International Journal of Theoretical and Applied Finance
Vol. 20, No. 1 (2017) 1750007 (29 pages)

DOI: 10.1142/S0219024917500078



# COHERENT FOREIGN EXCHANGE MARKET MODELS

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Received 6 January 2016
Accepted 3 January 2017
Published 3 February 2017

A model describing the dynamics of a foreign exchange (FX) rate should preserve the same level of analytical tractability when the inverted FX process is considered. We show that affine stochastic volatility models satisfy such a requirement. Such a finding allows us to use affine stochastic volatility models as a building block for FX dynamics that are functionally-invariant with respect to the construction of suitable products/ratios of rates, thus generalizing the model of [A. De Col, A. Gnoatto & M. Grasselli (2013) Smiles all around: FX joint calibration in a multi-Heston model, *Journal of Banking and Finance* **37** (10), 3799–3818.].

*Keywords*: FX options; foreign-domestic parity; affine processes.

## 1. Introduction

The foreign exchange (FX) market is the largest and most liquid financial market in the world. According to the Bank of International settlements (see Mallo 2010), the daily global FX market volume (or turnover) in 2010 was about USD 3981 bn. This large amount breaks down into spot transactions (1490 bn), FX swaps (1765 bn) and FX options (207 bn). These figures give an idea of the relevance of the market for FX products and FX options in particular and hence provide a reasonable grounding for questions focusing on it (see also Carr & Wu 2009).

When we look at the market for FX options we observe phenomena that may be summarized in two main categories: (a) stylized facts regarding the underlying securities and (b) stylized facts regarding the FX implied volatility. As far as the first category is concerned, we have that, unlike, e.g. in the equity market, we may consider both the underlying and its inverse. To be more specific, if $S$ denotes the EURUSD exchange rate (which is the price in dollars of 1 euro), the reciprocal of $S$, $1/S$, denotes then the USDEUR exchange rate, i.e. the price in Euros of 1 dollar. More generally, this kind of reasoning may be further extended as we will see, and hence suitable ratios/products of exchange rates are still exchange rates. This key feature of exchange rates has an implication on the set of requirements that a realistic model should satisfy. In particular, in the simple two-economy case, assuming that certain stochastic dynamics for the exchange rate $S$ have been postulated, it is not *a priori* clear if the process for the inverted exchange rate process $1/S$ shares the same level of analytical tractability of the original process for $S$.

Concerning the second problem, when we look at implied volatility of FX options we observe different values for different levels of moneyness/maturity, which are summarized in the so-called volatility surface. Extending our views to multiple currencies, and hence by looking at a variance covariance matrix of currencies, we observe that both variances and covariances are stochastic, pointing out the need for a model in which not only we have stochastic volatility on single exchange rates, but also stochastic correlations among them. An example of this phenomenon is visualized in Fig. 1, where we perform a very simple estimation on rolling windows of the variance covariance matrix of two liquid exchange rates EURUSD and EURJPY.

Combining the two categories of stylized facts above into a single model that is at the same time coherent under the inversion of the currency, while able to

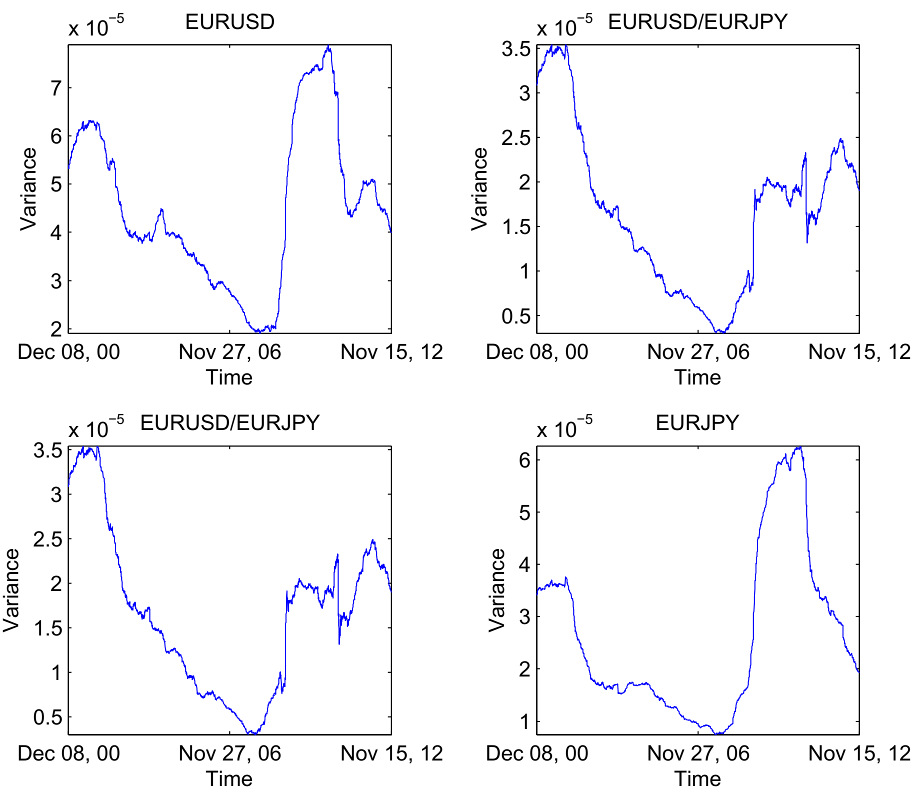


Fig. 1. Time series of variance and covariances for EURUSD and EURJPY. Estimation performed using a rolling sample of 500 data points on time series with daily frequency.

capture the features of the smile, is a nontrivial task. Ever since the contribution of Garman & Kohlhagen (1983), which represents an adaptation of the Black & Scholes (1973) model, it has become quite common to take a model initially imagined for, e.g. stock options, and employ it, with minor adjustments, for the evaluation of FX options. An example in this sense is the model of Heston (1993), whose FX adaptation is discussed in many references, e.g. Clark (2011), Janek *et al.* (2011). Other models that are employed in an FX setting are, e.g. Stein & Stein (1991) or Hull & White (1987), see the account in Lipton (2001). Our aim in the present paper is to identify a class of stochastic processes for the evolution of exchange rates that: (a) allows for the realistic description of stochastic volatility/correlation effects and discontinuous paths and (b) is closed under inversion or, more generally, under suitable products/ratios of processes.

This paper is outlined as follows: in Sec. 2, we introduce our main assumptions. In Sec. 3, we present our main result: we consider the class of affine stochastic volatility models, as introduced by Keller-Ressel (2011) and show that the process for the inverted exchange rate is still an affine process that is as analytically tractable as the starting model. Given this result, we can then look at more advanced situations and then consider triangles or more general geometric structures of FX rates. The idea of Sec. 4 is to use the previous findings in the two economy case, in order to present an example of model that is functionally symmetric under suitable product/ratios among exchange rates. The model is an extension of the multifactor stochastic volatility model introduced by De Col *et al.* (2013).

## 2. The Setting

### 2.1. *Basic traded assets and coherent models*

We specify a general market setting consisting of two economies. We assume the existence of a risk-free money market account for each currency area. We denote by $B^d(t), B^f(t)$ the domestic and the foreign money market accounts, respectively, which are solutions to the ODEs

$$dB^i(t) = r^i dt, \quad B^i(0) = 1, \quad i = d, f, \tag{2.1}$$

where we assume, for the sake of simplicity, that $r^i$, $i = d, f$ are real valued constants.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q}_i)$ $i = d, f$ be a filtered probability space, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions. We also let $\mathcal{F}_0$ be the trivial sigma algebra. Let us postpone for the moment the treatment of the family of probability measures $\mathbb{Q}_i$, $i = d, f$. On this probability space we will be considering in general two stochastic processes: $S = (S(t))_{t\geq 0}$ and $S^{-1} = (S^{-1}(t))_{t\geq 0}$. $S$ will be employed to model the FX rate in the usual FORDOM convention, i.e. if $S$ is a model for EURUSD and $S = 1.30$, then we say that one Euro is worth 1.30 dollars. In a dual way, we let $S^{-1}$ be a model for the USDEUR exchange rate, thus capturing the point of view of a European investor. Given the processes defined above, agents from the two economies may trade the following assets:

- The domestic agent may trade
  (1) In the domestic money market account, $B^d = (B^d(t))_{t\geq 0}$;
  (2) In the foreign money market account, $\tilde{B}^f = (B^f(t)S(t))_{t\geq 0}$.
- The foreign agent may trade
  (1) In the foreign money market account, $B^f = (B^f(t))_{t\geq 0}$;
  (2) In the domestic money market account, $\tilde{B}^d = (B^d(t)S^{-1}(t))_{t\geq 0}$.

We now concern ourselves with the viability of the market setting above. To this end, we introduce the following notation: whenever they exist, we denote via

- $\mathbb{Q}_d$ the probability measure such that

$$\mathbb{E}^{\mathbb{Q}_d}\left[\left.\frac{\tilde{B}^f(T)}{B^d(T)}\right|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_d}\left[\left.\frac{B^f(T)S(T)}{B^d(T)}\right|\mathcal{F}_t\right] = \frac{B^f(t)S(t)}{B^d(t)}, \tag{2.2}$$

  and call it domestic risk neutral measure;
- $\mathbb{Q}_f$ the probability measure such that

$$\mathbb{E}^{\mathbb{Q}_f}\left[\left.\frac{\tilde{B}^d(T)}{B^f(T)}\right|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_f}\left[\left.\frac{B^d(T)S^{-1}(t)}{B^f(T)}\right|\mathcal{F}_t\right] = \frac{B^d(t)S^{-1}(t)}{B^f(t)}, \tag{2.3}$$

  and call it foreign risk neutral measure.

We now introduce the following assumption.

**Assumption 2.1.** We assume that a $\mathbb{Q}_d$-risk neutral measure exists, i.e. we assume that the process

$$\mathcal{Z} = (\mathcal{Z}(t))_{t\geq 0} := \left(\frac{B^f(t)S(t)}{S(0)B^d(t)}\right)_{t\geq 0} \tag{2.4}$$

is a true $\mathbb{Q}_d$-martingale with $\mathcal{Z}(0) = 1$.

Under this assumption, we have that the market model we are considering is free of arbitrage. See Definition 9.2.8 and Theorem 14.1.1 in Delbaen & Schachermayer (2005). In general, we will not assume that $\mathbb{Q}_d$ is unique, as we will be concerned with stochastic volatility models possibly featuring jumps, which provide typical examples of incomplete markets. In such a setting the particular measure $\mathbb{Q}_d$ will be determined as the result of a calibration to market data.

