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Wishart Processes

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We propose some matrix generalizations of square Bessel processes and we indicate their first properties: hitting time of 0 of the smallest eigenvalue, additivity property, associated Martingales, distributions, which mainly extend the real-valued classical results. We explain why these processes are indecomposable and therefore differ from the real-valued ones. We conclude with some formulae concerning matrix quadratic functionals analogous to the Cameron Martin formula.

KEY WORDS: Wishart distribution; Bessel process; matrix diffusions; special matrix functions; Cameron Martin formulae.

1. INTRODUCTION

Let $B_t = (B_1(t),..., B_n(t))$ be an $\mathbb{R}^n$ Brownian motion; the process $X_t = B_1^2(t) + \cdots + B_n^2(t)$ is a real diffusion which satisfies the stochastic differential equation $dX_t = 2\sqrt{X_t}\, dB_t + n\, dt$, whose generator is the differential operator $2xD^2 + nD$, where $D = d/dx$.

More generally, a square Bessel process BESQ(α), with real index $\alpha \geqslant 0$, is a diffusion generated by $2xD^2 + \alpha D$ (see Refs. 22, 24, and 25). The density of a BESQ(α) process (X_t), with initial state $X_0 = x$, is a Bessel function with Laplace transform $E_x(e^{-\lambda X(t)}) = (1 + 2\lambda t)^{-\alpha/2} \exp(-\lambda x/(1 + 2\lambda t))$.

Bessel functions have classical matrix versions. One also knows that, for fixed t, the matrix analogue of X_t has been studied in multivariate statistics (Wishart[29]). Let $(B_1,..., B_n)$ be a sample of an $\mathbb{R}^p$ Gaussian vector, and B denote the $n \times p$ matrix whose ith line is the vector B_i. The matrix variable $B^T B$ (superscript T signifying transpose) has a Wishart distribution, which is a natural generalization of the χ^2 distribution, whose

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density is a Bessel matrix function. It is therefore natural to examine matrix versions of square Bessel processes.

Among the possible extensions, a natural one was considered in Ref. 3; it deals with pertubations of inertia in principal component analysis (see Ref. 4). It was observed there that the spectral analysis of the square $S_t = N_t^T N_t$ of an $n \times p$ Brownian matrix N_t can be done by techniques of stochastic matrix calculus (Williams[24,28]) and Martingale theory.

Notice that (S_t) satisfies the stochastic differential equation

$$dS_t = \sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t} + nI\, dt \tag{1.1}$$

where B_t is a $p \times p$ Brownian matrix and that its generator is

$$\operatorname{tr}(nD + 2xD^2) \tag{1.2}$$

It is natural to search $p \times p$ matrix diffusions, solutions of (1, 1) (or (1, 2)) when n is no more an integer $\geqslant p$ but a positive number or a fixed matrix and to call those possible solutions Wishart processes.

The object of this paper is to give a construction of Wishart processes (Theorem 2) and to indicate their first properties: hitting time of 0 of the smallest eigenvalue, additivity property, associated Martingales, distributions. As in the case of Wishart families (Letac[16]) it is not possible to find $p \times p$ Wishart processes in all cases. While real Bessel processes can be defined for all $\alpha \geqslant 0$, and thus have "infinitely decomposable" distributions (Shiga and Watanabe[25]), to force the eigenvalues of a Wishart process to be positive a.s. the index α must be large enough compared with the dimension p of the matrices, and this result gives another probabilistic interpretation to the Gindikin[11] theorem. However this indecomposability does not really affect the calculus; indeed we can show, for example, how the Laplace transforms of some "Wishart quadratic functionals" can be obtained by adapting to the matrix framework the real methods developped by Pitman and Yor.[22] These formulae are then extended to generalized Wishart processes among which we can find again, as particular case, squares of matrix Ornstein–Uhlenbeck processes considered by W. Kendall[15] in the theory of shapes (see also Ref. 14).

Some of the results here presented have been announced in a note (Ref. 5).

2. WISHART PROCESSES WITH INTEGER INDICES, SQUARE BROWNIAN MOTIONS

2.1. Definition

Throughout, a Brownian matrix will be a process taking its values in the set $\mathcal{M}(n, p)$ of *real-valued* $n \times p$ *matrices* whose components are independent Brownian motions. Let n and p be two integers $\geqslant 1$, and

$$N_t = (n_{ij}(t)) \qquad N_0 = (n_{ij}(0)) = C$$

be an $n \times p$ Brownian matrix, with initial state C.

Definition 1. A Wishart process, of dimension p, index n, and initial state s_0, denoted WIS(n, p, s_0), will be the matrix process

$$S_t = (s_{ij}(t)) = N_t^T N_t \qquad s_0 = C^T C \tag{2.1}$$

For fixed t, the r.v. S_t is a Wishart matrix which occurs naturally in multivariate statistics (see Refs. 2 and 10).

Example. If $p = 1$, WIS$(n, 1, s_0)$ is a square Bessel process of index n BESQ(n, s_0) (see Refs. 21, 23, and 24).

2.2. Properties of Wishart Processes with Integer Indices

Notations. Let $\tilde{\mathcal{S}}_p^+$ (resp. $\mathcal{S}_p^+$, $\mathcal{S}_p^-$, $\mathcal{S}_p$, $\hat{\mathcal{S}}_p^+$,...) denote the set of all symmetric positive definite (resp. symmetric positive, symmetric negative, symmetric, symmetric positive with distinct eigenvalues,...) $p \times p$ matrices, and D be the matrix operator $D = (D_{ij}) = (\partial/\partial x_{ij})$.

> Every Wishart process (S_t) with parameters n and p is a diffusion generated by the differential operator
> $$L_n = \operatorname{tr}(nD + 2xD^2) \qquad x \in \mathcal{S}_p \tag{2.2}$$

Indeed one can check (Faraut[8]) that, if f and F are C^2 functions defined respectively on $\mathcal{S}_p$ and on $\mathcal{M}(n, p)$ such that for all $y \in \mathcal{M}(n, p)$ we have

$$F(y) = f(y^T y)$$

then

$$\tfrac{1}{2} \Delta F = L_n f$$

where Δ is the Laplacian.

Remark 1. If $p = 1$, $L_n = n(d/dx) + 2x(d^2/dx^2)$ is the classical square Bessel differential operator.

Itô calculus applied to the relation (2.1) gives the following results:

$$dS_t = dN_t^T N_t + N_t^T \, dN_t + nI \, dt \tag{2.3}$$

where I is the identity matrix of $\mathbb{R}^p$.

For all $i, j, k, l \in \{1,\dots p\}$, we have the following:

$$(ds_{ij})(ds_{kl}) = (s_{ik}\delta_{jl} + s_{il}\delta_{jk} + s_{jk}\delta_{il} + s_{jl}\delta_{ik}) \, dt \tag{2.4}$$

($\mathrm{tr}\, S_t$) is a square Bessel process with index np, i.e.,

$$d(\mathrm{tr}\, S_t) = 2\sqrt{\mathrm{tr}\, S_t} \, dv_t + np \, dt \tag{2.5}$$

where v_t is a Brownian motion.

If $\tilde{S}_t$ is the comatrix of S_t

$$d(\det S_t) = \mathrm{tr}(\tilde{S}_t \, dS_t) + (1-p)\, \mathrm{tr}(\tilde{S}_t) \, dt \tag{2.6}$$

This last formula is helpful to define the stochastic differential equations which govern $\det S_t$ and $\ln(\det S_t)$ without the help of the eigenvalues. Indeed, for all t, S_t is a $p \times p$ symmetric matrix, with rank inferior or equal to $\min(n, p)$, so $\det S_t \geqslant 0$ a.s. If $t < \tau_0 = \inf\{s: \det S_s = 0\}$, since

$$\mathrm{tr}\, \tilde{S}_t = \det S_t \cdot \mathrm{tr}\, S_t^{-1} \qquad \text{and} \qquad \mathrm{tr}(\tilde{S}_t \, dS_t) = \det S_t \cdot \mathrm{tr}(S_t^{-1} \, dS_t)$$

together with (2.6) and (2.3), we have the following equations:

$$d(\det S_t) = 2 \det S_t \sqrt{\mathrm{tr}\, S_t^{-1}} \, dv_t + (n-p+1) \det S_t \,\mathrm{tr}\, S_t^{-1} \, dt \tag{2.7}$$

$$d(\det S_t)^\zeta = 2\zeta(\det S_t)^\zeta \sqrt{\mathrm{tr}\, S_t^{-1}} \, dv_t + \zeta(n-p+2\zeta-1)(\det S_t)^\zeta \,\mathrm{tr}\, S_t^{-1} \, dt \qquad \zeta \in \mathbb{R}^+ \tag{2.8}$$

$$d(\ln \det S_t) = 2\sqrt{\mathrm{tr}\, S_t^{-1}} \, dv_t + (n-p-1)\,\mathrm{tr}\, S_t^{-1} \, dt \tag{2.9}$$

2.3. Eigenvalues

In this paragraph we recall (see Ref. 4) the behavior of the eigenvalues of S_t, if $n \geqslant p$.

