An Explicitly Solvable Multi-Scale Heston FX Model with Stochastic Interest Rate

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Abstract

To be done.

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1 Introduction

1.1 Motivation and Research Background

After the collapse of the Bretton Woods system in 1971, U.S. dollar was no longer pegged to gold. Meanwhile, most of the world's currencies exited from the pegged exchange rate with respect to to U.S. dollar. The world economy has to face the risk of floating exchange rate, which can result in the possible investment loss in foreign exchange (FX) reserves and the associated derivatives.

Foreign exchange derivatives have been thought as efficient weapons against the FX rate risk by some scholars and policy makers. A growing number of FX derivatives have been generated in the derivatives market, such as Barrier option which adds an obstacle term to the European call or put option. Specifically, if the FX rate exceeds the barrier price before the option's maturity, the holder is unable to exercise.

Moreover, in the emerging financial Markets, for example China, FX derivatives are playing important roles in maintaining domestic economic stability. Since joining the World Trade Organisation (WTO) in 2001, the volume of Chinese international trade has explosively increased. Till the end of June 2015, FX reserves of China has amounted to 3.69 trillion U.S. dollar. Facing with the huge amount of FX reserves, and global financial instability since 2008 crisis, China needs efficient tools to hedge the risk of FX rate fluctuation. In addition, the explosive growth of China's emerging middle class stimulates domestic financial innovation in FX derivatives, e.g. FX options. Therefore, Bank of China has firstly launched FX options to individual investors since 2002, and the trading volume is continuously growing afterwards.

World economic integration increases the complexity of FX options, investment companies and banks take FX options as fundamental tools to hedge FX risk and stabilize returns of portfolios management. Therefore, from risk management point of view, how to accurately model the price of FX derivatives becomes particularly important today. Needless to say, precisely predicting the FX moment is a continuing demand by various financial sectors.

1.2 Literature Review and Chapter Outline

A model of a single FX spot underlying problem can be found in Wystep 2006, while Lipton 2001 and Clark 2011 study a multiple FX rates. These models can be seen as an extension of Black-Scholes model where the volatility is assumed to be constant. Volatility smile effect is studied by Carr and Wu 2007, where empirical evidence rejects the hypothesis of normal distribution of FX returns. Shiraya and Takahashi 2012 use SABR model to study the stochastic volatility effect, and obtain approximation formulas for pricing FX options. It is worth to note that FX rates has inversion and triangulation symmetries, e.g. EUR/USD can be derived from EUR/GBP and GBP/USD. SABR type model does not satisfy this symmetry property, since the inversion's volatility process has an additional drift term. Nevertheless, Heston-type model satisfy this symmetry property (see, Del Bano Rollin 2008).

Ahlip 2008 studies a model of the spot FX rate with stochastic volatility and stochastic domestic as well as foreign rates. Specifically, these rates are modelled by Ornstein-Uhlenbbeck processes, and the volatility by a mean-reverting Ornstein-Uhlenbeck process correlated with the spot FX rate. Ahlip also derives an analytical formula for the price of European call options on the spot FX rate. Despite the complexity and intractability, multiscale FX volatility models usually outperform the single dimensional volatility models with the advantage of well presenting real market data. For instance, Alvise De Col, Gnoatto and Grasselli 2013 propose a multi-factor Heston stochastic volatility model which is an (semi)-affine model whose analytical treatment is deduced using the approach proposed by Duffie, Pan and Singleton 2000. However, the model does not allow for stochastic interest rates. In order to overcome this problem, this chapter proposes the use of a stochastic process to describe the interest rate process (see, Section 2.2.1).

This chapter describes a Multiscale hybrid Heston model which is an extension of De Col, Gnoatto and Grasselli 2013. The contributions are twofold.

Firstly, the Multiscale Heston SDE model of De Col, Gnoatto and Grasselli 2013 is modified here in order to allow a stochastic interest rate. As highlighted in Chapter 2, 3 and 4, stochastic interest rate plays a fundamental role in better matching the market values of the options with the theoretical option prices.

Secondly, the analytical treatment of the model is described both under physical measure and risk neutral measure. In particular, closed-form formulas for FX rate approximation (under physical measure) and option pricing (under risk neutral measure) are obtained. In addition, an integral representation formula of the probability density function (pdf) of the stochastic process is derived by solving the backward Kolmogorov equation using some ideas illustrated in Fatone et al. 2009, 2013. This pdf has practical applications in calculating moment generating functions, model calibration, and empirical analysis which deserve further investigation.

This chapter is organized as follows. In Section 2.1, the Multiscale hybrid Heston SDE model is described to illustrate the main relevant formulas under physical measure. In Section 2.1.1, the analytical treatment and the corresponding formulas for the probability density function are deduced. Section 2.2 describes the approach of measurement change on Multiscale Hybrid Heston model. In Section 2.3, the model treatment under risk neutral measure is briefly described, and the corresponding formulas are deduced detailedly in Appendix C. In Section 2.4, analytical formulas are proposed to approximate the FX European vanilla call and put option prices as one-dimensional integrals of explicitly known functions.

2 The Multiscale Hybrid Heston SDE Model

This chapter considers the FX market where various currencies are mutually traded. Particularly, this chapter considers FX spot trading model under physical measure and the corresponding European vanilla options pricing model under risk neutral measure. This section presents the analytical treatment of Multiscale Heston hybrid model and some important results (i.e. transition probability density function, moment functions, etc) which are used to deduce option pricing formulas in the Section 2.4.

The proposed multiscale hybrid Heston model is inspired by De Col, Gnoatto and Grasselli 2013 model where the artificial currency is used as a universal numÃ©raire, and each values of currencies are considered in units of the artificial currency. Hereafter, this chapter denotes with $S_t^{i,j}$, $\mathbf{V}_t$ and $\mathbf{R}_t$ the exchange rate between currency i and j, d-dimensional independent volatility matrix and 2-dimensional independent interest rate matrix at time $t > 0$. Moreover, let us denote index $j = 0$ for artificial currency, however, the exact specification of the universal numÃ©raire is not necessary here. It is worth to highlight that, for formulating stochastic interest rate process, this chapter considers two fundamental SDE models, i.e. Cox Ingersoll Ross (CIR) model, and Hull-White (HW) model which allows negative values of interest rate. The dynamic of the process is described as follows:

$$\frac{dS_t^{0,i}}{S_t^{0,i}} = (r^0 - r^i)dt - (\mathbf{a}^i)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V}(t))}d\mathbf{W}_t^{p,v} - (\mathbf{b}^{0,i})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{0,i}(t))\right]^{\alpha} d\mathbf{W}_t^{p,r},$$
$$t > 0,\ i = 1, \ldots, N\,,\ \alpha = 0, \frac{1}{2}; \tag{1}$$
$$dv_n(t) = \chi_n(v_n^* - v_n(t))dt + \gamma_n\sqrt{v_n(t)}dW_{n,t}^v, \qquad n = 1, \ldots, d\,,\ t > 0; \tag{2}$$
$$dr_m(t) = \lambda_m(\theta_m - r_m(t))dt + \eta_m\, r_m^{\alpha}(t)dW_{m,t}^r, \qquad t > 0 \quad m = \{0, i\}\,; \tag{3}$$

where χ_n, v_n^*, γ_n, λ_m, η_m are positive constants. $\mathbf{a}^i = (a_1^i, \ldots, a_d^i)^{\mathrm{T}}$ and $\mathbf{a}^i \in \mathbb{R}^d$, $i = 1, \ldots, N$. $\sqrt{\mathrm{Diag}(\mathbf{V}(t))}$ (or $\sqrt{\mathrm{Diag}(\mathbf{V})}$) is a $d \times d$ dimensional diagonal matrix denoting the square root of the principle diagonal in the elements of the matrix $\mathbf{V}$, i.e. $\sqrt{\mathrm{Diag}(\mathbf{V})} = \begin{bmatrix} \sqrt{v_1} & 0 & \cdots & 0 \\ 0 & \sqrt{v_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{v_d} \end{bmatrix}$. $\mathbf{W}_t^{p,v}$ is a d-dimensional standard Wiener process with $\mathbf{W}_t^{p,v} = (W_{1,t}^{p,v}, \ldots, W_{d,t}^{p,v})^{\mathrm{T}}$, and $W_{n,t}^v$ is standard Wiener processes. Furthermore, $\mathbf{b}^{0,i} = (b_0, b_i)^{\mathrm{T}} \in \mathbb{R}^2$, and $[\mathrm{Diag}(\mathbf{R}_{0,i}(t))]^{\alpha}$ (or $[\mathrm{Diag}(\mathbf{R}_{0,i})]^{\alpha}$) is a 2×2 dimensional diagonal matrix which denotes the α-power of the principle diagonal in the elements of the vector $\mathbf{R}_{0,i}$, and $[\mathrm{Diag}(\mathbf{R}_{0,i})]^{\alpha} = \begin{bmatrix} r_0^{\alpha} & 0 \\ 0 & r_i^{\alpha} \end{bmatrix}$ where r_0 is the artificial currency rate in our universal numéraire. $\mathbf{W}_t^{p,r}$ is a 2-dimensional standard Wiener process with $\mathbf{W}_t^{p,r} = (W_{0,t}^{p,r}, W_{i,t}^{p,r})^{\mathrm{T}}$, and $W_{n,t}^v$ is standard Wiener processes. In contrary to the Del Col, Gnoatto and Grasselli 2013 model assuming deterministic interest rates, our proposed model assumes stochastic interest in Eq.(3) following CIR process($\alpha = \frac{1}{2}$) or

HW process ($\alpha = 0$). Moreover, the following correlation structure are assumed:

$$\begin{aligned}
E(dW^{p,r}_{m,t} dW^{p,v}_{n,t}) &= 0,\ m,n,t > 0, && (4)\\
E(dW^{r}_{m,t} dW^{v}_{n,t}) &= 0,\ m,n,t > 0, && (5)\\
E(dW^{p,v}_{n,t} dW^{v}_{n,t}) &= \rho_{n,v} dt,\ t > 0, && (6)\\
E(dW^{p,v}_{n,t} dW^{v}_{l,t}) &= 0,\ n \neq l, t > 0, && (7)\\
E(dW^{p,v}_{n,t} dW^{r}_{m,t}) &= 0,\ n,m,t > 0, && (8)\\
E(dW^{p,r}_{m,t} dW^{r}_{m,t}) &= \rho_{m,r} dt,\ t > 0, && (9)\\
E(dW^{p,r}_{m,t} dW^{r}_{m',t}) &= 0,\ m \neq m', t > 0, && (10)\\
E(dW^{p,r}_{m,t} dW^{v}_{n,t}) &= 0,\ m,n,t > 0, && (11)
\end{aligned}$$

where $E(\cdot)$ denotes the expected value, and $\rho_{n,v}$, $\rho_{m,r} \in [-1, 1]$ are constants known as correlation coefficients. Moreover, it is worth nothing that Feller condition i.e. $\forall n = 1, 2 \ldots, d$, $\frac{2\chi_n v_n^*}{\gamma_n^2} > 1$ should be satisfied in order to guarantee positive variance $v_n(t)$ with probability one for any $t > 0$ given that $v_n^* > 0$.

Before giving the main formulas derived in this chapter, let us rewrite the formula Eq.(1) in term of the log-price. Using Ito's lemma, we will obtain that the log-price $x_t^{0,i} = \ln S_t^{0,i}$, $t > 0$ satisfies the following dynamics:

$$\begin{aligned}
dx^{0,i} &= \left[(r^0 - r^i) - \frac{1}{2}(\mathbf{a}^i)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^i) - \frac{1}{2}(\mathbf{b}^{0,i})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{0,i})\right]^{2\alpha} (\mathbf{b}^{0,i}) \right] dt \\
&\quad -(\mathbf{a}^i)^{\mathrm{T}} \sqrt{\mathrm{Diag}\mathbf{V}} d\mathbf{W}_t^{p,v} - (\mathbf{b}^{0,i})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{0,i})\right]^{\alpha} d\mathbf{W}_t^{p,r}, \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2}, \qquad (12)
\end{aligned}$$

Substituting index i with j, we have the following FX stochastic process of $x_t^{0,j} = \ln S_t^{0,j}$.

$$\begin{aligned}
dx^{0,j} &= \left[(r^0 - r^j) - \frac{1}{2}(\mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^j) - \frac{1}{2}(\mathbf{b}^{0,j})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{r}_{0,j})\right]^{2\alpha} (\mathbf{b}^{0,j}) \right] dt \\
&\quad -(\mathbf{a}^j)^{\mathrm{T}} (\mathrm{Diag}\mathbf{V}) d\mathbf{W}_t^{p,v} - (\mathbf{b}^{0,j})^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{R}_{0,j})\right]^{2\alpha} d\mathbf{W}_t^{p,r}, \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2}, \qquad (13)
\end{aligned}$$

where $\mathbf{b}^{0,j} = (b_0, b_j)^{\mathrm{T}} \in \mathbb{R}^2$. Since $S^{i,j}$ denotes the exchange rate between different currencies i and j, and $S^{i,j} = \frac{S^{0,j}}{S^{0,i}}$ by definition, therefore we have the following triangular symmetry relationship:

$$x^{i,j} = \ln S^{i,j} = \ln S^{0,j} - \ln S^{0,i} = x^{0,j} - x^{0,i} \qquad (14)$$

as well as

$$dx^{i,j} = dx^{0,j} - dx^{0,i} \tag{15}$$

Substituting Eq.(12) into (13), we obtain:

$$\begin{aligned}
& dx^{i,j} \\
&= \Big\{ (r_i - r_j) + \frac{1}{2}\left[(\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^i + \mathbf{a}^j) + (b_i, -b_j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha}(b_i, b_j)^{\mathrm{T}} \right] \\
& \Big\} dt + (a^i - a^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}\mathbf{V}}\, d\mathbf{W}_t^{p,v} + (b_i, -b_j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r}, \qquad (16) \\
& \qquad t > 0,\ \alpha = 0, \frac{1}{2}
\end{aligned}$$

Lemma 2.1. *Assuming the log-price* $x_t^{i,j} = \ln S_t^{i,j}$*,* $t > 0$ *satisfies the SDE in Eq.(16), then spot price* $S_t^{i,j}$ *satisfies the following dynamic:*

$$\begin{aligned}
\frac{dS_t^{i,j}}{S_t^{i,j}} &= \left[r_i - r_j + (\boldsymbol{a}^i - \boldsymbol{a}^j)^{\mathrm{T}}(\mathrm{Diag}\,\boldsymbol{V})\boldsymbol{a}^i + \boldsymbol{b}_i^2 r_i^{2\alpha} \right] dt \\
&\quad + (\boldsymbol{a}^i - \boldsymbol{a}^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\boldsymbol{V})}\, d\boldsymbol{W}_t^{p,v} + (b_i, -b_j)\left[\mathrm{Diag}(\boldsymbol{R}_{i,j})\right]^{\alpha} d\boldsymbol{W}_t^{p,r}, \\
&\qquad t > 0,\ i,j = 1,\ldots,N\,,\ \alpha = 0, \frac{1}{2}
\end{aligned} \tag{17}$$

Proof. By definition in Eq.(1), the dynamic of underlying of FX rate $S_t^{i,j}$ satisfies a general geometric Brownian motion with drift coefficient $\mu^{i,j}$ as follows:

$$\begin{aligned}
\frac{dS_t^{i,j}}{S_t^{i,j}} &= \mu^{i,j} dt + (\mathbf{a}^{i,j})^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V}(t))}\, d\mathbf{W}_t^{p,v} + (\mathbf{b}^{i,j})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{i,j}(t))\right]^{\alpha} d\mathbf{W}_t^{p,r}, \\
&\qquad t > 0,\ i,j = 1,\ldots,N\,,\ \alpha = 0, \frac{1}{2}
\end{aligned} \tag{18}$$

where $\mathbf{b}^{i,j} = (b_i, -b_j)^{\mathrm{T}}$. Using Ito's Lemma in Eq.(18), we will obtain $x_t^{i,j} = \ln S_t^{i,j}$ as follows:

$$\begin{aligned}
dx^{i,j} &= \left[\mu^{i,j} - \frac{1}{2}(\mathbf{a}^{i,j})^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^{i,j}) - \frac{1}{2}(\mathbf{b}^{i,j})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha}(\mathbf{b}^{i,j}) \right] dt \\
&\quad + (\mathbf{a}^{i,j})^{\mathrm{T}}\sqrt{\mathrm{Diag}\mathbf{V}}\, d\mathbf{W}_t^{p,v} + (\mathbf{b}^{i,j})^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r}, \\
&\qquad t > 0,\ \alpha = 0, \frac{1}{2}
\end{aligned} \tag{19}$$