A direct consequence of Assumption 2.1, is that the process $\mathcal{Z}$ may be employed so as to define the density process of the risk neutral measure $\mathbb{Q}_f$. More explicitly, we have

$$\left.\frac{\partial\mathbb{Q}_f}{\partial\mathbb{Q}_d}\right|_{\mathcal{F}_t} = \frac{S(t)B^f(t)}{S(0)B^d(t)}, \tag{2.5}$$

(see Theorem 1 in Geman *et al.* 1995). The process above is the change of measure that is found in the classical literature on FX markets, for example in Sec. 2.9

of Brigo & Mercurio (2006). While this is a well established fact, we would like to underline, by means of the next examples, that the change of measure above may introduce significant changes in the model specification under different pricing measures.

**Example 2.1.** Let us consider the GARCH stochastic volatility model, which is studied in depth in Lewis (2000). The dynamics under the domestic risk neutral measure are given by

$$\begin{aligned} \frac{dS(t)}{S(t)} &= (r^d - r^f)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_d}(t), \\ dV(t) &= (\omega - \alpha V(t))dt + V(t)(\rho dW^{1,\mathbb{Q}_d}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)). \end{aligned} \tag{2.6}$$

The density process between the foreign and the domestic risk neutral measure is given by

$$\left.\frac{\partial \mathbb{Q}_f}{\partial \mathbb{Q}_d}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \sqrt{V(s)}dW_s^1 - \frac{1}{2}\int_0^t V(s)ds\right\}. \tag{2.7}$$

Note that this change of measure, which is typical for stochastic volatility models with nonzero correlations, implies that the second Brownian motion $W^2$ is not affected by the measure change, so $W^{2,\mathbb{Q}_f} = W^{2,\mathbb{Q}_d}$ (see Lewis (2000), p. 257). Under the foreign risk neutral measure the dynamics of the inverse exchange rate are now given by (compare with Lewis (2000), Eq. (3.6) p. 257)

$$\begin{aligned} \frac{dS^{-1}(t)}{S^{-1}(t)} &= (r^f - r^d)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_f}(t), \\ dV(t) &= (\omega - \alpha V(t) + \rho V(t)^{3/2})dt + V(t)(-\rho dW^{1,\mathbb{Q}_f}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_f}(t)), \end{aligned} \tag{2.8}$$

which is not a GARCH stochastic volatility model.

**Example 2.2.** Let us consider the Hull-White stochastic volatility model

$$\begin{aligned} \frac{dS(t)}{S(t)} &= (r^d - r^f)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_d}(t), \\ dV(t) &= \mu V(t)dt + \xi V(t)(\rho dW^{1,\mathbb{Q}_d}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)). \end{aligned} \tag{2.9}$$

Also, in this case, we have a pathological situation: under the foreign risk neutral measure the variance will not follow a geometric Brownian motion as under the starting measure

$$\begin{aligned} \frac{dS^{-1}(t)}{S^{-1}(t)} &= (r^f - r^d)dt + \sqrt{V(t)}dW^{1,\mathbb{Q}_f}(t), \\ dV(t) &= (\mu V(t) + \xi V(t)^{3/2}\rho)dt + \xi V(t)(-\rho dW^{1,\mathbb{Q}_f}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_f}(t)). \end{aligned} \tag{2.10}$$

By proceeding along the same lines, it is also possible to show that the SABR stochastic volatility model suffers from the same lack of symmetry. On the contrary, when we consider the stochastic volatility model of Heston (1993), we observe that the structure of the model is instead preserved, as shown in Del Baño Rollin (2008). More specifically, under the measure $\mathbb{Q}_d$, the dynamics of the instantaneous variance factor of $S$ are those of a square root process

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}(\rho dW^{1,\mathbb{Q}_d}(t) + \sqrt{1-\rho^2}dW^{2,\mathbb{Q}_d}(t)), \quad (2.11)$$

for $\sigma > 0, \kappa \in \mathbb{R}$, s.t. $\theta\kappa > 0$ whereas, under the measure $\mathbb{Q}_f$, the instantaneous variance of the inverted exchange rate is still a square root process where the parameters under $\mathbb{Q}_f$ are modified as follows:

$$\kappa^{\mathbb{Q}_f} = \kappa - \sigma\rho, \quad \theta^{\mathbb{Q}_f} = \frac{\kappa\theta}{\kappa^{\mathbb{Q}_f}}, \quad \rho^{\mathbb{Q}_f} = -\rho. \quad (2.12)$$

In view of the examples above we may say that an FX model is **coherent**,[a] if the processes for $S$ and $1/S$ belong to the same class.

### 2.2. *The foreign-domestic parity*

We can further extend the market setting by introducing European options written both on the exchange rate and its inverse. More specifically

**Assumption 2.2.** The domestic and the foreign agent may create positions in European call/put options written on $S$ and $S^{-1}$.

It is well known, see, e.g. Carr & Madan (1999) or Lewis (2000), that option prices are intimately linked to characteristic functions of log-prices. To this end, let us define, for fixed $t \geq 0$ and all $v \in \mathbb{C}$ such that the expectations exist, the characteristic functions

$$\begin{aligned} &\varphi_i : \mathbb{C} \to \mathbb{C}, \quad i = d, f, \\ &\varphi_d(v) := \mathbb{E}^{\mathbb{Q}_d}[e^{\mathtt{i}v \log S(t)}], \quad v \in \mathbb{C}, \\ &\varphi_f(v) := \mathbb{E}^{\mathbb{Q}_f}[e^{\mathtt{i}v \log S^{-1}(t)}], \quad v \in \mathbb{C}, \end{aligned} \quad (2.13)$$

where $\mathtt{i}$ denotes the imaginary unit. Let us define $F(t) := S(0)e^{(r^d - r^f)t}$. From Assumption 2.1, we can obtain the next result.

**Proposition 2.1.** *Under Assumption* 2.1 *the characteristic functions* $\varphi_d, \varphi_f$ *obey the relation*

$$\varphi_f(u) = F(t)^{-1}\varphi_d(-u - \mathtt{i}). \quad (2.14)$$

*for* $u \in \mathbb{R}$.

[a]We acknowledge that the term "coherent FX model" has already been used, with a different definition, in Brody & Hughston (2004).

**Proof.** Using the definitions of $\varphi_d, \varphi_f$ in (2.13), coupled with Assumption 2.1 and the Bayes rule, allows us to write

$$\begin{aligned}\varphi_f(u) &= \mathbb{E}^{\mathbb{Q}_f}[e^{\mathtt{i}u \log S^{-1}(t)}] \\ &= \frac{1}{Z(0)}\mathbb{E}^{\mathbb{Q}_d}[Z(t)e^{\mathtt{i}u \log S^{-1}(t)}] \\ &= \frac{B^d(0)B^f(t)}{S(0)B^f(0)B^d(t)}\mathbb{E}^{\mathbb{Q}_d}[e^{\mathtt{i}(-u-\mathtt{i}) \log S(t)}] \\ &= F(t)^{-1}\varphi_d(-u-\mathtt{i}).\end{aligned} \tag{2.15}$$

The finiteness of $\varphi_d(-u-\mathtt{i})$ is guaranteed by the martingale property of the density process $\mathcal{Z}$, hence the proof is complete. □

When the basic market model is enriched with the above derivative securities, we can investigate the foreign domestic parity, which is a no-arbitrage relationship, linking call options written on an FX rate to put options on the inverse FX rate (see Eberlein *et al.* 2008 and Eberlein *et al.* 2009). By foreign domestic parity, we mean the relation

$$\mathcal{CALL}(S(0), K, r^d, r^f, T) = S(0)K\mathcal{PUT}\left(\frac{1}{S(0)}, \frac{1}{K}, r^f, r^d, T\right). \tag{2.16}$$

To get an understanding of the relation we follow Wystup (2006) and take as an example a call on EURUSD (recall that EURUSD is quoted in FORDOM terms, meaning that we are looking at the dollar value of one euro and so we take the perspective of an American investor). The payoff $(S(T) - K)^+$ is worth $\mathcal{CALL}(S(0), K, r^d, r^f, T)$ for the American investor, hence $\mathcal{CALL}(S(0), K, r^d, r^f, T)/S(0)$ euros. This EUR-call may be viewed as the payoff $K(K^{-1} - S^{-1}(t))^+$. This payoff for a European investor is worth $K\mathcal{PUT}(S^{-1}(0), K^{-1}, r^f, r^d, T)$. Absence of arbitrage tells us that the two values must agree. As a Corollary to Proposition 2.1, we can state a corrected proof of the following result, originally proved by Del Baño Rollin (2008).

**Corollary 2.1.** *Under Assumption* 2.1 *then the foreign domestic parity holds.*

**Proof.** See Appendix. □

In the next section, we return to our main object of investigation, namely the identification of a wide class of models that are coherent.

## 3. Affine Stochastic Volatility Models

We consider affine stochastic volatility models, in the sense of Keller-Ressel (2008), Keller-Ressel (2011). More precisely, we consider an asset price $S = (S(t))_{t\geq 0}$ of the form

$$S(t) = \exp\{(r^d - r^f)t + X(t)\} \quad t \geq 0, \tag{3.1}$$

so that $X = (X(t))_{t\geq 0}$ is a model for the discounted stock log-price process. We let $V = (V(t))_{t\geq 0}$ be another process, with $V_0 > 0$ a.s., which may represent the instantaneous variance of $(X(t))_{t\geq 0}$ or may control the arrival rate of its jumps. assume that the joint process $(X, V) = (X(t), V(t))_{t\geq 0}$ satisfies Assumptions **A1, A2, A3, A4** in Keller-Ressel (2011) and call it affine stochastic volatility model. More precisely we assume that

- **A1** $(X, V)$ is a stochastically continuous, time-homogeneous Markov process with state space $\mathbb{R} \times \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} := [0, \infty)$,
- **A2** The cumulant generating function of $(X(t), V(t))$ is of a particular affine form: there exist functions $\phi(t, v, w)$ and $\psi(t, v, w)$ such that
$$\log \mathbb{E}^{\mathbb{Q}_d}[\exp(vX(t) + wV(t))] = \phi(t, v, w) + \psi(t, v, w)V_0 + vX_0 \tag{3.2}$$
for all $(t, v, w) \in \mathcal{U}$ for $\mathcal{U}$ defined as
$$\begin{aligned}\mathcal{U} := \{(t, v, w) \in \mathbb{R}_{\geq 0} \times \mathbb{C}^2 \mid & \mathbb{E}^{\mathbb{Q}_d}[|\exp(vX(t) + wV(t))|] \\ &= \mathbb{E}^{\mathbb{Q}_d}[\exp(\Re(v)X(t) + \Re(w)V(t))] < \infty\}.\end{aligned} \tag{3.3}$$

