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Discrete-Time Volatility Forecasting With Persistent Leverage Effect and the Link With Continuous-Time Volatility Modeling

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We first propose a reduced-form model in *discrete time* for S&P 500 volatility showing that the forecasting performance can be significantly improved by introducing a persistent leverage effect with a long-range dependence similar to that of volatility itself. We also find a strongly significant positive impact of lagged jumps on volatility, which however is absorbed more quickly. We then estimate *continuous-time* stochastic volatility models that are able to reproduce the statistical features captured by the discrete-time model. We show that a single-factor model driven by a fractional Brownian motion is unable to reproduce the volatility dynamics observed in the data, while a multifactor Markovian model fully replicates the persistence of both volatility and leverage effect. The impact of jumps can be associated with a common jump component in price and volatility. This article has online supplementary materials.

KEY WORDS: Fractional Brownian motion; Jumps; Leverage effect; Multifactor models; Volatility forecasting.

1. INTRODUCTION

The relevance of financial market volatility led to a very large literature trying to take into account its most salient dynamic features: clustering, slowly decaying autocorrelation, and asymmetric responses. The advent of high-frequency data, allowing for specification and estimation of models for realized volatility, elicited a considerable advancement in this field. However, a considerable gap still exists in the literature between models devised for volatility forecasting, which are commonly specified in discrete time, and volatility modeling in continuous time, which is used, among other things, for option pricing. This is particularly annoying for leverage effect, whose interpretation is completely different in discrete time, where it is typically interpreted as a negative correlation between lagged negative returns and volatility, and in continuous time, where the negative correlation between price and volatility shocks is contemporaneous.

This article contributes to this literature in two directions and aims at filling this apparent gap. In the first part of the article, we propose a new reduced-form model in *discrete time*, the *Leverage Heterogeneous Auto-Regressive with Continuous volatility and Jumps* (LHAR-CJ) model, which is able to provide a remarkable forecasting performance for volatility over a time horizon that ranges from 1 day to 1 month, along with a positive and significant risk-return trade-off. Our specification is extremely simple to implement and it is based on the incorporation of three effects. The first is the well-known volatility persistence, which is modeled with the heterogeneous autoregressive (HAR) specification of Corsi (2009). However, we do not restrict to lagged volatilities (at daily, weekly, and monthly frequencies) as possible sources of future volatility, but we also add jumps (as in Andersen, Bollerslev, and Diebold 2007) and, as a novel contribution of this article, negative returns over the past day, week, and month, thus imposing a common heterogeneous structure on the explanatory variables.

The empirical findings in the first part of the article are also relevant because of the important implications they bear on the set of continuous-time models consistent with the empirical features of financial data. In the second part of the article, we then estimate continuous-time models via indirect inference using the proposed discrete-time specifications as auxiliary models, thus reproducing the very same features captured by the discrete-time model. In particular, we show that a single-factor model is unable to reproduce the type of integrated volatility persistence, which is displayed by the data, even when allowing shocks to be driven by a fractional Brownian motion (Comte and Renault 1998), that is, by a genuine long-memory component. This induces us to investigate multifactor specifications. We find that a Markovian two-factor model is able to replicate not only the persistence of integrated volatility, but also the persistence in the leverage effect by correlating more than one volatility factor with the price shocks. The analyzed multifactor models do not need long-memory shocks to achieve their goal. While not ruling out the possible presence of long memory in volatility, this shows that the autocorrelation function of realized volatility is not necessarily a signature of genuine long memory in the data-generating process, and corroborates the framework

This article supersedes the previously circulating versions *Volatility determinants: heterogeneity, leverage and jumps and HAR volatility modelling with heterogenous leverage and jumps*. The daily variables used in this article are available from the authors upon request.

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according to which market volatility is generated by a superposition of different frequencies, as suggested in Corsi (2009) and Muller et al. (1997).

In the literature, it is well known that volatility tends to increase more after a negative shock than after a positive shock of the same magnitude; for example, see Christie (1982), Campbell and Hentschel (1992), Glosten, Jagannathan, and Runkle (1989), and more recently, Bollerslev, Litvinova, and Tauchen (2006), Bollerslev, Kretschmer, Pigorsch, and Tauchen (2009), and Martens, van Dijk, and de Pooter (2009). We extend the heterogeneous structure to the standard leverage effect by including lagged negative returns at different frequencies as explanatory variables to forecast volatility. This idea traces back to Corsi (2005) and can also be found in the concurrent work of Scharth and Medeiros (2009) and Allen and Scharth (2009) as well as in Fernandes, Medeiros, and Scharth (2009) to forecast the implied volatility. However, with this article, we are able to provide a novel and clear evidence on the fact that the impact of negative returns on future volatility of S&P 500 is also highly persistent and extends for a period of at least 1 month, thus displaying a long-range dependence similar to that of volatility itself.

With respect to the literature on jumps, we follow the separation of the quadratic variation in "continuous" volatility and jumps proposed in Andersen et al. (2007) for the explanatory variables and extended in Busch, Christensen, and Nielsen (2011) to the dependent variable. However, contrary to the above-mentioned studies, we do not use bipower variation (Barndorff-Nielsen and Shephard 2004) to measure the jumps contribution to quadratic variation, but we follow Corsi, Pirino, and Renò (2010) using *threshold bipower variation*, a measure of continuous quadratic variation that is able to crucially soften the small-sample issues of bipower variation. This provides superior forecasting performance, and allows to reveal that volatility does increase after a jump (both positive or negative) but that this shock is absorbed quickly in the volatility dynamics. Jumps are instead found to be almost unpredictable. When modeling in continuous time, the transient jump impact is captured by cojumps between price and a single volatility factor.

The article is organized as follows. Section 2 presents the main reduced-form model in discrete time. Section 3 contains the estimates and various robustness checks. In Section 4, we estimate continuous-time models via indirect inference. Section 5 concludes.

2. THE DISCRETE-TIME MODEL

This section is devoted to the specification of the reduced-form model in discrete time. We first define the data-generating process, the variables of interest, and their estimators. We then specify the LHAR-CJ model.

2.1 Construction of the Variables of Interest

We assume that the data-generating process X_t (the log-price) is a real-valued process that can be put, in a standard probability space, in the form of an Ito semimartingale:

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t, \tag{2.1}$$

where W_t is a standard Brownian motion, μ_t is predictable, and σ_t is càdlàg; $dJ_t = c_t dN_t$ is a jump process where N_t is a nonexplosive Poisson process whose intensity is an adapted stochastic process λ_t and c_t is the adapted random variable measuring the size of the jump at time t and satisfying, $\forall t \in [0, T]$, $\mathcal{P}(\{c_t = 0\}) = 0$.

While in Section 4 we propose possible specifications of model (2.1), including the possibility that σ_t is driven by a fractional Brownian motion with constant volatility (Comte and Renault 1998), here we concentrate on reduced-form models for quadratic variation, which is defined by

$$[X]_t^{t+T} = \int_t^{t+T} \sigma_s^2 ds + \sum_{j=N_t}^{N_{t+T}} c_{\tau_j}^2, \tag{2.2}$$

where we denote by τ_j the times in which jumps occur.

These quantities are not directly observable and they have to be replaced with consistent realized estimators, which we denote by $\widehat{\mathsf{V}}_t$ (for $[X]_t^{t+T}$), $\widehat{\mathsf{C}}_t$ (for $\int_t^{t+T} \sigma_s^2 ds$), and $\widehat{\mathsf{J}}_t$ (for $\sum_{j=N_t}^{N_{t+T}} c_{\tau_j}^2$). We use $T = 1$ day and we denote the daily (close-to-close) return by r_t. We remark that realized volatility models need both the specification of the dynamics of quadratic variation and the choice of small-sample estimators. For example, two models can share the same dynamics (e.g., the HAR model for the total quadratic variation) but be different just because quadratic variation estimators (e.g., realized volatility versus two-scale estimator) are different.

In order to mitigate the impact of microstructure effects on our estimates, $\widehat{\mathsf{V}}_t$ is the two-scale estimator (TSRV_t) proposed by Zhang, Mykland, and Aït-Sahalia (2005), which is consistent also in the presence of jumps. Details on the construction of the estimator are provided in a related online Appendix. Aït-Sahalia and Mancini (2008) showed that using the two-scale estimator instead of standard realized volatility measures yields significant gains in volatility forecasting.

