Rev Deriv Res (2018) 21:1–43
https://doi.org/10.1007/s11147-017-9132-8

A multivariate stochastic volatility model with applications in the foreign exchange market

Marcos Escobar[1] · **Christoph Gschnaidtner**[2]

Published online: 20 March 2017

Abstract The main objective of this paper is to study the behavior of a daily calibration of a multivariate stochastic volatility model, namely the principal component stochastic volatility (PCSV) model, to market data of plain vanilla options on foreign exchange rates. To this end, a general setting describing a foreign exchange market is introduced. Two adequate models—PCSV and a simpler multivariate Heston model—are adjusted to suit the foreign exchange setting. For both models, characteristic functions are found which allow for an almost instantaneous calculation of option prices using Fourier techniques. After presenting the general calibration procedure, both the multivariate Heston and the PCSV models are calibrated to a time series of option data on three exchange rates—*USD-SEK*, *EUR-SEK*, and *EUR-USD*—spanning more than 11 years. Finally, the benefits of the PCSV model which we find to be superior to the multivariate extension of the Heston model in replicating the dynamics of these options are highlighted.

Keywords Stochastic volatility models · Multivariate models · PCSV model · FX options · Calibration · Triangular relation

JEL Classification F31 · G13 · G15

✉ Christoph Gschnaidtner
christoph.gschnaidtner@tum.de

Marcos Escobar
marcos.escobar@uwo.ca

1 Statistical and Actuarial Sciences, Western University, 1151 Richmond Street, London, ON N6A 3K7, Canada

2 TUM School of Management, Technische Universität München, Arcisstr. 21, 80333 Munich, Germany

1 Introduction

Models for pricing options on foreign exchange (FX) rates have a long history dating back to Feiger and Bertrand (1979). They develop a pricing strategy for foreign exchange rate options using currency option bonds. At that time, the currency option bond market was well-organized in contrast to the foreign exchange market. While the ansatz and the resulting formula of Feiger and Bertrand (1979) rely on the existence of currency option bonds, Biger and Hull (1983), Garman and Kohlhagen (1983), and Grabbe (1983) develop almost simultaneously the same closed-form expression exclusively for valuing options on foreign exchange rates. Replacing the continuously payed dividend yield in the Black-Scholes-Merton model with the foreign risk-free interest rate their formula is a simple extension of the model developed in the seminal papers of Black and Scholes (1973) and Merton (1973). Since then various articles have been published on modeling and pricing contingent claims on foreign exchange rates (see e.g. Melino and Turnbull 1990).

A major breakthrough in the pricing of foreign exchange rate options was the publication of the Heston model (Heston 1993). Motivating the derivation of his model by explicitly referring to options on foreign exchange rates, Steve Heston overcame the problem of constant volatility inherent to the Black-Scholes-Merton model and its modifications by introducing stochastic volatility. At the same time, he provided an analytic expression for the characteristic function of the Heston model which allows for an almost instantaneous pricing of options via Fourier transform. Modeling the volatility by using an additional stochastic process, it was now possible to capture an important characteristic of the foreign exchange rate option market not incorporated in the Garman-Kohlhagen model; the volatility smile, or smirk, of foreign exchange rate options. After the publication of the Heston model, other authors allowed for further flexibility by assuming stochastic interest rates (Ahlip 2008), stochastic jumps in the underlying exchange rate (Bates 1996; Bannör and Scherer 2013), or multifactor models (Carr and Wu 2007; Christoffersen et al. 2009) better capturing the volatility smile, to mention just a few.

All of these articles have one thing in common: they are univariate (one-asset) models. However, as will be shown throughout this article, it does not suffice to consider foreign exchange rates in isolation. Instead, it is inevitable to follow a multivariate (multi-asset) approach when trying to capture all characteristics of the foreign exchange market. While it is rather trivial to extend the standard foreign exchange rate model by Garman and Kohlhagen (1983) to a multivariate setting, it is a much more complex endeavor to establish a multivariate setting where the volatility is stochastic (e.g. Lipton 2001 or De Col et al. 2013). There have been various proposals to add to the literature of pricing in stochastic covariance but mainly outside FX. For example, Dempster and Hong (2000) introduce a three-factor stochastic volatility model, in which several underlyings share one common stochastic factor. This ansatz permits them to introduce a correlation among the underlyings, allowing them to price spread options more realistically. Recchioni and Scoccia (2014) model the correlation among the assets by introducing an additional Brownian motion to the price processes, which is shared by all assets. Further model proposals include Carr and Crosby (2010), Branger and Muck (2012), Barndorff-Nielsen and Stelzer (2013), Da Fonseca et al. (2014), or Shiraya and Takahashi (2014).

There also exist few multivariate approaches that specifically aim at modeling foreign exchange rates. Bannör et al. (2015) extend the existing literature with a multi-currency jump-diffusion model which allows for two-sided jumps and accounts for stochastic volatility. Doust (2012) and De Col et al. (2013) also consider multivariate extensions of option pricing models. Both models hereby assume the existence of a natural numéraire as described by Flesaker and Hughston (1997). The natural numéraire can be thought of as a universal measure which determines the intrinsic value of each currency independently. In some way, the natural numéraire resembles the idea behind the *Big Mac Index* published by *The Economist* to determine the intrinsic value of a currency. Adopting the idea of the natural numéraire, both models initially do not model the exchange rates themselves but the value of each currency in terms of the natural numéraire.

Despite following the same aim as De Col et al. (2013), namely the extension of the univariate Heston model to a multivariate setting, our ansatz is rather different from the paths chosen by Doust (2012) and De Col et al. (2013). In the presented model, the Principal Component Stochastic Volatility (PCSV) model developed by Escobar et al. (2010) is used to replicate the dynamics of each exchange rate directly. Hereby, each foreign exchange rate follows a two-factor Heston type price process whose volatility factors (which are, in the context of the Principal Component Analysis (PCA), referred to as eigenvalue processes) are scaled by exchange rate specific parameters and shared among all underlyings. This model avoids identifiability issues (lack of uniqueness in the estimation of parameters) and imposes a flexible dependency structure on the exchange rates that varies stochastically across time allowing for switching algebraic signs of the correlation coefficients between the different exchange rates. Moreover, it accounts for the stochastic nature of the correlation between each exchange rate and its variance. As in De Col et al. (2013), the model is adjusted for a setting with three different, yet interdependent exchange rates resulting in a model which will be referred to as extended PCSV model. A methodology is developed which allows for an almost instantaneous pricing of foreign exchange rate options within this setting. This enables an empirical examination of the model using real foreign exchange market data. Within this study, the extended PCSV model is calibrated to three exchange rates on a daily basis, similar to Bakshi et al. (1997) who calibrate different option pricing models using equity options, for a period of more than 11 years. Despite the existence of many different univariate or multivariate option pricing models, only a small number of articles have conducted a similar extensive model calibration including option time series data spanning several years. One example is Doust (2012) who calibrates his multivariate SABR-style model to several foreign exchange rates over a period of 5 years. However, in contrast to this article, he does not perform a pure calibration but rather a hybrid fitting procedure which includes besides calibration the principle of maximum entropy.[1] Instead, many articles calibrate their models to individual days. These include among many others the articles of Carr and Crosby (2010), Muhle-

[1] See Neri and Schneider (2012) for more detail on maximum entropy.

Karbe et al. (2012), De Col et al. (2013), Bannör et al. (2015), or van der Stoep et al. (2015). Yet, a calibration of option pricing models to a time series of substantial length has many advantages. It provides new insights into the characteristics of option markets and their variation over time; it confirms the validity of a model with its core assumptions of which are indeed the parameters (not varying over time); it points out at possible breaks in market behaviour not compatible with the model structure. In general, such an analysis allows for a better assessment of the quality and suitability of the selected model. Hence, this article contributes to the literature in three main directions:

– It develops a multivariate stochastic volatility model, namely the extended PCSV model, for pricing options in a foreign exchange setting with more than one exchange rate. Unlike the recent paper by De Col et al. (2013), the extended PCSV model is statistically identifiable and it does not assume the existence of a universal numéraire. This model is flexible in terms of the number of model parameters needed, hence, allowing for a faster and/or more accurate (in terms of a lower calibration uncertainty as defined by Escobar and Gschnaidtner 2016) calibration to market data.
– It calibrates a multivariate Heston model (benchmark model referred to as extended Heston model) and the extended PCSV model to an extensive time series of daily option data on the exchange rates of three currencies, the Swedish krona (*SEK*), the euro (*EUR*), and the US dollar (*USD*), i.e. on the *USD-SEK*, *EUR-SEK*, and *EUR-USD* exchange rates. The calibration spans a period of more than 11 consecutive years (the largest period on foreign exchange data in the literature) allowing for a comprehensive, robust analysis of the examined models.
– Based on Fourier pricing techniques for correlation options, closed-form expressions for an almost instantaneous calculation of foreign exchange rate options in a trivariate setting for cases in which the conditional characteristic function of the option pricing model is known in closed form are derived.

The rest of the article has the following structure: the second section introduces the theoretical background of a foreign exchange market and discusses the risk-neutral pricing of FX options traded on that market. Besides some preliminaries on exchange rates and on the general formulas for determining the prices of plain vanilla European FX options, two specific models, namely the Heston model and the Principal Component Stochastic Volatility model are presented and analyzed. Since it is possible to derive the characteristic function in analytic form for both models, Sect. 2.3 adapts the general Fourier transform, based on which the formulas for the option prices are determined, to the specific foreign exchange setting.

The empirical part starts with a presentation of the data. Next, the methodologies chosen for calibrating the two models to the available market data are described (Sect. 4).

At the heart of the article lie the calibrations of the extended Heston and PCSV models to options on the exchange rates between the US dollar, the euro, and the Swedish krona. Section 5 illustrates and analyzes the empirical results in great detail. Furthermore, particular emphasis is placed on the interpretation of the calibrated model parameters to ensure that the reader is able to correctly assess the quality of the calibrations. The article concludes by revisiting its main achievements.

2 Pricing of options on foreign exchange rates

This section first presents a general framework for options on foreign exchange markets. To allow for a consistent option pricing, next two suitable models, the Heston model and the Principal Component Stochastic Volatility model, are adapted to the FX setting. For both models, the conditional characteristic function exists. This is of eminent importance as the existence of the conditional characteristic function enables the use of Fourier pricing methods entailing enormous speed advantages in the pricing of options. Suitable Fourier pricing methods, specifically intended for the FX setting, will be introduced in the last part of this section.

2.1 Foreign exchange rate preliminaries

Foreign exchange has various features which differentiate it from other asset classes such as stocks or commodities. These differences also prevail in the associated FX option market. Therefore, they must be accounted for in option pricing models being considered for use within a foreign exchange setting. The differences between FX option markets and other asset option markets can loosely be divided into differences regarding the option data provided by the market (see Sect. 3) and economic/structural differences. The remainder of this section is devoted to a general overview on the structural differences and their various implications for our models when used in pricing foreign exchange rate options.

2.1.1 Foreign exchange market

The main difference between the foreign exchange market and other asset markets is that the price of the underlying asset in the foreign exchange market, namely the exchange rate at which two currencies are exchanged, can be given in terms of either of the two currencies involved in a trade. Given a domestic economy with domestic currency *dom* and a foreign economy with foreign currency *for*, it is possible to quote the exchange rate between the two currencies in units of the domestic currency per 1 unit of the foreign currency ($S^{for\text{-}dom}$, according to Bessembinder 1994 referred to as *American terms*) or, vice versa, in units of the foreign currency per 1 unit of the domestic currency ($S^{dom\text{-}for}$, *European terms*). In the presented article, the quotation method common among practitioners in the foreign exchange market is applied: The exchange rate between any two currencies is stated as the amount of the domestic currency necessary to obtain 1 unit of the foreign currency, i.e. the exchange rate at time t is denoted as $S^{for\text{-}dom}(t)$. Hereby, as Reiswich and Wystup (2010) point out, *domestic* and *foreign* do not refer to the geographical region of the market agents but rather to a common convention on the numéraire. The classification of a currency as *domestic* or *foreign* follows a hierarchy described by Clark (2011) and depends on the specific currency pair. In the case of the example above, the US dollar is referred to as

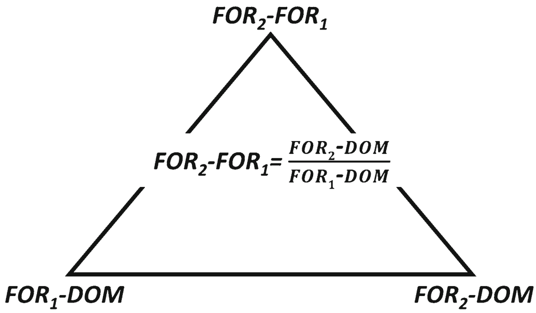

Fig. 1 Implied triangular relationship for foreign exchange rates

the domestic currency and the euro as the foreign currency (see Wystup 2007 or Clark 2011).

If the focus does not only lie on two economies, but a third is under scrutiny, a further characteristic specific to the foreign exchange world can be witnessed: If the first economy is defined as the domestic economy with currency DOM, the second economy as the foreign economy 1 with currency FOR_1, and the third economy as the foreign economy 2 with currency FOR_2, it is possible to establish a *triangular relationship* between the three currency pairs FOR_1-DOM, FOR_2-DOM, and FOR_2-FOR_1.[2] This relation states that it is possible to write the FOR_2-FOR_1 exchange rate as the ratio of the FOR_1-DOM and FOR_2-DOM exchange rates, i.e. it holds at any time t that

$$S^{FOR_2\text{-}FOR_1}(t) = \frac{S^{FOR_2\text{-}DOM}(t)}{S^{FOR_1\text{-}DOM}(t)}.$$

The triangular relationship is also depicted in Fig. 1. If this relation does not hold, market participants are able to realize an arbitrage opportunity, known as *triangular arbitrage* or *cross-currency arbitrage*. Adopting common practice, this article assumes arbitrage-free markets which, in particular, comprises the non-existence of triangular arbitrage opportunities.

As pointed out in the introduction, major drivers of the exchange rate between two currencies are the current as well as the expected (risk-free) interest rates in the respective countries. The interest rate of a country with, for example, currency DOM is denoted as r_{DOM}. In the following, the simplifying assumption of constant interest rates, both in the domestic and the foreign economies, is adopted. This assumption can be relaxed by introducing deterministic or even stochastic interest rates.[3]

Given these stylized features of foreign exchange rates, it is now possible to formally define a foreign exchange market. The following generalized definition of an arbitrage-

[2] Lipton (2001) also refers to the triangular relation as the *cross-currency rule*.

[3] See, for example, Grabbe (1983), Ahlip (2008), Grzelak and Oosterlee (2011), who include stochastic interest rates into the Heston framework, or Amin and Jarrow (1991), who use a two-factor model similar to Heath et al. (1990) for the interest rates. However, stochastic interest rates within a foreign exchange setting must be considered with caution as there exist causal or even reverse causal effects between exchange rates and their corresponding interest rates.

free foreign exchange market will serve as the foundation for the main theoretical results in the rest of this article and enables the derivation of a consistent theory for pricing options within a foreign exchange setting. To capture all characteristics entitative in the FX market we only consider the general, yet sufficient case of a foreign exchange market consisting of three economies, each with a different currency.

Definition 1 (*Arbitrage-free foreign exchange market consisting of three economies*) Given is a foreign exchange market consisting of three economies; a domestic economy with currency DOM and two foreign economies with currency FOR_1 and FOR_2, respectively. Each economy is described by a risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q}_{DOM})$, $(\Omega, \mathcal{F}, \mathbb{Q}_{FOR_1})$, or $(\Omega, \mathcal{F}, \mathbb{Q}_{FOR_2})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the associated filtration with σ-algebra $\mathcal{F}_t$ representing the (global) information available at time t. In each economy the corresponding currency serves as the numéraire and there exist deterministic risk-free money market accounts with time t values $e^{r_{DOM}\cdot t}$, $e^{r_{FOR_1}\cdot t}$, and $e^{r_{FOR_2}\cdot t}$ denominated in the local currency. Here, r_{DOM}, r_{FOR_1}, and r_{FOR_2} are the constant risk-free interest rates of each country. The exchange rates at time t between the currencies of the economies are given by $S^{FOR_1\text{-}DOM}(t)$, $S^{FOR_2\text{-}DOM}(t)$, and $S^{FOR_2\text{-}FOR_1}(t)$. Furthermore, each non-local money market account is a locally tradeable asset.[4]

Then, the foreign exchange market is called arbitrage-free, if the following two conditions are satisfied:

1. For any tuple (dom, for) of the three economies, with $dom, for \in \{DOM, FOR_1, FOR_2\}$, $dom \neq for$, and for all $T \geq t \geq 0$ the uncovered interest rate parity holds, i.e.

$$\mathrm{E}_{\mathbb{Q}_{dom}}\left[S^{for\text{-}dom}(T)\middle|\mathcal{F}_t\right] = S^{for\text{-}dom}(t)\cdot e^{\left(r_{dom}-r_{for}\right)\cdot(T-t)}. \tag{D1}$$

2. For all $t \geq 0$, the triangular relationship for foreign exchange rates given by

$$S^{FOR_2\text{-}FOR_1}(t) = \frac{S^{FOR_2\text{-}DOM}(t)}{S^{FOR_1\text{-}DOM}(t)} \tag{D2}$$

is fulfilled.

