Mathematical Finance, Vol. 18, No. 1 (January 2008), 135–153

SOLVABLE AFFINE TERM STRUCTURE MODELS

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An *Affine Term Structure Model* (ATSM) is said to be solvable if the pricing problem has an explicit solution, i.e., the corresponding Riccati ordinary differential equations have a regular globally integrable flow. We identify the parametric restrictions which are necessary and sufficient for an ATSM with continuous paths, to be solvable in a state space $\mathcal{D}_+ \times \mathbb{R}^{n-m}$, where $\mathcal{D}_+$, the domain of positive factors, has the geometry of a symmetric cone. This class of state spaces includes as special cases those introduced by Duffie and Kan (1996), and Wishart term structure processes discussed by Gourieroux and Sufana (2003). For all solvable models we provide the procedure to find the explicit solution of the Riccati ODE.

KEY WORDS: Affine Terms Structure Models, Riccati ODE, Lie algebra, symmetric cone

1. INTRODUCTION

The class of multifactor Affine Term Structure Models (ATSM hereafter), has the following financial appealing properties:

1. The sensitivities of the zero coupon yield curve to the stochastic factors are deterministic, as discussed in Brown and Schaefer (1994);
2. For a given state space $\mathcal{D}$ it is possible to identify explicit parametric restrictions, called *admissibility conditions,* granting the existence of a regular affine process with state space $\mathcal{D}$. The problem of admissibility has been introduced by Duffie and Kan (1996), Dai and Singleton (2000), Duffie et al. (2003), and Filipović (2005);
3. The pricing problem can be reduced to the solution of a system of ordinary differential equations (ODE) as discussed in Duffie, Pan, and Singleton (2000). In fact, the explicit expression of the conditional discounted characteristic function of the factors can be specified in terms of the solutions of Riccati ODE for any admissible ATSM (see, e.g., Duffie et al. 2003).

In this paper we characterize the class of admissible ATSM, with factors following continuous paths, for which the pricing problem has an explicit solution, i.e., the

We wish to thank the co-editor Tomas Björk and the referee for useful comments which greatly improved the final version of the paper; the usual disclaimer applies.

Manuscript received June 2005; final revision received June 2006.

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non-linear Riccati ODE can be linearized and integrated. We term this class of models Solvable Affine Term Structure Models (SATSM). The main result of the paper is the characterization of necessary and sufficient conditions for an ATSM to be solvable when the state space of positive factors has the geometry of a symmetric cone. An excellent introduction to the properties and applications of symmetric cones can be found in the book of Faraut and Korányi (1994). This class of state spaces virtually includes all those previously considered in financial applications and provides possible extensions yet unexplored.

The explicit solution for all SATSM can be obtained by solving linear ODE. Thus the notion of solvability characterizes the class of multivariate dynamic affine factor models which, while relaxing the constant volatility hypothesis, preserves the possibility to obtain the explicit expression of the conditional characteristic functions solving only linear ordinary differential equations.

The procedure to find the explicit solution is carried out for all solvable models within the Duffie and Kan (1996) state space $\mathcal{D}= \mathbb{R}_+^m \times \mathbb{R}^{n-m}$, for Wishart models (see Gourieroux and Sufana 2003) where $\mathcal{D} = Pos_+(r, \mathbb{R})$ and for a new class, corresponding to Lorentz cone (rigourously defined below) state space $\mathcal{D} = \Lambda_4$.

The paper is organized as follows: in Section 2 we introduce the pricing problem in ATSM and we discuss the admissibility and solvability conditions. In Section 3, we provide the procedure to integrate the ODE when the model is solvable. In Section 4, we apply the above results to the class of models introduced in Duffie and Kan (1996) as well as to the class of Wishart term structure models. In conclusion we explicitly determine the conditional characteristic function for a novel example of solvable ATSM obtained considering as state space the four dimensional Lorentz cone. We leave in the Appendix the technical results on Jordan algebras which are used in the proofs of our main results.

2. PRICING IN AN ATSM

We follow the definition of regular affine processes given in Duffie et al. (2003) in the case where the Markovian process $(X_t)_{t\in\mathbb{R}_+} \in \mathcal{D} \subseteq V$, where $\mathcal{D}$ is the state space, and V is a finite dimensional real vector space, dim $V = n$, with the standard scalar product denoted by $\langle,\rangle$. With a slight abuse of notation we denote with V also the complex extension of the vector space on $\mathbb{C}$; correspondingly $\langle,\rangle$ will denote the Hermitian extension of the scalar product.

Being interested in pricing contingent claims, we follow Duffie et al. (2000), Bakshi and Madan (2000) and compute the discounted characteristic function of the factors X_t, conditional on the information at time $t \geq 0$:

DEFINITION 2.1. The regular (continuous) Markov process X is said to be affine if for every $\tau = (T - t) \in \mathbb{R}_+$ the "discounted" conditional characteristic function has exponential-affine dependence on the initial condition x. That is,

$$\Psi_X(u, x_t, t, \tau) = \mathbb{E}_t \left[\exp\left(-\int_t^{t+\tau} (\eta_0 + \langle\eta, X_s\rangle)\, ds\right) \exp(i\langle u, X_T\rangle)\right] \tag{2.1}$$

$$= \exp\left(\mathcal{V}^0(\tau, iu) - \langle\mathcal{V}(\tau, iu), x_t\rangle\right),$$

where $(\tau, u) \in \mathbb{R}_+ \times V$, $\mathcal{V}^0 : \mathbb{R}_+ \times V \to \mathbb{C}$, $\mathcal{V} : \mathbb{R}_+ \times V \to V$, and $\eta_0 \in \mathbb{R}_+$, $\eta \in \mathcal{D}$.

Typically the term $R(X_t) = (\eta_0 + \langle \eta, X_t \rangle)$ represents the short term rate and consistently the zero-coupon bond price is given by:

$$P(t, t+\tau) = \exp\left(\mathcal{V}^0(\tau) - \langle \mathcal{V}(\tau), X_t \rangle\right) = \Psi_X(0, x, t, \tau) \tag{2.2}$$

$$\mathcal{V}(0) = 0.$$

A standard argument, see, e.g., Duffie and Kan (1996), shows that the infinitesimal generator for a regular affine diffusion process, has necessarily the following functional form:

$$\mathcal{A} = \frac{1}{2} Tr\left[(\Sigma(x) + \Sigma_0) D^T D\right] + \langle \Omega^0 + \Omega(x), D^T \rangle - (\eta_0 + \langle \eta, x \rangle), \tag{2.3}$$

where D denotes the (row vector) gradient operator, Tr the trace over $M_n(V)$, $D^T D$ the Hessian matrix, $\Sigma(x), \Sigma_0 \in Sym_n(V)$ are positive semidefinite, $\Omega(x), \Omega^0 \in V$. The relevant property is that $\Sigma(x)$, $\Omega(x)$ are linear functions of $x \in V$, hence:

$$\Sigma(x) = \sum_{k=1}^{n} C_{i,j}^k x_k, \quad \Omega(x) = \sum_{k=1}^{n} \Omega_l^k x_k.$$

