

## An Affine Multicurrency Model with Stochastic Volatility and Stochastic Interest Rates\*

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**Abstract.** We introduce a tractable multicurrency model with stochastic volatility and correlated stochastic interest rates that takes into account the smile in the foreign exchange (FX) market and the evolution of yield curves. The pricing of vanilla options on FX rates can be efficiently performed through the FFT methodology thanks to the affine property of the model. Our framework is also able to describe many nontrivial links between FX rates and interest rates: a calibration exercise highlights the ability of the model to simultaneously fit FX implied volatilities while being coherent with interest rate products.

**Key words.** FX options, long-dated FX, Wishart process

**AMS subject classifications.** 91G20, 91G30, 60H30

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**1. Introduction.** The foreign exchange (FX) market is the largest and most liquid financial market in the world. The daily volume of FX option transactions in 2010 was about 207 billion USD, according to Mallo [58]. The stylized facts concerning FX options may be ascribed to two main categories: features of the underlying exchange rates and the implied volatilities. The first and most important feature of FX rates is that the inverse of an FX rate is still an FX rate, so if  $S^{d,f}(t)$  is a model for the EURUSD exchange rate, i.e., the price in dollars of one euro, thus reflecting the point of view of an American investor, then  $S^{f,d}(t) = 1/S^{d,f}(t)$  represents the USDEUR rate, i.e., the price in euros of one dollar, hence representing the perspective of a European investor. This basic observation may be further generalized so as to construct, e.g., triangles of currencies where the no-arbitrage relation  $S^{f,d}(t) = S^{f,k}(t)S^{k,d}(t)$  must hold. This particular property of FX rates must also be coupled with the presence of a volatility smile for each FX rate involved in a currency triangle, in such a way that we can consider a model that is able to jointly capture relations among different underlyings and their respective implied volatilities.

Since the 2007 financial crisis, investors have looked for products with a long time horizon that are supposed to be less sensitive to short-term market fluctuations. Following Clark [21], the risk involved in such structures may be intuitively understood in terms of a simple example. The value of an at-the-money (ATM) call option, with maturity  $T$ , in a Black–

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Scholes setting, is usually approximated by practitioners by means of the formula  $0.4\sigma\sqrt{T}$ ; see Bachelier [5] and Schachermayer and Teichmann [65]. Under this approximation, the vega of such a position is simply given by  $0.4\sqrt{T}$  and hence scales as the square root of  $T$ . If we look instead at the rho risk, it can be shown that it scales as  $T$ . This heuristic observation suggests that when we look at short-dated FX options, the volatility smile is a dominant factor, whereas, as the maturity increases, the interest rate uncertainty plays an increasingly important role. As a consequence, when we consider long-dated FX products, a model that is simultaneously able to take into account volatility and interest rate risk is preferable.

**1.1. Related literature.** Historically, it has been quite a standard practice to employ, for the pricing of FX options, models originally designed, e.g., for equity options, modulo minor changes. This approach was initiated by Garman and Kohlhagen [36], who adapted the model by Black and Scholes [9]; see Wystup [73]. An important example is given by the Heston model, whose FX adaptation is presented, e.g., in Clark [21] and Janek et al. [47]. Other examples are the Stein and Stein [70] and Hull and White [45] models, as described, e.g., in Lipton [56]. We also mention the recent contribution of Leung, Wong, and Ng [53], where the Wishart multifactor stochastic volatility model introduced by Da Fonseca, Grasselli, and Tebaldi [25] is applied to a single exchange rate setting. With a view toward quanto options, Branger and Muck [11] also employ the Wishart process.

The typical limitation of the previous approaches is that they neglect the relationships among multiple exchange rates. In fact, the joint presence of triangular relationships and volatility smiles makes it difficult to model all FX rates in a triangle of currencies. In the literature, there exists a stream of contributions that try to recover the risk neutral probability distribution of the cross exchange rate, by means of either joint densities or copulas. These approaches represent an evolution of the Breeden and Litzenberger [12] approach. Among others, we recall Bliss and Panigirtzoglou [10], Schlögl [66], and Austing [4]. Copulas have been employed in Bennett and Kennedy [7], Salmon and Schleicher [64], and Hurd, Salmon, and Schleicher [46]. In the presence of a stochastic volatility model of Heston type, Carr and Verma [17] try to solve the joint valuation problem of FX options by specifying the dynamics of two rates influenced by a common stochastic volatility factor. This rather restrictive approach seems, however, difficult to extend. Asymptotic formulas for a SABR specification are provided in Shiraya and Takahashi [68]. The approach that interests us is presented in De Col, Gnoatto, and Grasselli [26], where a multifactor stochastic volatility model of Heston type is introduced. The model is coherent with respect to triangular relationships among currencies and allows for a simultaneous calibration of the volatility surfaces of the FX rates involved in a triangle, such as EUR/USD/JPY. The idea of De Col, Gnoatto, and Grasselli [26] is inspired by the work of Heath and Platen [43], who consider a model for FX rates of the form

$$(1.1) \quad S^{i,j}(t) = \frac{\mathcal{G}^i(t)}{\mathcal{G}^j(t)},$$

where  $\mathcal{G}^i, \mathcal{G}^j$  represent the value of the growth optimal portfolio under the two currencies. Flesaker and Hughston [34] first introduced the idea of a *natural numéraire*, the value of which can be expressed in different currencies, thus leading to consistent expressions for the

FX rates as ratios. A similar approach, known as the intrinsic currency valuation framework, has been independently proposed in Doust [29, 30].

In the industry, the evaluation of long-dated FX products is usually performed by coupling Hull and White [44] models for the short rates in each monetary area with a stochastic process for the FX rate, which is usually assumed to be a geometric Brownian motion; see Clark [21]. Many authors have studied the problem of combining uncertain FX rates or stocks with stochastic interest rates. Such a model represents a starting point for the approach presented in Piterbarg [63], where a local volatility effect in the FX process is also introduced. Van Haastrecht et al. [71] derive closed-form pricing formulas under the same kind of three-factor model, also coupled with a stochastic volatility process of Schöbel and Zhu [67] type; see also Ahlip [2] and Ahlip and King [3]. This kind of setting implies that both interest rates and volatility may become negative. Van Haastrecht and Pelsser [72] extend the previous approach and also consider approximate solutions when the instantaneous variance follows a square root process. Deelstra and Rayée [27] first present a local volatility framework, and then extend it to a stochastic volatility setting and provide approximations by means of a Markovian projection.

A desirable feature, when we couple interest rates and FX rates with stochastic volatility, is that we would like to use models such that the instantaneous variance and the interest rates remain positive. A natural choice in this context is given by the square root process. However, introducing a nonzero correlation among square root processes breaks the analytical tractability of the model when we work with affine processes on the canonical state space  $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$ . Grzelak and Oosterlee [41] attack the problem by providing approximations of the nonaffine terms in the Kolmogorov PDE satisfied by the characteristic function of the forward exchange rate.

**1.2. Main results of the article.** In this paper we propose an extension of De Col, Gnoatto, and Grasselli [26] to the case where the stochastic factors driving the volatilities of the exchange rates belong to the matrix state space  $S_d^+$ , the cone of positive semidefinite  $d \times d$  matrices. Stochastically continuous Markov processes on  $S_d^+$  with exponential affine dependence on the initial state space have been characterized in Cuchiero et al. [22]; see also Mayerhofer [60]. We will see that all good analytical properties of the approach presented in De Col, Gnoatto, and Grasselli [26] are preserved, while we allow for a more general dynamics. Our model is at the same time an affine multifactor stochastic volatility model for the FX rate, where the instantaneous variance is driven by a Wishart process (see Bru [14] and Da Fonseca, Grasselli, and Tebaldi [25]), and a Wishart affine short-rate model (see Gourieroux and Sufana [39], Grasselli and Tebaldi [40], Buraschi, Cieslak, and Trojani [15], Chiarella, Hsiao, and To [19], and Gnoatto [38]). The model has many interesting features; namely,

- it can be jointly calibrated on different FX volatility smiles;
- it is coherent with triangular relationships among FX rates;
- it allows for closed-form solutions for both FX options and basic interest rate derivatives;
- it allows for nontrivial correlations between interest rates and volatilities.

The first two interesting features are shared with the model of De Col, Gnoatto, and Grasselli [26]. The other results constitute a novel contribution of the current paper, which is outlined as

follows. In section 2, we set up our modelling framework and provide an example highlighting the flexibility of the approach. Section 3 is devoted to a complete characterization of all risk-neutral measures associated to the different economies, while section 4 shows that the present setting allows for stochastic correlations among many economic quantities. Section 5 presents all closed-form pricing formulas for FX and interest rates derivatives, together with asymptotic expansions of FX implied volatilities. The numerical treatment of our model is the topic of section 5, where we first perform a joint calibration to three implied volatility surfaces of a currency triangle and then fit the model to an FX surface and two yield curves. Finally, section 8 summarizes our findings. Technical proofs are gathered in the appendix (section 9).

**2. The model.** We consider an FX market in which  $N$  currencies are traded between each other via standard FX spot and FX vanilla option transactions. The value of each of these currencies  $i = 1, \dots, N$  in units of a universal numéraire is denoted by  $S^{0,i}(t)$ . (Note that  $S^{0,i}(t)$  can itself be thought of as an exchange rate, between the currency  $i$  and the artificial currency  $i = 0$ : we will see that the results are independent of the specification of the universal numéraire.)

We assume the existence of  $N$  money market accounts (one for each monetary area), whose values are driven by locally deterministic ODEs of the type

$$(2.1) \quad dB^i(t) = r^i(t)B^i(t)dt, \quad i = 1, \dots, N,$$

and we denote with  $r^0$  the interest rate corresponding to the artificial currency.

We assume the existence of a common multivariate stochastic factor  $\Sigma$  driving both the interest rates and the volatilities of the exchange rates  $S^{0,i}(t)$ . We model the stochastic factor  $\Sigma$  as a matrix Wishart process (see Bru [14]) evolving as

$$(2.2) \quad \begin{aligned} d\Sigma(t) &= (\beta Q^\top Q + M\Sigma(t) + \Sigma(t)M^\top)dt + \sqrt{\Sigma(t)}dW(t)Q + Q^\top dW(t)^\top, \\ \Sigma(0) &= \Sigma_0 \in S_d^{++}, \end{aligned}$$

where  $S_d^{++}$  is the interior of  $S_d^+$ , i.e., the cone of symmetric  $d \times d$  positive definite matrices. We assume that  $M \in M_d$ , the set of  $d \times d$  real matrices,  $Q \in Gl(d)$ , the set of  $d \times d$  real invertible matrices,  $W = (W_t)_{t \geq 0} \in M_d$  is a matrix Brownian motion (i.e., a  $d \times d$  matrix whose components are independent Brownian motions). The dimension  $d$  can be chosen according to the specific problem and may result from a principal component analysis. In order to ensure the typical mean reverting behavior of the process we assume that  $M$  is negative semidefinite (see Da Fonseca, Grasselli, and Tebaldi [24]); moreover, we assume  $\beta \geq d + 1$ . This last condition ensures the existence of a unique strong solution to the SDE (2.2), according to Corollary 3.2 in Mayerhofer, Pfaffel, and Stelzer [61].

We model each of the  $S^{0,i}(t)$  via a multifactor Wishart stochastic volatility model in the spirit of Da Fonseca, Grasselli, and Tebaldi [25]:

$$(2.3) \quad \frac{dS^{0,i}(t)}{S^{0,i}(t)} = (r^0(t) - r^i(t))dt - Tr \left[ A_i \sqrt{\Sigma(t)} dZ(t) \right], \quad i = 1, \dots, N,$$

where  $Z_t \in M_d$  is a matrix Brownian motion,  $A_i \in S_d$ ,  $i = 1, \dots, N$ , for  $S_d$  indicating the set of  $d \times d$  symmetric matrices, and finally  $Tr$  denotes the trace operator. The diffusion term

exhibits a structure that is completely analogous to that introduced in Heath and Platen [43] and De Col, Gnoatto, and Grasselli [26]: in the present case we have that the dynamics of the exchange rate is driven by a linear projection of the variance factor  $\sqrt{\Sigma(t)}$  along a direction parametrized by the symmetric matrix  $A_i$ . As a consequence the total instantaneous variance is  $\text{Tr}[A_i \Sigma(t) A_i] dt$ .

The stochastic factor  $\Sigma$  also drives the short interest rates:

$$(2.4) \quad r^0 = h^0 + \text{Tr}[H^0 \Sigma(t)],$$

$$(2.5) \quad r^i = h^i + \text{Tr}[H^i \Sigma(t)]$$

for  $h^k > 0$ ,  $H^k \in S_d^+$ ,  $k = 0, \dots, N$ .

We assume a correlation structure between the two matrix Brownian motions  $Z(t)$  and  $W(t)$  by means of an invertible matrix  $R$  according to the following relationship:

$$(2.6) \quad W(t) = Z(t)R^\top + B(t)\sqrt{I_d - RR^\top},$$

where  $B(t)$  is a matrix Brownian motion independent of  $Z(t)$ . We denote by  $S^{i,j}(t)$ ,  $i, j = 1, \dots, N$ , the exchange rate between currencies  $i$  and  $j$ . By Ito's lemma we have that  $S^{i,j}(t) = S^{0,j}(t)/S^{0,i}(t)$  has the following dynamics:

$$(2.7) \quad \begin{aligned} \frac{dS^{i,j}(t)}{S^{i,j}(t)} &= (r^i(t) - r^j(t))dt + \text{Tr}[(A_i - A_j)\Sigma(t)A_i]dt \\ &\quad + \text{Tr}[(A_i - A_j)\sqrt{\Sigma(t)}dZ(t)]. \end{aligned}$$

The additional drift term in (2.7) can be understood as a quanto adjustment between the currencies  $i$  and  $j$ .

We try to provide some intuition concerning the flexibility of the approach by considering an introductory example.

*Example 1.* Let  $d = 4$ . Consider the case

$$\begin{aligned} r^i &= h^i + \text{Tr} \left[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Sigma(t) \right] = h^i + \Sigma_{11}(t), \\ r^j &= h^j + \text{Tr} \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Sigma(t) \right] = h^j + \Sigma_{22}(t). \end{aligned}$$

Moreover, we let the matrices  $A^i, A^j$  be partitioned as follows:

$$A^i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11}^i & A_{12}^i \\ 0 & 0 & A_{12}^i & A_{22}^i \end{pmatrix}.$$

The idea is that the first two diagonal elements of the Wishart process are mainly responsible for the dynamics of the short rates, whereas the third and fourth drive the volatilities. The elements of the Wishart process are related to each other in a nontrivial way; in fact,

$$(2.8) \quad d\langle \Sigma_{ij}, \Sigma_{jk} \rangle_t = \left( \Sigma_{ik} (Q^\top Q)_{jl} + \Sigma_{il} (Q^\top Q)_{jk} + \Sigma_{jl} (Q^\top Q)_{ik} + \Sigma_{jk} (Q^\top Q)_{il} \right) dt.$$

Consequently, in the present four-dimensional hybrid FX-short-rate model, the covariation between, e.g., the elements  $\Sigma_{11}$  (which is mainly responsible for the dynamics of the first short rate) and  $\Sigma_{33}$  (which drives the volatility of the FX rate) are stochastic and given by

$$d\langle \Sigma_{11}, \Sigma_{33} \rangle_t = 4\Sigma_{13} (Q^\top Q)_{13} dt.$$

From the discussion above, we realize that we are in the presence of a hybrid model, allowing for nontrivial links among interest rates and FX rates while preserving full analytical tractability, as we will see in what follows. This interesting feature, to the best of our knowledge, is not shared by other existing hybrid models, which usually rely on approximations of the solution of the Kolmogorov PDE involving nonaffine terms (see, e.g., Grzelak and Oosterlee [41] and Grzelak et al. [42]).

**3. Risk-neutral probability measures.** Up to now we have worked under the risk-neutral measure defined by our artificial currency. In practical pricing applications, it is more convenient to change the numéraire to any of the currencies included in our FX multidimensional system. Without loss of generality, let us consider the risk-neutral measure defined by the  $i$ th money market account  $B^i(t)$  and derive the dynamics for the standard FX rate  $S^{i,j}(t)$ , its inverse  $S^{j,i}(t)$ , and a generic cross  $S^{j,l}(t)$ .