To define $\widehat{\mathsf{C}}_t$ and $\widehat{\mathsf{J}}_t$ we use the following approach. We first pretest the data for jumps using the C-Tz statistics proposed in Corsi et al. (2010) and formally defined in the online Appendix. The C-Tz test, which is distributed as a standard normal in the absence of jumps, is computed daily. When the null is not rejected (namely when C-Tz < 3.0902, corresponding to the 99.9% significance level), we set $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t = \mathsf{TSRV}_t$ and $\widehat{\mathsf{J}}_t = 0$. When instead the test rejects the null, we set $\widehat{\mathsf{C}}_t = \mathsf{TBPV}_t$ and $\widehat{\mathsf{J}}_t = \max(\mathsf{TSRV}_t - \mathsf{TBPV}_t, 0)$, where TBPV_t is the threshold bipower variation estimator introduced in Corsi et al. (2010):

$$\mathsf{TBPV}_t = \frac{\pi}{2}\frac{M}{M-2}\sum_{j=0}^{M-2} |\Delta_{t,j}X| \cdot |\Delta_{t,j+1}X| \times I_{\{|\Delta_{t,j}X|^2 \le \vartheta_{j-1}\}} I_{\{|\Delta_{t,j+1}X|^2 \le \vartheta_j\}}, \tag{2.3}$$

and $\Delta_{t,j}X$ is the jth intraday return of day t (we use 5-minute returns here), with $j = 1, \ldots, M$; $I_{\{\cdot\}}$ is the indicator function and ϑ_j is a threshold function (see the online Appendix for its precise definition), which is designed to remove jumps from the returns time series (Mancini 2009). We have $\widehat{\mathsf{V}}_t = \widehat{\mathsf{C}}_t + \widehat{\mathsf{J}}_t$ provided $\mathsf{TSRV}_t > \mathsf{TBPV}_t$ in days with jumps, which is always the case empirically. Equation (2.3) shows that TBPV_t is very similar to the bipower variation (BPV_t) of Barndorff-Nielsen and Shephard (2004), with the difference being the two indicator functions that remove returns larger than the threshold

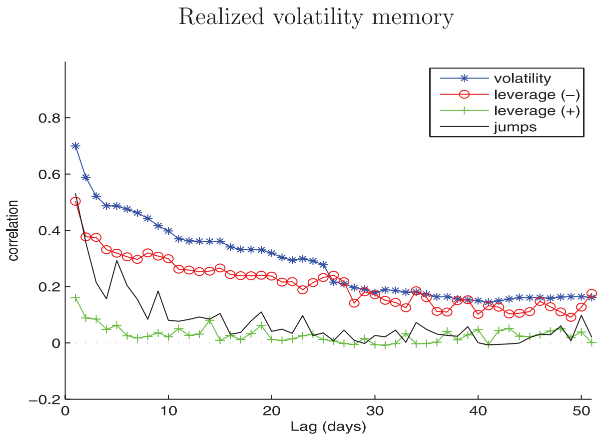

Figure 1. Lagged correlation function between past values Y_{t-h} and current, daily, integrated variance estimates TSRV_t as a function of h, with Y_{t-h} being TSRV_{t-h} itself, negative returns, positive returns, and jumps quadratic variation $\widehat{\mathsf{J}}_{t-h}$ for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The online version of this figure is in color.

ϑ_j. While this difference is irrelevant asymptotically, it has been shown by Corsi et al. (2010) to be crucial in small samples (e.g., we often have $\mathsf{TSRV}_t < \mathsf{BPV}_t$ in days with jumps because of the bias of BPV_t).

Figure 1 shows, for the S&P 500 series studied in this article, the lagged correlation function between the two-scale estimator TSRV_t and TSRV_{t-h} itself, negative returns, positive returns, and $\widehat{\mathsf{J}}_{t-h}$. The autocorrelation of TSRV_t decays very slowly, as it is well known. The lagged correlation between TSRV_t and negative returns shows the leverage effect: volatility is correlated with lagged negative returns. Figure 1 also shows that the impact of negative returns on future volatility is slowly decaying as well. Also jumps have a positive and large impact (as large as the negative returns when $h = 1$), which however decays more rapidly. Finally, positive returns have a very small and negligible impact on future volatility.

The slowly decaying impact of negative returns might well be a by-product of the slowly decaying autocorrelation function of volatility. However, since the same phenomenon is not observed with the jump component, it can also be suggestive of the fact that leverage effect might be very persistent, a possibility that has been seldom investigated so far.

Persistence in the leverage effect can be induced, in continuous time, by making the leverage effect an explicit function of volatility, as in Bandi and Renò (2011b). Our reduced-form model, which is in discrete time, explores an alternative possibility. We follow Corsi (2009) in modeling the slowly decaying autocorrelation function by means of a heterogeneous structure induced by a volatility cascade, and we extend this structure to negative returns and jumps.

2.2 The LHAR-CJ Model

Combining heterogeneity in realized volatility, leverage, and jumps, we construct the LHAR-CJ model. As it is common in practice, we use daily, weekly, and monthly frequencies. Then, by using variables specified in logs, we introduce averaged variables, which are defined over an integer number h of days as (jumps are aggregated instead of averaged)

$$\log \widehat{\mathsf{V}}_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} \log \widehat{\mathsf{V}}_{t-j+1}, \quad \log \widehat{\mathsf{C}}_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} \log \widehat{\mathsf{C}}_{t-j+1},$$

$$r_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} r_{t-j+1}, \quad \widehat{\mathsf{J}}_t^{(h)} = \sum_{j=1}^{h} \widehat{\mathsf{J}}_{t-j+1}.$$

To model the leverage effect at different frequencies, we define $r_t^{(h)-} = \min(r_t^{(h)}, 0)$. The proposed model reads:

$$\begin{aligned}\log \widehat{\mathsf{V}}_{t+h}^{(h)} &= c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ &\quad + \alpha^{(d)} \log\left(1 + \widehat{\mathsf{J}}_t\right) + \alpha^{(w)} \log\left(1 + \widehat{\mathsf{J}}_t^{(5)}\right) \\ &\quad + \alpha^{(m)} \log\left(1 + \widehat{\mathsf{J}}_t^{(22)}\right) + \gamma^{(d)} r_t^{-} + \gamma^{(w)} r_t^{(5)-} \\ &\quad + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)}, \end{aligned} \qquad (2.4)$$

with real parameters $\{c, \beta^{(d,w,m)}, \alpha^{(d,w,m)}, \gamma^{(d,w,m)}\}$ and where $\varepsilon_t^{(h)}$ is iid noise. Model (2.4) nests other models that have been successfully used for realized volatility. When $\alpha^{(d,w,m)} = \gamma^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$, the model becomes the HAR model of Corsi (2009). When $\gamma^{(d,w,m)} = 0$, we get the HAR-CJ model proposed by Andersen et al. (2007), which separately includes continuous and discontinuous component as explanatory variables. When $\alpha^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$, the model is referred to as the LHAR model. The model can also be specified directly for $\widehat{\mathsf{V}}_t$ and for $\sqrt{\widehat{\mathsf{V}}_t}$, as in Andersen et al. (2007) and Corsi et al. (2010).

We estimate model (2.4) and its variants, with h ranging from 1 to 22 to make multiperiod predictions, by using ordinary least squares (OLS) with Newey–West covariance correction for serial correlation.

3. EMPIRICAL EVIDENCE

The purpose of this section is to empirically analyze the performance of the LHAR-CJ model (2.4) and related ones, both in-sample and out-of-sample. Our dataset covers a long time span of almost 28 years of high-frequency data for the S&P 500 futures from 28 April 1982 to 5 February 2009. We leave out from the sample the week of the 1987 October crash (when included, results are qualitatively very similar but less clear-cut) and days with less than 500 trades. We are left with 6,669 days. All the quantities of interest are computed on an annualized base. Figure 2 reports the relative contribution of the quadratic variation of jumps with respect to the total quadratic variation, computed on a 3-month and 1-year moving window. In line with the results in Andersen et al. (2007) and Huang and Tauchen (2005), we find a jump contribution varying between 2% and 30% of the total variation (with an overall sample mean of about 6%).