Following Duffie et al. (2003), we look for the conditions on the parameters which allow to define uniquely a regular affine (continuous path) process $\forall x \in \mathcal{D}, \forall \tau \in \mathbb{R}^+$. By applying the Feynman–Kac argument, we have

$$-\frac{\partial \Psi_X}{\partial t} + \mathcal{A}\Psi_X = 0 \quad \Psi_X(u, x, T, 0) = \exp\left(i \langle u, x \rangle\right),$$

that is

$$-\frac{\partial \Psi_X}{\partial t} + \frac{1}{2} Tr\left[(\Sigma(x) + \Sigma_0) D^T D\right]\Psi_X + \left\langle \Omega^0 + \Omega(x), D^T \Psi_X \right\rangle - (\eta_0 + \langle \eta, x \rangle)\Psi_X = 0$$

and by replacing the exponential form and changing time variable $t \to \tau = T - t$ we obtain (notice that $D^T \Psi = -\mathcal{V}_\Psi$)

$$-\frac{\partial \mathcal{V}^0}{\partial \tau} + \left\langle \frac{\partial \mathcal{V}}{\partial \tau}, x \right\rangle = \frac{1}{2} Tr\left[(\Sigma(x) + \Sigma_0)\mathcal{V}\mathcal{V}^T\right] - \langle \Omega^0 + \Omega(x), \mathcal{V} \rangle - (\eta_0 + \langle \eta, x \rangle) \tag{2.4}$$

$$\mathcal{V}^0(0, iu) = 0 \quad \mathcal{V}(0, iu) = iu.$$

This equality implies that the vector of factor sensitivities $\mathcal{V}$ solves a (quadratic) ordinary differential equation. Given the solution for $\mathcal{V}$, the determination of $\mathcal{V}^0$ can be obtained for any ATSM by direct integration, hence we are going to concentrate our discussion on the ODE for $\mathcal{V}$. However, as a by product of our analysis we will illustrate in the applications section a computational scheme to obtain an explicit expression for $\mathcal{V}^0(\tau)$ for those models which admit a matrix representation. In the following section, we are going to investigate the parametric restrictions granting the existence of the Markov semigroup, or equivalently the global existence of a non singular flow for the ODE (admissibility conditions).

2.1. Admissible ATSM in a Symmetric Cone State Space

DEFINITION 2.2. An ATSM model is admissible in a state space $\mathcal{D} \subseteq V$ if the generator (2.3) and the corresponding regular affine Markov semigroup exist and are unique for any initial condition in $\mathcal{D}$, or equivalently if Ψ_X is uniquely defined by (2.1) $\forall x \in \mathcal{D}$ and $\forall \tau \in \mathbb{R}^+$.

According to the definition, an "Admissible ATSM" is no more than a "regular affine process with admissible parameters" in the terminology of Duffie et al. (2003). Notice that our definition, however, is still implicit since it does not identify the parametric restrictions granting admissibility. Theorem 2.1 and Proposition 2.2 will partially answer to this problem by giving sufficient conditions on the parameters. In fact, our goal will be the complete characterization of the subclass of admissible models for which the Riccati ODE can be explicitly solved.

Now we recall the notion of symmetric cone domain, which generalizes the notion of "positive factors' state space."

DEFINITION 2.3. A symmetric cone $C \subset V$ is an open convex cone which has the two following properties: (i) C is self dual, i.e., $C = \{y \in V; \langle x, y\rangle > 0, \ \forall x \in \overline{C}\backslash\{0\}\}$ (ii) C is homogeneous, i.e., the automorphism group $G(C) = \{g \in GL(V); g(C) = C\}$ operates transitively on C.

A symmetric cone is said to be irreducible if it cannot be decomposed in a Cartesian product of two symmetric cones. Among (irreducible) symmetric cones $C = Pos_r(\mathbb{R}) \subset Sym_r(\mathbb{R})(m = r(r+1)/2)$ is prototypical and will be the state space for the class of Wishart models. The Duffie et al. (2003) state space $\bar{C} = \mathbb{R}^m_+$ is (the closure of) a reducible symmetric cone and is obtained considering m copies of $\mathbb{R}_+$, i.e., $Pos_1(\mathbb{R})$.

The geometry of a symmetric cone domain is directly related to the algebraic notion of Euclidean Jordan Algebra (EJA):

DEFINITION 2.4 (Faraut and Korányi 1994). $(V, \langle,\rangle, *)$ forms an EJA if V is a finite dimensional real vector space with a scalar product $\langle,\rangle$ *and* $* : V \times V \to$ V is a bilinear commutative (not necessarily associative) product with unity e such that the following properties hold: (i) Jordan algebra property: $u^2 * (v * u) = (u^2 * v) * u$, (ii) Euclidean property: $\langle u * v, w\rangle = \langle v, u * w\rangle$.

The following objects can be naturally defined within each EJA:

(i) the linear map $L(u) \in M_n(V)$ defined by $L(u)\,v =: u * v$,
(ii) the quadratic representation: $P(u) = 2L^2(u) - L(u^2)$, which satisfies:

$$P(gu) = g\,P(u)g^T, \quad \forall g \in G(C) \tag{2.5}$$

(iii) the related bilinear form: $P(u, v) = [P(u+v) - P(u) - P(v)]/2$.

The following results reduce the classification of symmetric cones to the classification of simple EJA:

PROPOSITION 2.1 (Faraut and Korányi 1994). *(i) if $C \subseteq V$ is a symmetric cone then V carries the structure of a Euclidean Jordan Algebra and the closure $\bar{C}$ is the set of squares $u * u, u \in V$; (ii) the symmetric cone is irreducible if and only if the associated EJA is simple, i.e., it has no non-trivial ideals, where an ideal U of the EJA V is a subspace such* *that* $\forall\, u \in U, v \in V$, *then* $u * v \in U$; *(iii) there's a one-to-one correspondence between the decomposition of a symmetric cone in a product of irreducible symmetric cones and the decomposition of an EJA as a direct sum of simple EJA.*

All possible finite dimensional irreducible symmetric cones are:

1. The families of positive definite matrix cones $Pos(r, \mathbb{R})$, $Herm_+(r, \mathbb{C})$, $Herm_+(r, \mathbb{H})$($\mathbb{H}$ indicates the quaternions field),
2. The Lorentz cones $\Lambda_n = \{x \in \mathbb{R}^n : x_1^2 - \sum_{i=2}^n x_i^2 > 0, x_1 > 0\}$,
3. The exceptional cone (27 dimensional cone of 3×3 "positive definite" matrices over the Cayley algebra).

For example, when $C = Pos(r, \mathbb{R})$, the corresponding EJA is given by $Sym_r(\mathbb{R})$ with (Jordan) product: $A * B = (AB + BA)/2$.

Now we state our first result which proves a remarkable connection between the conditional covariance of an admissible ATSM (on an irreducible symmetric cone state space $\mathcal{D}$) and the relative quadratic representation $P(\mathcal{V})$:

THEOREM 2.1. *Consider an irreducible symmetric cone state space* $\mathcal{D} \subset V$ *and let* $(V, \langle,\rangle, *)$ *be the corresponding EJA. Then, for any positive definite covariance matrix* $\Sigma_\alpha(.)$ *of an admissible ATSM in* $\mathcal{D}$, *there exists* $\alpha_\Sigma \in \mathcal{D}$ *such that:*

$$Tr\left[\Sigma_\alpha(x)\mathcal{V}\mathcal{V}^T\right] = \langle x, P(\mathcal{V})\alpha_\Sigma\rangle \quad \forall x \in \mathcal{D}, \forall \mathcal{V} \in V \tag{2.6}$$

holds.

Proof. See the Appendix.

Notice that relation (2.6) provides a one-to-one correspondence between the quadratic term of the Riccati ODE and the related quadratic representation of the EJA: as we shall see in the next proposition, this relation can be extended to the complete vector field, thus defining a sufficient condition for an ATSM to be admissible.

