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Analytical pricing formulae for variance and volatility swaps with a new stochastic volatility and interest rate model

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ABSTRACT

We introduce an additional factor in the Heston-CIR model to form a new hybrid model in this paper. This new model features a more general correlation structure with the interest rate being correlated with the underlying asset, while still preserving the analytical tractability. We derive a series solution to the forward characteristic function when solving a time-dependent Riccati equation, after the measure transform is performed, so that variance and volatility swap prices can be finally written in an analytical form. The theoretical results are also accompanied with some numerical results demonstrating its potential of complementing the usage of the currently adopted non-correlated Heston-CIR model.

1. Introduction

Investors in today's financial markets are often making choices between large possible return with high risk and small possible return with low risk. As the market risk is often measured by volatility, the volatility index, or VIX in short, has become a focus of research interests in the past decade. Shortly after the introduction of VIX, VIX-related derivatives, including VIX futures and options among many others were launched and started to gain popularity. Therefore, market practitioners are keen to manage their volatility risk, and volatility derivatives have gained popularity since they enable the investors to be exposed to the volatility of the assets without investing into them. This actually results in an increasing demand for accurately pricing volatility derivatives.

In particular, variance as well as volatility swaps, as two typical examples of volatility derivatives, have attracted research attention widely. As is well-known, the sampling method is an important factor that can significantly affect the obtained variance (volatility) prices, since their payoff explicitly depends on the realized variance (volatility), and continuous and discrete sampling are the two main categories. The former category was initially very popular, since one is even able to obtain some model independent results (Carr & Lee, 2007, 2008). Unfortunately, continuous sampling is not consistent with common practice and can only be regarded as an approximation to the real world. Thus, it is not satisfactory given the potential pricing biases (Bernard & Cui, 2014; Elliott & Lian, 2013; Little & Pant, 2001). Therefore, this has prompted the development of the research in the latter category, the pricing performance of which depends heavily on the selected models.

In fact, stochastic volatility models are clearly the first choice since the nature of the volatility is random, and the model proposed by Heston (1993) is quite successful, as it is the one that is able to reflect some basic properties shown by real data (Beckers, 1983). Thus, under this particular model, the pricing of various financial derivatives has been considered (He & Zhu, 2016a; Zhu & Chen, 2011), and the two swap contracts under consideration have been shown to admit analytical solutions (Zhu & Lian, 2011, 2015).

Of course, the Heston model could not provide a perfect match to real data either, as a result of which more sophisticated models have been established, such as the (local) regime-switching models (He & Zhu, 2017, 2018b; Zhu et al., 2012), time-dependent Heston models (Forde & Jacquier, 2010) and regime-switching Heston models (Elliott & Lian, 2013; He & Zhu, 2016a). In addition, the performance of stochastic volatility models can be significantly improved if the relaxation of the constant interest rate assumption is conducted (Abudy & Izhakian, 2013). The single-factor Heston-CIR model belongs to this category, under which both of the volatility and interest rate follow the CIR process (Grzelak & Oosterlee, 2011; He & Zhu, 2018a). However, Chen et al. (2012) argued that the zero correlation between the interest and underlying would significantly affect the accuracy of the determined prices; such kind of correlation is rarely introduced when the interest rate in the Heston model is stochastic since the analytical tractability cannot be preserved in these cases (van Haastrecht & Pelsser, 2011).

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Empirically, Christoffersen et al. (2009) demonstrated that multi-factor stochastic volatility models can better capture volatility smile. Considering this and the importance of making the underlying and interest rate be correlated, we incorporate the interest rate as an addition factor into the underlying price process to establish a new two-factor Heston-CIR model. This model makes it possible to simultaneously consider the impact of the interest rate on underlying, as well as the correlation between these two. We derive closed-form series solutions to variance and volatility swaps through the numeraire change technique. The convergence of the solutions is theoretically guaranteed by providing the radius of convergence.

We organize the remaining of the paper as follows. Section 2 displays the proposed hybrid model. Section 3 shows the derivation of swap pricing formulae. Section 4 conducts numerical experiments to show the numerical implementation of the derived formulae. Section 5 concludes.

2. A two-factor Heston-CIR model

We now present a new two-factor model for the underlying price, whose first factor is dominated by the Heston stochastic volatility and the second one follows the CIR interest rate process, so that the interaction between the underlying and both factors can be captured. This model under a risk-neutral world is

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t dt + \sqrt{v_t} dW_{1,t}^S + \xi\sqrt{r_t} dW_{2,t}^S, \\ dv_t &= k(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v, \\ dr_t &= \alpha(\beta - r_t)dt + \eta\sqrt{r_t} dW_t^r, \end{aligned} \tag{2.1}$$

where we use S_t to represent the underlying price, r_t to denote the interest rate, and v_t to stand for the volatility. $W_{1,t}^S$, $W_{2,t}^S$, W_t^v and W_t^r are four standard Brownian motions. S_t is correlated to both v_t and r_t, through the correlation between $W_{1,t}^S$ and W_t^v (with a correlation parameter ρ_1) and that between $W_{2,t}^S$ and W_t^r (with a correlation parameter ρ_2), respectively. Other pairs are independent. By making use of four independent Brownian motions, $W_{1,t}^Q$, $W_{2,t}^Q$, $W_{3,t}^Q$ and $W_{4,t}^Q$, the model dynamics can be further represented as

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ dv_t \\ dr_t \end{bmatrix} = \mu^Q dt + \Sigma \times C \times \begin{bmatrix} dW_{1,t}^Q \\ dW_{2,t}^Q \\ dW_{3,t}^Q \\ dW_{4,t}^Q \end{bmatrix}, \tag{2.2}$$

where $\mu^Q = \left[r_t,\ k(\theta - v_t),\ \alpha(\beta - r_t)\right]^T$, and

$$\Sigma = \begin{bmatrix} \sqrt{v_t} & \xi\sqrt{r_t} & 0 & 0 \\ 0 & 0 & \sigma\sqrt{v_t} & 0 \\ 0 & 0 & 0 & \eta\sqrt{r_t} \end{bmatrix}. \tag{2.3}$$

C is given as

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \rho_1 & 0 & \sqrt{1-\rho_1^2} & 0 \\ 0 & \rho_2 & 0 & \sqrt{1-\rho_2^2} \end{bmatrix}, \tag{2.4}$$

such that the correlation matrix is

$$CC^T = \begin{bmatrix} 1 & 0 & \rho_1 & 0 \\ 0 & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{bmatrix}.$$

The following section is going to discuss how the two swap contacts can be priced under the new two-factor model.

3. Pricing variance and volatility swaps

We adopt a two-stage solution procedure to price variance and volatility swaps. The first stage carries out the pricing of the two swaps by presenting the general formulae through the change of numeraire, with the forward characteristic function being unsolved. The second stage provides an analytical series solution to the forward characteristic function, which is proved to be convergent, and thus the obtained pricing formulae are fully analytical.

3.1. A general pricing approach

There are some specific contract parameters of the consider two swaps, i.e., the notional amount (L), the annualized realized variance/volatility (RV_{var}/RV_{vol}) and the strike prices of variance/volatility swaps (K_{var}/K_{vol}). With these, the values of the two contracts can be evaluated through

$$\begin{aligned} V_{var} &= E^Q\left[e^{-\int_0^T r_t dt}(RV_{var} - K_{var})L|S_0, v_0, r_0\right], \\ V_{vol} &= E^Q\left[e^{-\int_0^T r_t dt}(RV_{vol} - K_{vol})L|S_0, v_0, r_0\right], \end{aligned}$$

which respectively represent the values of variance swaps and those of volatility swaps.