3.1 In-Sample Analysis

The results of the estimation of the LHAR-CJ on the S&P 500 sample with $h = 1, 5, 10$, and 22 are reported in Table 1, together with their statistical significance evaluated with the

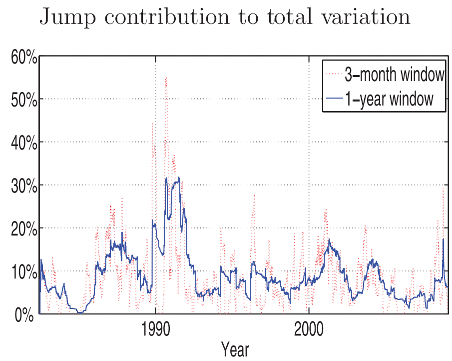

Figure 2. Percent contribution of daily jump to the total quadratic variation measured over a moving window of 3-month (dotted line) and 1-year (solid line) for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations) excluding the October 1987 crash. The C-Tz statistics is computed with a confidence interval of $\alpha = 99.9\%$. The online version of this figure is in color.

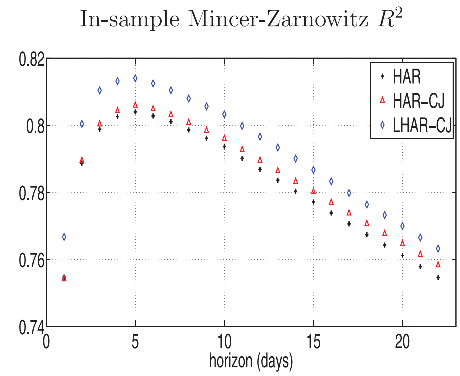

Figure 3. R^2 of Mincer-Zarnowitz regressions for static in sample one-step ahead forecasts for horizons ranging from 1 day to 1 month of the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The forecasting models are the standard HAR model with only heterogeneous volatility, the HAR-CJ model with heterogeneous jumps, and the LHAR-CJ model. The online version of this figure is in color.

Newey–West robust t statistic. The forecasts of the different models are evaluated on the basis of the adjusted R^2 of the regressions, and the heteroscedasticity-adjusted root mean square error (HRMSE) proposed by Bollerslev and Ghysels (1996).

As usual, all the coefficients of the three continuous volatility components are positive and, in general, highly significant. The impact of daily and weekly volatilities decreases with the forecasting horizon of future volatility, while the impact of monthly volatility increases. The coefficient that measures the impact of monthly volatility on future daily volatility is approximately double than that of daily volatility on future monthly volatility. This finding is consistent with Corsi (2009).

Table 1. LHAR-CJ estimates

S&P 500 LHAR in-sample regression, period 1982–2009				
Variable	1 day	1 week	2 weeks	1 month
c	0.442*	0.549*	0.662*	0.858*
	(10.699)	(9.258)	(8.525)	(7.756)
$\widehat{\mathsf{C}}$	0.307*	0.201*	0.154*	0.116*
	(16.983)	(14.158)	(12.984)	(10.590)
$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.359*	0.332*	0.286*
	(13.908)	(11.251)	(9.166)	(6.784)
$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.319*	0.370*	0.415*
	(10.958)	(10.913)	(10.198)	(9.344)
$\widehat{\mathsf{J}}$	0.043*	0.020*	0.017*	0.012*
	(7.057)	(4.453)	(4.485)	(3.804)
$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.013*	0.011*	0.010
	(3.373)	(3.112)	(2.256)	(1.913)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.008*	0.010*	0.014*
	(2.199)	(2.106)	(2.205)	(2.336)
r^-	−0.007*	−0.005*	−0.004*	−0.003*
	(−9.669)	(−10.435)	(−8.298)	(−5.518)
$r^{(5)-}$	−0.008*	−0.006*	−0.008*	−0.007*
	(−4.412)	(−3.059)	(−4.012)	(−3.472)
$r^{(22)-}$	−0.009*	−0.012*	−0.009	−0.004
	(−2.845)	(−2.314)	(−1.481)	(−0.467)
$\bar{R}^2$	0.7664	0.8137	0.8030	0.7629
HRMSE	0.2168	0.1692	0.1699	0.1796

NOTES: OLS estimates of LHAR-CJ regressions, model (2.4), for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The LHAR-CJ model is estimated with $h = 1$ (1 day), $h = 5$ (1 week), $h = 10$ (2 weeks), and $h = 22$ (1 month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parentheses are t statistics based on Newey–West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. An asterisk denotes 95% significance.

Estimation of model (2.4) also reveals the strong significance (with an economically sound negative sign) of the negative returns at all the daily, weekly, and monthly aggregation frequencies, which unveils a heterogeneous structure in the leverage effect as well. Not only daily negative returns affect the next day volatility (the well-known leverage effect) but, in addition, also the negative returns of the past week and month have an impact on forthcoming volatility. This novel finding suggests that the market might aggregate daily, weekly, and monthly memory, observing and reacting to price declines happened in the past week and month, revealing a persistent leverage effect.

A similar heterogeneous structure is present in the impact of jumps on future volatility. However, while the daily and weekly jumps are highly significant and positive, their impact decreases with the forecasting horizon at a fast rate. The monthly jump component is also slightly significant over all forecasting horizons, with its impact increasing with the horizon.

Figure 3 shows the Mincer-Zarnowitz R^2 for different models at various horizons, which obtains its maximum at 1 week. Moreover, Figure 3 shows unambiguously that the inclusion of both the heterogeneous jumps and the heterogeneous leverage effects considerably improves the forecasting performance of the S&P 500 volatility at any forecasting horizon. In particular, the inclusion of the heterogeneous leverage effect provides the most relevant overall benefit in the in-sample performance. We confirm this result out-of-sample in Section 3.6.

3.2 Forecasting Jumps and Continuous Volatility

Following Busch et al. (2011), we can use the continuous and jump component of the total quadratic variation as dependent variables as well, and investigate the possibility of forecasting them separately. We then specify the LHAR-C-CJ model for forecasting the continuous quadratic variation as

$$\begin{aligned}\log \widehat{\mathsf{C}}_{t+h}^{(h)} = {} & c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ & + \alpha^{(d)} \log(1 + \widehat{\mathsf{J}}_t) + \alpha^{(w)} \log\left(1 + \widehat{\mathsf{J}}_t^{(5)}\right) \\ & + \alpha^{(m)} \log\left(1 + \widehat{\mathsf{J}}_t^{(22)}\right) + \gamma^{(d)} r_t^{-} + \gamma^{(w)} r_t^{(5)-} \\ & + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)}, \end{aligned} \tag{3.1}$$

and the LHAR-J-CJ model for forecasting jumps as

$$\begin{aligned}\log\left(1 + \widehat{\mathsf{J}}_{t+h}^{(h)}\right) = {} & c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ & + \alpha^{(d)} \log(1 + \widehat{\mathsf{J}}_t) + \alpha^{(w)} \log\left(1 + \widehat{\mathsf{J}}_t^{(5)}\right) \\ & + \alpha^{(m)} \log\left(1 + \widehat{\mathsf{J}}_t^{(22)}\right) + \gamma^{(d)} r_t^{-} + \gamma^{(w)} r_t^{(5)-} \\ & + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)}. \end{aligned} \tag{3.2}$$

Corresponding models with $\alpha^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$ on the right-hand side are named LHAR-C and LHAR-J, respectively. Estimation results for the daily horizon ($h = 1$) are presented in Table 2. We can see that, as already recognized in the literature, the jump component is essentially unpredictable, with an adjusted R^2 of just 1.52%. We find a strong significant impact on future jumps only for the monthly jump component, which is a clear indication of jump clustering. Also, the daily and weekly volatilities are significant, but with opposite signs. The impact of monthly jumps on $\widehat{\mathsf{C}}_t$ is instead not significant, confirming that the impact of jumps on volatility is quite transitory in nature.