Theorem 1. If at time $t = 0$, the p eigenvalues of $s_0 = C^T C$ are distinct, labeled

$$\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$$

then for all $t \geqslant 0$, the p eigenvalues of S_t are distinct

$$\lambda_1(t) > \cdots > \lambda_p(t) \geqslant 0 \quad \text{a.s.}$$

the process $(\lambda_1(t),..., \lambda_p(t))$ is a diffusion, solution of the stochastic differential system:

$$d\lambda_i = 2\sqrt{\lambda_i}\, dv_i + \left(n + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j}\right) dt \qquad 1 \leqslant i \leqslant p \tag{2.10}$$

where $v_1(t),..., v_p(t)$ are p independent Brownian motions, adapted to the natural filtration $(\mathscr{F}_t)_{t \geqslant 0}$ associated to the process (S_t).

Remark 2. (a) Relation (2.10) shows that

$$d\lambda_i = (2\sqrt{\lambda_i}\, dv_i + n\, dt) + \sum_{j \neq i} \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \tag{2.11}$$

The eigenvalues $\lambda_1(t),..., \lambda_p(t)$ behave like square Bessel processes of index n submitted to repulsion forces which prohibit all collision. This fact, which recalls classical results on symmetric Brownian matrices and Brownian motions of ellipsoids (see Refs. 20, 21, and 28) has been derived independently by W. Kendall[15] (§3.1).

(b) When $n = p$, the process (S_t) is similar to the Brownian motions of ellipsoids studied by Norris, Rogers, and Williams,[21] who consider Dynkin's Brownian motion on $\mathscr{S}_n^+$

$$Y_t = (y_{ij}(t)) = G_t^T G_t$$

where G_t is the right-invariant Brownian motion on GL(n), the group of invertible $n \times n$ matrices, which solves the Stratonovich differential equation

$$\partial G_t = (\partial B_t) G_t \qquad G_0 = I$$

where B_t is an $n \times n$ Brownian matrix, and I the identity $n \times n$ matrix.

Y_t is a Markov process with the characteristic relation

$$(dy_{ij})(dy_{kl}) = 2(y_{ik}\, y_{jl} + y_{il}\, y_{jk})\, dt$$

which recalls relation (2.4), and whose differential operator is given by

$$\mathscr{G}^Y = \operatorname{tr}[(n+1)\, yD + 2yDyD] \qquad D = (D_{ij}) = \left(\frac{\partial}{\partial x_{ij}}\right)$$

which is analogous to the differential operator (2.2).

The eigenvalues $\lambda_1(t),..., \lambda_n(t)$ of Y_t do not collide and satisfy the stochastic differential equation

$$\frac{1}{2} d(\ln \lambda_i) = d\beta_i + \frac{1}{2} \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} dt \tag{2.12}$$

where $\beta_1(t),..., \beta_n(t)$ are independent Brownian motions, while the eigenvalues of S_t are solution of the system

$$\frac{1}{2} d(\ln \lambda_i) = \frac{1}{\sqrt{\lambda_i}} d\beta_i + \frac{1}{\lambda_i} \left[\frac{n-2}{2} + \frac{1}{2} \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \right] dt \qquad 1 \leqslant i \leqslant n$$

When $n = p = 1$, $Y_t = G_t^2$, where G_t is the "multiplicative Brownian motion" on $\mathbb{R}^+ - \{0\}$

$$G_t = \exp B_t \qquad Y_t = \exp 2B_t$$

while $S_t = B_t^2$, where B_t is the ordinary "additive Brownian motion" on $\mathbb{R}$. So $\frac{1}{2} \ln Y_t$ is a Brownian motion while S_t is a square Bessel process. And in the matrix case the same difference appears on the eigenvalues (2.11) and (2.12).

(c) From (2.10) we can deduce (2.7), (2.8), and (2.9) if we notice that

$$\det S_t = \prod_{i=1}^{p} \lambda_i(t) \qquad \operatorname{tr} S_t^{-1} = \sum_{i=1}^{p} \frac{1}{\lambda_i(t)}$$

and

$$\sum_i \sum_{j \neq i} \frac{1}{\lambda_i} \cdot \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} = -(p-1) \sum_i \frac{1}{\lambda_i}$$

Let us observe that

$$\sum_i \sum_{j \neq i} \frac{1}{\lambda_i^2} \cdot \frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} = -(p-2) \sum_i \frac{1}{\lambda_i^2} - \left(\sum_i \frac{1}{\lambda_i} \right)^2$$

so

$$d(\operatorname{tr} S_t^{-1}) = 2\sqrt{\operatorname{tr} S_t^{-3}}\, dv_t + ((2-n+p) \operatorname{tr} S_t^{-2} + (\operatorname{tr} S_t^{-1})^2)\, dt \tag{2.13}$$

(d) If $p = 1$, $\xi = \zeta = \frac{1}{2}$, (2.8) is $d\sqrt{S_t} = dv_t + (n-1)/2\sqrt{S_t}\, dt$,

which means that $\sqrt{S_t}$ is a Bessel process. On the other hand, for any integer p, the process $\sqrt{S_t}$ does not seem to verify any simple stochastic differential equation.

If $n<p$, by applying Theorem 1 to $\hat{S}_t = N_t N_t^T$, we obtain the following:

Corollary 1. If $n<p$, and if at time $t=0$

$$\lambda_1(0) > \cdots > \lambda_n(0) \geqslant \lambda_{n+1}(0) = \cdots = \lambda_p(0) = 0$$

then for all $t \geqslant 0$, the n first eigenvalues of S_t are distinct

$$\lambda_1(t) > \cdots > \lambda_n(t) \geqslant \lambda_{n+1}(t) = \cdots = \lambda_p(t) = 0 \quad \text{a.s.}$$

and for $1 \leqslant i \leqslant n$, $\lambda_i(t)$ are governed by the stochastic differential equations (2.10) of Theorem 1.

Proof. The n first eigenvalues of S_t are also eigenvalues of $\hat{S}_t$.

Example. If $n=1$, then for any p, the unique non null eigenvalues of $S_t = (n_i(t)\, n_j(t))$, [where $(n_1(t),..., n_p(t))$ is an $\mathbb{R}^p$ Brownian motion] is the square Bessel process BESQ(p) $\lambda_1(t) = \sum_i n_i^2(t)$, verifying

$$d\lambda_1(t) = 2\sqrt{\lambda_1(t)}\, dv_t + p\, dt$$

2.4. Martingales Associated with the Hitting Time of 0 of the Smallest Eigenvalue

Let $S_t = N_t^T N_t$, $s_0 \in \mathscr{S}_p^+$, be a WIS(n, p, s_0) process; det S_t cancels with the smallest eigenvalue λ_p whose behavior is as follows:

Proposition 1. If $n<p$, a.s., for all t, $\lambda_p(t) = 0$.
If $n=p$, $\{t/\lambda_p(t)=0\}$ is a.s. of Lebesgue measure zero.
If $n \geqslant p+1$, a.s., for all t, $\lambda_1(t) > \cdots > \lambda_p(t) > 0$.

Proof. If $n<p$, det $S_t = 0$ and $\lambda_p(t) = 0$ a.s.
If $n=p$, $\{t\colon \lambda_p(t)=0\} = \{t\colon \det S_t = 0\}$, but $\det S_t = (\det N_t)^2$, so

$$d(\det N_t) = \sum_{i,j} \tilde{n}_{ij}(t)\, dn_{ij}(t) \qquad (d \det N_t)^2 = \sum_{i,j} \tilde{n}_{ij}^2(t)\, dt$$

We just need to verify that a.s., $t \mapsto \int_0^t \sum_{i,j} \tilde{n}_{ij}^2(s)\, ds$ is strictly increasing, which is obvious if for all (i, j) $\{t\colon \tilde{n}_{ij}(t) = 0\}$ is of Lebesgue measure zero. But $(\tilde{n}_{ij}(t))$ is a matrix of the same type as N_t at order $n-1$. The result can be established by induction, the case $n=p=1$ being classical.

If $n=p+1$, and if $\tau_0 = \inf\{t\colon \det S_t = 0\}$, $\mathbb{U}(t) = \ln(\det S_t)$ is a local Martingale on $[0, \tau_0[$ which verifies [cf. Eq. (2.9)]

$$d(\ln \det S_t) = 2\sqrt{\operatorname{tr} S_t^{-1}}\, dv_t$$

McKean's[20] (p. 47) argument gives the conclusion: Suppose we have $\tau_0 < +\infty$, the mapping $t \mapsto \ln \det S_t$, being continuous, is bounded above on $[0, \tau_0[$, and

$$\lim_{t \nearrow \tau_0} \mathbb{U}(t) = -\infty$$

This cannot occur because $\mathbb{U}(t)$, being a local Martingale, is a time-changed Brownian motion and therefore cannot tend to infinity without infinite oscillations.