The Girsanov change of measure that transfers to the  $\mathbb{Q}^i$  risk-neutral measure (i.e., the risk-neutral measure in the  $i$ th country) is simply determined by assuming that under  $\mathbb{Q}^i$  the drift of the exchange rate  $S^{i,j}(t)$  is given by  $r^i - r^j$ . The associated Radon–Nikodym derivative is given by

$$\frac{d\mathbb{Q}^i}{d\mathbb{Q}^0} \Big|_{\mathcal{F}_t} = \exp \left( - \int_0^t \text{Tr}[A_i \sqrt{\Sigma(s)} dZ(s)] - \frac{1}{2} \int_0^t \text{Tr}[A_i \Sigma(s) A_i] ds \right).$$

As we shall see in the following, the measure change above has no impact on the parameter  $\beta$ , so that we have  $\beta \geq d+1$  under both probability measures. Using Theorem 4.1 in Mayerhofer [59], we immediately conclude that the stochastic exponential above is a true martingale. In the following, we proceed along the lines of De Col, Gnoatto, and Grasselli [26]. The possibility of buying the foreign currency and investing it at the foreign short rate of interest is equivalent to the possibility of investing in a domestic asset with price process  $\tilde{B}_j^i = B^j S^{i,j}$ , where  $i$  is the domestic economy and  $j$  is the foreign one. Then

$$\begin{aligned} d\tilde{B}_j^i(t) &= d(B^j(t) S^{i,j}(t)) \\ &= \tilde{B}_j^i(t) \left( r^i(t) dt + \text{Tr}[(A_i - A_j) \Sigma(t) A_i] dt + \text{Tr}[(A_i - A_j) \sqrt{\Sigma(t)} dZ(t)] \right) \\ &= \tilde{B}_j^i(t) \left( r^i(t) dt + \text{Tr}[(A_i - A_j) \sqrt{\Sigma(t)} dZ^{\mathbb{Q}^i}(t)] \right), \end{aligned}$$



where the matrix Brownian motion under  $\mathbb{Q}^i$  is given by

$$dZ^{\mathbb{Q}^i} = dZ + \sqrt{\Sigma(t)}A_i dt,$$

so that the  $\mathbb{Q}^i$  risk-neutral dynamics of the exchange rate is of the form

$$\begin{aligned} dS^{i,j}(t) &= d\left(\frac{\tilde{B}_j^i(t)}{B_j(t)}\right) \\ &= S^{i,j}(t) \left( (r^i(t) - r^j(t))dt + Tr \left[ (A_i - A_j) \sqrt{\Sigma(t)} dZ^{\mathbb{Q}^i}(t) \right] \right). \end{aligned}$$

The measure change, however, also has an impact on the variance processes, via the correlation matrix  $R$  introduced in (2.6). The component of  $dB(t)$  that is orthogonal to the spot driver  $dZ(t)$  is not affected by the measure change; this is a typical feature of FX stochastic volatility models, implying that the model is consistent with the foreign-domestic symmetry; see De Col, Gnoatto, and Grasselli [26] and Del Baño Rollin [28]. We are now able to derive the risk-neutral dynamics of the factor process  $\Sigma(t)$  governing the volatility of the exchange rates under  $\mathbb{Q}^i$ , which is given by

$$(3.1) \quad dW^{\mathbb{Q}^i}(t) = \left( dZ(t) + \sqrt{\Sigma(t)}A_i dt \right) R^\top + dB(t) \sqrt{I_d - RR^\top}.$$

From (2.2) and (2.6) we derive the  $\mathbb{Q}^i$  risk-neutral dynamics of  $\Sigma$  as follows:

$$\begin{aligned} d\Sigma(t) &= (\Omega\Omega^\top + M\Sigma(t) + \Sigma(t)M^\top)dt \\ &\quad + \sqrt{\Sigma(t)} \left( \left( dZ(t) + \sqrt{\Sigma(t)}A_i dt \right) R^\top + dB(t) \sqrt{I_d - RR^\top} \right) Q \\ &\quad + Q^\top \left( R \left( dZ_t^\top + A_i \sqrt{\Sigma(t)}dt \right) + \sqrt{I_d - RR^\top}^\top dB^\top \right) \sqrt{\Sigma(t)} \\ &\quad - \Sigma(t)A_i R^\top Q dt - Q^\top R A_i \Sigma(t) dt. \end{aligned}$$

Now define

$$M^{\mathbb{Q}^i} := M - Q^\top R A_i,$$

so that using (3.1) we can finally write

$$\begin{aligned} d\Sigma(t) &= (\Omega\Omega^\top + M^{\mathbb{Q}^i}\Sigma(t) + \Sigma(t)M^{\mathbb{Q}^i,\top})dt \\ &\quad + \sqrt{\Sigma(t)}dW^{\mathbb{Q}^i}(t)Q + Q^\top dW^{\mathbb{Q}^i,\top}(t)\sqrt{\Sigma(t)}, \end{aligned}$$

from which we deduce the following relations among the parameters:

$$(3.2) \quad \begin{aligned} R^{\mathbb{Q}^i} &= R, \\ Q^{\mathbb{Q}^i} &= Q, \\ M^{\mathbb{Q}^i} &= M - Q^\top R A_i. \end{aligned}$$

We observe that, as in the multi-Heston case of De Col, Gnoatto, and Grasselli [26], the functional form of the model is invariant under the measure change between  $\mathbb{Q}^0$  and the  $i$ th risk-neutral measure. The inverse FX rate under the  $\mathbb{Q}^i$  risk-neutral measure follows from Ito calculus, recalling that  $S^{j,i} = (S^{i,j})^{-1}$ :

$$\begin{aligned} \frac{dS^{j,i}(t)}{S^{j,i}(t)} &= S^{i,j}(t) d\left(\frac{1}{S^{i,j}(t)}\right) \\ &= (r^j(t) - r^i(t) + \text{Tr}[(A_j - A_i)\Sigma(t)(A_j - A_i)]) dt \\ &\quad + \text{Tr}\left[(A_j - A_i)\sqrt{\Sigma(t)}dZ^{\mathbb{Q}^i}(t)\right], \end{aligned}$$

which includes the self-quanto adjustment. Similarly, the SDE of a generic cross FX rate becomes

$$\begin{aligned} \frac{dS^{j,l}(t)}{S^{j,l}(t)} &= \frac{S^{i,j}(t)}{S^{i,l}(t)} d\left(\frac{S^{i,l}(t)}{S^{i,j}(t)}\right) \\ &= (r^j(t) - r^l(t) + \text{Tr}[(A_j - A_l)\Sigma(t)(A_j - A_l)]) dt \\ &\quad + \text{Tr}\left[(A_j - A_l)\sqrt{\Sigma(t)}dZ^{\mathbb{Q}^i}(t)\right]. \end{aligned}$$

The additional drift term is the quanto adjustment as described by the current model choice. By applying Girsanov's theorem again, this time switching to the  $\mathbb{Q}^j$  risk-neutral measure, the term is absorbed in the  $\mathbb{Q}^j$ -Brownian motion, while the Wishart parameters change according to the following fundamental transformation rules:

$$\begin{aligned} (3.3) \quad R^{\mathbb{Q}^j} &= R^{\mathbb{Q}^i}, \\ Q^{\mathbb{Q}^j} &= Q^{\mathbb{Q}^i}, \\ M^{\mathbb{Q}^j} &= M^{\mathbb{Q}^i} - Q^{\mathbb{Q}^i, \top} R^{\mathbb{Q}^i} (A_j - A_i). \end{aligned}$$

#### 4. Features of the model.

**4.1. Functional symmetry of the model.** Recall that a crucial property of the FX market requires that products or ratios of exchange rates in a triangle are also exchange rates, meaning that the dynamics of the exchange rates must be functionally symmetric with respect to which FX pairs we choose to be the main ones and which one the cross. That is, it is not a priori trivial to obtain a model such that the dynamics for the inverse of an exchange rate shares the same functional form of its coefficients. Or, equivalently, the dynamics of  $(S^{i,l}/S^{l,j})$  computed by applying Ito's rule to the product  $(S^{0,l}/S^{0,i}) \times (S^{0,j}/S^{0,l})$  must give the dynamics of a process that shares the same functional form of both  $S^{i,l}$  and  $S^{l,j}$ . This symmetry property is fundamental in being able to joint calibrate and consistently price multicurrency options (see, e.g., De Col, Gnoatto, and Grasselli [26] in a multifactor stochastic volatility framework and Eberlein, Papapantoleon, and Shiryaev [32] in a general semimartingale setting).

**Proposition 4.1.** *The dynamics of the exchange rates (2.7) satisfies the triangular relation; namely, the model is functional symmetric.*

*Proof.* For the proof see the appendix (section 9). ■



**4.2. Stochastic skew.** In analogy with De Col, Gnoatto, and Grasselli [26], if we calculate the infinitesimal correlation between the log returns of  $S^{i,j}$  and their variance, we find that it is stochastic. This is a nice feature of the model since it implies that the skewness of vanilla options on  $S^{i,j}$  is stochastic, which is a well-known stylized fact in the FX market (see, e.g., Carr and Wu [18]).

Let us consider the infinitesimal variance of  $S^{i,j}$ ,

$$d\langle \log S^{i,j}, \log S^{i,j} \rangle_t = \text{Tr}[(A_i - A_j)\Sigma(t)(A_i - A_j)]dt,$$

so that we can write

$$\begin{aligned} \frac{dS^{i,j}(t)}{S^{i,j}(t)} &= (r^i - r^j)dt + \text{Tr}[(A_i - A_j)\sqrt{\Sigma(t)}dZ^{\mathbb{Q}^i}(t)] \\ (4.1) \quad &= (r^i - r^j)dt + \sqrt{\text{Tr}[(A_i - A_j)\Sigma(t)(A_i - A_j)]}dB_1(t), \end{aligned}$$

where we defined

$$dB_1(t) := \frac{\text{Tr}[(A_i - A_j)\sqrt{\Sigma(t)}dZ^{\mathbb{Q}^i}(t)]}{\sqrt{\text{Tr}[(A_i - A_j)\Sigma(t)(A_i - A_j)]}}.$$

By the Lévy characterization theorem, the process  $B_1 = (B_1(t))_{t \geq 0}$  is a scalar Brownian motion, and we still denote by  $S^{i,j}$  the weak solution to the SDE (4.1).

The dynamics of the variance are given by

$$\begin{aligned} &d\text{Tr}[(A_i - A_j)\Sigma(t)(A_i - A_j)] \\ &= \left( \text{Tr}[(A_i - A_j)\Omega\Omega^\top(A_i - A_j)] + 2\text{Tr}[(A_i - A_j)M\Sigma(t)(A_i - A_j)] \right) dt \\ &\quad + 2\text{Tr}[(A_i - A_j)\sqrt{\Sigma}dW^{\mathbb{Q}^i}(t)Q(A_i - A_j)]. \end{aligned}$$

In order to determine the scalar Brownian motion driving the variance process we shall compute the following quadratic variation:

$$\begin{aligned} &d\langle \text{Tr}[(A_i - A_j)\Sigma(t)(A_i - A_j)] \rangle_t \\ &= 4d\left\langle \int_0^\cdot \text{Tr}[(A_i - A_j)\sqrt{\Sigma(v)}dW^{\mathbb{Q}^i}(v)Q(A_i - A_j)] \right. \\ &\quad \left. \int_0^\cdot \text{Tr}[(A_i - A_j)\sqrt{\Sigma(u)}dW^{\mathbb{Q}^i}(u)Q(A_i - A_j)] \right\rangle_t \\ &= 4d\left\langle \int_0^\cdot \sum_{a,b,c,e,f=1}^d (A_i - A_j)_{ab}\sqrt{\Sigma(v)}_{bc}dW_{ce}^{\mathbb{Q}^i}(v)Q_{ef}(A_i - A_j)_{fa}, \right. \\ &\quad \left. \int_0^\cdot \sum_{p,q,r,s,l=1}^d (A_i - A_j)_{pq}\sqrt{\Sigma(u)}_{qr}dW_{rs}^{\mathbb{Q}^i}(u)Q_{sl}(A_i - A_j)_{lp} \right\rangle_t \\ &= 4 \sum_{a,b,f,p,q,r,s,t=1}^d (A_i - A_j)_{ab}\sqrt{\Sigma(t)}_{br}Q_{sf}(A_i - A_j)_{fa} \end{aligned}$$

$$\begin{aligned}
& \times (A_i - A_j)_{pq} \sqrt{\Sigma(t)}_{qr} Q_{sl} (A_i - A_j)_{lp} dt \\
& = 4 \sum_{a,b,f,p,q,r,s,t=1}^d (A_i - A_j)_{fa} (A_i - A_j)_{ab} \sqrt{\Sigma(t)}_{br} \sqrt{\Sigma(t)}_{qr} \\
& \quad \times (A_i - A_j)_{pq} (A_i - A_j)_{lp} Q_{sl} Q_{sf} dt \\
& = 4 \text{Tr} \left[ (A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q \right] dt.
\end{aligned}$$

Then we can use the same arguments as before and express the dynamics of the variance as follows:

$$\begin{aligned}
& d\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)] \\
& = (\dots) dt + 2 \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]} dB_2(t),
\end{aligned}$$

where

$$dB_2(t) := \frac{\text{Tr} [(A_i - A_j) \sqrt{\Sigma} dW^{\mathbb{Q}^i}(t) Q (A_i - A_j)]}{\sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}},$$

which allows us to compute the covariation between the two noises. The skewness is then related to the quadratic covariation between the two noises  $B_1, B_2$ :

(4.2)

$$\begin{aligned}
& d\langle B_1, B_2 \rangle_t \\
& = \frac{d \left\langle \int_0^t \text{Tr} [(A_i - A_j) \sqrt{\Sigma(t)} dZ_t^{\mathbb{Q}^i}] , \int_0^t \text{Tr} [(A_i - A_j) \sqrt{\Sigma(t)} dW^{\mathbb{Q}^i}(t) Q (A_i - A_j)] \right\rangle_t}{\sqrt{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}} \\
& = \frac{d \left\langle \int_0^t \text{Tr} [(A_i - A_j) \sqrt{\Sigma(t)} dZ_t^{\mathbb{Q}^i}(t)] , \int_0^t \text{Tr} [(A_i - A_j) \sqrt{\Sigma(t)} dZ_t^{\mathbb{Q}^i}(t) R^\top Q (A_i - A_j)] \right\rangle_t}{\sqrt{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}} \\
& = \frac{\sum_{a,b,c,e,f,g,p,q,r=1}^d (A_i - A_j)_{pq} \sqrt{\Sigma(t)}_{qr} dZ_{rp}^{\mathbb{Q}^i}(t) (A_i - A_j)_{ab} \sqrt{\Sigma(t)}_{bc} dZ_{ce}^{\mathbb{Q}^i}(t) R_{ef}^\top Q_{fg} (A_i - A_j)_{ga}}{\sqrt{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}} \\
& = \frac{\sum_{a,b,c,e,f,q=1}^d (A_i - A_j)_{eq} \sqrt{\Sigma(t)}_{qc} \sqrt{\Sigma(t)}_{cb} (A_i - A_j)_{ba} (A_i - A_j)_{ag} Q_{gf}^\top R_{fe}}{\sqrt{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}} dt \\
& = \frac{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)^2 Q^\top R]}{\sqrt{\text{Tr} [(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr} [(A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q]}} dt.
\end{aligned}$$

In conclusion, our model generates a stochastic skew, a desirable feature for any FX model, as inferred by Carr and Wu [18].