Table 2. Continuous volatility and jumps forecasting

LHAR-CJ, LHAR-C-CJ, and LHAR-J-CJ regression			
	Dependent variables		
Variable	$\widehat{\mathsf{V}}$	$\widehat{\mathsf{C}}$	$\widehat{\mathsf{J}}$
c	0.442*	0.440*	−0.287*
	(10.699)	(10.995)	(−2.770)
$\widehat{\mathsf{C}}$	0.307*	0.324*	−0.097*
	(16.983)	(17.803)	(−2.388)
$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.354*	0.121*
	(13.908)	(13.406)	(2.089)
$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.214*	0.065
	(10.958)	(10.822)	(1.261)
$\widehat{\mathsf{J}}$	0.043*	0.037*	0.040
	(7.057)	(5.941)	(1.829)
$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.010*	0.000
	(3.373)	(3.051)	(0.029)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.001	0.026*
	(2.199)	(0.461)	(4.313)
r^{-}	−0.007*	−0.007*	−0.002
	(−9.669)	(−10.173)	(−1.128)
$r^{(5)-}$	−0.008*	−0.007*	−0.006
	(−4.412)	(−4.151)	(−1.241)
$r^{(22)-}$	−0.009*	−0.010*	0.014
	(−2.845)	(−2.869)	(1.341)
$\bar{R}^2$	0.7664	0.7594	0.0152

NOTES: OLS estimates for the LHAR-CJ model using dependent variables $\log \widehat{\mathsf{V}}_t$, $\log \widehat{\mathsf{C}}_t$, and $\log(1 + \widehat{\mathsf{J}}_t)$, and daily forecasting horizon for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parentheses are t statistics based on Newey–West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. An asterisk denotes 95% significance.

Table 3. OLS estimates of the regression of daily returns on daily variance forecasts $\widetilde{\widetilde{\mathsf{V}}}_t$ obtained with different models

Risk-return regression: $r_t = c + \beta \widetilde{\widetilde{\mathsf{V}}}_t + \varepsilon_t$			
Model	Slope $\hat{\beta}$	t statistics	$\bar{R}^2$
HAR	0.0026018	3.4257	0.16%
HAR-CJ	0.0025964	3.4113	0.16%
LHAR-CJ	0.0033497	6.7266	0.66%

While the inability of forecasting jumps has also been signaled by Busch et al. (2011), they find, contrary to our analysis, that the impact of daily jumps on the future daily continuous quadratic variation is significantly negative—a result which would imply, on average, a volatility decrease after a jump. Corsi et al. (2010) showed that this result is induced by the small-sample bias of bipower variation measures. Building on their work, we use threshold bipower variation and uncover the positive (and transitory) impact of jumps on future volatility also in the presence of a persistent leverage effect.

3.3 Risk-Return Trade-Off

Our volatility forecasts can also be evaluated in terms of the implied risk-return trade-off, since economic theory posits that there should be a positive relation between returns and perceived risk. The literature on the risk-return trade-off is very large. Recent research on this topic includes Ghysels, Santa-Clara, and Valkanov (2005), Christensen and Nielsen (2007), Bandi and Perron (2008), and Bollerslev, Tauchen, and Zhou (2008). Here, we use as a measure of risk the daily volatility forecast of (1) the standard HAR model, (2) the HAR-CJ model, and (3) the LHAR-CJ model. All models are specified in the logarithmic form. We regress the return on the variance forecasts $\widetilde{\widetilde{\mathsf{V}}}_t$, that is, on the exponential of the logarithmic forecasts $\log \widetilde{\mathsf{V}}_t$. Estimation results, obtained via OLS, are shown in Table 3.

The results are in agreement with those in Bali and Peng (2006), who analyzed the same data until 2002. With all models, we find a significant impact of volatility forecasts on returns, which is compatible with economic theory, even if we have a very low R^2, as it is common in this kind of applications. The inclusion of jumps is not beneficial to return forecasting. Instead, the inclusion of the leverage component increases the slope coefficient and almost doubles the significance of the effect. Similar results are obtained by regressing r_t on $\sqrt{\widetilde{\widetilde{\mathsf{V}}}_t}$, or replacing $\widetilde{\widetilde{\mathsf{V}}}_t$ with the volatility forecasts of $\widetilde{\widetilde{\mathsf{C}}}_t$ obtained with the LHAR-C-CJ model discussed in Section 3.2.

3.4 Is Leverage Effect Induced by Jumps?

An open research question is whether, and to what extent, the leverage effect is induced by jumps; see Bandi and Renò (2011a). In our setting, we investigate this issue by separating

Table 4. The impact of jump sign

HAR-CJ$^+$ regression					LHAR-CJ$^+$ regression				
	1 day	1 week	2 weeks	1 month		1 day	1 week	2 weeks	1 month
c	0.232*	0.377*	0.505*	0.747*	c	0.442*	0.549*	0.661*	0.858*
	(5.774)	(6.217)	(6.418)	(6.736)		(10.724)	(9.277)	(8.531)	(7.778)
$\widehat{\mathsf{C}}$	0.398*	0.265*	0.214*	0.165*	$\widehat{\mathsf{C}}$	0.307*	0.201*	0.154*	0.116*
	(21.521)	(18.225)	(16.084)	(12.442)		(16.972)	(14.185)	(13.007)	(10.608)
$\widehat{\mathsf{C}}^{(5)}$	0.366*	0.368*	0.346*	0.291*	$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.359*	0.332*	0.286*
	(13.889)	(11.697)	(9.750)	(7.327)		(13.885)	(11.237)	(9.144)	(6.777)
$\widehat{\mathsf{C}}^{(22)}$	0.190*	0.291*	0.338*	0.390*	$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.319*	0.370*	0.415*
	(9.470)	(9.743)	(9.059)	(8.875)		(10.914)	(10.905)	(10.183)	(9.336)
$\widehat{\mathsf{J}}^{+}$	0.044*	0.018*	0.016*	0.013*	$\widehat{\mathsf{J}}^{+}$	0.044*	0.018*	0.015*	0.012*
	(6.099)	(3.264)	(3.000)	(2.538)		(6.176)	(3.182)	(2.819)	(2.395)
$\widehat{\mathsf{J}}^{-}$	0.074*	0.040*	0.039*	0.027*	$\widehat{\mathsf{J}}^{-}$	0.043*	0.019*	0.020*	0.011*
	(6.909)	(6.833)	(6.658)	(5.351)		(4.598)	(3.387)	(3.633)	(2.285)
$\widehat{\mathsf{J}}^{(5)}$	0.009*	0.012*	0.010*	0.010	$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.013*	0.011*	0.010
	(2.645)	(2.724)	(2.028)	(1.799)		(3.372)	(3.110)	(2.254)	(1.914)
$\widehat{\mathsf{J}}^{(22)}$	0.005	0.007	0.009*	0.014*	$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.008*	0.010*	0.014*
	(1.845)	(1.875)	(2.026)	(2.242)		(2.200)	(2.104)	(2.203)	(2.337)
–					r^{-}	−0.007*	−0.005*	−0.004*	−0.003*
						(−9.772)	(−10.057)	(−7.804)	(−5.341)
–					$r^{(5)-}$	−0.008*	−0.006*	−0.008*	−0.007*
						(−4.409)	(−3.068)	(−4.020)	(−5.341)
–					$r^{(22)-}$	−0.009*	−0.012*	−0.008	−0.004
						(−2.844)	(−2.315)	(−1.484)	(−0.467)
$\bar{R}^2$	0.7543	0.8060	0.7960	0.7582	$\bar{R}^2$	0.7664	0.8137	0.8030	0.7629
HRMSE	0.2201	0.1721	0.1722	0.1812	HRMSE	0.2168	0.1692	0.1698	0.1796

NOTES: OLS estimates for the LHAR-CJ$^+$ and the HAR-CJ$^+$ model in which we separate daily jumps into positive and negative for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The models are estimated with $h = 1$ (1 day), $h = 5$ (1 week), $h = 10$ (2 weeks), and $h = 22$ (1 month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parentheses are t statistics based on Newey–West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. An asterisk denotes 95% significance.

the daily jump contribution to quadratic variation in a positive and negative part. To this purpose, we define

$$\widehat{\mathsf{J}}_t^{+} = \widehat{\mathsf{J}}_t \cdot I_{\{r_t>0\}},$$
$$\widehat{\mathsf{J}}_t^{-} = \widehat{\mathsf{J}}_t \cdot I_{\{r_t<0\}},$$

and we insert $\widehat{\mathsf{J}}_t^{+}$ and $\widehat{\mathsf{J}}_t^{-}$ in the LHAR model in place of $\widehat{\mathsf{J}}_t$, denoting by LHAR-CJ$^+$ the newly obtained model. We also estimate the HAR-CJ$^+$ model, which is the same without leverage terms. Results are reported in Table 4. Given the evidence provided by Todorov and Tauchen (2011) and Bandi and Renò (2011a), with different statistical methods, of a strong negative correlation between price and volatility jumps, we expect the coefficient of $\widehat{\mathsf{J}}_t^{-}$ to be larger than that of $\widehat{\mathsf{J}}_t^{+}$.