If $n \geqslant p+2$, the result follows from the same argument if we remark that

$$\mathbb{U}(t) = (\det S_t)^{(p+1-n)/2}$$

is a local Martingale on $[0, \tau_0[$, as can be seen with (2.9):

$$d((\det S_t)^{(p+1-n)/2}) = (p+1-n)(\det S_t)^{(p+1-n)/2} \sqrt{\operatorname{tr} S_t^{-1}}\, dv_t \qquad (2.14)$$

$\square$

Particular Case. If $n=3$, $p=1$, $B_t^T = (B_1(t), B_2(t), B_3(t))$ is a Brownian motion on $\mathbb{R}^3$, and $S_t = B_1^2(t) + B_2^2(t) + B_3^2(t)$, we know then that $1/\sqrt{S_t}$ is a positive local Martingale which is not a Martingale (see Ref. 27, p. 179, and Ref. 24, p. 375).

More generally, if $p=1$, we find the well-known results: $S_t^{1-n/2}$ if $n \geqslant 3$, and $\ln S_t$ if $n=2$, are local Martingales (see Refs. 12, 20, and 24).

2.5. Additivity Property

If (S_t) and (Σ_t) are two independent Wishart processes $\mathrm{WIS}(n, p, s_0)$ and $\mathrm{WIS}(m, p, \sigma_0)$ respectively, then $(S_t + \Sigma_t)$ is a Wishart process $\mathrm{WIS}(n+m, p, s_0 + \sigma_0)$. (2.15)

Proof. If $S_t = N_t^T N_t$ and $\Sigma_t = P_t^T P_t$ where N_t and P_t are, respectively, $n \times p$ and $m \times p$ independent Brownian motions, $E_t = \binom{N_t}{P_t}$ is an $(n+m) \times p$ matrix of independent Brownian motions, and

$$S_t + \Sigma_t = N_t^T N_t + P_t^T P_t = E_t^T E_t \qquad \square$$

Remark 3. We know that this property is determinant in Bessel process theory, (see Refs. 22 and 24).

We shall now introduce a definition of Wishart processes which gives a matrix extension to square Bessel processes with noninteger index α.

3. GENERALIZATION. THE WIS(α, p, s_0) PROCESSES

If $S_t = N_t^T N_t \in \mathrm{WIS}(n, p, S_0)$ with $n > p$ and $\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$, let $\sqrt{S_t}$ represent the symmetric positive square root of S_t. We can easily check that

$$dB_t = (\sqrt{S_t})^{-1} N_t^T \, dN_t$$

is a $p \times p$ Brownian matrix, and that (S_t) is governed by the stochastic differential equation

$$dS_t = \sqrt{S_t} \, dB_t + dB_t^T \sqrt{S_t} + nI \, dt \tag{3.1}$$

When $p = 1$, (S_t) is a square Bessel process: $dS_t = 2\sqrt{S_t} \, dB_t + nI \, dt$. By analogy with the real case we propose to study Eq. (3.1) when B_t is a $p \times p$ Brownian matrix, but n is not an integer.

First we give a general existence theorem in which we suppose that the initial state s_0 of (S_t) is in $\hat{\mathscr{S}}_p^+$, which means that all the eigenvalues of s_0 are distinct: $\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$. This hypothesis is then lifted.

Theorem 2. If $(B_t)_{t \geqslant 0}$ is a $p \times p$ Brownian matrix, then for every $p \times p$ symmetric matrix $s_0 = (s_{ij}(0)) \in \mathscr{S}_p^+$ with distinct eigenvalues labeled

$$\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$$

the stochastic differential equation

$$dS_t = \sqrt{S_t} \, dB_t + dB_t^T \sqrt{S_t} + \alpha I \, dt \tag{3.2}$$

has (1) a unique solution in $\mathscr{S}_p^+$ (in the sense of probability law) if $\alpha \in \left]p-1, p+1\right[$, and (2) a unique strong solution in $\tilde{\mathscr{S}}_p^+$ if $\alpha \geqslant p+1$. The eigenvalues of such a solution never collide: almost surely, for all $t > 0$, $\lambda_1(t) > \cdots > \lambda_p(t) \geqslant 0$, with $\lambda_p(t) > 0$ if $\alpha \geqslant p+1$ and satisfy the stochastic differential system

$$d\lambda_i = 2\sqrt{\lambda_i} \, dv_i + \left(\alpha + \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}\right) dt \qquad 1 \leqslant i \leqslant p \tag{3.3}$$

where $v_1(t), \ldots, v_p(t)$ are independent Brownian motions.

Remarks 4. (a) If $\alpha \in \{1, \ldots, p-1\}$, the results of Sec. 2 show that (3.2) has a solution in $\mathscr{S}_p^+$ whose α first eigenvalues satisfy (3.3); uniqueness is a consequence of (3.8).

(b) The proof of the existence of a strong solution of (3.2) when $\alpha \geqslant p+1$ does not require that s_0 has all its eigenvalues distinct; we just need the positive definiteness of s_0.

(c) Any process (S_t) solution of (3.2) is a diffusion generated by the same differential operator as in (2.2) where α takes the place of n:

$$L_\alpha = \operatorname{tr}(\alpha D + 2xD^2) \qquad D = \left(\frac{\partial}{\partial x_{ij}}\right) \tag{3.4}$$

Definition 1′. A matrix process on $\mathscr{S}_p^+$, governed by (3.2) and such that $S_0 = s_0$, is called a Wishart process with index α, dimension p, initial state s_0, and is denoted by WIS(α, p, s_0).

The proof of Theorem 2 is the object of this paragraph. When $\alpha \geqslant p+1$ and s_0 has all its eigenvalues distinct and strictly positive, the proof is quite simple and is given first; when $\alpha \in]p-1, p+1[$, the proof is a little involved and given later

Proof of Theorem 2 when $\alpha \in [p+1, +\infty[$ and $\lambda_1(0) > \cdots > \lambda_p(0) > 0$. The mapping $s \mapsto \sqrt{s}$ is analytic in $\tilde{\mathscr{S}}_p^+$ (Rogers and Williams,[24] p. 134), so Eq. (3.2) has a unique strong solution as long as $t < \tau_0 = \inf\{s\colon \det S_s = 0\}$ (see Ikeda and Watanabe,[12] p. 164).

It is easy to check that, up to time τ_0, any solution (S_t) of (3.2) verifies relation (2.4). So as in Sec. 2, $\det S_t$ is governed on $[0, \tau_0[$ by the stochastic differential equation (2.7):

$$d \det S_t = 2 \det S_t \sqrt{\operatorname{tr} S_t^{-1}}\, dv_t + (\alpha - p + 1) \det S_t \operatorname{tr} S_t^{-1}\, dt$$

If $\alpha > p+1$ (resp. $\alpha = p+1$), $\mathbb{U}(t) = (\det S_t)^{(p+1-\alpha)/2}$ (resp. $\mathbb{U}(t) = \ln \det S_t$) is a local Martingale, and hence a time-changed Brownian motion on $[0, \tau_0[$. The argument of McKean already used in Norris, Rogers, and Williams[21] and in Ref. 4 can be applied here: $\det S_t$ cannot tend to $+\infty$ without infinite spinning; so $\tau_0 = +\infty$ a.s.

If now $\tau = \inf\{s\colon \lambda_i(s) = \lambda_j(s) \text{ for some } (i,\ j)\}$ is the first collision time, the same argument with

$$\mathbb{V}(t) = \sum_{i<j} \ln(\lambda_i(t) - \lambda_j(t))$$

which is a local Martingale on $[0, \tau[$, shows that $r = +\infty$ a.s. □

The Process S^+

If s is a $p \times p$ symmetric matrix, let s^+ denote the symmetric matrix $\max(s, 0)$. The eigenvalues of s^+ are $\lambda_i^+ = \max(\lambda_i, 0)$, when $(\lambda_i)_{1 \leqslant i \leqslant p}$ are the eigenvalues of s (e.g., Farrell[9] or Marshall and Olkin[18]).

Proposition 2. For all α in $\mathbb{R}$, the stochastic differential equation

$$dS_t = \sqrt{S_t^+}\, dB_t + dB_t^T \sqrt{S_t^+} + \alpha I\, dt \qquad S_0 = s_0 \in \mathscr{S}_p \tag{3.5}$$

has a solution S_t, $t \geqslant 0$, in $\mathscr{S}_p$.

