.2202	0.06	0.05	0.01
1.23	0.2300	0.2301	0.05	0.03	0.00
1.24	0.2400	0.2401	0.04	0.03	0.00

at V_0. This provides the following two estimates of the pricing bias:

$$\frac{p_1 + p_3 - 2q_1}{2} \quad \text{and} \quad \frac{p_2 + p_4 - 2q_2}{2}.$$

These estimates are averaged over a large number of simulations.

This procedure uses the antithetic variable technique (twice) and the control variate technique. Both are described in Hammersley and Handscomb [8]. The principle of the control variate technique is that the difference between the values of the two variables can often be obtained most accurately for a given number of simulations when both are calculated using the same random number streams. Furthermore, this is often true even when the value of one of the variables can be calculated analytically.

This procedure is applicable to a wider range of situations than the first one but is not as efficient. For the mean-reverting model example considered above, the standard error of the pricing bias using $n = 90$ and 1000 simulations was 0.000041 (compared with 0.000014 for the first procedure). Also, approximately three times as much computer time was consumed.

III. Properties of the Option Price

In this section, the properties of the option price given by the series solution in equation (9) and the numerical solutions of Section II are examined. The principal finding is that, when the volatility is uncorrelated with the stock price, the option price is depressed relative to the B-S price for near-the-money options. When the volatility is correlated with the stock price, this at-the-money price depression continues on into the money for positive correlation and out of the money for negative correlation. As might be expected, these effects are exaggerated as the volatility, σ, the volatility of the volatility, ξ, or the time to maturity, $T - t$, increases. The surprising result of this is that longer term options have lower implied volatilities, as calculated by the B-S equation, than do shorter term options whenever the B-S price overprices the option.

Consider first the case in which the volatility is uncorrelated with the stock price and μ and ξ are constant. Figure 1 shows the general relationship between the B-S price and the correct option price. The option being priced has 180 days to maturity; the volatility of the underlying asset is initially fifteen percent per annum; $\mu = 0$ and $\xi = 1$. The B-S price is too low deep in and out of the money and, surprisingly, too high at the money. The largest absolute price differences occur at or near the money. The actual magnitude of the pricing error is quite small and is magnified twenty-five-fold to make it visible in Figure 1.

The choice of a value of ξ is not obvious. It is possible to estimate ξ by examining the changes in volatilities implied by option prices. Alternatively, ξ can be estimated from changes in estimates of the actual variance. For currencies and currency options listed on the Philadelphia exchange, Hull and White [9] found that the estimates of ξ using both methods ranged from 1 to 4. Both of the estimation methods have weaknesses. Using the implied volatilities is at best an indirect procedure for estimating ξ. It is also contaminated by the fact that the

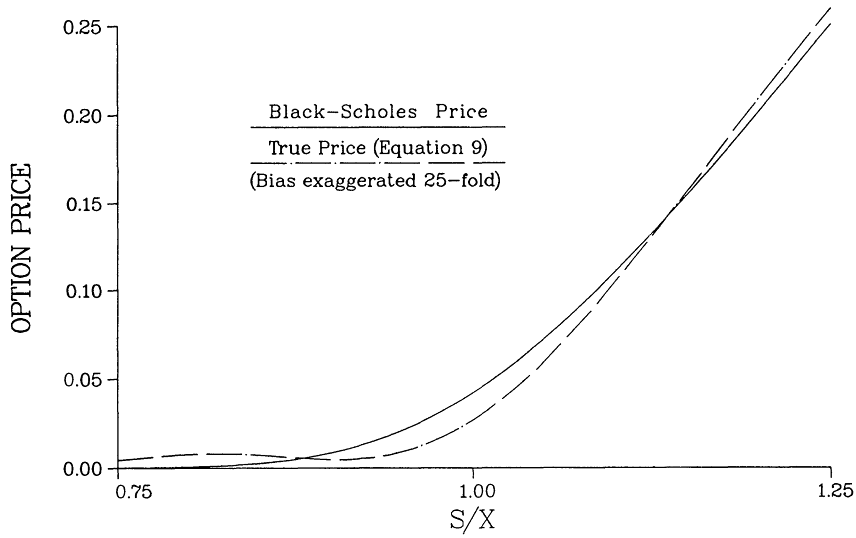

Figure 1. Pricing Bias When $\mu = 0$, $r = 0$, $\sigma_t = 15\%$, $\xi = 1$, $T - t = 180$ Days