**4.3. A stochastic variance-covariance matrix.** If we take the point of view of currency  $i$ , the variance-covariance matrix relative to any pair  $S^{i,j}, S^{i,l}$  is stochastic and well defined, in the sense that it is positive definite. In fact, consider the  $2 \times 2$  candidate covariance matrix:

$$(4.3) \quad \begin{pmatrix} \langle \ln S^{i,j} \rangle_t & \langle \ln S^{i,j}, \ln S^{i,l} \rangle_t \\ \langle \ln S^{i,j}, \ln S^{i,l} \rangle_t & \langle \ln S^{i,l} \rangle_t \end{pmatrix}.$$

We know that

$$(4.4) \quad d\langle \ln S^{i,j} \rangle_t = \text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_j)] dt,$$

$$(4.5) \quad d\langle \ln S^{i,j}, \ln S^{i,l} \rangle_t = \text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_l)] dt.$$

We first look at (4.4). We recall that we assumed  $A_i, A_j, A_l \in S_d$ . Note that the cone  $S_d^+$  is self-dual, meaning that

$$S_d^+ = \{u \in S_d \mid \text{Tr}[uv] \geq 0 \quad \forall v \in S_d^+\}.$$

Let  $O$  be an orthogonal matrix; then we may write  $(A_i - A_j) = O\Lambda O^\top$ , where  $\Lambda$  is a diagonal matrix containing the eigenvalues of  $(A_i - A_j)$  on the main diagonal. Then we have

$$\begin{aligned} \text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_j)] &= \text{Tr}[O\Lambda O^\top \Sigma(t) O\Lambda O^\top] \\ &= \text{Tr}[\Sigma(t) O\Lambda^2 O^\top] \geq 0 \end{aligned}$$

by self-duality. This shows that variances are positive. Now let us check that the variance-covariance matrix is in  $S_d^+$ . Let

$$\begin{aligned} \mathcal{M}(t) &= (A_i - A_j) \sqrt{\Sigma(t)}, \\ \mathcal{N}(t) &= (A_i - A_l) \sqrt{\Sigma(t)}. \end{aligned}$$

Then, using the Cauchy–Schwarz inequality for matrices, we have

$$\begin{aligned} &\text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_j)] \text{Tr}[(A_i - A_l) \Sigma(t) (A_i - A_l)] \\ &= \text{Tr}[\mathcal{M}(t) \mathcal{M}^\top(t)] \text{Tr}[\mathcal{N}(t) \mathcal{N}^\top(t)] \\ &\geq \text{Tr}[\mathcal{M}(t) \mathcal{N}^\top(t)]^2 \\ &= \text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_l)]^2. \end{aligned}$$

This implies that the determinant of the instantaneous variance-covariance matrix is nonnegative, so we conclude that the variance-covariance matrix is positive semidefinite, and, as a side effect, we have the usual bound for correlations, i.e., all correlations are bounded by one (in absolute value). In particular, no additional constraints on correlations are required in a calibration procedure.

**4.4. Stochastic correlation between short rates and FX rates.** Our model allows for nontrivial dependencies between short rates and exchange rates. In fact, in this paragraph we will show that the instantaneous correlation between the short rate  $r^i$  and the log-FX rate  $\log S^{ij}(t)$  is stochastic and given by

$$(4.6) \quad \rho_{r^i, \log S^{i,j}(t)} = \frac{2\text{Tr}[(A_i - A_j) \Sigma(t) H^i Q^\top R]}{\sqrt{\text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_j)]} \sqrt{\text{Tr}[QH^i \Sigma(t) H^i Q^\top]}}.$$

Let us first recall that under the  $\mathbb{Q}^i$  risk-neutral measure, the covariation between the short rate and the log-FX rate is given by

$$\begin{aligned} d\langle \log S^{i,j}, r^i \rangle_t &= d\left\langle \int_0^\cdot Tr \left[ (A_i - A_j) \sqrt{\Sigma(u)} dZ(u) \right], \int_0^\cdot 2Tr \left[ H^i \sqrt{\Sigma(u)} dZ(u) R^\top Q \right] \right\rangle_t \\ &= 2d\left\langle \int_0^\cdot \sum_{a,b,c=1}^d (A_i - A_j)_{ab} \sigma_{bc}(u) dZ_{ca}(u), \int_0^\cdot \sum_{p,q,r,s,n=1}^d H_{pq}^i \sigma_{qr}(u) dZ_{rs}(u) R_{ns} Q_{np} \right\rangle_t \\ &= 2(A_i - A_j)_{sb} \sigma_{br} \sigma_{rq} H_{qp}^i Q_{np} R_{ns} dt \\ &= 2Tr \left[ (A_i - A_j) \Sigma(t) H^i Q^\top R \right] dt, \end{aligned}$$

where we introduced  $\sigma_{bc}$ ,  $1 \leq b, c \leq d$ , so as to denote the  $bc$ th element of the matrix square root of  $\Sigma$ . For the short rate we have

$$(4.7) \quad d\langle r^i, r^i \rangle_t = Tr \left[ Q H^i \Sigma(t) H^i Q^\top \right] dt.$$

Given the instantaneous quadratic variation of the log-FX rate and the short rate, we get the result.

**4.5. Stochastic correlation between short rates and the variance of the FX rates.** The richness of our model specification may be further appreciated when we look at the correlation between any of the short rates and the variance of the log-FX rates, which is not usually captured in the literature (see Grzelak et al. [42], where a constant correlation is assumed).

**Proposition 4.2.** *The instantaneous correlation between the short rate  $r^i$  and the variance of the log-FX rate is stochastic and given by*

$$(4.8) \quad \rho_{r^i, \log Var^{i,j}} = \frac{2Tr \left[ (A_i - A_j) \Sigma(t) H^i Q^\top Q (A_i - A_j) \right]}{\sqrt{Tr \left[ (A_i - A_j)^2 \Sigma(t) (A_i - A_j)^2 Q^\top Q \right]} \sqrt{Tr \left[ Q H^i \Sigma(t) H^i Q^\top \right]}}.$$

*Proof.* Recall that the noise of the scalar instantaneous variance process is

$$dTr \left[ (A_i - A_j) \Sigma(t) (A_i - A_j) \right] = (\dots) dt + 2Tr \left[ (A_i - A_j) \sqrt{\Sigma(t)} dW Q (A_i - A_j) \right],$$

and observe that the noise of the short rate is simply given by

$$dr^i(t) = (\dots) dt + 2Tr \left[ H^i \sqrt{\Sigma(t)} dW(t) Q \right];$$

consequently,

$$\begin{aligned} d\langle r^i, Tr \left[ (A_i - A_j) \Sigma (A_i - A_j) \right] \rangle_t &= d\left\langle \int_0^\cdot 2Tr \left[ H^i \sqrt{\Sigma(u)} dW(u) Q \right], \int_0^\cdot 2Tr \left[ (A_i - A_j) \sqrt{\Sigma(u)} dW(u) Q (A_i - A_j) \right] \right\rangle_t \end{aligned}$$

$$\begin{aligned}
&= 4d \left\langle \int_0^\cdot \sum_{a,b,c,e=1}^d H_{ab} \sigma_{bc}(u) dW_{ce}(u) Q_{ea}, \int_0^\cdot \sum_{p,q,r,s,n=1}^d (A_i - A_j)_{pq} \sigma_{qr}(u) dW_{rs}(u) Q_{sn} (A_i - A_j)_{np} \right\rangle_t \\
&= 4 \sum_{a,b,p,q,r,s,n=1}^d H_{ab} \sigma_{br}(t) Q_{sa} (A_i - A_j)_{pq} \sigma_{qr}(t) Q_{sn} (A_i - A_j)_{np} dt \\
&= 4 \sum_{a,b,p,q,r,s,n=1}^d (A_i - A_j)_{pq} \sigma_{qr}(t) \sigma_{rb}(t) H_{ba} Q_{as}^\top Q_{sn} (A_i - A_j)_{np} dt \\
&= 4 \text{Tr} \left[ (A_i - A_j) \Sigma(t) H Q^\top Q (A_i - A_j) \right] dt.
\end{aligned}$$

Combining the above results with the quadratic variation of the instantaneous variance and the quadratic variation of the short rate, we conclude. ■

**5. Pricing of derivatives.** A distinctive feature of our model is the ability to price in closed-form derivatives written on different underlyings, meaning that we can consider, in a unified approach, more than one market simultaneously. In particular, we can jointly consider the FX and the fixed-income market. Basic European products like calls on FX rates and interest rate products related to different monetary areas can be analyzed together in a single model. In principle, this feature allows the desk to be jointly fitted to different markets by means of a single model. In this perspective, we provide fast pricing formulas in semiclosed form, up to Fourier integrals.

**5.1. European FX options.** We first provide the calculation of the discounted conditional Fourier–Laplace transform of  $x^{i,j}(t) := \ln S^{i,j}(t)$ , which will be useful for option pricing purposes. Let us consider a call option  $C(S^{i,j}(t), K^{i,j}, \tau)$ ,  $i, j = 1, \dots, N$ ,  $i \neq j$ , on a generic FX rate  $S^{i,j}(t) = \exp(x^{i,j}(t))$  with strike  $K^{i,j}$ , maturity  $T$  ( $\tau = T - t$  is the time to maturity), and face equal to one unit of the foreign currency. For ease of notation, we set  $R^{\mathbb{Q}^i} = R$  and  $Q^{\mathbb{Q}^i} = Q$ , and we will use the shorthand  $M^{\mathbb{Q}^i} = \tilde{M}$ . Let us introduce the following conditional expectation:

$$(5.1) \quad \phi^{i,j}(\lambda, t, \tau, x, \Sigma) = \mathbb{E}_t^{\mathbb{Q}^i} \left[ e^{-\int_t^T r_s^i ds} e^{i\lambda x^{i,j}(T)} | x^{i,j}(t) = x, \Sigma(t) = \Sigma \right]$$

for  $i = \sqrt{-1}$ . For  $\lambda = u \in \mathbb{R}$  we will use the terminology *discounted characteristic function*, whereas, for  $\lambda \in \mathbb{C}$  such that the expectation exists, the function  $\phi^{i,j}$  will be called *generalized discounted characteristic function*. Now, from the usual risk-neutral argument, the initial price of a call option can be written as a (domestic) risk-neutral expected value:

$$C(S^{i,j}(t), K^{i,j}, \tau) = \mathbb{E}_t^{\mathbb{Q}^i} \left[ e^{-\int_t^T r_s^i ds} \left( e^{x^{i,j}(T)} - K^{i,j} \right)^+ \right].$$

Following Lewis [54], we know that option prices may be interpreted as a convolution of the payoff and the probability density function of the (log-)underlying. As a consequence, the pricing of a derivative may be solved in Fourier space by relying on the Plancherel–Parseval identity (see Lewis [54]): for  $f, g \in L^2(\mathbb{R}, \mathbb{C})$ ,

$$\int_{-\infty}^{\infty} \overline{f(x)} g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{f}(u)} \hat{g}(u) du$$

for  $u \in \mathbb{R}$  and  $\hat{f}, \hat{g}$  denoting the Fourier transforms of  $f, g$ , respectively. Applying the above reasoning in an option pricing setting requires some additional care. In fact, most pay-off functions do not admit a Fourier transform in the classical sense; for example, it is well known that for the call option

$$\Phi(\lambda) = \int_{\mathbb{R}} e^{i\lambda x} (e^x - K^{i,j})^+ dx = -\frac{(K^{i,j})^{i\lambda+1}}{\lambda(\lambda - i)},$$

provided that we let  $\lambda \in \mathbb{C}$  with  $\Im(\lambda) > 1$ , meaning that  $\Phi(\lambda)$  is the Fourier transform of the pay-off function in the generalized sense. Such restrictions must be coupled with those which identify the domain where the generalized characteristic function of the log-price is well defined. The reasoning we just reported is developed in Theorem 3.2 in Lewis [54], where the following general formula is presented:

$$(5.2) \quad C(S^{i,j}(t), K^{i,j}, \tau) = \frac{1}{2\pi} \int_{\mathcal{Z}} \phi^{i,j}(-\lambda, t, \tau, x, \Sigma) \Phi(\lambda) d\lambda$$

for  $\mathcal{Z}$  denoting the line in the complex plane, parallel to the real axis, where the integration is performed. Carr and Madan [16] followed a different procedure by introducing the concept of a dampened option price; however, as Lewis [54] and Lee [52] point out, this alternative approach is just a particular case of the first. In Lee [52] the Fourier representation of option prices is extended to the case where interest rates are stochastic. Moreover, the shifting of contours pioneered by Lewis [54] is employed to prove Theorem 5.1, where the following general option pricing formula is presented:

$$(5.3) \quad C(S^{i,j}(t), K^{i,j}, \tau) = R(S^{i,j}(t), K^{i,j}, \alpha) + \frac{1}{2\pi} \int_{-\infty - i\alpha}^{\infty - i\alpha} e^{-i\lambda k^{i,j}} \frac{\phi^{i,j}(\lambda - i, t, \tau, x, \Sigma)}{-\lambda(\lambda - i)} d\lambda,$$

where  $k^{i,j} = \log K^{i,j}$ ,  $\alpha$  denotes the contour of integration, and the term  $R(S^{i,j}(t), K^{i,j}, \alpha)$ , coming from the application of the residue theorem, is given by

$$(5.4) \quad R(S^{i,j}(t), K^{i,j}, \alpha) = \begin{cases} \phi^{i,j}(-i, t, \tau, x, \Sigma) - K^{i,j} P^i(t, T) & \text{if } \alpha < -1, \\ \phi^{i,j}(-i, t, \tau, x, \Sigma) - \frac{K^{i,j}}{2} P^i(t, T) & \text{if } \alpha = -1, \\ \phi^{i,j}(-i, t, \tau, x, \Sigma) & \text{if } -1 < \alpha < 0, \\ \frac{1}{2} \phi^{i,j}(-i, t, \tau, x, \Sigma) & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha > 0, \end{cases}$$

where the zero coupon bond price is given by  $P^i(t, T) = \phi^{i,j}(0, t, \tau, x, \Sigma)$ . In other words, the pricing problem is solved once the (conditional) discounted characteristic function of the log-exchange rate and its strip of analyticity are known. The pricing formula above is very general: for example, equations (4.12) and (4.13) in Filipovic and Mayerhofer [33] can be recovered as special cases. In the following we will proceed along the following steps:

1. We first compute  $\phi^{i,j}(u, t, \tau, x, \Sigma)$ ,  $u \in \mathbb{R}$ . In this case the Fourier transform is always defined.



2. After that we investigate existence and uniqueness for the Riccati ODEs arising from the computation of  $\phi^{i,j}(\lambda, t, \tau, x, \Sigma)$ ,  $\lambda \in \mathbb{i}\mathbb{R}$ , and derive conditions for the existence of real moments of  $S^{i,j}$ .
3. We finally use the results of Keller-Ressel and Mayerhofer [49] to provide an analytic extension to the complex plane which allows us to employ formula (5.3) to price FX options.

The first step is accomplished via the following result. For notational convenience let us introduce  $\omega$  as a placeholder for  $\mathbf{i}u$ , with  $u \in \mathbb{R}$ .

**Proposition 5.1.** *Let  $\omega = \mathbf{i}u$ ,  $u \in \mathbb{R}$ . The discounted characteristic function of  $x^{i,j}(t) := \log S^{i,j}(t)$ , evaluated in  $u \in \mathbb{R}$ , is given by*

$$(5.5) \quad \phi^{i,j}(u, t, T, x, \Sigma) = \exp[\omega x + \mathcal{A}(\tau) + Tr[\mathcal{B}(\tau)\Sigma]],$$

where the functions

$$\begin{aligned} \mathcal{A} : \mathbb{R}_{\geq 0} &\rightarrow \mathbb{C}, \quad \tau \rightarrow \mathcal{A}(\tau), \\ \mathcal{B} : \mathbb{R}_{\geq 0} &\rightarrow S_d + \mathbf{i}S_d, \quad \tau \rightarrow \mathcal{B}(\tau), \end{aligned}$$

satisfy the system of Riccati ODEs

$$(5.6) \quad \begin{aligned} \frac{\partial}{\partial \tau} \mathcal{B}(\tau) &= \mathcal{B}(\tau) \left( \tilde{M} + \omega Q^\top R (A_i - A_j) \right) + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \mathcal{B}(\tau) \\ &\quad + 2\mathcal{B}(\tau) Q^\top Q \mathcal{B}(\tau) + \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j, \quad \mathcal{B}(0) = 0, \end{aligned}$$

$$(5.7) \quad \frac{\partial}{\partial \tau} \mathcal{A}(\tau) = \omega (h^i - h^j) - h^i + Tr[\Omega \Omega^\top \mathcal{B}(\tau)], \quad \mathcal{A}(0) = 0.$$

The solution of the system of ODEs is given by

$$(5.8) \quad \mathcal{A}(\tau) = (\omega (h^i - h^j) - h^i) \tau - \frac{\beta}{2} Tr \left[ \log B_{22}(\tau) + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \tau \right],$$

$$(5.9) \quad \mathcal{B}(\tau) = B_{22}(\tau)^{-1} B_{21}(\tau),$$

where  $B_{22}(\tau)$ ,  $B_{21}(\tau)$  are submatrices in

$$\begin{aligned} &\begin{pmatrix} B_{11}(\tau) & B_{12}(\tau) \\ B_{21}(\tau) & B_{22}(\tau) \end{pmatrix} \\ &= \exp \tau \begin{bmatrix} \tilde{M} + \omega Q^\top R (A_i - A_j) & -2Q^\top Q \\ \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j & -\left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \end{bmatrix}. \end{aligned}$$

*Proof.* For the proof see the appendix (section 9). ■

**5.2. Regularity of the affine transform formula.** A direct inspection of (5.3) shows that, to compute option prices, the valuation of the characteristic function at a point lying in the complex plane is required. It turns out that extending the validity of the affine formulas that we reported above is not a trivial issue. There exists a branch of literature which focuses on

the problem of the existence of exponential moments for affine processes; see Filipovic and Mayerhofer [33], Kallsen and Muhle-Karbe [48], Spreij and Veerman [69], Glasserman and Kim [37], and Keller-Ressel and Mayerhofer [49]. It is not a priori clear, for example, whether the transform formula holds for any time horizon  $T$  or whether we can legitimately extend the affine formula to general real or complex moments.