Assumptions **A1, A2** make the process $(X(t), V(t))_{t\geq 0}$ affine in the sense of Definition 2.1 in Duffie *et al.* (2003). For our purposes, this implies that the characteristic function of the logarithmic exchange rate is of a particularly nice form
$$\varphi_d(u) = \mathbb{E}^{\mathbb{Q}_d}[e^{\mathrm{i}u \log S(t)}] = e^{\mathrm{i}u(r^d - r^f)t + \phi(t,\mathrm{i}u,0) + V_0\psi(t,\mathrm{i}u,0) + X_0\mathrm{i}u} \tag{3.4}$$
for $u \in \mathbb{R}$. The functions $\phi, \psi$ may be easily characterized. In fact, from Theorem 2.1 in Keller-Ressel (2011), we know that $\phi, \psi$ satisfy, for $(t, v, w) \in \mathcal{U}$, the generalized Riccati equations:
$$\begin{aligned}\frac{\partial}{\partial t}\phi(t, v, w) &= F(v, \psi(t, v, w)), \quad \phi(0, v, w) = 0, \\ \frac{\partial}{\partial t}\psi(t, v, w) &= R(v, \psi(t, v, w)), \quad \psi(0, v, w) = w.\end{aligned} \tag{3.5}$$

The results presented in Duffie *et al.* (2003) imply that the RHS in the system of ODE above, i.e. the functions $F(v, \psi(t, v, w))$, and $R(v, \psi(t, v, w))$ are of Lévy–Khintchine form, i.e.
$$\begin{aligned}F(v, \psi) &= (v, \psi)\frac{a}{2}\begin{pmatrix} v \\ \psi \end{pmatrix} + b\begin{pmatrix} v \\ \psi \end{pmatrix} \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - v\frac{x}{1 + x^2}\right) m(dx, dy), \\ R(v, \psi) &= (v, \psi)\frac{\alpha}{2}\begin{pmatrix} v \\ \psi \end{pmatrix} + \beta\begin{pmatrix} v \\ \psi \end{pmatrix} \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} \left(e^{vx+\psi y} - 1 - v\frac{x}{1 + x^2} - \psi\frac{y}{1 + y^2}\right) \mu(dx, dy).\end{aligned} \tag{3.6}$$

Moreover the set of parameters $(a, \alpha, b, \beta, m, \mu)$ satisfy the admissibility conditions

- $a, \alpha$ are positive semi-definite $2 \times 2$ matrices with $a_{12} = a_{21} = a_{22} = 0$,
- $b \in \mathbb{R} \times \mathbb{R}_{\geq 0}$, $\beta \in \mathbb{R}^2$,
- $m, \mu$ are Lévy measures on $\mathbb{R} \times \mathbb{R}_{\geq 0}$, such that

$$\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} ((x^2 + y) \wedge 1) m(dx, dy) < \infty. \tag{3.7}$$

To gain an intuition on the role of the parameters, we observe that $F$ and $R$ represent respectively the constant and the state-dependent characteristics of the vector process $(X(t), V(t))_{t\geq 0}$. More precisely $a + \alpha V(t)$ is the instantaneous covariance matrix, $b + \beta V(t)$ is the drift and $m + \mu V(t)$ is the Lévy measure.

We will consider a simplified version of the system of ODEs (3.5) and (3.6). In particular, we will assume that we have at most jumps of finite variation for the positive (variance) component. Moreover, having applications in mind, we parametrize the linear diffusion coefficient $\alpha$ by means of a coefficient $\rho \in [-1, 1]$ and we include, in the jump transform, the term coming from the martingale condition for the asset price. Finally, we will assume $b_1 = -\frac{a_{11}}{2}$ and $\beta_1 = -\frac{\alpha_{11}}{2}$, which correspond to the condition $F(1, 0) = R(1, 0) = 0$ in Theorem 2.5 in Keller-Ressel (2011). In summary, we will be assuming that the process is conservative and is a martingale, and also $R(u, 0) \neq 0$ for some $u \in \mathbb{R}$, which excludes models where the distribution of the log asset price does not depend on the variance. These are Assumptions **A3, A4** in Keller-Ressel (2011). The functions $F, R$ are then of the form

$$\begin{aligned} F(v,\psi) &= v^2\frac{a_{11}}{2} + b\begin{pmatrix} v \\ \psi \end{pmatrix} \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{vx+\psi y} - 1 - vx) - v(e^x - 1 - x) m(dx, dy), \\ R(v,\psi) &= \frac{1}{2}\begin{pmatrix} v & \psi \end{pmatrix}\begin{pmatrix} \alpha_{11} & \rho\sqrt{\alpha_{11}\alpha_{22}} \\ \rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22} \end{pmatrix}\begin{pmatrix} v \\ \psi \end{pmatrix} + \beta\begin{pmatrix} v \\ \psi \end{pmatrix} \\ &\quad + \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{vx+\psi y} - 1 - vx) - v(e^x - 1 - x)\mu(dx, dy). \end{aligned} \tag{3.8}$$

We proceed to state our main result on the coherency of affine stochastic volatility models. For the sake of clarity, let us introduce the notation $\phi^{\mathbb{Q}_i}, \psi^{\mathbb{Q}_i}$, $i = d, f$ so as to denote the cumulant generating function under the foreign and the domestic risk neutral measure. Similarly, we also introduce $F^{\mathbb{Q}_i}, R^{\mathbb{Q}_i}$ $i = d, f$.

**Theorem 3.1.** *Let $(X, V) = (X(t), V(t))_{t\geq 0}$ be an affine stochastic volatility model for the exchange rate process $S$ of the form* (3.1) *whose affine representation is given by* (3.8)*. Then the foreign risk-neutral martingale measure $\mathbb{Q}_f$ has the following* *density process with respect to* $\mathbb{Q}_d$

$$\left.\frac{\partial \mathbb{Q}_f}{\partial \mathbb{Q}_d}\right|_{\mathcal{F}_t} = e^{X_t - X_0}, \tag{3.9}$$

*and the model for the inverted exchange rate* $S^{-1}$ *is still an affine stochastic volatility model with*

$$\phi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \phi^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), w), \quad \psi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \psi^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), w), \tag{3.10}$$

*and corresponding characteristics*

$$F^{\mathbb{Q}_f}(\mathtt{i}u, \psi) = F^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), \psi), \quad R^{\mathbb{Q}_f}(\mathtt{i}u, \psi) = R^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), \psi), \tag{3.11}$$

*for* $(t, \mathtt{i}(-u - \mathtt{i}), w) \in \mathcal{U}$ *with* $u \in \mathbb{R}$.

**Proof.** The form of the density process (3.9) and the martingale property are immediate given our assumptions. It remains to show that the model for the inverted exchange rate is still an affine stochastic volatility model with characteristics (3.11) and for this part we follow the arguments of (Keller-Ressel (2008), Theorem 4.14). We have that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_f}\left[e^{\mathtt{i}u \log S^{-1}(t) + wV(t)}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}u \log S^{-1}(t) + wV(t)} \mathcal{Z}(t)\right] \frac{1}{\mathcal{Z}(0)} \\
&\quad = e^{\mathtt{i}u(r^f - r^d)t} \mathbb{E}^{\mathbb{Q}_d}\left[e^{\mathtt{i}(-u-\mathtt{i})X(t) + wV(t)}\right] e^{-X(0)} \\
&\quad = e^{\mathtt{i}u(r^f - r^d)t + \phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w) + \psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w)V(0) + \mathtt{i}(-u-\mathtt{i})X(0) - X(0)} \\
&\quad = e^{\mathtt{i}u(\log S^{-1}(0) + (r^f - r^d)t) + \phi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w) + \psi^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}), w)V(0)} \\
&\quad = e^{\mathtt{i}u(\log S^{-1}(0) + (r^f - r^d)t) + \phi^{\mathbb{Q}_f}(\mathtt{i}u, w) + \psi^{\mathbb{Q}_f}(\mathtt{i}u, w)V(0)},
\end{aligned} \tag{3.12}$$

where we set $\phi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \phi^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), w)$ and $\psi^{\mathbb{Q}_f}(\mathtt{i}u, w) = \psi^{\mathbb{Q}_d}(\mathtt{i}(-u - \mathtt{i}), w)$. From the system of generalized Riccati ODEs (3.5), we finally obtain (3.11) upon direct inspection. □

**Corollary 3.1.** *Under the Assumption of Theorem* 3.1, *the admissible parameter sets for* $S$ *and* $S^{-1}$ *are related as follows*:

(1) $b_1^{\mathbb{Q}_f} = -a_{11} - b_1$,
(2) $\rho^{\mathbb{Q}_f} = -\rho$,
(3) $\beta_1^{\mathbb{Q}_f} = -\alpha_{11} - \beta_1$,
(4) $\beta_2^{\mathbb{Q}_f} = \beta_2 + \rho\sqrt{\alpha_{11}\alpha_{22}}$,
(5) $1_A(x, y)m^{\mathbb{Q}_f}(dx, dy) = 1_A(-x, y)e^x m(dx, dy)$,
(6) $1_A(x, y)\mu^{\mathbb{Q}_f}(dx, dy) = 1_A(-x, y)e^x \mu(dx, dy)$,

*for* $A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}_{\geq 0} \backslash \{0\})$, *and* $\mathcal{B}(\cdot)$ *denoting the Borel* $\sigma$*-field.*

**Proof.** From Theorem 3.1, we know that

$$\begin{aligned} F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi) &= F^{\mathbb{Q}_f}(\mathtt{i}u,\psi),\\ R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi) &= R^{\mathbb{Q}_f}(\mathtt{i}u,\psi). \end{aligned} \tag{3.13}$$

In the steps below, we analyze separately drift, diffusion and jumps coefficient of affine stochastic volatility models.

*Step* 1: *Constant diffusion and drift coefficients.* We compute $F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the no-jump case

$$\begin{aligned} F^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi) &= \frac{\mathtt{i}^2(-u-\mathtt{i})^2}{2}a_{11} + \mathtt{i}(-u-\mathtt{i})b_1 + \psi b_2\\ &= \frac{1}{2}\mathtt{i}^2u^2a_{11} + (-a_{11}-b_1, b_2)\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix} + \frac{a_{11}}{2} + b_1. \end{aligned} \tag{3.14}$$

We note that (3.11) is satisfied if and only if $b_1^{\mathbb{Q}_f} = -a_{11} - b_1$ and $b_1 = -\frac{a_{11}}{2}$, where the second condition is dictated by the martingale property for the log-exchange rate and is then satisfied by assumption. These conditions can be easily understood by looking at the Black–Scholes model.