A well-known feature of a swap contract is that its price is 0 when it is initially entered into, which implies $V_{var} = 0$ and $V_{vol} = 0$, leading to

$$\begin{aligned} E^Q\left[e^{-\int_0^T r_t dt}(RV_{var} - K_{var})L|S_0, v_0, r_0\right] &= 0, \\ E^Q\left[e^{-\int_0^T r_t dt}(RV_{vol} - K_{vol})L|S_0, v_0, r_0\right] &= 0. \end{aligned}$$

To further simplify the above equations so as to get an expression for the target K_{var} and K_{vol}, we make use of the forward measure $\mathbb{Q}^T$ (Brigo & Mercurio, 2007) so that

$$\begin{aligned} &E^Q\left[e^{-\int_0^T r_t dt}(RV_{var} - K_{var})L|S_0, v_0, r_0\right] \\ &= P(r, 0, T)E^{Q^T}\left[(RV_{var} - K_{var})L|S_0, v_0, r_0\right], \\ &E^Q\left[e^{-\int_0^T r_t dt}(RV_{vol} - K_{vol})L|S_0, v_0, r_0\right] \\ &= P(r, 0, T)E^{Q^T}\left[(RV_{vol} - K_{vol})L|S_0, v_0, r_0\right], \end{aligned}$$

where $P(r,t,T)$ is equal to the bond price paying no coupons expiring at T under $\mathbb{Q}$, the solution to which can be written in the form of[1]

$$P(r, t, T) = e^{A_1(t,T) - A_2(t,T)r}. \tag{3.1}$$

Here,

$$\begin{aligned} A_1(t,T) &= -\alpha\beta\left\{\frac{4}{(l-\alpha)(l+\alpha)}\ln\left[\frac{2l + (l+\alpha)(e^{l(T-t)} - 1)}{2l}\right] + \frac{2}{\alpha - l}(T-t)\right\}, \\ A_2(t,T) &= \frac{2(e^{l(T-t)} - 1)}{2l + (\alpha + l)(e^{l(T-t)} - 1)}, \end{aligned}$$

with $l = \sqrt{\alpha^2 + 2\eta^2}$. Therefore, we can obtain

$$K_{var} = E^{Q^T}[RV_{var}|S_0, v_0, r_0], \quad K_{vol} = E^{Q^T}[RV_{vol}|S_0, v_0, r_0].$$

Clearly, the first thing is to figure out the expression of the annualized realized variance (volatility), and one measure that has been widely adopted (Elliott & Lian, 2013; Howison et al., 2004) is given by

$$\begin{aligned} RV_{var} &= \frac{100^2}{T}\sum_{i=1}^{N}\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2, \\ RV_{vol} &= 100\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|, \end{aligned}$$

[1] We refer interested readers to Brigo and Mercurio (2007) and He and Zhu (2018a) for more details.

with the assumption of a uniform division for $[0,T]$. With such an set-up, we can compute K_{var} as well as K_{vol} through

$$\begin{aligned} K_{var} &= \frac{100^2}{T}\sum_{i=1}^{N} E^{Q^T}\left[(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}})^2\Big|S_0,v_0,r_0\right], \\ K_{vol} &= 100\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N} E^{Q^T}\left[|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}|\Big|S_0,v_0,r_0\right]. \end{aligned} \tag{3.2}$$

One should appreciate that once the $2N$ expectations presented in the above formulae are worked out, the target delivery prices of the two swaps can be directly obtained. By noticing the fact that these expectation should be computed under $\mathbb{Q}^T$, it is necessary to transform the specified model dynamics under $\mathbb{Q}$ to that under $\mathbb{Q}^T$, the details of which are illustrated below.

In order to conduct measure transform, it needs to be pointed out that $N_{1,t} = e^{\int_0^t r(s)ds}$ is the numeraire of the original risk-neutral measure, with

$$dN_{1,t} = N_{1,t}r(t)dt, \tag{3.3}$$

while $N_{2,t} = P(r,t,T)$ represents the numeraire under the forward measure, yielding

$$dN_{2,t} = N_{2,t}\left\{\left[\frac{dA}{dt} - r\frac{dB}{dt} - \alpha(\beta-r)B - \frac{1}{2}\eta^2 rB^2\right]dt - \eta\sqrt{r}BdW_{3,t}\right\}. \tag{3.4}$$

As a result, the volatility term of $N_{1,t}$ can be derived as $\sigma^{N_{1,t}} = (0,0,0,0)^T$, while the volatility term of $N_{2,t}$ can be specified as $\sigma^{N_{2,t}} = (0,0,0,-\eta\sqrt{r}N_{2,t}B)^T$. From the theory presented in Brigo and Mercurio (2007), the drift term μ^{Q_T} can be computed through

$$\begin{aligned} \mu^{Q_T} &= \mu^{Q} - \Sigma\times\rho\times(\frac{\sigma^{N_{1,t}}}{N_{1,t}} - \frac{\sigma^{N_{2,t}}}{N_{2,t}}) \\ &= \begin{bmatrix} r_t \\ k(\theta-v_t) \\ \alpha(\beta-r_t) \end{bmatrix} - \begin{bmatrix} \sqrt{v_t} & \xi\sqrt{r_t} & 0 & 0 \\ 0 & 0 & \sigma\sqrt{v_t} & 0 \\ 0 & 0 & 0 & \eta\sqrt{r_t} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & \rho_1 & 0 \\ 0 & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ \eta\sqrt{r_t}B \end{bmatrix} \\ &= \begin{bmatrix} (1-\rho_2\eta\xi B)r_t \\ k(\theta-v_t) \\ \alpha\beta - [\alpha+B\eta^2]r_t \end{bmatrix}, \end{aligned}$$

with which we obtain

$$\begin{bmatrix} \frac{dS_t}{S_t} \\ dv_t \\ dr_t \end{bmatrix} = \begin{bmatrix} (1-\rho_2\eta\xi B)r_t \\ k(\theta-v_t) \\ \alpha\beta - [\alpha+B\eta^2]r_t \end{bmatrix} dt + \Sigma\times C\times \begin{bmatrix} dW_{1,t}^{Q^T} \\ dW_{2,t}^{Q^T} \\ dW_{3,t}^{Q^T} \\ dW_{4,t}^{Q^T} \end{bmatrix}. \tag{3.5}$$

Eq. (3.5) displays the model formulation under $\mathbb{Q}^T$, which is needed in dealing with the $2N$ expectations contained in Eq. (3.2). We now start with the variance swap, rewriting K_{var} as

$$\begin{aligned} K_{var} &= \frac{100^2}{T}\sum_{i=1}^{N} E^{Q^T}[(e^{y_{t_{i-1},t_i}}-1)^2|S_0,v_0,r_0] \\ &= \frac{100^2}{T}\sum_{i=1}^{N} E^{Q^T}[e^{2y_{t_{i-1},t_i}} - 2e^{y_{t_{i-1},t_i}} + 1|S_0,v_0,r_0], \end{aligned}$$

with $y_{t,T} = \ln(\frac{S_T}{S_t})$. This can further lead to

$$K_{var} = \frac{100^2}{T}\sum_{i=1}^{N}[f(-2j,t_{i-1},t_i;S_0,v_0,r_0) - 2f(-j,t_{i-1},t_i;S_0,v_0,r_0) + 1], \tag{3.6}$$

if j represents the imaginary unit, and $f(\phi,t,T;S_0,v_0,r_0)$ denotes the forward characteristic function, the formula for which is

$$f(\phi,t,T;S_0,v_0,r_0) = E^{Q^T}[e^{j\phi y_{t,T}}|S_0,v_0,r_0]. \tag{3.7}$$

The simplification for K_{vol} associated with volatility swaps requires some further calculations, as it involves the absolute value of a random variable in the target expectations. By denoting $p(y_{t_{i-1},t_i})$ as the forward density function of y_{t_{i-1},t_i}, we can easily derive

$$\begin{aligned} E^{Q^T}\left[|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}|\Big|S_0,v_0,r_0\right] &= \int_0^{+\infty}(e^{y_{t_{i-1},t_i}}-1)p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &\quad + \int_{-\infty}^{0}(1-e^{y_{t_{i-1},t_i}})p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &= -\int_0^{+\infty} p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &\quad + \int_{-\infty}^{0} p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &\quad + \int_0^{+\infty} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &\quad - \int_{-\infty}^{0} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}. \end{aligned} \tag{3.8}$$