The main problem is given by the possibility that the matrix Riccati ODE explodes in finite time. For affine processes on the canonical state space and on  $S_d^+$ , the recent paper by Keller-Ressel and Mayerhofer [49] addresses this issue. In Keller-Ressel and Mayerhofer [49] the affine property is initially defined on the set  $\mathcal{U}$  such that the exponent is bounded; see formula (2.1) therein. Keller-Ressel and Mayerhofer [49] first show that the affine transform formula holds for general real moments up to the maximal lifetime of the solution of the Riccati ODE and finally perform the extension for complex exponents; see Theorems 2.14 and 2.26 in their paper. This discussion motivates our search for conditions such that the Riccati ODE has a unique global solution, as we proceed to show.

We recall that the nonlinear (quadratic) ODE is locally Lipschitz, hence locally solvable. Due to the presence of the quadratic term, the global existence of a solution is not guaranteed. Global existence and uniqueness results for matrix Riccati ODEs are well known, e.g., from the literature on control and system theory; see Abou-Kandil et al. [1]. We adopt the version of Cuchiero et al. [22].

**Proposition 5.2.** *Let  $\alpha, \gamma, u \in S_d^+$  and  $B : S_d \rightarrow S_d$  be of the form  $B(x) = \mathcal{M}^\top x + x\mathcal{M}$ ,  $\mathcal{M} \in M_d$ . Then there exists a unique global solution to the matrix Riccati ODE*

$$(5.10) \quad \frac{\partial}{\partial \tau} \psi(\tau, u) = -2\psi(\tau, u)\alpha\psi(\tau, u) + B^\top(\psi(\tau, u)) + \gamma, \quad \psi(0, u) = u.$$

*Proof.* For the proof see, e.g., Cuchiero et al. [22]. ■

In the following, we apply the previous result in order to ascertain the existence of real moments of  $S^{i,j}$ , so that, in contrast to Proposition 5.1, we let  $\omega \in \mathbb{R}$ . The matrix Riccati ODE we are interested in is

$$(5.11) \quad \frac{\partial}{\partial \tau} \mathcal{B} = \mathcal{B}(\tau)\mathcal{M} + \mathcal{M}^\top \mathcal{B}(\tau) + 2\mathcal{B}(\tau)Q^\top Q\mathcal{B}(\tau) - \gamma,$$

where

$$\begin{aligned} -\gamma &= \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j, \\ \mathcal{M} &= \tilde{M} + \omega Q^\top R(A_i - A_j). \end{aligned}$$

By performing the change  $\bar{\mathcal{B}} = -\mathcal{B}$  we get

$$\frac{\partial}{\partial \tau} \bar{\mathcal{B}} = \bar{\mathcal{B}}(\tau)\mathcal{M} + \mathcal{M}^\top \bar{\mathcal{B}}(\tau) - 2\bar{\mathcal{B}}(\tau)Q^\top Q\bar{\mathcal{B}}(\tau) + \gamma;$$

hence we can apply Proposition 5.2 and infer the existence of a unique global solution to the matrix Riccati ODE, provided that the conditions on the coefficients are satisfied. Since  $Q^\top Q \in S_d^+$ ,  $\bar{\mathcal{B}}(0) = 0$  and given the form of the linear coefficient, it remains to check that

$$(5.12) \quad \gamma \in S_d^+ \Leftrightarrow -\frac{\omega^2 - \omega}{2} (A_i - A_j)^2 - (\omega - 1)H^i + \omega H^j \in S_d^+.$$

For  $\omega \in [0, 1]$  and  $H^j - H^i \in S_d^+$  the condition is indeed satisfied. We can summarize our findings in the following

**Lemma 5.3.** *Let  $\omega \in [0, 1]$  and  $H^j - H^i \in S_d^+$ . Then all  $\omega$ th-order moments of the log-stock price exist for any time horizon.*

*Proof.* The proof follows from Theorem 2.14 in [49] in combination with our Proposition 5.2. ■

In the case of deterministic interest rates (i.e., when  $H^i = H^j = 0$ ), this condition is consistent with the one coming from the one-dimensional Heston model; see Filipovic and Mayerhofer [33]. In particular, for  $\omega = 1$  and deterministic interest rates, we obtain the martingale condition for the asset price; see Filipovic and Mayerhofer [33]. As an immediate consequence of Lemma 5.3 in conjunction with Theorem 2.26 in [49], or using Lukacs [57, Theorem 7.1.1], we can extend the domain of the characteristic function to the strip

$$(5.13) \quad \Lambda = \{\lambda \in \mathbb{C} \mid \Im(\lambda) \in [-1, 0]\}.$$

**Proposition 5.4.** *Let  $-1 < \alpha < 0$ , and assume  $H^j - H^i \in S_d^+$ . Then the price of a European call option on the FX rate  $S^{i,j}$  with strike  $K^{i,j}$  and maturity  $T$  at time  $t$  is*

$$(5.14) \quad C(S^{i,j}(t), K^{i,j}, \tau) = \phi^{i,j}(-\mathbf{i}, t, \tau, x, \Sigma) + \frac{1}{2\pi} \int_{-\infty - \mathbf{i}\alpha}^{\infty - \mathbf{i}\alpha} e^{-\mathbf{i}\lambda K^{i,j}} \frac{\phi^{i,j}(\lambda - \mathbf{i}, t, \tau, x, \Sigma)}{-\lambda(\lambda - \mathbf{i})} d\lambda.$$

*Proof.* Simply observe that, for  $\alpha = \Im(\lambda) \in (-1, 0)$ , we have  $1 - \Im(\lambda) \in (0, 1)$ ; hence  $\phi^{i,j}(\lambda - \mathbf{i}, t, \tau, x, \Sigma)$  is an analytic generalized characteristic function. ■

**5.3. Expansions.** The calibration of our model relies on a standard nonlinear least-squares procedure. This will be employed to minimize the distance between model- and market-implied volatilities. Model-implied volatilities are extracted from the prices produced by an FFT routine. This procedure for the Wishart model is more demanding from a numerical point of view than the analogue for the multi-Heston case, e.g., in De Col, Gnoatto, and Grasselli [26]. An alternative approach is to fit implied volatilities via a simpler function. A possibility is to find a relationship between the prices produced by the model and those produced by the standard Black–Scholes formula. The next result states that it is possible to approximate the prices of options under the Wishart model, via a suitable expansion of the standard Black–Scholes formula and its derivatives, analogously to what has been done in De Col, Gnoatto, and Grasselli [26]. The proof, which is reported in the appendix (section 9), relies on arguments that may be found in Lewis [55] and Da Fonseca and Grasselli [23] (we drop all currency indices; it is intended that we are considering the  $(i, j)$  FX pair). We work under the assumption that interest rates are deterministic. A proxy for the risk-free rate may be given by an overnight rate. Define  $\tau := T - t$ , and define the real deterministic functions  $\tilde{B}^0, \tilde{B}^1, \tilde{B}^{20}, \tilde{B}^{21}$  as follows:

$$(5.15) \quad \begin{aligned} \tilde{B}^0 &= \int_0^\tau e^{(\tau-u)\tilde{M}^\top} (A_i - A_j) e^{(\tau-u)\tilde{M}} du, \\ \tilde{B}^1 &= \int_0^\tau e^{(\tau-u)\tilde{M}^\top} \left( \tilde{B}^0(u) Q^\top R (A_i - A_j) \right. \\ (5.16) \quad &\quad \left. + (A_i - A_j) R^\top Q \tilde{B}^0(u) \right) e^{(\tau-u)\tilde{M}} du, \end{aligned}$$

$$(5.17) \quad \tilde{\mathcal{B}}^{20} = \int_0^\tau e^{(\tau-u)\tilde{M}^\top} 2\tilde{\mathcal{B}}^0(u)Q^\top Q\tilde{\mathcal{B}}^0(u)e^{(\tau-u)\tilde{M}} du,$$

$$(5.18) \quad \begin{aligned} \tilde{\mathcal{B}}^{21} = \int_0^\tau e^{(\tau-u)\tilde{M}^\top} & \left( \tilde{\mathcal{B}}^1(u)Q^\top R(A_i - A_j) \right. \\ & \left. + (A_i - A_j)R^\top Q\tilde{\mathcal{B}}^1(u) \right) e^{(\tau-u)\tilde{M}} du. \end{aligned}$$

Moreover, the real deterministic scalar functions  $\tilde{\mathcal{A}}^0(\tau)$ ,  $\tilde{\mathcal{A}}^1(\tau)$ ,  $\tilde{\mathcal{A}}^{20}(\tau)$ ,  $\tilde{\mathcal{A}}^{21}(\tau)$  are given by

$$(5.19) \quad \tilde{\mathcal{A}}^0(\tau) = \text{Tr} \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^0(u) du \right],$$

$$(5.20) \quad \tilde{\mathcal{A}}^1(\tau) = \text{Tr} \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^1(u) du \right],$$

$$(5.21) \quad \tilde{\mathcal{A}}^{20}(\tau) = \text{Tr} \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^{20}(u) du \right],$$

$$(5.22) \quad \tilde{\mathcal{A}}^{21}(\tau) = \text{Tr} \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^{21}(u) du \right].$$

**Proposition 5.5.** Assume that all interest rates are constant and the vol-of-vol matrix  $Q$  has been scaled by the factor  $\alpha > 0$ . Let

$$(5.23) \quad v = \sigma^2 \tau = \tilde{\mathcal{A}}^0(\tau) + \text{Tr} \left[ \tilde{\mathcal{B}}^0(\tau) \Sigma \right]$$

be the integrated variance. Then the call price  $C(S(t), K, \tau)$  in the Wishart-based exchange model can be approximated in terms of the vol-of-vol scale factor  $\alpha$  by differentiating the Black-Scholes formula  $C_{B\&S}(S(t), K, \sigma, \tau)$  with respect to the log exchange rate  $x(t) = \ln S(t)$  and the integrated variance  $v$ :

$$(5.24) \quad \begin{aligned} C(S(t), K, \tau) \approx C_{B\&S}(S(t), K, \sigma, \tau) & \\ + \alpha \left( \tilde{\mathcal{A}}^1(\tau) + \text{Tr} \left[ \tilde{\mathcal{B}}^1(\tau) \Sigma(t) \right] \right) \partial_{xv}^2 C_{B\&S}(S(t), K, \sigma, \tau) & \\ + \alpha^2 \left( \tilde{\mathcal{A}}^{20}(\tau) + \text{Tr} \left[ \tilde{\mathcal{B}}^{20}(\tau) \Sigma(t) \right] \right) \partial_{v^2}^2 C_{B\&S}(S(t), K, \sigma, \tau) & \\ + \alpha^2 \left( \tilde{\mathcal{A}}^{21}(\tau) + \text{Tr} \left[ \tilde{\mathcal{B}}^{21}(\tau) \Sigma(t) \right] \right) \partial_{x^2 v}^3 C_{B\&S}(S(t), K, \sigma, \tau) & \\ + \frac{\alpha^2}{2} \left( \tilde{\mathcal{A}}^1(\tau) + \text{Tr} \left[ \tilde{\mathcal{B}}^1(\tau) \Sigma(t) \right] \right)^2 \partial_{x^2 v^2}^4 C_{B\&S}(S(t), K, \sigma, \tau). & \end{aligned}$$

**Proof.** For the proof see the appendix (section 9). ■

Finally, we can state another formula, which does not involve the computation of option prices and constitutes an approximation of the implied volatility surface for a short time to maturity. This formula may be a useful alternative for getting a quicker calibration for short maturities.

**Proposition 5.6.** Assume that all interest rates are constant. For a short time to maturity the implied volatility expansion in terms of the vol-of-vol scale factor  $\alpha$  in the Wishart-based

exchange model is given by

$$\begin{aligned}
 \sigma_{imp}^2 \approx & Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)] + \alpha \frac{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)^2 Q^\top R]}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]} m_f \\
 & + \alpha^2 \frac{m_f^2}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]^2} \left[ \frac{1}{3} Tr[(A_i - A_j) Q^\top Q (A_i - A_j) \Sigma(t)] \right. \\
 & + \frac{1}{3} Tr\left[\left[(A_i - A_j) Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)\right] \right. \\
 (5.25) \quad & \left. \left. \times Q^\top R (A_i - A_j) \Sigma(t)\right] - \frac{5}{4} \frac{Tr[(A_i - A_j) Q^\top R (A_i - A_j) \Sigma(t)]^2}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]} \right],
 \end{aligned}$$

where  $m_f = \log\left(\frac{S^{i,j}(t)e^{(r_i - r_j)\tau}}{K}\right)$  denotes the log-moneyness.

**Proof.** For the proof see the appendix (section 9). ■

Notice that in the previous formula we recognize the proportionality of the linear term to the skew by looking at the numerator of formula (4.2). The formulas we presented in this section could also serve as a means of providing an initial guess to a standard calibration routine. Empirical tests on implied volatility approximations are featured in Da Fonseca and Grasselli [23].

**5.4. The pricing of zero-coupon bonds.** The flexibility of the Wishart hybrid model opens up the possibility of performing a simultaneous calibration of interest rate-related and FX-related products. This is a preliminary step that guarantees a coherent framework for the evaluation of payoffs depending on several interest rate curves under different currencies. Examples of such products may be found, e.g., in Brigo and Mercurio [13] and Clark [21]. We provide a closed-form formula for the price of a zero-coupon bond, which constitutes a building block for many other linear interest rate products. The following proposition may be easily proved along the same lines as Proposition 5.1.

**Proposition 5.7.** *The price of a zero-coupon bond at time  $t$ , with maturity  $T$ , under a generic risk-neutral measure  $\mathbb{Q}_d$ , is given by*

$$(5.26) \quad \mathbb{E}^{\mathbb{Q}_d} \left[ e^{-\int_t^T (h^d + Tr[H^d \Sigma(s)]) ds} \middle| \mathcal{F}_t \right] = \exp \left\{ \mathcal{A}^{ZC}(\tau) + Tr[\mathcal{B}^{ZC}(\tau) \Sigma(t)] \right\},$$

where the deterministic functions  $\mathcal{A}^{ZC}$ ,  $\mathcal{B}^{ZC}$  satisfy the system of matrix ODEs

$$\begin{aligned}
 \frac{\partial \mathcal{A}^{ZC}}{\partial \tau} &= Tr[\beta Q^\top Q \mathcal{B}^{ZC}(\tau)] - h^d, \quad \mathcal{A}^{ZC}(0) = 0, \\
 \frac{\partial \mathcal{B}^{ZC}}{\partial \tau} &= \mathcal{B}^{ZC}(\tau) M^{\mathbb{Q}_d} + M^{\mathbb{Q}_d, \top} \mathcal{B}^{ZC}(\tau) \\
 &\quad + 2\mathcal{B}^{ZC}(\tau) Q^\top Q \mathcal{B}^{ZC}(\tau) - H^d, \quad \mathcal{B}^{ZC}(0) = 0,
 \end{aligned}$$

whose solution is given by

$$(5.27) \quad \mathcal{A}^{ZC}(\tau) = -\frac{\beta}{2} Tr \left[ \log B_{22}^{ZC}(\tau) + \tau M^{\mathbb{Q}_d} \right] - h^d \tau,$$

$$(5.28) \quad \mathcal{B}^{ZC}(\tau) = \mathbf{B}_{22}^{ZC}(\tau)^{-1} \mathbf{B}_{21}^{ZC}(\tau),$$

where  $\mathbf{B}_{22}^{ZC}(\tau)$ ,  $\mathbf{B}_{21}^{ZC}(\tau)$  are submatrices defined as follows:

$$(5.29) \quad \begin{pmatrix} \mathbf{B}_{11}^{ZC}(\tau) & \mathbf{B}_{12}^{ZC}(\tau) \\ \mathbf{B}_{21}^{ZC}(\tau) & \mathbf{B}_{22}^{ZC}(\tau) \end{pmatrix} = \exp \left[ \tau \begin{pmatrix} M^{\mathbb{Q}_d} & -2Q^\top Q \\ -H^d & -M^{\mathbb{Q}_d, \top} \end{pmatrix} \right].$$

Given this simple formula for a zero-coupon bond, we may consider a joint calibration to the FX smile and to the risk-free curve of two different economies, so as to capture simultaneously the information coming from the FX smile and the interest rate curve, which plays an important role for long maturities. More precisely, a direct inspection of formula (5.26) reveals that the Wishart short-rate model belongs to the class of affine term structure models, meaning that the yield curve is of a particularly simple form. For fixed  $\Sigma(t)$  we define the yield curve as the function  $Y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  with

$$(5.30) \quad Y(\tau) = -\frac{1}{\tau} \left( \mathcal{A}^{ZC}(\tau) + Tr \left[ \mathcal{B}^{ZC}(\tau) \Sigma(t) \right] \right).$$

In section 7.2 we provide a concrete calibration example involving the simultaneous fit of the FX surface and two yield curves.

