SIAM J. CONTROL
Vol. 6, No. 4, 1968
Printed in U.S.A.

ON A MATRIX RICCATI EQUATION OF STOCHASTIC CONTROL*

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1. Introduction. The object of this paper is to discuss a generalized version of the matrix Riccati and matrix quadratic equations, which arise in problems of stochastic control and filtering. The properties obtained include existence, uniqueness and asymptotic behavior, and contain as special cases some (but not all) of the results reported in [1], [2]. We refer in particular to [2] for a detailed review and bibliography of the "standard" equation for the linear regulator problem.

The present generalization consists in the addition of a linear positive operator to the linear terms of the standard Riccati equation and, in certain instances, a weakening of the usual hypothesis of complete observability to observability of unstable modes (detectability).

The proofs given here are simple applications of Bellman's principle of quasi-linearization ("approximation in policy space") [5] and of a known monotone convergence property of symmetric matrices. In this way the discussion becomes unified and straightforward. Applications to control and filtering are indicated in §6.

2. Notation and summary. In the following, all vectors and matrices have real elements except where otherwise stated. A, B, C, K are matrices of dimension respectively $n \times n$, $n \times m$, $p \times n$, and $m \times n$; N, P, Q denote symmetric matrices of dimension respectively $m \times m$, $n \times n$, and $n \times n$; it will always be assumed that N is positive definite. A' denotes the transpose of A, and I is the identity matrix. Matrix functions of time t which are assumed as data are Lebesgue measurable and bounded in norm on every finite subinterval of their domain of definition. In particular $N(t)^{-1}$ is so bounded.

If P is positive (semi-)definite, we write $P > 0$ $(P \geqq 0)$; $P > Q$ means $P - Q > 0$, etc. If P is symmetric, the Euclidean norm $|P|$ is the absolute value of the numerically largest eigenvalue of P; thus $-|P|I \leqq P \leqq |P|I$.

Π will denote a (possibly t-dependent) positive linear map of the class of symmetric $n \times n$ matrices into itself: that is, $\Pi = \Pi(t, P)$ is measurable in (t, P), linear in P, and $P \geqq 0$ implies $\Pi(t, P) \geqq 0$. In the case where A, B, K

* Received by the editors November 3, 1967, and in revised form March 8, 1968.
† National Aeronautics and Space Administration, Electronics Research Center, Cambridge, Massachusetts 02139. This research was supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant AF-AFOSR-693-67, and in part by the National Science Foundation, Engineering, under Grant GK-967.

and Π are independent of t, the condition

$$\inf_K \left| \int_0^\infty e^{t(A-BK)'}\Pi(I)e^{t(A-BK)}\, dt \right| < 1 \tag{2.1}$$

will be important later. It expresses the fact that Π is not too large.

Let A, B be constant. The *controllability matrix* of (A, B) is the $n \times mn$ matrix

$$\Gamma(A, B) = [B, AB, \cdots, A^{n-1}B].$$

The pair (A, B) is *controllable* if the rank of Γ is n. If (A, B) is controllable, so is $(A - BK, B)$ for every matrix K.

The pair of constant matrices (A, B) is *stabilizable* if there exists a contant matrix K such that $A - BK$ is stable (i.e., all its eigenvalues have negative real parts). Let the minimum polynomial $\phi(\lambda)$ of A be factored as $\phi(\lambda) = \phi^+(\lambda)\phi^-(\lambda)$, where all zeros of $\phi^+(\lambda)$ lie in the closed right-half complex plane and all zeros of $\phi^-(\lambda)$ lie in the open left-half plane. It is well known that n-space E can be written as a direct sum $E = E_A^+ \oplus E_A^-$, where

$$E_A^+ = \{x : \phi^+(A)x = 0\}, \qquad E_A^- = \{x : \phi^-(A)x = 0\}.$$

E_A^+ thus represents the "unstable modes" of A. It is known [3] that (A, B) is stabilizable if and only if the range of $\Gamma(A, B)$ contains E_A^+.

Dual to the concept of controllability is that of observability: the pair of constant matrices (C, A) is *observable* if (A', C') is controllable. A weaker but useful property is that (at least) the unstable modes of A be observable. Precisely, (C, A) is *detectable* if (A', C') is stabilizable.

Controllability and observability are well-known concepts (cf. [7]); stabilizability is discussed in [3]; detectability in its present meaning originates here.

Of primary interest will be the Riccati equation

$$\begin{aligned} &\frac{dP(t)}{dt} + A(t)'P(t) + P(t)A(t) + \Pi\,[t, P(t)] \\ &\qquad - P(t)B(t)N(t)^{-1}B(t)'P(t) + C(t)'C(t) = 0, \end{aligned} \tag{2.2a}$$

$$t_0 \leqq t \leqq T,$$

subject to the terminal condition

$$P(T) = P_T \geqq 0. \tag{2.2b}$$

In the constant parameter case we also consider the quadratic equation

$$A'P + PA + \Pi(P) - PBN^{-1}B'P + C'C = 0. \tag{2.3}$$

The main result is the following theorem.

THEOREM 2.1. *There exists a matrix $P(t)$ with the following properties*:

(i) *P is defined and absolutely continuous on $[t_0, T]$ and satisfies* (2.2a) (*almost everywhere*) *and* (2.2b).

(ii) *$P(t) \geqq 0$, $t_0 \leqq t \leqq T$, and $P(t)$ is the unique solution of* (2.2a, b).