When we estimate the HAR-CJ$^+$ model, this is exactly what we find: the coefficient of negative jumps is almost double than that of positive jumps, and this is true for all the considered forecasting horizons, ranging from 1 day to 1 month. However, when we estimate the full LHAR-CJ$^+$ model, which includes all the leverage terms (which are also affected by the jump component), the impact of positive and negative jumps is estimated to be roughly the same, again at all the considered horizons. Our interpretation of this result is that the number of cojumps is likely too small to allow for the joint detection of the continuous leverage effect and the covariance part genuinely due to jumps.

3.5 Robustness to Other Volatility Measures

In the literature, many volatility measures have been proposed as explanatory variables for the volatility dynamics. Forsberg and Ghysels (2007) proposed the use of realized absolute variation (RAV), which shows a more persistent dynamics than realized volatility being more robust to microstructure noise and jumps. The range, that is, the difference between the highest and the lowest price within a day, has also been found to be significant by many authors; for example, see Brandt and Jones (2006) and Engle and Gallo (2006). Recently, Barndorff-Nielsen, Kinnebrock, and Shephard (2010) proposed the realized semivariance as the sum of square negative returns to capture the impact on volatility of downward price pressures. Visser (2008) combined RAV and semivariance by taking the sum of negative absolute squared returns.

In the spirit of Forsberg and Ghysels (2007), we compare the relative explanatory power of different volatility measures by estimating a set of models (for space concerns we limit ourselves to the 1-day horizon) obtained by adding explanatory variables to model (2.4). Estimation results are reported in Table 5. In line with previous literature, we find that the RAV computed at 5-minute frequency and the range have a significant impact on future volatility. However, they seem to be mainly substitutes for continuous volatility and jumps, which is not totally surprising since they are estimators (though noisy) of the total quadratic variation. Indeed, for instance, when the range

Table 5. Alternative specifications

S&P 500 in-sample estimates, period 1982–2009						
Variable	LHAR-CJ	LHAR-CJ-RAV	LHAR-CJ-Range	LHAR-CJ-SemiRV	LHAR-CJ-SemiRAV	LHAR-CJ-All
Constant	0.442*	0.593*	0.444*	0.470*	0.554*	0.616*
	(10.699)	(12.419)	(10.860)	(11.029)	(9.869)	(8.914)
$\widehat{C}$	0.307*	0.116*	0.207*	0.249*	0.252*	0.085*
	(16.983)	(3.355)	(10.035)	(9.588)	(9.489)	(2.449)
$\widehat{C}^{(5)}$	0.369*	0.374*	0.384*	0.372*	0.372*	0.389*
	(13.908)	(14.125)	(14.568)	(14.074)	(14.032)	(14.693)
$\widehat{C}^{(22)}$	0.222*	0.221*	0.223*	0.223*	0.222*	0.223*
	(10.958)	(10.943)	(11.071)	(11.003)	(10.963)	(11.035)
$\widehat{J}$	0.043*	0.018*	0.024*	0.033*	0.037*	0.010
	(7.057)	(2.519)	(3.893)	(4.850)	(5.620)	(1.331)
$\widehat{J}^{(5)}$	0.011*	0.012*	0.012*	0.011*	0.011*	0.012*
	(3.373)	(3.717)	(3.789)	(3.409)	(3.413)	(3.959)
$\widehat{J}^{(22)}$	0.005*	0.006*	0.005*	0.006*	0.006*	0.006*
	(2.199)	(2.334)	(2.207)	(2.266)	(2.277)	(2.329)
r^-	−0.007*	−0.007*	−0.006*	−0.006*	−0.006*	−0.005*
	(−9.669)	(−10.156)	(−8.193)	(−8.197)	(−7.792)	(−5.630)
$r^{(5)-}$	−0.008*	−0.007*	−0.008*	−0.008*	−0.008*	−0.008*
	(−4.412)	(−4.147)	(−4.847)	(−4.368)	(−4.278)	(−4.582)
$r^{(22)-}$	−0.009*	−0.009*	−0.009*	−0.010*	−0.010*	−0.010*
	(−2.845)	(−2.687)	(−2.868)	(−2.980)	(−2.951)	(−2.853)
RAV		0.185*				0.077
		(6.533)				(1.867)
Range			0.088*			0.086*
			(9.364)			(7.911)
Semi*RV*				0.058*		−0.011
				(3.305)		(−0.330)
Semi*RAV*					0.054*	0.056
					(3.141)	(1.455)
$\bar{R}^2$	0.7664	0.7681	0.7696	0.7668	0.7668	0.7704
HRMSE	0.2168	0.2158*	0.2148*	0.2165	0.2165	0.2142*
DM		(2.7060)	(4.6323)	(1.5020)	(1.8539)	(4.6778)

NOTES: Estimated parameters, adjusted R^2, and HRMSE of alternative specifications of the baseline LHAR-CJ model for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations); t statistic and DM test for HRMSE are given in parentheses. An asterisk denotes 95% significance.

replaces the jumps (LHAR-Range model, not reported), the coefficients of daily continuous volatility almost halve. The adjusted R^2 of the two competing regressions (LHAR-Range and LHAR-CJ) is practically the same. When the range is inserted together with the jumps (LHAR-CJ-Range), the coefficients of both daily volatility and jumps decrease, although they remain highly significant. The significance of the heterogeneous leverage effect is instead unaffected by the presence of RAV and range. We thus conclude that the RAV and the range, while partially proxying for both volatility and jumps, are also able to capture some other (small) effect, which is not captured by the other variables in the model. However, the adjusted R^2 of the encompassing regression increases only marginally. The realized semivariance (semiRV) of Barndorff-Nielsen et al. (2010) and the downward absolute power variation of Visser (2008) (semiRAV) have a weaker impact. Realized semivariance and semipower variation are significant in explaining future volatility, and, again, they are correlated with both the daily two-scale estimator and the jumps (typically depleting the significance of the corresponding coefficients without totally removing it), while unrelated to the leverage effect. However, their contribution to the model performance is not significant [as measured by the Diebold-Mariano (DM) test]. Moreover, when they are included in the all-encompassing model, they both remain insignificant.

In summary, the results of this section show that when the other volatility measures proposed in the literature are inserted in the baseline LHAR-CJ model, they either do not contribute significantly or only marginally contribute to the performance of the model. Moreover, they mainly act as substitutes for continuous volatility and jumps. Hence, we conclude that the LHAR-CJ model seems to capture the main determinants of volatility dynamics.

3.6 Out-of-Sample Analysis

In this section, we evaluate the performance of the LHAR-CJ model on the basis of a genuine out-of-sample analysis. For the out-of-sample forecast of $\widehat{V}_t$ on the $[t, t+h]$ interval, we keep the same forecasting horizons ranging from 1 day to 1 month and reestimate the model at each day t on an increasing window of all the observations available up to time $t-1$. The out-of-sample forecasting performance for $\log \widehat{V}_t$ in terms of Mincer-Zarnowitz R^2 is reported in Figure 4, together with the DM test

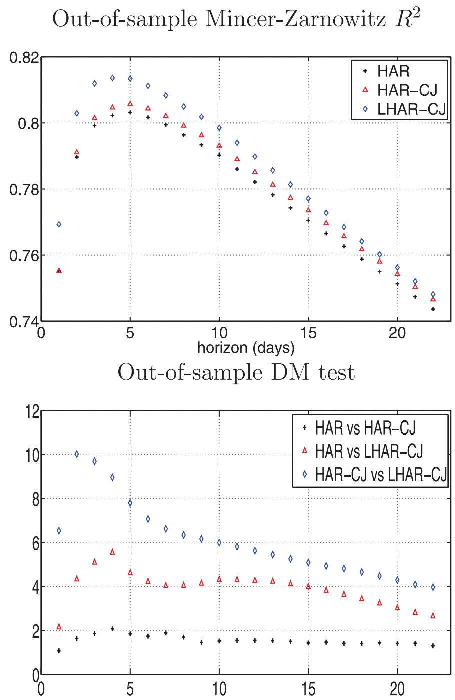

Figure 4. Top: R^2 of Mincer-Zarnowitz regressions for out-of-sample forecasts. Bottom: DM test for the out-of-sample HRMSE. Horizons range from 1 day to 1 month for the S&P 500 futures from 28 April 1982 to 5 February 2009 (6,669 observations, the first 2,500 observations are used to initialize the models). The forecasting models are the standard HAR model with only heterogeneous volatility, the HAR-CJ model with heterogeneous jumps, and the LHAR-CJ model. The comparison of the HAR-CJ and the LHAR-CJ models is made by employing the Clark and West (2007) adjustment to the DM test for nested models. In both cases, the test is asymptotically standard normal under the null. The online version of this figure is in color.

computed for the HRMSE loss function at all the considered horizons.