**5.5. The pricing of a cap.** For the sake of completeness, we also report the pricing of an interest rate cap. We would like to point out that the formula presented here is derived under the assumption of a single-curve interest rate framework. A generalization involving the recent developments in the context of interest rate modelling, such as multiple-curve models, is beyond the scope of the paper.

Suppose that we have a sequence of resetting dates  $\{T_\gamma, \dots, T_{\beta-1}\}$  and payment dates  $\{T_{\gamma+1}, \dots, T_\beta\}$  at which the Libor rate  $L(T_i, T_i, T_{i+1}) = L(T_i, T_{i+1})$  observed at time  $T_i$  for the time span between  $T_i$  and  $T_{i+1}$  is paid.<sup>1</sup> We let  $\tau_i = T_i - T_{i-1}$  be the tenor length. The price at time  $t < T_\gamma$  of an interest rate cap with notional  $N$  and strike price  $K$  is given by

$$C(t, K) = \sum_{i=\gamma+1}^{\beta} N \tau_i \mathbb{E}^{\mathbb{Q}_d} \left[ e^{-\int_t^{T_i} r_s ds} (L(T_{i-1}, T_i) - K)^+ \middle| \mathcal{F}_t \right].$$

It is well known that the expectation above may be conveniently rewritten as a put option on a zero-coupon bond:

$$\begin{aligned} Cap(t, K) &= \sum_{i=\gamma+1}^{\beta} N (1 + \tau_i K) \\ &\quad \times \mathbb{E}^{\mathbb{Q}_d} \left[ e^{-\int_t^{T_{i-1}} r_s ds} \left( \frac{1}{1 + \tau_i K} - P(T_{i-1}, T_i) \right)^+ \middle| \mathcal{F}_t \right]. \end{aligned}$$

<sup>1</sup>We recall that we work under a generic domestic risk-neutral measure  $\mathbb{Q}_d$ , and for notational simplicity here  $i$  is a tenor index.



Given the nice analytical properties of the Wishart process, we would like to compute this expectation using Fourier methods. The standard approach, in the presence of such an expectation, would involve a change to a forward risk-neutral measure. While this is also possible in this case, we remark that the resulting dynamics of the Wishart process would involve time varying coefficients, which in turn would lead to the solution of time-dependent matrix Riccati differential equations, a highly nontrivial problem. Fortunately, in the present setting, it is possible to compute the discounted characteristic function of the logarithmic-zero-coupon bond in full analogy with Proposition 5.1. Given such a result, it is possible to solve the pricing problem via Fourier techniques in complete analogy with section 5.1. As in Proposition 5.1, we let  $\omega = iu, u \in \mathbb{R}$ .

**Proposition 5.8.** *The time  $t$  conditional discounted characteristic function of the logarithmic zero-coupon bond price with maturity  $T_i$ , at time  $T_{i-1}$ , evaluated at the point  $\omega$  is given by*

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_d} \left[ e^{-\int_t^{T_{i-1}} (h^d + \text{Tr}[H^d \Sigma(S)]) ds + \omega \log P(T_{i-1}, T_i)} \middle| \mathcal{F}_t \right] \\ = \exp \left\{ \omega \mathcal{A}^{ZC}(T_i - T_{i-1}) + \mathcal{A}^{LZC}(T_{i-1} - t) \right. \\ \left. + \text{Tr} [\mathcal{B}^{LZC}(T_{i-1} - t) \Sigma(t)] \right\}, \end{aligned} \quad (5.31)$$

where the deterministic functions  $\mathcal{A}^{LZC}$ ,  $\mathcal{B}^{LZC}$  satisfy the system of matrix ODEs

$$\begin{aligned} \frac{\partial \mathcal{A}^{LZC}}{\partial \tau} &= \text{Tr} [\beta Q^\top Q \mathcal{B}^{LZC}(\tau)] - h^d, \quad \mathcal{A}^{LZC}(0) = 0, \\ \frac{\partial \mathcal{B}^{LZC}}{\partial \tau} &= \mathcal{B}^{LZC}(\tau) M^{\mathbb{Q}_d} + M^{\mathbb{Q}_d, \top} \mathcal{B}^{LZC}(\tau) \\ &\quad + 2\mathcal{B}^{LZC}(\tau) Q^\top Q \mathcal{B}^{LZC}(\tau) - H^d, \quad \mathcal{B}^{LZC}(0) = \omega \mathcal{B}^{ZC}(T_i - T_{i-1}), \end{aligned}$$

whose solution is given by

$$\begin{aligned} \mathcal{A}^{LZC}(\tau) &= -\frac{\beta}{2} \text{Tr} \left[ \log \mathbf{B}_{22}^{LZC}(\tau) + \tau M^{\mathbb{Q}_d} \right] - h^d \tau, \\ \mathcal{B}^{LZC}(\tau) &= (\omega \mathcal{B}^{ZC}(T_i - T_{i-1}) \mathbf{B}_{12}^{LZC}(\tau) + \mathbf{B}_{22}^{LZC}(\tau))^{-1} \\ &\quad \times (\omega \mathcal{B}^{ZC}(T_i - T_{i-1}) \mathbf{B}_{11}^{LZC}(\tau) + \mathbf{B}_{21}^{LZC}(\tau)), \end{aligned} \quad (5.32)$$

and  $\mathbf{B}_{22}^{LZC}(\tau)$ ,  $\mathbf{B}_{21}^{LZC}(\tau)$  are submatrices in

$$\begin{pmatrix} \mathbf{B}_{11}^{LZC}(\tau) & \mathbf{B}_{12}^{LZC}(\tau) \\ \mathbf{B}_{21}^{LZC}(\tau) & \mathbf{B}_{22}^{LZC}(\tau) \end{pmatrix} = \exp \left[ \tau \begin{pmatrix} M^{\mathbb{Q}_d} & -2Q^\top Q \\ -H^d & -M^{\mathbb{Q}_d, \top} \end{pmatrix} \right]. \quad (5.33)$$

With the previous result, caps and floors can be easily priced within our framework.

**6. Relation with the multi-Heston model of De Col, Gnoatto, and Grasselli [26].** In this section we follow Benabid, Bensusan, and El Karoui [6] in order to show that when all matrices  $M$ ,  $Q$ ,  $R$ ,  $\Sigma(0)$ ,  $A_i$ ,  $H^i$  are *diagonal*, for a particular specification of the noise  $Z$  driving the exchange rates, we can introduce a version of the multi-Heston model of De Col,

Gnoatto, and Grasselli [26]. In fact, in this particular case the dynamics of the elements of the Wishart process take the simpler form

$$d\Sigma_{pp}(t) = (\beta Q_{pp}^2 + 2M_{pp}\Sigma_{pp}(t)) dt + 2Q_{pp} \sum_{k=1}^d \sqrt{\Sigma(t)}_{pk} dW_{kp}^{\mathbb{Q}^i}(t)$$

for  $p = 1, \dots, d$  and

$$\begin{aligned} d\Sigma_{pm}(t) &= \Sigma_{pm}(t) (M_{pp} + M_{mm}) dt + Q_{pp} \sum_{k=1}^d \sqrt{\Sigma(t)}_{km} dW_{kp}^{\mathbb{Q}^i}(t) \\ &\quad + Q_{mm} \sum_{k=1}^d \sqrt{\Sigma(t)}_{pk} dW_{km}^{\mathbb{Q}^i}(t) \end{aligned}$$

for the off-diagonal elements. We notice that

$$d\langle \Sigma_{pp}(t), \Sigma_{pp}(t) \rangle = 4Q_{pp}\Sigma_{pp}(t)dt,$$

so the  $d$ -dimensional process  $\tilde{W} = (\tilde{W}(t))_{t \geq 0}$ , with

$$d\tilde{W}_p^{\mathbb{Q}^i}(t) = \frac{1}{\sqrt{\Sigma_{pp}}} \sum_{k=1}^d \sqrt{\Sigma(t)}_{pk} dW_{kp}^{\mathbb{Q}^i}(t),$$

is a vector of independent Brownian motions. As a consequence, we may conveniently express the dynamics of a generic diagonal element as

$$d\Sigma_{pp}(t) = (\beta Q_{pp}^2 + 2M_{ii}\Sigma_{pp}(t)) dt + 2Q_{pp}\sqrt{\Sigma_{pp}(t)}d\tilde{W}_p^{\mathbb{Q}^i}(t).$$

Note that the diagonal elements  $\Sigma_{pp}$ ,  $p = 1, \dots, d$ , follow a multi-CIR process with independent factors, which represents the starting point of the multi-Heston model of De Col, Gnoatto, and Grasselli [26]. In analogy with their approach, we can now introduce a vector Brownian motion  $Z^{\mathbb{Q}^i} = (Z^{\mathbb{Q}^i}(t))_{t \geq 0}$  that is correlated with the vector Brownian motion  $\tilde{W}^{\mathbb{Q}^i}$  via the elements of the diagonal matrix  $R$ . In this case the dynamics of the exchange rate  $S^{i,j}$  becomes

$$\frac{dS^{i,j}(t)}{S^{i,j}(t)} = (r^i(t) - r^j(t))dt + (\mathbf{a}_i - \mathbf{a}_j)^\top \sqrt{\text{diag}(\Sigma(t))} d\tilde{Z}^{\mathbb{Q}^i}(t),$$

where  $\mathbf{a}_i, \mathbf{a}_j$  are column vectors (corresponding to the diagonal elements of  $A_i, A_j$ ),  $Z^{\mathbb{Q}^i}(t)$  is a  $d$ -dimensional vector Brownian motion, and  $\sqrt{\text{diag}(\Sigma(t))}$  denotes a diagonal matrix, featuring the elements  $\sqrt{\Sigma_{pp}(t)}$ ,  $p = 1, \dots, d$ , along the main diagonal. The short rate processes may be equivalently expressed by means of a scalar product between a vector featuring the elements of the main diagonal of  $H^i$  and a second vector featuring the elements of the main diagonal of  $\Sigma$ . From this informal discussion, we also deduce an extension of the approach of De Col, Gnoatto, and Grasselli [26] featuring stochastic interest rates.

## 7. Numerical illustration.

**7.1. Simultaneous calibration to market data on a triangle.** We perform the calibration along the same lines as in De Col, Gnoatto, and Grasselli [26]. This means that we will be minimizing the squared distance between market- and model-implied volatilities on a book of vanilla quotes in a single trading day. We consider implied volatility surfaces for EURUSD, USDJPY, and finally EURJPY. With these currencies we are able to construct a triangular relation between rates. We consider market data as of July 22, 2010. Our sample includes expiry dates ranging from 1 day to 10 years. It is important to stress that, in the FX market-implied volatilities, surfaces are expressed in terms of maturity and delta: the market practice is to quote volatilities for strangles and risk reversals that can then be employed to reconstruct a whole surface of implied volatilities via an interpolation method. The conversion between deltas and strikes can be easily performed along the lines of Clark [21] or Wystup and Reiswich [74]. For the calibration of the model we use a nonlinear least-squares optimizer to minimize the following function:

$$\sum_i \left( \sigma_{i,mkt}^{imp} - \sigma_{i,model}^{imp} \right)^2.$$

The choice of this norm constitutes the market practice. From Christoffersen, Heston, and Jacobs [20] we know that the price in a generic asset price model of a plain vanilla European call option may be thought of as a first-order expansion with respect to the volatility parameter, around the Black–Scholes price. This first-order approximation may be simply recast to obtain that a norm in volatility is approximately equal to a weighted norm in price, where the weight is given by the Black–Scholes vega. The use of an unweighted norm in price should be avoided as the numerical range for option prices may be large, thus introducing a bias in the optimization, as discussed in Da Fonseca and Grasselli [23]. To be more precise, the construction of the objective function is performed along the following steps. First we consider a function implementing the Fourier–Laplace transform, i.e., involving formula (5.5). The Fourier–Laplace transform is then invoked by an FFT pricing routine which returns a surface of prices for different moneyness and maturities. Concerning the choice of the Fourier pricing formula, we can employ, e.g., formulation (5.14). In line with Da Fonseca and Grasselli [23], we also tested the classical Carr and Madan formulation [16] with  $N = 4096$  integration points, grid spacing  $\eta = 0.18$ , and damping factor  $\alpha = 1.2$ , which turned out to be numerically stable. As a final step we construct the model-implied volatility surface by using the surface of model prices as input for a standard Black–Scholes-implied volatility solver.

We report in Table 1 the parameters that we obtained by performing the calibration on a subsample of maturities ranging from 3 weeks up to 1 year. We were able to obtain a very good fit of market-implied volatilities. The result of the fit for the subsample we consider can be appreciated by looking at Figures 1–4. This is a result that we expected since the present model is a generalization of the framework introduced in De Col, Gnoatto, and Grasselli [26]. It is possible to perform a calibration on a reduced form of the model, as in De Col, Gnoatto, and Grasselli [26], in order to increase the stability of the parameters. For the sake of brevity we skip this test as well as the other robustness tests in De Col, Gnoatto, and Grasselli [26] that can be easily repeated in this framework and lead to similar results.

Table 1

This table reports the results of the calibration of the FX Wishart model to a triangle of currencies under the assumption of deterministic short rates.