It is worth nothing that Eqs.(16) and (19) should be identical. Thus, we have the following equalities:

$$\mathbf{a}^{i,j} = \mathbf{a}^i - \mathbf{a}^j \in \mathbb{R}^d\,,\ \text{or}\ \mathbf{a}^{i,j} = (\mathbf{a}_1^{i,j},\ldots,\mathbf{a}_d^{i,j})^{\mathrm{T}}\,,\text{with}\ \mathbf{a}_k^{i,j} = \mathbf{a}_k^i - \mathbf{a}_k^j\,, \qquad k = 1,2,\ldots d \tag{20}$$

$$\mathbf{b}^{i,j} = (b_i, -b_j)^{\mathrm{T}} \in \mathbb{R}^2\,, \tag{21}$$

$$\mu^{i,j} = r_i - r_j + (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathrm{V})\mathbf{a}^i + \underbrace{(b_i,-b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{2\alpha}\,(b_i,0)^{\mathrm{T}}}_{=b_i^2 r_i^{2\alpha}} \tag{22}$$

Substituting Eqs.(20)-(22) into Eq.(16), we obtain

$$\begin{aligned}\frac{dS_t^{i,j}}{S_t^{i,j}} &= \left[r_i - r_j + (\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})\mathbf{a}^i + \mathbf{b}_i^2 r_i^{2\alpha}\right] dt \\ &\quad +(\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V})}d\mathbf{W}_t^{p,v} + (b_i,-b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{\alpha}\,d\mathbf{W}_t^{p,r}, \\ &\qquad t>0,\ i,j=1,\ldots,N\,,\ \alpha = 0,\ \frac{1}{2}\end{aligned}$$

Without using matrix components, we could re-write Eq.(23) in the following linear equations:

$$\frac{dS_t^{i,j}}{S_t^{i,j}} = \left[r_i - r_j + \sum_{k=1}^{d}(a_k^i - a_k^j)a_k^i v_k + b_i^2 r_i^{2\alpha}\right] dt \tag{23}$$

$$+\sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r}\,, \tag{24}$$

$$t>0,\ i,j=1,\ldots,N\,,\ \alpha = 0,\ \frac{1}{2}$$

□

In order to simplify the following computations, we can re-write the matrix components of Eq.(16) with expression of linear components as follows:

$$\begin{aligned}&(\mathbf{a}^i - \mathbf{a}^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(\mathbf{a}^i + \mathbf{a}^j) \\ = \ &[a_1^i - a_1^j, a_2^i - a_2^j, \ldots, a_d^i - a_d^j]\cdot\begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_d\end{bmatrix}\cdot\begin{bmatrix} a_1^i + a_1^j \\ a_2^i + a_2^j \\ \vdots \\ a_d^i + a_d^j\end{bmatrix} \\ = \ &\sum_{k=1}^{d}[(a_k^i)^2 - (a_k^j)^2]\cdot v_k\end{aligned} \tag{25}$$

and

$$
\begin{aligned}
& (b_i, -b_j) \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{2\alpha} (b_i, b_j)^{\mathrm{T}} \\
= & \begin{bmatrix} b_i, -b_j \end{bmatrix} \begin{bmatrix} r_i^{2\alpha} & 0 \\ 0 & r_j^{2\alpha} \end{bmatrix} \begin{bmatrix} b_i \\ b_j \end{bmatrix} \\
= & b_i^2 \cdot r_i^{2\alpha} - b_j^2 \cdot r_j^{2\alpha}
\end{aligned} \tag{26}
$$

Thus, Eq.(19) can be re-written in the following form:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j + \frac{1}{2}\sum_{k=1}^{d}[(a_k^i)^2 - (a_k^j)^2]v_k + \frac{1}{2}(b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}) \right] dt \\
&+ \sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r}\,, \qquad (27) \\
& t > 0,\ \alpha = 0, \frac{1}{2},
\end{aligned}
$$

Combining Eqs.(2), (3) and (27) into a system of equations, we obtain:

$$
\begin{aligned}
dx^{i,j} &= \left[r_i - r_j + \frac{1}{2}\sum_{k=1}^{d}[(a_k^i)^2 - (a_k^j)^2]v_k + \frac{1}{2}(b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}) \right] dt \\
&+ \sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v} + b_i r_i^{\alpha} dW_i^{p,r} - b_j r_j^{\alpha} dW_j^{p,r}\,, \qquad (28) \\
& t > 0,\ \alpha = 0, \frac{1}{2}, \\
dv_k &= \chi_k(v_k^* - v_k)dt + \gamma_k\sqrt{v_k}dW_{k,t}^v, \qquad k = 1, \dots, d\,,\ t > 0; \qquad (29) \\
dr_i &= \lambda_i(\theta_i - r_i)dt + \eta_i\, r_i^{\alpha} dW_i^r, \qquad (30) \\
dr_j &= \lambda_j(\theta_j - r_j)dt + \eta_j\, r_j^{\alpha} dW_j^r. \qquad (31)
\end{aligned}
$$

2.1 The Model Treatment Under Physical Measure

First of all, let us equip Eqs.(28), (29), (30) and (31) with the initial conditions as follows:

$$
\begin{aligned}
x^{i,j}(0) &= \tilde{x}_0^{i,j}\,, \qquad (32) \\
v_k(0) &= \tilde{v}_{k,0}\,, \quad k = 1, \dots, d \qquad (33) \\
r_m(0) &= \tilde{r}_{m,0}\,, \quad m = i, j \qquad (34)
\end{aligned}
$$

where $\tilde{x}_0^{i,j}$, $\tilde{v}_{k,0}$, $\tilde{r}_{m,0}$ are random variables that are assumed to be concentrated in a point with probability one. For simplicity, we identify the random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{k,0}$,

$\tilde{r}_{m,0}$ with the points where they are concentrated. This chapter assumes $\tilde{v}_{k,0}$, $\tilde{r}_{m,0}$, χ_k, λ_m, γ_k, η_m, v_k^*, θ_m to be positive constant. In addition, this chapter assumes $\frac{2\chi_k v_k^*}{\gamma_k^2} > 1$ and $\frac{2\lambda_m\theta_m}{\eta_m^2} > 1$ (Feller condition). Hence, we mainly consider CIR type (i.e. $\alpha = 1/2$) volatility and interest rate processes where positive values of volatility and interest rate are guaranteed under the above assumptions. Nevertheless, in case of negative values of interest rate, the model can be explicitly extended by choosing a HW type interest rate process (i.e. $\alpha = 0$), and relaxing the assumptions of the second Feller condition for interest rate. This deserves further investigation.

Let $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$, $(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2$, t, $t' \geq 0$, $t' - t > 0$ be the transition probability density function associated with the stochastic differential system (28), (29), (30), (31). That is, the probability density function having $x' = x_{t'}^{i,j}$, $\underline{v}' = (v_1', \ldots, v_d')^{\mathrm{T}}$, $\underline{r}' = (r_i', r_j')^{\mathrm{T}}$ given that $x = x_t^{i,j}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, when $t' - t > 0$. In analogy with Lipton (2001)(pages 602–605), this transition probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$ as a function of the "past" variables $(x, \underline{v}, \underline{r}, t)$ satisfies the following backward Kolmogorov equation as follows:

$$
\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f}{\partial v_n^2} \\
&+\frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_f}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_i^{2\alpha}} \\
&-\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_j} + \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_f}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_f}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_f}{\partial r_j} \\
&+\left((r_i - r_j) + \frac{1}{2}\sum_{n=1}^{d}\left[(a_n^i)^2 - (a_n^j)^2\right]v_n + \frac{1}{2}[b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}]\right)\frac{\partial p_f}{\partial x} \\
&\quad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \leq t < t', \qquad (35)
\end{aligned}
$$

with final condition:

$$
\begin{aligned}
p_f(t, x, \underline{v}, \underline{r}, t', x', \underline{v}', \underline{r}'|t' = t) &= \delta(x' - x)\prod_{m\in\{i,j\}}\delta(r_m' - r_m)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n) \\
&= \delta(x' - x)\cdot\delta(r_i' - r_i)\delta(r_j' - r_j)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n)\,, \qquad (36)
\end{aligned}
$$

$$\forall n \in \{1, \ldots, d\} \text{ and } m \in \{i, j\}\,, \quad (x, v_n, r_m), (x', v_n', r_m') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t \geq 0,$$

and the appropriate boundary conditions. Defining $\tau = t' - t$, the function p_b is defined as follows:

$$
\begin{aligned}
&p_b(\tau, x, \underline{v}, r, x', \underline{v'}, r') = p_f(t, x, \underline{v}, r, t', x', \underline{v'}, r'),\\
&\qquad (x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2,\ t' = t + \tau,\ \tau > 0\,.
\end{aligned}
\tag{37}
$$

The representation (37) holds since the coefficients of the Kolmogorov backward equation and condition (36) are invariant by time translation. Using the change of the time variable $\tau = t - t'$ and equation (35), it is easy to see that p_b is the solution of the following problem:

$$
\begin{aligned}
\frac{\partial p_b}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_b}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_b}{\partial v_n^2}\\
&+\frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_b}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_b}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_b}{\partial x\partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_b}{\partial x \partial r_i^{2\alpha}}\\
&-\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_b}{\partial x\partial r_j} + \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_b}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_b}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_b}{\partial r_j}\\
&+\left((r_i - r_j) + \frac{1}{2}\sum_{n=1}^{d}\left[(a_n^i)^2 - (a_n^j)^2\right]v_n + \frac{1}{2}[b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha}]\right)\frac{\partial p_b}{\partial x}\\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t',
\end{aligned}
\tag{38}
$$

with the initial condition:

$$
\begin{aligned}
p_b(0, x, \underline{v}, r, x', \underline{v}', r') &= \delta(x' - x)\prod_{m=i}^{j}\delta(r'_m - r_m)\cdot\prod_{n=1}^{d}\delta(v'_n - v_n)\\
&= \delta(x' - x)\cdot\delta(r'_i - r_i)\delta(r'_j - r_j)\cdot\prod_{n=1}^{d}\delta(v'_n - v_n)\,,\\
&\qquad (x, v_n, r_m), (x', v'_n, r'_m) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t \ge 0,
\end{aligned}
\tag{39}
$$

and with the appropriate boundary conditions. For the convenience of later computation, let us consider the following change of dependent variable with a 'regularization' parameter q, which enables us to derive elementary formulas for the marginal probability density function and the FX option prices in Section 2.4.

$$
\begin{aligned}
&p_b(\tau, x, \underline{v}, r, x', \underline{v}', r') = e^{q(x-x')}p_q(\tau, x, \underline{v}, r, x', \underline{v}', r')\\
&\qquad (x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R}\times(\mathbb{R}^+)^d\times(\mathbb{R}^+)^2,\ t' = t + \tau,\ \tau > 0\,.
\end{aligned}
\tag{40}
$$

Substituting Eq.(40) into (38) and (39), it is easy to get that p_q is the solution of the following problem:

$$
\begin{aligned}
\frac{\partial p_q}{\partial \tau} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_q}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_q}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_q}{\partial x\partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_q}{\partial x\partial r_i} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_q}{\partial x\partial r_j} \\
&+\sum_{n=1}^{d}\left[\chi_n(\overline{v}_n - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial p_q}{\partial v_n} + \left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial p_q}{\partial r_i} \\
&+\left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial p_q}{\partial r_j} + \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right. \\
&+\left.\left(r_i + \frac{r_i^{2\alpha}}{2}b_i^2(2q+1)\right) + \left(-r_j + \frac{r_j^{2\alpha}}{2}b_j^2(2q-1)\right)\right]\frac{\partial p_q}{\partial x} + \left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right. \\
&+\left.\left(q r_i + \frac{r_i^{2\alpha}}{2}b_i^2(q^2+q)\right) + \left(q(-r_j) + \frac{r_j^{2\alpha}}{2}b_j^2(q^2-q)\right)\right]p_q \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t', \qquad (41)
\end{aligned}
$$

with the initial condition:

$$
\begin{aligned}
p_q(0, x, \underline{v}, r, x', \underline{v}', r') &= e^{q(x-x')}\delta(x'-x)\prod_{m=i}^{j}\delta(r_m' - r_m)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n) \\
&= e^{q(x-x')}\delta(x'-x)\cdot\delta(r_i'-r_i)\delta(r_j'-r_j)\cdot\prod_{n=1}^{d}\delta(v_n'-v_n), \qquad (42) \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ \tau = 0,
\end{aligned}
$$

Now let us consider the following representation formula for p_q with a Fourier transform:

$$
\begin{aligned}
&p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)} f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k), \\
&(\underline{v}, \underline{r}) \in (\mathbb{R}^+)^d, (\underline{v}', \underline{r}') \times (\mathbb{R}^+)^2,\ k\in\mathbb{R},\ \tau > 0. \qquad (43)
\end{aligned}
$$

This is possible since the coefficients (38) and the initial condition (39) are independent of translation in the log-price variable. Substituting Eq.(43) into (38), we obtain that the function f is the solution of the following problem:

$$\frac{\partial f}{\partial \tau} = \frac{-k^2}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right] f + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 f}{\partial r_i^2}$$
$$+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 f}{\partial r_j^2} + \sum_{n=1}^{d}(-\imath k)\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial f}{\partial v_n} + (-\imath k)\rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial f}{\partial r_i}$$
$$-(-\imath k)\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial f}{\partial r_j} + \sum_{n=1}^{d}\left[\chi_n(v_n^* - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial f}{\partial v_n}$$
$$+\left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial f}{\partial r_i} + \left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial f}{\partial r_j}$$
$$+\left\{\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left[\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right) - \imath k\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right]\right.$$
$$+\left(r_i(q - \imath k) + \frac{r_i^{2\alpha}}{2}b_i^2\left[(q^2 + q) - \imath k(2q + 1)\right]\right)$$
$$\left.+\left(r_j(-q + \imath k) + \frac{r_j^{2\alpha}}{2}b_j^2\left[(q^2 - q) - \imath k(2q - 1)\right]\right)\right\} f \tag{44}$$
$$(x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \le t < t'.$$

with the initial condition:

$$f(0, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \delta(r_i' - r_i)\delta(r_j' - r_j)\prod_{n=1}^{k}\delta(v_i' - v_i)\,, \tag{45}$$
$$(v_i, r_i, r_j), (v_i', r_i', r_j') \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+,\ k \in \mathbb{R}.$$

Now let us represent f as the inverse Fourier transform of the future variables $(\underline{v}', \underline{r}')$ whose conjugate variables are denoted by $(\underline{l}, \underline{\xi})$, that is:

$$f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \left(\frac{1}{2\pi}\right)^{d+2}\prod_{n=1}^{d}\int_{\mathbb{R}} dl_n\, e^{\imath l_n v_n'} \cdot \prod_{m\in\{i,j\}}\int_{\mathbb{R}} d\xi_m\, e^{\imath \xi_m r_m'}\, g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}),$$
$$m = i, j\,,\ (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0. \tag{46}$$

It is easy to see that the function g satisfies Eq.(44) with the following initial condition:

$$g(0, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = \prod_{m\in\{i,j\}} e^{-\imath \xi_m r_m}\prod_{n=1}^{k} e^{-\imath l_n v_n}\,, \tag{47}$$
$$= e^{-\imath \xi_i r_i} e^{-\imath \xi_j r_j}\prod_{n=1}^{k} e^{-\imath l_n v_n}, \tag{48}$$
$$m \in \{i, j\}\,,\ (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0.$$