*Step* 2: *Linear diffusion and drift coefficients.* Let us compute, under $\mathbb{Q}_d$, the function $R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the no-jump case.

$$\begin{aligned} R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi) &= \frac{1}{2}(\mathtt{i}(-u-\mathtt{i}),\psi)\begin{pmatrix}\alpha_{11} & \rho\sqrt{\alpha_{11}\alpha_{22}}\\ \rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\mathtt{i}(-u-\mathtt{i})\\ \psi\end{pmatrix}\\ &\quad + (\beta_1,\beta_2)\begin{pmatrix}\mathtt{i}(-u-\mathtt{i})\\ \psi\end{pmatrix}\\ &= \frac{1}{2}(\mathtt{i}u,\psi)\begin{pmatrix}\alpha_{11} & -\rho\sqrt{\alpha_{11}\alpha_{22}}\\ -\rho\sqrt{\alpha_{11}\alpha_{22}} & \alpha_{22}\end{pmatrix}\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix}\\ &\quad +(-\alpha_{11}-\beta_1, \beta_2+\rho\sqrt{\alpha_{11}\alpha_{22}})\begin{pmatrix}\mathtt{i}u\\ \psi\end{pmatrix} + \frac{\alpha_{11}}{2} + \beta_1. \end{aligned} \tag{3.15}$$

The functional form of the model is preserved if and only if we have $\rho^{\mathbb{Q}_f} = -\rho$, $\beta_1^{\mathbb{Q}_f} = -\alpha_{11} - \beta_1$, $\beta_2^{\mathbb{Q}_f} = \beta_2 + \rho\sqrt{\alpha_{11}\alpha_{22}}$ and $\beta_1 = -\frac{\alpha_{11}}{2}$, where again the final condition is dictated by the martingale property for the log-exchange rate and is then satisfied by assumption. These conditions may be easily visualized by considering the Heston (1993) model and where first obtained by Del Baño Rollin (2008).

*Step* 3: *Constant and linear jump coefficients.* We concentrate on the linear part of the cumulant generating function, since the procedure is completely analogous for the constant part. We compute $R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi)$ in the pure-jump case

$$\begin{aligned}R^{\mathbb{Q}_d}(\mathtt{i}(-u-\mathtt{i}),\psi) &= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{\mathtt{i}(-u-\mathtt{i})x+\psi y} - 1 - \mathtt{i}(-u-\mathtt{i})x) \\ &\quad - \mathtt{i}(-u-\mathtt{i})(e^x - 1 - x)\mu(dx,dy) \\ &= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{-\mathtt{i}ux+\psi y} - 1 - \mathtt{i}ue^{-x} + \mathtt{i}u)e^x\mu(dx,dy). \end{aligned} \tag{3.16}$$

We look then at the jump transform of the inverted exchange rate under the foreign risk neutral measure

$$\begin{aligned}R^{\mathbb{Q}_f}(\mathtt{i}u,\psi) &= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{\mathtt{i}ux+\psi y} - 1 - \mathtt{i}ux) \\ &\quad - \mathtt{i}u(e^x - 1 - x)\mu^{\mathbb{Q}_f}(dx,dy) \\ &= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\}} (e^{\mathtt{i}ux+\psi y} - 1 - \mathtt{i}ue^{x} + \mathtt{i}u)\mu^{\mathbb{Q}_f}(dx,dy). \end{aligned} \tag{3.17}$$

Using (3.13), we conclude that for every set $A \in \mathcal{B}(\mathbb{R}\times\mathbb{R}_{\geq 0}\setminus\{0\})$ we have $1_A(x,y)\mu^{\mathbb{Q}_f}(dx,dy) = 1_A(-x,y)e^x\mu(dx,dy)$. This condition is in line with (Eberlein *et al.* (2008), Theorem 3.4). The result on the constant jump coefficient is completely analogous and this completes the proof. □

**Remark 3.1.** The results of Theorem 3.1 and Corollary 3.1 can be directly extended along two directions: first, it is possible to choose a multivariate volatility factor process $V = (V(t))_{t\geq 0}$ taking values in $\mathbb{R}^d_+$ or even a matrix variate volatility driver taking values on the cone of positive semidefinite $d \times d$ matrices (see Gnoatto & Grasselli (2014) for an example based on the Wishart process). Second, time inhomogeneous Lévy or affine processes may be also considered, along the lines of Eberlein & Koval (2006).

## 4. Example: A Multi-Currency and Functionally Symmetric Model

The discussion so far focused on the problem of finding models such that the process of the inverted exchange rate is coherent, i.e. belongs to the same class as the original process for the starting FX rate. This represents a first requirement that any FX market model should satisfy. However, when we look at the FX market, more complicated situations may arise. In fact we may not only compute the inverted exchange rate, but we may also construct new exchange rates via products/ratios of exchange rates. The simplest situations we can think of in this sense are given by currency triangles or tetrahedra[b] as in Fig. 2.

[b]To the best of our knowledge, currency triangles and tetrahedra are first mentioned in Hughston (1993).

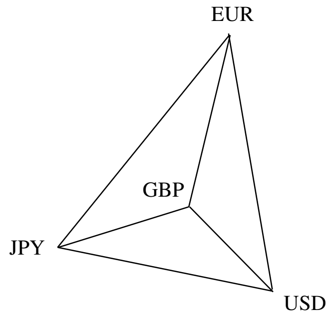


Fig. 2. A currency tetrahedron.

The fact that a model is coherent, does not guarantee that products of exchange rates preserve the functional form of the coefficients of the associated SDEs. An example in this sense is given by the basic Heston model, as evidenced by Doust (2012).

Even though a model is not fully functionally symmetric, it is possible to use coherent models as building blocks for processes that are stable under suitable multiplications/ratios. Building such models is possible if we change the starting point of the analysis: instead of specifying directly a generic exchange rate as a given state variable, the idea is to consider a family of primitive processes and then construct any exchange rate as a product/ratio of these primitive processes. In the literature, this kind of procedure has been undertaken, in a stochastic volatility setting, by De Col *et al.* (2013). The idea, developed in that paper, is inspired by the work of Heath & Platen (2006), who consider the following model for a generic exchange rate

$$S^{i,j}(t) = \frac{D^i(t)}{D^j(t)}, \tag{4.1}$$

where $D^i, D^j$ are the values of the growth optimal portfolio under currencies $i, j$. The same construction has been previously developed in Flesaker & Hughston (2000). This idea rephrases a classical concept from economics, namely the law of one price. Alternatively, the processes $D^i, D^j$ may be thought of as the values with respect to currencies $i, j$ of gold, or, using the terminology of De Col *et al.* (2013) a universal numéraire. This is the same principle independently followed by Doust (2007) and Doust (2012) who terms this approach intrinsic currency valuation framework.

We consider a FX market where $N$ currencies are traded and, as in De Col *et al.* (2013) and Heath & Platen (2006), we start by considering the value of each of these currencies in units of an artificial currency that can be viewed as a universal numéraire. We work under the risk neutral measure defined by the artificial currency and call $S^{0,i}(t)$ the value at time $t$ of one unit of the currency $i$ in terms of our artificial currency (so that $S^{0,i}$ can itself be thought as an exchange rate, between the artificial currency and the currency $i$). Let $T^\star > 0$ denote a fixed time horizon and let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T^\star}, \mathbb{Q}_0)$ be a filtered probability space where the filtration $\mathbb{F}$ satisfies the usual assumptions. The structure of the filtration $\mathbb{F}$ will be specified in the sequel. Let $d, f, g \in \mathbb{N}$. On this probability space we proceed to introduce several stochastic processes. We introduce $d$-dimensional Brownian motions $\mathbf{Z}^0 = (Z_1^0(t), \dots, Z_d^0(t))_{0\le t\le T^\star}$ and $\mathbf{W} = (W_1(t), \dots, W_d(t))_{0\le t\le T^\star}$ together with the correlation structure

$$\begin{aligned} d\langle Z_k^0, W_h\rangle_t &= \rho_k \delta_{kh} dt, \quad \rho_k \in [-1,1], \\ d\langle Z_k^0, Z_h^0\rangle_t &= \delta_{kh} dt, \\ d\langle W_k, W_h\rangle_t &= \delta_{kh} dt \end{aligned} \tag{4.2}$$

with $k, h = 1, \dots, d$, for $\delta_{xy}$ denoting the Dirac delta function of the set $\{x = y\}$ with respect to the filtration

$$\mathcal{F}_t^{\mathbf{Z}^0,\mathbf{W}} := \mathcal{F}_t^{Z_1^0,W_1} \vee \cdots \vee \mathcal{F}_t^{Z_d^0,W_d}. \tag{4.3}$$

We also introduce $f$ $g$-dimensional mutually independent Brownian motions $\mathbf{Z}^l = (Z_1^l(t), \dots, Z_d^l(t))_{0\le t\le T^\star}$ with respect to the filtration

$$\mathcal{F}_t^{\mathbf{Z}^1,\dots,\mathbf{Z}^f} := \mathcal{F}_t^{Z_1^1} \vee \cdots \vee \mathcal{F}_t^{Z_g^f}. \tag{4.4}$$

Such processes are independent with respect to $\mathbf{Z}^0$ and $\mathbf{W}$. Finally, given a family of measurable spaces of the form $(\mathbb{R}^g, \mathcal{B}(\mathbb{R}^g,), m_s)$, $l = 1, \dots, f$, for $\mathcal{B}(\cdot)$ denoting the Borel $\sigma$-field and $m_l$ are Lévy measures, we introduce a family of Poisson random measures

$$\begin{aligned} N_l : \Omega \times \mathcal{B}([0,T^\star] \times \mathbb{R}^g \setminus \{0\}) &\mapsto \mathbb{N} \\ (\omega, [0,t] \times A) &\mapsto N_l(A, [0,t]), \quad l = 1, \dots, f \end{aligned} \tag{4.5}$$

with associated filtration

$$\mathcal{F}_t^N = \mathcal{F}_t^{N_1} \vee \cdots \vee \mathcal{F}_t^{N_f}. \tag{4.6}$$

The Lévy measures are assumed to satisfy the moment condition: $\exists\, M > 1$ and $\epsilon > 0$ s.t.