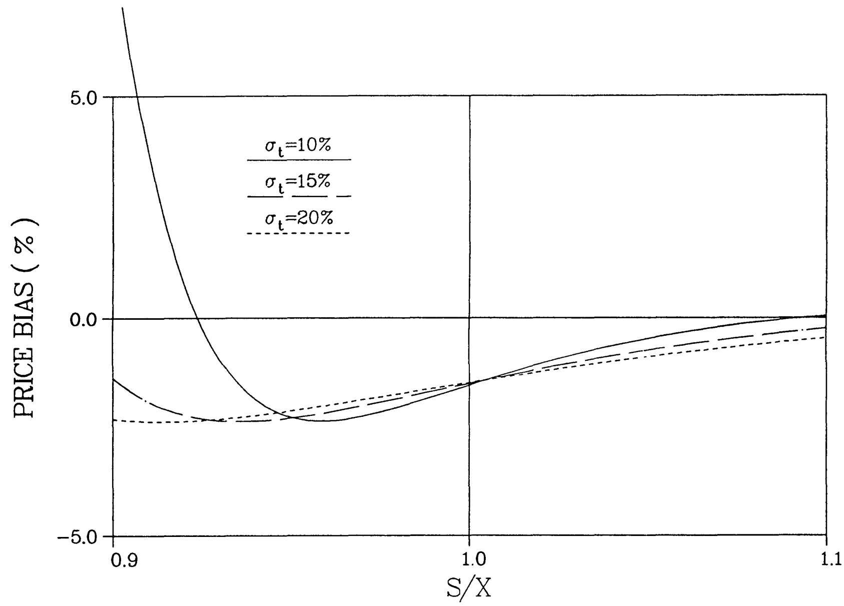

Figure 2. Effect of Varying σ_t When $\mu = 0$, $r = 0$, $\xi = 1$, $T - t = 180$ Days

changes in implied volatility are, at least to some extent, a result of pricing errors in the options. The problem with using estimates of the actual variance is that it requires very large amounts of data. Because of these weaknesses, the low end of the range for ξ was chosen as a conservative estimate.

In Figure 2, the effect of changing σ_t is shown, and, in Figure 3, the effect of changing ξ is shown. While the absolute magnitude of the price bias is very small,

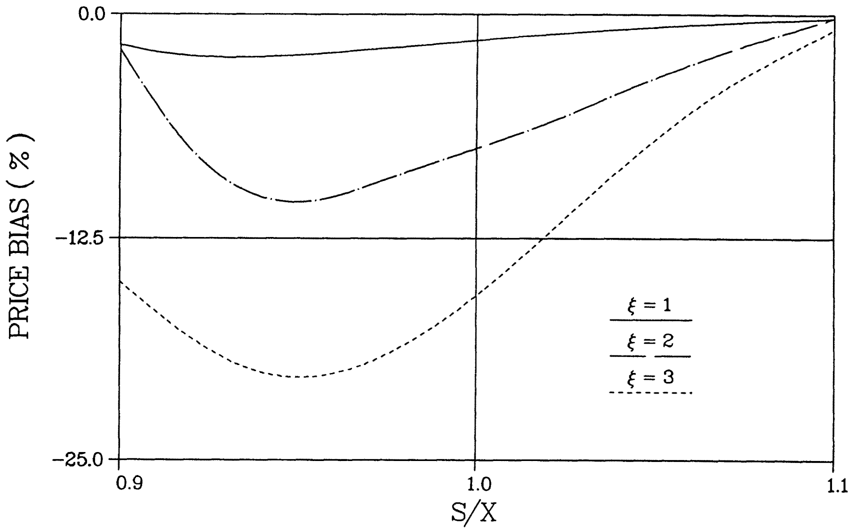

Figure 3. Effect of Varying ξ When $\mu = 0$, $r = 0$, $\sigma_t = 15\%$, $T - t = 180$ Days