PROPOSITION 2.2. *Consider the following Riccati ODE defined in an irreducible symmetric cone* $\mathcal{D}$:

$$-\frac{d\mathcal{V}}{d\tau} = -\frac{1}{2}P(\mathcal{V})\alpha_\Sigma + \Omega^T(\mathcal{V}) + \eta, \tag{2.7}$$

where $\alpha_\Sigma \in \mathcal{D}$ *is given in (2.6),* $\eta \in \mathcal{D}$ *and*

$$\Omega^T \in g(\mathcal{D}) = \left\{H \in M_n(V) \mid 2P(H(\mathcal{V}), \mathcal{V}) = HP(\mathcal{V}) + P(\mathcal{V})H^T\right\}. \tag{2.8}$$

Moreover, let η_0 *be a positive constant,* $\Sigma_0 \in Sym_r(V)$ *be positive semidefinite and* $\Omega_0 = k\alpha_\Sigma$, $\frac{k}{2} > \frac{n}{r} - 1$, *where r is the rank of the EJA. Then the corresponding ATSM given by (2.3) is admissible in the state space* $\mathcal{D}$.

The proof of Proposition 2.2 given in Appendix is grounded on a construction due to Kantor, Koecher, and Tits (KKT) (see the Appendix for a short introduction), which shows that each Jordan Algebra can be represented as a particular (quadratic) polynomial Lie algebra. While Theorem 2.1 states that there's a strong relationship between the Jordan Algebra structure and the covariance term, Proposition 2.2 shows that the vector field of the Riccati ODE is an element of the polynomial Lie algebra obtained using the KKT construction.

The extension of the above results to general (non-irreducible) symmetric cone state spaces is immediate. In fact, if the symmetric cone is not irreducible, it decomposes as $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_k$, where $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are the corresponding irreducible symmetric cones. Then the system of Riccati ODE corresponding to an ATSM admissible on $\mathcal{D}$ separates into a direct sum of independent systems of Riccati ODE, each corresponding to an admissible ATSM in $\mathcal{D}_i$.

2.2. Solvable ATSM in a Symmetric Cone State Space

The above analysis has shown that it is possible to provide a constructive test to select a class of admissible ATSM in any symmetric cone state space $\mathcal{D}$. The conditions in Proposition 2.2 are sufficient but not necessary, as it is easy to realize by comparing with the ones of Duffie et al. (2003), in the special case of $\bar{\mathcal{D}} = \mathbb{R}^m_+$ (see also Section 4 below). However, in this subsection we shall see that the class of admissible models which are selected by Proposition 2.2 is completely characterized by the important property that *the solution flow of the corresponding Riccati ODE can be linearized, and then explicitly determined through exponentiation.*

First of all let us denote by $(A(V, V), [,])$ the Lie algebra of all analytic maps from V into V, where the commutator is defined by:

$$[f, g](\mathcal{V}) = \nabla g(\mathcal{V}) f(\mathcal{V}) - \nabla f(\mathcal{V}) g(\mathcal{V}), \quad \mathcal{V} \in V, \ f, g \in A(V, V).$$

Within Lie approach to the integration of an ODE, the ability to get an explicit solution is essentially equivalent to its linearizability, which is formally defined by the following:

DEFINITION 2.5 (Walcher 1991). Consider the parametrized family of ODE:

$$\frac{d\mathcal{V}}{d\tau} = \mathcal{R}_\alpha(\mathcal{V}, \tau) \tag{2.9}$$

$$\mathcal{R}_\alpha(\mathcal{V}, \tau) = \sum_{i=1}^{K} \alpha_i(\tau) f^{(i)}(\mathcal{V}),$$

where the parameters $\alpha_1, \ldots, \alpha_K : R_+ \to \mathrm{R}$ are continuous functions of time and $L = \{f^{(i)}\}_{i=1\ldots r}$ is a family of autonomous maps $L \subset A(V, V)$. The family of ODE $\mathcal{R}_\alpha(\mathcal{V}, \tau)$ is linearizable if: (i) there exists a linear ODE in W:

$$\frac{d\mathcal{W}}{d\tau} = \mathcal{R}^\Phi_\alpha(\mathcal{W}, \tau); \tag{2.10}$$

(ii) there exists an analytic map $\Phi : W \to V$ which is solution preserving, i.e., Φ maps solutions curves $\mathcal{W}(\tau) \subset W, \tau \geq 0$ of the linear ODE to solutions $\mathcal{V}(\tau) \subset V, \tau \geq 0$ for the ODE (2.9).

Roughly speaking, Φ is a linearizing map if Φ^{-1} maps a non-linear ODE into a linear one. Notice however, that the correspondence will not be one to one in the general case, thus Φ^{-1} could possibly be multiply defined.

Now we illustrate a constructive approach to explicitly determine the linearized ODE (2.10). First of all one observes (see, e.g., Walcher 1991, proposition 8.7) that the existence of such a change of coordinates implies the existence of a finite dimensional Lie subalgebra, $\langle L \rangle$, which is the smallest Lie algebra containing $L \subset A(V, V)$. Since any finite dimensional Lie algebra is isomorphic to a matrix algebra (Ado's theorem), any element $f^{(i)} \in \langle L \rangle$ will correspond to a matrix $M(f^{(i)})$; then (2.10) is given by:

$$\frac{d\mathcal{W}}{d\tau} = \sum_{i=1}^{K} \alpha_i(\tau) M\left(f^{(i)}\right) \mathcal{W}.$$

The correspondence between $M(f^{(i)})$ and $f^{(i)}$ is a Lie algebra homomorphism, and the exponential map transfers it from the Lie algebra (tangent vectors) to the solution flows (curves) for the two ODE and provides the linearizing map Φ. In particular the flow for the linear equation can be explicitly obtained through matrix exponentiation.

The decomposition of $\mathcal{R}_\alpha$, corresponding to the choice of L, is not unique. In particular, the larger the number of parametric restrictions on the expression of $\mathcal{R}_\alpha$ the smaller the dimension of $\langle L \rangle$.[1] We shall consider as acceptable only those (sufficiently large) finite dimensional Lie subalgebras which are compatible with any choice of (possibly time dependent) parameters. In fact, for the purpose of model estimation, any restriction beyond admissibility would appear as an artificial constraint and nullify the advantages of an explicit solution. We formalize the notion of *admissible parameters* using a definition in line with Filipović (2005):

DEFINITION 2.6. Let $\mathcal{R}_\alpha$ be the vector field of the Riccati ODE arising from an ATSM with state space $\mathcal{D}$, and $\{\alpha_i(\tau)\}_{i=1,\ldots,I}$ be the vector of all real parameters appearing in $\mathcal{R}_\alpha$. The vector field $\mathcal{R}_\alpha$ is said to be (strongly) admissible if for all i the functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}$ are continuous and for any fixed t the corresponding autonomous Riccati ODE has a differentiable flow on the positive line.

We can now introduce the following:

DEFINITION 2.7. An ATSM is said to be solvable in a state space $\mathcal{D}$ if for any (strongly) admissible choice of the parameters, the conditional characteristic function can be determined by solving a sequence of linear ODEs.

Hence, an ATSM in a symmetric cone state space $\mathcal{D}$ is solvable iff the corresponding Riccati ODE is linearizable.

We now state our main result, stating that the parametric conditions introduced in Proposition 2.2 granting admissibility are also necessary for solvability.

THEOREM 2.2. *An ATSM in an irreducible symmetric cone state space $\mathcal{D}$ is solvable iff the corresponding Riccati ODE (2.7) satisfies the conditions in Proposition 3.*

The proof of the theorem, given in Appendix, uses again the KKT construction: in fact, following Walcher (1986), we will show that the KKT approach classifies all linearizable quadratic ODE.