If $s_0 \in \mathscr{S}_p^+$ and has distinct eigenvalues $\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$ the eigenvalues $(\lambda_i^+(t))$ of S_t^+ on $[0, \tau[$ where

$$\tau = \inf\{s\colon \lambda_i(s) = \lambda_j(s) \text{ for some } (i, j)\}$$

are solution of the stochastic differential system

$$d\lambda_i = 2\sqrt{\lambda_i^+}\, dv_i + \left(\alpha + \sum_{k \neq i} \frac{\lambda_i^+ + \lambda_k^+}{\lambda_i - \lambda_k}\right) dt \qquad 1 \leqslant i \leqslant p \tag{3.6}$$

Proof. The mapping $s \mapsto \sqrt{s^+}$ is continuous on $\mathscr{S}_p$, so S_t exists up to its explosion time (Ikeda and Watanabe,[12] Theorem 2.3, p. 159). Furthermore, as for a large enough K, we have

$$\|\sqrt{s^+}\|^2 + \|\alpha I\|^2 \leqslant |\alpha|^2 + \|s\| \leqslant K(1 + \|s\|^2)$$

this explosion time is a.s. infinite (Ikeda and Watanabe,[12] Theorem 2.4, p. 163). Relation (3.6) can be shown in the same way as (3.3) (see Ref. 4), using

$$(ds_{ij})(ds_{kl}) = (s_{ik}^{(+)}\delta_{jl} + s_{il}^{(+)}\delta_{jk} + s_{jk}^{(+)}\delta_{il} + s_{jl}^{(+)}\delta_{ik})\, dt \qquad (s_{ij}^{(+)}) = S^+ \tag{3.7}$$

Proof of Theorem 2 when $\alpha \in]p-1, p+1[$ or when $\alpha \geqslant p+1$ and $\lambda_p(0) = 0$. We shall show that for a given solution $(\lambda_1(t),..., \lambda_p(t))$ of the stochastic differential system (3.6), if we consider the $(p-1)$ largest eigenvalues as parameters, the stochastic differential equation verified by the smallest eigenvalue λ_p

$$d\lambda_p = 2\sqrt{\lambda_p^+}\, dv_p + \left(\alpha + \sum_{k \neq p} \frac{\lambda_p^+ + \lambda_k^+}{\lambda_p - \lambda_k}\right) dt$$

has a unique strong solution (Proposition 3) and hence λ_p stays positive if initially positive (Proposition 4). The other eigenvalues of S_t: $\lambda_1,..., \lambda_{p-1}$ must then be positive and all solution of (3.5) with $s_0 \in \hat{\mathscr{S}}_p^+$ is solution of (3.2). Uniqueness (in the sense of probability law) is then a natural consequence of Theorem 3.

Proposition 3. Let $(\lambda_1(t),..., \lambda_p(t))$ be a solution of (3.6) with initial state $\lambda_1(0) > \cdots > \lambda_p(0) \geqslant 0$. For the given $\mathbb{R}^{p-1}$ valued process $(\lambda_1(t),..., \lambda_{p-1}(t))_{t \geqslant 0}$ satisfying

$$\begin{cases} d\lambda_i = 2\sqrt{\lambda_i^+}\, dv_i + \left(\alpha + \sum_{k \neq i} \dfrac{\lambda_i^+ + \lambda_k^+}{\lambda_i - \lambda_k}\right) dt \qquad 1 \leqslant i \leqslant p-1 \\ \lambda_1(0) > \cdots > \lambda_{p-1}(0) > 0 \end{cases} \tag{3.6'}$$

the stochastic differential equation

$$d\lambda_p = 2\sqrt{\lambda_p^+}\, dv_p + \left(\alpha + \sum_{k=1}^{p-1} \frac{\lambda_p^+ + \lambda_k^+}{\lambda_p - \lambda_k}\right) dt \qquad \lambda_p(0) \geqslant 0 \qquad (3.6'')$$

has a unique strong solution.

Proof. Following a method due to H. Doss and E. Lenglart,[7] we consider in (3.6″) $\lambda_1,\ldots,\lambda_{p-1}$ as parameters and show that the process $(\lambda_p(t))_{t\geqslant 0}$, solution of (3.6″) is unique. Indeed let $\lambda_p(t)$ and $\lambda'_p(t)$ be two solutions of (3.6″) for a same Brownian vector $(v_1(t),\ldots, v_p(t))$ and a same given system $(\lambda_1(t),\ldots, \lambda_{p-1}(t))$. After suitable localization, the Yamada–Watanabe (Ref. 12, p. 168) theorem shows that

$$\lambda_p(t) = \lambda'_p(t) \text{ a.s.} \qquad \text{if} \quad t \leqslant \tau = \inf\{s\colon \lambda_i(s) = \lambda_j(s) \text{ for some } (i, j)\}$$

But $\tau = +\infty$ a.s. because, as studied for $\alpha \geqslant p+1$, if one $[0, \tau[$ $\lambda_1(t) > \cdots > \lambda_p(t)$ are the eigenvalues of a solution S_t of (3.5)

$$\mathbb{U}(\lambda_1(t),\ldots, \lambda_p(t)) = \sum_{i<j} \ln(\lambda_i(t) - \lambda_j(t))$$

is a local Martingale. Indeed, by Itô calculus, we can show that (Ref. 3, p. 218)

$$\text{drift } d\mathbb{U}_t = \sum_{i<k<j} \frac{(\lambda_j^+\lambda_j - \lambda_i^+\lambda_i) + (\lambda_i^+\lambda_k - \lambda_k^+\lambda_i) + (\lambda_j^+\lambda_k - \lambda_k^+\lambda_j)}{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_i)}\, dt$$

In the following Proposition 4 we shall remark that up to time τ all the eigenvalues of S_t are positive, so

$$\text{drift } d\mathbb{U}_t = -\sum_{i<k<j} \frac{\lambda_i + \lambda_j}{(\lambda_j - \lambda_k)(\lambda_i - \lambda_k)}\, dt = 0$$

and the argument of McKean (Ref. 20, (p. 47) can be applied here again.

The proof of Proposition 3 will be complete if we show that the process $\lambda_t = (\lambda_1(t),\ldots, \lambda_p(t))$ remains positive if initially positive. □

Remark 5. We can see this fact on the stochastic differential equation governing the smallest eigenvalue λ_p:

$$d\lambda_p = 2\sqrt{\lambda_p^+}\, dv_p + \left(\alpha + \sum_{k\neq p} \frac{\lambda_p^+ + \lambda_k^+}{\lambda_p - \lambda_k}\right) dt$$

If at time $t_0 \geqslant 0$, $\lambda_p(t_0) = 0$, at time $t_0 + \Delta t$, we have $\lambda_p(t_0 + \Delta t) = (\alpha - p + 1)\,\Delta t$, so if $\alpha > p - 1$, at time $t_0 + \Delta t$, $\lambda_p(t_0 + \Delta t) > 0$. This naive presentation is proved correct in the following:

Proposition 4. If $(\lambda_1(t),..., \lambda_p(t))$ is solution of (3.5) with initial state $\lambda_1(0) > \cdots > \lambda_p(0 \geqslant 0$, and $\alpha > p-1$, we have

(i) $\forall t > 0, \lambda_p(t) \geqslant 0$ a.s.

(ii) $\{t: \lambda_p(t) = 0\}$ is of Lebesgue measure zero

Proof. (i) We show this in the same way as the comparison theorem which is given by Doss and Lenglart (Ref. 7, p. 197). Let

$$g(t, \omega, x) \\ = \left[\left(\alpha + \sum_{k=1}^{p-1} \frac{x^+ + \lambda_k^+(t, \omega)}{x - \lambda_k(t, \omega)}\right) 1_{\{x>0\}} + (\alpha - p + 1) 1_{\{x \leqslant 0\}}\right] 1_{\{\lambda_{p-1}(t,\omega) \neq x\}}$$

If $t < \tau$, we have

$$g(t, \omega, \lambda_p(\omega, t)) = \left(\alpha + \sum_{k=1}^{p-1} \frac{\lambda_p^+ + \lambda_k^+}{\lambda_p - \lambda_k}\right) 1_{\{\lambda_p > 0\}} + (\alpha - p + 1) 1_{\{\lambda_p \leqslant 0\}}$$

After localization, as in the proof of Proposition 3, we can define

$$\tilde{\lambda}_p(t) = \lambda_p(0) + 2 \int_0^t \sqrt{\tilde{\lambda}_p^+(s)}\, dv_p(s) + \int_0^t g(s, \tilde{\lambda}_p(s))\, ds$$

where $v_p(t)$ is the pth component of the Brownian vector $(v_1(t),..., v_p(t))$ given in (3.6). As in Ref. 7, we deduce

$$d\tilde{\lambda}_p = 2\sqrt{\tilde{\lambda}_p^+}\, dv_p + \left(\alpha + \sum_{k \neq p} \frac{\tilde{\lambda}_p^+ + \tilde{\lambda}_k^+}{\tilde{\lambda}_p - \tilde{\lambda}_k}\right) dt$$

whose solution for a given Brownian motion v_p and an initial value $\lambda_p(0) \geqslant 0$ is a.s. pathwise unique (Proposition 3). So on $[0, \tau[$ the unique solution of (3.6″) is positive and the noncollision demonstration of Proposition 4 is valid: For all $t \geqslant 0$, $\lambda_p(t) \geqslant 0$ a.s.

(ii) The fact that $\{t: \lambda_p(t) = 0\}$ is of Lebesgue measure zero is like in (i) nothing else than a generalization of the second part of Theorem 5 presented on p. 198 in Doss and Lenglart.[7]

This ends the proof of Proposition 4. □

Theorem 2 then follows from Propositions 3 and 4, as for $\alpha > p-1$ any solution of (3.5) is a solution of (3.2). Unicity is a trivial consequence of Theorem 3 below. □

Remark 6. It would be interesting to show that the stochastic differential equation (3.2) has a unique strong solution when $\alpha \in]p-1, p+1[$.