(iii) (Minimum property). *Let $\tilde{K}(t)$ be an arbitrary* (*bounded measurable*) *$m \times n$ matrix defined on $[t_0, T]$ and let $\tilde{P}(t)$ be the solution of the linear equation*

$$\text{(2.4a)} \qquad \frac{d\tilde{P}(t)}{dt} + [A(t) - B(t)\tilde{K}(t)]'\tilde{P}(t) + \tilde{P}(t)[A(t) - B(t)\tilde{K}(t)] + \Pi[t, \tilde{P}(t)] + C(t)'C(t) + \tilde{K}(t)'N(t)\tilde{K}(t) = 0,$$

$$\text{(2.4b)} \qquad \tilde{P}(T) = P_T.$$

If $P(t)$ is the solution of (2.2a, b), *then $P(t) \leqq \tilde{P}(t)$, $t_0 \leqq t \leqq T$.*

(iv) *Let A, B, C, N and Π be independent of t and consider* (2.2) *with $t_0 = -\infty$, $T = 0$. If (A, B) is stabilizable and Π satisfies* (2.1), *then $P(t)$ is bounded on $(-\infty, 0]$. If in addition (C, A) is observable, then*

$$P_\infty = \lim P(t), \qquad t \to -\infty,$$

exists and is positive definite. In that case P_∞ is the unique positive semidefinite solution of (2.3), *and the matrix*

$$A - BN^{-1}B'P_\infty$$

is stable.

A proof is given in §3–§5. Results for the quadratic equation (2.3) are summarized in Theorem 4.1.

3. Existence, uniqueness and minimality.

LEMMA 3.1 (Monotone convergence). *Let $\{P_\nu;\nu = 1, 2, \cdots\}$ be a sequence of $n \times n$ symmetric matrices such that $P_1 \leqq P_2 \leqq \cdots$ and $P_\nu \leqq P$, $\nu = 1, 2, \cdots$, for some P. Then $P_\infty = \lim P_\nu$, $\nu \to \infty$, exists and $P_\infty \leqq P$.*

The lemma is a special case of a result for positive operators in Hilbert space [4, p. 189]; the result holds also for a monotone decreasing sequence which is bounded below.

We turn to a proof of assertions (i) and (ii) of Theorem 2.1. Let

$$\psi(P, K) = (A - BK)'P + P(A - BK) + \Pi(t, P)$$

and

$$\text{(3.1a)} \qquad K(t) = N(t)^{-1}B(t)'P(t).$$

Then (2.2a, b) become

$$\frac{dP(t)}{dt} + \psi[P(t), K(t)] + C(t)'C(t) + K(t)'N(t)K(t) = 0, \qquad t_0 \leqq t \leqq T, \tag{3.1b}$$

$$P(T) = P_T. \tag{3.1c}$$

The key to solving (3.1a, b, c) is the observation that (3.1b) is linear in P and that the expression (3.1a) minimizes the left side of (3.1b), regarded as a function of K (cf. [5]). The latter statement results from the identity

$$\begin{aligned}(A - BK_0)'P + P(A - BK_0) + K_0'NK_0 \\ \equiv (A - BK)'P + P(A - BK) \\ + K'NK - (K - K_0)'N(K - K_0),\end{aligned} \tag{3.2}$$

where $K_0 = N^{-1}B'P$.

Now let $K(t)$ be arbitrary and let $\Phi(t, s)$ be the fundamental matrix associated with the matrix $A(t) - B(t)K(t)$, that is, Φ is determined by the equations

$$\begin{aligned}\frac{\partial \Phi(t, s)}{\partial t} &= [A(t) - B(t)K(t)]\Phi(t, s), \qquad t_0 \leqq s, t \leqq T, \\ \Phi(t, t) &= I.\end{aligned} \tag{3.3}$$

Recall that $\Phi(s, t) = \Phi(t, s)^{-1}$, whence

$$\frac{\partial \Phi(s, t)}{\partial t} = -\Phi(s, t)[A(t) - B(t)K(t)]. \tag{3.4}$$

It is then easily checked that (3.1b) and (3.1c) are equivalent to the equation

$$\begin{aligned}P(t) = \Phi(T, t)'P_T\Phi(T, t) + \int_t^T \Phi(s, t)'\{\Pi[s, P(s)] \\ + C(s)'C(s) + K(s)'N(s)K(s)\}\Phi(s, t)\, ds, \qquad t_0 \leqq t \leqq T.\end{aligned} \tag{3.5}$$

The Volterra equation (3.5) has a unique integrable solution $P(t)$ which can be found by successive approximation. Consider the approximation sequence $\{P_\nu : \nu = 1, 2, \cdots\}$ with $P_1(t) \equiv 0$. Recalling the positivity of Π we see that $P_\nu(t) \geqq P_1(t)$ for all t; hence $P(t) = \lim P_\nu(t) \geqq 0$.

We solve the simultaneous equations (3.1a) and (3.5) as follows: Denote the right side of (3.5) by $\mathfrak{J}(K, P, t)$. Choosing K_1 arbitrarily, define P_1 to be the unique solution of

$$P_1(t) = \mathfrak{J}(K_1, P_1, t), \qquad t_0 \leqq t \leqq T.$$

Having defined $K_1, \cdots, K_\nu$, let P_ν be the solution of

$$P_\nu(t) = \mathfrak{I}(K_\nu, P_\nu, t), \qquad t_0 \leqq t \leqq T, \tag{3.6}$$

and define

$$K_{\nu+1}(t) = N(t)^{-1}B(t)'P_\nu(t). \tag{3.7}$$

From what was said previously, the matrices K_ν and P_ν are well-defined, measurable and bounded on $[t_0, T]$. Next we exploit the minimum property (3.2). For brevity let us write (3.1b) as

$$\frac{dP(t)}{dt} + \Psi\{P(t), K(t)\} = 0,$$

where

$$\Psi(P, K) = \psi(P, K) + C'C + K'NK.$$

Then (3.2), (3.6) and (3.7) yield

$$\begin{aligned} \frac{dP_\nu(t)}{dt} + \Psi\{P_\nu(t), K_{\nu+1}(t)\} &\leqq \frac{dP_\nu(t)}{dt} + \Psi\{P_\nu(t), K_\nu(t)\} \\ &= 0 \\ &= \frac{dP_{\nu+1}}{dt}(t) + \Psi\{P_{\nu+1}(t), K_{\nu+1}(t)\}, \end{aligned}$$