The extension to a generic (possibly reducible) symmetric cone state space $\mathcal{D}$ is straightforward: in this case the subalgebra of analytic maps separates into a direct sum of subalgebras, hence the solution reduces to the solution of an independent ODE in each irreducible symmetric cone.

[1] The extreme case being, for example, a family of a single autonomous equation, which always admits the trivial one dimensional algebra $\{\alpha F(\mathcal{V})\}$, $\alpha \in \mathbb{R}$. We thank the referee for pointing out this example.

2.3. Extension to the State Space $\mathcal{D} = \mathcal{D}_+ \times \mathbb{R}^{n-m}$

Let $\mathcal{D} = \mathcal{D}_+ \times \mathbb{R}^{n-m}$, where $\mathcal{D}_+$ is a symmetric cone and $\mathbb{R}^{n-m}$ represents the state space of evolution of conditionally Gaussian factors. As discussed in Duffie et al. (2003) in an ATSM with state space $\mathcal{D} = \mathbb{R}^m_+ \times \mathbb{R}^{n-m}$, the factor sensitivities corresponding to real possibly negative factors can be determined solving an independent system of linear equations. In the next proposition we extend this result to an ATSM with state space $\mathcal{D}_+ \times \mathbb{R}^{n-m}$, requiring that $\mathcal{D}_+$ is an ideal for $\mathcal{R}(\mathcal{V}, \tau)$, i.e., for any $\mathcal{V}_+ \in \mathcal{D}_+$ then $\mathcal{R}(\mathcal{V}_+, \tau) \in \mathcal{D}_+$.

PROPOSITION 2.3. *Let $\mathcal{D}_+$ be an ideal of a quadratic algebra induced by the (quadratic) polynomial ODE:*

$$\frac{d}{d\tau}\mathcal{V} = \mathcal{R}(\mathcal{V}, \tau) \quad \mathcal{V}(\tau) \in \mathcal{D}_+ \times \mathbb{R}^{n-m},$$

and let $\pi : \mathcal{D}_+ \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ be the natural projection. Then the solution $\mathcal{V} = \left(\begin{smallmatrix} \mathcal{V}^{\mathcal{D}_+} \\ \mathcal{V}^G \end{smallmatrix}\right)$ can be obtained by solving $(n-m)$ linear ODEs:

$$\frac{d}{d\tau}\mathcal{V}^G = \mathcal{R}^G(\mathcal{V}^G, \tau) \quad \mathcal{V}^G(0) = \pi\mathcal{V}(0) \tag{2.11}$$

$$\mathcal{R}^G(\mathcal{V}^G, \tau) = \pi\mathcal{R}(\mathcal{V}, \tau), \quad \mathcal{V}^G = \pi\mathcal{V},$$

and the quadratic differential equation in $\mathcal{D}_+$:

$$\frac{d}{d\tau}\begin{pmatrix} \mathcal{V}^{\mathcal{D}_+} \\ 0 \end{pmatrix} = \mathcal{R}\left(\begin{pmatrix} \mathcal{V}^{\mathcal{D}_+} \\ \mathcal{V}^G \end{pmatrix}, \tau\right) - \frac{d}{d\tau}\mathcal{V}^G$$

$$\begin{pmatrix} \mathcal{V}^{\mathcal{D}_+}(0) \\ 0 \end{pmatrix} = \mathcal{V}(0) - \begin{pmatrix} 0 \\ \mathcal{V}^G(0) \end{pmatrix},$$

where $\overline{\mathcal{V}^G}$ denotes the solution to (2.11).

Proof. The result follows directly from theorem 2.5 of Walcher (1991) and by noting from (2.4) that the presence of conditionally Gaussian factors leads to linear ODEs for $\mathcal{V}^G$. □

The parametric restrictions on the Riccati ODE implied by the condition that $\mathcal{D}_+$ is an ideal for the whole Riccati ODE depend on the specific choice for $\mathcal{D}_+$, and will not be discussed here.

3. THE EXPLICIT SOLUTION OF A SATSM

The above definition of solvable ATSM is intended to provide an explicit path to compute the conditional characteristic function by mean of linear ODEs. The computational algorithm to find the general integral is the following:

1. Solve the ODE (2.11) corresponding to conditionally Gaussian factors;
2. Decompose the remaining ODE as a direct sum of Riccati ODE corresponding to factors living on irreducible symmetric cone state spaces;
3. Linearize each of these Riccati ODEs according to the classification of simple EJAs (a detailed classification of these algebras can be found in Faraut and Korányi 1994). In particular, we recall that each EJA, apart from the exceptional case, has a symmetric matrix realization M, i.e., it can be realized as a subalgebra

of a suitable EJA of symmetric matrices taking as Jordan product the symmetrized matrix product. The linearizing map Φ of Definition 2.5 is then the natural homomorphism between the quadratic polynomial realization and the matrix realization of the EJA (see also the examples in Section 4).

4. Solve by exponentiation the linearized system in M and use the solution preserving map Φ to recover the solution for the original ODE.

All these steps can be easily implemented with any numerical software and the only relevant task is the computation of the matrix exponential function which solves the linear system. While this computation is straightforward when the ODE is autonomous, in the presence of time dependent coefficients the solution to the linear system involves a so called *time ordered exponential*, a symbolic expression whose analytic computation is problematic. On the other hand, this situation is exactly that occurring for linear (non-autonomous) ODEs and nothing better could be expected. In the non-autonomous case, an alternative procedure to determine the general integral of the Riccati ODE is proposed by Walcher (1991, p.134). Notice however that this expression is explicit up to the knowledge of a particular solution, which has to be found case by case.

Now we specialize our results to the class of models most relevant for financial applications.

4. APPLICATIONS

4.1. The Duffie and Kan (1996) State Space $\mathbb{R}^m_+ \times \mathbb{R}^{n-m}$

Consider the familiar setup of Duffie and Kan (1996) ATSM in the specification proposed by Dai and Singleton (2000), which is essentially the same of Duffie et al. (2000), when factors are restricted to have continuous paths.

The dynamics of the factors are determined (up to a rescaling) by the following SDE:

$$dX_t = (\Omega X_t + \Omega^0)\,dt + diag\left[\left(C_i X_t + C_i^0\right)^{1/2}\right] dW_t, \quad t \geq 0, \tag{4.1}$$

$$X_0 = x \in \mathbb{R}^m_+ \times \mathbb{R}^{n-m},$$

(C_i denotes the $i - th$ row of the matrix C, $m = rank(C) \leq n$) where W_t is an n-dimensional Brownian motion and the parameters $\phi = (\eta, \eta^0, \Omega, \Omega^0, C, C^0)$ are defined as follows:

(i) the drift matrix Ω is given by

$$\Omega = \begin{pmatrix} \Omega^{DD}_{m\times m} & 0_{m\times(n-m)} \\ \Omega^{BD}_{(n-m)\times m} & \Omega^{BB}_{(n-m)\times(n-m)} \end{pmatrix}, \tag{4.2}$$

where the out of diagonal elements of Ω^{BB} are restricted to be nonnegative;

(ii) $\Omega^0 \in \mathbb{R}^m_+ \times \mathbb{R}^{n-m}$;

(iii) C and C^0 are given by

$$C = \begin{pmatrix} \mathbb{I}_{m\times m} & 0_{m\times(n-m)} \\ C^{BD}_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{pmatrix}, \tag{4.3}$$

$$C^0 = \begin{pmatrix} 0_{m\times 1} \\ 1_{(n-m)\times 1} \end{pmatrix}.$$

(iv) $\eta_0 \geq 0, \eta^D \in \mathbb{R}^m_+,\ \eta^B = 0.$

Observe that the state space of positive factors $\mathbb{R}^m_+$ is the closure of a reducible symmetric cone: the corresponding EJA is given by $\mathbb{R}^m$ endowed with (the natural scalar product and) the Jordan product $(a * b)_i = a_i b_i$, whose quadratic representation corresponds to $P(c) = \operatorname{diag}(c)$.