2.1.2 *Option pricing within a foreign exchange setting*

In their seminal paper Cox and Ross (1976) propose that the value of any derivative is equal to its risk-free discounted expected payoff(s) under the risk-neutral probability measure conditional on the current available information. This also applies to foreign exchange rate options. Given the data available for the empirical work conducted in this article and given that plain vanilla options have the highest liquidity among foreign exchange rate options (compare Bossens et al. 2010), in the subsequent elaborations only European plain vanilla options—particularly European call options—will be

[4] Here, *local* refers to the geographical location of the investor.

analyzed. The formulas for European put options follow analogously and are thus omitted. For $dom, for \in \{DOM, FOR_1, FOR_2\}$, $dom \neq for$, the value $V_{Call}^{for\text{-}dom}(t,T)$ at time t of a European call option on the exchange rate $S^{for\text{-}dom}(t)$ with maturity T, $t < T$, and payoff at maturity defined by the function $g_{Call}\colon \mathbb{R}^+ \to \mathbb{R},\ S^{for\text{-}dom}(T) \mapsto g_{Call}(S^{for\text{-}dom}(T))$ with

$$
\begin{aligned}
g_{Call}\left(S^{for\text{-}dom}(T)\right) &= \left(S^{for\text{-}dom}(T) - K\right)^+ \\
&= \max\left\{S^{for\text{-}dom}(T) - K, 0\right\},
\end{aligned}
$$

is given by

$$
V_{Call}^{for\text{-}dom}(t,T) = e^{-r_{dom}\cdot(T-t)} \cdot \mathrm{E}_{\mathbb{Q}_{dom}}\left[g_{Call}\left(S^{for\text{-}dom}(T)\right)\middle|\mathcal{F}_t\right].
$$

In the context of the foreign exchange market described by Definition 1, it follows that the value of a call option, depending on the option's underlying exchange rate, at time t is given by either of the following

$$
\begin{aligned}
V_{Call}^{FOR_1\text{-}DOM}(t,T) &= e^{-r_{DOM}\cdot(T-t)} \\
&\quad \cdot \mathrm{E}_{\mathbb{Q}_{DOM}}\left[\left(S^{FOR_1\text{-}DOM}(T) - K^{FOR_1\text{-}DOM}\right)^+\middle|\mathcal{F}_t\right], &(1)\\
V_{Call}^{FOR_2\text{-}DOM}(t,T) &= e^{-r_{DOM}\cdot(T-t)} \\
&\quad \cdot \mathrm{E}_{\mathbb{Q}_{DOM}}\left[\left(S^{FOR_2\text{-}DOM}(T) - K^{FOR_2\text{-}DOM}\right)^+\middle|\mathcal{F}_t\right], &(2)\\
V_{Call}^{FOR_2\text{-}FOR_1}(t,T) &= e^{-r_{FOR_1}\cdot(T-t)} \\
&\quad \cdot \mathrm{E}_{\mathbb{Q}_{FOR_1}}\left[\left(S^{FOR_2\text{-}FOR_1}(T) - K^{FOR_2\text{-}FOR_1}\right)^+\middle|\mathcal{F}_t\right]. &(3)
\end{aligned}
$$

It is important to notice that the values of the options described by Eqs. (1) and (2) are denominated in DOM, i.e. in the currency of the domestic economy, whereas in Eq. (3) the value of the option written on the FOR_2-FOR_1 exchange rate is given in terms of the currency of the first foreign economy. Moreover, while the expectations in Eqs. (1) and (2) are calculated under the risk-neutral probability measure with the domestic money market account as numéraire, the value of a contingent claim on the FOR_1-FOR_2 exchange rate is determined using the money market account of the first foreign economy as numéraire. As will be seen in Sects. 2.2 and 2.3 it is, however, for technical reasons necessary to calculate the expectation in Eq. (3) under the equivalent risk-neutral probability measure generated by the DOM currency and with the domestic money market account as numéraire. This can be achieved by using the fundamental change of numéraire theorem proposed by Geman et al. (1995). Since it is of particular interest to change from one numéraire currency (FOR_1) to another (DOM), an adapted change of numéraire theorem which can be found in Frey and

Sommer (1996), Schlögl (2002), Pelsser (2003), or Bannör and Scherer (2013) is presented.

Theorem 1 (Foreign exchange change of numéraire theorem, Pelsser (2003)) *Given an arbitrage-free foreign exchange market as described by Definition 1. For any tuple (dom, for) with dom, for* $\in \{DOM, FOR_1, FOR_2\}$*, dom* $\neq$ *for, it holds at time t that the expected value under* $\mathbb{Q}_{for}$ *of a random variable at time T can be written as its expectation under* $\mathbb{Q}_{dom}$ *times the Radon-Nikodým derivative*

$$\left.\frac{d\mathbb{Q}_{for}}{d\mathbb{Q}_{dom}}\right|_{\mathcal{F}_T} = \frac{\frac{e^{r_{for}\cdot(T-t)}\cdot S^{for\text{-}dom}(T)}{1\cdot S^{for\text{-}dom}(t)}}{\frac{e^{r_{dom}\cdot(T-t)}}{1}} = \frac{\frac{1\cdot S^{for\text{-}dom}(T)}{e^{-r_{for}\cdot(T-t)}\cdot S^{for\text{-}dom}(t)}}{\frac{1}{e^{-r_{dom}\cdot(T-t)}}}$$

i.e. it holds that

$$\begin{aligned} E_{\mathbb{Q}_{for}}\left[g(\cdot)\middle|\mathcal{F}_t\right] &= E_{\mathbb{Q}_{dom}}\left[g(\cdot)\cdot\frac{\frac{1\cdot S^{for\text{-}dom}(T)}{e^{-r_{for}\cdot(T-t)}\cdot S^{for\text{-}dom}(t)}}{\frac{1}{e^{-r_{dom}\cdot(T-t)}}}\middle|\mathcal{F}_t\right] \\ &= \frac{e^{-r_{dom}\cdot(T-t)}}{e^{-r_{for}\cdot(T-t)}\cdot S^{for\text{-}dom}(t)}\cdot E_{\mathbb{Q}_{dom}}\left[g(\cdot)\cdot S^{for\text{-}dom}(T)\middle|\mathcal{F}_t\right], \end{aligned}$$

with $g(\cdot)$ *an arbitrary function (e.g. the payoff function of an option) of a* $\mathcal{F}_T$*-measurable random variable and* $S^{for\text{-}dom}(t)$, $S^{for\text{-}dom}(T)$ *the realized foreign exchange rates at time t and T, respectively.*

Proof The proof of Theorem 1 is mainly based on the arguments provided by Geman et al. (1995) for the change of numéraire theorem.

The change of numéraire theorem is required to switch in Eq. (3) from the risk-neutral probability measure $\mathbb{Q}_{FOR_1}$ with the money market account of the first foreign economy (FOR_1) as the numéraire to the risk-neutral probability measure associated with the domestic economy. With $for = FOR_1$ and $dom = DOM$ it follows immediately from Theorem 1 that

$$\begin{aligned} V_{Call}^{FOR_2\text{-}FOR_1}(t,T) &= e^{-r_{FOR_1}\cdot(T-t)}\cdot \mathrm{E}_{\mathbb{Q}_{FOR_1}}\left[\left(S^{FOR_2\text{-}FOR_1}(T)-K^{FOR_2\text{-}FOR_1}\right)^+\middle|\mathcal{F}_t\right] \\ &= \frac{e^{-r_{DOM}\cdot(T-t)}}{S^{FOR_1\text{-}DOM}(t)}\cdot \mathrm{E}_{\mathbb{Q}_{DOM}}\Big[\left(S^{FOR_2\text{-}FOR_1}(T)-K^{FOR_2\text{-}FOR_1}\right)^+ \\ &\quad \cdot S^{FOR_1\text{-}DOM}(T)\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{4}$$

One quickly notices that, due to the change of numéraire, in Eq. (4) the payoff of the call option on the FOR_2-FOR_1 exchange rate is in contrast to (1) and (2) now multiplied by the FOR_1-DOM exchange rate at time T. Hence, whereas only the univariate probability distributions of the exchange rates at time T are needed to evaluate the expectations in the first two equations, Eq. (4) requires the joint distribution of the FOR_2-FOR_1 and FOR_1-DOM exchange rates. It is rather obvious that the two exchange rates are presumably not independent as an increase in the FOR_2-FOR_1 exchange rate (depreciation of the FOR_1 currency) is likely to be accompanied by a decrease in the FOR_1-DOM exchange rate (appreciation of the DOM currency). This implies that the joint distribution does not equal the product of the marginal distributions. Instead, to correctly reflect the dependence structure a sophisticated bivariate distribution is necessary. In addition, computing the expectation in Eq. (4) is a non-trivial task that requires existing methods, which are based on characteristic functions and allow for a numerically fast computation, to be adapted (Sect. 2.3).

2.2 Models

This article focuses on the multivariate stochastic volatility model suggested by Escobar et al. (2010) and developed further by Escobar and Olivares (2013). Despite working on different research topics (pricing of collateralized debt obligations and mountain range derivatives), both articles use a model inspired by the idea behind Principal Component Analysis (PCA). Similar to Principal Component Analysis, their time-continuous model, which they call the Principal Component Stochastic Volatility (PCSV) model allows for dimension reduction by identifying major drivers (principal components) and their impact on the (co-)variance matrix of the examined multivariate time series. Since their model belongs to the class of affine stochastic covariance models it is suitable for various applications in many financial areas. These include, in addition to pricing debt securities and multivariate derivatives, the modeling of foreign exchange rates being the field of application of the PCSV model within this article.

Besides the PCSV model, we use an additional, less complex stochastic volatility model as a benchmark for the examination of foreign exchange rates. As already reported in the literature, market implied volatility surfaces do not support the assumption of constant volatility across strike levels and time, this being particularly true for the foreign exchange market. Instead data on foreign exchange rate options encourage allowing for stochastic volatility, a view shared by various other authors including Lipton (2001), Wystup (2007), and Clark (2011). A natural first choice for a model with stochastic volatility is the Heston model (Heston 1993). In contrast to other stochastic volatility models such as the Hull and White model (Hull and White 1987) or the Stein and Stein model (Stein and Stein 1991), the Heston model allows for analytical Fourier transform methods (not the case for the Hull and White model) while keeping leverage constant (no sudden changes in sign as in the Stein and Stein model). Definition 2 states the FX version of the Heston model (referred to as *extended Heston model*) for a foreign exchange market with three economies as introduced in Definition 1.

Definition 2 (*Extended Heston model for foreign exchange rates*) Given is an arbitrage-free foreign exchange market as described in Definition 1. The exchange rates $S^{FOR_1\text{-}DOM}(t)$ and $S^{FOR_2\text{-}DOM}(t)$ follow the extended Heston model if the risk-neutral dynamics of the natural logarithm of the exchange rates $X^{FOR_i\text{-}DOM}(t) = ln(S^{FOR_i\text{-}DOM}(t)), i \in \{1, 2\}$, under the domestic risk-neutral probability measure $\mathbb{Q}_{DOM}$ are given by

$$dX^{FOR_i\text{-}DOM}(t) = \left(r_{DOM} - r_{FOR_i} - \frac{\lambda_i(t)}{2}\right)dt + \sqrt{\lambda_i(t)}dW_i(t),$$
$$d\lambda_i(t) = \kappa_i(\theta_i - \lambda_i(t))dt + \sigma_i\sqrt{\lambda_i(t)}dB_i(t),$$

with $W_i(t)$ and $B_i(t)$ dependent $\mathbb{Q}_{DOM}$-Brownian motions satisfying

$$\langle dW_i(t), dB_i(t)\rangle = \rho_i dt,$$

with $\rho_i \in [-1, 1]$, whereas

$$\langle dW_1(t), dW_2(t)\rangle = \langle dB_1(t), dB_2(t)\rangle = \langle dW_1(t), dB_2(t)\rangle = \langle dW_2(t), dB_1(t)\rangle = 0.$$

Moreover, the dynamics of the natural logarithm $X^{FOR_2\text{-}FOR_1}(t) = \ln\left(S^{FOR_2\text{-}FOR_1}(t)\right)$ of the implied exchange rate $S^{FOR_2\text{-}FOR_1}(t)$ are described by

$$dX^{FOR_2\text{-}FOR_1}(t) = \left(r_{FOR_1} - r_{FOR_2} - \frac{\lambda_2(t) - \lambda_1(t)}{2}\right)dt + \sqrt{\lambda_2(t)}dW_2 - \sqrt{\lambda_1(t)}dW_1(t),$$

with the dynamics of $\lambda_1(t)$ and $\lambda_2(t)$ as above.

Definition 2 has several implications noteworthy to mention. The dynamics of $X^{FOR_2\text{-}FOR_1}(t)$ are automatically imposed by the second no-arbitrage condition (D2). Yet, as can readily be established, its dynamics under the domestic risk-neutral probability measure are no longer equivalent to a Heston type process. Furthermore, in the extended Heston model the simplifying assumption is made that the exchange rate $S^{FOR_1\text{-}DOM}(t)$ is stochastically independent of $S^{FOR_2\text{-}DOM}(t)$. $S^{FOR_2\text{-}FOR_1}(t)$, in contrast, depends on the other two exchange rates with the instantaneous correlations of the log-exchange rates given by

$$\rho_{dX^{FOR_2\text{-}FOR_1}, dX^{FOR_i\text{-}DOM}}(t) = (-1)^i \cdot \frac{\sqrt{\lambda_i(t)}}{\sqrt{\lambda_1(t) + \lambda_2(t)}}.$$

For pricing purposes, it is desirable to obtain an analytic expression for the characteristic function of a stochastic process following the extended Heston model. A closed form of the characteristic function enables the application of Fourier pricing methods that allow for an almost instantaneous pricing of various options. Heston (1993) shows that it is possible to derive an explicit equation for the conditional characteristic function of the stochastic process described by his model. In the following proposition we extend the findings of Heston (1993) by deriving an analytically tractable formula for the characteristic function of the extended Heston model, i.e. of the three-dimensional stochastic process $(\mathbf{X}(t))_{t\geq 0}$ with

$$\mathbf{X}(t) = \begin{pmatrix} X^{FOR_1\text{-}DOM}(t) \\ X^{FOR_2\text{-}DOM}(t) \\ X^{FOR_2\text{-}FOR_1}(t) \end{pmatrix} = \begin{pmatrix} \ln\left(S^{FOR_1\text{-}DOM}(t)\right) \\ \ln\left(S^{FOR_2\text{-}DOM}(t)\right) \\ \ln\left(S^{FOR_2\text{-}FOR_1}(t)\right) \end{pmatrix}.$$

Proposition 1 *(*Trivariate conditional characteristic function of the extended Heston model) *Given is the same set-up as in Definition 2. Then, the characteristic function of the stochastic process $(\mathbf{X}(t))_{t\geq 0}$ at time T conditional on the information set at time t only depends on the univariate conditional characteristic functions of $X^{FOR_1\text{-}DOM}(t)$, $X^{FOR_2\text{-}DOM}(t)$ and is given by*

$$\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) = \Phi_{X^{FOR_1\text{-}DOM}}(w_1 - w_3, T \mid \mathcal{F}_t) \cdot \Phi_{X^{FOR_2\text{-}DOM}}(w_2 + w_3, T \mid \mathcal{F}_t),$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$ *is an arbitrary vector and* $\Phi_{X^{FOR_1\text{-}DOM}}$, $\Phi_{X^{FOR_2\text{-}DOM}}$ *the univariate conditional characteristic functions of* $X^{FOR_1\text{-}DOM}(t)$ *and* $X^{FOR_2\text{-}DOM}(t)$.[5]

Proof The proof is provided in the "Appendix".

A major shortcoming of the extended Heston model arises from the assumption of independence between the exchange rates. This limitation is overcome by the Principal Component Stochastic Volatility model. Allowing for stochastic dependence it captures possible interaction effects between the exchange rates. The extended PCSV model is a natural generalization of the simpler Heston model with an additional parameter ξ that uniquely defines the eigenvector matrix $\mathbf{A}$ theoretically resulting from a time-continuous PCA. The eigenvector matrix $\mathbf{A}$ plays a central part as it influences the dependence structure in the model. The following definition states the extended PCSV model adapting the version of Escobar and Olivares (2013) for the foreign exchange market with three economies.