It is worth to note that g is the Fourier transform with respect to the future variables $(\underline{v}', \underline{r}')$ of the function obtained by extending f, as a function of the variables $(\underline{v}, \underline{r})$, with zero when $v_n \notin \mathbb{R}^+$ and/or $r_m \notin \mathbb{R}^+$. The coefficients of the partial differential operator appearing on the right hand side of Eq.(38) are first degree polynomials in $\underline{v}$ and $\underline{r}$. Thus, we seek a solution of problem (44), (45) in the form (see Lipton, 2001):

$$\begin{aligned} g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) &= e^{A(\tau,k,\underline{l},\underline{\xi})} \prod_{m\in\{i,j\}} e^{-r_m B_{r_m}(\tau,k,\xi_m)} \prod_{n=1}^{d} e^{-v_n B_{v_n}(\tau,k,l_n)}, \\ &= e^{A(\tau,k,\underline{l},\underline{\xi})} e^{-r_i B_{r_i}(\tau,k,\xi_i)} e^{-r_j B_{r_j}(\tau,k,\xi_j)} e^{-\sum_{n=1}^{d} v_n B_{v_n}(\tau,k,l_n)}, \qquad (49) \\ & \quad (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, \underline{l}, \underline{\xi}) \in \mathbb{R} \times (\mathbb{R})^d \times (\mathbb{R})^2,\ \tau > 0. \end{aligned}$$

From now on, let us only focus on the CIR type interest rate process, i.e choosing $\alpha = \frac{1}{2}$. In this case, non-negative values of interest rate is guaranteed with probability one. Substituting Eq.(49) into Eq.(44) , the formulas $A(\tau, k, \underline{l}, \underline{\xi})$, $B_{v_n}(\tau, k, l_n)$, $B_{r_i}(\tau, k, \xi_i)$, and $B_{r_j}(\tau, k, \xi_j)$ must satisfy the following ordinary differential equations:

$$\begin{aligned} \frac{dA}{d\tau}(\tau, k, \underline{l}, \underline{\xi}) &= -\lambda_i \theta_i B_{r_i}(\tau, k, \xi_i) - \lambda_j \theta_j B_{r_j}(\tau, k, \xi_j) - \sum_{n=1}^{d} \chi_n v_n^* B_{v_n}(\tau, k, l_n) \\ & (k, l_n, \xi_i, \xi_j) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0, \qquad (50) \end{aligned}$$

$$\begin{aligned} & \frac{dB_{v_n}}{d\tau}(\tau, k, l_n) \\ &= \frac{k^2}{2}(a_n^i - a_n^j)^2 - \frac{(a_n^i - a_n^j)^2}{2}\left[\left(q^2 + q\frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right) - \imath k\left(2q + \frac{(a_n^i + a_n^j)}{(a_n^i - a_n^j)}\right)\right] \\ & \quad - \left[\chi_n + (\imath k - q)\gamma_n \rho_{n,v}(a_n^i - a_n^j)\right] B_{v_n} - \frac{\gamma_n^2}{2} B_{v_n}^2 \qquad (51) \\ &= \varphi_q^{v_n}(k)(a_n^{i,j})^2 - (\chi_n + (\imath k - q)\gamma_n \tilde{\rho}_{n,v}) B_{v_n}(\tau, k, l_n) - \frac{\gamma_n^2}{2} B_{v_n}^2(\tau, k, l_n), \qquad (52) \\ & \qquad (k, l_n) \in \mathbb{R} \times \mathbb{R}, \tau > 0 \end{aligned}$$

where $\varphi_q^{v_n}(k) = \frac{k^2}{2} - \frac{1}{2}\left[\left(q^2 + q\frac{\tilde{a}_n^{i,j}}{a_n^{i,j}}\right) - \imath k\left(2q + \frac{\tilde{a}_n^{i,j}}{a_n^{i,j}}\right)\right]$, $a_n^{i,j} = a_n^i - a_n^j$, $\tilde{a}_n^{i,j} = a_n^i + a_n^j$, and $\tilde{\rho}_{n,v} = \rho_{n,v}(a_n^i - a_n^j)$.

$$
\begin{aligned}
&\frac{dB_{r_i}}{d\tau}(\tau,k,\xi_i)\\
&=\frac{k^2}{2}b_i^2+(\imath k-q)-\frac{b_i^2}{2}\left[(q^2+q)-\imath k(2q+1)\right]-\left[\lambda_i+(\imath k-q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i)\\
&\quad-\frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i)\\
&=\varphi_q^{r_i}(k)b_i^2+(\imath k-q)-\left[\lambda_i+(\imath k-q)\eta_i\rho_{i,r}b_i\right]B_{r_i}(\tau,k,\xi_i)-\frac{\eta_i^2}{2}B_{r_i}^2(\tau,k,\xi_i) \qquad (53)\\
&\qquad\qquad (k,\xi_i)\in\mathbb{R}\times\mathbb{R},\tau>0,
\end{aligned}
$$

where $\varphi_q^{r_i}(k)=\frac{k^2}{2}-\frac{1}{2}\left[(q^2+q)-\imath k(2q+1)\right]$.

$$
\begin{aligned}
&\frac{dB_{r_j}}{d\tau}(\tau,k,\xi_i)\\
&=\frac{k^2}{2}b_j^2+(q-\imath k)-\frac{b_j^2}{2}\left[(q^2-q)-\imath k(2q-1)\right]-\left[\lambda_j+(\imath k-q)\eta_j\rho_{j,r}(-b_j)\right]B_{r_j}\\
&\quad-\frac{\eta_j^2}{2}B_{r_j}^2\\
&:=\varphi_q^{r_j}(k)b_j^2+(q-\imath k)-\left[\lambda_j+(q-\imath k)\eta_j\rho_{i,r}b_j\right]B_{r_j}(\tau,k,\xi_j)-\frac{\eta_j^2}{2}B_{r_j}^2(\tau,k,\xi_j)\\
&\qquad\qquad (k,\xi_j)\in\mathbb{R}\times\mathbb{R},\tau>0, \qquad (54)
\end{aligned}
$$

with $\varphi_q^{r_j}(k)=\frac{k^2}{2}-\frac{1}{2}\left[(q^2-q)-\imath k(2q-1)\right]$, and initial conditions:

$$
\begin{aligned}
&A(0,k,\underline{l},\underline{\xi})=0,\quad B_{v_n}(0,k,l_n)=\imath\, l_n,\quad B_{r_i}(0,k,\xi_i)=\imath\,\xi_i,\quad B_{r_j}(0,k,\xi_j)=\imath\,\xi_j,\\
&\qquad\text{with } (k,\underline{l},\underline{\xi})\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^2. \qquad (55)
\end{aligned}
$$

Eqs.(52)-(54) are Riccati equations that can be solved analytically. Then substituting their solutions into (50) and integrating with respect to τ, $A(\tau,k,\underline{l},\underline{\xi})$ can be obtained straightforward.

Let us firstly solve Eq.(52), and the solutions of (53) and (54) can be obtained analogously. Let us seek for the solution of Eq.(52) in the following form:

$$
B_{v_n}(\tau,k,l_n)=\frac{2}{\gamma_n^2}\frac{\frac{d}{d\tau}C_{v_n}}{C_{v_n}}. \qquad (56)
$$

Replacing Eq.(65) into (52), we can obtain that C_v^i must satisfy the following problem:

$$
\frac{d^2C_{v_n}}{d\tau^2}+(\chi_n+(\imath k-q)\gamma_n\tilde{\rho}_{n,v})\frac{dC_{v_n}}{d\tau}-\frac{\gamma_i^2}{2}\varphi_q^{v_n}(k)(a_n^{i,j})^2C_{v_n}=0, \qquad (57)
$$

$$C_{v_n}(0,k,l_n)=1, \qquad \frac{dC_{v_n}}{d\tau}(0,k,l_n)=\imath\, l_i\,\frac{\gamma_n^2}{2}, \quad (k,l_n)\in\mathbb{R}\times\mathbb{R}. \tag{58}$$

Please note that Eq.(57) is a second order ordinary differential equation with constant coefficients. Therefore, the solution is given by:

$$C_{v_n}(\tau,k,l_n)=C_{v_n}(\tau,k,l_n)=e^{(\mu_{q,v_n}+\zeta_{q,v_n})\tau}\left[\frac{s_{q,v_n,b}+\imath\, l_n\frac{\gamma_n^2}{2}s_{q,v_n,g}}{2\zeta_{q,v_n}}\right], \tag{59}$$

where

$$\mu_{q,v_n}=-\frac{1}{2}(\chi_n+(\imath k-q)\gamma_n\tilde{\rho}_{n,v}), \quad \zeta_{q,v_n}=\frac{1}{2}\left[4\mu_{v_n}^2+2\gamma_n^2\varphi_q^{v_n}(k)(a_n^{i,j})^2\right]^{1/2}, \tag{60}$$

$$s_{q,v_n,g}=1-e^{-2\zeta_{q,v_n}\tau}, \tag{61}$$

$$s_{q,v_n,b}=(\zeta_{q,v_n}+\mu_{q,v_n})e^{-2\zeta_{q,v_n}\tau}+(\zeta_{q,v_n}-\mu_{q,v_n}). \tag{62}$$

Furthermore, we can obtain:

$$\begin{aligned}
&\frac{dC_{v_n}}{d\tau}(\tau,k,l_n)=e^{(\mu_{q,v_n}+\zeta_{q,v_n})\tau}\,\cdot \\
&\left[\frac{(\mu_{q,v_n}-\zeta_{q,v_n})(\mu_{q,v_n}+\zeta_{q,v_n}-\frac{\gamma_n^2}{2}\imath\, l_n)e^{-2\zeta_{q,v_n}\tau}+(\mu_{q,v_n}+\zeta_{q,v_n})(\zeta_{q,v_n}-\mu_{q,v_n}+\frac{\gamma_n^2}{2}\imath\, l_n)}{2\zeta_{q,v_n}}\right] \\
&=\frac{e^{(\mu_{q,v_n}+\zeta_{q,v_n})\tau}}{2\zeta_{q,v_n}}\left[[\zeta_{q,v_n}^2-\mu_{q,v_n}^2]s_{q,v_n,g}+\frac{\gamma_n^2}{2}\imath\, l_n\, s_{q,v_n,d}\right],
\end{aligned} \tag{63}$$

where

$$s_{q,v_n,d}=(\zeta_{q,v_n}-\mu_{q,v_n})e^{-2\zeta_{q,v_n}\tau}+(\zeta_{q,v_n}+\mu_{q,v_n}). \tag{64}$$

Substituting Eqs.(59) and (63) into (65), we obtain:

$$B_{v_n}(\tau,k,l_n)=\frac{2}{\gamma_n^2}\frac{\left((\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}+\frac{\gamma_n^2}{2}\imath\, l_n\, s_{q,v_n,d}\right)}{s_{q,v_n,b}+\imath\, l_i\frac{\gamma_n^2}{2}s_{q,v_n,g}}. \tag{65}$$

Then, the solutions of Eqs.(53) and (54) are obtained analogously:

$$B_{r_m}(\tau,k,\xi_m)=\frac{2}{\eta_m^2}\frac{\left((\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}+\frac{\eta_m^2}{2}\imath\,\xi_m\, s_{q,r_m,d}\right)}{s_{q,r_m,b}+\imath\,\xi_m\frac{\eta_m^2}{2}s_{q,r_m,g}}, \quad m=i,j \tag{66}$$

where

$$s_{q,r_m,g}=1-e^{-2\zeta_{q,r_m}\tau}, \tag{67}$$

$$s_{q,r_m,b}=(\zeta_{q,r_m}+\mu_{q,r_m})e^{-2\zeta_{q,r_m}\tau}+(\zeta_{q,r_m}-\mu_{q,r_m}), \tag{68}$$

$$s_{q,r_m,d}=(\zeta_{q,r_m}-\mu_{q,r_m})e^{-2\zeta_{q,r_m}\tau}+(\zeta_{q,r_m}+\mu_{q,r_m}). \tag{69}$$

and for $m = i$

$$\mu_{q,r_i} = -\frac{1}{2}\left(\lambda_i + (\imath k - q)\eta_i\rho_{i,r}b_i\right), \tag{70}$$

$$\zeta_{q,r_i} = \frac{1}{2}\left[4\mu_{q,r_i}^2 + 2\eta_i^2(\varphi_q^{r_i}(k)b_i^2 - q + \imath k_i)\right]^{1/2}, \tag{71}$$

for $m = j$

$$\mu_{q,r_j} = -\frac{1}{2}\left(\lambda_j + (q - \imath k)\eta_i\rho_{i,r}b_i\right), \tag{72}$$

$$\zeta_{q,r_i} = \frac{1}{2}\left[4\mu_{q,r_i}^2 + 2\eta_i^2(\varphi_q^{r_i}(k)b_i^2 + q - \imath k_i)\right]^{1/2}, \tag{73}$$

where $\varphi_q^{v_n}(k)\,, \varphi_q^{r_i}(k)\,, \varphi_q^{r_j}(k)$ and $a_n^{i,j}\,, b_i\,, b_j$ are defined in Eqs.(52), (53) and (54).

2.1.1 Transition Probability Density Function Of Multiscale Heston CIR Model

Let us derive the joint transition probability density function p_f in the case of $\alpha = 1/2$, that is when the CIR interest rate model is considered. Considering integration on Eq.(50) for $A(\tau, k, \underline{l}, \underline{\xi})$, we obtain:

$$\begin{aligned} &A(\tau, k, \underline{l}, \underline{\xi}) \\ &= -\sum_{n=1}^{d} \frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau, k, l_n) - \sum_{m=i}^{j} \frac{2\lambda_m\theta_m}{\eta_m^2} \ln C_{r_m}(\tau, k, \xi_m), \\ &= -\sum_{n=1}^{d} \frac{2\chi_n v_n^*}{\gamma_n^2} \ln C_{v_n}(\tau, k, l_n) - \frac{2\lambda_i\theta_i}{\eta_i^2} \ln C_{r_i}(\tau, k, \xi_i) - \frac{2\lambda_j\theta_j}{\eta_j^2} \ln C_{r_j}(\tau, k, \xi_j) \end{aligned} \tag{74}$$

Hence the function $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ in Eq.(**??**) is given by:

$$\begin{aligned} g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) = &\prod_{n=1}^{d}\left(e^{-\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n)}\, e^{-\frac{2v_n}{\gamma_n^2}\frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}}\right)\prod_{m\in\{i,j\}}\Big(\\ &e^{-\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m)}\, e^{-\frac{2r_m}{\eta_m^2}\frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}}\Big), \\ &(v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, l_n, \xi_m) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau > 0. \end{aligned} \tag{75}$$

In order to get an explicit expression for $f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k)$ in Eq.(46), that is the inverse Fourier transform of $g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi})$ with respect to the variables $\underline{v}'$ and $\underline{r}'$, we have to compute the following integrals:

$$
\begin{aligned}
& L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl_n e^{\imath l_n v_n'}\, e^{-\frac{2\chi_n \overline{v}_n}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n)}\, e^{-\frac{2v_n}{\gamma_n^2}\frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \qquad (76)\\
& L_{r_m}(\tau, r_m, r_m', k \mid \underline{\Theta}_{r_m}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi_m\, e^{\imath\,\xi_m\, r_m'} e^{-\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m)}\, e^{-\frac{2r_m}{\eta_m^2}\frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \qquad (77)
\end{aligned}
$$

Let us show how to compute the integral appearing in (76) and (77) analytically by using Eqs.(59) and (63) with the change of variable $l_n' = -l_n \frac{(\gamma_n)^2}{2}\frac{s_{q,v_n,g}}{s_{q,v_n,b}}$, and the following equality:

$$
s_{q,v_n,d}s_{q,v_n,b} = \left(\zeta_{q,v_n}^2 - \mu_{v_n}^2\right) s_{q,v_n,g}^2 - \imath\, l_n' \left(\frac{8\zeta_{v_n}^2\, e^{-2\zeta_{v_n}\tau}}{1-2\imath\, l_n'}\right), \qquad (78)
$$