We apply our main theorem and we get the following

THEOREM 4.1. *The ATSM (4.1) is solvable in the state space $\mathbb{R}^m_+ \times \mathbb{R}^{n-m}$ if and only if the matrix Ω^{DD} is diagonal.*

Proof. The statement is a consequence of the reducibility of the EJA, which in turn implies that both C^{DD} and Ω^{DD} are diagonal. □

In conclusion, a necessary condition for an ATSM to be solvable in the state space $\mathbb{R}^m_+ \times \mathbb{R}^{n-m}$ is that the *positive risk factors are uncorrelated*.

REMARK 4.1. Even if the previous theorem represents a No Go result, there are some non-trivial SATSM (for example the Balduzzi et al. 1996 model, see Grasselli and Tebaldi 2004) which can be solved explicitly using our methodology.

For a SATSM, the Riccati ODEs become

$$(4.4)\quad -\frac{d}{d\tau}\mathcal{V}_i^D(\tau) = \eta_i^D + \Omega_{ii}^{DD}\mathcal{V}_i^D + \sum_{j=m+1}^{n}(\Omega^{BD})^T_{ij}\mathcal{V}_j^B - \frac{1}{2}\sum_{i=1}^{m}\left(\mathcal{V}_i^D(\tau)\right)^2 - \frac{1}{2}\sum_{j=m+1}^{n}(C^{BD})^T_{ij}\left(\mathcal{V}_j^B(\tau)\right)^2, \quad i = 1, \ldots, m$$

$$(4.5)\quad -\frac{d}{d\tau}\mathcal{V}_i^B(\tau) = \sum_{j=m+1}^{n}(\Omega^{BB})^T_{ij}\mathcal{V}_j^B(\tau), \quad i = m+1, \ldots, n,$$

with boundary condition $\mathcal{V}(T) = iu \in \mathbb{C}^n$.

The procedure of Section 3 in order to find the explicit solution in the case of a SATSM consists in the following two steps:

Step (1). Put $\tau = T - t$ and solve the $n - m$ linear ODEs (4.5):

$$(4.6)\quad \mathcal{V}^B(\tau) = \exp\{-\tau(\Omega^{BB})^T\}\mathcal{V}^B(0).$$

Step (2). Plug (4.6) into (4.4) and for all $i = 1, \ldots, m$ solve the 1-dimensional time-dependent Riccati equations

$$(4.7)\quad -\frac{d}{d\tau}\mathcal{V}_i^D(\tau) = \tilde{\gamma}_i(\tau) + \Omega_{ii}^{DD}\mathcal{V}_i^D(\tau) - \frac{1}{2}\mathcal{V}_i^D(\tau)^2, \quad i = 1, \ldots, m,$$

where $\tilde{\gamma}_i(\tau) = \eta_i + \sum_{j=m+1}^{n}((\Omega^{BD})^T_{ij}\mathcal{V}_j^B(\tau) - \frac{1}{2}(C^{BD})^T_{ij}\mathcal{V}_j^B(\tau)^2)$.

The only non-trivial step is the second one. However, now we show that although the coefficients $\tilde{\gamma}_i(\tau)$ are time dependent, the one dimensional Riccati ODE (4.7) can be linearized. In fact, the vector field

$$\mathcal{R}(\tau) = \left(\gamma(\tau) + \omega\mathcal{V} - \frac{1}{2}\mathcal{V}^2\right), \tag{4.8}$$

can be included in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with generators:

$$L_0 = \gamma \in \mathbb{R}, \quad L_1 = \mathcal{V}, \quad L_2 = \mathcal{V}^2,$$

obeying the commutation relations:

$$[L_0, L_1] = -L_0, [L_0, L_2] = -2L_1, [L_2, L_1] = L_2.$$

The consequence is that we can now rewrite our differential equation (4.7) in a coordinate free way, and we can move from the above non-linear representation to a more convenient space $W = \mathbb{R}^2$, where the group is linearly represented.

In fact, the generators L_0, L_1, L_2 have the following linear realization in $M_2(\mathbb{R})$:

$$M_2(L_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_2(L_1) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_2(L_2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and in the new space W the equation (4.9) can be written as

$$-\frac{d}{d\tau}\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix} = \left(\gamma(\tau)M_2(L_0) + \omega M_2(L_1) - \frac{1}{2}M_2(L_2)\right)\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix},$$

which becomes the linear ODE:

$$-\frac{d}{d\tau}\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\omega & \gamma(\tau) \\ -\frac{1}{2} & -\frac{1}{2}\omega \end{pmatrix}\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix}. \tag{4.9}$$

The solutions to this linear system are isomorphic to the solution space of the original non-linear Riccati ODE through the map:

$$\Phi\begin{pmatrix} \pi(\tau) \\ \lambda(\tau) \end{pmatrix} = \pi(\tau)\lambda(\tau)^{-1}.$$

In conclusion, for all $i = 1, \ldots, m$ we obtain the explicit solution for (4.7):

$$\mathcal{V}_i(\tau) = \frac{M_1^i(\tau)\mathcal{V}_i(0) + M_2^i(\tau)}{M_3^i(\tau)\mathcal{V}_i(0) + M_4^i(\tau)},$$

with

$$\begin{pmatrix} M_1^i(\tau) & M_2^i(\tau) \\ M_3^i(\tau) & M_4^i(\tau) \end{pmatrix} = T\exp\left\{\begin{pmatrix} -\tau\Omega_{ii}^{DD} & -\int_0^\tau \tilde{\gamma}_i(\tau')\,d\tau' \\ -\tau/2 & 0 \end{pmatrix}\right\},$$

where $T\exp$ denotes the time ordered exponential.

4.2. Wishart Affine Models: $\mathcal{D} = Pos_+(r, \mathbb{R})$

Wishart stochastic processes have been firstly investigated by Bru (1991), while Gourieroux and Sufana (2003) showed their potentialities for financial and econometric applications and their ability to model multivariate intertemporal correlations.

Within the Wishart framework, factors are assumed to evolve according to the following (symmetric matrix valued) process:

$$dX_t = \left(kQ^TQ + \Omega X_t + X_t\Omega^T\right)dt + X_t^{1/2}\,dW_t Q + Q^T(dW_t)^T X_t^{1/2}, \tag{4.10}$$

with $\Omega,\ Q \in M_r(\mathbb{R})$($Q$ invertible), $W_t \in M_r(\mathbb{R})$ is composed by r^2 independent Brownian motions and $\frac{k}{2} > \frac{n}{r} - 1 = \frac{r-1}{2}$ (here $\dim(V) = n = \frac{r(r+1)}{2}$).