To prove such a result we could establish a matrix form of the Yamada theorem (Ikeda and Watanabe,[12] Theorem 3-2, p. 168). For instance, the stochastic differential equation

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt \qquad X_0 \in \mathscr{S}_p$$

has a unique strong solution if b is Lipschitz-continuous and σ Hölder-continuous with exponent 1/2 in $\mathscr{S}_p^+$. Such a result would considerably shorten the previous construction. We do not know presently if such a "matrix Yamada theorem" is true, but nevertheless it is easy to show that the mapping $A \to \sqrt{A}$ is Hölder-continuous of exponent 1/2 in $\mathscr{S}_p^+$: In fact for every integer p, $\|A-B\|^2 \leqslant \|A^2 - B^2\|$, if $A, B \in \mathscr{S}_p^+$, where the norm of A is the largest absolute value of its eigenvalues.

Distributions

We characterize here the distribution of S_t when t is fixed. We obtain, by adapting standard martingale methods (Ikeda and Watanabe,[12] p. 222), an extension of the classical noncentral Wishart laws (e.g., Barra,[2] Chap. 8, No. 10).

Theorem 3. Let B_t be a $p \times p$ Brownian matrix; whenever the stochastic differential equation

$$dS_t = \sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t} + \alpha I\, dt \qquad S_0 = s_0$$

has a solution in $\mathscr{S}_p^+$, the distribution of S_t, for fixed t, is given by its Laplace transform:

$$E_{s_0}[\exp - \operatorname{tr} uS_t] = (\det(I + 2tu))^{-\alpha/2} \exp[-\operatorname{tr}(s_0(I+2tu)^{-1}u)))] \qquad (3.8)$$

for all $p \times p$ symmetric matrix $u \in \mathscr{S}_p^+$.

Proof. Let us write

$$\Delta_t = \det(I + 2tu) \qquad W_t = (I+2tu)^{-1}u \qquad V(t, s) = \operatorname{tr} sW_t$$

$$v(t, s) = \Delta_t^{-\alpha/2} \cdot e^{-V(t,s)}$$

□

Lemma 1. If L_α is the operator defined in (3.4) by

$$L_\alpha = \operatorname{tr}(\alpha D + 2xD^2) \qquad D = \left(\frac{\partial}{\partial x_{ij}}\right)$$

then we have $\partial v/\partial t = L_\alpha v$.

If we assume this lemma, then for all fixed t_1, $U(t, S_t) = v(t_1 - t, S_t)$ is a solution of $(\partial U/\partial t) + L_\alpha U = 0$ so U is a Martingale on $[0, t_1]$. Hence $E_{s_0}[v(0, S_{t_1})] = v(t_1, s_0)$, and the result follows.

Proof. To prove the lemma we just need to observe that

$$\frac{\Delta_t'}{\Delta_t} = \operatorname{tr}\left[(I + 2tu)^{-1}\frac{d}{dt}(I + 2tu)\right] = 2\operatorname{tr} W_t \qquad W_t' = -2W_t^2$$

so that $\partial v/\partial t = v(-\alpha \operatorname{tr} W_t + 2\operatorname{tr} sW_t^2)$.

On the other hand for all $i, j, k, l \in \{1,..., p\}$,

$$\frac{\partial v}{\partial s_{ij}} = -v\frac{\partial V}{\partial s_{ij}} \qquad \frac{\partial^2 v}{\partial s_{ij}\,\partial s_{kl}} = v\frac{\partial V}{\partial s_{ij}}\cdot\frac{\partial V}{\partial s_{kl}} \qquad \text{and} \qquad DV = W_t$$

so that $L_\alpha v = \alpha \operatorname{tr}(Dv) + 2\operatorname{tr}(sDv^2) = -v\alpha \operatorname{tr}(DV) + 2v \operatorname{tr} s(DV)^2$. □

Remark. This theorem confirms the uniqueness in law of solutions of (3.2), and gives a simple probabilistic proof of the following result: If $\alpha \in \{1,..., p-1\} \cup\,]p-1, +\infty[$ and t is fixed, the mapping defined on $\mathscr{S}_p^+$ [see (3.8)] is the Laplace transform of a probability distribution on $\mathscr{S}_p^+$; this result does not seem to figure explicitly in the classical statistical literature, apart from the case where α is an integer. S_t has then a non-central Wishart distribution (Barra,[2] p. 123).

Lemma 2. Let $\mathscr{C}_b(\mathscr{S}_p^+, \mathbb{R})$ be the set of all real, bounded, continuous functions defined on $\mathscr{S}_p^+$, and $\Phi = \{\phi_u : s \mapsto \operatorname{tr} -us/u \in \mathscr{S}_p\}$. The family $(P_t)_{t \geqslant 0}$ of operators defined on $\mathscr{C}_b(\mathscr{S}_p^+, \mathbb{R})$ whose restriction on Φ is given by

$$P_t\phi_u(s) = (\det(I + 2tu))^{-\alpha/2} \exp -\operatorname{tr}[s(I + 2tu)^{-1}u]$$

is a Feller semigroup with generator

$$L_\alpha = \operatorname{tr}(\alpha D + 2sD^2)$$

Proof. Straightforward. □

Study of the Case When $\alpha > p - 1$ and s_0 is Any Element in $\mathscr{S}_p^+$

We suppose here that $\lambda_1(0) \geqslant \cdots \geqslant \lambda_p(0) \geqslant 0$ and we propose to show that even in this case (3.2) has a solution.

We consider a sequence $(s_n)_{n \geqslant 0}$ in $\hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+$ (which means that $s_n > 0$ and has all its eigenvalues distinct) such that $s_n \to s_0$. The first part of the proof of Theorem 2 shows that for each n we have a diffusion $(S_t^n)_{t \geqslant 0}$, which is a solution of (3.2) with initial value s_n. If μ_n is the probability law on $\mathscr{C}_b(\mathscr{S}_p^+, \mathbb{R})$ of the process $(S_t^n)_{t \geqslant 0}$, its finite distributions $\Pi_{t_1,\ldots,t_k}\mu_n$ defined by their Laplace transforms

$$\Pi_{t_1,\ldots,t_k}\hat{\mu}_n(u_1,\ldots, u_k) = E\left[\exp -\operatorname{tr} \sum_i u_i S_{t_i}^n\right] \qquad (u_1,\ldots, u_k) \in (\mathscr{S}_p^+)^k$$

have a limit $\phi_{s_0}^{t_1,\ldots,t_k}(u_1,\ldots, u_k)$ which is the Laplace transform of the finite distribution $\Pi_{t_1,\ldots,t_k}\mu_0$ of a probability law μ_0. For this ϕ_{s_0}, the family $(P_t)_{t \geqslant 0}$ defined by

$$P_t \phi_u(s) = \phi_{s_0}^t(u) = (\det(I + 2tu))^{-\alpha/2} \exp -\operatorname{tr}[s_0(I + 2tu)^{-1}u] \qquad t > 0 \qquad u \in \mathscr{S}_p^+$$

is, according to Lemma 2, a Feller semigroup with generator L_α. Let $(S_t)_{t \geqslant 0}$ be a Markov process with initial value s_0 and probability law μ_0. It is natural to expect (Williams,[27] Theorem 28, p. 137) this process to be a diffusion. This can be seen by sticking together solutions of the stochastic differential equation (3.2).

More precisely, if (λ_i^s) are the eigenvalues of s, the set

$$\mathscr{S}_p^+ - \hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+ = \{s \in \mathscr{S}_p^+ : \lambda_p^s = 0, \text{ or } \exists (i \neq j), \lambda_i^s = \lambda_j^s\}$$

is of Lebesgue measure zero, as for all fixed t, the distribution of S_t has a density (this can be seen, for example, by adapting Ref. 13, p. 176) which, when $s_0 = 0$ has the following explicit form (see Refs. 2, 10, and 13):

$$\pi_t \mu_0(s, s + ds) = P_0\{S_t \in (s, s + ds)\} = K_t (\det s)^{\gamma - 1} e^{-\operatorname{tr} s/t}\, ds \qquad \gamma = (\alpha - p + 1)/2$$

So if $\alpha \geqslant p + 1$, as $s \in \hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+$, the processes $(S_t^n)_{t \geqslant 0}$ are in $\hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+$; hence for all $t > 0$, $P_{s_0}[S_t \in \hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+] = 1$. Takinig $t_0 > 0$ as the initial time, we have $\lambda_1(t_0) > \cdots > \lambda_p(t_0) > 0$ and we can apply the first part of the proof of Theorem 2 which ensures, for a given Brownian matrix $(B_t)_{t \geqslant 0}$, a unique strong solution of Eq. (3.2) on $[t_0, +\infty[$ taking its values in $\hat{\mathscr{S}}_p^+ \cap \tilde{\mathscr{S}}_p^+$. This solution S_t is a Markov process so, if we use this argument again with initial value $t_1 \in]0, t_0[$, by sticking together the solution of (3.2) on $[t_1, +\infty[$ with the former, they have the same restrictions on $[t_0, +\infty[$, and we can go on with this argument for $t_2 \in]0, t_1[,\ldots, t_n \in]0, t_{n-1}[$, etc., where $t_n \searrow 0$; we finally find a continuous version of S_t on $[0, +\infty[$ having s_0 for initial state, which is the unique solution (in law) of (3.2).