We can re-write $L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n})$ as follows:

$$
L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) = \frac{1}{2\pi} M_{q,v_n} e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n})+(\mu_{q,v_n}+\zeta_{q,v_n})\tau]}\,\cdot
$$

$$
e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}}\int_{-\infty}^{+\infty} dl_n'\, e^{-(\imath\, l_n')M_{q,v_n}} e^{-(2\chi_n v_i^*/\gamma_n^2)\ln(1-\imath\, l_n')} e^{\frac{(M_{q,v_n}\tilde{v}_{q,n})\imath\, l_n'}{1-\imath\, l_n'}}, \qquad (79)
$$

where

$$
M_{q,v_n} = \frac{2}{\gamma_n^2}\frac{s_{q,v_n,b}}{s_{q,v_n,g}}, \quad \tilde{v}_{q,n} = \frac{4(\zeta_{q,v_n})^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,b}^2},\ M_{q,v_n}\tilde{v}_{q,n} = \frac{8}{\gamma_n^2}\frac{\zeta_{q,v_n}^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,g}s_{q,v_n,b}}. \qquad (80)
$$

Now using formula n.34 on p.156 in Oberhettinger 1973, we obtain:

$$
\begin{aligned}
& L_{v_n}(\tau, v_n, v_n', k \mid \underline{\Theta}_{v_n}) \\
&= e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n})+(\mu_{q,v_n}+\zeta_{q,v_n})\tau]}\, e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}} M_{q,v_n} \\
& (M_{q,v_n}\tilde{v}_{q,n})^{-\nu_{q,v_n}/2} (M_{q,v_n}v_n')^{\nu_{q,v_n}/2}\, e^{-M_{q,v_n}\tilde{v}_{q,n}} e^{-M_{q,v_n}v_n'} I_{\nu_{q,v_n}}(2\tilde{M}_{q,v_n}(\tilde{v}_{q,n}v_n')^{1/2}), \\
& \quad ,\text{with}\,,\ v_n, v_n' > 0,\ k\in\mathbb{R}. \qquad (81)
\end{aligned}
$$

where $\nu_{q,v_n} = 2\chi_n v_n^*/\gamma_n^2 - 1$, and $I_{\nu_{q,v_n}}$ is the modified Bessel function of order ν_{q,v_n} (see, for example, Abramowitz and Stegun, 1970). Analogously, we obtain:

$$
\begin{aligned}
& L_{r_m}(\tau, r_m, r_m', k \mid \underline{\Theta}_{r_m}) \\
&= e^{-(2\lambda_m\theta_m/\eta_m^2)[\ln(s_{q,r_m,b}/2\zeta_{q,r_m})+(\mu_{q,r_m}+\zeta_{q,r_m})\tau]}\, e^{-(2r_m/\eta_m^2)(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/s_{q,r_m,b}} M_{q,r_m} \\
& (M_{q,r_m}\tilde{r}_{q,n})^{-\nu_{q,r_m}/2} (M_{q,r_m}r_m')^{\nu_{q,r_m}/2}\, e^{-M_{q,r_m}\tilde{r}} e^{-M_{q,r_m}r_m'} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,m}r_m')^{1/2}), \\
& \quad \text{with}\,,\ r_m, r_m' > 0,\ k\in\mathbb{R}, \qquad (82)
\end{aligned}
$$

where

$$M_{q,r_m} = \frac{2}{\eta_m^2}\frac{s_{q,r_m,b}}{s_{q,r_m,g}}, \quad \tilde{r}_{q,m} = \frac{4(\zeta_{q,r_m})^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,b}^2}, M_{q,r_m}\tilde{r}_{q,m} = \frac{8}{\eta_m^2}\frac{\zeta_{q,r_m}^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,g}s_{q,r_n,b}},$$

$$\text{and } \nu_{q,r_m} = 2\lambda_m\theta_m/\eta_m^2 - 1\,, \quad m = i, j \tag{83}$$

$$\tag{84}$$

From Eq.(45), we can obtain:

$$\begin{aligned}
& f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) \\
= & \prod_{n=2}^{d} L_{v_n}(\tau, v_n, v_n', k) \cdot L_{r_i}(\tau, r_i, r_i', k) L_{r_j}(\tau, r_j, r_j', k) \\
= & e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2} e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})\tau/\gamma_n^2} \\
& \cdot e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2} e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})\tau/\eta_m^2} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})} e^{-\sum\limits_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v_n')} \cdot \\
& \quad \prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v_n'}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2} \cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\, v_n')^{1/2}\right)\right]\Bigg\} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})} e^{-\sum\limits_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r_m')} \cdot \\
& \quad \prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\, r_m')^{1/2}\right)\right]\Bigg\}, \\
& (x, v_n, r_m),\ (x', v_n', r_m') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t, t' \geq 0,\ t'-t > 0.
\end{aligned} \tag{85}$$

Using Eq.(43), we obtain the probability density function $p_f(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')$ as follows:

$$
\begin{aligned}
& p_f(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t') = e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\cdot\Bigg\{ \\
& \cdot e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2} \\
& \cdot e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\, e^{-\sum\limits_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v_n')}\cdot \\
& \quad \prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v_n'}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2}\cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v_n')^{1/2}\right)\right]\Bigg\} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum\limits_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r_m')}\cdot \\
& \quad \prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r_m')^{1/2}\right)\right]\Bigg\}\Bigg\}, \qquad (86) \\
& (x,v_n,r_m),\ (x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0.
\end{aligned}
$$

The following results are remarkable (see Abramowitz and Stegun 1970 pp. 375 and 486):

$$
\begin{aligned}
& P_{p,v_n}(\tau,v_n,k) \\
&= \int_0^{+\infty} dv_n'(v_n')^{\nu_{q,v_n}/2} I_{\nu_{q,v_n}}(2M_{q,v_n}(\tilde{v}_{q,v_n}v_n')^{1/2})e^{-M_{q,v_n}\,v_n'} = \frac{(\tilde{v}_{q,n})^{\nu_{q,v_n}/2}}{M_{q,v_n}} e^{M_{q,v_n}\tilde{v}_{q,n}}, \\
& \qquad v_n>0,\ k\in\mathbb{R}, \qquad (87)
\end{aligned}
$$

$$
\begin{aligned}
& P_{p,r_m}(\tau,r_m,k) \\
&= \int_0^{+\infty} dr_m'(r_m')^{\nu_{q,r_m}/2} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,r_m}r_m')^{1/2})e^{-M_{q,r_m}\,r_m'} = \frac{(\tilde{r}_{q,m})^{\nu_{q,r_m}/2}}{M_{q,r_m}} e^{M_{q,r_m}\tilde{r}_{q,m}}, \\
& \qquad r_m>0,\ k\in\mathbb{R}, \qquad (88)
\end{aligned}
$$

Using Eq.(87), we integrate the joint probability density function over the future variance $\underline{v}'$ to find the marginal density for $(x',\underline{r}')$ as follows:

$$D_v(x,\underline{v},\underline{r},t,\underline{x}',\underline{r}',t') = \prod_{n=1}^{d}\left(\int_0^{+\infty} dv_n'\, p_f(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')\right)$$

$$= e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\cdot\Bigg\{$$

$$\cdot e^{-2\sum_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}$$

$$\cdot e^{-2\sum_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}$$

$$\cdot\Bigg\{ e^{-2\sum_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\Bigg\}$$

$$\cdot\Bigg\{ e^{-2\sum_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r_m')}\cdot$$

$$\prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r_m'}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\, r_m')^{1/2}\right)\right]\Bigg\}\Bigg\}, \tag{89}$$

$$(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\; t,t'\geq 0,\; t'-t>0.$$

2.2 Numéraire Invariance

In derivatives market, it is necessary to consider the risk neutral world and no-arbitrage condition. Thus, it is necessary to study SDE model under risk neutral measure.

Considering a Brownian motion vector $\mathbf{W}_t^{p,v(Q^i)}$ and $\mathbf{W}_t^{p,r(Q^i)}$ under measure Q^i by imposing the Q^i-martingale property and Girsanov Theorem (in Section 1.2.6), we obtain:

$$d\mathbf{W}_t^{p,v(Q^i)} = d\mathbf{W}_t^{p,v} + \sqrt{\mathrm{Diag}\mathbf{V}}\mathbf{a}^i dt \tag{90}$$

$$\begin{aligned} d\mathbf{W}_t^{p,r(Q^i)} &= d\mathbf{W}_t^{p,r} + [\mathrm{Diag}(\mathbf{R}_{i,j})]^{\alpha}\,(b_i,0)^{\mathrm{T}}dt \\ &= d\mathbf{W}_t^{p,r} + (r_i^{\alpha}b^i,0)^{\mathrm{T}}dt \end{aligned} \tag{91}$$

Since $d\mathbf{W}_t^{p,r(Q^i)} = \left(dW_i^{p,r(Q^i)}(t)\,,\ dW_j^{p,r(Q^i)}(t)\right)^{\mathrm{T}}$, Eq.(91) can be separated as follows:

$$dW_i^{p,r(Q^i)}(t) = dW_i^{p,r} + r_i^{\alpha}b_i dt\,, \tag{92}$$

$$dW_j^{p,r(Q^i)}(t) = dW_j^{p,r} \tag{93}$$

Substituting Eqs.(90) and (92) into Eq.(17), a SDE under risk measure Q^i is obtained as follows:

$$\frac{dS_t^{i,j}}{S_t^{i,j}} = \left(r^i - r^j\right) dt + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V})} d\mathbf{W}_t^{p,v(Q^i)} + (b_i, -b_j) \left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} d\mathbf{W}_t^{p,r(Q^i)}$$
$$= \left(r^i - r^j\right) dt + (a^i - a^j)^{\mathrm{T}} \sqrt{\mathrm{Diag}(\mathbf{V})} d\mathbf{W}_t^{p,v(Q^i)} + b_i r_i^{\alpha} dW_i^{p,r(Q^i)} - b_j r_j^{\alpha} dW_j^{p,r(Q^i)}, \qquad (94)$$
$$t > 0,\ i, j = 1, \ldots, N\,,\ \alpha = 0,\ \frac{1}{2}$$

It is worth to remark that Q^i-measure is clearly the risk neutral measure. Because in each monetary index i, the money market accounts accrues interest based on interest rate r^i . Thus, cash bond price $B^i(r^i, t)$ for currency i is defined as follows:

$$\frac{dB^i}{B^i}(t) = r_t^i\, dt \qquad (95)$$

hence

$$B^i(t) = e^{\int_0^t r_i(s) ds} \qquad (96)$$

Similarly, we could define cash bond pricing $B^j(t) = e^{\int_0^t r_j(s) ds}$ for currency j. Furthermore, let us define inter-currency cash bond price as follows:

$$\begin{aligned} B^{i,j}(t) &= \frac{B^i(t)}{B^i(t)} \\ &= e^{\int_0^t [r_i(s) - r_j(s)] ds} \end{aligned} \qquad (97)$$

Symmetrically, the following holds:

$$\begin{aligned} B^{j,i}(t) &= [B^{i,j}(t)]^{-1} \\ &= e^{\int_0^t -[r_i(s) - r_j(s)] ds} \end{aligned} \qquad (98)$$

hence

$$\frac{dB^{j,i}}{B^{j,i}} = -[r_i(t) - r_j(t)] dt \qquad (99)$$

Lemma 2.2. *Under the risk neutral measure Q^i, $S^{i,j}(t)/B^{i,j}(t)$ or $S^{i,j}(t) \cdot B^{j,i}(t)$ is a martingale process.*

Proof. Here is the simple proof that $S^{i,j}(t) \cdot B^{j,i}(t)$ is a martingale process, while the proof of $S^{i,j}(t)/B^{i,j}(t)$ can be deduced analogously.

$$\begin{aligned} \frac{d(S^{i,j} B^{j,i})}{S^{i,j} B^{j,i}} &= \frac{dS^{i,j} B^{j,i} + S^{i,j} dB^{j,i}}{S^{i,j} B^{j,i}} \\ &= \frac{dS^{i,j}}{S^{i,j}} + \frac{dB^{j,i}}{B^{j,i}} \\ &= \frac{dS^{i,j}}{S^{i,j}} - [r_i(t) - r_j(t)] dt \end{aligned} \qquad (100)$$

Substituting Eq.(94) into (100), we obtain

$$d(S^{i,j}B^{j,i}) = S^{i,j}(a^i - a^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V})}d\mathbf{W}_t^{p,v(Q^i)} + S^{i,j}(b_i, -b_j)\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha} dW_t^{p,r(Q^i)},$$
$$t > 0,\ i,j = 1,\ldots,N\,,\ \alpha = 0,\ \frac{1}{2} \tag{101}$$

Since $d\mathbf{W}_t^{p,v(Q^i)}$ and $d\mathbf{W}_t^{p,r(Q^i)}$ are new Brownian Motion vectors under Q^i measure, $S^{i,j}(t)\cdot B^{j,i}(t)$ is a Q^i-martingale process by imposing the Q^i-local martingale property.

□

The measurement change on Brownian Motion vectors also affect variance processes via the correlation coefficients ρ_n^v, $n = 1,\ldots,d$ as follows:

$$\begin{aligned} dW_{n,t}^{v(Q^i)} &= dW_{n,t}^v + \rho_n^v(\mathbf{e}^n)^{\mathrm{T}}\sqrt{\mathrm{Diag}(\mathbf{V(t)})}a^i dt \\ &= dW_{n,t}^v + \rho_n^v\sqrt{v_{n,t}}a_n^i dt \end{aligned} \tag{102}$$

where $\mathbf{e}^n$ is unit vector with $n-th$ element is 1. Moreover, $\{\mathbf{e}^n\,,\ n = 1,\ldots,d\}$ forms a set of d-dimensional mutually orthogonal unit vectors(or standard basis). Therefore, we obtain the following volatility process under Q^i measure:

$$\begin{aligned} dv_n(t) = \chi_n^{Q^i}(v_n^{*(Q^i)} - v_n(t))dt + \gamma_n\sqrt{v_n(t)}dW_{n,t}^{v(Q^i)}, \\ n = 1,\ldots,d\,,\ t > 0 \end{aligned} \tag{103}$$

with coefficients defined as follows:

$$\rho_n^{v(Q^i)} = \rho_n^v \tag{104}$$
$$\chi_n^{Q^i} = \chi_n + \gamma_n\rho_n^v a_n^i \tag{105}$$
$$v_n^{*(Q^i)} = v_n^*\frac{\chi^{Q^i}}{\chi_n^{Q^i}} \tag{106}$$
$$\tag{107}$$

Similarly, interest rate process $dr_m\,, r = i,j$ under risk neutral measure Q^i satisfies the following dynamics:

$$dW_{m,t}^{r(Q^i)} = dW_{m,t}^r + \rho_m^r(\mathbf{e}^m)^{\mathrm{T}}\left[\mathrm{Diag}(\mathbf{R}_{i,j})\right]^{\alpha}(b_i, 0)^{\mathrm{T}}dt \tag{108}$$

or

$$dW_{m,t}^{r(Q^i)} = \begin{cases} dW_{i,t}^r + \rho_i^r r_i^{\alpha} b_i dt & \text{if } m = i \\ dW_{j,t}^r & \text{if } m = j \end{cases}$$

where $\mathbf{e}^m$, $m = i, j$ is the unit vector, and ρ^r_m , $m = i, j$ is the correlation coefficient defined before. Therefore, the following interest rate process under Q^i measure are obtained:

$$dr_m(t) = \lambda_m^{Q^i}(\theta_m^{Q^i} - r_m(t))dt + \eta_m\, r_m^\alpha(t)dW_{m,t}^{r(Q^i)}, \qquad t > 0 \quad m \in \{i, j\} \tag{109}$$

with coefficients as follows:

- when $m = i$

$$\rho_i^{r(Q^i)} = \rho_i^r \tag{110}$$
$$\lambda_i^{Q^i} = \lambda_i + \eta_i \rho_i^r b_i \tag{111}$$
$$\theta_i^{Q^i} = \theta_i \frac{\lambda_i}{\lambda_i^{Q^i}} \tag{112}$$

- when $m = j$

$$\rho_j^{r(Q^i)} = \rho_j^r \tag{113}$$
$$\lambda_j^{Q^i} = \lambda_j \tag{114}$$
$$\theta_j^{Q^i} = \theta_j \tag{115}$$

From now on, let us write Q instead of Q^i for abbreviation. Moreover, the log-price of FX rate $x_t^{i,j} = \ln S_t^{i,j}$, $t > 0$ under Q measure satisfies the following SDE:

$$\begin{aligned} dx^{i,j} = &\left[(r_i - r_j) - \frac{1}{2}(a^i - a^j)^{\mathrm{T}}(\mathrm{Diag}\mathbf{V})(a^i - a^j) - \frac{1}{2}(b_i, -b_j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^{2\alpha}\,(b_i, -b_j)^{\mathrm{T}}\right]dt \\ &+(a^i - a^j)^{\mathrm{T}}\sqrt{\mathrm{Diag}\mathbf{V}}d\mathbf{W}_t^{p,v(Q)} + (b^i, -b^j)\,[\mathrm{Diag}(\mathbf{R}_{i,j})]^\alpha\, d\mathbf{W}_t^{p,r(Q)}, \\ &t > 0,\ \alpha = 0, \frac{1}{2}, \end{aligned} \tag{116}$$

Obviously, Eq.(116) can be written in the following form:

$$\begin{aligned} dx^{i,j} \;=\; &\left[r_i - r_j - \frac{1}{2}\sum_{n=1}^{d}[a_n^i - a_n^j]^2 v_k - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right] dt \\ &+\sum_{k=1}^{d}(a_k^i - a_k^j)\sqrt{v_k}dW_k^{p,v(Q)} + b_i r_i^\alpha dW_i^{p,r(Q)} - b_j r_j^\alpha dW_j^{p,r(Q)}, \\ &t > 0,\ \alpha = 0, \frac{1}{2}, \end{aligned} \tag{117}$$

Therefore, a system of equations defined on risk-neutral measure Q^i is obtained as follows:

$$\begin{aligned}
dx^{i,j} &= \left[r_i - r_j - \frac{1}{2}\sum_{n=1}^{d}[a_n^i - a_n^j]^2 v_k - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right] dt \\
&\quad + \sum_{n=1}^{d}(a_n^i - a_n^j)\sqrt{v_n} dW_k^{p,v(Q)} + b_i r_i^{\alpha} dW_i^{p,r(Q)} - b_j r_j^{\alpha} dW_j^{p,r(Q)}, \\
&\qquad\qquad t > 0,\ n = 1, \ldots, d\,,\ \alpha = 0, \frac{1}{2}, \qquad (118) \\
dv_n(t) &= \chi_n^Q(v_n^{*(Q)} - v_n(t))dt + \gamma_n \sqrt{v_n(t)} dW_{n,t}^{v(Q)}, \qquad n = 1, \ldots, d\,,\ t > 0; \qquad (119) \\
dr_m(t) &= \lambda_m^Q(\theta_m^Q - r_m(t))dt + \eta_m\, r_m^{\alpha}(t) dW_{m,t}^{r(Q)}, \qquad t > 0 \quad m = \{i, j\}\,; \qquad (120)
\end{aligned}$$

with the following correlation structure:

$$\begin{aligned}
E(dW_{n,t}^{p,v(Q)} dW_{m,t}^{p,r(Q)}) &= 0,\ \ n = 1, \ldots d\,, m = i, j,\ t > 0, && (121) \\
E(dW_{n,t}^{p,v(Q)} dW_{l,t}^{v(Q)}) &= 0,\ n \neq l,\ n, l = 1, \ldots, d,\ t > 0, && (122) \\
E(dW_{n,t}^{p,v(Q)} dW_{n,t}^{v(Q)}) &= \rho_{n,v} dt,\ n = 1, \ldots, d\,,\ t > 0, && (123) \\
E(dW_{n,t}^{p,v(Q)} dW_{m,t}^{r(Q)}) &= 0,\ m = i, j;\ n = 1, \ldots, d,\ t > 0, && (124) \\
E(dW_{m,t}^{p,r(Q)} dW_{m,t}^{r(Q)}) &= \rho_{m,r} dt,\ m = i, j\ t > 0, && (125) \\
E(dW_{m,t}^{p,r(Q)} dW_{m',t}^{r}) &= 0,\ m \neq m',\ m, m' = i, j\,,\ t > 0, && (126) \\
E(dW_{m,t}^{p,r(Q)} dW_{n,t}^{v(Q)}) &= 0,\ \ n = 1, \ldots d\,, m = i, j,\ t > 0, && (127) \\
E(dW_{m,t}^{r(Q)} dW_{n,t}^{v(Q)}) &= 0,\ n = 1, \ldots d\,, m = i, j,\ t > 0, && (128)
\end{aligned}$$

2.3 The Model Treatment Under Risk-Neutral Measure

In this section, the model are treated under risk neutral measure. First of all, let us equipped Eqs.(118)-(120) with the initial condition:

$$\begin{aligned}
x^{i,j}(0) &= \tilde{x}_0^{i,j}, && (129) \\
v_n(0) &= \tilde{v}_{n,0}, && (130) \\
r_m(0) &= \tilde{r}_{m,0}, && (131)
\end{aligned}$$

where $\tilde{x}_0^{i,j}$, $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ are random variables that are assumed to be concentrated in a point with probability one. For simplicity, we identify the random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{m,0}$, $\tilde{r}_{m,0}$ with the points where they are concentrated. $\tilde{v}_{m,0}$, $\tilde{r}_{m,0}$, χ_k^Q, λ_m^Q, γ_n, η_m, $v_n^{*(Q)}$, θ_m^Q are assumed to be positive constants. In addition, Feller condition is considered here, i.e. $\frac{2\chi_k v_n^*}{\gamma_k^2} > 1$ and $\frac{2\lambda_m \theta_m}{\eta_m^2} > 1$.

Let $p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')$, $(x,\underline{v},\underline{r}),(x',\underline{v}',\underline{r}')\in\mathbb{R}\times(\mathbb{R}^+)^d\times(\mathbb{R}^+)^2$, t, $t'\geq 0$, $t'-t>0$ be the transition probability density function under risk neutral measure Q associated with the stochastic differential system (118), (119), (120), and (31). That is, the probability density function having $x'=x_{t'}^{i,j}$, $\underline{v}'=(v_1',\dots,v_d')^{\mathrm{T}}$, $\underline{r}'=(r_i',r_j')^{\mathrm{T}}$, given that $x=x_t^{i,j}$, $\underline{v}=(v_1,\dots,v_d)^{\mathrm{T}}$, $\underline{r}=(r_i,r_j)^{\mathrm{T}}$, when $t'-t>0$.

In analogy with Lipton (2001) (pages 602–605), this transition probability density function $p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')$ as a function of the "past" variables $(x,\underline{v},\underline{r},t)$ satisfies the following backward Kolmogorov equation:

$$\begin{aligned}
-\frac{\partial p_f^Q}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i-a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f^Q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f^Q}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f^Q}{\partial r_i^2}\\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f^Q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i-a_n^j)v_n\frac{\partial^2 p_f^Q}{\partial x\partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f^Q}{\partial x\partial r_i^{2\alpha}} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f^Q}{\partial x\partial r_j}\\
&+\sum_{n=1}^{d}\chi_n(\overline{v}_n-v_n)\frac{\partial p_f^Q}{\partial v_n} + \lambda_i(\theta_i-r_i)\frac{\partial p_f^Q}{\partial r_i} + \lambda_j(\theta_j-r_j)\frac{\partial p_f^Q}{\partial r_j}\\
&+\left((r_i-r_j) - \frac{1}{2}\sum_{n=1}^{d}(a_n^i-a_n^j)^2 v_n - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right)\frac{\partial p_f^Q}{\partial x} \qquad (132)\\
&\quad (x,v_n,r_i,r_j)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ 0\leq t<t',
\end{aligned}$$

with final condition:

$$\begin{aligned}
p_f^Q(t,x,\underline{v},\underline{r},t',x',\underline{v}',\underline{r}'|t'=t) &= \delta(x'-x)\prod_{m\in\{i,j\}}\delta(r_m'-r_m)\cdot\prod_{n=1}^{d}\delta(v_n'-v_n)\\
&= \delta(x'-x)\cdot\delta(r_i'-r_i)\delta(r_j'-r_j)\cdot\prod_{n=1}^{d}\delta(v_n'-v_n)\,, \qquad (133)\\
\forall n\in\{1,\dots,d\}\,\text{and}\, m\in\{i,j\}\,, &\quad (x,v_n,r_m),(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0,
\end{aligned}$$

and the appropriate boundary conditions. The analytical treatment is similar to Section 2.1, and the detail deduction is shown in Appedix C, where the following transition probability density function is obtained using a suitable parametrization.

$$\begin{aligned}
&p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v}',\underline{r}',t')\\
&= e^{q(x-x')}\frac{1}{2\pi}\int_{\mathbb{R}}dk\, e^{\imath k(x'-x)}\underline{L_v^Q}(t'-t,\underline{v},\underline{v}',k\mid\underline{\Theta_v})\cdot\underline{L_r^Q}(t'-t,\underline{r},\underline{r}',k\mid\underline{\Theta_r})\,,\\
&(x,v_n,r_m),\ (x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0. \qquad (134)
\end{aligned}$$

where $\imath$ stands for the imaginary unit. $\underline{L_v^Q}(t'-t,\underline{v},\underline{v'},k \mid \underline{\Theta_v})$ and $\underline{L_r^Q}(t'-t,\underline{r},\underline{r'},k \mid \underline{\Theta_r})$ are explicitly known functions given in Eqs.(135) and (136) which both depend on the product of the modified Bessel functions, $\prod_n^d I_{\nu^Q_{q,v_n}}$ and $\prod_{m\in\{i,j\}} I_{\nu^Q_{q,r_m}}$ (see, for example, Abramowitz and Stegun, 1970). $\nu^Q_{q,v_n} = (2\chi v_n^{*(Q)}/\gamma_n^2)-1$ and $\nu_{q,r_m} = (\lambda\theta^Q_m/\eta^2_m)-1$ are positive real indices under the conditions $2\chi v_n^{*(Q)}/\gamma_n^2 > 1$ and $2\lambda\theta^Q_m/\eta^2_m > 1$.

As mentioned in Chapter 2 and Section 2.1, positive indices imply non negative modified Bessel functions which guarantee that the function p^Q_f given in Eq.(134) is a probability density function with respect to the future variables. Moreover, these conditions guarantee positive values of the variance and interest rate processes $\forall t > 0$ (with probability one) given that the initial stochastic conditions $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ are positive (with probability one).

$$\begin{aligned}
&\underline{L_v^Q}(t'-t,\underline{v},\underline{v'},k \mid \underline{\Theta_v}) = \prod_{n=1}^{d} L^Q_{v_n}(t'-t,v_n,v'_n,k \mid \underline{\Theta}_{v_n})\\
&= e^{-2\sum_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2}\\
&\cdot\left\{ e^{-2\sum_{n=1}^{d} v_n(\zeta^2_{q,v_n}-\mu^2_{q,v_n})s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})}\, e^{-\sum_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v'_n)}\,\cdot\right.\\
&\left.\prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v'_n}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2}\cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\,v'_n)^{1/2}\right)\right]\right\}
\end{aligned} \tag{135}$$

$$\begin{aligned}
&\underline{L_r^Q}(t'-t,\underline{r},\underline{r'},k \mid \underline{\Theta_r}) = \prod_{m\in\{i,j\}} L^Q_{r_m}(t'-t,r_m,r'_m,k \mid \underline{\Theta}_{r_m})\\
&= e^{-2\sum_{m\in\{i,j\}}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2}\, e^{-2\sum_{m\in\{i,j\}}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2}\\
&\cdot\left\{ e^{-2\sum_{m\in\{i,j\}} r_m(\zeta^2_{q,r_m}-\mu^2_{q,r_m})s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum_{m\in\{i,j\}} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)}\,\cdot\right.\\
&\left.\prod_{m\in\{i,j\}}\left[M_{q,r_m}\left(\frac{r'_m}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2}\cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\,r'_m)^{1/2}\right)\right]\right\},
\end{aligned} \tag{136}$$

Furthermore, let us remark that formula (134) can be interpreted as the inverse Fourier transform of the convolution of the probability density functions associated with the stochastic processes described by Eqs.(116)- (120). It is worth noting that the integrals of $\underline{L_v^Q}$ and $\underline{L_r^Q}$ with respect the future variables $\underline{v'}$ and $\underline{r'}$ are given by elementary functions $\underline{W^{z(Q)}_{v,q}}$ and $\underline{W^{z(Q)}_{r,q}}$ when $z=0$.

$$\underline{W^{z(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v}) = \prod_{n=1}^{d}\int_0^{+\infty} dv'_n\,(v'_n)^z L_{v,q}(t'-t,v_n,v'_n,k;\underline{\Theta_v}), \tag{137}$$

$$\underline{W^{z(Q)}_{r,q}}(t'-t,r,k;\underline{\Theta_r}) = \prod_{m\in\{i,j\}} \int_0^{+\infty} dr'_m\,(r'_m)^z L_{r,q}(t'-t,r,r',k;\underline{\Theta}_r), \tag{138}$$

Especially, when $z=0$, we obtain:

$$\underline{W^{0(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v}) = e^{-\sum_{n=1}^{d}(2\chi_n^Q v_n^{*(Q)}/\gamma_n^2)\ln(s_{q,v_n,b}/2\zeta_{q,v_n})}\, e^{-\sum_{n=1}^{d}(2\chi_n^Q v_n^{*(Q)}/\gamma_n^2)(\mu_{q,v_n}+\zeta_{q,v_n})T}\cdot$$
$$e^{-\sum_{n=1}^{d}(2v_n(0)/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}}, \tag{139}$$

$$\underline{W^{0(Q)}_{r,q}}(t'-t,\underline{r},k;\underline{\Theta_r}) = e^{-\sum_{m\in\{i,j\}}(2\lambda_m^Q\theta_m^Q/\eta_m^2)\ln(s_{q,r_m,b}/2\zeta_{q,r_m})}\, e^{-\sum_{m\in\{i,j\}}(2\lambda_m^Q\theta_m^Q/\eta_m^2)(\mu_{q,r_m}+\zeta_{q,r_m})T}\cdot$$
$$e^{-\sum_{m\in\{i,j\}}(2r_m(0)/\eta_m^2)(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/s_{q,r_m,b}}, \tag{140}$$

Let us highlight that formulas (139) and (140) are elementary formulas that do not involve integrals. These functions and the function $L_{r,q}$ can be used to get an integral representation formula for the marginal probability density function, $D^Q_{v,q}(x,\underline{v},\underline{r},t,x',\underline{r'},t')$, of the future variables $(x',\underline{r'})$.

$$D^Q_{v,q}(x,\underline{v},\underline{r},t,x',\underline{r'},t') = \prod_{n=0}^{d}\int_0^{+\infty} dv'_n\, p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v'},\underline{r'},t')$$
$$= e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}\underline{W^{z(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v})\underline{L^Q_r}(t'-t,\underline{r},\underline{r'},k\mid\underline{\Theta_r}), \tag{141}$$

and for the marginal probability density function, $D^Q_{v,r,q}(x,\underline{v},\underline{r},t,x',,t')$, of the price variable x':

$$D^Q_{v,r,q}(x,\underline{v},\underline{r},t,x',t') = \prod_{m\in\{i,j\}}\int_0^{+\infty} dr'_m \prod_{n=0}^{d}\int_0^{+\infty} dv'_n\, p_f^Q(x,\underline{v},\underline{r},t,x',\underline{v'},\underline{r'},t')$$
$$= e^{q(x-x')}\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{\imath k(x'-x)}\underline{W^{z(Q)}_{v,q}}(t'-t,\underline{v},k;\underline{\Theta_v})\underline{W^{z(Q)}_{r,q}}(t'-t,r,k;\underline{\Theta_r})\,. \tag{142}$$

It is worth to highlight that in formulas (141) and (142) the variables x, v, r are the initial values of the log-price, the stochastic variance and the stochastic interest rate respectively. The marginal probability density function, $D_{v,q}$, can be used to price European call and put options with payoff functions independent of the variance process in the framework of stochastic interest rates. Thus, $D^Q_{v,q}$ is used to deduce formulas for European call and put vanilla options.