$$t_0 \leqq t \leqq T.$$

Setting $Q(t) = P_\nu(t) - P_{\nu+1}(t)$, we have

$$\frac{dQ(t)}{dt} + \psi[Q(t), K_{\nu+1}(t)] + R(t) = 0$$

for a suitable matrix $R(t) \geqq 0$; and from this we obtain, as before, $Q(t) \geqq 0$. It follows that for each $t \in [t_0, T]$ the sequence of nonnegative matrices $\{P_\nu(t)\}$ is monotone nonincreasing and hence, by Lemma 3.1,

$$P(t) = \lim_{\nu \to \infty} P_\nu(t)$$

exists. Since

$$| P_\nu(t) | \leqq \sup \{| P_1(s) | : t_0 \leqq s \leqq T\}, \qquad \nu = 1, 2, \cdots,$$

it follows by (3.7) that the sequence $\{| K_\nu(t) |\}$ is uniformly bounded. Let $\Phi_A(t, s)$ be the fundamental matrix (cf. (3.3)) determined by A.

Then (3.6) is equivalent to

$$
\begin{aligned}
P_\nu(t) = \Phi_A(T, t)'P_T\,\Phi_A(T, t) + \int_t^T \Phi_A(s, t)'\{\Pi[s, P_\nu(s)]& \\
- K_\nu(s)'B(s)'P_\nu(s) - P_\nu(s)B(s)K_\nu(s) + C(s)'C(s)& \\
+ K_\nu(s)'N(s)K_\nu(s)\}\Phi_A(s, t)\, ds.&
\end{aligned}
\tag{3.8}
$$

Applying the dominated convergence theorem to the integral in (3.8), we conclude that (3.8) holds with P_ν, K_ν replaced by P, K, where

$$
\begin{aligned}
K(t) &= \lim_{\nu\to\infty} K_\nu(t) \\
&= N(t)^{-1}B(t)'P(t).
\end{aligned}
\tag{3.9}
$$

Equations (3.8) (with $P_\nu = P$, $K_\nu = K$) and (3.9) are equivalent to (3.1 a, b, c); hence existence of an absolutely continuous solution of (2.2 a, b) is established.

Uniqueness of the solution results from the fact that the function

$$\Psi(P, K) = \Psi(P, N^{-1}B'P)$$

satisfies a uniform Lipschitz condition in P in every domain $t_0 \leqq t \leqq T$, $|P| <$ const.

Assertions (i) and (ii) of Theorem 2.1 have now been proved.

The minimum property (iii) is proved by using (3.2) in the same manner as before. Thus from the inequality

$$\Psi[P(t), K(t)] \leqq \Psi[P(t), \tilde{K}(t)]$$

together with (2.4 a, b) and (3.2), there follows

$$0 = \frac{d\tilde{P}(t)}{dt} + \Psi[\tilde{P}(t), \tilde{K}(t)] \leqq \frac{dP(t)}{dt} + \Psi[P(t), \tilde{K}(t)].$$

If $Q(t) = \tilde{P}(t) - P(t)$, then

$$\frac{dQ(t)}{dt} + \Psi[\tilde{P}(t), \tilde{K}(t)] - \Psi[P(t), \tilde{K}(t)] \leqq 0; \tag{3.10}$$

hence for a suitable matrix $R(t) \geqq 0$,

$$\frac{dQ(t)}{dt} + \psi[Q(t), \tilde{K}(t)] + R(t) = 0, \tag{3.11a}$$

$$Q(T) = 0. \tag{3.11b}$$

If (3.11 a, b) are written as an integral equation (cf. (3.5)) and solved, as before, by successive approximation, we obtain a sequence $Q_\nu(t)$ such that $Q_\nu(t) \to Q(t)$, $\nu \to \infty$. Since we can choose $Q_0(t) \equiv 0$ it is easily seen that $Q_\nu(t) \geqq 0$ for all t, ν. This completes the proof of Theorem 2.1 (iii).

4. Solution of the quadratic equation (2.3). In this section we assume that the parameter matrices A, B, C, N and the operator Π are independent of t, and consider exclusively the quadratic equation (2.3).

THEOREM 4.1. *If (A, B) is stabilizable and (C, A) is detectable, and if Π satisfies condition* (2.1), *then* (2.3) *has at least one solution $\bar{P}$ in the class of positive semidefinite matrices. The matrix $A - BN^{-1}B'\bar{P}$ is stable. If in addition (C, A) is observable, then $\bar{P}$ is unique and $\bar{P} > 0$.*

For the proof we need three auxiliary results.

LEMMA 4.1. *Let*

$$C'C + D'D = F'F$$

and let G be an arbitrary matrix of suitable dimension.

(i) *If (C, A) is observable, then $(F, A + GD)$ is observable.*

(ii) *If (C, A) is detectable, then $(F, A + GD)$ is detectable.*

Proof. Let $\{\cdot\}$ denote the range of a matrix and $\mathfrak{N}(\cdot)$ the null space. It is easily seen that $\{\Gamma(A' + \hat{F}', F')\} = \{\Gamma(A', F')\}$ whenever $\{\hat{F}'\} \subset \{F'\}$. Also, if $x'F'Fx = 0$, then $x'C'Cx = x'D'Dx' = 0$, so that $\mathfrak{N}(F) \subset \mathfrak{N}(C) \cap \mathfrak{N}(D)$; taking orthogonal complements, we have

$$\{C'\} + \{D'\} \subset \{F'\}.$$

Thus $\{D'G'\} \subset \{F'\}$, so that

$$\{\Gamma(A' + D'G', F')\} = \{\Gamma(A', F')\} \supset \{\Gamma(A', C')\},$$

proving (i). For (ii), write $\hat{D}' = D'G'$ and let $A' + C'R'$ be stable. Since $\{C'R' - \hat{D}'\} \subset \{F'\}$, a matrix S' can be chosen such that

$$A' + \hat{D}' + F'S' = A' + C'R'.$$

The proof is complete.