The superiority of the LHAR-CJ model at all horizons, with respect to the HAR and the HAR-CJ models, is statistically significant, thus validating the importance of including both the heterogeneous leverage effects and the jumps in the forecasting model. The out-of-sample exercise confirms that the maximum R^2 is obtained at a forecasting horizon of 1 week.

The superiority of the HAR-CJ model versus the HAR model is instead milder, but the reason is that the improvements appear only in days that follow a jump (368 out of 6,669), and thus on a small subsample. However, it is important to note that the inclusion of the jump component also helps in forecasting longer horizon volatility; see the results in the online Appendix and Andersen et al. (2007).

4. CONTINUOUS-TIME MODELS

The main motivation of this section is to provide a continuous-time model that delivers the stylized facts documented in the previous sections and captured by the newly proposed discrete-time model. Such a continuous-time model would then, at the same time, not only provide a more accurate statistical representation of the data, but also bridge the gap between continuous-time and discrete-time modeling. Since the inception of the GARCH literature, indeed, volatility forecasting is mostly set up in discrete time, and the LHAR-CJ model is no exception. However, models used in practice, for example, for option pricing, are often specified in continuous time. In the literature, the link between GARCH-like models and continuous-time models is well established; see Nelson (1990), Duan (1997), and Corradi (2000). However, the link between continuous-time models and HAR-like models is unclear.

We achieve this goal by estimating continuous-time models via *indirect inference* [Gourieroux, Monfort, and Renault 1993; see Bollerslev, Gallant, Pigorsch, Pigorsch, and Tauchen (2006) for an application similar to ours] using the (L)HAR (-CJ) specification as an auxiliary model. The idea of the indirect inference approach is to estimate the auxiliary model both on actual data and on data simulated from the structural model, and then to minimize the distance (labeled by χ^2), as a function of the structural model parameters, between the estimated coefficients weighted with the inverse variance-covariance matrix of the estimates. Details are provided in the online Appendix.

It is usually suggested that the long-range autocorrelation function of realized volatility is generated by a long-memory model. For this reason, we start by estimating the Comte and Renault (1998) continuous-time model:

$$\begin{aligned} dX_t &= \sigma_t dW_t, \\ d\log\sigma_t &= k(\omega - \log\sigma_t)dt + \eta dW_t^{(d)}, \end{aligned} \tag{4.1}$$

where W_t is a standard Brownian motion and $dW_t^{(d)}$ is an independent *fractional Brownian motion* with memory parameter $d \in [0, 0.5]$ (ensuring stationarity). The value $d = 0$ corresponds to the standard Brownian motion, while a higher d corresponds to a longer memory in the time series. Estimation of model (4.1) has only been performed, to the best of our knowledge, in Casas and Gao (2008) using spectral methods. For simulation studies, see Nielsen and Frederiksen (2008) and Rossi and Spazzini (2010). A discrete-time specification of model (4.1) is instead estimated more routinely (see Comte and Renault 1996; Christensen and Nielsen 2007). To assess the impact on the results of the fractional difference parameter d, we first estimate model (4.1) for different fixed values of d, and then estimate the four parameters (k, ω, η, d) jointly.

The results, reported in Table 6, show that model (4.1) is substantially unable to reproduce the coefficients of the HAR model. The best fit is obtained with a value of $d = 0.491$, which is very close to nonstationarity. However, even for this fit, the implied daily coefficient of the HAR model is still too high, and the implied weekly coefficient is still too low. To understand the motivation of this failure, it is interesting to look at the estimates obtained for fixed and increasing values of d. When $d = 0$, the model is assimilable to an $AR(1)$ specification and thus is unsurprisingly unable to reproduce the HAR coefficients. As d increases, persistence comes from two terms: the mean-reverting term $k(\omega - \log\sigma_t^2)dt$ and the fractional Brownian motion $\eta dW_t^{(d)}$. However, these two components can vary only in a rigid fashion. For example, when d increases, the mean-reversion parameter k has to increase sharply because of the mean reversion observed in the volatility series, which would not be reproduced by $\eta dW_t^{(d)}$ alone. Note that the ratio

Table 6. Estimates (daily units) via indirect inference of the long-memory model (4.1), using the HAR model as an auxiliary model, and implied HAR coefficients

Structural model:
$$d\log\sigma_t = k(\omega - \sigma_t)dt + \eta dW_t^{(d)}$$

Parameter	Estimates $d=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$	$d=0.49$	Unconstrained d
k	0.144	0.325	0.750	1.917	7.706	145.081	144.853
ω	−0.166	−0.218	−0.268	−0.290	−0.313	−0.446	−0.447
η	0.248	0.272	0.352	0.625	2.512	118.350	123.746
d	0.000	0.100	0.200	0.300	0.400	0.490	0.491
χ^2	1109.4187	1137.1615	931.5612	527.4070	178.8777	43.2839	42.6094

Auxiliary model:
$$\log\widehat{C}_{t+1} = c + \beta^{(d)}\log\widehat{C}_{t+1} + \beta^{(w)}\log\widehat{C}_{t+1}^{(5)} + \beta^{(m)}\log\widehat{C}_{t+1}^{(22)} + \varepsilon_{t+1}$$

Parameter	Estimated	Implied $d=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$	$d=0.49$	Unconstrained d
c	0.208	0.661	0.875	1.019	0.966	0.688	0.440	0.436
$\beta^{(d)}$	0.388	0.913	0.877	0.756	0.575	0.449	0.420	0.421
$\beta^{(w)}$	0.368	−0.044	−0.083	−0.059	0.057	0.203	0.281	0.281
$\beta^{(m)}$	0.203	0.004	0.035	0.099	0.173	0.209	0.211	0.211
σ_ε^2	0.198	0.188	0.191	0.195	0.198	0.199	0.198	0.197

$k_\infty = \eta d/k$ remains approximately constant when changing d. This rigidity makes model (4.1) unable to reproduce the HAR model.

Given the failure of the single-factor long-memory model, we get inspiration from the very nature of the HAR model, which reproduces a slowly decaying autocorrelation function via the aggregation of different frequencies, and estimate affine multifactor models with jumps:

$$\begin{aligned} dX_t &= \textstyle\sum_{i=1}^N \sqrt{V_t^i}\,dW_t^i + dJ_t^X, \\ dV_t^i &= \kappa_i(\omega_i - V_t^i)dt + \eta_i\sqrt{V_t^i}\,dW_t^{i+N} + dJ_t^i, \\ &\qquad i = 1,\dots,N, \\ \mathrm{corr}&(dW^i, dW^{i+N}) = \rho_i, \end{aligned} \tag{4.2}$$

where $W^1,\dots,W^{2N}$ is a multivariate (possibly correlated) Brownian motion and $\mathbf{J} = \{J^X, J^1, \dots, J^N\}$ is a multivariate (possibly correlated) Poisson process with constant intensities, normal jump sizes in the prices, and exponential jump sizes in volatility. In the case of no jumps, when $N = 1$, this is the well-known Heston (1993) model; Duffie, Pan, and Singleton (2000) and Pan (2002) included jumps as in the Eraker, Johannes, and Polson (2003) model considered earlier. With $N = 2$, this model has been used, for example, by Bates (2000) and, more recently, by Christoffersen, Heston, and Jacobs (2009).