2.4 Integral formulas for Vanilla Foreign Exchange Call and Put Options

Foreign exchange (FX) future (or currency future) is an important futures contract in derivatives market. The holders of FX future are obliged to exchange one currency for another at a fixed price (exchange rate) on a specified date. Thus, the corresponding call and put options on FX future are popular risk hedging tools in fluctuated FX market. In the framework of the model (117)-(128), integral formulas are derived to approximate the prices of the corresponding European call and put options with strike price $E > 0$ and maturity time T. This is done using the no arbitrage pricing theory associated with one of the currencies involved, for instance currency i. As illustrated in De Col,Gnoatto and Grasselli 2013, the option price is computed as the expected value of a discounted payoff with respect to an equivalent martingale measure known as a risk-neutral measure (see, for example, Duffie, 2001; Schoutens, 2003, Wong, 2006, Grzelak, 2011). That is, let S_0 be the spot price (future) at time zero we can compute the prices of European call and put options with strike price E and maturity time T as follows:

$$C(S_0^{i,j}, T, E, \underline{r}(0), \underline{v}(0)) = E^Q \left(\frac{(S_0^{i,j} e^{x_T} - E)_+}{e^{\int_0^T r_i(t)\, dt}} \right), \tag{143}$$

$$T > 0, S_0^{i,j} > 0, \; v_n(0)\,, n = 1, \ldots, d\,, r_m(0) > 0, m = i, j\,, \tag{144}$$

$$P(S_0^{i,j}, T, E, \underline{r}(0), \underline{v}(0)) = E^Q \left(\frac{(E - S_0^{i,j} e^{x_T})_+}{e^{\int_0^T r_i(t)\, dt}} \right), \tag{145}$$

$$T > 0, S_0^{i,j} > 0, \; v_n(0)\,, n = 1, \ldots, d\,, r_m(0) > 0, m = i, j \tag{146}$$

where $(\,\cdot\,)_+ = max\{\,\cdot\,,\, 0\}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$ and the expectation is taken under the risk-neutral measure Q. In contrast to physical measure, another advantage of using risk-neutral measure lies in the fact that it is not necessary introduce the risk premium parameters.

Please note that $v_n(0)\,, n = 1, \ldots d$ and $r_m(0)\,, m = i, j$ are not observable in the market so that we consider them as model parameters that must be estimated. The numerical evaluation of formula (143) is very time consuming. We get a formula to evaluate these prices approximating the stochastic integral that defines the discount factor as follows:

$$e^{-\int_0^T r_i(t) dt} \approx e^{-r_i(t) \frac{T}{(1+e^{\lambda_i T})} - r_T \frac{T e^{\lambda_i T}}{(1+e^{\lambda_i T})}}. \tag{147}$$

Roughly speaking, formula (147) has been obtained approximating r_t as a suitable weighted sum of short rate r_t at $t = 0$ and $t = T$. The choice of these weights is inspired by the analytical expression of zero-coupon bond given in Eq.(**??**). As shown in Section **??**, this approximation works well also for long maturity. Furthermore, the use of formula (147) allows us to reduce the computation of the option prices to the evaluation of a one dimensional integral.

In fact, let $p_f^Q(x, \underline{v}, \underline{r}, t, x', \underline{v}'\underline{r}', t')$, $(x, \underline{v}, \underline{r}, t), (x', \underline{v}', \underline{r}', t') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$, $t, t' \geq 0$, $\tau = t' - t > 0$, be the transition probability density function of the stochastic process described by Eqs.(117)–(128). Using formula (134) for p_f^Q and (147), we obtain:

$$C_A(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \int_{\ln(E/S_0^{i,j})}^{+\infty} dx'\, e^{-qx'}(S_0^{i,j}e^{x'} - E)$$
$$\int_0^{+\infty} dr_i' e^{-r_i' \frac{Te^{\lambda_i T}}{(1+e^{\lambda_i T})}} \int_0^{+\infty} dr_j' D_v(0, \underline{v_0}, \underline{r_0}, 0, \underline{x}', \underline{r}', T), \tag{148}$$
$$S_0^{i,j},\, T,\, E > 0\,, \quad \underline{r_0} = (r_i(0), r_j(0))^{\mathrm{T}} \in \mathbb{R}^{+2}, \;\; \underline{v_0} = (v_1(0), \ldots, v_d(0))^{\mathrm{T}} \in \mathbb{R}^{+d}, \;\; q > 1,$$

$$P_A(S_0^{i,j}, T, E, \underline{r_0}, \underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \int_{-\infty}^{\ln(E/S_0^{i,j})} dx'\, e^{-qx'}(E - S_0^{i,j}e^{x'})$$
$$\int_0^{+\infty} dr_i' e^{-r' \frac{Te^{\lambda_i T}}{(1+e^{\lambda_i T})}} \int_0^{+\infty} dr_j' D_v(0, \underline{v_0}, \underline{r_0}, 0, x', r', T), \tag{149}$$
$$S_0^{i,j},\, T,\, E > 0\,, \underline{r_0} = (r_i(0), r_j(0))^{\mathrm{T}} \in \mathbb{R}^{+2}, \;\; \underline{v_0} = (v_1(0), \ldots, v_d(0))^{\mathrm{T}} \in \mathbb{R}^{+d}, \;\; q < -1,$$

where D_v is given in (141). Choosing $q > 1$, we obtain:

$$\int_{\ln(E/S_0^{i,j})}^{+\infty} dx'\, e^{-q\,x'}(S_0^{i,j}e^{x'} - E)e^{\imath\,k\,x'} = \frac{S_0^{i,j}\left(\frac{S_0^{i,j}}{E}\right)^{q-1-\imath\,k}}{-k^2 - (2q-1)\imath\,k + q(q-1)}, \tag{150}$$

and choosing $q < -1$ we obtain:

$$\int_{-\infty}^{\ln(E/S_0^{i,j})} dx'\, e^{-q\,x'}(E - S_0^{i,j}e^{x'})e^{\imath\,k\,x'} = \frac{S_0^{i,j}\left(\frac{S_0^{i,j}}{E}\right)^{q-1-\imath\,k}}{-k^2 - (2q-1)\imath\,k + q(q-1)}\,. \tag{151}$$

In addition, the following formula holds (see Erdely et al. Vol I, 1954, p. 197 formula (18)).

$$\int_0^{\infty} t^{\frac{\nu}{2}} I_\nu(2\alpha^{\frac{1}{2}}t^{\frac{1}{2}})e^{-p\cdot t}dt = \alpha^{\frac{\nu}{2}}p^{-\nu-1}e^{\frac{\alpha}{p}}, \tag{152}$$

and this implies the following equality:

$$\int_0^{+\infty} dr_i'\,(r')^{\nu_{r_i}/2}\,e^{-(M_{q,r_i}+b)r'} I_{\nu_{r_i}}(2M_{q,r_i}(\tilde{r}_{q,i}r_i')^{1/2})$$
$$= \;\; [(M_{q,r_i})^2\tilde{r}_{q,i}]^{\frac{\nu_{r_i}}{2}}(M_{q,r_i}+b)^{-\nu_{r_i}-1}e^{\frac{(M_{q,r_i})^2\tilde{r}_{q,i}}{(M_{q,r_i}+b)}}\,, \;\; b \in \mathbb{R}. \tag{153}$$

Now using the expression of D_v given in Eq.(141), and Eq.(153), (150) with $q=2$, we obtain the approximation C_A of the call option price C given in (143), that is:

$$C_A(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \frac{S_0^{i,j}}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0^{i,j}}{E}\right)^{(1-\imath k)}}{-k^2-3\imath k+2} \underline{W_{v,q}^{0(Q)}}(t'-t,\underline{v},k;\underline{\Theta_r})$$

$$\underline{W_{r,q}^{0(Q)}}(t'-t,r,k;\underline{\Theta_r})\left(\frac{M_{q,r_i}}{M_{q,r_i}+\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)^{\nu_{r_i}+1} e^{-\left(\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}\right)\left(\frac{M_{q,r_i}\tilde{r}_{q,i}}{M_{q,r_i}+\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)}, \tag{154}$$

$n=1,\ldots,d\,, m=i,j\,,\,(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0\,, q=2,$

Proceeding in a similar way, we obtain the following approximation, P_A, of the put option price P:

$$P_A(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)\frac{T}{(1+e^{\lambda_i T})}} \frac{S_0^{i,j}}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0^{i,j}}{E}\right)^{-(3+\imath k)}}{-k^2+5\imath k+6} \underline{W_{v,q}^{0(Q)}}(t'-t,\underline{v},k;\underline{\Theta_r})$$

$$\underline{W_{r,q}^{0(Q)}}(t'-t,r,k;\underline{\Theta_r})\left(\frac{M_{q,r_i}}{M_{q,r_i}+\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)^{\nu_{r_i}+1} e^{-\left(\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}\right)\left(\frac{M_{q,r_i}\tilde{r}_{q,i}}{M_{q,r_i}+\frac{Te^{\lambda_i T}}{1+e^{\lambda_i T}}}\right)}, \tag{155}$$

$n=1,\ldots,d\,, m=i,j\,,\,(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0\,, q=-2,$

Taking the limit $a^{i,j}\to 0^+$, $\lambda_m\to 0^+$, $\eta_n\to 0^+$ in Eqs.(154) and (155), we derive the following exact formulas for the price of the European call and put options of the Heston model:

$$C_H(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)T}\,e^{2r_i(0)T}\frac{S_0^{i,j}}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0^{i,j}}{E}\right)^{(1-\imath k)} e^{-\imath k r_i(0)T}}{-k^2-3\imath k+2}\underline{W_{v,q}^{0(Q)}}(t'-t,\underline{v},k;\underline{\Theta_r}), \tag{156}$$

$n=1,\ldots,d\,, m=i,j\,,\,(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t,t'\geq 0,\ t'-t>0\,, q=2,$

$$P_H(S_0^{i,j},T,E,\underline{r_0},\underline{v_0}) = e^{-r_i(0)T}\,e^{-2r_i(0)T}\frac{S_0^{i,j}}{2\pi}\int_{-\infty}^{+\infty} dk \frac{\left(\frac{S_0^{i,j}}{E}\right)^{-(3+\imath k)} e^{-\imath k r_i(0)T}}{-k^2+5\imath k+6}\underline{W_{v,q}^{0(Q)}}(t'-t,\underline{v},k;\underline{\Theta_r}), \tag{157}$$

$n=1,\ldots,d\,, m=i,j\,,\,(x,v_n,r_m),\,(x',v_n',r_m')\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,$

$t,t'\geq 0,\ t'-t>0\,, q=-2,$

It is worthy of note that the integrand functions appearing in formulas (154)–(157) are smooth functions whose integration do not require a specific care. This regularity is due to the specific approach used to derive them. The future study will focus on the application of this theoretical model. This includes the simulation study to compare the performance of different models in interpreting real data.

3 Simulation study

4 Conclusion

5 Appendix: The Analytical Treatment Of Mutiscale Hybrid Heston Model Under Risk Neutral Measure

In Appendix C, let us derive an integral representation formula under risk neutral measure for the transition probability density function of the process described by Eqs.(118)-(120) and initial conditions (129) where we identify the initial condition of random variables $\tilde{x}_0^{i,j}$, $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$ with the points where they are concentrated. We assume $\tilde{v}_{n,0}$, $\tilde{r}_{m,0}$, χ_n, λ_m, γ_k, η_m, v_n^*, θ_m to be positive constant. Moreover, we assume $\frac{2\chi_n v_n^*}{\gamma_n^2} > 1$ and $\frac{2\lambda_m\theta_m}{\eta_m^2} > 1$. In addition, we deduce the moments of the volatility variables and the mixed moments. For simplicity, here we just delete the index Q from every parameter and Brownian motion, i.e. $W_{n,t}^{p,v} = W_{n,t}^{p,v(Q)}$, $W_{m,t}^{p,r} = W_{m,t}^{p,r(Q)}$, $\chi_n = \chi_n^Q$, $v_n^* = v_n^{*(Q)}$ and so on so forth.

Let $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$, $(x, \underline{v}, \underline{r}), (x', \underline{v}', \underline{r}') \in \mathbb{R} \times (\mathbb{R}^+)^d \times (\mathbb{R}^+)^2$, $t, t' \geq 0$, $t' - t > 0$, be the transition probability density function associated with the stochastic differential system (118),(119),(120), and (31), that is, the probability density function of having $x' = x_{t'}^{i,j}$, $\underline{v}' = (v_1', \ldots, v_d')^{\mathrm{T}}$, $\underline{r}' = (r_i', r_j')^{\mathrm{T}}$ given that $x = x_t^{i,j}$, $\underline{v} = (v_1, \ldots, v_d)^{\mathrm{T}}$, $\underline{r} = (r_i, r_j)^{\mathrm{T}}$, when $t' - t > 0$. In analogy with Lipton (2001), this transition probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v}', \underline{r}', t')$ as a function of the "past" variables $(x, \underline{v}, \underline{r}, t)$ satisfies the following backward Kolmogorov equation:

$$
\begin{aligned}
-\frac{\partial p_f}{\partial t} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_f}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_f}{\partial r_i^2} \\
&+ \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_f}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_f}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_i^{2\alpha}} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_f}{\partial x \partial r_j} \\
&+ \sum_{n=1}^{d}\chi_n(\overline{v}_n - v_n)\frac{\partial p_f}{\partial v_n} + \lambda_i(\theta_i - r_i)\frac{\partial p_f}{\partial r_i} + \lambda_j(\theta_j - r_j)\frac{\partial p_f}{\partial r_j} \\
&+ \left((r_i - r_j) - \frac{1}{2}\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n - \frac{1}{2}(b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha})\right)\frac{\partial p_f}{\partial x} \qquad (158) \\
&\qquad (x, v_n, r_i, r_j) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \ 0 \leq t < t',
\end{aligned}
$$

with final condition:

$$\begin{aligned} p_f(t,x,\underline{v},r,t'=t,x',\underline{v}',r') &= \delta(x'-x)\prod_{m=i}^{j}\delta(r'_m-r_m)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n) \\ &= \delta(x'-x)\cdot\delta(r'_i-r_i)\delta(r'_j-r_j)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n)\,, \qquad (159) \\ & \quad (x,v_n,r_m),(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0, \end{aligned}$$

and the appropriate boundary conditions. Letting $\tau = t'-t$, we can introduce the function p_b defined as follows:

$$\begin{aligned} &p_b(\tau,x,\underline{v},r,x',\underline{v}',r') = p_f(t,x,\underline{v},r,t',x',\underline{v}',r'), \\ &\quad (x,\underline{v},\underline{r}),(x',\underline{v}',\underline{r}')\in\mathbb{R}\times(\mathbb{R}^+)^d\times(\mathbb{R}^+)^2,\ t'=t+\tau,\ \tau>0\,. \end{aligned} \qquad (160)$$