The following remark will be useful: if (C, A) is detectable, then either A is stable or the matrix

$$W_t(A, C) = \int_0^t e^{s}{}_{A}{}'C'Ce^{sA}\,ds$$

is unbounded on $0 \leqq t < \infty$. For if A is not stable, let λ be an eigenvalue of A with $\operatorname{Re}\lambda \geqq 0$ and eigenvector ξ. If ξ^* is the conjugate transpose of ξ,

$$\xi^* W_t(A, C)\xi = \int_0^t e^{2s\operatorname{Re}\lambda} \mid C\xi \mid^2 ds.$$

Suppose the integral is bounded. Then $C\xi = 0$, i.e., $CA^{\nu-1}\xi = \lambda^{\nu-1}C\xi = 0$,

$\nu = 1, \cdots, n$, so that

$$\operatorname{Re}\xi, \operatorname{Im}\xi \in \mathfrak{N}(\Gamma(A', C')').$$

If (C, A) is detectable, $\{\Gamma(A', C')\} \supset E_{A'}^{+}$, and taking orthogonal complements, we have

$$\mathfrak{N}\{\Gamma(A', C')'\} \subset (E_{A'}^{+})^{\perp} = E_A^{-}.$$

Thus, $\operatorname{Re}\xi, \operatorname{Im}\xi \in E_A^{+} \cap E_A^{-}$, namely $\xi = 0$, a contradiction.

LEMMA 4.2. *Let (C, A) be detectable and suppose the equation*

$$A'P + PA + \Pi(P) + C'C = 0 \tag{4.1}$$

has a solution $P \geqq 0$. Then A is stable. Let $\mathfrak{J}(R)$ be defined by

$$\mathfrak{J}(R) = \int_0^\infty e^{tA'}\Pi(R)e^{tA}\,dt,$$

and put $\mathfrak{J}^\nu(R) = \mathfrak{J}(\mathfrak{J}^{\nu-1}(R))$, $\mathfrak{J}^0(R) = R$, $\nu = 1, 2, \cdots$. If (C, A) is observable, then the series

$$\sum_{\nu=0}^{\infty} \mathfrak{J}^\nu(R) = (\mathfrak{I} - \mathfrak{J})^{-1}(R) \tag{4.2}$$

converges for every symmetric $n \times n$ matrix R. In that case, the solution P of (4.1) is unique and is given by

$$P = (\mathfrak{I} - \mathfrak{J})^{-1}\left(\int_0^\infty e^{tA'}C'Ce^{tA}\,dt\right).$$

Here $\mathfrak{I}$ denotes the identity operator.

Proof. From (4.1) there results the identity

$$P = e^{tA'}Pe^{tA} + \int_0^t e^{sA'}[\Pi(P) + C'C]e^{sA}\,ds, \qquad t \geqq 0. \tag{4.3}$$

Since (C, A) is detectable, the integral

$$Q(t) = \int_0^t e^{sA'}C'Ce^{sA}\,ds, \qquad t \geqq 0,$$

is bounded only if A is stable. Since $P - Q(t) \geqq 0$, the first assertion is proved. To prove the second, let $t \to \infty$ in (4.3) and write $Q = Q(\infty)$ to obtain $P = \mathfrak{J}(P) + Q$. Hence

$$P = \mathfrak{J}^{\nu+1}(P) + \sum_{k=0}^{\nu} \mathfrak{J}^k(Q), \qquad \nu = 1, 2, \cdots. \tag{4.4}$$

Since $Q \geqq 0$ there follows $\mathfrak{J}^k(Q) \geqq 0$; thus the last written sum is dominated by P, and the series of nonnegative matrices converges. Now suppose

(C, A) is observable. Then $Q > 0$, and convergence of the series with arbitrary $R \geqq 0$ in place of Q follows by linearity of $\mathfrak{J}$ and the fact that $R \leqq \rho Q$ for some $\rho < \infty$. Finally, every symmetric matrix R can be written in the form $R = R^+ - R^-$, where $R^+ \geqq 0$, $R^- \geqq 0$.[1] Since the operators $\mathfrak{J}^\nu$ are linear, convergence of (4.2) for arbitrary symmetric R is established. Uniqueness of P follows by invertibility of $\mathfrak{g} - \mathfrak{J}$.

LEMMA 4.3 (Minimum property). *Let $P \geqq 0$ satisfy (2.3). Let $Q \geqq 0$ and suppose that for some matrix J,*

$$(4.5) \quad (A - BJ)'Q + Q(A - BJ) + \Pi(Q) + C'C + J'NJ = 0.$$

If (C, A) is detectable, then $A - BJ$ is stable, and if (C, A) is observable, then $P \leqq Q$.