The first integral can be viewed as the probability that y_{t_{i-1},t_i} is greater than 0, and thus we can obtain

$$\int_0^{+\infty} p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} RE\left[\frac{f(\phi,t_{i-1},t_i;S_0,v_0,r_0)}{j\phi}\right]d\phi, \tag{3.9}$$

from which the second integral can be straightforwardly derived as

$$\int_{-\infty}^{0} p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} RE\left[\frac{f(\phi,t_{i-1},t_i;S_0,v_0,r_0)}{j\phi}\right]d\phi. \tag{3.10}$$

The third integral can be firstly transformed into

$$\begin{aligned} \int_0^{+\infty} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} &= f(-j,t_{i-1},t_i;S_0,v_0,r_0) \\ &\quad \times \int_0^{+\infty}\frac{e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})}{f(-j,t_{i-1},t_i;S_0,v_0,r_0)}dy_{t_{i-1},t_i}, \end{aligned}$$

as $\frac{e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})}{f(-j,t_{i-1},t_i;S_0,v_0,r_0)}$ is another forward density function according to the following identity

$$\int_{-\infty}^{+\infty} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} = f(-j,t_{i-1},t_i;S_0,v_0,r_0).$$

With the forward characteristic function corresponding to the density $\frac{e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})}{f(-j,t_{i-1},t_i;S_0,v_0,r_0)}$ derived as $\frac{f(\phi,t,T;S_0,v_0,r_0)}{f(-j;t,T;S_0,v_0,r_0)}$ by performing the Fourier transform of its density, we can thus represent the third integral using

$$\begin{aligned} &\int_0^{+\infty} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} \\ &= f(-j,t_{i-1},t_i;S_0,v_0,r_0) \\ &\times\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} RE\left[\frac{f(\phi-j,t_{i-1},t_i;S_0,v_0,r_0)}{j\phi\cdot f(-j,t_{i-1},t_i;S_0,v_0,r_0)}\right]d\phi\right\}. \end{aligned} \tag{3.11}$$

Given that the last integral can be expressed as

$$\begin{aligned} \int_{-\infty}^{0} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i} &= f(-j,t_{i-1},t_i;S_0,v_0,r_0) \\ &\quad \times \int_{-\infty}^{0}\frac{e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})}{f(-j,t_{i-1},t_i;S_0,v_0,r_0)}dy_{t_{i-1},t_i}, \end{aligned}$$

it can be further derived as

$$\int_{-\infty}^{0} e^{y_{t_{i-1},t_i}}p(y_{t_{i-1},t_i})dy_{t_{i-1},t_i}$$

$$= f(-j, t_{i-1}, t_i; S_0, v_0, r_0) \times \left\{ \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} RE\left[\frac{f(\phi - j, t_{i-1}, t_i; S_0, v_0, r_0)}{j\phi \cdot f(-j, t_{i-1}, t_i; S_0, v_0, r_0)} \right] d\phi \right\}. \tag{3.12}$$

Combining the expression of the four integrals finally yields

$$E^{Q^T}\left[\left| \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right| \Big| S_0, v_0, r_0 \right] = \frac{2}{\pi} \int_0^{+\infty} RE\left[\frac{f(\phi - j, t_{i-1}, t_i; S_0, v_0, r_0) - f(\phi, t_{i-1}, t_i; S_0, v_0, r_0)}{j\phi} \right] d\phi, \tag{3.13}$$

resulting in

$$K_{vol} = 100\sqrt{\frac{2}{\pi NT}} \int_0^{+\infty} \sum_{i=1}^{N} RE \times \left[\frac{f(\phi - j, t_{i-1}, t_i; S_0, v_0, r_0) - f(\phi, t_{i-1}, t_i; S_0, v_0, r_0)}{j\phi} \right] d\phi. \tag{3.14}$$

Considering the fact that both formulae, (3.6) and (3.14), for the delivery prices of the two swaps involve only one unknown term, i.e., $f(\phi, t, T; S_0, v_0, r_0)$, in the following subsection, we will provide the details on how to derive this particular forward characteristic function, whose dynamics are given in Eq. (3.5).

3.2. The forward characteristic function

The first stage of this subsection is to assume that the current time is t and presents the formula for the forward characteristic function. With the results in the first stage, we remove the assumption to make 0 as the current time, and compute the expectation of the characteristic function, which forms the main task of the second stage.

Applying the chain rule yields

$$\begin{aligned} f(\phi, t, T; S_0, v_0, r_0) &= E^{Q^T}[e^{j\phi y_{t,T}} | S_0, v_0, r_0] \\ &= E^{Q^T}\left\{ e^{-j\phi z_t} E^{Q^T}[e^{j\phi z_T} | z_t, v_t, r_t] \Big| z_0, v_0, r_0 \right\}, \end{aligned} \tag{3.15}$$

with $z_t = \ln(S_t)$. Therefore, the inner expectation should be firstly computed. If we denote

$$m(\phi; \tau, z_t, v_t, r_t) = E^{Q^T}[e^{j\phi z_T} | z_t, v_t, r_t], \ \tau = T - t,$$

its derivation process is presented below.

Theorem 1. *With Eq. (3.5) showing the underlying model, we obtain*

$$m = e^{C(\phi;\tau) + D(\phi;\tau)v_t + E(\phi;\tau)r_t + j\phi z_t}, \tag{3.16}$$

where

$$\begin{aligned}
D &= \frac{d - (j\phi\rho_1\sigma - k)}{\sigma^2} \cdot \frac{1 - e^{d\tau}}{1 - ge^{d\tau}}, \\
E &= -\frac{2\sum_{n=0}^{+\infty}(n+1)\hat{a}_{n+1}\tau^n}{\eta^2 \sum_{n=0}^{+\infty}\hat{a}_n\tau^n}, \\
C &= \frac{k\theta}{\sigma^2}\{[d - (j\phi\rho_1\sigma - k)]\tau - 2\ln(\frac{1 - ge^{d\tau}}{1 - g}) \\
&\quad - \alpha\beta \int_0^{\tau} \frac{2\sum_{n=0}^{+\infty}(n+1)\hat{a}_{n+1}t^n}{\eta^2 \sum_{n=0}^{+\infty}\hat{a}_n t^n} dt\}, \\
d &= \sqrt{(j\phi\rho_1\sigma - k)^2 + \sigma^2(j\phi + \phi^2)}, \ g = \frac{(j\phi\rho_1\sigma - k) - d}{(j\phi\rho_1\sigma - k) + d},
\end{aligned}$$

$$\begin{aligned}
\hat{a}_{n+2} &= -\frac{\hat{I}}{2l(n+1)(n+2)}, n \geq 0, \ \hat{a}_0 = 1, \ \hat{a}_1 = 0, \\
\hat{I} &= 2l(\alpha - j\phi\rho_2\eta\xi)(n+1)\hat{a}_{n+1} + \eta^2 l(j\phi - \frac{1}{2}j\phi\xi^2 - \frac{1}{2}\xi^2\phi^2)\hat{a}_n \\
&\quad + (\alpha + l)\sum_{i=1}^{n}(n+2-i)(n+1-i)c_i\hat{a}_{n+2-i} \\
&\quad + [(\alpha - j\phi\rho_2\eta\xi)(\alpha + l) + 2\eta^2]\sum_{i=1}^{n}(n+1-i)c_i\hat{a}_{n+1-i} \\
&\quad + \frac{1}{2}\eta^2[(j\phi - \frac{1}{2}j\phi\xi^2 - \frac{1}{2}\xi^2\phi^2)(\alpha + l) - 2j\phi\rho_2\eta\xi]\sum_{i=1}^{n}c_i\hat{a}_{n-i},
\end{aligned}$$

with $c_i = \frac{l^n}{n!}$.