The representation (160) holds since the coefficients of the Kolmogorov backward equation and condition (159) are invariant by time translation. Substituting the change of the time variable $\tau = t-t'$ into Eq.(158), it is worth nothing that p_b is the solution of the following problem:

$$\begin{aligned} \frac{\partial p_b}{\partial\tau} &= \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i-a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_b}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n\frac{\partial^2 p_b}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_b}{\partial r_i^2} \\ &+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_b}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i-a_n^j)v_n\frac{\partial^2 p_b}{\partial x\partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_b}{\partial x\partial r_i^{2\alpha}} - \rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_b}{\partial x\partial r_j} \\ &+\sum_{n=1}^{d}\chi_n(\overline{v}_n-v_n)\frac{\partial p_b}{\partial v_n} + \lambda_i(\theta_i-r_i)\frac{\partial p_b}{\partial r_i} + \lambda_j(\theta_j-r_j)\frac{\partial p_b}{\partial r_j} \\ &+\left((r_i-r_j) - \frac{1}{2}\sum_{n=1}^{d}(a_n^i-a_n^j)^2 v_n - \frac{1}{2}(b_i^2 r_i^{2\alpha} - b_j^2 r_j^{2\alpha})\right)\frac{\partial p_b}{\partial x} \qquad (161) \\ &\quad (x,v_n,r_i,r_j)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0\leq t<t', \end{aligned}$$

with the initial condition:

$$\begin{aligned} p_b(0,x,\underline{v},r,x',\underline{v}',r') &= \delta(x'-x)\prod_{m=i}^{j}\delta(r'_m-r_m)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n) \\ &= \delta(x'-x)\cdot\delta(r'_i-r_i)\delta(r'_j-r_j)\cdot\prod_{n=1}^{d}\delta(v'_n-v_n)\,, \qquad (162) \\ & \quad (x,v_n,r_m),(x',v'_n,r'_m)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t\geq 0, \end{aligned}$$

and with the appropriate boundary conditions. For later convenience, let us consider the following change of dependent variable:

$$\begin{aligned} &p_b(\tau,x,\underline{v},r,x',\underline{v}',r') = e^{q(x-x')}p_q(\tau,x,\underline{v},r,x',\underline{v}',r') \qquad (163) \\ &\quad (x,\underline{v},\underline{r}),(x',\underline{v}',\underline{r}')\in\mathbb{R}\times(\mathbb{R}^+)^d\times(\mathbb{R}^+)^2,\ t'=t+\tau,\ \tau>0\,. \end{aligned}$$

Substituting Eq.(163) into (161)– (162), we obtain that p_q is the solution of the following problem:

$$\frac{\partial p_q}{\partial \tau} = \frac{1}{2}\left[\sum_{n=1}^{d}(a_n^i - a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right]\frac{\partial^2 p_q}{\partial x^2} + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 p_q}{\partial v_n^2}$$
$$+\frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 p_q}{\partial r_i^2} + \frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 p_q}{\partial r_j^2} + \sum_{n=1}^{d}\rho_{n,v}\gamma_n(a_n^i - a_n^j)v_n\frac{\partial^2 p_q}{\partial x \partial v_n} + \rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial^2 p_q}{\partial x \partial r_i}$$
$$-\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial^2 p_q}{\partial x \partial r_j} + \sum_{n=1}^{d}\left[\chi_n(\overline{v}_n - v_n) + q\gamma_n\rho_{n,v}(a_n^i - a_n^j)v_n\right]\frac{\partial p_q}{\partial v_n}$$
$$+\left[\lambda_i(\theta_i - r_i) + q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial p_q}{\partial r_i} + \left[\lambda_j(\theta_j - r_j) + q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial p_q}{\partial r_j}$$
$$+\left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2(2q-1) + \left(r_i + \frac{r_i^{2\alpha}}{2}b_i^2(2q-1)\right) + \left(-r_j + \frac{r_j^{2\alpha}}{2}b_j^2(2q-1)\right)\right]\frac{\partial p_q}{\partial x}$$
$$+\left[\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i - a_n^j)^2\left(q^2 - q\right) + \left(qr_i + \frac{r_i^{2\alpha}}{2}b_i^2(q^2-q)\right) + \left(q(-r_j) + \frac{r_j^{2\alpha}}{2}b_j^2(q^2-q)\right)\right]p_q$$
$$(x, v_n, r_i, r_j) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0 \le t < t', \tag{164}$$

with the initial condition:

$$\begin{aligned} p_q(0, x, \underline{v}, r, x', \underline{v}', r') &= e^{q(x-x')}\delta(x'-x)\prod_{m=i}^{j}\delta(r_m' - r_m)\cdot\prod_{n=1}^{d}\delta(v_n' - v_n) \\ &= e^{q(x-x')}\delta(x'-x)\cdot\delta(r_i'-r_i)\delta(r_j'-r_j)\cdot\prod_{n=1}^{d}\delta(v_n'-v_n), \qquad (165) \\ &\quad (x, v_n, r_i, r_j)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ \tau = 0, \end{aligned}$$

Now we consider the following representation formula for p_q with a Fourier transform:

$$p_q(\tau, x, \underline{v}, r, x', \underline{v}', r') = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x'-x)}\, f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k),$$
$$(\underline{v}, \underline{r}) \in (\mathbb{R}^+)^d, (\underline{v}', \underline{r}') \times (\mathbb{R}^+)^2,\ k\in\mathbb{R},\ \tau > 0. \tag{166}$$

This is possible since the coefficients (161) and the initial condition (162) are independent of translation in the log-price variable. Substituting Eq.(166) into (161), we obtain that the function f is the solution of the following problem:

$$
\begin{aligned}
\frac{\partial f}{\partial \tau} &= \frac{-k^2}{2}\left[\sum_{n=1}^{d}(a_n^i-a_n^j)^2 v_n + b_i^2 r_i^{2\alpha} + b_j^2 r_j^{2\alpha}\right] f + \frac{1}{2}\sum_{n=1}^{d}\gamma_n^2 v_n \frac{\partial^2 f}{\partial v_n^2} + \frac{1}{2}\eta_i^2 r_i^{2\alpha}\frac{\partial^2 f}{\partial r_i^2} \\
&+\frac{1}{2}\eta_j^2 r_j^{2\alpha}\frac{\partial^2 f}{\partial r_j^2} + \sum_{n=1}^{d}(-\imath k)\rho_{n,v}\gamma_n(a_n^i-a_n^j)v_n\frac{\partial f}{\partial v_n} + (-\imath k)\rho_{i,r}\eta_i b_i r_i^{2\alpha}\frac{\partial f}{\partial r_i} \\
&-(-\imath k)\rho_{j,r}\eta_j b_j r_j^{2\alpha}\frac{\partial f}{\partial r_j} + \sum_{n=1}^{d}\left[\chi_n(v_n^*-v_n) + q\gamma_n\rho_{n,v}(a_n^i-a_n^j)v_n\right]\frac{\partial f}{\partial v_n} \\
&+\left[\lambda_i(\theta_i-r_i)+q\eta_i\rho_{i,r}b_i r_i^{2\alpha}\right]\frac{\partial f}{\partial r_i} + \left[\lambda_j(\theta_j-r_j)+q\eta_i\rho_{j,r}(-b_j)r_j^{2\alpha}\right]\frac{\partial f}{\partial r_j} \\
&+\left\{\sum_{n=1}^{d}\frac{v_n}{2}(a_n^i-a_n^j)^2\left[(q^2-q)-\imath k\,(2q-1)\right] + \left(r_i(q-\imath k)+\frac{r_i^{2\alpha}}{2}b_i^2\left[(q^2-q)\right.\right.\right. \\
&\left.\left.\left.-\imath k(2q-1)\right]\right) + \left(r_j(-q+\imath k)+\frac{r_j^{2\alpha}}{2}b_j^2\left[(q^2-q)-\imath k(2q-1)\right]\right)\right\} f \qquad (167) \\
&\quad (x,v_n,r_i,r_j)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ 0\le t<t',
\end{aligned}
$$

with the initial condition:

$$
f(0,\underline{v},\underline{r},\underline{v}',\underline{r}',k) = \delta(r_i'-r_i)\delta(r_j'-r_j)\prod_{n=1}^{k}\delta(v_i'-v_i)\,, \qquad (168)
$$
$$
(v_i,r_i,r_j),(v_i',r_i',r_j')\in\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+,\ k\in\mathbb{R}.
$$

Now let us represent f as the inverse Fourier transform of the future variables $(\underline{v}',\underline{r}')$ whose conjugate variables are denoted by $(\underline{l},\underline{\xi})$, that is:

$$
f(\tau,\underline{v},\underline{r},\underline{v}',\underline{r}',k) = \left(\frac{1}{2\pi}\right)^{d+2}\prod_{n=1}^{d}\int_{\mathbb{R}} dl_n\, e^{\imath l_n v_n'}\cdot\prod_{m=i}^{j}\int_{\mathbb{R}} d\xi_m\, e^{\imath\xi_m r_m'}\, g(\tau,\underline{v},\underline{r},k,\underline{l},\underline{\xi}),
$$
$$
(v_n,r_m)\in\mathbb{R}^+\times\mathbb{R}^+,\ (k,l_n,\xi_m)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau>0. \qquad (169)
$$

It is easy to see that the function g satisfies Eq.(167) with the following initial condition:

$$
\begin{aligned}
g(0,\underline{v},\underline{r},k,\underline{l},\underline{\xi}) &= e^{-\imath\xi_i r_i}e^{-\imath\xi_j r_j}\prod_{n=1}^{k}e^{-\imath l_n v_n}, &(170)\\
&= \prod_{m=i}^{j}e^{-\imath\xi_m r_m}\prod_{n=1}^{k}e^{-\imath l_n v_n}\,, &(171)
\end{aligned}
$$
$$
(v_n,r_m)\in\mathbb{R}^+\times\mathbb{R}^+,\ (k,l_n,\xi_m)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau>0.
$$

Please note that g is the Fourier transform with respect to the future variables $(\underline{v}',\underline{r}')$ of the function obtained by extending f, as a function of the variables $(\underline{v},\underline{r})$, with zero when $v_n \notin \mathbb{R}^+$ and/or $r_m \notin \mathbb{R}^+$. The coefficients of the partial differential operator appearing on the right hand side of (161) are first degree polynomials in $\underline{v}$ and $\underline{r}$ so that we seek a solution of problem (167), (168) in the form (see Lipton, 2001):

$$
\begin{aligned}
g(\tau, \underline{v}, \underline{r}, k, \underline{l}, \underline{\xi}) &= e^{A(\tau,k,\underline{l},\underline{\xi})} \prod_{m=i}^{j} e^{-r_m B_{r_m}(\tau,k,\xi_m)} \prod_{n=1}^{d} e^{-v_n B_{v_n}(\tau,k,l_n)}, \\
&= e^{A(\tau,k,\underline{l},\underline{\xi})} e^{-r_i B_{r_i}(\tau,k,\xi_i)} e^{-r_j B_{r_j}(\tau,k,\xi_j)} e^{-\sum_{n=1}^{d} v_n B_{v_n}(\tau,k,l_n)}, \qquad (172)\\
&\quad (v_n, r_m) \in \mathbb{R}^+ \times \mathbb{R}^+,\ (k, \underline{l}, \underline{\xi}) \in \mathbb{R} \times (\mathbb{R})^d \times (\mathbb{R})^2,\ \tau > 0.
\end{aligned}
$$

Substituting Eq.(172) into Eq.(167) and setting $\alpha = \dfrac{1}{2}$, we obtain that the functions $A(\tau, k, \underline{l}, \underline{\xi})$, $B_{v_n}(\tau, k, l_n)$, and $B_{r_i}(\tau, k, \xi_i)$, $B_{r_j}(\tau, k, \xi_j)$ must satisfy the following ordinary differential equations:

$$
\begin{aligned}
\frac{dA}{d\tau}(\tau, k, \underline{l}, \underline{\xi}) &= -\lambda_i\, \theta_i\, B_{r_i}(\tau, k, \xi_i) - \lambda_j\, \theta_j\, B_{r_j}(\tau, k, \xi_j) - \sum_{n=1}^{d} \chi_n v_n^* B_{v_n}(\tau, k, l_n) \\
&\quad (k, l_n, \xi_i, \xi_j) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \tau > 0, \qquad (173)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{dB_{v_n}}{d\tau}(\tau, k, l_n) \\
&= \frac{k^2}{2}(a_n^i - a_n^j)^2 - \frac{(a_n^i - a_n^j)^2}{2}\left[\left(q^2 - q\right) - \imath k\,(2q - 1)\right] \\
&\quad - \left[\chi_n + (\imath k - q)\gamma_n \rho_{n,v}(a_n^i - a_n^j)\right] B_{v_n} - \frac{\gamma_n^2}{2} B_{v_n}^2 \\
&= \varphi_q^{v_n}(k)(a_n^{i,j})^2 - (\chi_n + (\imath k - q)\gamma_n \tilde{\rho}_{n,v}) B_{v_n}(\tau, k, l_n) - \frac{\gamma_n^2}{2} B_{v_n}^2(\tau, k, l_n), \qquad (174)\\
&\qquad (k, l_n) \in \mathbb{R} \times \mathbb{R}, \tau > 0
\end{aligned}
$$

where $\varphi_q^{v_n}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2 - q) - \imath k\,(2q - 1)\right]$, $\tilde{\rho}_{n,v} = \rho_{n,v}(a_n^i - a_n^j)$.

$$
\begin{aligned}
\frac{dB_{r_i}}{d\tau}(\tau, k, \xi_i) &= \frac{k^2}{2} b_i^2 + (\imath k - q) - \frac{b_i^2}{2}\left[(q^2 - q) - \imath k(2q - 1)\right] \\
&\quad - \left[\lambda_i + (\imath k - q)\eta_i \rho_{i,r} b_i\right] B_{r_i}(\tau, k, \xi_i) - \frac{\eta_i^2}{2} B_{r_i}^2(\tau, k, \xi_i) \\
&= \varphi_q^{r_i}(k) b_i^2 + (\imath k - q) - \left[\lambda_i + (\imath k - q)\eta_i \rho_{i,r} b_i\right] B_{r_i}(\tau, k, \xi_i) - \frac{\eta_i^2}{2} B_{r_i}^2(\tau, k, \xi_i) \\
&\quad (k, \xi_i) \in \mathbb{R} \times \mathbb{R}, \tau > 0, \qquad (175)
\end{aligned}
$$

where $\varphi_q^{r_i}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right]$.

$$
\begin{aligned}
\frac{dB_{r_j}}{d\tau}(\tau,k,\xi_i) &= \frac{k^2}{2}b_j^2 + (q-\imath k) - \frac{b_j^2}{2}\left[(q^2-q) - \imath k(2q-1)\right] \\
&\quad - \left[\lambda_j + (\imath k - q)\eta_j\rho_{j,r}(-b_j)\right] B_{r_j} - \frac{\eta_j^2}{2}B_{r_j}^2 \\
= \varphi_q^{r_j}(k)b_j^2 &+ (q-\imath k) - \left[\lambda_j + (q-\imath k)\eta_j\rho_{i,r}b_j\right] B_{r_j}(\tau,k,\xi_j) - \frac{\eta_j^2}{2}B_{r_j}^2(\tau,k,\xi_j) \\
&\qquad (k,\xi_j) \in \mathbb{R}\times\mathbb{R}, \tau > 0,
\end{aligned} \tag{176}
$$

where $\varphi_q^{r_j}(k) = \dfrac{k^2}{2} - \dfrac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right]$. It is worth to highlight that, in comparison with the analytical treatment under physical measure in Section 5.2, the following equality holds. Indeed, this equality simplifies our computation of the FX option pricing formulas as well as correlating SDEs.

$$
\varphi_q(k) := \varphi_q^{v_n}(k) = \varphi_q^{r_i}(k) = \varphi_q^{r_j}(k) = \frac{k^2}{2} - \frac{1}{2}\left[(q^2-q) - \imath k(2q-1)\right] \tag{177}
$$

with initial condition:

$$
\begin{aligned}
&A(0,k,\underline{l},\underline{\xi}) = 0, \quad B_{v_n}(0,k,l_n) = \imath\, l_n, \quad B_{r_i}(0,k,\xi_i) = \imath\,\xi_i, \quad B_{r_j}(0,k,\xi_j) = \imath\,\xi_j, \\
&\qquad\qquad \text{with } (k,\underline{l},\underline{\xi}) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^2.
\end{aligned}
$$

Eqs.(174), (175) and (176) are Riccati equdations that can be solved elementarily by substituting their solutions into Eq.(173) and integrating with respect to τ to obtain $A(\tau,k,\underline{l},\underline{\xi})$. The following steps are exactly the same with Section 2.1. We could get the joint probability density function (pdf) in Eq.(191) and the pdf over the future variance $\underline{v}'$ to find the marginal density for $(x',\underline{r}')$ in Eq.(192).