Proof. Let $C'C + J'NJ = F'F$. By Lemma 4.1 (with $D = N^{1/2}J$ and $G = -BN^{-1/2}$), the pair $(F, A - BJ)$ is detectable. Applying Lemma 4.2, we conclude that $A - BJ$ is stable. Setting $Q - P = V$ and using (3.2) we obtain

$$(4.6) \qquad (A - BJ)'V + V(A - BJ) + \Pi(V) + S = 0$$

for some $S \geqq 0$. Then

$$(4.7) \qquad V = \mathfrak{J}(V) + R,$$

where $\mathfrak{J}$ is defined as in Lemma 4.2 (with $A - BJ$ in place of A, and $R = \int_0^\infty e^{\alpha(A-BJ)'}Se^{\alpha(A-BJ)}\,d\alpha$). Again, by Lemma 4.1, observability of (C, A) implies observability of $(F, A - BJ)$, and then Lemma 4.2 applied to (4.5) shows that

$$\sum_0^\infty \mathfrak{J}^\nu(I)$$

converges. Since

$$-|V|\mathfrak{J}^\nu(I) \leqq \mathfrak{J}^\nu(V) \leqq |V|\mathfrak{J}^\nu(I),$$

there follows $\mathfrak{J}^\nu(V) \to 0$, $\nu \to \infty$. Hence (4.7) yields

$$V = \mathfrak{J}^{\nu+1}(V) + \sum_{k=0}^{\nu} \mathfrak{J}^k(R), \qquad \nu = 1, 2, \cdots,$$

$$\to \sum_{k=0}^{\infty} \mathfrak{J}^k(R) \quad \text{as} \quad \nu \to \infty.$$

[1] Choose T so that $T'RT = D$, where $D = \operatorname{diag}(d_1, \cdots, d_n)$. Define $d_i^+ = \max(d_i, 0)$, $d_i^- = -\min(d_i, 0)$, $i = 1, \cdots, n$, $D^+ = \operatorname{diag}(d_i^+)$, $D^- = \operatorname{diag}(d_i^-)$. Then $R^+ = TD^+T'$, $R^- = TD^-T'$.

Thus $V \geqq 0$, and Lemma 4.3 is proved.

Turning to the proof of Theorem 4.1 we use, as before, quasi-linearization and successive approximations. Equation (2.3) is equivalent to the pair of equations

$$K = N^{-1}B'P, \tag{4.8a}$$

$$(A - BK)'P + P(A - BK) + \Pi(P) + C'C + K'NK = 0. \tag{4.8b}$$

If $K_0 = N^{-1}B'P$ and K is arbitrary, (3.2) yields the inequality

$$\begin{aligned} &(A - BK_0)'P + P(A - BK_0) + K_0'NK_0 \\ &\qquad \leqq (A - BK)'P + P(A - BK) + K'NK. \end{aligned} \tag{4.9}$$

First we solve (4.8b) for a suitable fixed matrix K. If $A - BK$ is stable, then (4.8b) is equivalent to

$$P = \int_0^\infty e^{t(A-BK)'}[\Pi(P) + C'C + K'NK]e^{t(A-BK)}\, dt. \tag{4.10}$$

Denote the right side of (4.10) by $f(K, P)$. Since

$$-|P|\Pi(I) \leqq \Pi(P) \leqq |P|\Pi(I),$$

condition (2.1) implies that for some K,

$$\left| \int_0^\infty e^{t(A-BK)'}\Pi(P)e^{t(A-BK)}\, dt \right| \leqq \theta\, | P |, \tag{4.11}$$

where $\theta \in (0, 1)$ is independent of P. Hence for this K the function $f(K, P)$ is a contraction mapping in P, and so (4.10) has a unique solution. For later reference we note that the approximating sequence $\{P_\nu\}$, defined by

$$P^{(1)} = 0, \qquad P^{(\nu)} = f(K, P^{(\nu-1)}), \qquad \nu = 2, 3, \cdots,$$

is monotone nondecreasing.

We can now solve the pair of equations (4.8a, b). By assumption there exists K_1 such that $A - BK_1$ is stable and (4.11) is true. Let P_1 be the solution of $P = f(K_1, P)$ and define $K_2 = N^{-1}B'P_1$. Next solve the equation

$$P = f(K_2, P)$$

by successive approximations. To see that this is possible observe that (4.9) and Lemma 4.3 imply that $A - BK_2$ is stable; hence $f(K_2, P)$ is defined. Now set

$$P_2^{(1)} = 0, \qquad P_2^{(\kappa+1)} = f(K_2, P_2^{(\kappa)}), \quad \kappa = 1, 2, \cdots.$$

As before it follows that $P_2^{(\kappa)} \geqq 0$, $\kappa = 2, 3, \cdots$, and $\{P_2^{(\kappa)}\}$ is nondecreasing. We shall show that

$$P_2^{(\kappa)} \leqq P_1. \tag{4.12}$$

The inequality (4.9) (with $K_0 = K_2$ and $P = P_1$) implies

$$f(K_2, P_2^{(\kappa)}) \leqq -\int_0^\infty e^{t(A-BK_2)'}[(A - BK_2)'P_1 + P_1(A - BK_2) + \Pi(P_1) - \Pi(P_2^{(\kappa)})]e^{t(A-BK_2)}\, dt$$
$$= P_1 - \int_0^\infty e^{t(A-BK_2)'}\Pi(P_1 - P_2^{(\kappa)})e^{t(A-BK_2)}\, dt.$$

Thus if $P_2^{(\kappa)} \leqq P_1$, then $P_2^{(\kappa+1)} = f(K_2, P_2^{(\kappa)}) \leqq P_1$; since $P_2^{(1)} = 0$, (4.12) is true. It follows by Lemma 3.1 that the limit

$$P_2 = \lim_{\kappa\to\infty} P_2^{(\kappa)}$$

exists, and $0 \leqq P_2 \leqq P_1$.

Repeating this procedure we obtain sequences $\{K_\mu\}$, $\{P_\mu\}$ with $K_{\mu+1} = N^{-1}B'P_\mu$ and $0 \leqq P_{\mu+1} \leqq P_\mu$. Then

$$\bar{P} = \lim_{\mu\to\infty} P_\mu$$

exists. If

$$\bar{K} = \lim_{\mu\to\infty} K_\mu = N^{-1}B'\bar{P},$$

it is clear that $\bar{K}$, $\bar{P}$ satisfy (4.8a, b), and (4.10) shows that $\bar{P} \geqq 0$.