Definition 3 (*Extended PCSV model for foreign exchange rates*) Given is an arbitrage-free foreign exchange market as described in Definition 1. The exchange rates $S^{FOR_1\text{-}DOM}(t)$ and $S^{FOR_2\text{-}DOM}(t)$ follow the extended PCSV model if for $i \in \{1, 2\}$ the risk-neutral dynamics of their natural logarithm under $\mathbb{Q}_{DOM}$ are given by

$$\begin{aligned} dX^{FOR_i\text{-}DOM}(t) &= \left(r_{DOM} - r_{FOR_i} - \frac{1}{2} \sum_{j=1}^{2} a_{ij}^2 \lambda_j(t) \right) dt \\ &\quad + \sum_{j=1}^{2} a_{ij} \sqrt{\lambda_j(t)} dW_j(t), \\ d\lambda_j(t) &= \kappa_j(\theta_j - \lambda_j(t))dt + \sigma_j \sqrt{\lambda_j(t)} dB_j(t), \end{aligned}$$

where $(\lambda_j(t))_{t\geq 0}$ is the jth eigenvalue process with CIR parameters κ_j, θ_j, and σ_j. The ith component a_{ij} of the corresponding jth eigenvector is defined according to

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix},$$

[5] See Heston (1993) or Albrecher et al. (2007) for a specification of the univariate conditional characteristic function of the Heston model.

with $\xi \in [0, \pi]$. For $j, j' \in \{1, 2\}$, the Brownian motions $W_j(t)$ and $B_j(t)$ are subject to

$$\left\langle dW_j(t), dW_{j'}(t)\right\rangle = \left\langle dB_j(t), dB_{j'}(t)\right\rangle = \begin{cases} dt, & \text{if } j = j' \\ 0, & \text{otherwise} \end{cases}$$

and, with correlation parameters $\rho_1, \rho_2 \in [-1, 1]$, subject to

$$\left\langle dW_j(t), dB_{j'}(t)\right\rangle = \begin{cases} \rho_j dt, & \text{if } j = j' \\ 0, & \text{otherwise.} \end{cases}$$

With (D2) it follows for the $\mathbb{Q}_{DOM}$-dynamics of $X^{FOR_2\text{-}FOR_1}(t) = \ln\left(S^{FOR_2\text{-}FOR_1}(t)\right)$ that

$$\begin{aligned} dX^{FOR_2\text{-}FOR_1}(t) = &\left(r_{FOR_1} - r_{FOR_2} - \frac{1}{2}\sum_{j=1}^{2}\left(a_{2j}^2 - a_{1j}^2\right)\lambda_j(t)\right) dt \\ &+ \sum_{j=1}^{2}\left(a_{2j} - a_{1j}\right)\sqrt{\lambda_j(t)}dW_j(t). \end{aligned}$$

In the context of the PCSV model the variance processes $(\lambda_j(t))_{t\geq 0}$, $j \in \{1, 2\}$, are referred to as eigenvalue processes. This is to be understood in a rather loose sense: Due to the stochastic nature of the independent CIR processes the usual ordering of the instantaneous eigenvalues $\lambda_1(t) > \lambda_2(t)$ cannot be guaranteed at all times.[6]

Similar to the extended Heston model, it is trivial to show that the dynamics of $X^{FOR_2\text{-}FOR_1}(t)$ do not follow the PCSV model. However, the extended PCSV model does not assume independence between the exchange rates $S^{FOR_1\text{-}DOM}(t)$ and $S^{FOR_2\text{-}DOM}(t)$. Instead, it exhibits stochastic correlation between all exchange rates with the instantaneous correlation coefficients for $i \in \{1, 2\}$ given by

$$\begin{aligned} \rho_{dX^{FOR_2\text{-}FOR_1}, dX^{FOR_i\text{-}DOM}}(t) &= \frac{\sum_{j=1}^{2} a_{ij}\cdot\left(a_{2j} - a_{1j}\right)\cdot\lambda_j(t)}{\sqrt{\sum_{j=1}^{2} a_{ij}^2\cdot\lambda_j(t)}\cdot\sqrt{\sum_{j=1}^{2}\left(a_{2j} - a_{1j}\right)^2\cdot\lambda_j(t)}}, \\ \rho_{dX^{FOR_2\text{-}DOM}, dX^{FOR_1\text{-}DOM}}(t) &= \frac{\sum_{j=1}^{2} a_{1j}\cdot a_{2j}\cdot\lambda_j(t)}{\sqrt{\sum_{j=1}^{2} a_{1j}^2\cdot\lambda_j(t)}\cdot\sqrt{\sum_{j=1}^{2} a_{2j}^2\cdot\lambda_j(t)}}. \end{aligned}$$

Thus, the nature of the correlation coefficients depends on the eigenvector matrix $\mathbf{A}$, i.e. on the parameter ξ, and on the current levels of the eigenvalue processes.

Besides stochastic correlation between different exchange rates, another stylized fact commonly observed for foreign exchange rates is the time varying skew or leverage effect indicating that the correlation between the spot exchange rate and its volatility, commonly referred to as spot-vol correlation, is also stochastic (see Bakshi and Chen

[6] The empirical findings, however, strongly suggest that an interpretation as eigenvalues is appropriate.

1997, Carr and Wu 2007, or Da Fonseca et al. 2008). In contrast to the extended Heston model where the spot-vol correlation is only time varying for the implied exchange rate and constant else, the structure of the extended PCSV model allows to replicate stochastic leverage effects for all exchange rates with the instantaneous spot-vol correlation parameters given by

$$
\begin{aligned}
&\rho_{dX^{FOR_i\text{-}DOM},d\sigma^{FOR_i\text{-}DOM}}(t)\\
&\quad=\frac{\sum_{j=1}^{2} a_{ij}^{3}\cdot\sigma_j\cdot\rho_j\cdot\lambda_j(t)}{\sqrt{\sum_{j=1}^{2} a_{ij}^{2}\cdot\lambda_j(t)}\cdot\sqrt{\sum_{j=1}^{2} a_{ij}^{4}\cdot\sigma_j^{2}\cdot\lambda_j(t)}},\\
&\rho_{dX^{FOR_2\text{-}FOR_1},d\sigma^{FOR_2\text{-}FOR_1}}(t)\\
&\quad=\frac{\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{3}\cdot\sigma_j\cdot\rho_j\cdot\lambda_j(t)}{\sqrt{\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{2}\cdot\lambda_j(t)}\cdot\sqrt{\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{4}\cdot\sigma_j^{2}\cdot\lambda_j(t)}}.
\end{aligned}
$$

For $i\in\{1,2\}$, $\left(\sigma^{FOR_i\text{-}DOM}(t)\right)_{t\geq 0}$ and $\left(\sigma^{FOR_2\text{-}FOR_1}(t)\right)_{t\geq 0}$ denote the volatility processes of the FOR_i-DOM and FOR_2-FOR_1 exchange rates whose dynamics are given by

$$
\begin{aligned}
d\sigma^{FOR_i-DOM}(t)=&\frac{1}{2\cdot\sqrt{\sum_{j=1}^{2} a_{ij}^{2}\cdot\lambda_j(t)}}\\
&\left[\left(\sum_{j=1}^{2} a_{ij}^{2}\cdot\kappa_j\cdot\left(\theta_j-\lambda_j(t)\right)-\frac{\sum_{j=1}^{2} a_{ij}^{4}\cdot\sigma_j^{2}\cdot\lambda_j(t)}{4\cdot\sum_{j=1}^{2} a_{ij}^{2}\cdot\lambda_j(t)}\right)dt\right.\\
&\left.-\sum_{j=1}^{2} a_{ij}^{2}\cdot\sigma_j\cdot\sqrt{\lambda_j(t)}dB_j(t)\right],\\
d\sigma^{FOR_2-FOR_1}(t)=&\frac{1}{2\cdot\sqrt{\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{2}\cdot\lambda_j(t)}}\\
&\left[\left(\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{2}\cdot\kappa_j\cdot\left(\theta_j-\lambda_j(t)\right)\right.\right.\\
&\left.-\frac{\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{4}\cdot\sigma_j^{2}\cdot\lambda_j(t)}{4\cdot\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{2}\cdot\lambda_j(t)}\right)dt\\
&\left.-\sum_{j=1}^{2}\left(a_{2j}-a_{1j}\right)^{2}\cdot\sigma_j\cdot\sqrt{\lambda_j(t)}dB_j(t)\right].
\end{aligned}
$$

Despite the higher complexity of the PCSV model, it is still possible to derive the conditional characteristic function in closed form as was shown by Escobar and

Olivares (2013). In Proposition 2, the FX specific trivariate conditional characteristic function is presented. In contrast to the extended Heston model, in the PCSV model the log-exchange rates $X^{FOR_1\text{-}DOM}(t)$ and $X^{FOR_2\text{-}DOM}(t)$ are usually not independent—independence occurs if $\xi \in \{0, \frac{\pi}{2}\}$ in which case the extended PCSV model reduces to the extended Heston model—complicating the derivation of the trivariate characteristic function.

Proposition 2 (FX specific trivariate conditional characteristic function of the extended PCSV model) *Given is the same set-up as in Definition 3. Then the characteristic function $\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t)$ of the stochastic process $(\mathbf{X}(t))_{t\geq 0}$ at time T conditional on the information set $\mathcal{F}_t$ is given by*

$$\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) = e^{i\mathbf{w}^{*\prime}\mathbf{X}^*(t)} \cdot \prod_{j=1}^{2} e^{C_j(1,T-t)+D_j(1,T-t)\cdot\lambda_j(t)\cdot c_j^2(\mathbf{w})},$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$, $\mathbf{w}^* = \begin{pmatrix} w_1 - w_3 \\ w_2 + w_3 \end{pmatrix}$, $\mathbf{X}^*(t) = \begin{pmatrix} X^{FOR_1\text{-}DOM}(t) \\ X^{FOR_2\text{-}DOM}(t) \end{pmatrix}$, *and*

$$\begin{aligned}
C_j(\phi, T-t) &= r(\mathbf{w})\phi i \cdot (T-t) + \frac{\kappa_j\theta_j}{\sigma_j^2}\Bigg\{(\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j)\cdot(T-t) \\
&\quad - 2\ln\left[\frac{1 - g_j\cdot e^{-d_j\cdot(T-t)}}{1-g_j}\right]\Bigg\}, \\
D_j(\phi, T-t) &= \frac{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j}{c_j^2(\mathbf{w})\sigma_j^2}\left[\frac{1-e^{-d_j\cdot(T-t)}}{1-g_j\cdot e^{-d_j\cdot(T-t)}}\right], \\
g_j &= \frac{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j}{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i + d_j}, \\
d_j &= \sqrt{(\rho_j c_j(\mathbf{w})\sigma_j\phi i - \kappa_j)^2 + \sigma_j^2\left(-2b_j(\mathbf{w})\phi i + c_j^2(\mathbf{w})\phi^2\right)}, \\
c_j(\mathbf{w}) &= a_{1j}\cdot(w_1 - w_3) + a_{2j}\cdot(w_2 + w_3), \\
b_j(\mathbf{w}) &= -\frac{1}{2}\left(a_{1j}^2\cdot(w_1-w_3) + a_{2j}^2\cdot(w_2+w_3)\right), \\
r(\mathbf{w}) &= \frac{r_{DOM} - r_{FOR_1}}{2}\cdot(w_1 - w_3) + \frac{r_{DOM} - r_{FOR_2}}{2}\cdot(w_2+w_3).
\end{aligned}$$

Proof The proof is provided in the "Appendix".

Independent of the model, the trivariate characteristic function allows for a straightforward construction of the univariate conditional characteristic function for each component of the multivariate stochastic process $(\mathbf{X}(t))_{t\geq 0}$ by choosing $\mathbf{w}$ appropriately. Moreover, based on the trivariate version it is also an effortless task to derive any bivariate characteristic function. In particular, as pointed out above, to be able to calculate option prices for the FOR_2-FOR_1 exchange rate using Eq. (3), the analytic expression for the bivariate conditional characteristic function of $(\mathbf{X}^*(t))_{t\geq 0}$ with $\mathbf{X}^*(t) = \begin{pmatrix} X^{FOR_1\text{-}DOM}(t) \\ X^{FOR_2\text{-}FOR_1}(t) \end{pmatrix}$ is required. It can be obtained by setting $w_2 = 0$ in Propositions 1 and 2.

2.3 Fourier pricing methods within a foreign exchange setting

The analytic tractability of the characteristic function is important as it enables the use of various formulas for rapid option pricing. To compute the price of a European style option which is given by its risk-free discounted expected payoff it is only necessary to know the distribution of the underlying's price at the time of maturity. While often there exists no closed form formula for the actual density function, the corresponding characteristic function can still be analytically manageable. In these cases, it is possible to derive the option price via its Fourier transform which depends only on the characteristic function of the underlying price process. To obtain the option price we deliberately follow the wide spread ansatz developed by Carr and Madan (1999). They derive the Fourier transform of a modified call option price and apply the Fourier inversion formula to obtain the corresponding prices. There are several reasons that justify this choice: First, the methodology proposed by Carr and Madan (1999) allows for the application of fast Fourier transform (FFT) methods, which ensure almost instant calculations of option prices. Second, in contrast to other methods, it only involves the evaluation of a single integral and thus, it reduces the number of sources of discretization errors. Third, and most importantly, there exist enhancements of the method of Carr and Madan (1999), proposed by Dempster and Hong (2000), that are valuable for the pricing of FX options in the context of the triangular relationship. The expansion by Dempster and Hong (2000) will be specifically helpful in evaluating the conditional expectation of Eq. (3).

First, however, by adapting the formula proposed by Carr and Madan (1999) to the foreign exchange context, Corollary 1 derives the price of a usual European call option for the case that the underlying is an exchange rate.

Corollary 1 (Fourier pricing formula for foreign exchange options) *Let $X^{for-dom}(t)$ with $dom, for \in \{DOM, FOR_1, FOR_2\}$, $dom \neq for$, be the logarithmus naturalis of the exchange rate $S^{for-dom}(t)$. It follows that the price of a European style call option on the exchange rate $S^{for-dom}(t)$ with strike $K^{for-dom}$ and time of maturity T is given by*

$$V_{Call}^{for-dom}(t,T) = \frac{e^{-\alpha k}}{\pi} \int_0^\infty \mathcal{R}e \left[e^{-ivk} \cdot \frac{e^{-r_{dom}\cdot(T-t)} \cdot \Phi_{X^{for-dom}}(v - i\cdot(\alpha+1), T \mid \mathcal{F}_t)}{\alpha^2 + \alpha - v^2 + i\cdot(2\alpha+1)\cdot v} \right] dv, \tag{5}$$

where $\alpha > 0$ is a damping parameter, $k = \ln(K^{for-dom})$, and $\Phi_{X^{for-dom}}(\phi, T \mid \mathcal{F}_t)$ the characteristic function of $X^{for-dom}(T)$ at the time of expiry T conditional on the information available at time t.

Proof The results of Corollary 1 follow directly from Carr and Madan (1999) when setting the risk-free interest rate r equal to the domestic interest rate r_{dom}.

The damping parameter α ensures that the call price is absolutely integrable and hence, the Fourier inversion formula can be applied. Usually, a range of values satisfies the upper and lower boundaries for α that are established, for example, by Carr and Madan (1999) or Levendorskiy (2012) to guarantee the existence of the Fourier inversion. Yet, no indications on the optimal choice of α are given in the literature and it is left to the user to pick a specific value. Since Escobar and Gschnaidtner (2016) point out that the choice for α has only a marginal (for options with maturities $\leq$1 month) or even insignificant (longer maturities) impact on the resulting option prices, we follow Levendorskiy (2012) and set α equal to 1 without conducting further investigations.