Let us derive the joint transition probability density function p_f in the case $\alpha = 1/2$, that is when the CIR interest rate model is considered. Considering integration on Eq.(173) for $A(\tau,k,\underline{l},\underline{\xi})$, we obtain:

$$
\begin{aligned}
&A(\tau,k,\underline{l},\underline{\xi}) \\
&= -\sum_{n=1}^{d}\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n) \; - \sum_{m=i}^{j}\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m), \\
&= -\sum_{n=1}^{d}\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n) \; - \frac{2\lambda_i\theta_i}{\eta_i^2}\ln C_{r_i}(\tau,k,\xi_i) - \frac{2\lambda_j\theta_j}{\eta_j^2}\ln C_{r_j}(\tau,k,\xi_j)
\end{aligned} \tag{178}
$$

Hence, the function $g(\tau,\underline{v},\underline{r},k,\underline{l},\underline{\xi})$ in Eq.(170) is given by:

$$
\begin{aligned}
g(\tau,\underline{v},\underline{r},k,\underline{l},\underline{\xi}) = \prod_{n=1}^{d} & \left(e^{-\frac{2\chi_n v_n^*}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n)}\, e^{-\frac{2v_n}{\gamma_n^2}\frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \right) \prod_{m\in\{i,j\}} \Bigg(\\
& e^{-\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2}\frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \Bigg), \qquad (179) \\
& (v_n,r_m)\in\mathbb{R}^+\times\mathbb{R}^+,\ (k,l_n,\xi_m)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R},\ \tau>0.
\end{aligned}
$$

In order to obtain an explicit expression for $f(\tau,\underline{v},\underline{r},\underline{v}',\underline{r}',k)$ in Eq.(169), that is the inverse Fourier transform of $g(\tau,\underline{v},\underline{r},k,\underline{l},\underline{\xi})$ with respect to the variable $\underline{v}'$ and $\underline{r}'$, we have to compute the following integrals:

$$
\begin{aligned}
& L_{v_n}(\tau,v_n,v_n',k\mid\underline{\Theta}_{v_n}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dl_n e^{\imath l_n v_n'}\, e^{-\frac{2\chi_n\overline{v}_n}{\gamma_n^2}\ln C_{v_n}(\tau,k,l_n)}\, e^{-\frac{2v_n}{\gamma_n^2}\frac{dC_{v_n}}{d\tau}(\tau,k,l_n)/C_{v_n}} \qquad (180) \\
& L_{r_m}(\tau,r_m,r_m',k\mid\underline{\Theta}_{r_m}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi_m\, e^{\imath\,\xi_m\,r_m'} e^{-\frac{2\lambda_m\theta_m}{\eta_m^2}\ln C_{r_m}(\tau,k,\xi_m)} e^{-\frac{2r_m}{\eta_m^2}\frac{dC_{r_m}}{d\tau}(\tau,k,\xi_m)/C_{r_m}} \qquad (181)
\end{aligned}
$$

Let us show how to compute the integral appearing in (180) and (181) analogously by using Eq.(59) and (63) with the change of variable $l_n' = -l_n\frac{(\gamma_n)^2}{2}\frac{s_{q,v_n,g}}{s_{q,v_n,b}}$ and the following equality:

$$
s_{q,v_n,d}s_{q,v_n,b} = \left(\zeta_{q,v_n}^2-\mu_{v_n}^2\right)s_{q,v_n,g}^2 - \imath\, l_n' \left(\frac{8\zeta_{v_n}^2 e^{-2\zeta_{v_n}\tau}}{1-2\imath\, l_n'}\right), \qquad (182)
$$

Thus, $L_{v_n}(\tau,v_n,v_n',k\mid\underline{\Theta}_{v_n})$ can be written as follows:

$$
\begin{aligned}
& L_{v_n}(\tau,v_n,v_n',k\mid\underline{\Theta}_{v_n}) = \frac{1}{2\pi}M_{q,v_n}e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n})+(\mu_{q,v_n}+\zeta_{q,v_n})\tau]}\,\cdot \\
& e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}} \int_{-\infty}^{+\infty} dl_n'\, e^{-(\imath\,l_n')M_{q,v_n}} e^{-(2\chi_n v_i^*/\gamma_n^2)\ln(1-\imath\,l_n')} e^{\frac{(M_{q,v_n}\tilde{v}_{q,n})\imath\,l_n'}{1-\imath\,l_n'}}, \qquad (183)
\end{aligned}
$$

where

$$
M_{q,v_n} = \frac{2}{\gamma_n^2}\frac{s_{q,v_n,b}}{s_{q,v_n,g}},\quad \tilde{v}_{q,n} = \frac{4(\zeta_{q,v_n})^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,b}^2},\ M_{q,v_n}\tilde{v}_{q,n} = \frac{8}{\gamma_n^2}\frac{\zeta_{q,v_n}^2 v_n e^{-2\zeta_{q,v_n}\tau}}{s_{q,v_n,g}s_{q,v_n,b}}. \qquad (184)
$$

Now using formula n.34 on p.156 in Oberhettinger 1973, we obtain:

$$
\begin{aligned}
& L_{v_n}(\tau,v_n,v_n',k\mid\underline{\Theta}_{v_n}) \\
&= e^{-(2\chi_n v_n^*/\gamma_n^2)[\ln(s_{q,v_n,b}/2\zeta_{q,v_n})+(\mu_{q,v_n}+\zeta_{q,v_n})\tau]}\, e^{-(2v_n/\gamma_n^2)(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/s_{q,v_n,b}} M_{q,v_n} \\
& (M_{q,v_n}\tilde{v}_{q,n})^{-\nu_{q,v_n}/2}(M_{q,v_n}v_n')^{\nu_{q,v_n}/2} e^{-M_{q,v_n}\tilde{v}_{q,n}} e^{-M_{q,v_n}v_n'} I_{\nu_{q,v_n}}(2\tilde{M}_{q,v_n}(\tilde{v}_{q,n}v_n')^{1/2}), \\
& \quad ,\text{with},\ v_n,v_n'>0,\ k\in\mathbb{R}. \qquad (185)
\end{aligned}
$$

where $\nu_{q,v_n} = 2\chi_n v_n^*/\gamma_n^2 - 1$ and $I_{\nu_{q,v_n}}$ is the modified Bessel function of order ν_{q,v_n} (see, for example, Abramowitz and Stegun, 1970). Similarly, we obtain:

$$
\begin{aligned}
& L_{r_m}(\tau, r_m, r'_m, k \mid \underline{\Theta}_{r_m}) \\
&= e^{-(2\lambda_m\theta_m/\eta_m^2)[\ln(s_{q,r_m,b}/2\zeta_{q,r_m})+(\mu_{q,r_m}+\zeta_{q,r_m})\tau]}\, e^{-(2r_m/\eta_m^2)(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/s_{q,r_m,b}} M_{q,r_m} \\
& \quad (M_{q,r_m}\tilde{r}_{q,n})^{-\nu_{q,r_m}/2} (M_{q,r_m} r'_m)^{\nu_{q,r_m}/2} e^{-M_{q,r_m}\tilde{r}} e^{-M_{q,r_m} r'_m} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,m} r'_m)^{1/2}), \\
& \quad \text{with}\,,\ r_m, r'_m > 0,\ k \in \mathbb{R},
\end{aligned} \tag{186}
$$

where

$$
\begin{aligned}
& M_{q,r_m} = \frac{2}{\eta_m^2}\frac{s_{q,r_m,b}}{s_{q,r_m,g}}, \quad \tilde{r}_{q,m} = \frac{4(\zeta_{q,r_m})^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,b}^2}\,, M_{q,r_m}\tilde{r}_{q,m} = \frac{8}{\eta_m^2}\frac{\zeta_{q,r_m}^2 r_m e^{-2\zeta_{q,r_m}\tau}}{s_{q,r_m,g}s_{q,r_n,b}}\,, \\
& \text{and } \nu_{q,r_m} = 2\lambda_m\theta_m/\eta_m^2 - 1\,, \quad m = i, j
\end{aligned} \tag{187}
$$

The following results are remarkable (see Abramowitz and Stegun 1970 pp. 375 and 486):

$$
\begin{aligned}
& P_{p,v_n}(\tau, v_n, k) \\
&= \int_0^{+\infty} dv'_n (v'_n)^{\nu_{q,v_n}/2} I_{\nu_{q,v_n}}(2M_{q,v_n}(\tilde{v}_{q,v_n} v'_n)^{1/2}) e^{-M_{q,v_n}\, v'_n} = \frac{(\tilde{v}_{q,n})^{\nu_{q,v_n}/2}}{M_{q,v_n}} e^{M_{q,v_n}\tilde{v}_{q,n}}, \\
& \qquad v_n > 0,\ k \in \mathbb{R},
\end{aligned} \tag{188}
$$

$$
\begin{aligned}
& P_{p,r_m}(\tau, r_m, k) \\
&= \int_0^{+\infty} dr'_m (r'_m)^{\nu_{q,r_m}/2} I_{\nu_{q,r_m}}(2M_{q,r_m}(\tilde{r}_{q,r_m} r'_m)^{1/2}) e^{-M_{q,r_m}\, r'_m} = \frac{(\tilde{r}_{q,m})^{\nu_{q,r_m}/2}}{M_{q,r_m}} e^{M_{q,r_m}\tilde{r}_{q,m}}\,, \\
& \qquad r_m > 0,\ k \in \mathbb{R},
\end{aligned} \tag{189}
$$

Substituting Eqs.(185) and (186) into Eq.(168), we obtain:

$$
\begin{aligned}
& f(\tau, \underline{v}, \underline{r}, \underline{v}', \underline{r}', k) = \prod_{n=2}^{d} L_{v_n}(\tau, v_n, v'_n, k) \cdot L_{r_i}(\tau, r_i, r'_i, k) L_{r_j}(\tau, r_j, r'_j, k) \\
& e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2}\, e^{-2\sum\limits_{n=1}^{d}\chi_n v_n^*(\mu_{q,v_n}+\zeta_{q,v_n})\tau/\gamma_n^2}\, e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2} \\
& \cdot e^{-2\sum\limits_{m=i}^{j}\lambda_m\theta_m(\mu_{q,r_m}+\zeta_{q,r_m})\tau/\eta_m^2} \cdot \Bigg\{ e^{-2\sum\limits_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})} \cdot \\
& e^{-\sum\limits_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v'_n)} \prod_{n=1}^{d}\left[M_{q,v_n}\left(\frac{v'_n}{\tilde{v}_{q,n}}\right)^{\nu_{q,v_n}/2} \cdot I_{\nu_{q,v_n}}\left(2M_{q,v_n}(\tilde{v}_{q,n}\, v'_n)^{1/2}\right)\right]\Bigg\} \\
& \cdot\Bigg\{ e^{-2\sum\limits_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})}\, e^{-\sum\limits_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)} \cdot \\
& \prod_{m=i}^{j}\left[M_{q,r_m}\left(\frac{r'_m}{\tilde{r}_{q,m}}\right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}}\left(2M_{q,r_m}(\tilde{r}_{q,m}\, r'_m)^{1/2}\right)\right]\Bigg\}, \\
& \quad (x, v_n, r_m),\ (x', v'_n, r'_m) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+,\ t, t' \geq 0,\ t'-t > 0.
\end{aligned} \tag{190}
$$

Substituting Eq.(190) into Eq.(166), we obtain the probability density function $p_f(x, \underline{v}, \underline{r}, t, x', \underline{v'}, \underline{r'}, t')$ as follows:

$$
\begin{aligned}
& p_f(\underline{x}, \underline{v}, \underline{r}, t, \underline{x'}, \underline{v'}, \underline{r'}, t') = e^{q(x-x')} \frac{1}{2\pi} \int_{\mathbb{R}} dk \, e^{\imath k(x'-x)} \cdot \Bigg\{ \\
& \cdot e^{-2\sum_{n=1}^{d} \chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2} e^{-2\sum_{n=1}^{d} \chi_n v_n^* (\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2} \\
& \cdot e^{-2\sum_{m=i}^{j} \lambda_m \theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2} e^{-2\sum_{m=i}^{j} \lambda_m \theta_m (\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2} \\
& \cdot \Bigg\{ e^{-2\sum_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})} e^{-\sum_{n=1}^{d} M_{q,v_n}(\tilde{v}_{q,n}+v'_n)} \cdot \\
& \quad \prod_{n=1}^{d} \left[M_{q,v_n} \left(\frac{v'_n}{\tilde{v}_{q,n}} \right)^{\nu_{q,v_n}/2} \cdot I_{\nu_{q,v_n}} \left(2M_{q,v_n} (\tilde{v}_{q,n} \, v'_n)^{1/2} \right) \right] \Bigg\} \\
& \cdot \Bigg\{ e^{-2\sum_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})} e^{-\sum_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)} \cdot \\
& \quad \prod_{m=i}^{j} \left[M_{q,r_m} \left(\frac{r'_m}{\tilde{r}_{q,m}} \right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}} \left(2M_{q,r_m} (\tilde{r}_{q,m} \, r'_m)^{1/2} \right) \right] \Bigg\} \Bigg\}, \qquad (191) \\
& (x, v_n, r_m), \, (x', v'_n, r'_m) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \, t, t' \geq 0, \, t'-t > 0.
\end{aligned}
$$

Thanks to Eq.(188), we integrate the joint probability density function in Eq.(191) over the future variance $\underline{v}'$ to find the marginal density for $(x', \underline{r}')$ as follows:

$$
\begin{aligned}
& D_v(x, \underline{v}, \underline{r}, t, \underline{x}', \underline{r}', t') = \prod_{n=1}^{d} \left(\int_0^{+\infty} dv'_n \, p_f(x, \underline{v}, \underline{r}, t, x', \underline{v'}, \underline{r}', t') \right) \\
& = e^{q(x-x')} \frac{1}{2\pi} \int_{\mathbb{R}} dk \, e^{\imath k(x'-x)} \cdot \Bigg\{ \\
& \cdot e^{-2\sum_{n=1}^{d} \chi_n v_n^* \ln(s_{q,v_n,b}/2\zeta_{q,v_n})/\gamma_n^2} e^{-2\sum_{n=1}^{d} \chi_n v_n^* (\mu_{q,v_n}+\zeta_{q,v_n})(t'-t)/\gamma_n^2} \\
& \cdot e^{-2\sum_{m=i}^{j} \lambda_m \theta_m \ln(s_{q,r_m,b}/2\zeta_{q,r_m})/\eta_m^2} e^{-2\sum_{m=i}^{j} \lambda_m \theta_m (\mu_{q,r_m}+\zeta_{q,r_m})(t'-t)/\eta_m^2} \\
& \cdot \Bigg\{ e^{-2\sum_{n=1}^{d} v_n(\zeta_{q,v_n}^2-\mu_{q,v_n}^2)s_{q,v_n,g}/(\gamma_n^2 s_{q,v_n,b})} \Bigg\} \\
& \cdot \Bigg\{ e^{-2\sum_{m=i}^{j} r_m(\zeta_{q,r_m}^2-\mu_{q,r_m}^2)s_{q,r_m,g}/(\eta_m^2 s_{q,r_m,b})} e^{-\sum_{m=i}^{j} M_{q,r_m}(\tilde{r}_{q,m}+r'_m)} \cdot \\
& \quad \prod_{m=i}^{j} \left[M_{q,r_m} \left(\frac{r'_m}{\tilde{r}_{q,m}} \right)^{\nu_{q,r_m}/2} \cdot I_{\nu_{q,r_m}} \left(2M_{q,r_m} (\tilde{r}_{q,m} \, r'_m)^{1/2} \right) \right] \Bigg\} \Bigg\}, \qquad (192) \\
& (x, v_n, r_m), \, (x', v'_n, r'_m) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \, t, t' \geq 0, \, t'-t > 0.
\end{aligned}
$$

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