If $\alpha \in]p - 1, p + 1[$, the argument is the same, if we just notice that, $(\lambda_1^n(t),\ldots, \lambda_p^n(t))_{t \geqslant 0}$ being the eigenvalues of $(S_t^n)_{t \geqslant 0}$, for all $\varepsilon > 0$, and for all n, $N_\varepsilon^n = \{t \in]0, \varepsilon[: \lambda_p^n(t) = 0\}$ is of Lebesgue measure zero (cf. Proposition 4), hence for all $\varepsilon > 0$ there is a $t_0 \in]0, \varepsilon[$ such that S_t has all its eigenvalues distinct and >0 a.s.

The diffusion (S_t) solution of (3.2) on $[t_0, +\infty[$ is built as previously; then for $\varepsilon \searrow 0$ we obtain the desired solution □

Remark 8. Let (S_t) be a WIS$(\alpha, p, 0)$ diffusion with $\alpha > p-1$, the law of S_t when t is fixed, is a member of "the exponential Wishart families" defined by Letac[16]:

$$P_0\{S_t \in (s, s+ds)\} = K_t (\det s)^{\gamma - 1} e^{-\operatorname{tr} s/t}\, ds \qquad \gamma = (\alpha - p + 1)/2$$

This gives a simple probabilistic interpretation of these densities which did not seem to be known up to now (see Ref. 16, p. 76).

4. SOME GENERAL RESULTS

If S_t is a Wishart process with parameters (α, p, s_0), we have the following:

If $\alpha > p+1$, $(\det S_t)^{(p+1-\alpha)/2}$ is a local Martingale on $\mathbb{R}_+$.

If $\alpha \geqslant p+1$, $d(\det S_t) = 2 \det S_t \sqrt{\operatorname{tr} S_t^{-1}}\, dv_t + (\alpha - p + 1) \det S_t \cdot \operatorname{tr} S_t^{-1}\, dt$

and $d(\ln \det S_t) = 2\sqrt{\operatorname{tr} S_t^{-1}}\, dv_t + (\alpha - p - 1) \operatorname{tr} S_t^{-1}\, dt$ (4.1)

The stochastic differential equation (3.3) shows that all the relations given in Sec. 2 [Eqs. (2.5), (2.7), (2.8), and (2.9)] are also true when α is not an integer, as long as the Wishart process (S_t) exists.

In particular if x is a vector of $\mathbb{R}^p$ with euclidean norm 1, $x^T S x$ is then a BESQ$(\alpha, x^T S_0 x)$ Bessel process, and if u is a $p \times p$ symmetric determinist matrix

$$d(\operatorname{tr} u S_t) = 2\sqrt{\operatorname{tr} u^2 S_t}\, dv_t + \alpha \operatorname{tr} u\, dt \tag{4.2}$$

Additivity Property

If (S_t) and (Σ_t) are two independent Wishart process WIS(α, p, s_0) and WIS(β, p, σ_0), respectively, then $(S_t + \Sigma_t)$ is a Wishart process WIS$(\alpha + \beta, p, s_0 + \sigma_0)$. (4.3)

Proof. This property can be proved with the help of a representation theorem [cf. (5.10)] consequence of the classical representation theorem of semi-Martingales (Ikeda and Watanabe,[12] p. 90, with $d = p^2$).

Remark 9. If S_t is generated by L_α, and if we fix time t, the Laplace transform of S_t is given by (3.8). Now let $s_0 \to 0$ in $\mathscr{S}_p^+$; Lévy's theorem would show that $(\det(I+2tu))^{-\alpha/2}$ is the Laplace transform of a probability law on $\mathscr{S}_p^+$, which is not possible if α is not an integer; indeed, as shown in the Gindikin[11,16] theorem, for $s_0=0$ this expression can be the Laplace transform of a probability distribution on $\mathscr{S}_p^+$ only if α is an integer (Gindikin[11] and Letac[16]). So we have the following:

For all $s_0 \in \mathscr{S}_p^+$ and all $\alpha \in \mathbb{R}$, the following results are equivalent:

(i) $\alpha \in \Delta_p = \{1,..., p-1\} \cup]p-1, +\infty[$.

(ii) There exists a unique diffusion in $\mathscr{S}_p^+$ with initial state $s_0 \in \mathscr{S}_p^+$, which is a solution of the stochastic differential equation

$$dS_t = \sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t} + \alpha I\, dt$$

where $(B_t)_{t \geqslant 0}$ is a Brownian matrix.

(iii) There exists a unique diffusion in $\mathscr{S}_p^+$ with initial state $s_0 \in \mathscr{S}_p^+$, whose differential operator is $L_\alpha = \operatorname{tr}(\alpha D + 2sD^2)$.

If $\alpha = 1$, $p=2$, and $s_0 = 0$, a classical result of P. Lévy[17] shows that the distribution of S_t is indecomposable. Indeed if there were Wishart processes with $0 < \alpha < 1$, the additivity property would make that distribution infinitely divisible.

Let us now give another approach of Theorem 3 by adapting methods considered by Pitman and Yor.[22]

Let μ be a Radon measure in $\mathscr{S}_p^+$ with compact support. If $(S_t)_{t \geqslant 0}$ and $(\Sigma_t)_{t \geqslant 0}$ are two independent Wishart processes with parameters (α, p, s_0) and (β, p, s_0), respectively, the mapping $\mathbb{R} \times \mathscr{S}_p^+ \to \mathbb{R}$, $(\alpha, s_0) \mapsto \phi_\mu(\alpha, s_0) = E[\exp - \operatorname{tr} \int S_t \mu(dt)]$ is measurable, and the additivity property (4.3) gives

$$\phi_\mu(\alpha + \beta, s + \sigma) = \phi_\mu(\alpha, s)\, \phi_\mu(\beta, \sigma) \tag{4.4}$$

Hence there exists a real constant A_μ and a $p \times p$ symmetric matrix V_μ such that

$$E_{s_0}\left[\exp - \operatorname{tr} \int S_t \mu(dt)\right] = A_\mu^\alpha \exp \operatorname{tr}[s_0 V_\mu] \tag{4.5}$$

We propose to find A_μ and V_μ for some specific μ. As in Pitman and Yor,[22] it is easy to show the following:

Proposition 5. If $\Phi: \mathbb{R}_+ \to \bar{\mathscr{S}}_p^+$ is continuous, constant on $[t, +\infty[$, and such that its right derivative (in the distribution sense) $\Phi'_d: \mathbb{R}_+ \to \mathscr{S}_p^-$

is continuous, with $\Phi(0)=I$, and $\Phi'_d(t)=0$, then for every Wishart process $S_t \in \mathrm{WIS}(\alpha, p, s_0)$ we have

$$E_{s_0}\left[\exp -\tfrac{1}{2}\operatorname{tr}\int_0^t \Phi''_s \Phi_s^{-1} S_s\, ds\right] = (\det \Phi_t)^{\alpha/2} \exp \tfrac{1}{2}\operatorname{tr}(s_0 \Phi_0^+) \tag{4.6}$$

where

$$\Phi_0^+ = \lim_{t \searrow 0} \Phi'_t$$

From the special case where δ_t is the dirac measure at point t, and

$$\Phi''_s \Phi_s^{-1}\, ds = v 1_{[0,t[}(s)\, ds + w\delta_t(s)$$

we deduce the following:

Matrix Cameron-Martin Formula

For $(S_t) \in \mathrm{WIS}(\alpha, p, s_0)$, $v \in \tilde{\mathscr{S}}_p^+$ and $w \in \mathscr{S}_p^+$ we have

$$E_{s_0}\left[\exp -\tfrac{1}{2}\operatorname{tr}\left(wS_t + \int_0^t vS_s\, ds\right)\right] = (\det \sqrt{v^{-1}}(\sqrt{v}\operatorname{ch}\sqrt{v}t + w \operatorname{sh}\sqrt{v}t)^{-\alpha/2}$$
$$\cdot \exp -\tfrac{1}{2}\operatorname{tr}[s_0\sqrt{v}(\sqrt{v}\operatorname{ch}\sqrt{v}t + w\operatorname{sh}\sqrt{v}t)^{-1}(\sqrt{v}\operatorname{sh}\sqrt{v}t + w\operatorname{ch}\sqrt{v}t)] \tag{4.7}$$

In particular (1) for $t=1$, $w=0$, $u \in \mathscr{S}_p^+$ we find the (nearly) usual formula

$$E_{s_0}\left[\exp -\tfrac{1}{2}\int_0^1 \operatorname{tr} uS_s\, ds\right] = (\det \operatorname{ch}\sqrt{u})^{-\alpha/2} \exp -\tfrac{1}{2}\operatorname{tr}[s_0\sqrt{u}\operatorname{th}\sqrt{u}] \tag{4.8}$$

and (2) for $v=0$ we find as announced the Laplace transform (3.8).