Parameter	Value	Parameter	Value
$\Sigma(t)(1, 1)$	0.0271	$A_{us}(1, 1)$	1.1703
$\Sigma(t)(1, 2)$	0.0079	$A_{us}(1, 2)$	0.0094
$\Sigma(t)(2, 2)$	0.0851	$A_{us}(2, 2)$	0.9229
$M(1, 1)$	-0.9953	$A_{eur}(1, 1)$	1.5543
$M(1, 2)$	-0.0457	$A_{eur}(1, 2)$	-0.2456
$M(2, 1)$	-0.2761	$A_{eur}(2, 2)$	1.2388
$M(2, 2)$	-0.9511	$A_{jpy}(1, 1)$	0.9936
$Q(1, 1)$	0.2057	$A_{jpy}(1, 2)$	0.1851
$Q(1, 2)$	0.3249	$A_{jpy}(2, 2)$	1.2357
$Q(2, 1)$	0.2558	$\beta$	3.1344
$Q(2, 2)$	0.0000		
$R(1, 1)$	1.3024		
$R(1, 2)$	-1.0416		
$R(2, 1)$	0.5416		
$R(2, 2)$	-1.1418		

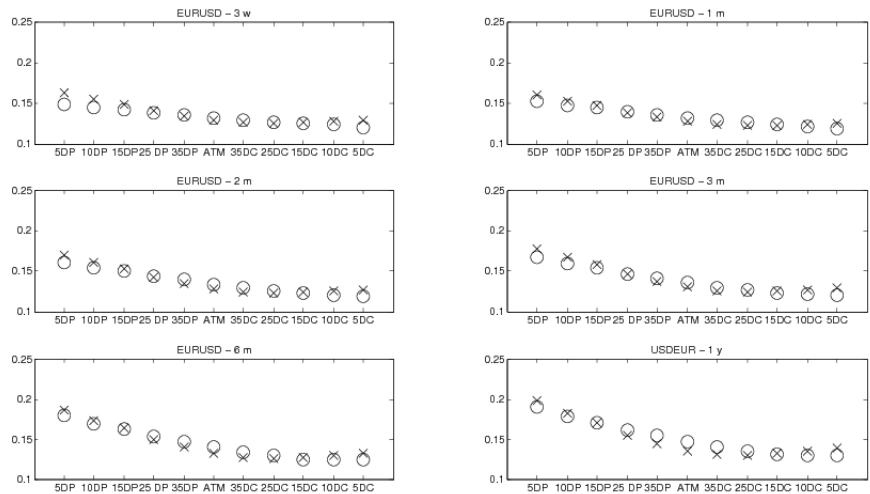
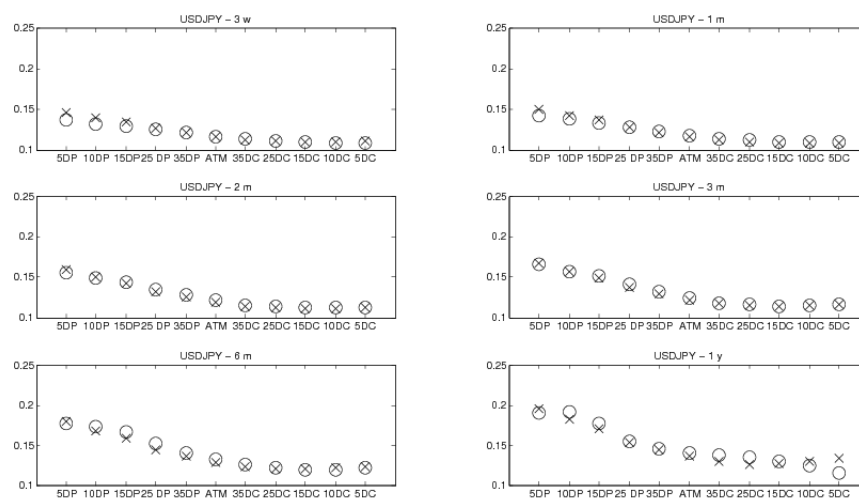
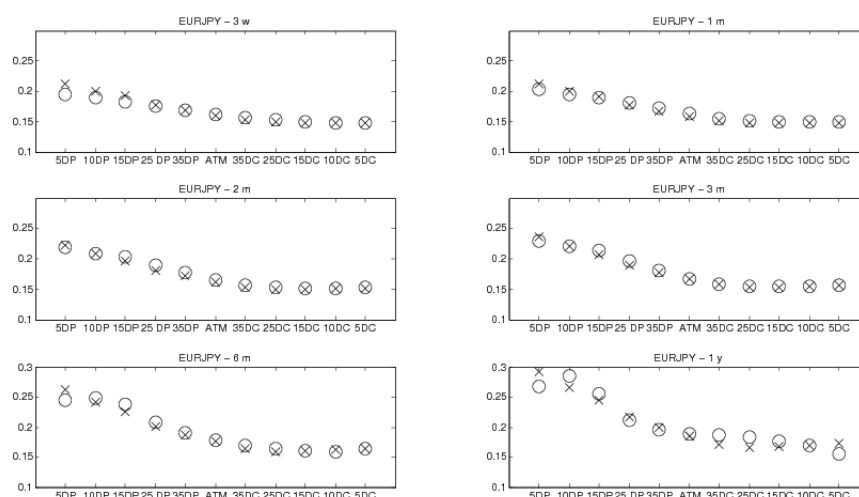


Figure 1. Calibration of the EURUSD-implied volatility surface for the full sample. Circles and crosses indicate model- and market-implied volatilities, respectively.

Note, however, that restrictions on the parameters are not required: even when all matrices are fully populated, a closed-form expression for the characteristic function, which does not require any numerical integration, is always available. As a consequence, the calibration can always be performed with a reasonable amount of computational effort, in contrast with the  $S_d^+$ -valued jump-based approach of Muhle-Karbe, Pfaffel, and Stelzer [62], which does not admit a closed-form solution for the characteristic function in the general case.

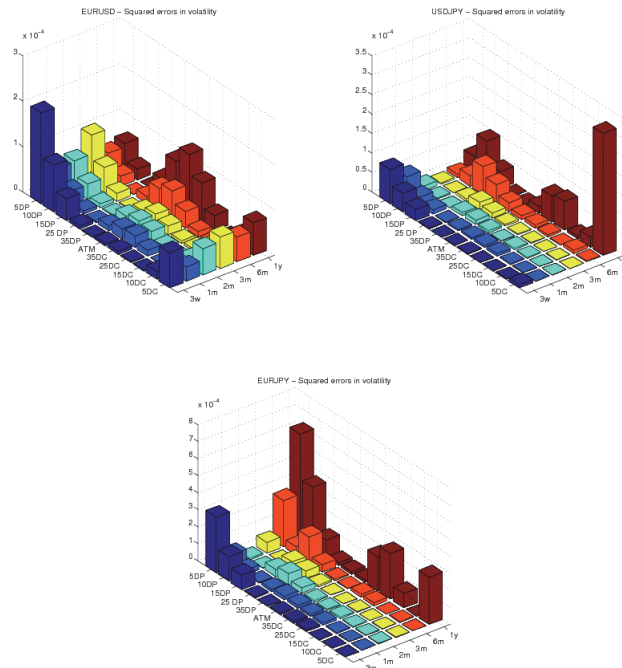


**Figure 2.** Calibration of the USDJPY-implied volatility surface for the full sample. Circles and crosses indicate model- and market-implied volatilities, respectively.



**Figure 3.** Calibration of the EURJPY-implied volatility surface for the full sample. Circles and crosses indicate model- and market-implied volatilities, respectively.

**7.2. Joint calibration of the FX-interest rate hybrid model.** In this subsection we report the results of a simultaneous calibration of an FX option volatility surface and of the two yield curves of the economies linked by the FX rate. To be more specific, we consider an implied volatility surface of EURUSD and the yield curves of EUR and USD economies. We consider market data on January 15, 2013. The data set features an implied volatility surface for options written on EURUSD. Moneyness ranges from  $5\Delta$  put up to  $5\Delta$  call. Maturities range from 1 day to 15 years. As far as the interest rate market is concerned, we obtained the yield

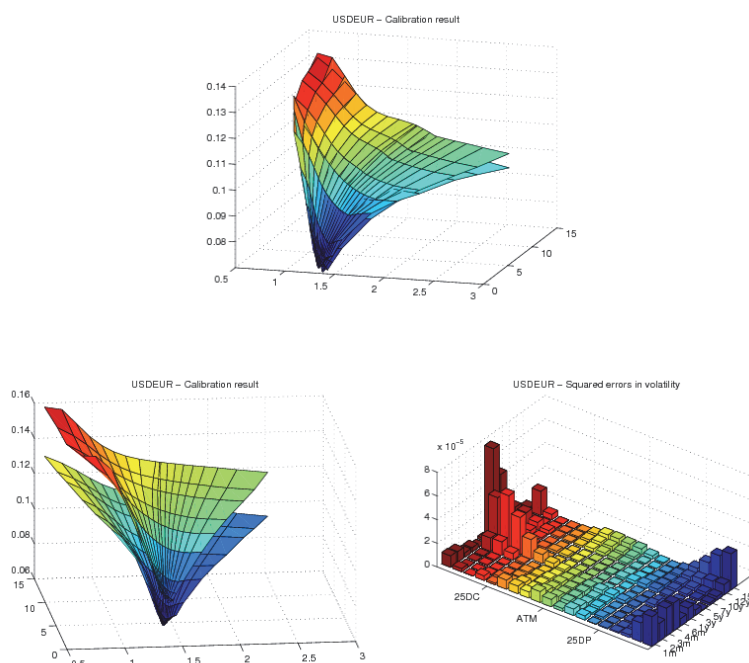


**Figure 4.** Volatility squared errors for the calibration experiment from section 7.1.

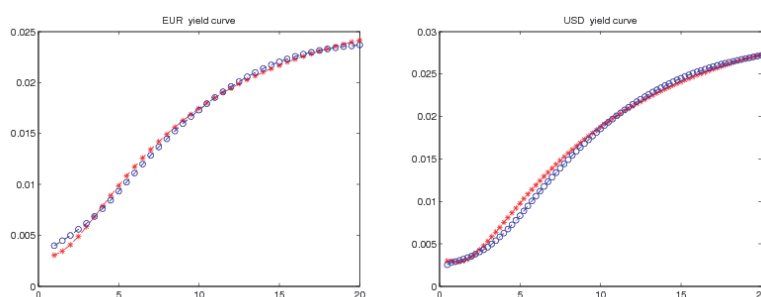
curves for both EUR and USD. In this case we considered maturities up to 20 years for both curves. The idea of the present calibration is to try to simultaneously fit the following data: the market FX-implied volatility surface of EURUSD and the EUR and USD yield curves. Our calibration is thought of as instrumental for the evaluation of typically long-dated FX products such as power reverse dual currency notes; see Clark [21]. For this reason, the range of expiries that we consider is much larger. As far as the yield curves are concerned, we fit market data up to 20 years. Concerning the implied volatility surface, we consider maturities ranging from 1 month up to 15 years (the longest maturity at which traders are allowed to trade). The penalty function is constructed first by considering the distance between market- and model-implied volatilities for FX options and then by also looking at the distance between market yields and model yields computed according to formula (5.30).

Figures 5 and 6 report the results of our calibration. The fit is very satisfactory across different strikes and maturities for FX options. The top part of Figure 5 shows the actual distance between the two surfaces, which are almost overlapping. For the sake of readability, in the bottom left part of Figure 5 we multiply by a factor  $> 1$  all model-implied volatilities so as to ease the comparison. The bottom right part of Figure 5 provides a more precise view on the quality of the calibration of the FX surface, by plotting for each point in the maturity/delta space the squared difference between model- and market-implied volatilities. Figure 6 reports a comparison between market- and model-implied yield curves. Red stars and blue circles denote model- and market-implied yields, respectively. Even in this market we are able to obtain a satisfactory fit. Recall that we perform a joint calibration procedure,





**Figure 5.** We simultaneously consider a joint calibration of the implied volatility surface of USDEUR and of the two yield curves of USD and EUR. The top figure compares the implied volatility surface to observed market data. As the fit is very good and the two surfaces are almost indistinguishable, in the bottom left figure we shift upwards the model-implied volatility by multiplying each point by 1.1 so as to increase readability. In the bottom right figure we report squared errors in implied volatility for each point of the surface.



**Figure 6.** We simultaneously consider a joint calibration of the implied volatility surface of USDEUR and of the two yield curves of USD and EUR. Stars and circles denote model- and market-implied yields, respectively.

meaning that the same set of model parameters allows us to obtain the results of Figures 5 and 6.

The values of the parameters arising from the calibration are reported in Table 2. By observing the values of the parameters we obtained, we notice that for the parameter  $\beta$  we have  $\beta \geq d + 1$ ; hence for the fitted model we conclude that the dollar and the euro risk-

Table 2

This table reports the results of the calibration of the FX-interest rate Wishart hybrid model.

Parameter	Value	Parameter	Value
$\Sigma(t)(1, 1)$	0.1688	$A_{us}(1, 1)$	0.7764
$\Sigma(t)(1, 2)$	0.1708	$A_{us}(1, 2)$	0.4837
$\Sigma(t)(2, 2)$	0.3169	$A_{us}(2, 2)$	0.9639
$M(1, 1)$	-0.5213	$A_{eur}(1, 1)$	0.6679
$M(1, 2)$	-0.3382	$A_{eur}(1, 2)$	0.6277
$M(2, 1)$	-0.4940	$A_{eur}(2, 2)$	0.8520
$M(2, 2)$	-0.4389	$h_{us}$	-0.2218
$Q(1, 1)$	0.2184	$h_{eur}$	-0.1862
$Q(1, 2)$	0.0957	$H_{us}(1, 1)$	0.2725
$Q(2, 1)$	0.2483	$H_{us}(1, 2)$	0.0804
$Q(2, 2)$	0.3681	$H_{us}(2, 2)$	0.4726
$R(1, 1)$	-0.5417	$H_{eur}(1, 1)$	0.1841
$R(1, 2)$	0.1899	$H_{eur}(1, 2)$	0.0155
$R(2, 1)$	-0.1170	$H_{eur}(2, 2)$	0.4761
$R(2, 2)$	-0.4834	$\beta$	3.1442

neutral measures are well-posed martingale measures. As far as the values of  $h_{EUR}$  and  $h_{USD}$  are concerned, they are negative, which is not in line with our starting assumption. In a first calibration experiment, we imposed the constraint  $h_i > 0$ ,  $i = USD, EUR$ , but observed that the model was not able to replicate the observed yield curve shape. Recall that in the present framework we are using (projections of) the Wishart process for the fit on both the FX-implied volatility surface and the yield curves. This fact results in a tradeoff between the parameters of the model. In other words, a small value of the initial state variables  $\Sigma$  conflicts with the ability of the model to fit the short-term smile. This is in line with the findings of Chiarella, Hsiao, and To [19], who also found negative values for  $h$ , thus implying a distribution of the short rate that can become negative. In any case, upon relaxation of the positivity constraint we obtained the very satisfactory fit that we reported in the present section.

**8. Conclusions.** In this paper we introduced a novel hybrid model that allows a joint evaluation of interest rate products and FX derivatives. The model is based on the Wishart process, and so it retains the same level of analytical tractability typical of the affine class, as in the multi-Heston version introduced in De Col, Gnoatto, and Grasselli [26]. In analogy with De Col, Gnoatto, and Grasselli [26], we have a model that is theoretically consistent with the triangular relationship among FX rates and other stylized symmetries that are commonly observed in the FX market. Moreover, the model allows for interest rate risk since there is the presence of stochastic interest rates that are nontrivially correlated with the volatility structure of the exchange rates. A successful calibration of the EURUSD-implied volatility surface and of the EUR and USD yield curves suggests that the framework we propose is a suitable instrument for the evaluation of long-dated FX products, where a joint description of interest rates and FX markets is required.

There is ample room for future research. For example, the introduction of  $S_d^+$ -valued jump processes, as in Muhle-Karbe, Pfaffel, and Stelzer [62], if coupled with GPU/parallel computing, could help in capturing the highly skewed implied volatilities that are usually observed for

short maturities. Another interesting direction is the generalization to the multicurve setting that is emerging in interest rate modelling as a response to the recent financial crisis. Several authors have attempted a solution in the FX context in order to model different levels of risks between, e.g., EONIA and EURIBOR curves (see, e.g., Bianchetti [8], Fries [35], Kenyon [50], Kijima, Tanaka, and Wong [51], and references therein). We believe that a Wishart-based approach would give very interesting insights into this multivariate puzzle.