as a percentage of the B-S price it is quite significant. The principal result of increasing σ_t^2 is to make the percentage price bias for out-of-the-money (in-the-money) options more positive (negative). When one looks sufficiently far out of the money, this effect is reversed, with higher σ_t^2 causing smaller biases. The effect on at- or in-the-money options is small. The main effect of increasing ξ is to lower the price of (i.e., to make the bias more negative for) near-the-money options. Although not evident from Figure 3, it is true that, for sufficiently deep out-of-the-money options, the reverse is true; increasing ξ increases a positive bias.

Figures 1 and 2 were produced using the series solution in equation (9). For Figure 3, when $\xi = 2$ and 3, it was found that the series solution did not converge quickly, and the Monte Carlo simulation approach was used. This was also used to investigate the results for the mean-reverting process in (10). As might be expected, the results for this process show biases that are similar to but less pronounced than those for the case when μ and ξ are constant. The effect of moving to a mean-reverting process from a process where μ and ξ are constant is to reduce the variance of $\bar{V}$. It is similar to the effect of reducing ξ.

The effect of a nonzero ρ when both μ and ξ are constant was investigated using the Monte Carlo simulation approach in equation (11). The results are shown in Table II. When the volatility is positively correlated with the stock price, the option price has a bias relative to the B-S price, which tends to decline as the stock price increases. Out-of-the-money options are priced well above the B-S price, while the price of in-the-money options is below the B-S price. The crossing point, the point at which the B-S price is correct, is slightly below being at the money. When the volatility is negatively correlated with the stock price, the reverse is true. Out-of-the-money options are priced below the B-S price, while in-the-money options have prices above the B-S price. The crossing point

Table II

Price Bias as a Percentage of the Black-Scholes Price for Varying Values of S/X and Correlation, ρ, between the Volatility and the Stock Price; Option Parameters: $\sigma_0 = 15\%$, $r = 0$, $\xi = 1$, and $\mu = 0$

		S/X				
T (Days)	ρ	0.90	0.95	1.00	1.05	1.10
90	−1.0	−66.06	−22.68	−2.13	1.84	1.56
		(1.98)	(0.51)	(0.23)	(0.12)	(0.08)
	−0.5	−31.55	−10.89	−1.62	0.91	0.89
		(1.14)	(0.32)	(0.13)	(0.07)	(0.04)
	0.0	3.72	−0.98	−0.92	−0.25	0.07
		(0.50)	(0.13)	(0.05)	(0.03)	(0.02)
	0.5	39.37	7.70	−0.53	−1.68	−0.85
		(1.12)	(0.28)	(0.12)	(0.07)	(0.04)
	1.0	72.24	15.62	−0.84	−3.12	−1.56
		(2.42)	(0.61)	(0.25)	(0.14)	(0.09)
180	−1.0	−56.22	−22.49	−4.77	0.94	1.79
		(1.23)	(0.55)	(0.31)	(0.21)	(0.15)
	−0.5	−25.96	−11.50	−2.93	0.27	1.29
		(0.80)	(0.35)	(0.20)	(0.13)	(0.09)
	0.0	0.63	−2.25	−1.87	−0.82	−0.09
		(0.42)	(0.17)	(0.09)	(0.06)	(0.04)
	0.5	24.04	5.30	−1.10	−2.57	−1.61
		(0.78)	(0.32)	(0.19)	(0.11)	(0.09)
	1.0	45.99	12.43	−1.11	−4.58	−4.05
		(1.69)	(0.77)	(0.40)	(0.27)	(0.18)
270	−1.0	−53.32	−23.12	−7.53	−0.20	2.01
		(1.11)	(0.58)	(0.39)	(0.28)	(0.21)
	−0.5	−25.33	−12.33	−5.29	−0.44	0.62
		(0.73)	(0.39)	(0.25)	(0.17)	(0.13)
	0.0	−1.88	−3.56	−2.45	−1.37	−0.52
		(0.40)	(0.21)	(0.14)	(0.09)	(0.07)
	0.5	17.87	4.36	−1.77	−2.81	−2.37
		(0.69)	(0.39)	(0.24)	(0.17)	(0.14)
	1.0	33.41	8.94	−1.09	−6.21	−5.07
		(1.64)	(0.87)	(0.55)	(0.34)	(0.26)