$$\int_{|x|>1} e^{u^\top x} m_l(dx) < \infty, \tag{4.7}$$

$\forall\, u \in [-(1+\epsilon)M, (1+\epsilon)M]^g$, $\forall\, l = 1, \dots, f$, which ensures the existence of exponential moments. In summary, the initially postulated filtration $\mathcal{F}_t$ is of the form

$$\mathcal{F}_t := \mathcal{F}_t^{\mathbf{Z}^0,\mathbf{W}} \vee \mathcal{F}_t^{\mathbf{Z}^1,\dots,\mathbf{Z}^f} \vee \mathcal{F}_t^N \tag{4.8}$$

so that all stochastic processes we introduce are mutually independent with respect to their own filtrations, up to the exception in (4.2). To construct the set of financial quantities we are interest in, let us also introduce

- A family of constant vectors $\mathbf{a}^i = (a_1^i, \dots, a_d^i)^\top$ for $i = 1, \dots, N$;

- A family of deterministic vector functions $\sigma_l^i : [0, T^\star] \mapsto \mathbb{R}^g$, $t \mapsto (\sigma_{l,1}^i(t), \ldots, \sigma_{l,g}^i(t))^\top$, $i = 1, \ldots, N$, $l = 1, \ldots, f$ satisfying
$$\int_0^{T^\star} (\sigma_{l_1}^{i_1}(s))^\top \sigma_{l_2}^{i_2}(s) ds < \infty, \tag{4.9}$$
for all $i_1, i_2 = i = 1, \ldots, N$, $l_1, l_2 = 1, \ldots, f$, which will be referred to as *local volatility functions*;
- A family of constant vectors $\mathbf{b}_l^i = (b_{l,1}^i, \ldots, b_{l,g}^i)^\top$, satisfying
$$\sup_{\substack{i_1, i_2 = 1, \ldots, N \\ l_1, l_2 = 1, \ldots, f \\ l_3 = 1, \ldots, g}} |b_{l_1, l_3}^{i_1} - b_{l_2, l_3}^{i_2}| < (1 + \epsilon) M, \tag{4.10}$$
where $M$ and $\epsilon$ are the constants defined in (4.7). Such vectors will be referred to as *jump projections*.

Given the above objects, we model each of the $S^{0,i}$ via three main mutually independent stochastic drivers: the first is a multi-variate stochastic volatility term of Heston (1993) type, with $d$ independent Cox *et al.* (1985) components, $\mathbf{V}(t) \in \mathbb{R}_+^d$. We further assume that these stochastic volatility components are *common* between the different $S^{0,i}$. This part corresponds to the model of De Col *et al.* (2013). We generalize the framework by including a time dependent (local) volatility term and jumps. For $0 \le t \le T^\star$ we write

$$\begin{aligned} \frac{dS^{0,i}(t)}{S^{0,i}(t^-)} &= (r^0 - r^i)dt - (\mathbf{a}^i)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(t))} d\mathbf{Z}^0(t) \\ &\quad + \sum_{l=1}^{f} \left( -(\sigma_l^i(t))^\top d\mathbf{Z}^l(t) + \int_{\mathbb{R}^g} (e^{-(\mathbf{b}_l^i)^\top x} - 1)(N_l(dx, dt) - m_l(dx)dt) \right), \\ dV_k(t) &= \kappa_k(\theta_k - V_k(t))dt + \xi_k \sqrt{V_k(t)} dW_k(t), \quad k = 1, \ldots, d. \end{aligned} \tag{4.11}$$

As far as the stochastic volatility part is concerned, $\kappa_k, \theta_k, \xi_k \in \mathbb{R}$ are parameters in a CIR dynamics, whereas $\sqrt{\mathrm{Diag}(\mathbf{V})}$ denotes the diagonal matrix with the square root of the elements of the vector $\mathbf{V}$ in the main diagonal. This term is multiplied with the linear vector $\mathbf{a}^i \in \mathbb{R}^d$ ($i = 1, \ldots, N$), in consequence, the dynamics of the exchange rate is also driven by a linear projection of the variance factor $\mathbf{V}$ along a direction parametrized by $\mathbf{a}^i$, so that the total instantaneous variance arising from the stochastic volatility term is $(\mathbf{a}^i)^\top \mathrm{Diag}(\mathbf{V}(t)) \mathbf{a}^i dt$. In each monetary area $i$, the money-market account accrues interest based on the deterministic risk free rate $r^i$,
$$dB^i(t) = r^i B^i(t) dt, \quad i = 1, \ldots, N; \tag{4.12}$$
in our universal numéraire analogy $r^0$ is the artificial currency rate. We also notice that the jumps sizes are multiplied by a $\mathbf{b}_l^i \in \mathbb{R}^g$, $i = 1, \ldots, N$, $l = 1, \ldots, f$, consequently, the dynamics of the exchange rate is also driven by a mixture of multifactor jump processes.

### 4.1. *Products and ratios of FX rates*

The aim of this section is to show that the general model that we introduced above gives rise to exchange rates that are closed under product/ratios. The model describes primitive exchange rates, i.e. exchange rates with respect to an artificial currency, to which an artificial risk-neutral measure $\mathbb{Q}_0$ is associated. Exchange rates among real currencies are constructed by performing two steps: we apply first the Ito formula for semi-martingales in order to deduce the dynamics of $S^{i,0}(t) = (S^{0,i}(t))^{-1}$, and then we compute the dynamics of $S^{i,j} = S^{i,0}(t)S^{0,j}(t)$ by relying on the product rule. Similarly, one can show that the resulting process $S^{i,j}$ may be used to construct different exchange rates of the same model class, meaning that, e.g. $S^{i,j}$ has the same function form as $S^{i,l}S^{l,j}$, i.e. the coefficients of the SDE preserve their functional form.

**Proposition 4.1.** *The dynamics of the exchange rate $S^{i,j} = (S^{i,j}(t))_{t\geq 0}$ under the $\mathbb{Q}_0$ risk neutral measure are given by*

$$\begin{aligned}\frac{dS^{i,j}(t)}{S^{i,j}(t-)} &= (r^i - r^j)dt \\ &\quad + (\mathbf{a}^i - \mathbf{a}^j)^\top \mathrm{Diag}(\mathbf{V}(t))\mathbf{a}^i dt + (\mathbf{a}^i - \mathbf{a}^j)^\top \mathrm{Diag}(\sqrt{\mathbf{V}(t)})d\mathbf{Z}^0(t) \\ &\quad + \sum_{l=1}^{f}\Bigg[(\sigma_l^i(t) - \sigma_l^j(t))^\top(\sigma_l^i(t))dt + (\sigma_l^i(t) - \sigma_l^j(t))^\top d\mathbf{Z}^l(t) \\ &\quad + \int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1)(N_l(dx,dt) - e^{-(\mathbf{b}_l^i)^\top x}m_l(dx)dt)\Bigg]. \qquad (4.13)\end{aligned}$$

*Moreover, the dynamics of the exchange rate is functionally invariant under products/ratios of exchange rates.*

**Proof.** The dynamics of the inverted exchange rate are given by

$$\begin{aligned}dS^{i,0}(t) &= d\left(\frac{1}{S^{0,i}(t)}\right) \\ &= -\frac{1}{S^{0,i}(t-)}\Bigg((r^0 - r^i)dt - (\mathbf{a}^i)^\top\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^0(t) \\ &\quad + \sum_{l=1}^{f}\left(-(\sigma_l^i(t))^\top d\mathbf{Z}^l(t) + \int_{\mathbb{R}^g}(e^{-(\mathbf{b}_l^i)^\top x} - 1)(N_l(dx,dt) - m_l(dx)dt)\right)\Bigg) \\ &\quad + \frac{1}{S^{0,i}(t-)}\left((\mathbf{a}^i)^\top\mathbf{V}(t)(\mathbf{a}^i) + \sum_{l=1}^{f}(\sigma_l^i(t))^\top(\sigma_l^i(t))\right)dt \\ &\quad + \frac{1}{S^{0,i}(t-)}\sum_{l=1}^{f}\int_{\mathbb{R}^g}\frac{(e^{-(\mathbf{b}_l^i)^\top x} - 1)^2}{e^{-(\mathbf{b}_l^i)^\top x}}N_l(dx,dt), \qquad (4.14)\end{aligned}$$

hence

$$dS^{i,0}(t) = S^{i,0}(t^-)\Bigg((r^i - r^0)dt + (\mathbf{a}^i)^\top\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^0(t) + (\mathbf{a}^i)^\top\mathbf{V}(t)(\mathbf{a}^i)dt$$
$$+ \sum_{l=1}^{f}\Bigg[(\sigma_l^i(t))^\top(\sigma_l^i(t))dt + (\sigma_l^i(t))^\top d\mathbf{Z}^l(t)\int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i)^\top x} - 1)(N_l(dx, dt)$$
$$- e^{-(\mathbf{b}_l^i)^\top x}m_l(dx)dt)\Bigg]\Bigg) \tag{4.15}$$

from which we can deduce the dynamics of a generic *final* exchange rate under the $\mathbb{Q}_0$ risk neutral measure.

$$\begin{aligned} dS^{i,j}(t) &= d(S^{i,0}(t)S^{0,j}(t)) \\ &= S^{i,j}(t^-)\Bigg((r^i - r^0)dt + (\mathbf{a}^i)^\top\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^0(t) + (\mathbf{a}^i)^\top\mathbf{V}(t)(\mathbf{a}^i)dt \\ &\quad + \sum_{l=1}^{f}\Bigg[(\sigma_l^i(t))^\top(\sigma_l^i(t))dt + (\sigma_l^i(t))^\top d\mathbf{Z}^l(t)\int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i)^\top x} - 1)(N_l(dx, dt) \\ &\quad - e^{-(\mathbf{b}_l^i)^\top x}m_l(dx)dt)\Bigg] + (r^0 - r^j)dt - (\mathbf{a}^j)^\top\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{Z}^0(t) \\ &\quad + \sum_{l=1}^{f}\Bigg[-(\sigma_l^j(t))^\top d\mathbf{Z}^l(t) + \int_{\mathbb{R}^g}(e^{-(\mathbf{b}_l^j)^\top x} - 1)(N_l(dx, dt) - m_l(dx)dt)\Bigg] \\ &\quad - (\mathbf{a}^j)^\top\mathbf{V}(t)(\mathbf{a}^i)dt - \sum_{l=1}^{f}\Bigg[(\sigma_l^j(t))^\top(\sigma_l^i(t))dt \\ &\quad + \int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i)^\top x} - 1)(e^{-(\mathbf{b}_l^j)^\top x} - 1)N(dx, dt)\Bigg]\Bigg). \end{aligned} \tag{4.16}$$