Based on Corollary 1 and independent of the model choice, the option prices (1) and (2) can now easily be calculated by plugging the respective expressions of the characteristic functions into (5). Unfortunately, Eq. (5) is not suitable for determining the price of a call option on the induced exchange rate $S^{FOR_2\text{-}FOR_1}(t)$ given by Eq. (3). Instead, we require a formula for the valuation of an option written on two underlyings, $S^{FOR_2\text{-}FOR_1}(t)$ and $S^{FOR_1\text{-}DOM}(t)$, with maturity T, strike $K^{FOR_2\text{-}FOR_1}$ and payoff given by

$$\left(S^{FOR_2\text{-}FOR_1}(T) - K^{FOR_2\text{-}FOR_1}\right)^+ \cdot S^{FOR_1\text{-}DOM}(T). \tag{6}$$

Since (6) depends on the evolution of more than one underlying, the univariate approach of Carr and Madan (1999) is no longer applicable. Instead, a multivariate ansatz similar to Dempster and Hong (2000), who derive the Fourier transform for a correlation option, or Hurd and Zhou (2010) is required. A correlation option is a contingent claim that depends on the price of two different securities (here: $S^{FOR_2\text{-}FOR_1}(t)$, $S^{FOR_1\text{-}DOM}(t)$) and whose payoff is according to Bakshi and Madan (2000) equivalent to the product of two European call options

$$\left(S^{FOR_2\text{-}FOR_1}(T) - K^{FOR_2\text{-}FOR_1}\right)^+ \cdot \left(S^{FOR_1\text{-}DOM}(T) - K^{FOR_1\text{-}DOM}\right)^+, \tag{7}$$

with strikes $K^{FOR_2\text{-}FOR_1}$, $K^{FOR_1\text{-}DOM}$ and maturity T. Expression (7) is almost identical to the required payoff, the only difference being that $K^{FOR_1\text{-}DOM} = 0$. This is well defined as $S^{FOR_1\text{-}DOM}(t)$ is, by definition, non-negative and thus $\left(S^{FOR_1\text{-}DOM}(T) - 0\right)^+ = S^{FOR_1\text{-}DOM}(T)$ leading to the required payoff (6). We refer to a contingent claim with payoff given by Eq. (6) as a reduced correlation option. In the following corollary, the Fourier formula for a reduced correlation option is derived.

Corollary 2 (Fourier pricing formula for reduced foreign exchange rate correlation otpions) *With* $X^{FOR_1\text{-}DOM}(t) = \ln\left(S^{FOR_1\text{-}DOM}(t)\right)$ *and* $X^{FOR_2\text{-}FOR_1}(t) = \ln\left(S^{FOR_2\text{-}FOR_1}(t)\right)$, *it follows that the price denoted in the DOM currency of an option expiring at time* T, *with strike* $K^{FOR_2\text{-}FOR_1}$ *and with payoff* $\left(S^{FOR_2\text{-}FOR_1}(T) - K^{FOR_2\text{-}FOR_1}\right)^+ \cdot S^{FOR_1\text{-}DOM}(T)$ *is given by*

$$V_{corCall}^{FOR_2\text{-}FOR_1,DOM}(t,T) = \frac{e^{-\alpha k}}{\pi}\int_0^{\infty}\mathcal{R}e$$
$$\left[e^{-ivk}\cdot\frac{e^{-r_{DOM}\cdot(T-t)}\cdot\Phi_{\mathbf{X}^*}((-i, v-i\cdot(1+a))', T\mid\mathcal{F}_t)}{\alpha^2+\alpha-v^2+i\cdot(2\alpha+1)\cdot v}\right]dv. \tag{8}$$

Here, $\alpha > 0$ is a damping parameter, $k = \ln(K^{FOR_2\text{-}FOR_1})$, and $\Phi_{\mathbf{X}^*(T)}\left((\phi_1,\phi_2)', T\mid\mathcal{F}_t\right)$ the bivariate characteristic function of $\mathbf{X}^*(T) = \begin{pmatrix} X^{FOR_1\text{-}DOM}(T) \\ X^{FOR_2\text{-}FOR_1}(T)\end{pmatrix}$ at the time of expiry T conditional on the available information at time t.

Proof The results follow immediately from Dempster and Hong (2000).

In Corollary 2 there are several important things that have to be kept in mind. Within the foreign exchange setting the reduced correlation option is, in fact, just a call option on the exchange rate between the economies with currencies FOR_1 and FOR_2 for which the payoff at the time of expiry is, however, not paid as usually in terms of the FOR_1 currency but in terms of the DOM currency. Hence, the payoff depends on two factors, namely the call option ending up in the money (i.e. $S^{FOR_2\text{-}FOR_1}(T) > K^{FOR_2\text{-}FOR_1}$) and the value of the exchange rate $S^{FOR_1\text{-}DOM}(T)$ at the time of expiry of the option. In addition, viewing the reduced correlation option as a simple call option where the payoff is only paid in another currency, it has also to be considered that the price $V_{corCall}^{FOR_2\text{-}FOR_1,DOM}(t,T)$ is denoted in terms of the domestic currency rather than in terms of the FOR_1 currency. To obtain the price of the call option in terms of the FOR_1 currency, it needs to be divided by the current exchange rate resulting in

$$V_{Call}^{FOR_2\text{-}FOR_1}(t,T) = \frac{V_{corCall}^{FOR_2\text{-}FOR_1,DOM}(t,T)}{S^{FOR_1\text{-}DOM}(t)}. \tag{9}$$

Using (9) in combination with Eq. (3) results in

$$\begin{aligned}\frac{V_{corCall}^{FOR_2\text{-}FOR_1,DOM}(t,T)}{S^{FOR_1-DOM}(t)} &= V_{Call}^{FOR_2\text{-}FOR_1}(t,T,K^{FOR_2\text{-}FOR_1})\\ &= \frac{e^{-r_{DOM}\cdot(T-t)}}{S^{FOR_1\text{-}DOM}(t)}\\ &\quad\cdot \mathrm{E}_{\mathbb{Q}_{DOM}}\Bigg[\left(S^{FOR_2\text{-}FOR_1}(T)-K^{FOR_2\text{-}FOR_1}\right)^+\\ &\quad\cdot S^{FOR_1\text{-}DOM}(T)\Big|\mathcal{F}_t\Bigg]\end{aligned}$$

and finally in

$$\begin{aligned}V_{corCall}^{FOR_2\text{-}FOR_1,DOM}(t,T) = e^{-r_{DOM}\cdot(T-t)}\cdot\mathrm{E}_{\mathbb{Q}_{DOM}}\Bigg[\left(S^{FOR_2\text{-}FOR_1}(T)-K^{FOR_2\text{-}FOR_1}\right)^+\\ \cdot S^{FOR_1\text{-}DOM}(T)\Big|\mathcal{F}_t\Bigg].\end{aligned} \tag{10}$$

Table 1 Currency naming convention for foreign exchange rates

Economy	Theor. label	Empir. label	Currency	Interest rate	Proxy
Sweden	DOM	SEK	Swedish krona	r_{SEK}	SEK OIS
U.S. of America	FOR_1	USD	US dollar	r_{USD}	USD OIS
Eurozone	FOR_2	EUR	euro	r_{EUR}	EONIA OIS

Equation (10) hints at another important detail: The bivariate conditional characteristic function of the log-exchange rates $X^{FOR_1\text{-}DOM}(T)$ and $X^{FOR_2\text{-}FOR_1}(T)$ must be provided under $\mathbb{Q}_{DOM}$. Our definitions of the extended Heston and PCSV models ensure that this is indeed the case.

3 Data

In the empirical analysis, the extended Heston and PCSV models are calibrated to European style plain vanilla options written on three exchange rates. The calibration period covers more than 11 years, ranging from January 2004 to July 2015. Similar to Bannör et al. (2015), the exchange rates between the Swedish economy, the US economy, and the eurozone economy, i.e. the combined economy of all countries which adopted the euro as their national currency, are examined. In conformance with the international naming convention described by Wystup (2007) and Clark (2011), the Swedish economy is chosen as the domestic (DOM), the US as the first foreign (FOR_1), and the eurozone as the second foreign (FOR_2) economy (see also Table 1).

Besides options, the data necessary for the calibration comprises spot exchange rates and (risk-free) interest rates matching the options' maturities. In addition, FX forward outright points are collected. They are not necessary for the actual calibration but will be used for validity checks during the data preparation.

All data used during the calibration was obtained from the financial data provider Bloomberg. For the purpose of cross-checking the quantities, the equivalent data, if available, was also collected from Thomson Reuters' Datastream. The option data comprises market quoted implied volatilities for all three exchange rates with 11 different times to maturity ranging from 1 month to 3 years and five different levels of moneyness ($10\Delta P$, $25\Delta P$, ATM, $25\Delta C$, and $10\Delta C$). The implied volatilities across all levels of moneyness are presented exemplary for a maturity of one year on the left in Fig. 2 (the corresponding quantities in terms of $\%moneyness = \frac{K}{S(t)}$ can be found on the right in Fig. 2). In addition, the implied volatilities for the $10\Delta P$ and $10\Delta C$ levels are plotted separately in Fig. 3 (together with the implied volatility curves for different times to maturities on December 20th, 2012) to indicate if the structure of the implied volatility curve either resembles a symmetric smile (mostly before the 2008 financial crisis) or an upward/downward sloping smirk (particularly after the turmoil of 2008).

Since the option quotes are given as Delta-Volatility pairs they must undergo a transformation before being suitable for model calibration. The procedure is described in detail by Bisesti et al. (2005), Wystup (2007), Reiswich and Wystup (2010), DeRosa

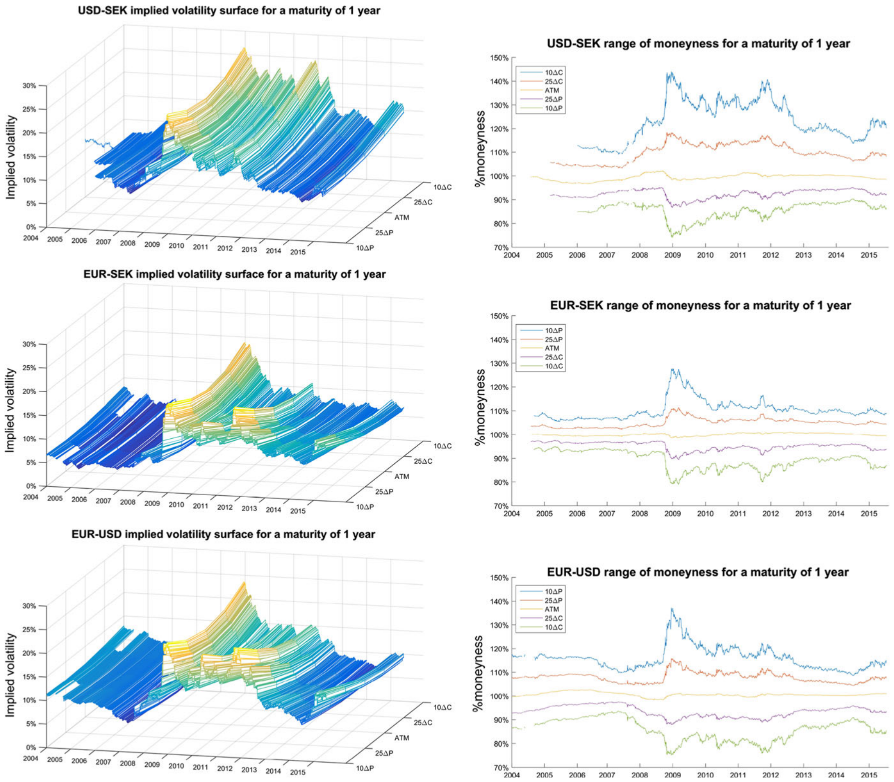

Fig. 2 Implied volatilities (*left*) for the different delta levels and the corresponding *%moneyness* (*right*) over the entire calibration period for options with a maturity of 1 year

(2011), and Clark (2011). With the exception of the transformation of the option data, no major issues arise for the option quotes and the spot exchange rates. This is not the case for the interest rates which are retrieved for the same maturities as the option prices. Particularly after the 2008 financial crisis, the correct choice for the risk-free interest rate remains a challenge, broadly discussed among the financial community: Before the crisis, practitioners and academics alike used the currency specific LIBOR or, in the eurozone, the EURIBOR rate as proxies for the risk-free interest rates. This however changed during and after the turmoil of the sub-prime crisis. Instead the corresponding OIS rates are now increasingly applied as best approximations for the risk-free interest rates (compare Bianchetti and Carlicchi 2013 or Hull and White 2013 for a broader discussion). This article adopts the same ansatz and uses the SEK OIS, the USD OIS, and the EONIA OIS rates as approximations for the respective risk-free interest rates (see Table 1).

In the data obtained from Bloomberg, several incorrect values were found. An *incorrect data point* is defined as an observation which could not have occurred on the examined market and thus, must have arisen due to a measurement error or similar. All incorrect values were detected manually due to unusually high jumps or

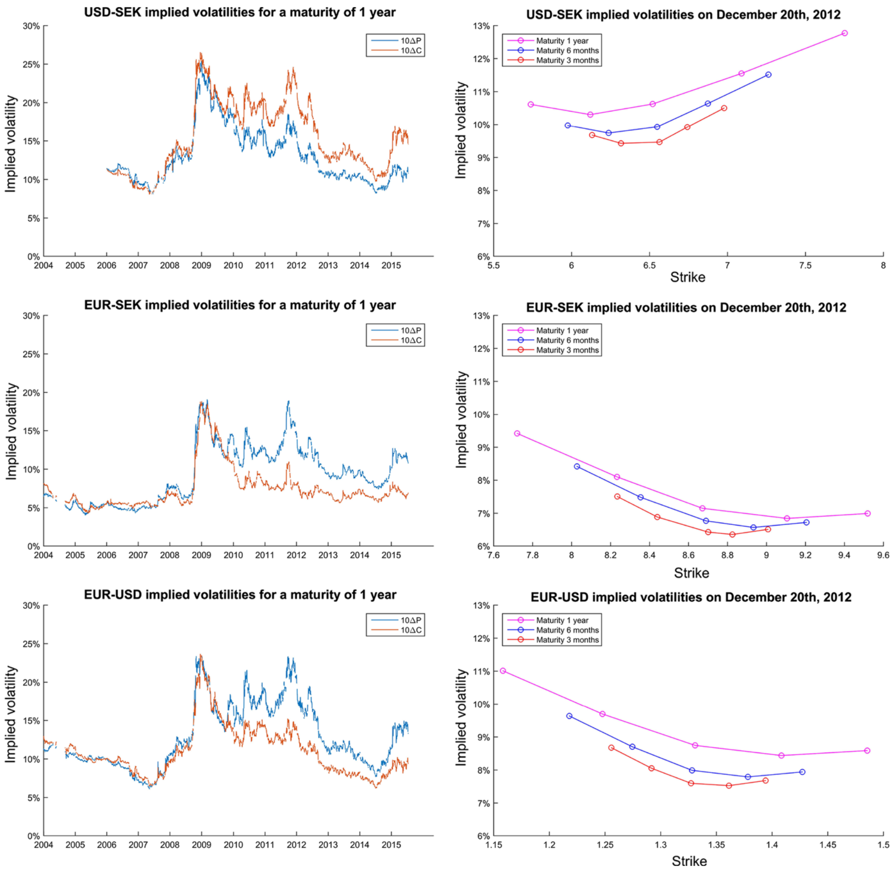

Fig. 3 *Left* Implied volatilities for the $10\Delta P$ and $10\Delta C$ levels indicating if the implied volatility curve forms a smile ($\sigma_{10\Delta P} = \sigma_{10\Delta C}$) or rather an upward ($\sigma_{10\Delta P} < \sigma_{10\Delta C}$) or downward ($\sigma_{10\Delta P} > \sigma_{10\Delta C}$) sloping smirk. *Right* Implied volatility curves on December 20th, 2012 for different maturities

other abnormalities in the data that cannot be explained by market activity.[7] To verify their incorrectness, the identified data points were, in addition, cross-checked with the corresponding data obtained from Thomson Reuter's Datastream. If a value was definitely found to be incorrect, it was replaced by the value of the same data point on the previous day. It was desisted from using the values provided by Datastream to ensure consistency in the sense of using data coming from only one source. Errors were identified for interest rates and options.

The sample available for the calibration comprises overall 425,339 individual calibration instruments (i.e. option implied volatilities) of which 113,045 are options on the *USD-SEK* exchange rate, 118,315 on the *EUR-SEK* exchange rate, and 193,979

[7] On August 16th, 2007 there is also an unusually large increase in the implied volatility for options on the *EUR-USD* exchange rate. This data point, however, is not removed, because the reason for this sharp rise in volatility was the announcement of Countrywide Financial, which was at the time the biggest mortgage lender in the US, expressing concerns over its liquidity.

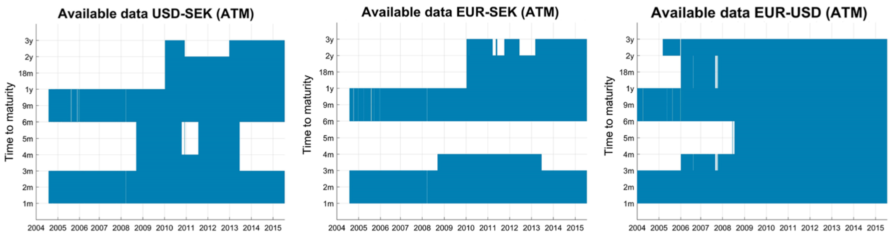

Fig. 4 ATM implied volatilities available for the calibration

with the *EUR-USD* exchange rate as the underlying. Hence, there is data available on 2,822 consecutive trading days for the calibration of the models to options on all 3 exchange rates. Notice, however, that the number of calibration instruments available on each day varies over the calibration period as implied volatilities do not necessarily exist on each day for all maturities and levels of moneyness. This can exemplary be seen in Fig. 4. It displays the available ATM implied volatilities for options on all three exchange rates. Particularly at the beginning of the calibration period, the missing data leads to the sample of calibration instruments being rather small. However, as Escobar and Gschnaidtner (2016) point out, too few calibration instruments can have strong negative effects on the calibration results. The reason for this lies in the model having too many parameters given the available calibration instruments and small changes in the data can have significant effects on the calibration results.