is slightly in the money. When ρ is zero, the bias is a combination of these two effects. The price is above the B-S price for in- and out-of-the-money options and below the B-S price at the money. For all values of ρ, the absolute percentage bias tends to zero as S/X tends to infinity. These general observations appear to be true for all maturities.

The intuition behind these effects can be explained by the impact that the correlation has on the terminal distribution of stock prices. First, consider the case in which the volatility is positively correlated with the stock price. High stock prices are associated with high volatilities; as stock prices rise, the probability of large positive changes increases. This means that very high stock prices become more probable than when the volatility is fixed. Low stock prices are associated with low volatilities; if stock prices fall, it becomes less likely that large changes take place. Low stock prices become like absorbing states, and it becomes more likely that the terminal stock price will be low. The net effect is that the terminal stock price distribution is more positively skewed than the lognormal distribution arising from a fixed volatility. When volatility changes are negatively correlated with stock price changes, the reverse is true. Price increases reduce the volatility so that it is unlikely that very high stock prices will result. Price decreases increase volatility, increasing the chance of large positive price changes; very low prices become less likely. The net effect is that the terminal stock price distribution is more peaked than the usual lognormal distribution.

One phenomenon arising from these results might be called the time-to-maturity effect. If the time to maturity is increased with all else being held constant, the effect is the same as increasing both σ_t and ξ. Thus, longer term near-the-money options have a price that is lower (relative to the B-S price) than that of shorter term options. Because the B-S price is approximately linear with respect to volatility, these proportional price differences map into equivalent differences in implied volatilities. If the B-S equation is used to calculate implied volatilities, longer term near-the-money options will exhibit lower implied volatilities than shorter term options. This effect occurs whenever the B-S formula overprices the option. Table III shows the effects of changing terms on the implied volatilities for an option with an expected volatility of fifteen percent, $\xi = 1$, $\mu = 0$, and $r = 0$ for different values of ρ and S/X. The time-to-maturity effect is clear. In the worst case, it changes the implied volatility by almost one half of one percent. The effect increases as ξ increases and as the initial volatility increases.

This time-to-maturity effect is counterintuitive. One might expect that uncertainty about the volatility would increase uncertainty about the stock price, hence raising the option price, and that longer times to maturity would exacerbate this. The actual result is just the opposite. Wherever the B-S formula overprices the option, it is due to the local concavity of the B-S price with respect to σ. Because of the concavity of the option price with respect to volatility, increases in volatility do not increase the option price as much as decreases in volatility decrease the price. Thus, the average of the B-S prices for a stochastic volatility with a given mean lies below the B-S price for a fixed volatility with the same mean for all near-the-money options. As the time to maturity increases, the variance of the stochastic volatility increases, exacerbating the effect of the curvature of the option price with respect to volatility. Wherever the B-S price underprices the option, the reverse effect is observed.