5. FIVE-PARAMETER WISHART PROCESSES; SQUARE ORNSTEIN–UHLENBECK PROCESSES

5.1. Real Case

Whereas the $\mathrm{WIS}(\alpha, p, s_0)$ are generalizations of squares of Brownian matrices $\Sigma_t = B_t^T B_t$, we now study the same type of generalizations for squares of Ornstein–Uhlenbeck matrices (Kendall[15]), denoted by $\mathrm{WIS}(\alpha, \beta, \gamma, p, s_0)$. Let X_t be an $n \times p$ matrix diffusion solution of the stochastic differential equation

$$dX_t = \gamma\, dN_t + \beta X_t\, dt \qquad X_0 = x_0 \tag{5.1}$$

where N_t is an $n \times p$ Brownian matrix, x_0 is an $n \times p$ determinist matrix, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}_-$. Let $S_t = X_t^T X_t$, $s_0 = x_0^T x_0$; then $dB_t = \sqrt{S_t^{-1}}\, X_t^T\, dN_t$ is clearly a $p \times p$ Brownian matrix and (S_t) solves the stochastic differential equation

$$dS_t = \gamma(\sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t}) + 2\beta S_t\, dt + n\gamma^2 I\, dt \qquad S_0 = s_0 \tag{5.2}$$

With the help of a generalized time-change formula we can deduce from Secs. 2 and 3 the study of such processes (S_t) where the real α takes the place of n.

Time-Change Formula. (See Pitman and Yor[22] for the real case.) We have the following:

If (X_t) is a solultion of (5.1), and $s_0 = x_0^T x_0$, there exists $(\Sigma_t) \in$ WIS(α, p, s_0) such that

$$S_t = X_t^T X_t = e^{2\beta t} \Sigma\left(\gamma^2 \frac{1 - e^{-2\beta t}}{2\beta}\right) \tag{5.3}$$

This time-change formula yields the following:

Theorem 2′. If $(B_t)_{t \geq 0}$ is a $p \times p$ Brownian matrix, then for all $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $s_0 \in \mathcal{S}_p^+$ with distinct eigenvalues labeled $\lambda_1(0) > \cdots > \lambda_p(0) \geq 0$, the stochastic differential equation

$$dS_t = \gamma(\sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t}) + 2\beta S_t\, dt + \alpha\gamma^2 I\, dt \tag{5.4}$$

has (1) a unique solution in $\mathcal{S}_p$ (in the sense of probability law) if $\alpha \in \,]p-1, p+1[$, and (2) a unique strong solution in $\tilde{\mathcal{S}}_p^+$ if $\alpha \geq p+1$.

The eigenvalues of such a solution never collide: a.s. for all $t > 0$

$$\lambda_1(t) > \cdots > \lambda_p(t) \geq 0, \qquad \lambda_p(t) > 0 \quad \text{if } \alpha \geq p+1$$

and satisfy the stochastic differential system

$$d\lambda_i = 2\sqrt{\lambda_i}\, dv_i + \alpha\gamma^2\, dt + 2\beta\lambda_i\, dt + \gamma^2 \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}\, dt \tag{5.5}$$

where $v_1(t),..., v_p(t)$ are independent Brownian motions.

Remarks 10. (a) When α is an integer and $\gamma = 1$ this last result has been established by W. Kendall[15] in shape theory.

(b) If $\alpha \in \{1,..., p-1\}$ the same results hold for the α first eigenvalues.

Proof of (5.5). This relation is a consequence of (5.3). $S_t \in \hat{\mathcal{S}}_p^+$ being symmetric can be diagonalized via an orthogonal matrix of eigenvectors H_t which, as well as the eigenvalues Λ_t^S and Δ_t^Σ of $S_t \in \mathrm{WIS}(\alpha, \beta, \gamma, p, s_0)$ and $\Sigma_t \in \mathrm{WIS}(\alpha, p, s_0)$, respectively, can be chosen to be semi-Martingales as long as these eigenvalues do not collide. Obviously, we have

$$\Lambda_t^S = e^{2\beta t} H_t \Sigma_{\phi_t} H_t \qquad H_t^T H_t = H_t H_t^T = I$$

so $H_t \Sigma_{\phi_t} H_t$ is diagonal and hence nothing else then the matrix of eigenvalues of Σ after the time-change: $\Lambda_t^S = e^{2\beta t} \Lambda^\Sigma(\phi_t)$. □

5.2. Matrix Case

Let us now replace β and α by $p \times p$ matrices $b = (b_{ij})$, and $a = (a_{ij})$. If (X_t) is governed by the stochastic differential equation

$$dX_t = dN_t a + X_t b\, dt \qquad X_0 = x_0 \tag{5.6}$$

where (N_t) is an $n \times p$ Brownian matrix; let $S_t = X_t^T X_t$, $s_0 = x_0^T x_0$, and

$$dB_t = \sqrt{S_t^{-1}}\, X_t^T\, dN_t a(\sqrt{a^T a})^{-1}$$

(B_t) is a $p \times p$ Brownian matrix and (S_t) is a solution of

$$dS_t = \sqrt{S_t}\, dB_t \sqrt{a^T a} + \sqrt{a^T a}\, dB_t^T \sqrt{S_t} + (bS_t + S_t b)\, dt + n a^T a\, dt \quad S_0 = s_0 \tag{5.7}$$

In order not to lengthen this paragraph we just give a very particular and simple case.

Theorem 2″. If $\alpha \in \Delta_p = \{1,..., p-1\} \cup]p-1, +\infty[$, a is in the group $\mathrm{GL}(p)$, $b \in \mathcal{S}_p^-$, s_0 is in $\mathcal{S}_p^+$ and has all its eigenvalues distinct, and (B_t) is a $p \times p$ Brownian matrix, then on $[0, \tau[$ (τ first time of collision), the stochastic differential equation

$$dS_t = \sqrt{S_t}\, dB_t \sqrt{a^T a} + \sqrt{a^T a}\, dB_t^T \sqrt{S_t} + (bS_t + S_t b)\, dt + \alpha \sqrt{a^T a}\, dt \qquad S_0 = s_0 \tag{5.8}$$

has a unique solution if b and $\sqrt{a^T a}$ commute.

Remark 11. The eigenvalues of such a solution S_t (if they are not null) are solutions on $[0, \tau[$ of the stochastic differential equation

$$d\lambda_i = \sqrt{\lambda_i (H^T a H)_{ii}}\, dv_i + 2\lambda_i (H^T b H)_{ii} + \alpha (H^T a H)_{ii}\, dt$$
$$+ \sum_{k \neq i} \frac{\lambda_i (H^T a H)_{kk} + \lambda_k (H^T a H)_{ii}}{\lambda_i - \lambda_k}\, dt \tag{5.9}$$

where $v_1(t),..., v_p(t)$ are independent Brownian motions, and H_t is for $t < \tau$ a continuous orthogonal semi-Martingale matrix of eigenvectors of S_t:

$$\Lambda_t = H_t^T S_t H_t \qquad H_t^T H_t = H_t H_t^T = I$$

Definition 1″. A matrix process of $\mathscr{S}_p^+$, governed by (5.8) [resp. Eq. (5.4)] and such that $S_0 = s_0$, is called a Wishart process with index α, dimension p, initial state s_0, and matrix parameters b and a [resp. real parameters β and γ], and is denoted by $\mathrm{WIS}(\alpha, b, a, p, s_0)$ [resp. $\mathrm{WIS}(\alpha, \beta, \gamma, p, s_0)$].

5.3. Some General Results

Characterization of the WIS(α, b, a, p, s_0) *Processes*

The two following results are equivalent:

(i) S_t is solution of (5.7).

(ii) M_t is a local Martingale and

$$\begin{cases} dS_t = dM_t + (bS_t + S_t b)\, dt + \alpha a^T a\, dt \\ (ds_{ij})(ds_{kl}) = (s_{ik}(a^T a)_{jl} + s_{il}(a^T a)_{jk} + s_{jk}(a^T a)_{il} + s_{jl}(a^T a)_{ik})\, dt \end{cases} \tag{5.10}$$

Denoting $D = (\partial / \partial x_{ij})$, such processes have the following:

Differential Operator

$$\text{In the real case } L_\alpha^{\beta,\gamma} = \mathrm{tr}[2\beta x D + \alpha \gamma^2 D + 2\gamma^2 x D^2] \tag{5.11}$$

$$\text{In the matrix case } L_\alpha^{b,a} = \mathrm{tr}((bx + xb + \alpha a^T a) D + 2x D a^T a D) \tag{5.12}$$

Trace

$$d\, \mathrm{tr}\, S_t = 2\sqrt{\mathrm{tr}\, a^T a S_t}\, dv_t + (2\, \mathrm{tr}\, bS_t + \alpha\, \mathrm{tr}\, a^T a)\, dt \tag{5.13}$$

Determinant

$$d \det S_t = 2 \det S_t \sqrt{\operatorname{tr} a^T a S_t^{-1}}\, dv_t$$
$$+ \det S_t((\alpha - p + 1) \operatorname{tr} a^T a S_t^{-1} + 2 \operatorname{tr} b)\, dt \quad (5.14)$$

$$d(\det S_t^{(p+1-\alpha)/2}) = (p + 1 - \alpha) \det S_t^{(p+1-\alpha)/2} \sqrt{\operatorname{tr} a^T a S_t^{-1}}\, dv_t$$
$$+ (p + 1 - \alpha) \det S_t^{(p+1-\alpha)/2} \operatorname{tr} b\, dt \quad (5.15)$$

$$d(\ln \det S_t) = 2\sqrt{\operatorname{tr} a^T a S_t^{-1}}\, dv_t$$
$$+ ((\alpha - p - 1) \operatorname{tr} a^T a S_t^{-1} + 2 \operatorname{tr} b)\, dt \quad (5.16)$$

5.4. Proof of Theorem 2″.

Part 1. If $b = 0$, we have the following:

(i) For $\alpha \geqslant p + 1$, the argument of Theorem 2 can be applied here as $s_0 \in \mathscr{S}_p^+$. The stochastic differential equation

$$dS_t = \sqrt{S_t}\, dB_t \sqrt{a^T a} + \sqrt{a^T a}\, dB_t^T \sqrt{S_t} + \alpha a^T a\, dt \qquad S_0 = s_0 \quad (5.17)$$

has a unique strong solution on $[0, \tau_0[$ (τ_0 first hitting time of 0 of $\det S_t$). Equalities (5.14) and (5.15) written with $b = 0$ show, as in Theorem 2, that when $\alpha \geqslant p + 1$ we have $\tau_0 = +\infty$ a.s.