4 Empirical methodology

Considering the findings of Escobar and Gschnaidtner (2016), the calibrations of the extended Heston and PCSV models are performed by minimizing on each day of the calibration period the Relative Squared Volatility Error (RSVE) measure

$$\text{RSVE} = \sum_{i=1}^{N_C} \left(\frac{\sigma_{i,Model}(\Gamma) - \sigma_{i,Market}}{\sigma_{i,Market}} \right)^2, \tag{11}$$

where N_C is the number of calibration instruments available on the respective day, $\sigma_{i,Market}$ the implied volatilities observed on the market, and $\sigma_{i,Model}(\Gamma)$ the corresponding model quantities for parameter set Γ.

Since, however, the number of calibration instruments varies over time, in Sect. 5 we mainly report and illustrate the Mean Relative Squared Volatility Error (MRSVE) measure

$$\text{MRSVE} = \frac{1}{N_C} \sum_{i=1}^{N_C} \left(\frac{\sigma_{i,Model}(\Gamma) - \sigma_{i,Market}}{\sigma_{i,Market}} \right)^2, \tag{12}$$

i.e. the RSVE measure adjusted for the number of calibration instruments N_C. This allows for comparison of the resulting errors over time. Note, that switching from the

RSVE to the MRSVE measure does not exert any effects on the parameters resulting from the calibration.

In addition to the RSVE and the MRSVE, we also calculate and report the Mean Absolute Squared Volatility Error (MASVE)

$$\text{MASVE} = \frac{1}{N_C} \sum_{i=1}^{N_C} \left(\sigma_{i,Model}(\Gamma) - \sigma_{i,Market} \right)^2$$

and the Mean Absolute Squared Price Error (MASPE)

$$\text{MASPE} = \frac{1}{N_C} \sum_{i=1}^{N_C} \left(V_{i,Model}(\Gamma) - V_{i,Market} \right)^2,$$

where $V_{i,Model}(\Gamma)$, $V_{i,Market}$ are the model implied and market observed (call) option prices, respectively. To exclude currency effects and to permit a comparison between the errors of options written on different exchange rates, the prices are all quoted in the same currency (domestic currency *SEK*).

The latter two error measures are typically used for calibrations to equity options. They are reported to allow for a better classification and comparison of our results with calibrations, either to FX or equity options, performed by other authors.

Similar to De Col et al. (2013), for both the extended Heston and PCSV models not the complete but a reduced set of parameters is calibrated. Coinciding with the findings of Escobar and Gschnaidtner (2016), high variations in the calibrated values of the parameters, particularly for the mean reversion speed κ, are observed in preliminary calibration runs with the full set of model parameters. This improved drastically when fixing the value of κ without suffering a significant loss in the calibration accuracy. Moreover, fixing κ has the additional advantage of reducing the number of model parameters diminishing the possibility of a model overfitting. This applies particularly to the beginning of the calibration period where the number of calibration instruments available is rather low. Hence, κ is set in all calibrations equal to 2 reducing the number of model parameters from 10 to 8 for the extended Heston model and from 11 to 9 for the extended PCSV model.

A particular characteristic of the foreign exchange market is the triangular relationship (illustrated for the specific case of the *USD-SEK*, *EUR-SEK*, and *EUR-USD* exchange rates in Fig. 5) which implies that the dynamics of the third exchange rate are induced by the dynamics of the other two exchange rates. This allows for two different ways of calibrating a model (compare also Doust 2012 and De Col et al. 2013):

- Calibration of the model only to option data of the first two exchange rates. The values of options on the third, induced exchange rate can then be derived using the calibrated model parameters. In the following, this approach is referred to as a *bivariate* calibration.
- A calibration which uses the available information of options on all three exchange rates during the fitting process. This *trivariate* calibration may be able to capture dependencies which are not accounted for by the bivariate calibration.

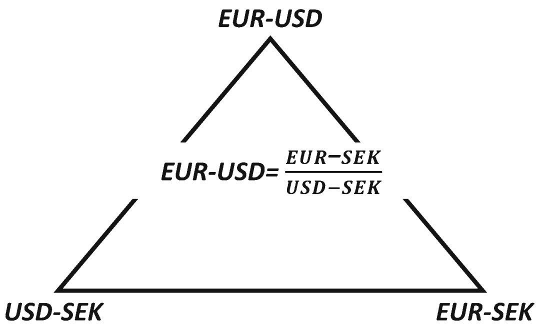

Fig. 5 Implied triangular relationship for foreign exchange rates in the specific case of the SEK, USD, and EUR currencies

Table 2 Overview of the calibration scenarios which were conducted within the scope of this study

Scenario code	Model				Initial parameters	
	Heston model		PCSV model		Previous day	Other model
	Bivariate	Trivariate	Bivariate	Trivariate		
HestonBiv1	×				×	
HestonTriv1		×			×	
HestonTriv2		×				×
PCSVBiv1			×		×	
PCSVBiv2			×			×
PCSVTriv1				×	×	
PCSVTriv2				×		×

Whereas a bivariate calibration only requires data on *USD-SEK* and *EUR-SEK* exchange rate options, for a trivariate calibration also data for options on the *EUR-USD* exchange rate is necessary. Since option data exists for all three exchange rates, both a bivariate and a trivariate calibration is possible. Hereby, the bivariate calibration is expected to lead to better fittings for the first two exchange rates (*USD-SEK* and *EUR-SEK*) but to rather high deviations for the induced exchange rate (*EUR-USD*). The trivariate calibration, on the other hand, is expected to have smaller overall discrepancies between the market and the model data for all three exchange rates but higher errors when only considering the first two exchange rates. Due to this distinction four different calibration scenarios originate when implementing the bivariate and the trivariate approaches. These are, among others, summarized in Table 2.

Escobar and Gschnaidtner (2016) point out that another important aspect of any calibration to empirical data is a careful choice for the initial parameters submitted to the optimization algorithm. Particularly when conducting the calibration with a local optimizer proper initial parameters are crucial for the success of the calibration. The initial parameters need to be close to the global minimum of the optimization problem to avoid ending up in one of the many local minima. We decided to determine suitable initial parameters in a rather simple, yet intuitive way: Instead of applying complex methodological approaches (see e.g. Da Fonseca and Grasselli 2011 or De Col et al. 2013), approximations for the individual parameters are derived based on historical data and, in case of the extended PCSV model, on a PCA performed on the log-returns of the exchange rates. The corresponding formulas and the resulting initial parameters are given in Table 3.

Conducting the model calibration on every day with the same initial values is, however, not efficient. Instead, the approximations in Table 3 are only used on the first day of the calibration period as initial parameters. Afterwards, we implement two different approaches for determining the initial values for each subsequent calibration day: First, the initial values are set equal to the calibrated parameters of the previous day. This ansatz significantly helps to increase the speed of the calibration and, at the same time, it is economically reasonable. Assuming the suitability of the models for pricing FX options, the calibrated parameters are expected to vary only slightly over the calibration period. Hence, the best approximation for the model parameters on the next day are the current day's parameters.

Besides using the calibrated parameters of the previous day it is also possible, due to the high proximity of the models, to use the resulting calibrated parameters of one of the other model set-ups as approximation for the initial parameters. Having conducted the bivariate calibration of the extended Heston model, we use the resulting parameters as initials for the trivariate calibration of the extended Heston model and for the bivariate calibration of the extended PCSV model.[8] In turn, the results of the bivariate PCSV model are then used for the trivariate calibration of the extended PCSV model. Again, this allows for a significant increase in the calibration speed because the initial parameters already include a considerable amount of information on the current market situation. Yet, both approaches have a significant disadvantage: The probability of a transmission of a possible local minima from one day to the next or from one model to the other is high. For this reason we introduce several mechanisms which prevent the calibration to stay at or close to a local minimum for an extended period. First, if the value of the RSVE error function becomes too large, the initial parameters are set back to their original values given in Table 3. An unusually high value of the error function as compared to the previous day's value serves as a good indicator that the optimization ended up in a local rather than a global optimum. Second, if the calibrated parameters of the previous day show an abnormal behavior, as in the values significantly differ from the calibrated parameters of the day before the previous day, the initial parameters are also set back to the original values. Both mechanisms prevent a remaining in a possible local minimum. They are conducted for all calibration scenarios increasing the number of calibrations scenarios to a total of 7 (see Table 2).

All calibrations are implemented using MATLAB and the built-in functions of the Optimization Toolbox. These include among others the *trapz*$(\cdot)$ function (trapezoidal rule) which is used to evaluate the integrals in (5) and (8) and the optimization function *lsqnonlin*$(\cdot)$ implementing the Trust-Region-Reflective algorithm of Coleman and Li

[8] An anonymous referee pointed out that, instead of setting the initial parameters of the extended PCSV model equal to the parameters obtained for the extended Heston model, it might be a better strategy to choose the initial parameters for the PCSV model such that the spot-vol correlations of the two models coincide. However, the empirical results presented below suggest that the effects of changing the initial parameters are negligible.

Table 3 Approximations of the initial values for the model parameters

Parameter	ext. Heston model			ext. PCSV model			Boundaries	
	Formula	$i=1$	$i=2$	Formula	$i=1$	$i=2$	Lower	Upper
$\lambda_i(t)$	$(\sigma_{ATM})^2$	0.0127	0.0032	$250 \cdot \lambda_i(t)$	0.0260	0.0013	0.00001	3
κ_i	–	2	2	–	2	2	–	–
ξ	–	–	–	ξ	0.4072		0	π
θ_i	$250 \cdot \bar{\sigma}^2_{hist}$	0.0165	0.0053	$250 \cdot \bar{\lambda}_i(t)$	0.0191	0.0027	0.00001	3
							0.00001	3
σ_i	$250 \cdot \sqrt{Var[\sigma^2_{hist}]}$	0.7899	0.5306	$250 \cdot \sqrt{Var[\lambda_i(t)]}$	0.3347	0.0366	0.00001	3
ρ_i	determ.	0	0	determ.	0	0	−0.99	0.99

For the initial variance the implied volatility of an ATM option with maturity 3 months (ext. Heston model) and the annualized eigenvalues resulting from a PCA on the log-returns of the exchange rates (ext. PCSV model) are used. The initial value of ξ is also obtained via a PCA. The θ's are approximated by the annualized average of the historical variances (Heston) respectively by the annualized average of the eigenvalues (PCSV) calculated on the basis of 3 months rolling time windows. The initials for the volatility of the variance and eigenvalue processes are set equal to their sample variances determined over the entire calibration period

Table 4 Total, average, maximum, and minimum calibration times (in s) for selected calibration scenarios

(in s)	HestonBiv1	HestonTriv1	PCSVBiv1	PCSVTriv1
Total	49,413	289,131	489,248	378,448
Average	18	103	175	135
Max	1473	8595	6393	6049
Min	6	12	17	20

(1994) for which we obtained within reasonable time the most reliable results in preliminary calibrations. The actual calibration times of selected scenarios for the entire period along with daily averages and respective maxima and minima are presented in Table 4.

The original constraints of the parameters in the extended Heston and PCSV models are changed to avoid numerical issues during the calibration (this holds particularly for the lower bounds) as well as speeding up the calibration (upper bounds). The modifications are listed in Table 3. Moreover, similar to De Col et al. (2013), we decided not to enforce the Feller condition on the parameters to prevent alterations of the calibration results due to too many restrictions on the parameters. The Feller condition, after Feller (1951), states that

$$2\kappa_j\theta_j > \sigma_j^2. \tag{13}$$

It prevents the jth variance process (extended Heston model) and the jth eigenvalue process (extended PCSV model) from reaching 0 or becoming even negative. Instead, the outcome of the calibrations will be examined for violations of the Feller condition at the end of the presentation of the empirical results in the next section.

5 Empirical results

Subsequently, the results of the calibration of the extended Heston and PCSV models to FX option data are presented and compared. Hereby, mostly the calibration scenarios with *previous day* initial parameters are discussed in detail as the calibrations with initial parameters based on the other models yield almost identical results. At the same time, this observation strengthens our believes that the calibrations, despite different initials, yield parameters that are at or close to the minimum of problem (11).

Figures 6 and 7 depict the parameters resulting from the different calibrations over the entire period. As predicted by Escobar and Gschnaidtner (2016), the parameters $\lambda(t)$ and θ—to improve the readability, parameter subindices are omitted where not necessary—seem to be very well calibrated across all models as their values evolve steadily over time (see Fig. 6). At the same time, showing elevated values during the financial crisis, the parameters accurately reflect the macroeconomic developments which occurred within the calibration period. Moreover, the higher values of $\lambda_1(t)$ and θ_1, as compared to their respective counterparts with subindex $i = 2$, also coincide with empirical observations. For the extended Heston model, in which each variance

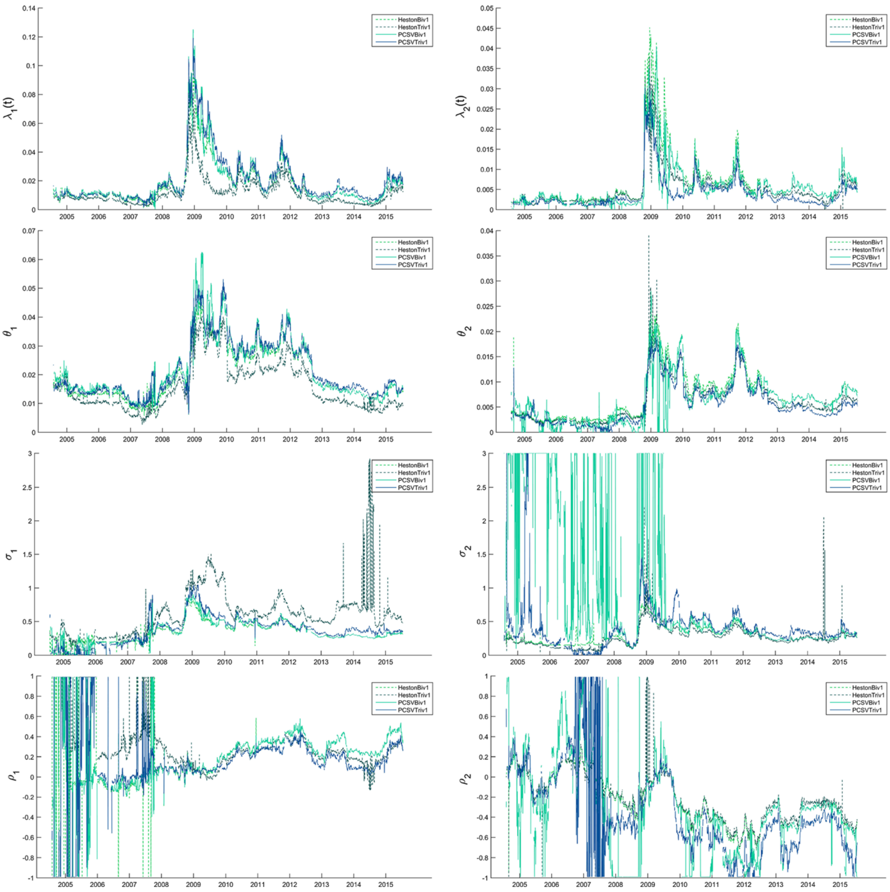

Fig. 6 Calibrated parameters resulting from the different calibration scenarios. On the *left*, all parameters with subindex $i = 1$ can be found and on the *right*, all parameters with subindex $i = 2$

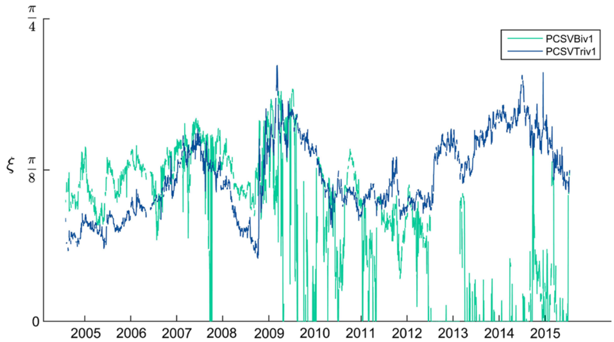

Fig. 7 Values of the parameter ξ resulting from the PCSV scenarios $\left(\frac{\pi}{8} \approx 0.39\right)$

process is exclusively assigned to the dynamics of one exchange rate, the consistently higher values of $\lambda(t)$ and θ in the first variance process correctly reflect the empirically observed (implied) volatility of the *USD-SEK* exchange rate which is substantially higher than the (implied) volatility of the *EUR-SEK* exchange rate. For the extended PCSV model, similar to a PCA, a unique assignment of the eigenvalue processes to a specific exchange rate is not possible. Nonetheless, the calibrated values of $\lambda(t)$ and θ in the *PCSVBiv*1 and *PCSVTriv*1 scenarios correctly reproduce the results of a PCA conducted on the log-returns of the *USD-SEK* and *EUR-SEK* exchange rates: The first eigenvalue (process) is significantly higher than the second eigenvalue (process) and explains already most of the variation in the data. Hereby, the impact of the eigenvalues on the respective exchange rates depends on the eigenvector matrix $\mathbf{A}$ whose entries are, in case of the extended PCSV model, uniquely determined by the parameter ξ. As depicted in Fig. 7, for scenarios *PCSVBiv1* and *PCSVTriv1* the calibrated value of the parameter ξ mostly varies around $\frac{\pi}{8} \approx 0.39$ implying

$$\mathbf{A} = \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{pmatrix} = \begin{pmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{pmatrix}.$$

Hence, the first eigenvalue process mostly affects the *USD-SEK* exchange rate and the second eigenvalue process the *EUR-SEK* exchange rate confirming the empirical observations for the respective (implied) volatilities.