## 9. Appendix. Proofs.

**9.1. Proof of Proposition 4.1.** Application of the Ito formula to the product  $S^{i,l}(t)S^{l,j}(t)$  (using the property  $\text{Tr}[dW(t)A]\text{Tr}[dW(t)B] = \text{Tr}[AB]dt$ ) leads to (2.7). We have the following formulas:

$$\begin{aligned}\frac{dS^{i,l}(t)}{S^{i,l}(t)} &= (r^i - r^l)dt + \text{Tr}[(A_i - A_l)\Sigma(t)A_i]dt + \text{Tr}[(A_i - A_l)\sqrt{\Sigma(t)}dZ(t)], \\ \frac{dS^{l,j}(t)}{S^{l,j}(t)} &= (r^l - r^j)dt + \text{Tr}[(A_l - A_j)\Sigma(t)A_l]dt + \text{Tr}[(A_l - A_j)\sqrt{\Sigma(t)}dZ(t)], \\ dS^{i,j}(t) &= dS^{i,l}(t)S^{l,j}(t) + S^{i,l}(t)dS^{l,j}(t) + d\left\langle S^{i,l}, S^{l,j} \right\rangle_t \\ &= S^{l,j}(t)S^{i,l}(t)\left((r^i - r^l)dt + \text{Tr}[(A_i - A_l)\Sigma(t)A_i]dt + \text{Tr}[(A_i - A_l)\sqrt{\Sigma(t)}dZ(t)]\right) \\ &\quad + S^{l,j}(t)S^{i,l}(t)\left((r^l - r^j)dt + \text{Tr}[(A_l - A_j)\Sigma(t)A_l]dt + \text{Tr}[(A_l - A_j)\sqrt{\Sigma(t)}dZ(t)]\right) \\ &\quad + d\left\langle \int_0^\cdot S^{i,l}(v)\text{Tr}[(A_i - A_l)\sqrt{\Sigma(v)}dZ(v)], \int_0^\cdot S^{l,j}(v)\text{Tr}[(A_l - A_j)\sqrt{\Sigma(v)}dZ(v)] \right\rangle_t.\end{aligned}$$

We concentrate on the covariation term:

$$\begin{aligned}&d\left\langle \int_0^\cdot S^{i,l}(v)\text{Tr}[(A_i - A_l)\sqrt{\Sigma(v)}dZ(v)], \int_0^\cdot S^{l,j}(v)\text{Tr}[(A_l - A_j)\sqrt{\Sigma(v)}dZ(v)] \right\rangle_t \\ &= d\left\langle \int_0^\cdot S^{i,l}(v) \sum_{p,q,r=1}^d (A_i - A_l)_{pq}\sqrt{\Sigma(v)}_{qr}dZ(v)_{rp}, \int_0^\cdot S^{l,j}(v) \sum_{s,t,u=1}^d (A_l - A_j)_{st}\sqrt{\Sigma(v)}_{tu}dZ(v)_{us} \right\rangle_t \\ &= \sum_{p,q,r,s,t,u=1}^d S^{i,l}(t)(A_i - A_l)_{pq}\sqrt{\Sigma(t)}_{qr}dZ(t)_{rp} \\ &\quad \times S^{l,j}(t)(A_l - A_j)_{st}\sqrt{\Sigma(t)}_{tu}dZ(t)_{us}\delta_{r=u}\delta_{p=s} \\ &= S^{i,j}(t) \sum_{s,q,u,t=1}^d (A_i - A_l)_{sq}\sqrt{\Sigma(t)}_{qu}(A_l - A_j)_{st}\sqrt{\Sigma(t)}_{tu}dt \\ &= S^{i,j}(t) \sum_{s,q,u,t=1}^d (A_i - A_l)_{sq}\sqrt{\Sigma(t)}_{qu}\sqrt{\Sigma(t)}_{ut}(A_l^\top - A_j^\top)_{ts}dt.\end{aligned}$$

By assuming that the matrices  $A_i, A_l, A_j$  are symmetric, we get

$$d\left\langle \int_0^\cdot S^{i,l}(t)\text{Tr}[(A_i - A_l)\sqrt{\Sigma(v)}dZ(v)], \int_0^\cdot S^{l,j}(t)\text{Tr}[(A_l - A_j)\sqrt{\Sigma(v)}dZ(v)] \right\rangle_t$$

$$= S^{i,j}(t) \text{Tr}[(A_i - A_l) \Sigma(t) (A_l - A_j)].$$

Finally, using the fact that terms in the trace commute, we obtain

$$\begin{aligned} \frac{dS^{i,j}(t)}{S^{i,j}(t)} &= (r^i - r^j)dt + \text{Tr}[(A_i - A_j) \Sigma(t) A_i]dt \\ &\quad + \text{Tr}[(A_i - A_j) \sqrt{\Sigma(t)} dZ(t)], \end{aligned}$$

as desired.

**9.2. Proof of Proposition 5.1.** We follow closely Da Fonseca, Grasselli, and Tebaldi [25], so we first write the PDE satisfied by  $\phi^{i,j}$ , which requires the dynamics of  $x(t) = x^{i,j}(t)$  under the measure  $\mathbb{Q}^i$ :

$$\begin{aligned} d \log S^{i,j}(t) &= \left( (r^i - r^j) - \frac{1}{2} \text{Tr}[(A_i - A_j) \Sigma(t) (A_i - A_j)] \right) dt \\ (9.1) \quad &\quad + \text{Tr}[(A_i - A_j) \sqrt{\Sigma(t)} dZ^{\mathbb{Q}^i}(t)], \end{aligned}$$

where the short rates are driven by the Wishart process in line with (2.5). The characteristic function solves the following PDE in terms of  $\tau = T - t$ :

$$(9.2) \quad \frac{\partial}{\partial \tau} \phi^{i,j} = \mathcal{A}_{x,\Sigma} \phi^{i,j} - r^i \phi^{i,j},$$

$$(9.3) \quad \phi^{i,j}(\omega, T, 0, x, \Sigma) = e^{\omega x}.$$

Since we are simply considering the PDE associated to the characteristic function, the general characterization of affine processes in Duffie, Filipović, and Schachermayer [31] and Cuchiero et al. [22] suffices to claim the uniqueness of the solution to the Cauchy problem above. Following Da Fonseca, Grasselli, and Tebaldi [25], the PDE satisfied by  $\phi^{i,j}$  is

$$\begin{aligned} \frac{\partial \phi^{i,j}}{\partial \tau} &= \left( (r^i - r^j) - \frac{1}{2} \text{Tr}[(A_i - A_j) \Sigma (A_i - A_j)] \right) \frac{\partial \phi^{i,j}}{\partial x} \\ &\quad + \frac{1}{2} \text{Tr}[(A_i - A_j) \Sigma (A_i - A_j)] \frac{\partial^2 \phi^{i,j}}{\partial x^2} \\ &\quad + \text{Tr} \left[ \left( \Omega \Omega^\top + \tilde{M} \Sigma + \Sigma \tilde{M}^\top \right) D G + 2 \left( \Sigma D Q^\top Q D \right) \phi^{i,j} \right] \\ (9.4) \quad &\quad + 2 \text{Tr} \left[ \Sigma (A_i - A_j) R^\top Q D \right] \frac{\partial \phi^{i,j}}{\partial x} - h^i - \text{Tr} [H^i \Sigma], \end{aligned}$$

where  $D$  is the differential operator:

$$D_{pt} = \frac{\partial}{\partial \Sigma_{pt}}.$$

Since the Wishart process  $\Sigma$  is affine, we guess the following form for the solution:

$$(9.5) \quad \phi^{i,j}(\omega, t, \tau, x, \Sigma) = \exp[\omega x + \mathcal{A}(\tau) + \text{Tr}[\mathcal{B}(\tau) \Sigma]],$$

with  $\mathcal{A} \in \mathbb{C}$  and  $\mathcal{B} \in S_d \times \mathbb{I}S_d$ . Moreover, these functions satisfy the following terminal conditions:

$$\begin{aligned}\mathcal{A}(0) &= 0 \in \mathbb{R}, \\ \mathcal{B}(0) &= 0 \in S_d.\end{aligned}$$

We substitute the candidate (9.5) into (9.4) and obtain

$$\begin{aligned}(9.6) \quad \frac{\partial}{\partial \tau} \mathcal{B} &= \mathcal{B}(\tau) \left( \tilde{M} + \omega Q^\top R (A_i - A_j) \right) + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \mathcal{B}(\tau) \\ &\quad + 2\mathcal{B}(\tau) Q^\top Q \mathcal{B}(\tau) + \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j\end{aligned}$$

and the final ODE, which may then be solved upon direct integration:

$$(9.7) \quad \frac{\partial}{\partial \tau} \mathcal{A} = \omega (h^i - h^j) - h^i + \text{Tr} \left[ \beta Q^\top Q \mathcal{B}(\tau) \right].$$

Following Grasselli and Tebaldi [40], it is possible to linearize (9.6) by writing

$$(9.8) \quad \mathcal{B}(\tau) = \mathcal{F}^{-1}(\tau) \mathcal{G}(\tau)$$

for  $\mathcal{F}(\tau) \in Gl(d)$  and  $\mathcal{G}(\tau) \in M_d$ . Then we have

$$\begin{aligned}\frac{\partial}{\partial \tau} [\mathcal{F}(\tau) \mathcal{B}(\tau)] &= \frac{\partial}{\partial \tau} [\mathcal{F}(\tau)] \mathcal{B}(\tau) + \mathcal{F}(\tau) \frac{\partial}{\partial \tau} \mathcal{B}(\tau) \\ &= \frac{\partial}{\partial \tau} [\mathcal{F}(\tau)] \mathcal{B}(\tau) \\ &\quad + \mathcal{F}(\tau) \left( \mathcal{B}(\tau) \left( \tilde{M} + \omega Q^\top R (A_i - A_j) \right) \right. \\ &\quad + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \mathcal{B}(\tau) \\ &\quad + 2\mathcal{B}(\tau) Q^\top Q \mathcal{B}(\tau) \\ &\quad \left. + \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j \right),\end{aligned}$$

which gives rise to the following system of ODEs:

$$(9.9) \quad \frac{\partial}{\partial \tau} \mathcal{F} = -\mathcal{F}(\tau) \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) - 2\mathcal{G}(\tau) Q^\top Q,$$

$$(9.10) \quad \begin{aligned}\frac{\partial}{\partial \tau} \mathcal{G} &= \mathcal{G}(\tau) \left( \tilde{M} + \omega Q^\top R (A_i - A_j) \right) \\ &\quad + \mathcal{F}(\tau) \left( \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j \right),\end{aligned}$$

with  $\mathcal{F}(0) = I_d$  and  $\mathcal{G}(0) = \mathcal{B}(0)$ . The solution of the above system is

$$(\mathcal{B}(0), I_d) \begin{pmatrix} \mathcal{B}_{11}(\tau) & \mathcal{B}_{12}(\tau) \\ \mathcal{B}_{21}(\tau) & \mathcal{B}_{22}(\tau) \end{pmatrix} = (\mathcal{B}(0), I_d)$$

$$\times \exp \tau \begin{bmatrix} \tilde{M} + \omega Q^\top R (A_i - A_j) & -2Q^\top Q \\ \frac{\omega^2 - \omega}{2} (A_i - A_j)^2 + (\omega - 1)H^i - \omega H^j & -(\tilde{M}^\top + \omega (A_i - A_j) R^\top Q) \end{bmatrix},$$

so that the solution for  $\mathcal{B}(\tau)$  is

$$\mathcal{B}(\tau) = (\mathcal{B}(0)\mathbf{B}_{12}(\tau) + \mathbf{B}_{22}(\tau))^{-1} (\mathcal{B}(0)\mathbf{B}_{11}(\tau) + \mathbf{B}_{21}(\tau)).$$

Since  $\mathcal{B}(0) = 0$ , we finally get  $\mathcal{B}(\tau) = \mathbf{B}_{22}(\tau)^{-1}\mathbf{B}_{21}(\tau)$ . Now starting from (9.9) we write

$$-\frac{1}{2} \left( \frac{\partial}{\partial \tau} \mathcal{F} + \mathcal{F}(\tau) \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \right) (Q^\top Q)^{-1} = \mathcal{G}(\tau).$$

We plug this into (9.8); then we insert the resulting formula for  $\mathcal{B}(\tau)$  into (9.7) and obtain

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{A} &= \omega (h^i - h^j) - h^i \\ &\quad + \text{Tr} \left[ -\frac{\beta}{2} \left( \mathcal{F}^{-1}(\tau) \frac{\partial}{\partial \tau} \mathcal{F} + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \right) \right], \end{aligned}$$

whose solution is

$$\mathcal{A} = (\omega (h^i - h^j) - h^i) \tau - \frac{\beta}{2} \text{Tr} \left[ \log \mathcal{F}(\tau) + \left( \tilde{M}^\top + \omega (A_i - A_j) R^\top Q \right) \tau \right].$$

To conclude, notice that  $\mathcal{F}(\tau) = \mathbf{B}_{22}(\tau)$ .

**9.3. Proof of Proposition 5.5.** The matrix Riccati ODE (9.6) may be rewritten as follows after replacing  $Q$  by a small perturbation  $\alpha Q$ ,  $\alpha \in \mathbb{R}$ :

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{B} &= \mathcal{B}(\tau) \left( \tilde{M} + \alpha \omega Q^\top R (A_i - A_j) \right) + \left( \tilde{M}^\top + \omega \alpha (A_i - A_j) R^\top Q \right) \mathcal{B}(\tau) \\ &\quad + 2\alpha^2 \mathcal{B}(\tau) Q^\top Q \mathcal{B}(\tau) + \frac{\omega^2 - \omega}{2} (A_i - A_j)^2, \end{aligned} \quad (9.11)$$

$$\mathcal{B}(0) = 0. \quad (9.12)$$

We now consider an expansion in terms of  $\alpha$  of the form  $\mathcal{B} = \mathcal{B}^0 + \alpha \mathcal{B}^1 + \alpha^2 \mathcal{B}^2$ . We substitute this expansion and identify terms by powers of  $\alpha$ . We obtain the following ODEs:

$$\frac{\partial}{\partial \tau} \mathcal{B}^0 = \mathcal{B}^0(\tau) \tilde{M} + \tilde{M}^\top \mathcal{B}^0(\tau) + \frac{\omega^2 - \omega}{2} (A_i - A_j)^2, \quad (9.13)$$

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{B}^1 &= \mathcal{B}^1(\tau) \tilde{M} + \tilde{M}^\top \mathcal{B}^1(\tau) + \mathcal{B}^0(\tau) Q^\top R (A_i - A_j) \omega \\ &\quad + \omega (A_i - A_j) R^\top Q \mathcal{B}^0(\tau), \end{aligned} \quad (9.14)$$

$$\begin{aligned} \frac{\partial}{\partial \tau} \mathcal{B}^2 &= \mathcal{B}^2(\tau) \tilde{M} + \tilde{M}^\top \mathcal{B}^2(\tau) + \mathcal{B}^1(\tau) Q^\top R (A_i - A_j) \omega \\ &\quad + \omega (A_i - A_j) R^\top Q \mathcal{B}^1(\tau) + 2\mathcal{B}^0(\tau) Q^\top Q \mathcal{B}^0(\tau). \end{aligned} \quad (9.15)$$



Let  $\gamma := \frac{\omega^2 - \omega}{2}$ ; then these equations admit the following solutions:

$$\begin{aligned} \mathcal{B}^0(\tau) &= \frac{\omega^2 - \omega}{2} \int_0^\tau e^{(\tau-u)\tilde{M}^\top} (A_i - A_j)^2 e^{(\tau-u)\tilde{M}} du \\ &:= \gamma \tilde{\mathcal{B}}^0(\tau), \end{aligned} \quad (9.16)$$

$$\begin{aligned} \mathcal{B}^1(\tau) &= \gamma\omega \int_0^\tau e^{(\tau-u)\tilde{M}^\top} \left( \tilde{\mathcal{B}}^0(u) Q^\top R (A_i - A_j) \right. \\ &\quad \left. + (A_i - A_j) R^\top Q \tilde{\mathcal{B}}^0(u) \right) e^{(\tau-u)\tilde{M}} du \\ &:= \gamma\omega \tilde{\mathcal{B}}^1(\tau), \end{aligned} \quad (9.17)$$

$$\begin{aligned} \mathcal{B}^2(\tau) &= \gamma\omega^2 \int_0^\tau e^{(\tau-u)\tilde{M}^\top} \left( \tilde{\mathcal{B}}^1(u) Q^\top R (A_i - A_j) \right. \\ &\quad \left. + (A_i - A_j) R^\top Q \tilde{\mathcal{B}}^1(u) \right) e^{(\tau-u)\tilde{M}} du \\ &\quad + \gamma^2 \int_0^\tau e^{(\tau-u)\tilde{M}^\top} 2\tilde{\mathcal{B}}^0(u) Q^\top Q \tilde{\mathcal{B}}^0(u) e^{(\tau-u)\tilde{M}} du \\ &:= \gamma^2 \tilde{\mathcal{B}}^{20}(\tau) + \gamma\omega^2 \tilde{\mathcal{B}}^{21}(\tau), \end{aligned} \quad (9.18)$$

whereby we implicitly defined the matrices  $\tilde{\mathcal{B}}^0(\tau)$ ,  $\tilde{\mathcal{B}}^1(\tau)$ ,  $\tilde{\mathcal{B}}^{20}(\tau)$ ,  $\tilde{\mathcal{B}}^{21}(\tau)$ . We can now write the function  $\mathcal{B}(\tau)$  as follows:

$$(9.19) \quad \mathcal{B}(\tau) = \gamma \tilde{\mathcal{B}}^0(\tau) + \alpha\gamma\omega \tilde{\mathcal{B}}^1(\tau) + \alpha^2\gamma^2 \tilde{\mathcal{B}}^{20}(\tau) + \alpha^2\gamma\omega^2 \tilde{\mathcal{B}}^{21}(\tau).$$