Proof. With the definition of the characteristic function,

$$m(\phi; \tau, z_t, v_t, r_t) = E^{Q^T}[e^{j\phi z_T} | y_t, v_t, r_t],$$

the Feynman–Kac theorem demonstrates that $m(\phi; \tau, z_t, v_t, r_t)$ should satisfy the following PDE (partial differential equation)

$$\begin{aligned}
\frac{\partial m}{\partial \tau} &= (\frac{1}{2}v + \frac{1}{2}\xi^2 r)\frac{\partial^2 m}{\partial z^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 m}{\partial v^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 m}{\partial r^2} \\
&\quad + \rho_1\sigma v\frac{\partial^2 m}{\partial z \partial v} + \rho_2\eta\xi r\frac{\partial^2 m}{\partial z \partial r} \\
&\quad + (r - \rho_2\eta\xi Br - \frac{1}{2}v - \frac{1}{2}\xi^2 r)\frac{\partial m}{\partial z} + k(\theta - v)\frac{\partial m}{\partial v} \\
&\quad + [\alpha\beta - (\alpha + B\eta^2)r]\frac{\partial m}{\partial r},
\end{aligned} \tag{3.17}$$

with

$$m|_{\tau=0} = e^{j\phi y_t}.$$

Following Heston (1993) and He and Zhu (2016b), we assume the form of $m(\phi; \tau, z_t, v_t, r_t)$ is as given in Eq. (3.16), so that PDE (3.17) reduces to the following systems

$$\begin{aligned}
\frac{dD}{d\tau} &= \frac{1}{2}\sigma^2 D^2 + (j\phi_1\rho\sigma - k)D - \frac{1}{2}(j\phi + \phi^2), \\
\frac{dE}{d\tau} &= \frac{1}{2}\eta^2 E^2 + [j\phi\rho_2\eta\xi - (\alpha + \eta^2 B(\tau))]E + j\phi - \frac{1}{2}\xi^2\phi^2 \\
&\quad - \frac{1}{2}j\phi\xi^2 - j\phi\rho_2\eta\xi B(\tau), \\
\frac{dC}{d\tau} &= k\theta D + \alpha\beta E,
\end{aligned}$$

with $C(\phi; 0) = D(\phi; 0) = E(\phi; 0) = 0$. Although the first two ordinary differential equations (ODEs) are Riccati equations, the derivation of the solution to $E(\phi; \tau)$ is not as easy as that of $D(\phi; \tau)$, since variable coefficients can be identified in the governing equation of $E(\phi; \tau)$. To solve for $E(\phi; \tau)$, if we write

$$E(\phi; \tau) = -\frac{2u'(\tau)}{\eta^2 u(\tau)}, \tag{3.18}$$

the ODE for $E(\phi; \tau)$ can be transformed into

$$u'' + [j\phi\rho_2\eta\xi - (\alpha + \eta^2 B(\tau))]u' + \frac{1}{2}\eta^2[j\phi - \frac{1}{2}\xi^2\phi^2 - \frac{1}{2}j\phi\xi^2 - j\phi\rho_2\eta\xi B(\tau)] = 0. \tag{3.19}$$

The obtained second-order linear ODE contains time-dependent coefficients, the existence of an analytical solution to which is not guaranteed[2], and thus we try to find one in series form. Specifically, we write

$$u = \sum_{n=0}^{+\infty} a_n \tau^n, \tag{3.20}$$

with which Eq. (3.19) can be reformulated as

$$\begin{aligned}
&[2l + (\alpha + l)(e^{l\tau} - 1)]\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}\tau^n \\
&+ \{(\alpha - j\phi\rho_2\eta\xi)[2l + (\alpha + l)(e^{l\tau} - 1)] + 2\eta^2(e^{l\tau} - 1)\}\sum_{n=0}^{+\infty}(n+1)a_{n+1}\tau^n \\
&+ \frac{1}{2}\eta^2\{(j\phi - \frac{1}{2}\xi^2\phi^2 - \frac{1}{2}j\phi\xi^2)[2l + (\alpha + l)(e^{l\tau} - 1)] - 2j\phi\rho_2\eta\xi(e^{l\tau} - 1)\}
\end{aligned}$$

[2] One should notice that if a similar transform is made for the ODE governing $D(\phi; \tau)$, it will be turned into an ODE whose coefficients are all constant. That means $D(\phi; \tau)$ can be easily solved and in this sense, we will only show how $E(\phi; \tau)$ can be analytically obtained while omitting the details related to the derivation process of $D(\phi; \tau)$.

$$\times \sum_{n=0}^{+\infty} a_n\tau^n = 0. \tag{3.21}$$

With the Taylor series of $e^{l\tau}$ being $e^{l\tau} = \sum_{n=0}^{+\infty} c_n\tau^n$, Eq. (3.21) can be further expressed as

$$\begin{aligned}
&[2l+(\alpha+l)\sum_{n=1}^{+\infty} c_n\tau_n]\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}\tau^n \\
&+\{2l(\alpha-j\phi\rho_2\eta\xi)+[(\alpha-j\phi\rho_2\eta\xi)(\alpha+l)+2\eta^2]\sum_{n=1}^{+\infty} c_n\tau_n\}\sum_{n=0}^{+\infty}(n+1)a_{n+1}\tau^n \\
&+\{\eta^2 l(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)+\frac{1}{2}\eta^2[(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)(\alpha+l) \\
&-2j\phi\rho_2\eta\xi]\sum_{n=1}^{+\infty} c_n\tau_n\}\sum_{n=0}^{+\infty} a_n\tau^n = 0.
\end{aligned}$$

As the above equation should hold for any τ so as to ensure a solution to the original ODE, we should set

$$\begin{aligned}
&2l(n+1)(n+2)a_{n+2}+2l(\alpha-j\phi\rho_2\eta\xi)(n+1)a_{n+1} \\
&+\eta^2 l(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)a_n \\
&+(\alpha+l)\sum_{i=1}^{n}(n+2-i)(n+1-i)c_i a_{n+2-i}+[(\alpha-j\phi\rho_2\eta\xi)(\alpha+l)+2\eta^2] \\
&\times\sum_{i=1}^{n}(n+1-i)c_i a_{n+1-i} \\
&+\frac{1}{2}\eta^2[(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)(\alpha+l)-2j\phi\rho_2\eta\xi]\sum_{i=1}^{n} c_i a_{n-i} = 0.
\end{aligned} \tag{3.22}$$

Therefore, we can finally arrive at

$$a_{n+2} = -\frac{I}{2l(n+1)(n+2)}, \tag{3.23}$$

with

$$\begin{aligned}
I \;=\; & 2l(\alpha-j\phi\rho_2\eta\xi)(n+1)a_{n+1}+\eta^2 l(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)a_n \\
&+(\alpha+l)\sum_{i=1}^{n}(n+2-i)(n+1-i)c_i a_{n+2-i}+[(\alpha-j\phi\rho_2\eta\xi)(\alpha+l)+2\eta^2] \\
&\times\sum_{i=1}^{n}(n+1-i)c_i a_{n+1-i} \\
&+\frac{1}{2}\eta^2[(j\phi-\frac{1}{2}\xi^2\phi^2-\frac{1}{2}j\phi\xi^2)(\alpha+l)-2j\phi\rho_2\eta\xi]\sum_{i=1}^{n} c_i a_{n-i} = 0.
\end{aligned}$$