Lemma 4.3 implies that $A - B\bar{K}$ is stable. If (C, A) is observable, uniqueness of $\bar{P}$ in the class $P \geqq 0$ is an immediate result of the minimum property. Finally, Lemma 4.1 shows that $((C'C + K'NK)^{1/2}, A - BK)$ is observable if (C, A) is observable; then it is clear from (4.10) that $\bar{P} > 0$. Theorem 4.1 is proved.

5. Proof of Theorem 2.1 (iv): asymptotic behavior of the solution. In this section we prove assertion (iv) of Theorem 2.1. As in §4, the parameter matrices A, B, C, N and the operator Π are independent of t.

LEMMA 5.1. *Let (C, A) be observable, let $P_0 \geqq 0$, and suppose $P(t)$ satisfies the differential equation*

$$\frac{dP(t)}{dt} + A'P(t) + P(t)A + \Pi[P(t)] + C'C = 0, \qquad t \leqq 0, \tag{5.1a}$$

$$P(0) = P_0. \tag{5.1b}$$

If there exists a constant matrix $P^* \geqq 0$ *such that*

$$A'P^* + P^*A + \Pi(P^*) + C'C = 0, \tag{5.2}$$

then

$$P(t) \to P^* \quad \textit{as} \quad t \to -\infty. \tag{5.3}$$

Proof. By Lemma 4.2, P^* is the unique solution of (5.2). Let $\mathcal{L}$ denote the operator defined by

$$\mathcal{L}(P) = A'P + PA + \Pi(P).$$

From Lemma 4.2 we know that $\mathcal{L}$ (regarded as a linear transformation on the $n \times n$ symmetric matrices) is nonsingular, and that $-\mathcal{L}^{-1}(Q) \geqq 0$ if $Q \geqq 0$.

Since $\mathcal{L}$ is linear and independent of t, it is enough to consider the homogeneous equation

$$\begin{aligned} \frac{dQ(t)}{dt} + \mathcal{L}[Q(t)] &= 0, \qquad t \leqq 0, \\ Q(0) &= P_0, \end{aligned} \tag{5.4}$$

and to show that

$$Q(t) \to 0 \quad \text{as} \quad t \to -\infty. \tag{5.5}$$

For this let

$$R(t) = \int_t^0 Q(s)\, ds, \qquad t \leqq 0,$$

and let $\bar{R}$ be the (unique) solution of

$$\mathcal{L}(\bar{R}) + P_0 = 0. \tag{5.6}$$

It will be shown that $0 \leqq R(t) \uparrow \bar{R}$ ($t \downarrow -\infty$). In fact, $Q(t) \geqq 0$ by (3.5), so that $R(t)$ is nondecreasing as t decreases. Integration of (5.4) yields

$$\frac{dR(t)}{dt} + \mathcal{L}[R(t)] + P_0 = 0. \tag{5.7}$$

Setting $F(t) = \bar{R} - R(t)$, we obtain from (5.6) and (5.7),

$$\begin{aligned} \frac{dF(t)}{dt} + \mathcal{L}[F(t)] &= 0, \qquad t \leqq 0, \\ F(0) &= \bar{R}. \end{aligned} \tag{5.8}$$

Since $\bar{R} = -\mathcal{L}^{-1}(P_0) \geqq 0$, it is clear from (5.8) (cf. (3.5)) that $F(t) \geqq 0$, $t \leqq 0$, that is, $0 \leqq R(t) \leqq \bar{R}$ and $R(-\infty) = \lim R(t)$, $t \to -\infty$, exists.

Next, (5.7) shows that dR/dT is bounded and then that d^2R/dt^2 is bounded. Since

$$R(-\infty) = \int_{-\infty}^{0} \frac{dR(t)}{dt}\, dt,$$

it follows that $Q(t) = dR(t)/dt \to 0$ as $t \to -\infty$. The proof is complete.

Next, consider the Riccati equation

$$\text{(5.9a)} \quad \frac{dP(t)}{dt} + [A - BK(t)]'P(t) + P(t)[A - BK(t)] + \Pi[P(t)] + C'C + K(t)'N(t)K(t) = 0, \qquad t \leqq 0,$$

$$\text{(5.9b)} \quad K(t) = N^{-1}B'P(t),$$

$$\text{(5.9c)} \quad P(0) = P_0 \geqq 0.$$

From §3 we know that (5.9a, b, c) have a unique solution $P(t) \geqq 0$.

LEMMA 5.2. (i) *If* (A, B) *is stabilizable and if* Π *satisfies* (2.1), *then the solution* $P(t)$ *of* (5.9a, b, c) *is bounded on* $(-\infty, 0]$.

(ii) *If* $P_0 = 0$, *then* $P(t)$ *is monotone nondecreasing as* t *decreases.*

Proof. Let $\hat{K}$ be a constant matrix such that $\hat{A} = A - B\hat{K}$ is stable, and let $\hat{P}(t)$ be the solution of (5.9a) and (5.9c) with $K(t) = \hat{K}$. By the minimum property (Theorem 2.1(iii)), $P(t) \leqq \hat{P}(t)$. It will be shown that $\hat{P}(t)$ is bounded for suitable $\hat{K}$. Now

$$\text{(5.10)} \quad \hat{P}(t) = e^{-t\hat{A}'}P_0\, e^{-t\hat{A}} + \int_t^0 e^{-(t-s)\hat{A}'}\{\Pi[\hat{P}(s)] + C'C + \hat{K}'N\hat{K}\}e^{-(t-s)\hat{A}}\, ds.$$