By rearranging terms we arrive at

$$\begin{aligned} \frac{dS^{i,j}(t)}{S^{i,j}(t^-)} &= (r^i - r^j)dt + (\mathbf{a}^i - \mathbf{a}^j)^\top\mathrm{Diag}(\mathbf{V}(t))\mathbf{a}^i dt \\ &\quad + (\mathbf{a}^i - \mathbf{a}^j)^\top\mathrm{Diag}(\sqrt{\mathbf{V}(t)})d\mathbf{Z}^0(t) \\ &\quad + \sum_{l=1}^{f}\Bigg[(\sigma_l^i(t) - \sigma_l^j(t))^\top(\sigma_l^i(t))dt + (\sigma_l^i(t) - \sigma_l^j(t))^\top d\mathbf{Z}^l(t) \\ &\quad + \int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1)(N_l(dx, dt) - e^{-(\mathbf{b}_l^i)^\top x}m_l(dx)dt)\Bigg] \end{aligned} \tag{4.17}$$

which represents the dynamics of a generic exchange rate between currency $i$ and currency $j$ under the $\mathbb{Q}_0$ risk neutral measure. By means of an application of the product rule for semi-martingales it can be shown, using the same techniques as above, that the functional form of the process remains unchanged under products/ratios of exchange rates. □

### 4.2. *The $\mathbb{Q}_i$-risk-neutral process for the FX rate*

So far, the specification of the model has been performed under the risk neutral measure $\mathbb{Q}_0$ associated to the artificial numéraire. The aim of the present section is to present the risk neutral dynamics of the exchange rates together with a precise statement of the relation among the parameters of the model under different measures. This is a key step because a precise understanding of the relationship among parameters under different measure is necessary, e.g. to perform a joint calibration of the model to different volatility surfaces simultaneously, as shown in De Col *et al.* (2013) and Gnoatto & Grasselli (2014).

Under the assumptions of the fundamental theorem of asset pricing (see Björk (2009), Chap. 13 and 14), investing into the foreign money market account gives a traded asset with value $S^{i,j}B^j/B^i$, and its value has to be a $\mathbb{Q}^i$-martingale. Hence

$$\begin{aligned}\frac{d\left(\dfrac{S^{i,j}(t)B^j(t)}{B^i(t)}\right)}{\dfrac{S^{i,j}(t^-)B^j(t^-)}{B^i(t^-)}} &= (\mathbf{a}^i-\mathbf{a}^j)^\top \mathrm{Diag}(\sqrt{\mathbf{V}(t)})d(\mathbf{Z}^0)^{\mathbb{Q}_i}(t)\\ &\quad+\sum_{l=1}^{f}\Bigg[(\sigma_l^i(t)-\sigma_l^j(t))^\top d(\mathbf{Z}^l)^{\mathbb{Q}_i}(t)\\ &\quad+\int_{\mathbb{R}}(e^{(\mathbf{b}_l^i-\mathbf{b}_l^j)^\top x}-1)(N_l(dx,dt)-(m_l)^{\mathbb{Q}_i}(dx)dt)\Bigg].\end{aligned}\tag{4.18}$$

In the last line, we implicitly defined the new Brownian motion vectors $(\mathbf{Z}^0)^{\mathbb{Q}_i}$, $(\mathbf{Z}^l)^{\mathbb{Q}_i}$, $l=1,\ldots,f$, together with the exponentially tilted Lévy measures $(m_l)^{\mathbb{Q}_i}$, $l=1,\ldots,f$ under the measure $\mathbb{Q}_i$ from the constraint of having a $\mathbb{Q}_i$-local martingale and by Girsanov theorem:

$$\begin{aligned}d(\mathbf{Z}^0)(t)^{\mathbb{Q}_i} &= d\mathbf{Z}^0(t)+\sqrt{\mathrm{Diag}(\mathbf{V}(t))}\mathbf{a}^i dt, \quad i=1,\ldots,N,\\ d(\mathbf{Z}^l)(t)^{\mathbb{Q}_i} &= d\mathbf{Z}^l(t)+\sigma_j^i(t)dt, \quad l=1,\ldots,f \;\; i=1,\ldots,N,\\ (m_l)^{\mathbb{Q}_i}(dx) &= e^{-(\mathbf{b}_l^i)^\top x}m_l(dx).\end{aligned}\tag{4.19}$$

If we denote by $\mathbb{Q}_0$ the risk neutral measure associated with the universal numéraire, the Radon–Nikodym derivative corresponding to the change of measure from $\mathbb{Q}_0$ to $\mathbb{Q}_i$ reads

$$\begin{aligned}
\left.\frac{d\mathbb{Q}^i}{d\mathbb{Q}^0}\right|_t &= \mathcal{Z}^{i,0}(t) \\
&= \exp\Bigg(-\int_0^t (\mathbf{a}^i)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(s))}d\mathbf{Z}^0(s) - \frac{1}{2}\int_0^t (\mathbf{a}^i)^\top \mathrm{Diag}(\mathbf{V}(s))\mathbf{a}^i ds \\
&\quad \times \sum_{l=1}^f \Bigg[-\int_0^t (\sigma_l^i(s))^\top d\mathbf{Z}^l(s) - \frac{1}{2}\int_0^t (\sigma_l^i(s))^\top \sigma_l^i(s)ds \\
&\quad - \int_0^t \int_{\mathbb{R}^d} (\mathbf{b}_l^i)^\top x(N_l(dx,ds) - m_l(dx)ds) \\
&\quad - \int_0^t \int_{\mathbb{R}^d} (e^{-(\mathbf{b}_l^i)^\top x} - 1 + (\mathbf{b}_l^i)^\top x) m_l(dx)ds\Bigg]\Bigg),
\end{aligned} \tag{4.20}$$

hence under the $\mathbb{Q}_i$-risk-neutral measure the exchange rate has the dynamics

$$\begin{aligned}
\frac{dS^{i,j}(t)}{S^{i,j}(t^-)} &= (r^i - r^j)dt + (\mathbf{a}^i - \mathbf{a}^j)^\top \mathrm{Diag}(\sqrt{\mathbf{V}(t)})d(\mathbf{Z}^0)^{\mathbb{Q}_i}(t) \\
&\quad + \sum_{l=1}^f \Bigg[(\sigma_l^i(t) - \sigma_l^j(t))^\top d(\mathbf{Z}^l)^{\mathbb{Q}_i}(t) \\
&\quad + \int_{\mathbb{R}} (e^{(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1)(N_l(dx,dt) - (m_l)^{\mathbb{Q}_i}(dx)dt)\Bigg],
\end{aligned} \tag{4.21}$$

as desired.

Given our assumption on the correlation structure in (4.2), we can write the following factorization under $\mathbb{Q}^0$

$$dW_k(t) = \rho_k dZ_k^0(t) + \sqrt{1-\rho_k^2}dZ_k^\perp(t), \quad k = 1, \ldots, d, \tag{4.22}$$

where $\mathbf{Z}^\perp$ is a Brownian motion independent of $\mathbf{Z}^0$. Hence the measure change has also an impact on the variance processes, via the correlations $\rho_k$, $k = 1, \ldots, d$,

$$dW_k^{\mathbb{Q}^i}(t) = dW_k(t) + \rho_k(\mathbf{e}^k)^\top \sqrt{\mathrm{Diag}(\mathbf{V}(t))}\mathbf{a}^i dt. \tag{4.23}$$

We finally obtain the dynamics of the instantaneous variance process under the new measure by means of a redefinition of the CIR parameters

$$\begin{aligned}
\rho_k^{\mathbb{Q}^i} &= \rho_k, \\
\kappa_k^{\mathbb{Q}^i} &= \kappa_k + \xi_k \rho_k a_k^i, \\
\theta_k^{\mathbb{Q}^i} &= \theta_k \frac{\kappa_k}{\kappa_k^{\mathbb{Q}^i}},
\end{aligned} \tag{4.24}$$

so that we can re-express the variance SDE in its original form

$$dV_k(t) = \kappa_k^{\mathbb{Q}^i}(\theta_k^{\mathbb{Q}^i} - V_k(t))dt + \xi_k\sqrt{V_k(t)}dW_k^{\mathbb{Q}^i}(t). \tag{4.25}$$

The change of measure above is well posed once we can prove that the stochastic exponential in (4.20) is a martingale. This is the object of the next result.

**Proposition 4.2.** *Under the preceding assumptions we have*

$$\mathbb{E}^{\mathbb{Q}_0}[\mathcal{Z}^{i,0}(t)] = 1, \quad 0 \le t \le T^\star. \tag{4.26}$$

**Proof.** In view of applying a multivariate extension of Theorem 2.1 in Cherny (2006), we rewrite the stochastic exponential (4.20) as

$$\begin{aligned}
\mathcal{Z}^{i,0}(t) &= \prod_{k=1}^d \exp\bigg\{-\int_0^t a_k^i\sqrt{V_k(s)}dZ_k^0(s) - \frac{1}{2}\int_0^t a_k^i V_k(s) a_k^i ds\bigg\}\\
&\quad\times\prod_{l=1}^f \exp\bigg\{-\int_0^t (\sigma_l^i(s))^\top dZ^l(s) - \frac{1}{2}\int_0^t \left(\sigma_l^i(s)\right)^\top \sigma_l^i(s)ds\\
&\quad - \int_0^t\int_{\mathbb{R}^d}(\mathbf{b}_l^i)^\top x(N_l(dx,ds) - m_l(dx)ds)\\
&\quad - \int_0^t\int_{\mathbb{R}^d}(e^{-(\mathbf{b}_l^i)^\top x} - 1 + (\mathbf{b}_l^i)^\top x)m_l(dx)ds\bigg\}\\
&= \prod_{k=1}^d \mathcal{Z}^{\mathrm{SV},k}(t)\prod_{l=1}^f \mathcal{Z}^{\mathrm{LVJ},l}(t)
\end{aligned} \tag{4.27}$$