Under certain circumstances, namely $\xi \in \{0, \frac{\pi}{2}\}$, the extended PCSV model reduces to the simpler extended Heston model. Based on Fig. 7, $\xi = \frac{\pi}{2}$ can be excluded for both scenarios and $\xi = 0$ or close to 0 mostly occurs towards the end of the calibration period and only in case of the *PCSVBiv1* scenario. In these instances the calibrated parameters of the extended PCSV model are almost identical to the parameters found for *HestonBiv1*. A possible explanation for this behavior lies in the existence of 0 as a lower boundary for ξ silently assuming that the parameters of the eigenvalue processes are able to adjust accordingly if the parameter ξ is about to become negative during the calibration. However, it seems that the optimization algorithm is not able to utilize this flexibility of the parameters. Instead, it sets ξ (close) to 0 and converges to the same optimum of problem (11) as for the extended Heston model. Hence, it might be advisable to allow the parameter ξ to become negative in future calibrations of the extended PCSV model.

The varying levels of volatility observed on the markets also affect the parameters with increased values during the financial crisis and slightly elevated values afterwards. However, the generally sound calibration results for σ are distorted by various jumps in the parameter values at the beginning of the calibration period. This occurs particularly for σ_1 in *HestonBiv1* as well as for σ_2 in *PCSVBiv1*. A high number of jumps in the values can also be identified for the correlation parameters ρ. This is in so far conspicuous as they often take values at or close to -1 and 1, i.e. at their artificially imposed boundaries. Hence, it can be speculated that the optimization routine, as a crucial part of the calibration process, did not converge to a global minimum in these instances. Instead, it ended up at mathematically reasonable, yet artificially enforced boundaries and thus, most likely in a local minimum. A possible explanation for the

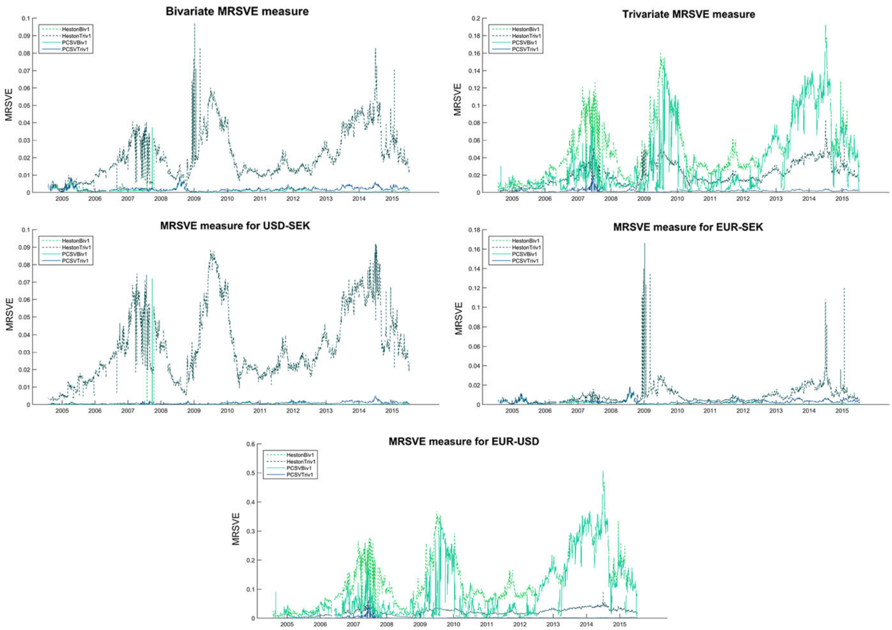

Fig. 8 *Top* Bivariate and trivariate MRSVE. *Bottom* MRSVE broken down for each exchange rate

high variation in the parameters can be found in Escobar and Gschnaidtner (2016): Even for an optimal setting, the calibration of the original Heston model to only few data points is generally accompanied by low calibration success, specifically for the parameters σ and ρ. Since the data available at the beginning of the calibration period is scarce, we hypothesize that the small number of calibration instruments is responsible for not reaching the global minimum of the calibration problem and thus, for the high variation in the parameters.

To draw decisive statements on the quality of a calibration, usually the calibration errors are considered. Figure 8 depicts the MRSVE for all four scenarios over the entire calibration period, on an aggregated (top two diagrams) and on an individual (bottom three diagrams) level.[9] In the top part of Fig. 8 the bivariate and the trivariate MRSVE measures are illustrated. With the exception of *HestonTriv1*, the bivariate MRSVE measure is very small indicating successful model calibrations. In contrast, the trivariate error measures show significantly higher values for almost all scenarios. Only for *PCSVTriv1* it is at a reasonable low level. Hence, while the bivariate calibrations of both models fit the implied volatility surfaces of *USD-SEK* and *EUR-SEK* options exceptionally well, they are not able to maintain this level of precision when the calibrated parameters are used for extrapolation to FX options on the induced *EUR-USD* exchange rate. Instead, the trivariate MRSVE measure is unacceptably high for *HestonBiv1* and *PCSVBiv1*. On the other side, the trivariate scenarios *HestonTriv1*

[9] To allow for comparisons between the bivariate and the trivariate calibrations, the respective other quantities as well as the further error measures are calculated using the calibrated parameters.

Table 5 Average error measures for the entire calibration period

	Heston		PCSV	
	HestonBiv1	HestonTriv1	PCSVBiv1	PCSVTriv1
RSVE				
Biv.	0.0741	1.6804	0.0659	0.1402
Triv.	7.0626	2.7566	4.6293	0.2198
MRSVE				
Biv.	0.0011	0.0212	0.0010	0.0019
Triv.	0.0531	0.0206	0.0341	0.0018
MASVE				
Biv.	7.473E−06	0.0003	6.913E−06	1.532E−05
Triv.	0.0006	0.0003	0.0003	1.485E−05
MASPE				
Biv.	3.121E−05	0.0025	2.952E−05	6.060E−05
Triv.	0.0036	0.0035	0.0020	6.262E−05

and *PCSVTriv1*, which consider data on all three exchange rates during the calibrations, lead to significant improvements in the trivariate MRSVE measure, particularly for the latter scenario. However, the better overall fitting comes at the expense of a slightly (*PCSVTriv1*) or significantly (*HestonTriv1*) worse fitting of the *USD-SEK* and *EUR-SEK* implied volatility surfaces. Thus, it can be deduced from the bivariate and trivariate MRSVE that only *PCSVTriv1*, the extended PCSV model combined with the trivariate calibration scenario, results in an adequate or even almost perfect fitting of the implied volatility surfaces for all three exchange rates.

These findings are confirmed by Table 5 which provides an overview of the averages of the different error measures over the entire calibration period. For all error measures, *PCSVTriv1* yields the lowest values for the trivariate case without suffering much accuracy in the bivariate case.

Similar conclusions can also be drawn from the calibration errors when split up for each exchange rate in the bottom part of Fig. 8. The extended Heston model, independent of the calibration method, and the extended PCSV model with a bivariate calibration seem to systematically miss-price *EUR-USD* FX options. Only the MRSVE of the trivariate PCSV model calibration shows across all exchange rates low values.

Hence, while the extended Heston model is obviously not able to adequately reproduce the FX dynamics and their dependencies in a setting with three exchange rates, the findings for the extended PCSV model are twofold: A bivariate calibration of the extended PCSV model does not capture the entire dependency structure inherent to a market with three exchange rates, a problem which is however overcome by a trivariate calibration.

The correct representation of the dependency structure as the most likely explanation for the superiority of the extended PCSV model is confirmed by Fig. 9 which depicts the empirical and model implied correlation coefficients. By definition, the extended Heston model implies a correlation of 0 between the log-returns of the *USD-*

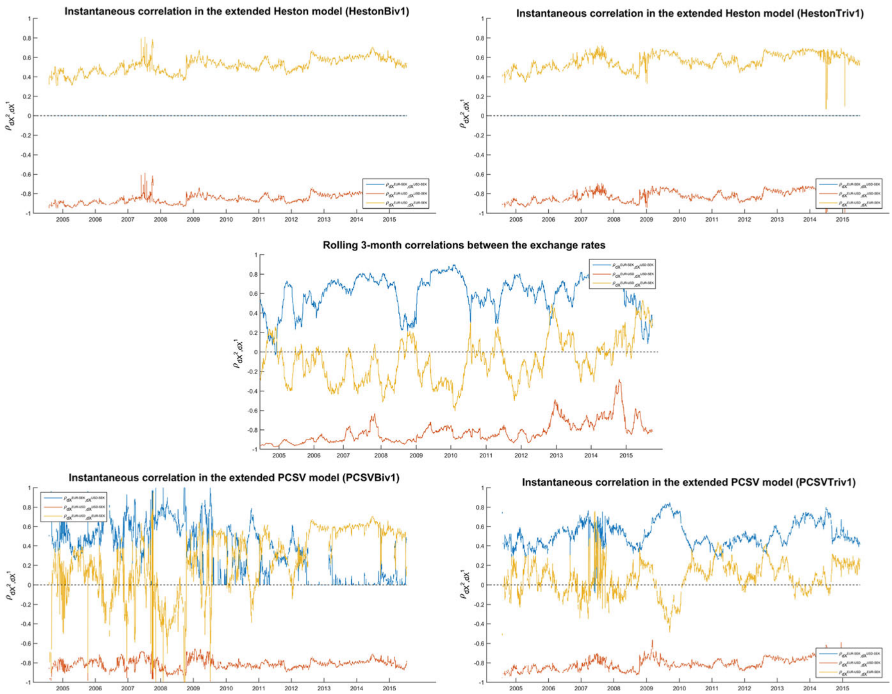

Fig. 9 Instantaneous correlation coefficients for the calibrated parameters resulting from the *HestonBiv1* and *HestonTriv1* as well as from the *PCSVBiv1* and *PCSVTriv1* calibration scenarios. The empirical correlation coefficient is calculated on a rolling 3 month period

SEK and the *EUR-SEK* exchange rates and a strictly positive (negative) coefficient for *EUR-USD* and *EUR-SEK* (for *EUR-USD* and *USD-SEK*). A different picture emerges for the extended PCSV model which, due to its flexibility resulting from the additional parameter ξ, does not obligatorily assume a zero correlation. Instead, the calibrated parameters, particularly for *PCSVTriv1*, imply a positive (negative) correlation for *USD-SEK* and *EUR-SEK* (*USD-SEK* and *EUR-USD*) as well as a continuously switching algebraic sign of the correlation coefficient between the *EUR-SEK* and *EUR-USD* exchange rates, exactly as empirically observed.

The superiority of the extended PCSV model in combination with a trivariate calibration is also supported by the findings on December 20th, 2012 which are analyzed in more detail exemplary for the entire calibration period. In Figs. 10 and 11 the implied volatility curves, as determined by the models, are compared with the implied volatilities observed on the market. For both, the extended Heston model (green) and the extended PCSV model (blue) the volatility surfaces are determined on the basis of the bivariate as well as the trivariate calibrated parameters (Table 6). In addition, the differences (calculated using the RSVE measure) between the model and market implied volatility surfaces are displayed for each exchange rate using three dimensional barplots. Here, one quickly notices that the bivariate extended Heston and PCSV models almost perfectly reproduce the *USD-SEK* and *EUR-SEK* implied

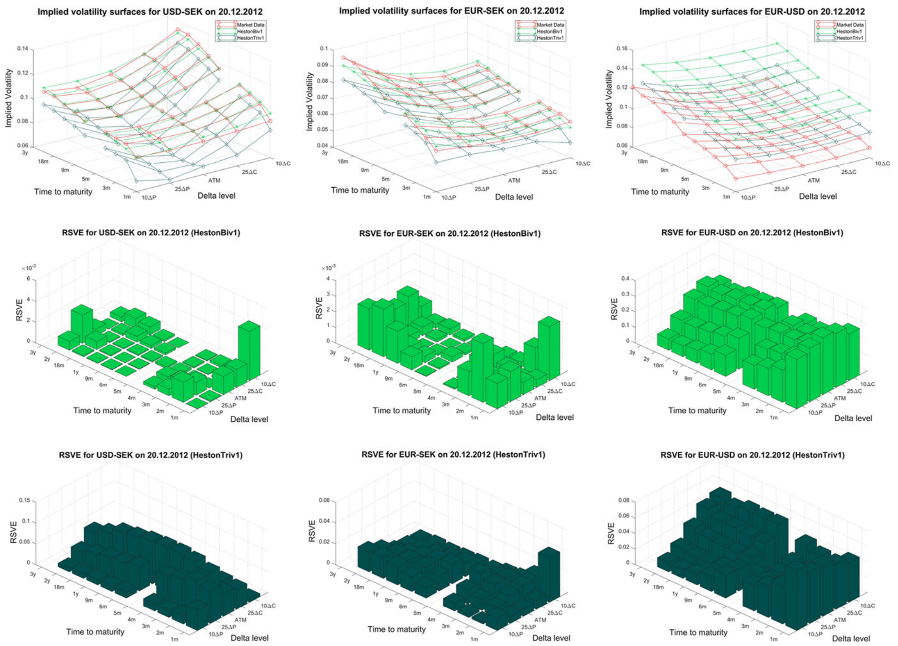

Fig. 10 Calibrated volatility surfaces and the respective RSVE measures on 20.12.2012 for the extended Heston model. Notice the different scales of the *vertical axes*. *Blank spaces* occur due to missing data

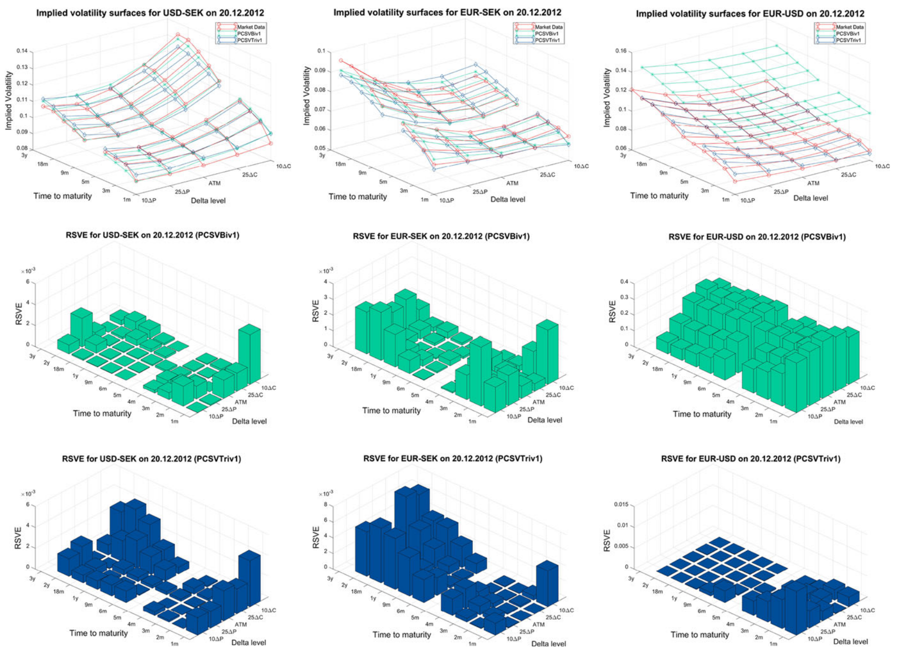

Fig. 11 Calibrated volatility surfaces and the respective RSVE measures on 20.12.2012 for the extended PCSV model. Notice the different scales of the *vertical axes*. *Blank spaces* occur due to missing data