The infinitesimal generator of the matrix process X_t is given by (see, e.g., Bru 1991):

$$\mathcal{A} = Tr\left[(kQ^TQ + 2\Omega x)D + 2xDQ^TQD\right], \tag{4.11}$$

with $D = (\frac{\partial}{\partial x_{ij}})$. The characteristic function is

$$\Psi_X(u, x, t, \tau) = \mathbb{E}_t\left[\exp\left(-\int_t^{t+\tau}(\eta_0 + Tr[\eta X_s])\,ds\right)\exp(Tr[iuX_T])\right] \tag{4.12}$$

$$= \exp(\mathcal{V}^0(\tau, iu) - Tr[\mathcal{V}(\tau, iu)x]), \tag{4.13}$$

and the corresponding (Matrix) Riccati ODE are given by

$$-\frac{d}{d\tau}\mathcal{V}(\tau) = \eta + \Omega\mathcal{V}(\tau) + \mathcal{V}(\tau)\Omega^T - 2\mathcal{V}(\tau)Q^TQ\mathcal{V}(\tau), \quad \mathcal{V}(0) = u \in Sym(r, \mathbb{R}). \tag{4.14}$$

This class of ATSM corresponds to the choice of state space $\mathcal{D} = Pos_+(r, \mathbb{R})$, which is an irreducible symmetric cone. The corresponding (simple) EJA is obtained by considering the vector space $V = Sym(r, \mathbb{R})$, with scalar product $\langle A, B\rangle = Tr[AB]$ and Jordan product $A * B = (AB + BA)/2$(AB indicates the matrix product). Observe that the quadratic representation of this algebra is given by: $P(A)\,S = ASA$, with $S \in Sym(r, \mathbb{R})$. According to our main theorem, the model is then solvable for all $Q^TQ,\ C \in Pos_+(r, \mathbb{R})$ and $\Omega \in M_r(\mathbb{R})$ (corresponding to the linear application $H(V) = \Omega\mathcal{V} + \mathcal{V}\Omega^T$). As a consequence, the (Matrix) Riccati ODE can be written as a linear combination of the generators for the Lie algebra $\mathfrak{sl}(2r, \mathbb{R})$ (see also Lafortune and Winternitz 1996).

The equation (4.14) for $\mathcal{V}(\tau)$ is a symmetric Riccati Matrix ODE, which corresponds to a linear flow in homogeneous coordinates (the r dimensional generalization of projective spaces), so in analogy with the one dimensional case, let us put $\mathcal{V}(\tau) = G(\tau)F(\tau)^{-1}$ with $F(\tau) \in GL(r, \mathbb{R})$, $G(\tau) \in M_r(\mathbb{R})$, then we get the linear ODE:

$$-\frac{d}{d\tau}\begin{pmatrix} F(\tau) \\ G(\tau) \end{pmatrix} = \begin{pmatrix} -\Omega^T & 2Q^TQ \\ \eta & \Omega \end{pmatrix}\begin{pmatrix} F(\tau) \\ G(\tau) \end{pmatrix}, \tag{4.15}$$

which is solved by

$$\begin{pmatrix} F(\tau) \\ G(\tau) \end{pmatrix} = \exp\left[-\tau\begin{pmatrix} -\Omega^T & 2Q^TQ \\ \eta & \Omega \end{pmatrix}\right]\begin{pmatrix} F(0) \\ G(0) \end{pmatrix} =: \begin{pmatrix} M_1(\tau) & M_2(\tau) \\ M_3(\tau) & M_4(\tau) \end{pmatrix}\begin{pmatrix} F(0) \\ G(0) \end{pmatrix},$$

where $M_i(\tau)$, $i = 1, \ldots 4$ are $r \times r$ matrices. In conclusion we get:

$$\begin{aligned} \mathcal{V}(\tau) &= G(\tau)F(\tau)^{-1} \\ &= (M_1(\tau)\mathcal{V}(0) + M_2(\tau))\,(M_3(\tau)\mathcal{V}(0) + M_4(\tau))^{-1}. \end{aligned} \tag{4.16}$$

Notice that the last expression is completely explicit and does not involve any integration: it is therefore more efficient than the alternative expression one obtains using the method proposed by Walcher (1991) (see, e.g., Gourieroux and Sufana 2003).

Quite remarkably, it turns out as a by-product of our analysis that in the Wishart matrix case also the expression for $\mathcal{V}^0$ can be made explicit[2] : in fact, from the ODE for $\mathcal{V}^0$

$$\frac{d\mathcal{V}^0(\tau)}{d\tau} = kTr[Q^T Q\mathcal{V}(\tau)] + \eta_0$$

and using equation (4.16) and the first equation in system (4.15), we obtain:

$$\frac{d\mathcal{V}^0(\tau)}{d\tau} = \frac{k}{2}Tr\left[-\left(\frac{d}{d\tau}F(\tau)\right)F(\tau)^{-1} + \Omega^T\right] + \eta_0$$

$$\mathcal{V}^0(\tau) - \mathcal{V}^0(0) = -\frac{k}{2}Tr[\log F(\tau) - \Omega^T\tau] + \eta_0\tau.$$

4.3. Lorentz Cone Affine Processes

Symmetric cone domains different from the above examples, produce new solvable ATSM models. In particular, in this subsection we discuss the class of solvable models which corresponds to a Lorentz cone domain.

Let us consider the solvable ATSM with state space $\Lambda_4 \subset \mathbb{R}^4$, the Lorentz cone of four dimensional vectors:

$$\Lambda_4 = \left\{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_0 > 0, x_0 > \sqrt{x_1^2 + x_2^2 + x_3^2}\right\}.$$

For notational convenience it is customary to use the following notation:

$$x = (x_0, \overrightarrow{x}),$$

$$\overrightarrow{x} \in \mathbb{R}^3.$$

The corresponding Euclidean Jordan Algebra is defined introducing the Jordan product:

$$x * y = z = (z_0, \overrightarrow{z}),$$

with

$$z_0 = x_0 y_0 - \overrightarrow{x} \cdot \overrightarrow{y}$$

$$\overrightarrow{z} = y_0 \overrightarrow{x} + x_0 \overrightarrow{y},$$

where $\cdot$ denotes the scalar product. Notice that the above EJA admits a complex representation in terms of 2×2 symmetric matrices: in fact, if we introduce the following basis (Pauli matrices):

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

which are orthogonal with respect to the scalar product: $\langle x, y\rangle = Tr[XY]$, then the generic Lorentz vector will be represented by:

$$x \to X = \begin{bmatrix} x_0 + x_1 & x_2 - i x_3 \\ x_2 + i x_3 & x_0 - x_1 \end{bmatrix}. \tag{4.17}$$

[2] We thank J. Da Fonseca, private communication, for bringing this result to our attention.

Consistently, the Jordan product is mapped into the matrix symmetric product:

$$x * y \to \frac{XY + YX}{2};$$

in addition,

$$\det(X) = x_0^2 - \overrightarrow{x} \cdot \overrightarrow{x}$$

and accordingly we can define the Lorentz cone as follows:

$$\Lambda_4 = \{x \mid \det(X) > 0\}.$$

Finally, the quadratic form is obtained by the symmetric representation:

$$\mathbb{P}(x)a \to XAX.$$

We can define the solvable ATSM model with state space Λ_4 by specifying its conditional characteristic function: affinity implies that it will have the form:

$$\Psi_\Lambda(u, \Lambda_t, t, \tau) = \exp(\mathcal{W}(\tau, iu) - Tr[\mathcal{V}(\tau, iu), \Lambda_t]), \tag{4.18}$$

which is completely determined by the solution for $\mathcal{V}, \mathcal{W}$ and the initial values of the factors $\Lambda_t \in \Lambda_4$.