for SV denoting stochastic volatility and LVJ local volatility with jump components, respectively. Our aim is to show that each of these stochastic exponentials is a true martingale with respect to its own filtration. The conclusion then follows from Theorem 2.1 in Cherny (2006). Now, concerning the processes $\mathcal{Z}^{\mathrm{SV},k}(t)$ we argue as in Gnoatto (2012). We shall apply Theorem 2.1 in Mijatović & Urusov (2012). Using their notation, we have $Y_k = V_k$, $b_k(x) = -\rho_k\sqrt{x_k}a_k^i$ and $\sigma_k(x) = \sqrt{x_k}\xi_k$. If the Feller condition holds, i.e. if $2\kappa_k\theta_k \ge \xi_k^2$, then condition (a) and (c) in Theorem 2.1 of Mijatović & Urusov (2012) are satisfied. Now from our (4.25), we note that the boundary behavior of $V_k$ is the same under both $\mathbb{Q}_0$ and $\mathbb{Q}_i$. Suppose now that the Feller condition is violated. Condition (a) still holds while condition (c) does not but the boundary 0 is good, in the terminology of Mijatović & Urusov (2012). To see this we compute the functions $\rho_k(x), s_k(x)$ from Eqs. (16)–(18) in Mijatović & Urusov (2012) in the present setting:

$$\rho_k(x) = \alpha x^{-\frac{2\kappa_k\theta_k}{\xi_k^2}}\exp\bigg\{\frac{2\kappa_k x}{\xi_k^2}\bigg\}, \tag{4.28}$$

for $\alpha$ a positive constant and

$$s_k(x) = \int_c^1 \rho_k(y)dy = \alpha \int_c^1 y^{-\frac{2\kappa_k\theta_k}{\xi_k^2}} \exp\left\{\frac{2\kappa_k y}{\xi_k^2}\right\} dy, \tag{4.29}$$

for $c$ an arbitrary constant. We can confirm, using (26) in Mijatović & Urusov (2012), that 0 is good: we have $s_k(0) > -\infty$ (by the assumption $2\kappa_k\theta_k < \xi_k^2$) and the function

$$\frac{(s_k(x) - s_k(0))b_k^2(x)}{\rho_k(x)\sigma_k(x)} = \alpha(s_k(x) - s_k(0))x^{\frac{2\kappa_k\theta_k}{\xi_k^2}} \exp\left\{-\frac{2\kappa_k x}{\xi_k^2}\right\} \tag{4.30}$$

is locally integrable around 0 since $s_k(x)$ behaves like $x^{-\frac{2\kappa_k\theta_k}{\xi_k^2}+1}$, hence (26) in Mijatović & Urusov (2012) holds and it follows that each stochastic exponential $\mathcal{Z}^{\mathrm{SV},k}$ is a true martingale with respect to its own filtration. Let us now concentrate on the local volatility plus jumps component. We base our reasoning on Proposition 2.2.3 in Koval (2007), which is in turn based on the uniform integrability conditions for exponential local martingales in Kallsen & Shiryaev (2002). To this end, let us define the process $L_l = (L_l(t))_{0\le t\le T^\star}$ by setting

$$L_l(t) := -\int_0^t (\sigma_l^i(s))^\top dZ^l(s) - \int_0^t \int_{\mathbb{R}^d} (\mathbf{b}_l^i)^\top x(N_l(dx, ds) - m_l(dx)ds). \tag{4.31}$$

In line with Definition 2.14 in Kallsen & Shiryaev (2002), we introduce the associated exponential compensator

$$K^{L_l}(t) := \frac{1}{2}\int_0^t (\sigma_l^i(s))^\top \sigma_l^i(s)ds + \int_0^t \int_{\mathbb{R}^d} (e^{-(\mathbf{b}_l^i)^\top x} - 1 + (\mathbf{b}_l^i)^\top x)m_l(dx)ds \tag{4.32}$$

so that we can write each local martingale $\mathcal{Z}^{\mathrm{LVJ},l}$ in the form $\mathcal{Z}^{\mathrm{LVJ},l} = \exp\{L_l(t) - K^{L_l}(t)\}$. Concerning $K^{L_l}$, we notice that from (4.7) in conjunction with (4.7) we can use Lemma 2.13 in Kallsen & Shiryaev (2002) to conclude that

$$\int_0^t \int_{\mathbb{R}^d} (e^{-(\mathbf{b}_l^i)^\top x} - 1 + (\mathbf{b}_l^i)^\top x)m_l(dx)ds < \infty, \tag{4.33}$$

while the finiteness of $\int_0^t (\sigma_l^i(s))^\top \sigma_l^i(s)ds$ follows from (4.9). Now according to Corollary 3.10 in Kallsen & Shiryaev (2002), each process $\mathcal{Z}^{\mathrm{LVJ},l}$ is a uniformly integrable martingale if the following condition holds

$$\lim_{\delta\to 0} \sup_{t\in[0,T^\star]} \delta \log\left(\mathbb{E}^{\mathbb{Q}_0}\left[\exp\left\{\frac{1}{\delta}[(1-\delta)K^{L_l}(t) - K^{L_l}(t)|_{(1-\delta)}]\right\}\right]\right) = 0, \tag{4.34}$$

where $\delta \in (0,1)$ and

$$\begin{aligned} K^{L_l}(t)|_{(1-\delta)} := \frac{(1-\delta)^2}{2}\int_0^t (\sigma_l^i(s))^\top \sigma_l^i(s)ds \\ \times \int_0^t \int_{\mathbb{R}^d} (e^{-(1-\delta)(\mathbf{b}_l^i)^\top x} - 1 + (1-\delta)(\mathbf{b}_l^i)^\top x)m_l(dx)ds. \end{aligned} \tag{4.35}$$

The above condition (4.34) simplifies in our setting due to the fact that the functions $K^{L_l}(t)$ and $K^{L_l}(t)|_{(1-\delta)}$ are deterministic, hence the further steps allowing us to conclude that each $\mathcal{Z}^{\mathrm{LVJ},l}$ is a martingale follow along the lines of the calculation in the proof of Proposition 2.2.3 in Koval (2007). In summary, we have shown that all processes $\mathcal{Z}^{\mathrm{SV},k}$ and $\mathcal{Z}^{\mathrm{LVJ},l}$ are martingales with respect to their natural filtrations, hence a generalization of Theorem 2.1 in Cherny (2006) for multiple products of martingales allows us to conclude. □

### 4.3. *Stochastic covariations*

As we already mentioned, the model we present in this section is intended as an illustration of how our result on the coherency of FX market models may be applied in order to build a fully functional symmetric model for multiple FX rates. The setting we present however, is general enough in view of capturing some stylized facts in FX markets.

With respect to Fig. 1, we note that our model presents both a stochastic volatility for the single FX rates and a stochastic correlation among different rates. To see this, we can compute the covariation between two generic FX rates as in De Col *et al.* (2013).

$$\begin{aligned} d&\left[\int_0^{\cdot} \frac{dS^{i,j}(s)}{S^{i,j}(s^-)}, \int_0^{\cdot} \frac{dS^{i,l}(s)}{S^{i,l}(s^-)}\right](t) \\ &= (\mathbf{a}^i - \mathbf{a}^j)^\top \mathbf{V}(t)(\mathbf{a}^i - \mathbf{a}^l)dt + \sum_{o=1}^{f}\left[(\sigma_o^i(t) - \sigma_o^j(t))^\top(\sigma_o^i(t) - \sigma_o^l(t))dt\right. \\ &\qquad \left. + \int_{\mathbb{R}} (e^{(\mathbf{b}_o^i - \mathbf{b}_o^j)^\top x} - 1)(e^{(\mathbf{b}_o^i - \mathbf{b}_o^l)^\top x} - 1)N(dx, dt)\right]. \end{aligned} \tag{4.36}$$

The result above clearly implies that the variance covariance matrix among exchange rates is a stochastic process, a desirable feature given in the Sec. 1.

A second interesting feature is that the infinitesimal correlation between the logarithm of the FX rate and its infinitesimal variance, usually termed skewness, is also a stochastic process, as in De Col *et al.* (2013) Eq. (11). This is a feature arising from the fact that we are considering a multifactor stochastic volatility model. The importance of stochastic skewness is discussed, in the FX setting, e.g. in Bakshi *et al.* (2008).

### 4.4. *Analytical tractability*

A fundamental feature of the proposed model is the availability of a closed-form solution for the characteristic function of the log-exchange rate, allowing the application of standard Fourier techniques, (see, e.g. Lewis 2000, Lipton 2002, Sepp 2003, Carr & Madan 1999, Lee 2004, Eberlein *et al.* 2010). The next proposition provides such closed-form solution. For the sake of simplicity, we drop all superscripts indicating the measure and assume that the parameters have been transformed according to the relations illustrated in Sec. 4.2.

**Proposition 4.3.** *Define $\tau := T - t$ and $u \in \mathbb{R}$. The conditional characteristic function of the log-exchange rate is given by:*

$$\varphi_i(u) = \exp\Bigg[\mathtt{i}u\log S^{i,j}(t) + (r^i - r^j)\mathtt{i}u\tau + \sum_{k=1}^{d}(A_k^{i,j}(\tau) + B_k^{i,j}(\tau)V_k)$$

$$\times\,\tau\sum_{l=1}^{f}\Bigg(\frac{\mathtt{i}^2u^2\sigma_{l,AV}^2 - \mathtt{i}u\sigma_{l,AV}^2}{2}\int_{\mathbb{R}^g}(e^{\mathtt{i}u(\mathbf{b}_l^i-\mathbf{b}_l^j)^\top x} - 1 - \mathtt{i}u(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x)$$

$$- \mathtt{i}u(e^{(\mathbf{b}_l^i-\mathbf{b}_l^j)^\top x} - 1 - (\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x)m_l(dx)]\Bigg)\Bigg], \tag{4.37}$$

*where for $k = 1, \ldots, d$:*

$$\begin{aligned}
A_k^{i,j}(\tau) &= \frac{\kappa_k\theta_k}{\xi_k^2}\left[(Q_k - d_k)\tau - 2\log\frac{1 - c_ke^{-d_k\tau}}{1 - c_k}\right],\\
B_k^{i,j}(\tau) &= \frac{Q_k - d_k}{\xi_k^2}\frac{1 - e^{-d_k\tau}}{1 - c_ke^{-d_k\tau}},\\
d_k &= \sqrt{Q_k^2 - 4R_kP_k}, \quad c_k = \frac{Q_k - d_k}{Q_k + d_k},\\
P_k &= \frac{1}{2}\mathtt{i}^2u^2(a_k^i - a_k^j)^2 - \frac{1}{2}(a_k^i - a_k^j)^2\mathtt{i}u,\\
Q_k &= \kappa_k - \mathtt{i}u(a_k^i - a_k^j)\rho_k\xi_k, \quad R_k = \frac{1}{2}\xi_k^2
\end{aligned} \tag{4.38}$$