We solve (5.10) by successive approximation, setting $\hat{P}_0(t) \equiv 0$. Then

$$\text{(5.11)} \quad \hat{P}_{\nu+1}(t) \leqq \gamma I + \int_t^0 e^{-(t-s)\hat{A}'}\Pi[\hat{P}_\nu(s)]e^{-(t-s)\hat{A}}\, ds, \qquad \nu = 0, 1, \cdots,$$

where

$$\gamma = \sup_{t \leqq 0} |\, e^{-t\hat{A}'}P_0\, e^{-t\hat{A}} \,| + \left| \int_0^\infty e^{s\hat{A}'}(C'C + \hat{K}'N\hat{K})e^{s\hat{A}}\, ds \right|.$$

By (2.1), $\hat{K}$ can be chosen so that

$$\left| \int_0^\infty e^{s\hat{A}'}\Pi(I)e^{s\hat{A}}\, ds \right| = \theta < 1.$$

With this choice (5.11) yields, on iteration,

$$|\, \hat{P}_\nu(t) \,| \leqq \gamma(1 + \theta + \cdots + \theta^{\nu-1}) \leqq \gamma(1 - \theta)^{-1}, \qquad t \leqq 0, \quad \nu = 1, 2, \cdots.$$

Hence,

$$\hat{P}(t) = \lim_{\nu\to\infty} \hat{P}_\nu(t)$$

is a bounded function of t, and (i) follows.

To prove (ii) set $P_0 = 0$ and let $\Phi(t, s)$ denote the fundamental matrix associated with $A - BK(t)$ (cf. (3.3)). Then

$$P(t) = \int_t^0 \Phi(s, t)'\{\Pi[P(s)] + C'C + K(s)'N(s)K(s)\}\Phi(s, t)\, ds. \tag{5.12}$$

Let $\tau \geqq 0$ be fixed, and define

$$\tilde{K}(t) = K(t - \tau), \qquad t \leqq 0.$$

If $\tilde{\Phi}(t, s)$ is the fundamental matrix determined by $A - B\tilde{K}(t)$, then clearly

$$\tilde{\Phi}(t, s) = \Phi(t - \tau, s - \tau).$$

Let $\tilde{P}(t)$ be the solution of (5.9a) with K replaced by $\tilde{K}$ and $\tilde{P}(0) = 0$. Again by the minimum property (Theorem 2.1(iii)), $P(t) \leqq \tilde{P}(t)$, or

$$\begin{aligned} P(t) &\leqq \int_t^0 \tilde{\Phi}(s, t)'\{\Pi[\tilde{P}(s)] + C'C + \tilde{K}(s)'N(s)\tilde{K}(s)\}\tilde{\Phi}(s, t)\, ds \\ &= \int_{t-\tau}^{-\tau} \tilde{\Phi}(s + \tau, t)'\{\Pi[\tilde{P}(s + \tau)] + C'C \\ &\qquad + \tilde{K}(s + \tau)'N(s)\tilde{K}(s + \tau)\}\tilde{\Phi}(s + \tau, t)\, ds \\ &\leqq \int_{t-\tau}^0 \Phi(s, t - \tau)'\{\Pi[\tilde{P}(s + \tau)] + C'C \\ &\qquad + K(s)'N(s)K(s)\}\Phi(s, t - \tau)\, ds, \end{aligned} \tag{5.13}$$

where we have set $\tilde{P}(s) \equiv 0$ for $s \geqq 0$. Now

$$\begin{aligned} \tilde{P}(s + \tau) &= \int_{s+\tau}^0 \tilde{\Phi}(\sigma, s + \tau)'\{\Pi[\tilde{P}(\sigma)] + C'C + \tilde{K}(\sigma)'N(\sigma)\tilde{K}(\sigma)\}\tilde{\Phi}(\sigma, s + \tau)\, d\sigma \\ &= \int_{s+\tau}^0 \Phi(\sigma - \tau, s)'\{\Pi[\tilde{P}(\sigma)] + C'C \\ &\qquad + K(\sigma - \tau)'N(\sigma)K(\sigma - \tau)\}\Phi(\sigma - \tau, s)\, d\sigma \\ &\leqq \int_s^0 \Phi(\sigma, s)'\{\Pi[\tilde{P}(\sigma + \tau)] + C'C + K(\sigma)'N(\sigma)K(\sigma)\}\Phi(\sigma, s)\, d\sigma. \end{aligned}$$

Writing $Q(s) = P(s) - \tilde{P}(s + \tau)$ and using (5.12), we see that

$$Q(s) \geqq \int_s^0 \Phi(\sigma, s)'\Pi[Q(\sigma)]\Phi(\sigma, s)\, d\sigma. \tag{5.14}$$

Denote the right side of (5.14) by $\mathcal{S}Q(s)$. Defining $\mathcal{S}^\nu$ by iteration, there results

$$Q(s) \geqq \mathcal{S}^\nu Q(s)$$

and

$$| \mathcal{S}^\nu Q(s) | \leqq \alpha(\beta | s |)^\nu/\nu!, \qquad \nu = 1, 2, \cdots,$$

for suitable constants $\alpha > 0$, $\beta > 0$ (which may depend on s). Letting $\nu \to \infty$ we obtain $Q(s) \geqq 0$, or

$$\tilde{P}(s + \tau) \leqq P(s), \qquad s \leqq 0.$$

Substituting this result in (5.13) and comparing with (5.12), we conclude that

$$P(t) \leqq P(t - \tau), \qquad t \leqq 0, \quad \tau \geqq 0,$$

and the proof is complete.

LEMMA 5.3. *If* (A, B) *is stabilizable and* (C, A) *is detectable, if* Π *satisfies* (2.1), *and if* $P_0 = 0$, *then the solution* $P(t)$ *of* (5.9a, b, c) *has the property*

$$\lim_{t \to -\infty} P(t) = \bar{P}, \tag{5.15}$$

where $\bar{P}$ *is a positive semidefinite solution of* (2.3).