A direct substitution of (9.19) into (9.7) allows us to express the function  $\mathcal{A}(\tau)$  as

$$\begin{aligned} \mathcal{A}(\tau) &= \omega(r_i - r_j)\tau + \gamma Tr \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^0(u) du \right] + \alpha\gamma\omega Tr \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^1(u) du \right] \\ &\quad + \alpha^2\gamma^2 Tr \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^{20}(u) du \right] + \alpha^2\gamma\omega^2 Tr \left[ \Omega \Omega^\top \int_0^\tau \tilde{\mathcal{B}}^{21}(u) du \right] \\ &= \omega(r_i - r_j)\tau + \gamma \tilde{\mathcal{A}}^0(\tau) + \alpha\gamma\omega \tilde{\mathcal{A}}^1(\tau) \\ &\quad + \alpha^2\gamma^2 \tilde{\mathcal{A}}^{20}(\tau) + \alpha^2\gamma\omega^2 \tilde{\mathcal{A}}^{21}(\tau), \end{aligned} \quad (9.20)$$

having again implicitly defined the functions  $\tilde{\mathcal{A}}^0(\tau)$ ,  $\tilde{\mathcal{A}}^1(\tau)$ ,  $\tilde{\mathcal{A}}^{20}(\tau)$ ,  $\tilde{\mathcal{A}}^{21}(\tau)$ . We now consider the pricing in terms of the Fourier transform, i.e.,  $\omega = -i\lambda$ ,  $\lambda \in \mathbb{C}$ , as in (5.2). A Taylor–McLaurin expansion with respect to  $\alpha$  gives the following:

$$\begin{aligned} C(S(t), K, \tau) &\approx \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} e^{\omega(r_i - r_j)\tau + \omega x + \gamma(\tilde{\mathcal{A}}^0(\tau) + Tr[\tilde{\mathcal{B}}^0(\tau)\Sigma])} \Phi(\lambda) d\lambda \\ &\quad + \alpha \left( \tilde{\mathcal{A}}^1(\tau) + Tr[\tilde{\mathcal{B}}^1(\tau)\Sigma] \right) \\ &\quad \times \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} \gamma\omega e^{\omega(r_i - r_j)\tau + \omega x + \gamma(\tilde{\mathcal{A}}^0(\tau) + Tr[\tilde{\mathcal{B}}^0(\tau)\Sigma])} \Phi(\lambda) d\lambda \\ &\quad + \alpha^2 \left( \tilde{\mathcal{A}}^{20}(\tau) + Tr[\tilde{\mathcal{B}}^{20}(\tau)\Sigma] \right) \end{aligned}$$

$$\begin{aligned}
& \times \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} \gamma^2 e^{\omega(r_i-r_j)\tau + \omega x + \gamma(\tilde{\mathcal{A}}^0(\tau) + Tr[\tilde{\mathcal{B}}^0(\tau)\Sigma])} \Phi(\lambda) d\lambda \\
& + \alpha^2 \left( \tilde{\mathcal{A}}^{21}(\tau) + Tr \left[ \tilde{\mathcal{B}}^{21}(\tau)\Sigma \right] \right) \\
& \times \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} \gamma \omega^2 e^{\omega(r_i-r_j)\tau + \omega x + \gamma(\tilde{\mathcal{A}}^0(\tau) + Tr[\tilde{\mathcal{B}}^0(\tau)\Sigma])} \Phi(\lambda) d\lambda \\
& + \frac{\alpha^2}{2} \left( \tilde{\mathcal{A}}^1(\tau) + Tr \left[ \tilde{\mathcal{B}}^1(\tau)\Sigma \right] \right)^2 \\
& \times \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} \gamma^2 \omega^2 e^{\omega(r_i-r_j)\tau + \omega x + \gamma(\tilde{\mathcal{A}}^0(\tau) + Tr[\tilde{\mathcal{B}}^0(\tau)\Sigma])} \Phi(\lambda) d\lambda.
\end{aligned}$$

Recall now from (5.23) the definition of the integrated Black–Scholes variance. In the previous formula, in the first term we recognize the Black–Scholes price in terms of the characteristic function when the integrated variance is  $v = \sigma^2\tau$ :

$$(9.21) \quad C_{B\&S}(S(t), K, \sigma, \tau) = \frac{e^{-r_i\tau}}{2\pi} \int_{\mathcal{Z}} e^{\omega(r_i-r_j)\tau + \omega x + \gamma v} \Phi(\lambda) d\lambda,$$

so that the price expansion is of the form

$$\begin{aligned}
C(S(t), K, \tau) & \approx C_{B\&S}(S(t), K, \sigma, \tau) \\
& + \alpha \left( \tilde{\mathcal{A}}^1(\tau) + Tr \left[ \tilde{\mathcal{B}}^1(\tau)\Sigma \right] \right) \partial_{xv}^2 C_{B\&S}(S(t), K, \sigma, \tau) \\
& + \alpha^2 \left( \tilde{\mathcal{A}}^{20}(\tau) + Tr \left[ \tilde{\mathcal{B}}^{20}(\tau)\Sigma \right] \right) \partial_{v^2}^2 C_{B\&S}(S(t), K, \sigma, \tau) \\
& + \alpha^2 \left( \tilde{\mathcal{A}}^{21}(\tau) + Tr \left[ \tilde{\mathcal{B}}^{21}(\tau)\Sigma \right] \right) \partial_{x^2v}^3 C_{B\&S}(S(t), K, \sigma, \tau) \\
& + \frac{\alpha^2}{2} \left( \tilde{\mathcal{A}}^1(\tau) + Tr \left[ \tilde{\mathcal{B}}^1(\tau)\Sigma \right] \right)^2 \partial_{x^2v^2}^4 C_{B\&S}(S(t), K, \sigma, \tau),
\end{aligned}
\tag{9.22}$$

which completes the proof.

**9.4. Proof of Proposition 5.6.** We follow the procedure in Da Fonseca and Grasselli [23]. We suppose an expansion for the integrated implied variance of the form  $v = \sigma_{imp}^2\tau = \zeta_0 + \alpha\zeta_1 + \alpha^2\zeta_2$ , and we consider the Black–Scholes formula as a function of the integrated implied variance and the log exchange rate  $x = \log S$ :  $C_{B\&S}(S(t), K, \sigma, \tau) = C_{B\&S}(x(t), K, \sigma_{imp}^2\tau, \tau)$ . A Taylor–McLaurin expansion gives us the following:

$$\begin{aligned}
C_{B\&S}(x(t), K, \sigma_{imp}^2\tau, \tau) & = C_{B\&S}(x(t), K, \zeta_0, \tau) + \alpha\zeta_1\partial_v C_{B\&S}(x(t), K, \zeta_0, \tau) \\
& + \frac{\alpha^2}{2} (2\zeta_2\partial_v C_{B\&S}(x(t), K, \zeta_0, \tau) \\
& + \zeta_1^2\partial_{v^2}^2 C_{B\&S}(x(t), K, \zeta_0, \tau)).
\end{aligned}
\tag{9.23}$$

By comparing this with the price expansion (9.22) we deduce that the coefficients must be of the form

$$(9.24) \quad \zeta_0 = v_0,$$

$$\begin{aligned}
(9.25) \quad \zeta_1 &= \frac{(\tilde{\mathcal{A}}^1(\tau) + Tr[\tilde{\mathcal{B}}^1(\tau)\Sigma])\partial_{xv}^2 C_{B\&S}}{\partial_v C_{B\&S}}, \\
\zeta_2 &= \frac{-\zeta_1^2 \partial_{\xi^2}^2 C_{B\&S} + 2(\tilde{\mathcal{A}}^{20}(\tau) + Tr[\tilde{\mathcal{B}}^{20}(\tau)\Sigma])\partial_{v^2}^2 C_{B\&S}}{2\partial_v C_{B\&S}} \\
&\quad + \frac{2(\tilde{\mathcal{A}}^{21}(\tau) + Tr[\tilde{\mathcal{B}}^{21}(\tau)\Sigma])\partial_{x^2v}^3 C_{B\&S}}{2\partial_v C_{B\&S}} \\
(9.26) \quad &\quad + \frac{(\tilde{\mathcal{A}}^1(\tau) + Tr[\tilde{\mathcal{B}}^1(\tau)\Sigma])^2 \partial_{x^2v^2}^4 C_{B\&S}}{2\partial_v C_{B\&S}},
\end{aligned}$$

where the Black–Scholes formula  $C_{B\&S}(x(t), K, \sigma_{imp}^2 \tau, \tau)$  is evaluated at the point  $(x, K, v_0, \tau)$ . In order to find the values of  $\zeta_1, \zeta_2$ , we differentiate (5.15)–(5.18), thus obtaining the following ODEs:

$$\begin{aligned}
(9.27) \quad \frac{\partial}{\partial \tau} \tilde{\mathcal{B}}^0 &= \tilde{\mathcal{B}}^0(\tau) \tilde{M} + \tilde{M}^\top \tilde{\mathcal{B}}^0(\tau) + (A_i - A_j)^2, \\
(9.28) \quad \frac{\partial}{\partial \tau} \tilde{\mathcal{B}}^1 &= \tilde{\mathcal{B}}^1(\tau) \tilde{M} + \tilde{M}^\top \tilde{\mathcal{B}}^1(\tau) \\
&\quad + \tilde{\mathcal{B}}^0(\tau) Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q \tilde{\mathcal{B}}^0(\tau), \\
(9.29) \quad \frac{\partial}{\partial \tau} \tilde{\mathcal{B}}^{20} &= \tilde{\mathcal{B}}^{20}(\tau) \tilde{M} + \tilde{M}^\top \tilde{\mathcal{B}}^{20}(\tau) + 2\tilde{\mathcal{B}}^0(\tau) Q^\top Q \tilde{\mathcal{B}}^0(\tau), \\
(9.30) \quad \frac{\partial}{\partial \tau} \tilde{\mathcal{B}}^{21} &= \tilde{\mathcal{B}}^{21}(\tau) \tilde{M} + \tilde{M}^\top \tilde{\mathcal{B}}^{21}(\tau) \\
&\quad + \tilde{\mathcal{B}}^1(\tau) Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q \tilde{\mathcal{B}}^1(\tau).
\end{aligned}$$

We consider a Taylor–McLaurin expansion in terms of  $\tau$ :

$$\begin{aligned}
(9.31) \quad \tilde{\mathcal{B}}^0(\tau) &\approx (A_i - A_j)^2 \tau + \frac{\tau^2}{2} \left( (A_i - A_j)^2 \tilde{M} + \tilde{M}^\top (A_i - A_j)^2 \right), \\
\tilde{\mathcal{B}}^1(\tau) &\approx \frac{\tau^2}{2} \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right] \\
&\quad + \frac{\tau^3}{6} \left( \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right] \tilde{M} \right. \\
&\quad \left. + \tilde{M}^\top \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right] \right) \\
&\quad + \frac{\tau^3}{6} \left( \left[ (A_i - A_j)^2 \tilde{M} + \tilde{M}^\top (A_i - A_j)^2 \right] Q^\top R (A_i - A_j) \right. \\
(9.32) \quad &\quad \left. + (A_i - A_j) R^\top Q \left[ (A_i - A_j)^2 \tilde{M} + \tilde{M}^\top (A_i - A_j)^2 \right] \right),
\end{aligned}$$

$$\begin{aligned}
(9.33) \quad \tilde{\mathcal{B}}^{20}(\tau) &\approx \frac{\tau^3}{6} 4 (A_i - A_j)^2 Q^\top Q (A_i - A_j)^2, \\
\tilde{\mathcal{B}}^{21}(\tau) &\approx \frac{\tau^3}{6} \left( \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right] \right. \\
&\quad \left. \times Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q \right)
\end{aligned}$$

$$(9.34) \quad \times \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right].$$

Noting from (5.19)–(5.22) that  $\tilde{\mathcal{A}}^i(\tau)$  are one order in  $\tau$  higher than the corresponding  $\tilde{\mathcal{B}}^i(\tau)$ , the following approximations hold:

$$(9.35) \quad \tilde{\mathcal{A}}^0(\tau) + Tr \left[ \tilde{\mathcal{B}}^0(\tau) \Sigma \right] = Tr \left[ (A_i - A_j)^2 \Sigma \right] \tau + o(\tau),$$

$$(9.36) \quad \tilde{\mathcal{A}}^1(\tau) + Tr \left[ \tilde{\mathcal{B}}^1(\tau) \Sigma \right] = Tr \left[ (A_i - A_j)^2 Q^\top R (A_i - A_j) \Sigma \right] \tau^2 + o(\tau^2),$$

$$(9.37) \quad \tilde{\mathcal{A}}^{20}(\tau) + Tr \left[ \tilde{\mathcal{B}}^{20}(\tau) \Sigma \right] = \frac{2}{3} Tr \left[ (A_i - A_j)^2 Q^\top Q (A_i - A_j)^2 \right] \tau^3 + o(\tau^3),$$

$$(9.38) \quad \begin{aligned} \tilde{\mathcal{A}}^{21}(\tau) + Tr \left[ \tilde{\mathcal{B}}^{21}(\tau) \Sigma \right] = Tr \left[ \left( (A_i - A_j)^2 Q^\top R (A_i - A_j) \right. \right. \\ \left. \left. + (A_i - A_j) R^\top Q (A_i - A_j)^2 \right) \right. \\ \left. \times Q^\top R (A_i - A_j) \Sigma \right] \frac{\tau^3}{3} + o(\tau^3). \end{aligned}$$

We introduce two variables: the log-moneyness  $m_f = \log\left(\frac{S^{i,j}(t)e^{(r_i-r_j)\tau}}{K}\right)$  and the variance  $V = Tr \left[ (A_i - A_j) \Sigma(t) (A_i - A_j) \right] \tau$ . Then, from Lewis [55], we consider the following ratios among the derivatives of the Black–Scholes formula:

$$(9.39) \quad \frac{\partial_{x,v}^2 C_{B\&S}(x, K, V, \tau)}{\partial_v C_{B\&S}(x, K, V, \tau)} = \frac{1}{2} + \frac{m_f}{V},$$

$$(9.40) \quad \frac{\partial_{v^2}^2 C_{B\&S}(x, K, V, \tau)}{\partial_v C_{B\&S}(x, K, V, \tau)} = \frac{m_f^2}{2V^2} - \frac{1}{2V} - \frac{1}{8},$$

$$(9.41) \quad \frac{\partial_{x^2,v}^3 C_{B\&S}(x, K, V, \tau)}{\partial_v C_{B\&S}(x, K, V, \tau)} = \frac{1}{4} + \frac{m_f - 1}{V} + \frac{m_f^2}{V^2},$$

$$(9.42) \quad \frac{\partial_{x^2,v^2}^4 C_{B\&S}(x, K, V, \tau)}{\partial_v C_{B\&S}(x, K, V, \tau)} = \frac{m_f^4}{2V^4} + \frac{m_f^2(m_f - 1)}{2V^3}.$$

Upon substitution of (9.35)–(9.42) into (9.25)–(9.26) we obtain the values for  $\zeta_i$ ,  $i = 0, 1, 2$ , allowing us to express the expansion of the implied volatility:

$$(9.43) \quad \zeta_0 = Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)] \tau,$$

$$(9.44) \quad \zeta_1 = \frac{Tr[(A_i - A_j)^2 Q^\top R (A_i - A_j) \Sigma(t)] m_f \tau}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]},$$

$$(9.45) \quad \begin{aligned} \zeta_3 = \frac{m_f^2}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]^2} \tau \left[ \frac{1}{3} Tr[(A_i - A_j)^2 Q^\top Q (A_i - A_j)^2 \Sigma(t)] \right. \\ \left. + \frac{1}{3} Tr[(A_i - A_j)^2 Q^\top R (A_i - A_j) + (A_i - A_j) R^\top Q (A_i - A_j)^2] \right. \\ \left. \times Q^\top R (A_i - A_j) \Sigma(t) \right] - \frac{5}{4} \frac{Tr[(A_i - A_j)^2 Q^\top R (A_i - A_j) \Sigma(t)]^2}{Tr[(A_i - A_j) \Sigma(t) (A_i - A_j)]}. \end{aligned}$$

Plugging these expressions into (9.23) gives the result.

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