When using the HAR model as an auxiliary model, we set $N = 2$, $\rho_i = 0$, $\mathbf{J} = 0$, and to achieve identification $\omega_1 = \omega_2 = \omega$. Corresponding estimates, together with the implied HAR coefficients, are reported in Table 7. Contrary to the single-factor model with the fractional Brownian motion, the two-factor model is perfectly able to reproduce the HAR coefficients, thus obtaining an objective function χ^2 close to zero. Estimates are compatible with those typically encountered in the literature: the fit implies the presence of a fast mean-reverting factor with a half-life of less than 1 day, and a slowly mean-reverting factor with a half-life of nearly 200 days. The superposition of these two frequencies produces the desired effect in terms of volatility persistence. Lieberman and Phillips (2008) suggested that the usage of integrated volatility measures produces a longer memory than that implied in the dynamics of spot volatility. Our result also explains why multifactor model works so well in

Table 7. Estimates (daily units) via indirect inference of model (4.2) with $N = 2$ and $\omega_1 = \omega_2 = \omega$, using the HAR model as an auxiliary model, and implied HAR coefficients

Structural model:
$$\begin{aligned} dX_t &= \sqrt{V_t^1}\,dW_t^1 + \sqrt{V_t^2}\,dW_t^2 \\ dV_t^1 &= \kappa_1(\omega - V_t^1)dt + \eta_1\sqrt{V_t^1}\,dW_t^3 \\ dV_t^2 &= \kappa_2(\omega - V_t^2)dt + \eta_2\sqrt{V_t^2}\,dW_t^4 \end{aligned}$$

Parameter	Estimates
κ_1	2.1461
κ_2	0.0042
ω	0.4497
η_1	0.8513
η_2	0.3110
χ^2	0.0000026

Auxiliary model:
$$\log\widehat{C}_{t+1} = c + \beta^{(d)}\log\widehat{C}_{t+1} + \beta^{(w)}\log\widehat{C}_{t+1}^{(5)} + \beta^{(m)}\log\widehat{C}_{t+1}^{(22)} + \varepsilon_{t+1}$$

Parameter	Estimated	Implied
c	0.208	0.208
$\beta^{(d)}$	0.388	0.388
$\beta^{(w)}$	0.368	0.368
$\beta^{(m)}$	0.203	0.203
σ_ε^2	0.198	0.198

Table 8. Estimates (daily units) via indirect inference of model (4.2) with two and three factors, using the LHAR model as an auxiliary model, and implied LHAR coefficients

Structural model:

$$
\begin{aligned}
dX_t &= \sqrt{V_t^1}dW_t^1 + \sqrt{V_t^2}dW_t^2 + \sqrt{V_t^3}dW_t^3 + c_X dN_t \\
dV_t^1 &= \kappa_1(\omega_1 - V_t^1)dt + \eta_1\sqrt{V_t^1}dW_t^4 + c_\sigma dN_t \\
dV_t^2 &= \kappa_2(\omega_2 - V_t^2)dt + \eta_2\sqrt{V_t^2}dW_t^5 \\
dV_t^3 &= \kappa_3(\omega_3 - V_t^3)dt + \eta_3\sqrt{V_t^3}dW_t^6 \\
&\mathrm{corr}(dW^1, dW^4) = \rho_1 \\
&\mathrm{corr}(dW^2, dW^5) = \rho_2 \\
&\mathrm{corr}(dW^3, dW^6) = \rho_3 \\
&N_t \sim \mathrm{Poisson}(\lambda t), \quad c_X \sim \mathcal{N}(0, \sigma_J^2), \quad c_\sigma \sim \exp(\mu_\sigma)
\end{aligned}
$$

Parameter	Two factor	Three factor	Three factor with jumps
κ_1	8.1088	6.7647	5.7233
κ_2	0.0003	0.6556	0.8390
κ_3	–	0.0036	0.00004
ω_1	0.3030	0.2480	0.2468
ω_2	0.5165	0.1348	0.1740
ω_3	–	0.1894	0.1311
η_1	1.6348	2.0128	1.8970
η_2	0.3748	0.3880	0.4137
η_3	–	0.2849	0.3412
ρ_1	0.9847	0.3201	0.5040
ρ_2	−0.9807	−0.9949	−0.8947
ρ_3	–	−0.9173	−0.9714
λ	–	–	0.0129
σ_J	–	–	0.0254
μ_σ	–	–	0.1420
χ^2	123.301	0.221	28.369

Auxiliary model:

$$\log \widehat{\mathsf{C}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_{t+1} + \beta^{(w)} \log \widehat{\mathsf{C}}_{t+1}^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_{t+1}^{(22)} + \gamma^{(d)} r_{t+1}^{-} + \gamma^{(w)} r_{t+1}^{(5)-} + \gamma^{(m)} r_{t+1}^{(22)-} + \varepsilon_{t+1}$$

Parameter	Estimated	Two factor	Three factor
c	0.421	0.561	0.428
$\beta^{(d)}$	0.299	0.238	0.302
$\beta^{(w)}$	0.366	0.530	0.358
$\beta^{(m)}$	0.236	0.105	0.239
$\gamma^{(d)}$	−0.007	−0.004	−0.007
$\gamma^{(w)}$	−0.008	−0.014	−0.008
$\gamma^{(m)}$	−0.009	−0.010	−0.010
σ_ε^2	0.187	0.188	0.187

Auxiliary model:

$$\log \widehat{\mathsf{V}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_{t+1} + \beta^{(w)} \log \widehat{\mathsf{C}}_{t+1}^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_{t+1}^{(22)} + \alpha^{(d)} \log(1 + \widehat{\mathsf{J}}_{t+1}) + \alpha^{(w)} \log(1 + \widehat{\mathsf{J}}_{t+1}^{(5)}) + \alpha^{(m)} \log(1 + \widehat{\mathsf{J}}_{t+1}^{(22)})$$
$$+ \gamma^{(d)} r_{t+1}^{-} + \gamma^{(w)} r_{t+1}^{(5)-} + \gamma^{(m)} r_{t+1}^{(22)-} + \varepsilon_{t+1}$$

Parameter	Estimated	Three factor with jumps
c	0.446	0.384
$\beta^{(d)}$	0.304	0.306
$\beta^{(w)}$	0.369	0.349
$\beta^{(m)}$	0.222	0.260
$\alpha^{(d)}$	0.042	0.017
$\alpha^{(w)}$	0.011	0.011
$\alpha^{(m)}$	0.005	−0.000
$\gamma^{(d)}$	−0.007	−0.006
$\gamma^{(w)}$	−0.008	−0.006
$\gamma^{(m)}$	−0.009	−0.014
σ_ε^2	0.183	0.182