Table 6 Calibrated parameters and corresponding error measures on December 20th, 2012

(December 20th, 2012)	Heston			PCSV			
	HestonBiv1	HestonTriv1	HestonTriv2	PCSVBiv1	PCSVBiv2	PCSVTriv1	PCSVTriv2
λ_1	0.0078	0.0044	0.0044	0.0078	0.0078	0.0099	0.0099
λ_2	0.0040	0.0031	0.0031	0.0040	0.0040	0.0018	0.0018
ξ	–	–	–	1.023E−05	2.407E−05	0.4823	0.4823
θ_1	0.0158	0.0112	0.0112	0.0158	0.0158	0.0178	0.0178
θ_2	0.0076	0.0060	0.0060	0.0076	0.0076	0.0051	0.0051
σ_1	0.2664	0.5260	0.5260	0.2664	0.2664	0.3199	0.3199
σ_2	0.2249	0.2161	0.2161	0.2249	0.2249	0.2924	0.2924
ρ_1	0.2023	0.1741	0.1741	0.2023	0.2023	0.0922	0.0922
ρ_2	−0.2878	−0.2400	−0.2400	−0.2878	−0.2878	−0.4200	−0.4200
RSVE							
Biv.	0.0622	2.5990	2.5990	0.0622	0.0622	0.1384	0.1384
Triv.	10.6620	4.3617	4.3617	10.6618	10.6615	0.1883	0.1883
MRSVE							
Biv.	0.0007	0.0286	0.0286	0.0007	0.0007	0.0015	0.0015
Triv.	0.0730	0.0299	0.0299	0.0730	0.0730	0.0013	0.0013
MASVE							
Biv.	5.133E−06	0.0003	0.0003	5.133E−06	5.133E−06	1.256E−05	1.256E−05
Triv.	0.0005	0.0002	0.0002	0.0005	0.0005	1.000E−05	1.000E−05
MASPE							
Biv.	1.729E−05	0.0009	0.0009	1.729E−05	1.729E−05	4.754E−05	4.754E−05
Triv.	0.0029	0.0010	0.0010	0.0029	0.0029	3.178E−05	3.178E−05

volatility surfaces, with short and long dated options being fitted slightly worse than options with times to maturity between 3 months and 1 year. However, this no longer holds for the *EUR-USD* exchange rate: Both models systematically over-price options for a bivariate calibration. In particular, for the bivariate calibration the extended PCSV model reduces to the extended Heston model with ξ almost 0 leading to nearly identical parameters and error measures for both models (see Table 6). Confirming the overall findings above, it is only the extended PCSV model calibrated in a trivariate setting that significantly improves the calibration quality for *EUR-USD* options without suffering too much accuracy for the other two exchange rates.

Overall, all findings hint at a trivariate calibration capturing the dependency structure more accurately than a bivariate calibration and at the extended PCSV model outperforming the extended Heston model in fitting the implied volatility surfaces. This is attributed to the trivariate calibration procedure incorporating more information and to the extended PCSV model allowing for more flexibility in modeling the dependencies between the exchange rate dynamics. As a result, we question the suitability of the extended Heston model for pricing FX options in a trivariate setting as it seems that it fails to capture important characteristics prevailing in the foreign exchange market. This holds especially when only data for two exchange rates is available and the implied volatilities for the third exchange rate must be extrapolated based on the parameters from the bivariate calibration (cross-currency option pricing). While the trivariate calibration improves the overall pricing consistency in a setting with three exchange rates, the inflexibility of the extended Heston model prevents an even better fitting. This inadequacy arises from the critical assumption of independence between the *USD-SEK* and *EUR-SEK* exchange rates highlighted after Definition 2. Yet, this does not reflect the actual dependency structure between the two exchange rates which shows a strictly positive correlation coefficient during the entire calibration period. Instead, historical data analysis reveals that rather the *EUR-SEK* and *EUR-USD* exchange rates are (almost) uncorrelated. Furthermore, fundamental economic observations[10] suggest that the *EUR-USD* exchange rate is not induced by the *USD-SEK* and the *EUR-SEK* exchange rates. In contrast to Fig. 5, it is instead the *USD-SEK* exchange rate which is, due to its minor importance for the global FX market, implied by the *EUR-SEK* and the *EUR-USD* exchange rates.[11]

$$S^{USD\text{-}SEK}(t) = \frac{S^{EUR\text{-}SEK}(t)}{S^{EUR\text{-}USD}(t)}. \tag{14}$$

[10] The *Triennial Central Bank Survey: Global foreign exchange market turnover in 2013* conducted by the *Bank for International Settlements* states that *EUR-USD* spot market transactions account for 24.14% of the total activity on the global spot FX market manifesting the exchange rate's predominant role in the world. In contrast, the *USD-SEK* and *EUR-SEK* exchange rates only play a minor role on the FX markets contributing only 0.38%, respectively 0.69% to the global spot transactions. Despite the insignificance of these two exchange rates for the global FX market, they are nonetheless important for the Swedish FX market with the *USD-SEK* exchange rate accounting for 29.08% and the *EUR-SEK* exchange rate for 52.40% of all spot transactions with the Swedish krona on one side.

[11] Notice, that while from an economic perspective $S^{EUR\text{-}USD}(t) = \frac{S^{EUR\text{-}SEK}(t)}{S^{USD\text{-}SEK}(t)}$ is not confirmed, the mathematical validity of this relationship cannot be questioned.

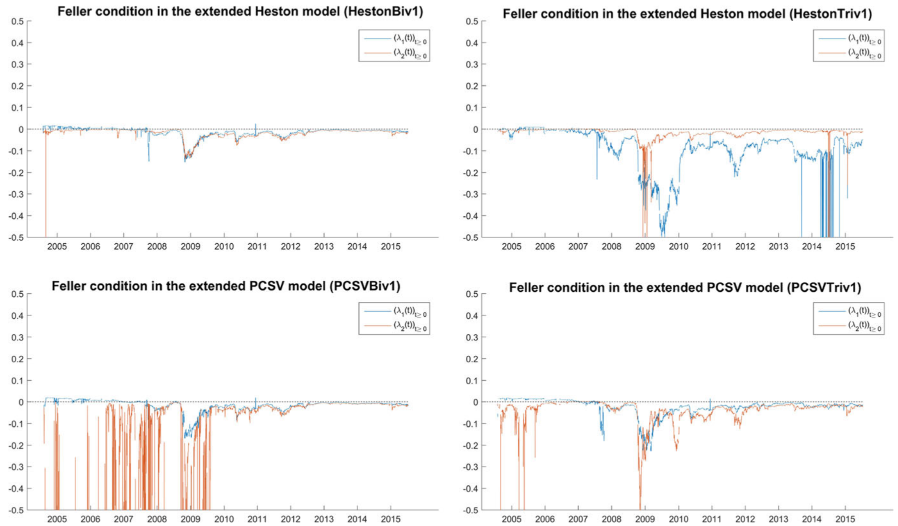

Fig. 12 Feller condition for the calibrated model parameters. The displayed values of F_j, $j \in \{1, 2\}$, are calculated using the formula $F_j = \theta_j - \frac{\sigma_j^2}{2 \cdot \kappa_j}$ with $\kappa_j = 2$. Positive values for F_j imply that the Feller condition is met, whereas negative values for F_j are identical to violations of the Feller condition

Accounting for these circumstances by modeling the *EUR-SEK* and *EUR-USD* exchange rates directly and inducing the dynamics of the *USD-SEK* exchange rate, the zero correlation assumption of the extended Heston model would no longer constitute a strong limitation. Yet, the technical implications of Eq. (14) are much more subtle. The beauty of the original version of the triangular relation (as in Fig. 1 or Eq. (D2)) lies in the exchange rates on the right having the same domestic currency (*SEK*) allowing for attractive theoretical results. This is not the case in Eq. (14). To still allow for a consistent pricing within our theoretical framework as well as to take advantage of the independence between the *EUR-SEK* and *EUR-USD* exchange rates, Eq. (14) could be slightly altered to $S^{SEK\text{-}USD}(t) = \frac{S^{SEK\text{-}EUR}(t)}{S^{USD\text{-}EUR}(t)}$ with the dynamics of the exchange rates being derived under the risk-neutral probability measure $\mathbb{Q}_{EUR}$.

The empirical part concludes with an analysis of the Feller condition (13) for the calibrated models. The Feller condition ensures that the variance processes (extended Heston model) and the eigenvalues processes (extended PCSV model), and thus the covariance matrices remain positive definite. If the Feller condition is not met, the covariance matrices might become eventually semi-positive definite resulting in the so-called moment explosion of the underlying price processes (see Andersen 2007 for a detailed description of the phenomenon and De Col et al. 2013 for exemplary calculations of the time of moment explosion). Non-finite moments, however, have severe implications on the reliability of the Fourier pricing method used in this article for calculating the prices of FX options. Figure 12 illustrates the Feller condition for selected calibration scenarios. Calculating the quantity $F_j = \theta_j - \frac{\sigma_j^2}{2 \cdot \kappa_j}$ for each variance or eigenvalue process, a negative F_j indicates a violation of and a positive value the case of the variance or eigenvalue process meeting the Feller condition. Hence, most of the time and across all models the Feller condition is indeed violated. Yet, as Clark (2011) and De Col et al. (2013) confirm, it is not unusual that the Feller condition is not fulfilled when calibrating the Heston model to foreign exchange data. It seems the extended Heston model as well as the extended PCSV model inherit this characteristic. A possible way for solving this problem is to impose the Feller condition upon the parameters either by adding a further constraint when performing the calibration with a constrained optimization routine or by implementing a convenient parametrization of the models allowing for an unconstrained optimization. Yet, both approaches will certainly lead to a decline in calibration performance.

6 Conclusion

The main objective of this article was the analysis and calibration of (multivariate) stochastic volatility models suitable for pricing foreign exchange rate options. For this purpose we first introduced a general framework for a foreign exchange market accounting for stylized facts affiliated to a setting with three economies. Next, the extended Heston model, as a benchmark, and the extended PCSV model, both incorporating stochastic volatility, were defined and adapted where necessary. Moreover, since the characteristic functions of the extended Heston and PCSV models exist we stated Fourier pricing techniques for determining the prices of plain vanilla FX options based on the respective models.

In the empirical part, first we shortly introduced the empirical data as well as the methodology used for calibrating the models. Hereafter, the results of the calibrations to *USD-SEK*, *EUR-SEK*, and *EUR-USD* FX options spanning a period of more than 11 years were displayed and analyzed for several scenarios. These scenarios were developed to overcome several possible calibration issues identified before. We found that overall the extended PCSV model combined with a trivariate calibration scenario, i.e. option data on all three exchange rates was used during the calibration procedure, yielded the best fitting to the empirical data. We detected further that the extended Heston model is generally not suitable for pricing FX options in a setting with three exchange rates and that cross-currency option pricing based on parameters obtained from a bivariate calibration results in a significant miss-pricing of FX options on the third exchange rate. Thus, option data on all three exchange rates should be used during the calibration to ensure an adequate fitting to the implied volatility surfaces of all three exchange rates involved.

The explanations for these findings are twofold: First, a trivariate calibration uses all the information available, which seems to eliminate important internal inconsistencies prevalent on a partial analysis including only two exchange rates (bivariate calibration). It fully captures the actual dynamics in the foreign exchange market. Second, the extended PCSV model emerges, due to its parametric-flexibility, as the superior model in comparison to the extended Heston model. By introducing one additional parameter, we are able to reproduce the dependency structure between the different exchange rates in a more appropriate fashion than the Heston model allows us to do.

As another indicator of the feasibility of PCSV on real data for long periods, we found the calibration of the trivariate PCSV to be the most steady during the crisis period of 2008–2009. This superior performance could be seen in terms of stability of the parameters as well as with respect to the level of calibration errors. We can conclude that the results of this study show that option pricing in a foreign exchange setting is anything but trivial. While a standard (extended) Heston model, adjusted to the foreign exchange setting is only partly able to capture the entire foreign exchange rate dynamics, this article provides with the (extended) PCSV model a model which is not only suitable for pricing foreign exchange options, but which is also adequate for practical applications as the number of parameters is relatively small and calibration time is reasonable.

Acknowledgements We want to thank two anonymous referees for their valuable comments and suggestions helping to improve this essay.

Appendix: Proofs

Proof (Proposition 1) The trivariate conditional characteristic function can be written as

$$\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) = E_{\mathbb{Q}_{DOM}}\left[e^{i\mathbf{w}'\mathbf{X}(T)}|\mathcal{F}_t\right], \tag{15}$$

where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$. With (D2) it holds for the log-exchange rates that

$$\mathbf{X}(t) = \begin{pmatrix} X^{FOR_1\text{-}DOM}(t) \\ X^{FOR_2\text{-}DOM}(t) \\ X^{FOR_2\text{-}FOR_1}(t) \end{pmatrix} = \begin{pmatrix} X^{FOR_1\text{-}DOM}(t) \\ X^{FOR_2\text{-}DOM}(t) \\ X^{FOR_2\text{-}DOM}(t) - X^{FOR_1\text{-}DOM}(t) \end{pmatrix}. \tag{16}$$

Using (15) and (16) we obtain the expression for the trivariate conditional characteristic function

$$\begin{aligned}
\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) &= E_{\mathbb{Q}_{DOM}}\left[e^{i\mathbf{w}'\mathbf{X}(T)}\middle|\mathcal{F}_t\right] \\
&= E_{\mathbb{Q}_{DOM}}\left[e^{i\left((w_1-w_3)\cdot X^{FOR_1\text{-}DOM}(T)+(w_2+w_3)\cdot X^{FOR_2\text{-}DOM}(T)\right)}\middle|\mathcal{F}_t\right] \\
&= E_{\mathbb{Q}_{DOM}}\left[e^{i(w_1-w_3)\cdot X^{FOR_1\text{-}DOM}(T)}\middle|\mathcal{F}_t\right] \\
&\quad\cdot E_{\mathbb{Q}_{DOM}}\left[e^{i(w_2+w_3)\cdot X^{FOR_2\text{-}DOM}(T)}\middle|\mathcal{F}_t\right] \\
&= \Phi_{X^{FOR_1\text{-}DOM}}(w_1 - w_3, T \mid \mathcal{F}_t) \cdot \Phi_{X^{FOR_2\text{-}DOM}}(w_2 + w_3, T \mid \mathcal{F}_t),
\end{aligned}$$

where $\Phi_{X^{FOR_1\text{-}DOM}}(w_1 - w_3, T \mid \mathcal{F}_t)$ and $\Phi_{X^{FOR_2\text{-}DOM}}(w_2 + w_3, T \mid \mathcal{F}_t)$ are the conditional characteristic functions of $X^{FOR_1\text{-}DOM}(T)$ and $X^{FOR_2\text{-}DOM}(T)$, respectively. The penultimate equation follows from the independence of $(X^{FOR_1\text{-}DOM}(t))_{t\geq 0}$ and $(X^{FOR_2\text{-}DOM}(t))_{t\geq 0}$.