The implications of these results for empirical tests of option pricing are interesting. Rubinstein [14] compared implied volatilities of matched pairs of options differing only in exerise price. In the period 1976–1977, he generally found that, as S/X increased, the implied volatility decreased. For the subsequent period 1977–1978, the reverse was true. Rubinstein also compared matched pairs of options differing only in time to maturity. He found that, in the 1976–1977 period, the shorter term options had higher implied volatilities for out-of-the-money options. For at-the-money and in-the-money options, the reverse is true.

Table III

Implied Volatility Calculated by Black-Scholes Formula from the Option Prices Given in Table II; Actual Expected Mean Volatility 15%; Option Parameters: $\sigma_0 = 15\%$, $r = 0$, $\xi = 1$, and $\mu = 0$

		S/X				
T (Days)	ρ	0.90	0.95	1.00	1.05	1.10
90	−1.0	11.94	13.38	14.68	15.69	16.63
		(0.13)	(0.04)	(0.03)	(0.04)	(0.08)
	−0.5	13.75	14.23	14.76	15.34	15.97
		(0.05)	(0.02)	(0.02)	(0.03)	(0.04)
	0.0	15.13	14.93	14.86	14.91	15.08
		(0.02)	(0.01)	(0.01)	(0.01)	(0.02)
	0.5	16.32	15.53	14.92	14.36	13.98
		(0.03)	(0.02)	(0.02)	(0.03)	(0.05)
	1.0	17.29	16.07	14.87	13.80	13.00
		(0.07)	(0.04)	(0.04)	(0.05)	(0.13)
180	−1.0	11.66	13.04	14.28	15.26	15.99
		(0.09)	(0.05)	(0.05)	(0.06)	(0.08)
	−0.5	13.59	14.01	14.56	15.08	15.72
		(0.05)	(0.03)	(0.03)	(0.04)	(0.05)
	0.0	15.03	14.81	14.72	14.77	14.94
		(0.02)	(0.01)	(0.01)	(0.02)	(0.02)
	0.5	16.20	15.45	14.83	14.27	14.06
		(0.04)	(0.03)	(0.03)	(0.03)	(0.05)
	1.0	17.23	16.05	14.83	13.70	12.50
		(0.08)	(0.06)	(0.06)	(0.08)	(0.13)
270	−1.0	11.38	12.79	13.87	14.95	15.85
		(0.09)	(0.06)	(0.06)	(0.07)	(0.09)
	−0.5	13.38	13.83	14.20	14.89	15.27
		(0.05)	(0.04)	(0.04)	(0.04)	(0.06)
	0.0	14.88	14.66	14.63	14.66	14.77
		(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
	0.5	16.07	15.41	14.73	14.30	13.96
		(0.04)	(0.04)	(0.04)	(0.04)	(0.06)
	1.0	16.97	15.84	14.84	13.44	12.70
		(0.09)	(0.08)	(0.08)	(0.09)	(0.13)

In the period 1977–1978, almost all options exhibited the property that shorter term options had higher implied volatilities.

The observed implied volatility patterns in relation to S/X are consistent with a situation in which, during the 1976–1977 period, the volatility was positively correlated with the stock price, while, in the 1977–1978 period, the correlation was negative. However, the results from comparing implied volatilities across different times to maturity are not consistent with this. If the volatility were positively correlated with the stock price, we would expect out-of-the-money options to exhibit increasing implied volatility with increasing time to maturity.

It is difficult to draw direct comparisons between Rubinstein's results and our model. As suggested by equation (9), the key element is the relationship between the stock price and the present value of the exercise price. Thus, when Rubinstein chooses pairs matched on the basis of exercise price, they are not truly matched in the variable of interest, the present value of the exercise price. Figure 4 illustrates the price biases for different times to maturity for the case in which volatility is uncorrelated with the stock price and the risk-free rate is not zero. The net effect of the nonzero risk-free rate is to lower the effective exercise price of longer term options. Figure 4 shows that increasing the time to maturity raises the implied volatility for almost all options except the very deep in-the-money options, in which case the effect is very small. When the volatility is positively correlated with the stock price, the effect is to enhance the time-to-maturity effect for all but very deep out-of-the-money options. When the correlation is negative, the result is a reduction of the time-to-maturity effect for out-of-the-money options and an enhancement of the tendency to observe higher implied volatilities in long-term in-the-money options. This latter effect is, however, very small. Thus, overall, we might expect the time-to-maturity effect to be strongest for out-of-the-money options and weakest for in-the-money options. This is exactly what Rubinstein found.