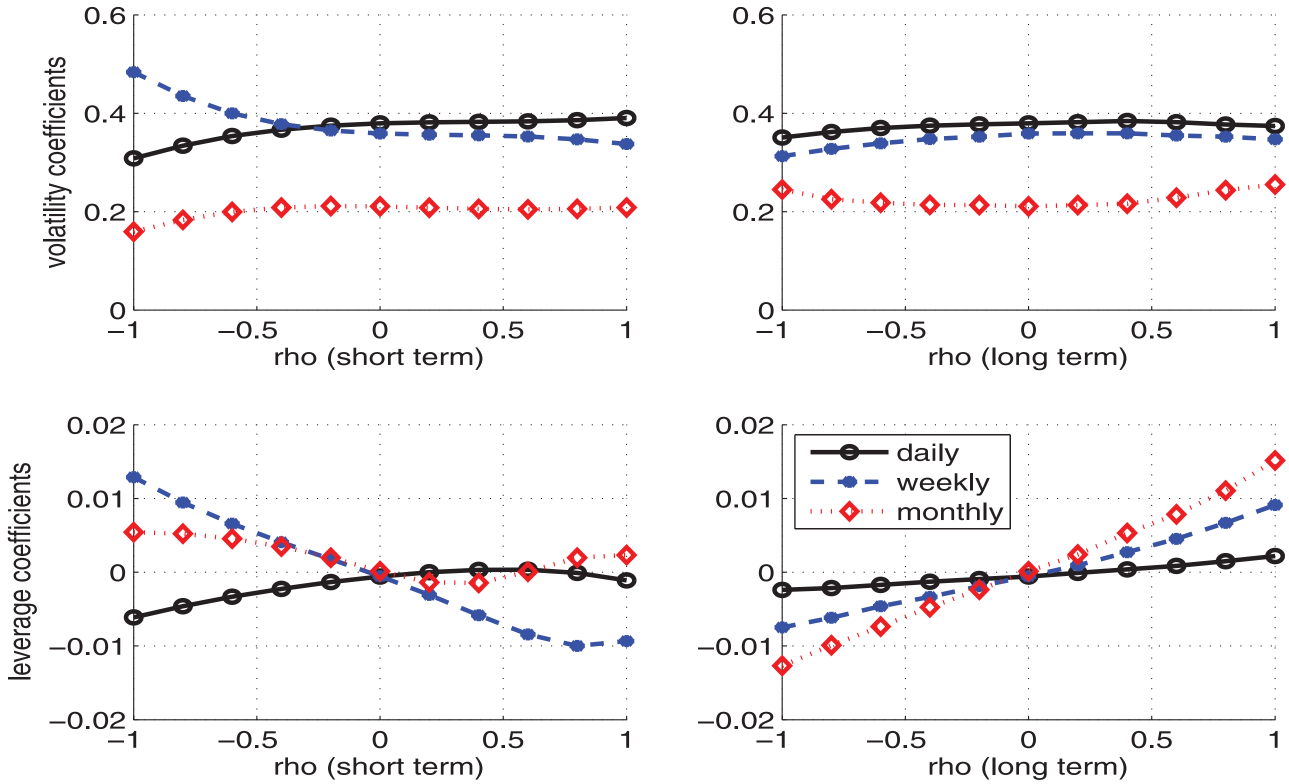

Figure 5. Sensitivity of the coefficients of the LHAR specification (top row: β coefficients of volatility; bottom row: γ coefficients of negative returns), the value of the leverage coefficients ρ_1 (correlation with the fast mean-reverting factor, left column), when $\rho_2 = 0$, and ρ_2 (correlation with the slowly mean-reverting factor, right column), when $\rho_1 = 0$. The online version of this figure is in color.

describing the dynamics of options (see Bates 2000), since they are able to reproduce the volatility dynamics under the natural probability. It also suggests that two factors might be redundant if the volatility dynamics is specified directly with a model similar to HAR: an attempt in this direction is the article of Corsi, Fusari, and La Vecchia (2011), which develops an option pricing model with HAR volatility dynamics providing a remarkable pricing performance with a single volatility factor. Finally, two volatility factors have also been shown to be priced in the cross section of expected returns; see Adrian and Rosenberg (2008).

When the auxiliary model is LHAR, the natural approach is to allow for nonzero correlation coefficients to introduce a leverage effect, again setting $\mathbf{J} = 0$. We report estimates of the two-factor model with leverage effect in Table 8. When fitting the two-factor model using the LHAR model as an auxiliary model, we find $\rho_1 > 0$, that is, a positive correlation coefficient between the fast mean-reverting volatility factor and the returns, while ρ_2 is negative. This fact is not totally surprising since it echoes the results of Chernov, Gallant, Ghysels, and Tauchen (2003) and Bollerslev, Litvinova, et al. (2006), who also estimated (among other models) a two-factor affine model on S&P 500 returns via an efficient method of moments (using an auxiliary GARCH model) and find the correlation coefficient associated with the fast mean-reverting volatility factor to be positive.

In the presence of two factors, the interpretation of the leverage effect is not trivial since, as Chernov et al. (2003) also explained, the average leverage can be negative even with a positive correlation coefficient. However, the reason why a positive correlation arises with the fastest volatility factor remained unclear. Figure 5 can help to provide a possible explanation for this occurrence. We simulate model (4.2) with the coefficients estimated in Table 7, and we vary ρ_1 (with $\rho_2 = 0$) and ρ_2 (with $\rho_1 = 0$) to evaluate the impact of the introduced correlations on the LHAR coefficients. When ρ_1 (the leverage effect of the fast mean-reverting factor) is different from zero, the impact of daily negative returns on future volatility follows the sign of ρ_1, but the opposite hold for the impact of weekly and monthly negative returns. For example, when ρ_1 is negative, we find $\gamma^{(d)} < 0$ but $\gamma^{(w)}, \gamma^{(m)} > 0$. This is due to an *overshooting* effect: a positive correlation at a higher frequency becomes negative at a slower one and vice versa. However, for the slowly mean-reverting factor, the sign of ρ_2 induces a leverage effect with the same sign on the daily, weekly, and monthly coefficients, with the impact increasing with the horizon. The effect of introducing correlations on volatility coefficients is instead marginal. Thus, with a positive ρ_1, we get the right sign for the weekly and monthly coefficients, while the daily coefficient is adjusted with a negative ρ_2.

Using the two-factor model, this mechanism is able to reproduce the LHAR model only partially: the signs are correct but the coefficients estimated on the data can be reproduced only to a limited extent. In order to get a satisfactory agreement with the LHAR model, we need to introduce a third factor (in this case, the parameters of the structural model are not identified). Estimates of the three-factor model are again reported in Table 8, and they show that with $\rho_1 > 0$ and $\rho_2, \rho_3 < 0$, we can reproduce completely the LHAR model.

Finally, we include jumps in the structural model with the aim of reproducing the results of the LHAR-CJ model. Extensive Monte Carlo analysis, not reported here for brevity but available in the online Appendix, shows that a possible mechanism explaining the significant impact of jumps on future volatility is given by the presence of contemporaneous jumps in price and volatility, a possibility that has been recently empirically confirmed by Todorov and Tauchen (2011) and Bandi and Renò (2011a). For this reason, in our last estimation, we introduce a single Poisson process N_t with constant intensity λ, and we set $dJ^X = c_X dN_t$, $dJ^1 = c_V dN_t$, and $J^2 = J^3 = 0$ with $c_X \sim \mathcal{N}(0, \sigma_J^2)$ and $c_V \sim \exp(\mu_\sigma)$, that is, we introduce cojumps in price and in a single volatility factor, namely the less persistent (also in this case, the structural model is not identified). Estimation results are reported in Table 8 and indicate that introducing cojumps provides a reasonable fit of the LHAR-CJ model, since we are able to reproduce both the short-range persistence of jumps and the long-range persistence of leverage.

In conclusion, we have seen that the LHAR-CJ model, and some of its relevant restrictions, is fully consistent with a multifactor Markovian volatility model. While it is certainly outside the scope of this article to provide a thorough interpretation of the mechanism that generated the stylized facts described by the estimated statistical models, both in discrete time and in continuous time, a possible interpretation of the empirical results goes as follows. Volatility is highly persistent, and this persistence can be generated by a superposition of factors with different frequencies. Negative returns and jumps are correlated with volatility. However, jumps are only correlated with a fast-reverting volatility factor via the mechanism of cojumps, so that their impact can only be short-lived. On the contrary, negative returns are correlated with *all* volatility factors through the correlation of the shocks. For this reason, negative returns can have a long-span impact on volatility, thus producing a persistent leverage effect.

5. CONCLUSIONS

In this article, we uncover new stylized facts about volatility dynamics. While it is well known that past negative returns are correlated with current volatility (leverage effect), we show that the forecasting power of past negative returns remains significant even when considering them over long horizons. The data also suggest that past jumps are (positively) correlated with current volatility, but the forecasting power of aggregated jumps is milder when the aggregation horizon is large. We then specify, both in discrete time and in continuous time, suitable models that are able to capture these novel stylized facts along with the well-established volatility features.

In the first stage, we propose a new discrete-time model for realized volatility measures, the LHAR-CJ model, which naturally identifies three main determinants of volatility dynamics, namely heterogeneous lagged continuous volatility, heterogeneous lagged negative returns, and heterogeneous lagged jumps. We find that each of the components in the discrete-time model plays a different role at different forecasting horizons, but all the three are highly significant and neglecting each one of them is detrimental to the forecasting performance.

In the second stage, we look for continuous-time models that reproduce the very same stylized facts, which are captured by the discrete-time specification. This is achieved by using the discrete-time model as a convenient statistical metric in an indirect inference framework. A multifactor Markovian specification is found to be consistent with the empirical results and compatible with the LHAR-CJ model. To reproduce the long-term impact of negative returns, all the volatility factors have to be correlated with the price shocks, while to reproduce the transient impact of jumps, it is enough to correlate price jumps with the jumps of one volatility factor only (cojumps).

We conclude by noting that our discrete-time model is very simple to implement, as it does not require sophisticated computational techniques. The estimation of the model parameters can be performed through a simple OLS regression, and the computation of the explanatory variables is trivial. We think that, for all the aforementioned reasons, the LHAR-CJ model may be effectively used for risk management.

SUPPLEMENTAL MATERIALS

Appendix: *http://www.econ-pol.unisi.it/∼reno/*

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