*and*

$$\sigma_{l,AV}^2 = \frac{1}{T - t}\int_t^T(\sigma_l^i(s) - \sigma_l^j(s))^\top(\sigma_l^i(s) - \sigma_l^j(s))ds, \tag{4.39}$$

*moreover, the characteristic function admits an analytic extension to the set of complex numbers*

$$\mathcal{S} = \{\zeta \in \mathbb{C} \,|\, \Im(\zeta) \in [-1, 0]\}. \tag{4.40}$$

**Proof.** Due to the mutual independence of the stochastic processes driving the exchange rate, the characteristic function admits the factorization

$$\varphi_i(u) = e^{\mathtt{i}u\log S^{i,j}(t) + (r^i - r^j)\mathtt{i}u\tau}\prod_{k=1}^{d}\varphi_k^{\mathrm{SV}}(u)\prod_{l=1}^{f}\varphi_l^{\mathrm{LVJ}}(u), \tag{4.41}$$

where

$$\varphi_k^{\mathrm{SV}}(u) := \mathbb{E}^{\mathbb{Q}_i}\Bigg[\exp\Bigg\{-\frac{(a_k^i - a_k^j)^2}{2}\int_t^T V_k(s)ds$$

$$+ (a_k^i - a_k^j)\int_t^T\sqrt{V_k(s)}dZ_k^0(s)\Bigg\}\Bigg|\,\mathcal{F}_t^{Z_k^0, W_k}\Bigg] \tag{4.42}$$

and

$$\varphi_l^{\mathrm{LVJ}}(u) := \exp\left\{-\tau \mathtt{i} u\left(\frac{\sigma_{l,AV}^2}{2} + \int_{\mathbb{R}^g}(e^{(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x} - 1 - (\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x) m_l(dx)\right)\right\}$$
$$\times \mathbb{E}^{\mathbb{Q}_0}\left[\exp\left\{\mathtt{i} u\left(\int_t^T (\sigma_l^i(s) - \sigma_l^j(s))^\top d(\mathbf{Z}^l)(s)\right.\right.\right.$$
$$\left.\left.\left.\left. + \int_t^T\int_{\mathbb{R}^g}(\mathbf{b}_l^i - \mathbf{b}_l^j)^\top x(N_l(dx,ds) - m_l(dx)ds)\right)\right\}\right| \mathcal{F}_t^{\mathbf{Z}^l, N_l}\right]. \tag{4.43}$$

Each term $\varphi_k^{\mathrm{SV}}$ represents the characteristic function of a Heston process, hence the computation leading to formulas (4.38) follows by means of standard arguments involving the solution of the associated system of Riccati ODEs (see, among others, Lord & Kahl 2010, Lewis 2000 or Albrecher *et al.* 2007). Concerning the terms $\varphi_l^{\mathrm{LVJ}}$, the explicit result in (4.3) is a direct consequence of the Lévy–Khintchine formula for (time-inhomogeneous) Lévy process, as stated, e.g. in Cont & Tankov (2004), Theorem 14.1. Concerning the extension of the domain of the characteristic function, we can first use Lemma 5.4(2) in Filipovic & Mayerhofer (2009) on the domain of definition of Riccati ODEs to show that all functions $\varphi_k^{\mathrm{SV}}$ are defined in $\mathcal{S}$. Concerning the functions $\varphi_l^{\mathrm{LVJ}}$, we note that on the one side, the characteristic function of the Brownian part is defined on the whole complex plane, on the other side, our conditions (4.7)–(4.10) imply that also the jump transform is well defined on $\mathcal{S}$, thus yielding the claim. □

As anticipated, the above result allows for the application of Fourier techniques for option pricing. For example, in line with Lee (2004) Theorem 5.1, we can compute the time $t = 0$ price of a European call option written on $S^{i,j}$ with strike $K^{i,j}$ and maturity $T$ by means of the formula

$$\mathcal{CALL}(S^{i,j}(0), K^{i,j}, r^d, r^f, T)$$
$$= e^{-r^d T}\varphi_i(-\mathtt{i}) + \frac{e^{-r^d T}}{2\pi}\int_{-\infty - \mathtt{i}\Im(\zeta)}^{+\infty - \mathtt{i}\Im(\zeta)} e^{-\mathtt{i}\zeta \log K^{i,j}}\frac{\varphi_i(\zeta - \mathtt{i})}{-\zeta(\zeta - \mathtt{i})} d\zeta, \tag{4.44}$$

for $-1 < \Im(\zeta) < 0$. Such formula can be easily implemented and the numerical integration can be performed by means of standard FFT algorithms, which return a vector of option prices for a fixed maturity over a grid of strike prices.

The paper by De Col *et al.* (2013) provides examples of simultaneous calibrations to a triangle of FX implied volatilities of the multifactor stochastic volatility nested in the present framework. A similar calibration experiment can be found in Gnoatto & Grasselli (2014), where Wishart dynamics are considered. Both papers consider the triangle EUR-USD-JPY for a typical trading day and fit a two-dimensional multifactor stochastic volatility model to the three volatility surfaces of

EURUSD, USDJPY and JPYEUR simultaneously. The results are promising and the reader is referred to these references for a deeper understanding of the fitting procedure and the consequent calibration performance.

## 5. Conclusions

In this paper, we investigated models for FX rates. We observed that we may simultaneously consider an FX rate and its inverse, and we looked for a model class that is coherent, i.e. functionally invariant under inversion of the FX rate while being rich enough in order to accommodate for stylized facts like the presence of volatility smiles in the market.

Our main result shows that affine stochastic volatility models represent the ideal candidates for FX modelling, when it comes to guarantee the requirement above. More generally, however, we may not only consider an FX rate and its inverse, but we may also construct FX rates by means of suitable products or ratios of FX rates. The simplest example in this sense is provided by an FX triangle like EUR-USD-JPY. Using a coherent FX model as a building block, we illustrated a possible way to construct a setting where FX rates are functionally symmetric under such compositions.

The results we presented in this paper are not restricted to the case where a risk-neutral measure exists and may be applied also in the context of the more general benchmark approach as in Baldeaux *et al.* (2015) and Gnoatto *et al.* (2016).

## Appendix A. Proof of Corollary 2.1

We follow Del Baño Rollin (2008). The prices of call and put options on $S$ under $\mathbb{Q}_d$ may be expressed, by means of the characteristic functions, via the following formulas:

$$\mathcal{CALL}(S(0), K, r^d, r^f, T) = e^{-r^d T}\left(\frac{1}{2}(F(T) - K) + \frac{1}{\pi}\int_0^\infty (F(T)f_1 - Kf_2)du\right), \tag{A.1}$$

$$\mathcal{PUT}(S(0), K, r^d, r^f, T) = e^{-r^d T}\left(\frac{1}{2}(F(T) - K) - \frac{1}{\pi}\int_0^\infty (F(T)f_1 - Kf_2)du\right), \tag{A.2}$$

where

$$f_1 = \Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u - \mathtt{i})}{\mathtt{i}uF(T)}\right), \tag{A.3}$$

$$f_2 = \Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right). \tag{A.4}$$

Looking now at the foreign domestic parity, we would like to check the agreement between the RHS and the LHS of the following

$$\begin{aligned}&\frac{1}{2}(F(T)-K)+\frac{1}{\pi}\int_0^\infty(F(T)f_1-Kf_2)d\lambda\\&\quad=F(T)K\left(\frac{1}{2}(F(T)^{-1}-K^{-1})-\frac{1}{\pi}\int_0^\infty(F(T)^{-1}f_1'-K^{-1}f_2')d\lambda\right).\end{aligned}\tag{A.5}$$

where $f_i'$ indicates that $f_1, f_2$ are now computed with respect to the foreign risk neutral measure $\mathbb{Q}_f$. We substitute the expressions for $f_1, f_2, f_1', f_2'$ so that

$$\begin{aligned}&F(T)-K\\&\quad=-\frac{1}{\pi}\int_0^\infty\left(F(T)\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF(T)}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\&\qquad-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}\varphi_f(u-\mathtt{i})}{\mathtt{i}uF(T)^{-1}}\right)-F(T)\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}\varphi_f(u)}{\mathtt{i}u}\right)\right)du.\end{aligned}\tag{A.6}$$

Now, under Assumption 2.1, the characteristic functions $\varphi_d, \varphi_f$ are related according to Proposition 2.1 , which allows us to write the following:

$$\begin{aligned}&F(T)-K\\&\quad=-\frac{1}{\pi}\int_0^\infty\left(F(T)\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF(T)}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\&\qquad-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}F(T)^{-1}\varphi_d(-u+\mathtt{i}-\mathtt{i})}{\mathtt{i}uF(T)^{-1}}\right)\right.\\&\qquad\left.-F(T)\Re\left(\frac{e^{-\mathtt{i}u\log K^{-1}}F(T)^{-1}\varphi_d(-u-\mathtt{i})}{\mathtt{i}u}\right)\right)du\\&\quad=-\frac{1}{\pi}\int_0^\infty\left(F(T)\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{\mathtt{i}uF(T)}\right)-K\Re\left(\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{\mathtt{i}u}\right)\right)du\\&\qquad-\frac{1}{\pi}\int_0^\infty\left(K\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u)}{\mathtt{i}u}\right)-F(T)\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u-\mathtt{i})}{\mathtt{i}uF(T)}\right)\right)du.\end{aligned}\tag{A.7}$$

We regroup terms and obtain:

$$\begin{aligned}F(T)-K=&+\frac{1}{\pi}\int_0^\infty\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u-\mathtt{i})}{\mathtt{i}uF(T)}+\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{-\mathtt{i}uF(T)}\right)duF(T)\\&-\frac{1}{\pi}\int_0^\infty\Re\left(\frac{e^{\mathtt{i}u\log K}\varphi_d(-u)}{\mathtt{i}u}+\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{-\mathtt{i}u}\right)duK.\end{aligned}\tag{A.8}$$

We apply the residue theorem to both integrals and obtain respectively

$$\frac{1}{\pi}\left(\frac{1}{2}2\pi\mathtt{i}\lim_{u\to 0} u\frac{e^{-\mathtt{i}u\log K}\varphi_d(u-\mathtt{i})}{-\mathtt{i}uF(T)}\right)F(T) = F(T) \tag{A.9}$$

$$\frac{1}{\pi}\left(\frac{1}{2}2\pi\mathtt{i}\lim_{u\to 0} u\frac{e^{-\mathtt{i}u\log K}\varphi_d(u)}{-\mathtt{i}u}\right)K = K \tag{A.10}$$

Thus showing that the foreign domestic parity is indeed satisfied. □

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