Although it seems to be very straightforward that all the coefficients $a_n, n \geq 2$ can be figured out starting with the value of a_0 and a_1, and $a_1 = 0$ can be directly derived through the initial condition for $E(\phi;\tau)$, we actually have no access to the value of a_0. In this case, we go another way by introducing new coefficients as the original ones divided by a_0, i.e., $\hat{a}_n = \frac{a_n}{a_0}$, so that

$$E = -\frac{2\sum_{n=0}^{+\infty}(n+1)\hat{a}_{n+1}\tau^n}{\eta^2\sum_{n=0}^{+\infty}\hat{a}_n\tau^n}, \tag{3.24}$$

with a new recurrence relationship being

$$\hat{a}_{n+2} = -\frac{\hat{I}}{2m(n+1)(n+2)}, \tag{3.25}$$

and $\hat{I} = \frac{I}{a_0}$. This new relationship enables all the coefficients to be derived since now we have $\hat{a}_0 = 1$ and $\hat{a}_1 = 0$. Finally, the derivation of $C(\phi;\tau)$ is straightforward through integration, which marks the last step of the proof. □

According to Eq. (3.15), the left work is to find the outer expectation, being exactly

$$\begin{aligned}
f(\phi,t,T;v_0,r_0) \;&=\; E^{Q^T}[e^{-j\phi z_t}m(\phi;\tau,z_t,v_t,r_t)|z_0,v_0,r_0], \\
&=\; e^{C(\phi;\tau)}E^{Q^T}\{e^{D(\phi;\tau)v_t+E(\phi;\tau)r_t}|v_0,r_0\},
\end{aligned} \tag{3.26}$$

which is resulted from the substitution of Eq. (3.16).[3] Clearly, one has to work out

$$w(\phi,t,T;v_s,r_s,s) = E^{Q^T}\{e^{D(\phi;\tau)v_t+E(\phi;\tau)r_t}|v_s,r_s\},\ \ s\in[0,t], \tag{3.27}$$

whose governing PDE system is

$$\begin{cases}
\frac{\partial w}{\partial s}+\frac{1}{2}\sigma^2 v\frac{\partial^2 w}{\partial v^2}+\frac{1}{2}\eta^2 r\frac{\partial^2 w}{\partial r^2}+k(\theta-v)\frac{\partial w}{\partial v} \\
\quad +\{\alpha\beta-[\alpha+B(s,T)]r\}\frac{\partial w}{\partial r}=0, \\
w|_{s=t} = e^{D(\phi;\tau)v_t+E(\phi;\tau)r_t}
\end{cases} \tag{3.28}$$

The assumption of

$$w(\phi,t,T;v_s,r_s,s) = e^{\bar{C}(\phi;t_s)+\bar{D}(\phi;t_s)v_s+\bar{E}(\phi;t_s)r_s},\ \ t_s = t-s \tag{3.29}$$

leads to the ODE systems below

$$\begin{aligned}
\frac{d\bar{D}}{dt_s} &= \frac{1}{2}\sigma^2\bar{D}^2-k\bar{D},\ \ \bar{D}(\phi;0)=D(\phi;\tau), \\
\frac{d\bar{E}}{dt_s} &= \frac{1}{2}\eta^2\bar{E}^2-[\alpha+B(s,T)\eta^2]E,\ \ \bar{E}(\phi;0)=E(\phi;\tau), \\
\frac{d\bar{C}}{dt_s} &= k\theta\bar{D}+\alpha\beta\bar{E},\ \ \bar{C}(\phi;0)=0,
\end{aligned}$$

As a Bernoulli's equation with all the coefficients being constant, the solution to the first ODE can be formulated as

$$\bar{D}(\phi;t_s) = \frac{2k}{\sigma^2}\frac{1}{1-[1-\frac{2k}{\sigma^2 D(\phi;\tau)}]e^{kt_s}}. \tag{3.30}$$

The second ODE is a Bernoulli's equation as well, but with time-dependent coefficients, and it can be transformed into

$$u'-[\alpha+\bar{B}(t_s,\tau)\eta^2]u = -\frac{1}{2}\eta^2,$$

with $u(t_s) = \frac{1}{\bar{E}(\phi;t_s)}$, and $\bar{B}(t_s,\tau)$ defined as

$$\bar{B}(t_s,\tau) = B(s,T) = \frac{2(e^{l(t_s+\tau)}-1)}{2l+(\alpha+l)(e^{l(t_s+\tau)}-1)}.$$

In this case, the expression of $u(t_s)$ can be presented as

$$u = \frac{\int_0^{t_s}-\frac{1}{2}\eta^2 e^{-\int_0^z \alpha+\bar{B}(x,\tau)\eta^2 dx}dz+c_0}{e^{-\int_0^{t_s}\alpha+\bar{B}(x,\tau)\eta^2 dx}}, \tag{3.31}$$

from which we have

$$\bar{E}(\phi;t_s) = \frac{1}{u(t_s)} = \frac{e^{-\int_0^{t_s}\alpha+\bar{B}(x,\tau)\eta^2 dx}}{\int_0^{t_s}-\frac{1}{2}\eta^2 e^{-\int_0^z \alpha+\bar{B}(x,\tau)\eta^2 dx}dz+c_0}. \tag{3.32}$$

The initial condition of its ODE gives $c_0 = \frac{1}{E(\phi;\tau)}$, and this further leads to

$$\bar{E}(\phi;t_s) = \frac{e^{-(\alpha+\frac{2\eta^2}{\alpha-l})t_s}\left\{\frac{2l+(l+\alpha)[e^{l(\tau+t_s)}-1]}{2l+(l+\alpha)(e^{l\tau}-1)}\right\}^{-\frac{4\eta^2}{(l-\alpha)(l+\alpha)}}}{-\frac{1}{2}\eta^2\int_0^{t_s}e^{-(\alpha+\frac{2\eta^2}{\alpha-l})x}\left\{\frac{2l+(l+\alpha)[e^{l(\tau+x)}-1]}{2l+(l+\alpha)(e^{l\tau}-1)}\right\}^{-\frac{4\eta^2}{(l-\alpha)(l+\alpha)}}dx+\frac{1}{E(\phi;\tau)}}. \tag{3.33}$$

Thus, $\bar{C}(\phi;t_s)$ can be derived as

$$\begin{aligned}
\bar{C}(\phi;t_s) = \frac{2k\theta}{\sigma^2}&\{kt_s-\ln[1-(1-\frac{2k}{\sigma^2 D(\phi;\tau)})e^{kt_s}]+\ln(\frac{2k}{\sigma^2 D(\phi;\tau)})\} \\
&+\alpha\beta\int_0^{t_s}\bar{E}(\phi;z)dz.
\end{aligned} \tag{3.34}$$

Finally, setting $s = 0$ yields

$$f(\phi,t,T;v_0,r_0) \;=\; e^{C(\phi;\tau)}w(\phi,t,T;v_0,r_0,0)$$

[3] The substitution here actually cancels the term z_t involved in the expectation, resulting in the disappearance of the underlying in (3.26).

$$= e^{C(\phi;\tau)+\bar{C}(\phi;t)+\bar{D}(\phi;t)v_0+\bar{E}(\phi;t)r_0}. \tag{3.35}$$

Till now, we have provided completely explicit swap pricing formulae. However, the new formulae are still not in analytical forms since we are not sure about the convergence at this stage. Fortunately, we manage to establish the radius of convergence after finding out the fact that we only need to consider the convergence of $u(\tau)$ defined in (3.20) when solving Eq. (3.19). The following theorem establishes its theoretical convergence.