The general expression of the ODE corresponding to the solvable model can be determined directly from the above EJA structure applying Theorem 2.2. In fact:

(i) the constant part will be defined by introducing $a^0 \in \Lambda_4$;
(ii) the linear part is obtained observing that the choice:

$$H(\mathcal{V}) = H^T(\mathcal{V}) = A^1\mathcal{V} + \mathcal{V}(A^1)^\#$$

$$A^1 = \begin{bmatrix} a+b & c-id \\ e+if & a-b \end{bmatrix},$$

$$(a, b, c, d, e, f) \in \mathbb{R}^6$$

(where the superscript$^\#$ indicates the Hermitian conjugate) verifies the condition of Theorem 2.2;
(iii) the covariance term is given by the quadratic representation of the EJA:

$$P(\mathcal{V})a^2, \qquad a^2 \in \Lambda_4.$$

In the symmetric matrix representation the Riccati ODE will be:

$$-\frac{d}{d\tau}\mathcal{V} = A^0 + A^1\mathcal{V} + \mathcal{V}(A^1)^\# - \frac{1}{2}P(\mathcal{V})A^2$$

$$\mathcal{V}(0) = iu.$$

Notice that the linearization procedure will work also in this case, allowing to determine explicitly the expression of $\mathcal{V}$ as $\mathcal{V}(\tau) = G(\tau)F(\tau)^{-1}$ satisfying the following linear ODE:

$$-\frac{d}{d\tau}\begin{pmatrix} F(\tau) \\ G(\tau) \end{pmatrix} = \begin{pmatrix} -(A^1)^\# & A^2/2 \\ A^0 & A^1 \end{pmatrix}\begin{pmatrix} F(\tau) \\ G(\tau) \end{pmatrix}.$$

Given the solution for the quadratic ODE, we can apply the same procedure used for the Wishart example, thus obtaining also the explicit term $\mathcal{W}(\tau)$:

$$\mathcal{W}(\tau) - \mathcal{W}(0) = -\frac{k}{2} Tr[\log F(\tau) - (A^1)^{\#}\tau] + A^0\tau.$$

Finally, k is constrained by the condition:

$$\frac{k}{2} > \frac{n}{r} - 1 = \frac{4}{2} - 1 = 1,$$

in fact in this case the EJA has rank 2.

This completes the definition of the conditional characteristic function of a diffusion process evolving in a Lorentz state space Λ_4 and whose infinitesimal generator is given by:

$$\mathcal{A} = Tr[(k(A^2)^{\#} A^2 + 2(A^1)^{\#} X)D + 2XD(A^2)^{\#} A^2 D],$$

where X is given by (4.17) and $D = (\frac{\partial}{\partial x_{ij}}),\ i, j = 1, 2$. At present it is under investigation whether the state space of this model can be useful as factor's space for some financial model.

APPENDIX

Proof of Theorem 2.1

Under the assumption that $\Sigma_\alpha(x)$ is non-singular $\forall x \in \mathcal{D}$, it is immediate to observe that the linearity of $\Sigma_\alpha(x)$ with respect to the argument x implies the relation:

$$\langle x, \alpha(\mathcal{V}^2)\rangle = Tr[\Sigma_\alpha(x)\mathcal{V}\mathcal{V}^T] \quad \forall x \in \mathcal{D}$$

which defines uniquely a vector $\alpha(\mathcal{V}^2)$. This vector belongs to $\mathcal{D}$, since $Tr[\Sigma_\alpha(x)\mathcal{V}\mathcal{V}^T] > 0$ for $x \in \bar{\mathcal{D}}/\{0\}$, and the domain is self dual. Now we claim that admissibility implies:

$$\alpha(\mathcal{V}^2) = P(\mathcal{V})\,\alpha_\Sigma, \tag{A.1}$$

where $P(\mathcal{V})$ is the (unique up to a proportionality factor) quadratic representation of the EJA corresponding to the irreducible symmetric cone $\mathcal{D}$ and $\alpha_\Sigma \in \mathcal{D}$.

Consider a coordinate transformation $x \to y = g\,x$ where $g \in G(\mathcal{D})$, the group of linear automorphism of $\mathcal{D}$ defined by

$$G(\mathcal{D}) = \{g \in GL(V) \mid g\mathcal{D} = \mathcal{D}\}.$$

Affinity and admissibility imply that the conditional covariance will transform like:

$$\Sigma_\alpha(x) = (g)^{-1}\Sigma_\alpha(gx)(g^T)^{-1} \quad \forall x \in \mathcal{D},\ \forall g \in G(\mathcal{D}),$$

so that

$$Tr[\Sigma_\alpha(x)\mathcal{V}\mathcal{V}^T] = Tr\big[\Sigma_\alpha(gx)((g^T)^{-1}\mathcal{V})((g^T)^{-1}\mathcal{V})^T\big],$$

which in turn implies

$$\langle x, \alpha(\mathcal{V}^2)\rangle = \langle gx, \alpha((g^T)^{-1}\mathcal{V}^2)\rangle = \langle x, g^T\alpha((g^T)^{-1}\mathcal{V}^2)\rangle.$$

Now let us define $\alpha_\Sigma = \alpha(e)$, where e is the identity element of the EJA: applying the previous relation with $\mathcal{V} = e$ yields

$$\left\langle x, g^T \alpha((g^T)^{-1} e) - \alpha_\Sigma \right\rangle = 0 \quad \forall x \in \mathcal{D}, \quad \forall g \in G(\mathcal{D}).$$

Since $P(e) = Id$, using (2.5) we get $g^T P((g^T)^{-1} e) g = Id \ \forall g \in G(\mathcal{D})$ and

$$\left\langle x, g^T \alpha((g^T)^{-1} e) - g^T P((g^T)^{-1} e) g \alpha_\Sigma \right\rangle = 0 \quad \forall x \in \mathcal{D}, \quad \forall g \in G(\mathcal{D}),$$

and applying the above equality for a system $x = e_i \ \in \mathcal{D}, i = 1, \dots r$, of orthogonal idempotents for the EJA (which generate the whole EJA V), we can conclude that $\forall g \in G(\mathcal{D})$

$$\alpha((g^T)^{-1} e) = P((g^T)^{-1} e) \, g \alpha_\Sigma .$$

Observe that $P(u) \in G(\mathcal{D})$ for $u \in \mathcal{D}$, therefore we can put $g = P(u)^{-1}$ in the above equality. By symmetry we obtain $(g^T)^{-1} = P(u) \in G(\mathcal{D})$ and using the fundamental property (2.5) we get

$$\alpha\left(P(u) e\right) = P(u) \alpha_\Sigma \quad \forall u \in \mathcal{D}.$$

Using the fact that $P(\mathcal{V})_e = \mathcal{V}^2$ we can conclude that (A.1) holds true. □

Proof of Proposition 2.2

In order to prove Proposition 2.2, first we provide an essential review of the KKT (Kantor, Koecher, and Tits) construction and its relationship with the ATSM problem. An extensive discussion of the geometric and algebraic aspects of these topics can be found in Faraut and Korányi (1994), Walcher (1991, 1986).

Let $C \subset V$ be a symmetric cone; observe that $G(C)$ is a Lie subgroup of $G(T_C)$, the group of holomorphic automorphisms of the "tube domain" $T_C = V + iC$.

The whole group of affine linear transformations on the tube domain, $z \to gz + a$, $g \in G(C)$, $a \in V$, can be obtained as the semidirect product of translations $N^+ = \{a \in V \mid z \to z + a \ \forall z \in T_C\}$ and automorphisms in $G(C)$. Adding the inversion $J{:}z \to -z^{-1}$, the whole group $G(T_C)$ is generated.