Proof. By Lemmas 3.1 and 5.2, the limit in (5.15) exists and is positive semidefinite. Since $P(t)$ is bounded, (5.9a, b, c) show that the same is true of $dP(t)/dt$ and $d^2P(t)/dt^2$; then convergence of the integral

$$\int_{-\infty}^{0} \frac{dP(t)}{dt} \tag{5.16}$$

shows that $dP/dt \to 0$ as $t \to \infty$. The conclusion follows by inspection of (2.3) and (5.9a, b, c).

We turn to a proof of assertion (iv) of Theorem 2.1. Set $T = 0$. Boundedness of $P(t)$ follows from Lemma 5.2. If (C, A) is observable, then, by Theorem 4.1, (2.3) has a unique solution $\bar{P} \geqq 0$. Set $\bar{K} = N^{-1}B'\bar{P}$, $\bar{A} = A - B\bar{K}$; Theorem 4.1 implies that $\bar{A}$ is stable. Denote by $P^*(t)$ the solution of (5.9a) and (5.9c) with $K(t) = \bar{K}$. By the minimum property (Theorem 2.1 (iii)), the solution $P(t)$ of (5.9a, b, c) satisfies

$$P(t) \leqq P^*(t), \qquad t \leqq 0, \tag{5.17}$$

and by Lemma 5.1 (with A replaced by $\bar{A}$, P^* by $\bar{P}$, and $C'C$ by $C'C + \bar{K}'N\bar{K}$),

$$P^*(t) \to \bar{P} \quad \text{as} \quad t \to -\infty. \tag{5.18}$$

On the other hand, if $P_*(t)$ denotes the solution of (5.9a) and (5.9b) with $P_*(0) = 0$, then by Lemma 5.3,

$$P_*(t) \to \bar{P} \quad \text{as} \quad t \to -\infty. \tag{5.19}$$

Hence the desired result will follow if we show that

$$P(t) \geqq P_*(t), \qquad t \leqq 0. \tag{5.20}$$

For this observe from (3.5) that

$$P(t) \geqq \int_t^0 \Phi(s, t)'\{\Pi[P(s)] + C'C + K(s)'N(s)K(s)\}\Phi(s, t)\, ds \tag{5.21}$$

Denote the right side of (5.21) by $Q(t)$. It will be shown that $Q(t) \geqq P_*(t)$. Write $K_*(t) = N^{-1}B'P_*(t)$ and $\Phi_*(t, s)$ for the fundamental matrix associated with $A - BK_*(t)$; and let $\tilde{P}(t)$ be the solution of (5.9a) with $K(t)$ as before and $\tilde{P}(0) = 0$. Then, by the minimum property,

$$P_*(t) \leqq \tilde{P}(t) \tag{5.22}$$

and

$$\tilde{P}(t) = \int_t^0 \Phi(s, t)'\{\Pi[\tilde{P}(s)] + C'C + K(s)'N(s)K(s)\}\Phi(s, t)\, ds. \tag{5.23}$$

Then (5.21) and (5.23) yield

$$P(t) - \tilde{P}(t) \geqq \int_t^0 \Phi(s, t)'\Pi[P(s) - \tilde{P}(s)]\Phi(s, t)\, ds$$

and this shows, as in the proof of Lemma 5.2 (ii), that

$$P(t) - \tilde{P}(t) \geqq 0, \qquad t \leqq 0. \tag{5.24}$$

Inequalities (5.22) and (5.24) yield (5.20). Combining (5.17) and (5.20), we have that

$$P_*(t) \leqq P(t) \leqq P^*(t), \qquad t \leqq 0. \tag{5.25}$$

Since the extreme terms of the inequality (5.25) both tend to $\bar{P}$ as $t \to -\infty$, the desired result is established.

6. Applications.

6.1. Stochastic control. An equation of type (2.2a) arises in optimal control of a linear system with state-dependent white noise and quadratic cost (cf. [6], where time-invariant control was discussed, leading to (2.3)). In this problem,

$$[\Pi(t, P)]_{ij} = \operatorname{tr}\{G_i(t)'PG_j(t)\}, \quad i, j = 1, \cdots, n,$$

for certain G_i. We mention that an obvious generalization to include control-dependent white noise leads to (2.2a) with Π replaced by

$$\Pi + PB(\Gamma + N)^{-1}\Gamma(\Gamma + N)^{-1}B'P. \tag{6.1}$$

In (6.1), $\Gamma = \Gamma(t, P)$ is a function of the same type as Π. This case can be discussed in exactly the same way, if (3.1a) is replaced by

$$K = (\Gamma + N)^{-1}B'P.$$

6.2. Linear filtering. A well-known linear filtering scheme [7] leads to the following equation for the covariance matrix:

$$\frac{dP}{dt} = AP + PA' + FF' - (PC' + FG')(GG')^{-1}(PC' + FG')', \quad t_1 \leqq t \leqq t_2, \tag{6.2}$$

$$P(t_1) \equiv P_0 \geqq 0.$$

It is clear that (6.2) is equivalent to (2.2a, b) after replacing in (6.2) t, t_1, t_2 by $T + t_0 - t$, t_0, T, respectively, setting $\Pi = 0$, and redefining matrices. Thus Theorem 2.1 shows that (6.2) uniquely determines the covariance matrix. If the parameter matrices are constants, then the limit property of Theorem 2.1 (iv) holds if $(A' - C'(GG')^{-1}GF', C')$ is stabilizable and $(H, A' - C'(GG')^{-1}GF')$ is observable, where

$$FF' - FG'(GG')^{-1}GF' = H'H.$$

In particular this is true if (C, A) is detectable, (A, F) is controllable, and

$$FF' - FG'(GG')^{-1}GF' \geqq \rho FF'$$

for some $\rho \in (0, 1]$. The latter result under strengthened hypotheses was reported in [7, §13.33].

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