Proof (Proposition 2) For the conditional characteristic function of $\mathbf{X}(t)$ it holds that

$$
\begin{aligned}
\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) &= E\left[e^{i\mathbf{w}'\mathbf{X}(T)} \middle| \mathcal{F}_t \right] \\
&= E\left[e^{i\left(w_1 \cdot X^{FOR_1\text{-}DOM}(T) + w_2 \cdot X^{FOR_2\text{-}DOM}(T) + w_3 \cdot X^{FOR_2\text{-}FOR_1}(T)\right)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot E\left[e^{i\left(w_1 \cdot (X^{FOR_1\text{-}DOM}(T) - X^{FOR_1\text{-}DOM}(t)) + w_2 \cdot (X^{FOR_2\text{-}DOM}(T) - X^{FOR_2\text{-}DOM}(t))\right.}\right. \\
&\qquad \left. {}^{\left.+ w_3 \cdot (X^{FOR_2\text{-}FOR_1}(T) - X^{FOR_2\text{-}FOR_1}(t))\right)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot E\left[e^{i\mathbf{w}'(\mathbf{X}(T) - \mathbf{X}(t))} \middle| \mathcal{F}_t \right].
\end{aligned}
$$

Here, the differences $X^{FOR_1\text{-}DOM}(T) - X^{FOR_1\text{-}DOM}(t)$, $X^{FOR_2\text{-}DOM}(T) - X^{FOR_2\text{-}DOM}(t)$, and $X^{FOR_2\text{-}FOR_1}(T) - X^{FOR_2\text{-}FOR_1}(t)$ can be rewritten as

$$
\begin{aligned}
X^{FOR_1\text{-}DOM}(T) - X^{FOR_1\text{-}DOM}(t) &= \sum_{j=1}^{2} \left(Z_j^{FOR_1\text{-}DOM}(T) - Z_j^{FOR_1\text{-}DOM}(t) \right), \\
X^{FOR_2\text{-}DOM}(T) - X^{FOR_2\text{-}DOM}(t) &= \sum_{j=1}^{2} \left(Z_j^{FOR_2\text{-}DOM}(T) - Z_j^{FOR_2\text{-}DOM}(t) \right), \\
X^{FOR_2\text{-}FOR_1}(T) - X^{FOR_2\text{-}FOR_1}(t) &= \sum_{j=1}^{2} \left(Z_j^{FOR_2\text{-}FOR_1}(T) - Z_j^{FOR_2\text{-}FOR_1}(t) \right),
\end{aligned}
$$

with the dynamics of $Z_j^{FOR_1\text{-}DOM}(t)$, $Z_j^{FOR_2\text{-}DOM}(t)$, and $Z_j^{FOR_2\text{-}FOR_1}(t)$ defined by

$$
\begin{aligned}
dZ_j^{FOR_1\text{-}DOM}(t) &= \left(\frac{r_{DOM} - r_{FOR_1}}{2} - \frac{1}{2} a_{1j}^2 \lambda_j(t) \right) dt + a_{1j} \sqrt{\lambda_j(t)} dW_j(t), \\
dZ_j^{FOR_2\text{-}DOM}(t) &= \left(\frac{r_{DOM} - r_{FOR_2}}{2} - \frac{1}{2} a_{2j}^2 \lambda_j(t) \right) dt + a_{2j} \sqrt{\lambda_j(t)} dW_j(t), \\
dZ_j^{FOR_2\text{-}FOR_1}(t) &= \left(\frac{r_{FOR_1} - r_{FOR_2}}{2} - \frac{1}{2} \left(a_{2j}^2 - a_{1j}^2 \right) \lambda_j(t) \right) dt \\
&\quad + \left(a_{2j} - a_{1j} \right) \sqrt{\lambda_j(t)} dW_j(t).
\end{aligned}
$$

Defining $Z_j^*(t)$ as $Z_j^*(t) := w_1 \cdot Z_j^{FOR_1\text{-}DOM}(t) + w_2 \cdot Z^{FOR_2\text{-}DOM}(t) + w_3 \cdot Z^{FOR_2\text{-}FOR_1}(t)$, it follows that $Z_j^*(t)$ is a Heston-type stochastic process with dynamics

$$dZ_j^*(t) = \left(r(\mathbf{w}) + \frac{b_j(\mathbf{w})}{c_j^2(\mathbf{w})}\lambda_j^*(t)\right)dt + \sqrt{\lambda_j^*(t)}dW_j(t),$$
$$d\lambda_j^*(t) = \kappa_j(c_j^2(\mathbf{w}) \cdot \theta_j - \lambda_j^*(t))dt + c_j(\mathbf{w}) \cdot \sigma_j \cdot \sqrt{\lambda_j^*(t)}dB_j(t),$$

where

$$\begin{aligned} c_j(\mathbf{w}) &:= (w_1 - w_3) \cdot a_{1j} + (w_2 + w_3) \cdot a_{2j}, \\ b_j(\mathbf{w}) &:= -\frac{1}{2}\left((w_1 - w_3) \cdot a_{1j}^2 + (w_2 + w_3) \cdot a_{2j}^2\right), \\ r(\mathbf{w}) &:= (w_1 - w_3) \cdot \frac{r_{DOM} - r_{FOR_1}}{2} + (w_2 + w_3) \cdot \frac{r_{DOM} - r_{FOR_2}}{2}, \end{aligned} \tag{17}$$

and $\lambda_j^*(t) := c_j^2(\mathbf{w}) \cdot \lambda_j(t)$.

Due to the fact that $\langle dW_j(t), dB_{j'}(t)\rangle = \langle dW_j(t), dW_{j'}(t)\rangle = \langle dB_j(t), dB_{j'}(t)\rangle = 0$ for all $j \neq j'$ it can be shown that $Z_j^*(t)$ and $Z_{j'}^*(t)$ are independent.

As a Heston-type stochastic process, the characteristic function of $Z_j^*(t)$ at time T conditional on $\mathcal{F}_t$ is given by

$$\begin{aligned} \Phi_{Z_j^*}(\phi, T \mid \mathcal{F}_t) &= E\left[e^{i\phi Z_j^*(T)} \middle| \mathcal{F}_t\right] \\ &= e^{C_j(\phi, T-t) + D_j(\phi, T-t)\cdot\lambda_j^*(t) + i\phi Z_j^*(t)}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} C_j(\phi, T-t) &= r(\mathbf{w})\phi i \cdot (T-t) + \frac{\kappa_j\theta_j}{\sigma_j^2} \\ &\quad \left\{(\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j) \cdot (T-t) - 2\ln\left[\frac{1 - g_j \cdot e^{-d_j\cdot(T-t)}}{1 - g_j}\right]\right\}, \\ D_j(\phi, T-t) &= \frac{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j}{c_j^2(\mathbf{w})\sigma_j^2}\left[\frac{1 - e^{-d_j\cdot(T-t)}}{1 - g_j \cdot e^{-d_j\cdot(T-t)}}\right], \\ g_j &= \frac{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i - d_j}{\kappa_j - \rho_j c_j(\mathbf{w})\sigma_j\phi i + d_j}, \\ d_j &= \sqrt{(\rho_j c_j(\mathbf{w})\sigma_j\phi i - \kappa_j)^2 + \sigma_j^2\left(-2b_j(\mathbf{w})\phi i + c_j^2(\mathbf{w})\phi^2\right)}, \end{aligned}$$

with $c_j(\mathbf{w})$, $b_j(\mathbf{w})$, and $r(\mathbf{w})$ as in (17). Hence, for the conditional characteristic function it follows that

$$
\begin{aligned}
\Phi_{\mathbf{X}}(\mathbf{w}, T \mid \mathcal{F}_t) &= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot E\left[e^{i\mathbf{w}'(\mathbf{X}(T)-\mathbf{X}(t))} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot E\left[e^{i\left(w_1\cdot\left(X^{FOR_1\text{-}DOM}(T)-X^{FOR_1\text{-}DOM}(t)\right)+w_2\cdot\left(X^{FOR_2\text{-}DOM}(T)-X^{FOR_2\text{-}DOM}(t)\right)\right.} \right. \\
&\qquad \left. {}^{\left.+w_3\cdot\left(X^{FOR_2\text{-}FOR_1}(T)-X^{FOR_2\text{-}FOR_1}(t)\right)\right)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot E\left[e^{i\left(\sum_{j=1}^{2} Z_j^*(T)-Z_j^*(t)\right)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot \prod_{j=1}^{2} E\left[e^{i\left(Z_j^*(T)-Z_j^*(t)\right)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot \prod_{j=1}^{2} e^{-iZ_j^*(t)} E\left[e^{iZ_j^*(T)} \middle| \mathcal{F}_t \right] \\
&= e^{i\mathbf{w}'\mathbf{X}(t)} \cdot \prod_{j=1}^{2} e^{C_j(1,T-t)+D_j(1,T-t)\cdot\lambda_j(t)\cdot c_j^2(\mathbf{w})},
\end{aligned}
$$

where the third equation holds due to the independence of $Z_1^*(t)$ and $Z_2^*(t)$. Moreover, with $\phi = 1$ in Eq. (18) the desired form of the characteristic function is obtained.

References

Ahlip, R. (2008). Foreign exchange options under stochastic volatility and stochastic interest rates. *International Journal of Theoretical and Applied Finance*, *11*(03), 277–294.

Albrecher, H., Mayer, P., Schoutens, W., & Tistaert, J. (2007). The Little Heston trap. *Wilmott Magazine*, *26*, 83–92.

Amin, K., & Jarrow, R. (1991). Pricing foreign currency options under stochastic interest rates. *Journal of International Money and Finance*, *10*(3), 310–329.

Andersen, L. B. G. (2007). *Efficient simulation of the Heston stochastic volatility model*. Available at SSRN 946405.

Bakshi, G., & Madan, D. (2000). Spanning and derivative-security valuation. *Journal of Financial Economics*, *55*(2), 205–238.

Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *The Journal of Finance*, *52*(5), 2003–2049.

Bakshi, G. S., & Chen, Z. (1997). Equilibrium valuation of foreign exchange claims. *The Journal of Finance*, *52*(2), 799–826.

Bannör, K. F., & Scherer, M. (2013). A BNS-type stochastic volatility model with two-sided jumps, with applications to FX options pricing. *Wilmott Magazine*, *65*, 58–69.

Bannör, K. F., Scherer, M., & Schulz, T. (2015). A two-sided BNS model for multicurrency FX markets. In K. Glau, M. Scherer, & R. Zagst (Eds.), *Innovations in Quantitative Risk Management* (pp. 93–107). Berlin: Springer.

Barndorff-Nielsen, O. E., & Stelzer, R. (2013). The multivariate supOU stochastic volatility model. *Mathematical Finance*, *23*(2), 275–296.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies*, *9*(1), 69–107.

Bessembinder, H. (1994). Bid-ask spreads in the interbank foreign exchange markets. *Journal of Financial Economics*, *35*(3), 317–348.

Bianchetti, M., & Carlicchi, M. (2013). Evolution of the markets after the credit crunch. In M. Bianchetti & M. Morini (Eds.), *Interest rate modelling after the financial crisis* (pp. 5–60). London: Risk Books.

Biger, N., & Hull, J. (1983). The valuation of currency options. *Financial Management*, *12*(1), 24–28.

Bisesti, L., Castagna, A., & Mercurio, F. (2005). Consistent pricing and hedging of an FX options book. *The Kyoto Economic Review*, *74*(1), 65–83.

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy*, *81*(3), 637–654.

Bossens, F., Rayée, G., Skantzos, N. S., & Deelstra, G. (2010). Vanna-Volga methods applied to FX derivatives: From theory to market practice. *International Journal of Theoretical and Applied Finance*, *13*(08), 1293–1324.

Branger, N., & Muck, M. (2012). Keep on smiling? The pricing of Quanto options when all covariances are stochastic. *Journal of Banking & Finance*, *36*(6), 1577–1591.

Carr, P., & Crosby, J. (2010). A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options. *Quantitative Finance*, *10*(10), 1115–1136.

Carr, P., & Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance*, *2*(4), 61–73.

Carr, P., & Wu, L. (2007). Stochastic skew in currency options. *Journal of Financial Economics*, *86*(1), 213–247.

Christoffersen, P., Heston, S., & Jacobs, K. (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science*, *55*(12), 1914–1932.

Clark, I. J. (2011). *Foreign exchange option pricing: A practitioners guide*. Chichester: Wiley.

Coleman, T. F., & Li, Y. (1994). On the convergence of interior-reflective Newton methods for nonlinear minimization subject to bounds. *Mathematical Programming*, *67*(2), 189–224.

Cox, J. C., & Ross, S. A. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, *3*(1–2), 145–166.

Da Fonseca, J., & Grasselli, M. (2011). Riding on the smiles. *Quantitative Finance*, *11*(11), 1609–1632.

Da Fonseca, J., Grasselli, M., & Tebaldi, C. (2008). A multifactor volatility Heston model. *Quantitative Finance*, *8*(6), 591–604.

Da Fonseca, J., Grasselli, M., & Ielpo, F. (2014). Estimating the Wishart affine stochastic correlation model using the empirical characteristic function. *Studies in Nonlinear Dynamics & Econometrics*, *18*(3), 253–289.

De Col, A., Gnoatto, A., & Grasselli, M. (2013). Smiles all around: FX joint calibration in a multi-Heston model. *Journal of Banking & Finance*, *37*(10), 3799–3818.

Dempster, M. A. H., & Hong, S. S. G. (2000). Spread option valuation and the fast Fourier transform. In: *Mathematical Finance—Bachelier Congress 2000* (pp. 203–220). Berlin: Springer.

DeRosa, D. F. (2011). *Options on foreign exchange*. Chichester: Wiley.

Doust, P. (2012). The stochastic intrinsic currency volatility model: A consistent framework for multiple FX rates and their volatilities. *Applied Mathematical Finance*, *19*(5), 381–445.

Escobar, M., & Gschnaidtner, C. (2016). Parameters recovery via calibration in the Heston model: A comprehensive review. *Wilmott Magazine*, *86*, 60–81.

Escobar, M., & Olivares, P. (2013). Pricing of mountain range derivatives under a principal component stochastic volatility model. *Applied Stochastic Models in Business and Industry*, *29*(1), 31–44.

Escobar, M., Götz, B., Seco, L., & Zagst, R. (2010). Pricing a CDO on stochastically correlated underlyings. *Quantitative Finance*, *10*(3), 265–277.

Feiger, G., & Bertrand, J. (1979). Currency option bonds, puts and calls on spot exchange and the hedging of contingent foreign earnings. *The Journal of Finance*, *34*(5), 1129–1139.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*, *54*(1), 173–182.

Flesaker, B., & Hughston, L. P. (1997). International models for interest rates and foreign exchange. *Net Exposure*, *3*, 55–79.

Frey, R., & Sommer, D. (1996). A systematic approach to pricing and hedging international derivatives with interest rate risk: Analysis of international derivatives under stochastic interest rates. *Applied Mathematical Finance*, *3*(4), 295–317.

Garman, M. B., & Kohlhagen, S. W. (1983). Foreign currency option values. *Journal of International Money and Finance*, *2*(3), 231–237.

Geman, H., Karoui, N. E., & Rochet, J. C. (1995). Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability*, *32*(2), 443–458.

Grabbe, J. O. (1983). The pricing of call and put options on foreign exchange. *Journal of International Money and Finance*, *2*(3), 239–253.

Grzelak, L. A., & Oosterlee, C. W. (2011). On the Heston model with stochastic interest rates. *SIAM Journal on Financial Mathematics*, *2*(1), 255–286.

Heath, D., Jarrow, R., & Morton, A. (1990). Contingent claim valuation with a random evolution of interest rates. *Review of Futures Markets*, *9*(1), 54–76.
Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, *6*(2), 327–343.
Hull, J., & White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, *42*(2), 281–300.
Hull, J., & White, A. (2013). LIBOR vs. OIS: The derivatives discounting dilemma. *The Journal of Investment Management*, *11*(3), 14–27.
Hurd, T., & Zhou, Z. (2010). A Fourier transform method for spread option pricing. *SIAM Journal on Financial Mathematics*, *1*(1), 142–157.
Levendorskiy, S. (2012). Efficient pricing and reliable calibration in the Heston model. *International Journal of Theoretical and Applied Finance*, *15*(07), 1250050.
Lipton, A. (2001). *Mathematical methods for foreign exchange: A financial engineer's approach*. Singapore: World Scientific.
Melino, A., & Turnbull, S. M. (1990). Pricing foreign currency options with stochastic volatility. *Journal of Econometrics*, *45*(1–2), 239–265.
Merton, R. C. (1973). Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, *4*(1), 141–183.
Muhle-Karbe, J., Pfaffel, O., & Stelzer, R. (2012). Option pricing in multivariate stochastic volatility models of OU type. *SIAM Journal on Financial Mathematics*, *3*(1), 66–94.
Neri, C., & Schneider, L. (2012). Maximum entropy distributions inferred from option portfolios on an asset. *Finance and Stochastics*, *16*(2), 293–318.
Pelsser, A. (2003). Mathematical foundation of convexity correction. *Quantitative Finance*, *3*(1), 59–65.
Recchioni, M. C., & Scoccia, A. (2014). An analytically tractable multi-asset stochastic volatility model. *Applied Mathematical Sciences*, *8*(27), 1339–1355.
Reiswich, D., & Wystup, U. (2010). A guide to FX options quoting conventions. *The Journal of Derivatives*, *18*(2), 58–68.
Schlögl, E. (2002). A multicurrency extension of the lognormal interest rate market models. *Finance and Stochastics*, *6*(2), 173–196.
Shiraya, K., & Takahashi, A. (2014). Pricing multiasset cross-currency options. *Journal of Futures Markets*, *34*(1), 1–19.
Stein, E. M., & Stein, J. C. (1991). Stock price distributions with stochastic volatility: An analytic approach. *The Review of Financial Studies*, *4*(4), 727–752.
van der Stoep, A. W., Grzelak, L. A., & Oosterlee, C. W. (2015). The time-dependent FX-SABR model: Effcient calibration based on effective parameters. *International Journal of Theoretical and Applied Finance*, *18*(06), 1550042.
Wystup, U. (2007). *FX options and structured products*. Oxford: Wiley.