Theorem 2. *The series solution $u = \sum_{n=0}^{+\infty} a_n \tau^n$ converges if*

$$\tau \le \frac{1}{l}\sqrt{[\ln(\frac{l-\alpha}{l+\alpha})]^2 + \pi^2}. \tag{3.36}$$

Proof. According to Bender and Orszag (1999), if the expansion point in the solution of a second order linear ordinary differential equation is ordinary, its minimum radius of convergence can be determined using its distance from the nearest singularity. $\tau = 0$ is our expansion point, and it is not difficult to find that it is ordinary, implying that we should find out its nearest singularity. Clearly, apart from the case where the denominator of the coefficient $2l + (\alpha + l)(e^{l\tau} - 1) = 0$, both $j\phi\rho_2\eta\xi - [\alpha + \eta^2 B(\tau)]$ and $\frac{1}{2}\eta^2[j\phi - \frac{1}{2}\xi^2\phi^2 - \frac{1}{2}j\phi\xi^2 - j\phi\rho_2\eta\xi B(\tau)]$ are analytic. This means that all the singularities are

$$\tau_k = \frac{1}{l}\ln(\frac{l-\alpha}{l+\alpha}) + j\frac{(2k+1)\pi}{l}, \ k = 0, 1, 2..., \tag{3.37}$$

as a consequence of which the target nearest singularity is found to be $\frac{1}{l}\ln(\frac{l-\alpha}{l+\alpha}) + j\frac{\pi}{l}$, and the result follows. □

Obviously, we have theoretically presented converged series solutions for valuing the two swaps with our two-factor Heston-CIR model. The corresponding pricing formulae are presented in (3.6) and (3.14), respectively, combined with the solution to $f(\phi, t, T; v_0, r_0)$ being given in (3.35).

3.3. Hedging ratios

Since variance and volatility swap prices have been obtained analytically, it is therefore quite straightforward to obtain the analytical expression of all the hedging ratios, or the Greeks, by differentiating the prices. In this subsection, we present some important hedging ratios under our model.

One of the most important hedging ratios is the delta of variance and volatility swaps (Broadie & Jain, 2008), which is defined as the first-order derivative of the fair strike prices with respect to the variance, v_0. The calculation of the delta is quite straightforward, by differentiating K_{var} and K_{vol} respectively defined in (3.6) and (3.14) with respect to v_0, so that we can obtain

$$\Delta_{var} = \frac{\partial K_{var}}{\partial v_0} = \frac{100^2}{T}\sum_{i=1}^{N}[\frac{\partial f(-2j, t_{i-1}, t_i; v_0, r_0)}{\partial v_0} - 2\frac{\partial f(-j, t_{i-1}, t_i; v_0, r_0)}{\partial v_0}], \tag{3.38}$$

and

$$\Delta_{vol} = \frac{\partial K_{vol}}{\partial v_0} = 100\sqrt{\frac{2}{\pi N T}}\int_0^{+\infty}\sum_{i=1}^{N} RE\{\frac{1}{j\phi}[\frac{\partial f(\phi - j, t_{i-1}, t_i; v_0, r_0)}{\partial v_0} - \frac{\partial f(\phi, t_{i-1}, t_i; v_0, r_0)}{\partial v_0}]\}d\phi, \tag{3.39}$$

where $\frac{\partial f(\phi,t,T;v_0,r_0)}{\partial v_0} = f(\phi, t, T; v_0, r_0) \cdot \bar{D}(\phi; t)$.

As our model introduces the correlation between the underlying price and interest rate, it is of interest to investigate the hedging ratio associated with the newly introduced correlation parameter ρ_2. In particular, we can compute

$$\Lambda_{var} = \frac{\partial K_{var}}{\partial \rho_2} = \frac{100^2}{T}\sum_{i=1}^{N}[\frac{\partial f(-2j, t_{i-1}, t_i; v_0, r_0)}{\partial \rho_2} - 2\frac{\partial f(-j, t_{i-1}, t_i; v_0, r_0)}{\partial \rho_2}], \tag{3.40}$$

and

$$\Lambda_{vol} = \frac{\partial K_{vol}}{\partial \rho_2} = 100\sqrt{\frac{2}{\pi N T}}\int_0^{+\infty}\sum_{i=1}^{N} RE\{\frac{1}{j\phi}[\frac{\partial f(\phi - j, t_{i-1}, t_i; v_0, r_0)}{\partial \rho_2} - \frac{\partial f(\phi, t_{i-1}, t_i; v_0, r_0)}{\partial \rho_2}]\}d\phi, \tag{3.41}$$

implying that we need to find $\frac{\partial f(\phi,t,T;v_0,r_0)}{\partial \rho_2}$, the derivation of which requires some further calculation. In fact, we have

$$\frac{\partial f(\phi, t, T; v_0, r_0)}{\partial v_0} = f(\phi, t, T; v_0, r_0) \cdot [\frac{\partial C(\phi;\tau)}{\partial \rho_2} + \frac{\partial \bar{C}(\phi;t)}{\partial \rho_2} + \frac{\partial \bar{E}(\phi;t)}{\partial \rho_2} r_0], \tag{3.42}$$

which requires the computation of three partial derivatives. Based on the expression of $C(\phi;\tau)$, $\bar{C}(\phi;t)$ and $\bar{E}(\phi;t)$, we obtain

$$\frac{\partial C(\phi;\tau)}{\partial \rho_2} = \alpha\beta\int_0^{\tau}\frac{\partial E(\phi;t)}{\partial \rho_2}dt, \tag{3.43}$$

$$\frac{\partial \bar{C}(\phi;t)}{\partial \rho_2} = \alpha\beta\int_0^{t}\frac{\partial \bar{E}(\phi;z)}{\partial \rho_2}dz, \tag{3.44}$$

$$\frac{\partial \bar{E}(\phi;t)}{\partial \rho_2} = \frac{e^{-(\alpha+\frac{2\eta^2}{\alpha-l})t}\left\{\frac{2l+(l+\alpha)[e^{l(\tau+t)}-1]}{2l+(l+\alpha)(e^{l\tau}-1)}\right\}^{-\frac{4\eta^2}{(l-\alpha)(l+\alpha)}}}{\left\{-\frac{1}{2}\eta^2\int_0^t e^{-(\alpha+\frac{2\eta^2}{\alpha-l})x}\left\{\frac{2l+(l+\alpha)[e^{l(\tau+x)}-1]}{2l+(l+\alpha)(e^{l\tau}-1)}\right\}^{-\frac{4\eta^2}{(l-\alpha)(l+\alpha)}}dx\cdot E(\phi;\tau)+1\right\}^2} \cdot \frac{\partial E(\phi;\tau)}{\partial \rho_2}, \tag{3.45}$$

where

$$\frac{\partial E(\phi;\tau)}{\partial \rho_2} = -\frac{2}{\eta^2}\frac{[\sum_{n=0}^{+\infty}(n+1)\frac{\partial \hat{a}_{n+1}}{\partial \rho_2}t^n](\sum_{n=0}^{+\infty}\hat{a}_n t^n) - [\sum_{n=0}^{+\infty}(n+1)\hat{a}_{n+1}t^n](\sum_{n=0}^{+\infty}\frac{\partial \hat{a}_n}{\partial \rho_2}t^n)}{(\sum_{n=0}^{+\infty}\hat{a}_n t^n)^2}.$$

Clearly, the only unknown terms contained in the three target partial derivatives are $\frac{\partial \hat{a}_n}{\partial \rho_2}, n \ge 0$. If we define $b_n = \frac{\partial \hat{a}_n}{\partial \rho_2}$, they can be determined from the following recursive relationship

$$b_{n+2} = -\frac{1}{2l(n+1)(n+2)}\frac{\partial \hat{I}}{\partial \rho_2}, n \ge 0, \ b_0 = 0, \ b_1 = 0, \tag{3.46}$$