The results of Rubinstein may not be inconsistent with the model presented in this paper, but neither do they seem to provide strong support. In order for them to support this model, it is necessary to posit that, from one year to the next, the correlation between stock prices and the associated volatility reversed sign. It is difficult to think of a convincing reason why this event should occur. It is tempting to suggest that the observed effect may be a sampling result that can occur if some stocks have positive correlations and some have negative correlations. In this case, by changing the relative numbers of each group in the sample from period to period, we could see the observed result. Unfortunately, Rubinstein found that the result also prevailed on a security-by-security basis.

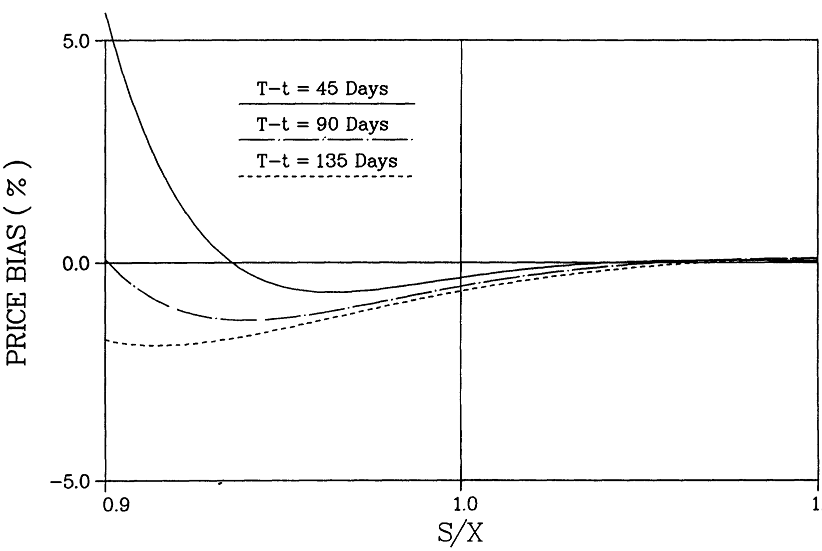

Figure 4. Effect of Varying $T - t$ When $\mu = 0$, $r = 10\%$, $\sigma_t = 15\%$, $\xi = 1$

IV. Conclusions

The general differential equation of Garman [6] is used to derive a series solution for the price of a call option on a security with a stochastic volatility that is uncorrelated with the security price. It is shown for such a security that the Black-Scholes price overvalues at-the-money options and undervalues deep in- and out-of-the-money options. The range over which overpricing by the B-S formula takes place is for stock prices within about ten percent of the exercise price. The magnitude of the pricing bias can be up to five percent of the B-S price.

The case in which the volatility is correlated with the stock price is examined using numerical methods. When there is a positive correlation between the stock price and its volatility, out-of-the-money options are underpriced by the B-S formula, while in-the-money options are overpriced. When the correlation is negative, the effect is reversed. These results can be used to explain the empirical observations of Rubinstein [14] but require the questionable assumption that the correlation between volatilities and stock prices reverses from one year to the next.

This paper has concentrated on the pricing of a European call option on a stock subject to a stochastic volatility. The results are directly transferable to European puts through the use of put-call parity. They are also transferable to American calls on non-dividend-paying stocks. This follows from Merton's [12] results. The pricing of American puts, however, cannot be easily determined.

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