Let $\mathbf{g}(T_C)$ be the Lie algebra of $G(T_C)$: each curve $\{g_t^{\mathcal{R}}\}_{t \in \mathbb{R}_+} \subset G(T_C)$ defines a vector field $\mathcal{R}(u)$ on V through the relation:

$$Df(z)\mathcal{R}(z) := \frac{d}{dt} f\left(g_t^{\mathcal{R}}(z)\right)_{|t=0} \quad \forall f \in \mathcal{C}^1(T_C).$$

For example, consider the subgroup $N^- = J \circ N^+ \circ J$ with elements $g_t^{N^-}(z) = (z^{-1} - tv)^{-1}$ when $V = \mathbb{R}$, $t \in \mathbb{R}^+$. Then direct computation shows:

$$\frac{d}{dt} f((z^{-1} - tv)^{-1})_{|t=0} = f'(z) z^2 v,$$

thus giving $\mathcal{R}_{-1}(z) = z^2 v = P(z) v$.

The Lie algebra of analytic maps generated by N^+, $G(C)$, N^- closes in the so-called KKT (finite dimensional Lie) algebra $\mathbf{g}(T_C)$:

Proposition A.1 (see Faraut and Korányi 1994, p. 208). $\mathbf{g}(T_C)$ *can be represented as follows:*

$$\mathbf{g}(T_C) = \mathbf{g}_{-1}(T_C) \oplus \mathbf{g}_0(T_C) \oplus \mathbf{g}_{+1}(T_C),$$

where (i) $\mathbf{g}_{-1}(T_C) = \{\mathcal{R}_-(z) = -P(z)v; z \in V, v \in V\}$ *generates the subgroup* N^- *with elements* $g_t^{N^-}(z) = (z^{-1} - tv)^{-1}$; (ii) $\mathbf{g}_0(T_C) = \{\mathcal{R}_0(z) = Tz : z \in V, T \in \mathbf{g}(C)\}$ *generates* $G(C)$ given by:

$$\mathbf{g}(C) = \{H \in M_n : 2P(\mathcal{V}, H\mathcal{V}) = HP(\mathcal{V}) + P(\mathcal{V})H^T\};$$

(iii) $\mathbf{g}_{+1}(T_C) = \{\mathcal{R}_+(z) = u, u \in V\}$ *generates* N^+, i.e., $g_t^{N^+}(z) = z + tu$.

Among the elements of $\mathbf{g}(T_C)$, we will be interested in those which generate subgroups preserving the domain C, i.e., $g_t(C) \subseteq C\ \forall t \in \mathbb{R}_+$; remarkably the following proposition provides the general expression of the Lie generators for a *compression* semigroup, i.e., a one parameter semigroup which leaves the symmetric cone C invariant.

PROPOSITION A.2 (see, e.g., section 7 of Lawson and Lim 2000). *Let* $G(T_C)$ *be the group of holomorphic automorphisms and* $\mathbf{g}(T_C)$ *its Lie algebra. Consider the semigroup of compressions* $\Gamma_C \subseteq G(\Gamma_C)$:

$$\Gamma_C := \{g \in G(T_C) : gC \subseteq C\}.$$

Each $g \in \Gamma_C$ *is generated by* $\mathcal{R} \in \mathbf{g}(T_C)$ *of the form:*

$$\mathcal{R}(u) = -P(u)a + Tu + b \quad \forall u \in T_C \qquad \text{(A.2)}$$
$$a, b \in C, \quad T \in \mathbf{g}_0(T_C).$$

Given the above results we can now prove Proposition 2.2.

Proof of Proposition 2.2. Following the argument of Proposition 7.4 in Duffie et al. (2003), an admissible regular ATSM exists provided that:

(i) the characteristic function of the Markov semigroup is well defined and
(ii) the flow of the Riccati ODE arising in (2.4) exists and is uniquely globally defined $\forall t \geq t_0,\ \forall x \in \mathcal{D}$.

Now if $\Omega_0 = ka_\Sigma$, with $k > \frac{n}{r} - 1$, where r is the rank of the EJA, then condition (i) is satisfied from theorem VII.1.3 in Faraut and Korányi (1994), while the global existence condition concerning the flow of the Riccati comes from sections 6 and 7 of Lawson and Lim (2000), where they provide the expression of the compression semigroup generators (A.2) for a generic symmetric cone. Repeating the arguments of Lawson and Lim (2006), we can conclude the global existence of the flow of the Riccati. □

Proof of Theorem 2.8

An important by-product of the KKT construction is the classification of all possible quadratic finite dimensional algebras, and in particular the linearizable Riccati ODE. Let us now briefly recall this result in order to prove our Theorem 2.8.

Let us consider a particular subalgebra of $A(V, V)$, namely, the (infinite dimensional) algebra of polynomial functions $Pol(V)$. We can define $P_k \subset Pol(V)$ as the subspace of homogeneous polynomials of degree k, for $k \geq 0$. Since $[P_j, P_k] \subset P_{j+k}$ for all $j, k \in \mathbb{N}$, it follows that

$$Pol(V) = \oplus_{k \in \mathbb{N}} P_k,$$

and it is a *graded subalgebra* of $A(V, V)$ (see Walcher 1991, p. 118). For example, the subspace P_0 contains all constant maps and can be identified with V, while P_1 can be identified with $Hom(V, V)$.

According to our definition of solvability and to Chapter 8 of Walcher (1991), the family L of analytic maps of the Riccati ODE in an EJA has to close in a finite dimensional subalgebra of $A(V, V)$, and in particular in a graded subalgebra of the type

$$\bar{L} = L_0 \oplus L_1 \oplus L_2,$$

where $L_0 = V$ (i.e., the algebra is transitive), $L_i \subset P_i$, $i = 1, 2$.

The following theorem shows that the classification of these subalgebras is easily related with the Kantor–Koecher–Tits class of algebras $\mathbf{g}(T_C)$:

THEOREM A.1 (Walcher 1991, theorem 8.6 b). *Let $\bar{L} = L_0 \oplus L_1 \oplus L_2$ be a transitive subalgebra of $Pol(V)$. Then $\bar{L}$ is a subalgebra of a Kantor–Koecher–Tits algebra* $\mathbf{g}(T_C)$ with $L_i \subset \mathbf{g}_{i-1}(T_C)$, $i = 0, 1, 2$.

In other terms, the KKT construction classifies all possible quadratic finite dimensional subalgebras, and in particular (the vector fields of) the linearizable Riccati ODE.

Proof of Theorem 2.8. Each admissible ATSM in an irreducible symmetric cone generates a quadratic polynomial ODE. Solvability requires that the set $f_i(\mathcal{V}) \in P_i$, $i = 0, 1, 2$ generate a graded transitive subalgebra, say $\bar{L}$. From the previous theorem, $\mathbf{g}(T_C)$ is maximal among graded transitive subalgebras, hence $\bar{L} \subseteq g(T_C)$. On the other hand, since $\bar{L}$ must contain the constant vector fields, we have that $V = \mathbf{g}_{-1}(T_C) \subset \bar{L}$, and by Theorem A.1 above $\bar{L}$ must also contain $\mathbf{g}_{+1}(T_C) = P(\mathcal{V})a$, $a \in V$. Now we can conclude that $\bar{L} \supseteq \mathbf{g}(T_C)$ since $\mathbf{g}_0(T_C) \subseteq [\mathbf{g}_{-1}(T_C), \mathbf{g}_{+1}(T_C)]$. Hence $\bar{L} = \mathbf{g}(T_C)$ and the theorem is proved. □

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