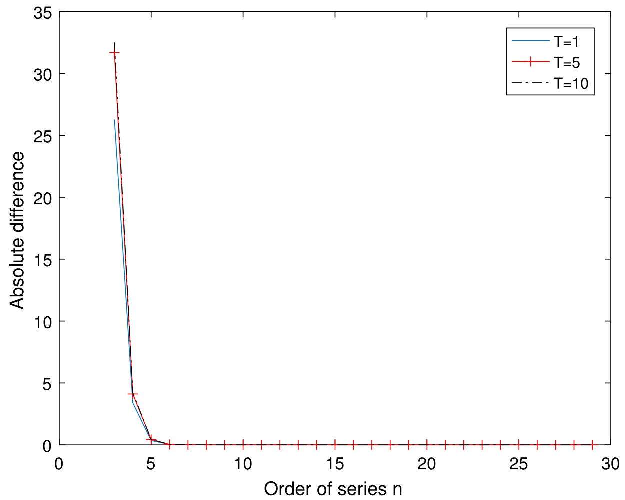
where

$$\begin{aligned}
\frac{\partial \hat{I}}{\partial \rho_2} &= 2l(\alpha - j\phi\rho_2\eta\xi)(n+1)b_{n+1} - 2lj\phi\eta\xi(n+1)\hat{a}_{n+1} \\
&+ \eta^2 l(j\phi - \frac{1}{2}j\phi\xi^2 - \frac{1}{2}\xi^2\phi^2)b_n - j\phi\eta^3\xi\sum_{i=1}^{n}c_i\hat{a}_{n-i} \\
&+ (\alpha + l)\sum_{i=1}^{n}(n+2-i)(n+1-i)c_i b_{n+2-i} \\
&+ [(\alpha - j\phi\rho_2\eta\xi)(\alpha + l) + 2\eta^2]\sum_{i=1}^{n}(n+1-i)c_i b_{n+1-i} \\
&- [j\phi\eta\xi(\alpha + l)]\sum_{i=1}^{n}(n+1-i)c_i\hat{a}_{n+1-i} \\
&+ \frac{1}{2}\eta^2[(j\phi - \frac{1}{2}j\phi\xi^2 - \frac{1}{2}\xi^2\phi^2)(\alpha + l) - 2j\phi\rho_2\eta\xi]\sum_{i=1}^{n}c_i b_{n-i}.
\end{aligned}$$

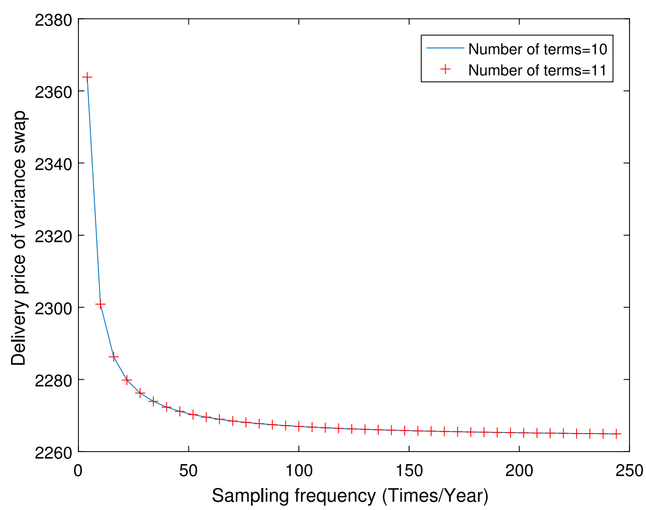
Thus, one can directly compute the hedging ratio associated with the newly introduced correlation parameter ρ_2 using Eqs. (3.40) and (3.41) for variance and volatility swaps, respectively.

4. Numerical implementation

This section firstly shows the computational efficiency of our pricing formulae using the speed of convergence as an indicator, and then



(a) Absolute difference between the $(n+1)$-term price and n-term price.



(b) The 10-term and 11-term variance swap prices.

Fig. 1. The convergence of our solution.

the results provided by our formulae are compared with the corresponding Monte Carlo results to validate our formulae,[4] after which the difference between the Heston-CIR model with and without the additional factor is demonstrated. Here are the default values for the model parameters when conducting the simulations in this section. The mean-reversion speed for the volatility and interest rate processes are $k = 10$, $\alpha = 2$, respectively, the two mean-reverting levels are $\theta = 0.2$, $\beta = 0.05$, and the volatility of the two processes are $\sigma = 0.1$, $\eta = 0.05$, with the initial values as $v_0 = 0.05$, $r_0 = 0.03$. The correlation coefficients ρ_1 and ρ_2 are respectively allocated as -0.5 and -0.8.[5] We also use 4 times per year for the sampling frequency N, and the expiry T is equal to 1 year. The parameter related to the additional factor, ξ, is 1.

Fig. 1(a) exhibits the absolute difference in the prices when using one more term for the computation. It can be easily noticed that the difference reduces very quickly and approaches 0 when there is a slight increase in the number of terms used. This phenomenon clearly confirms that our solution converges with a rapid speed and a few terms in the solution can lead to accurate results under this chosen set of parameters. The order of the absolute difference is already around 10^{-5} with only 10 terms, and adding one more term does not make a large difference, as shown in Fig. 1(b). Due to this reason, we choose 11 terms for computing the formulae for the rest of this section.

Table 1
Our price vs. Monte-Carlo price.

N	4	24	44	64	84	104
Ours	2363.82	2278.44	2271.51	2268.97	2267.65	2266.85
MC	2364.16 ($\pm$6.73)	2278.77 ($\pm$2.26)	2271.03 ($\pm$1.65)	2268.74 ($\pm$1.36)	2267.99 ($\pm$1.18)	2267.34 ($\pm$1.06)
RE (%)	0.014	0.015	0.021	0.010	0.015	0.022
N	124	144	164	184	204	224
Ours	2266.30	2265.91	2265.61	2265.38	2265.20	2265.05
MC	2266.40 ($\pm$0.97)	2266.06 ($\pm$0.90)	2265.56 ($\pm$0.85)	2265.55 ($\pm$0.80)	2264.99 ($\pm$0.77)	2264.79 ($\pm$0.73)
RE (%)	0.004	0.007	0.002	0.007	0.009	0.011

Table 1 is used to check the correctness of our pricing formulae, by comparing our prices (Ours) with those obtained using Monte Carlo (MC) simulations, which are also accompanied by a 98% confidence interval. One can clearly observe that both prices almost replicate each other with all of our prices falling within the corresponding confidence interval, and the relative difference between the two prices is less than 0.03%, which verifies the derived formula. It should be remarked here that although both approaches can yield accurate results, our solution is much more superior than the Monte Carlo simulation in terms of the computational efficiency; it only costs our pricing formula 0.32 s in producing one variance swap price with the sampling frequency being equal to 4 times per year, while 56.08 s (175 times more than ours) are needed to obtain a same swap price using the Monte Carlo approach.

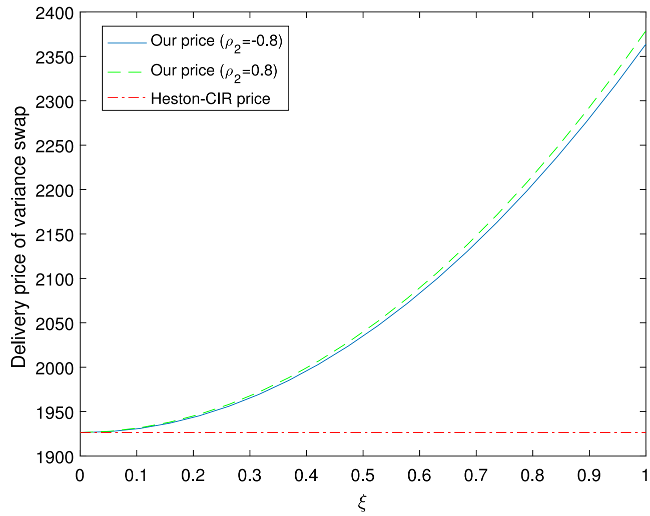
With the confidence gained in our pricing formulae, we are now ready to show the impact of the new second factor, by comparing the results from our formulae and the corresponding single-factor model. What are shown in Fig. 2 are the two model prices with respect to ξ, and the price under the single-factor model is always a constant when ξ changes because this parameter is not contained in the model. In contrast, our price increases with ξ, as a larger ξ increases the underlying volatility level and results in a potentially larger realized variance/volatility, contributing to a higher swap price. One can also notice that our price actually goes back to the model price without the second factor if ξ equals to zero, as a result of the second factor in our model disappearing in this case. When the correlation between S_t and r_t changes from positive to negative, both swap prices experience a decrease, if other parameters are kept the same. This is because a positive correlation implies that an increase in the stochastic interest rate would lead a rise in the underlying.

On the other hand, Fig. 3 displays how the two model prices vary with the initial interest rate value, and it is clear that both model prices are increasing functions of r_0. The only difference lies in the rate of change that our price increases faster when we enlarge the value of r_0. The main explanation is that the volatility as well as the realize variance/volatility is now positively affected by the interest rate. In summary, a clear phenomenon that can be observed here is that the newly introduced factor can significantly influence swap prices, justifying the introduction of this new factor.

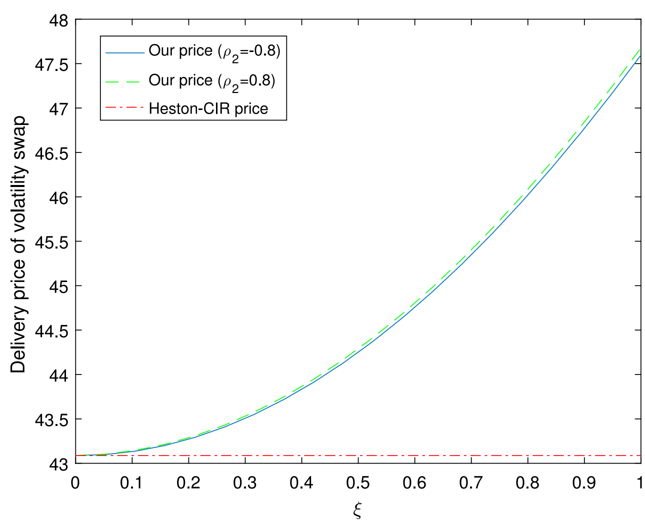
One may also be interested in how the hedging ratios derived in Section 3.3 can be used in practice. Thus, following Broadie and Jain (2008), we provide a simple numerical example about delta-hedging here. Since the volatility itself is not a tradable asset, the hedging of a volatility derivative should be conducted using another

[4] To illustrate the convergence speed and computational accuracy of both formulae, we just need to use variance swap prices, as the series solution is only involved in the forward characteristic function.

[5] The choice of negative values for the correlation parameters is according to the so-called leverage effects (Bakshi et al., 1997; Jacquier et al., 2004).

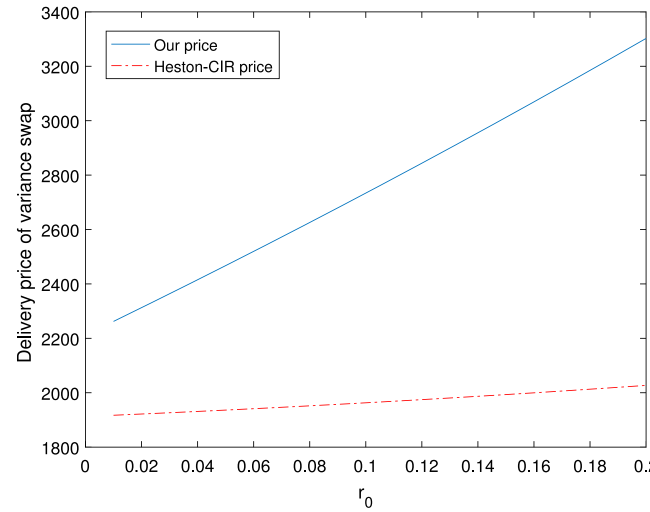


(a) Our variance swap price vs the corresponding Heston-CIR price.

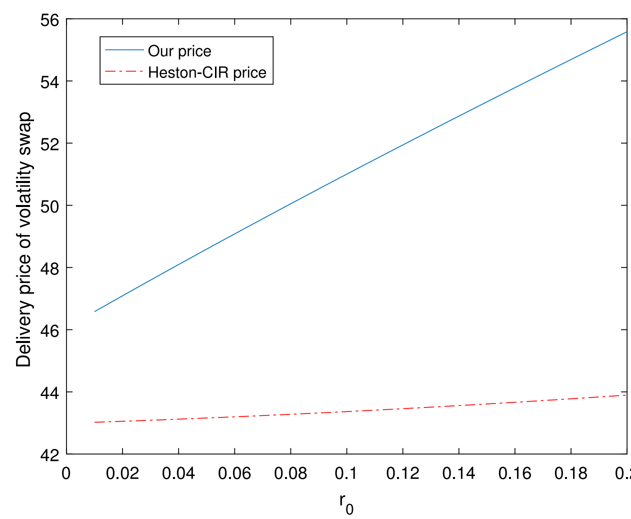


(b) Our volatility swap price vs the corresponding Heston-CIR price.

Fig. 2. The effect of ξ and ρ_2.



(a) Our variance swap price vs the corresponding Heston-CIR price.



(b) Our volatility swap price vs the corresponding Heston-CIR price.

Fig. 3. The effect of r_0.

Table 2
A delta-hedging example.

v_{t_1}	0.101	0.105	0.11	0.12
Delta-hedging (%)	4.9E−3	4.1E−3	1.9E−3	7.0E−3
No-hedging (%)	2.4E−2	9.8E−2	5.5E−1	2.5E−1

derivative. In fact, a volatility swap contract can be hedged using variance swaps, and we can form a portfolio consisting of one unit of a volatility swap and β units of variance swaps at time $t_0 = 0$, where the hedging ratio β is equal to $-\Delta_{vol}/\Delta_{var}$. As both variance and volatility swap contracts are worth the value of zero at the initial time, the value of the constructed portfolio is also zero. Consider a volatility change at $t_1 = 1/52$ (a week later), and one can deduce that the profit (loss) of this hedged portfolio at that time is equal to $e^{-r(T-t_1)}\left[\beta(Z_{var,t_1} - K_{var}) - (Z_{vol,t_1} - K_{vol})\right]$, where Z_{var,t_1} and Z_{var,t_1} respectively denote the delivery prices of variance and volatility swaps at the current time t_1. In contrast, the profit (loss) of the unhedged volatility swap contract should be $e^{-r(T-t_1)}(Z_{vol,t_1} - K_{vol})$. With $v_0 = 0.1$, the comparison results for the absolute profit (loss) (in percentage) of two strategies with and without hedging are provided in Table 2. One can see clearly that the delta-hedging strategy is able to hedge the risk associated with the change in the volatility values, and the value of the hedged portfolio is much more stable than that of the unhedged portfolio, especially when the change in the volatility becomes larger.

5. Conclusion

In this article, we show analytical solutions to variance as well as volatility swap prices written as convergent series solutions under the proposed two-factor Heston-CIR model. Through numerical experiments, our formulae are shown to be accurate and able to converge very quickly, implying their strong potential in practical applications. The introduction of the second factor related to the interest rate has resulted in a large difference between swap prices under the single and two-factor Heston-CIR model, which demonstrates the importance to make the underlying and interest rate be correlated.

CRediT authorship contribution statement

Xin-Jiang He: Methodology, Investigation, Software, Validation, Writing – original draft. **Song-Ping Zhu:** Conceptualization, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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