Remark 12. $\mathbb{V}(t) = \sum_{i,j} \ln(\lambda_i - \lambda_j)$ is a local Martingale if

$$\sum_{k \neq i} \frac{\lambda_i (H^T a H)_{kk} + \lambda_k (H^T a H)_{ii}}{\lambda_i - \lambda_k} = 0$$

But this latter equality is not trivial, so the argument showing that the eigenvalues do not collide cannot be directly applied here.

(ii) If $\alpha \in]p - 1, p + 1[$, as $s_0 \in \mathscr{S}_p^+$ and has all its eigenvalues distinct, it is clear by Theorem 2 that

$$d\Sigma_t = \sqrt{\Sigma_t}\, dN_t + dN_t^T \sqrt{\Sigma_t} + \alpha I\, dt \qquad \Sigma_0 = \sigma_0 = (a^T)^{-1} s_0 a^{-1} \in \mathscr{S}_p^+$$

has a unique solution. Applyinig (5.10) to $S_t = a^T \Sigma_t a$, there exists a Brownian matrix B_t such that:

$$dS_t = \sqrt{S_t}\, dB_t \sqrt{a^T a} + \sqrt{a^T a}\, dB_t^T \sqrt{S_t} + \alpha a^T a\, dt \qquad S_0 = s_0$$

Remark 13. The class $\mathrm{WIS}(\alpha, a, p, s_0)$ of all those processes is hence invariant under transformations T_a: $x \mapsto a^T \Sigma_t a$, $a \in \mathrm{Gl}(p)$. For fixed t, the law of S_t is given by its Laplace transform:

$$\begin{aligned} E_{s_0}[\exp - \operatorname{tr} uS_t] &= E_{(a^T)^{-1} s_0 a^{-1}}[\exp - \operatorname{tr}(ua^T \Sigma_t a] \\ &= \det(I + 2tua^T a)^{-\alpha/2} \\ &\quad \cdot \exp - \operatorname{tr}[s_0(I + 2tua^T a)^{-1} u] \end{aligned} \tag{5.18}$$

which is nothing else than (5.21), which follows, when $b = 0$.

G. Letac[16] shows that the Wishart exponential families (corresponding to $s_0 = 0$ and t fixed) are invariant by T_a and are essentially the only ones to be so among the exponential families.

Part 2. If $b \neq 0$, the proof requires the

Girsanov transformation: If $\mathscr{C}(\mathbb{R}_+, \tilde{\mathscr{S}}_p^+)$ is the set of all continuous functions of $\tilde{\mathscr{S}}_p^+$, and if ${}^pW_{s_0}^{\alpha, a}$ is the law of $(\Sigma_t) \in \mathrm{WIS}(\alpha, a, p, s_0)$ on $\mathscr{C}(\mathbb{R}_+, \tilde{\mathscr{S}}_p^+)$, then the law ${}^pW_{s_0}^{\alpha, a, b}$ of $(S_t) \in \mathrm{WIS}(\alpha, b, a, p, s_0)$ is equivalent to ${}^pW_{s_0}^{\alpha, a}$ (Pitman and Yor,[22] p. 455, for the real case).

$$\begin{aligned} L_t^b = \frac{d^p W_{s_0}^{\alpha, a, b}}{d^p W_{s_0}^{\alpha, a}} &= (\det e^{bt})^{-\alpha/2} \cdot \exp - \tfrac{1}{2} \operatorname{tr}[s_0 (a^T a)^{-1} b] \\ &\quad \cdot \exp - \tfrac{1}{2} \operatorname{tr}\left[-(a^T a)^{-1} \left[bS_t + \int_0^t b^2 S_s \, ds \right] \right] \end{aligned} \tag{5.19}$$

End of the Proof of Theorem 2″. A solution of (5.17) after change of drift gives a solution of

$$dS_t = \sqrt{S_t} \, dB_t \sqrt{a^T a} + \sqrt{a^T a} \, dB_t^T \sqrt{S_t} + (bS_t + S_t b) \, dt + \alpha a^T a \, dt \qquad S_0 = s_0$$

Unicity in law is the immediate consequence of (5.22) which follows. □

The characterization (5.10) of the $\mathrm{WIS}(\alpha, b, a, p, s_0)$ processes gives

the *Additivity Property.* If $(S_t) \in \mathrm{WIS}(\alpha, b, a, p, \sigma_0)$ are two independent Wishart processes, then $(S_t + \Sigma_t) \in \mathrm{WIS}(\alpha + \alpha', b, a, p, s_0 + \sigma_0)$.

These generalized Wishart processes are thus additive. If $p = 1$, it is shown (Shiga and Watanabe[25]) that they are essentially the only ones to be so.

Matrix Cameron–Martin Formula. If $(S_t) \in \mathrm{WIS}(\alpha, b, a, p, s_0)$ where $\alpha \in \Delta_p = \{1, ..., p-1\} \cup]p-1, +\infty[$, $a \in \mathrm{GP}(p)$, $b \in \mathscr{S}_p^-$ such that b and $\sqrt{a^T a}$ commute (resp. $\beta \in \mathbb{R}_-$, $\gamma \in \mathbb{R}$), s_0, v, $w \in \mathscr{S}_p^+$, we have the following:

In the matrix case:

$$E_{s_0}\left[\exp - \tfrac{1}{2}\operatorname{tr}\left(wS_t + \int_0^t vS_s\,ds\right)\right]$$
$$= (\det[e^{bt}\sqrt{a^Tav+b^2}^{-1}(\sqrt{a^Tav+b^2}\operatorname{ch}\sqrt{a^Tav+b^2}t$$
$$+ (a^Taw-b)\operatorname{sh}\sqrt{a^Tav+b^2}t)])^{-\alpha/2}$$
$$\cdot\exp - \tfrac{1}{2}\operatorname{tr}[s_0(a^Ta)^{-1}[b+\sqrt{a^Tav+b^2}(\sqrt{a^Tav+b^2}\operatorname{ch}\sqrt{a^Tav+b^2}t$$
$$+ (a^Taw-b)\operatorname{sh}\sqrt{a^Tav+b^2}t)^{-1}(\sqrt{a^Tav+b^2}\operatorname{sh}\sqrt{a^Tav+b^2}t$$
$$+ (a^Taw-b)\operatorname{ch}\sqrt{a^Tav+b^2}t)]] \tag{5.20}$$

In the real case:

$$E_{s_0}\left[\exp - \frac{1}{2}\operatorname{tr}\left(wS_t + \int_0^t vS_s\,ds\right)\right]$$
$$= (\det[e^{\beta t}\sqrt{\gamma^2v+\beta^2I}^{-1}(\sqrt{\gamma^2v+\beta^2I}\operatorname{ch}\sqrt{\gamma^2v+\beta^2I}t$$
$$+ (\gamma^2w-\beta I)\operatorname{sh}\sqrt{\gamma^2v+\beta^2I}t)])^{-\alpha/2}$$
$$\cdot\exp - \frac{1}{2\gamma^2}\ \operatorname{tr} s_0[\beta I+\sqrt{\gamma^2v+\beta^2I}(\sqrt{\gamma^2v+\beta^2I}\operatorname{ch}\sqrt{\gamma^2v+\beta^2I}t$$
$$+ (\gamma^2w-\beta I)\operatorname{sh}\sqrt{\gamma^2v+\beta^2I}t)^{-1}(\sqrt{\gamma^2v+\beta^2I}\operatorname{sh}\sqrt{\gamma^2v+\beta^3I}t$$
$$+ (\gamma^2w-\beta I)\operatorname{ch}\sqrt{\gamma^2v+\beta^2I}t))]] \tag{5.21}$$

The following is a particular case of (5.20).

Laplace Transform. If $(S_t)\in \mathrm{WIS}(\alpha, b, a, p, s_0)$ (resp. $(S_t)\in \mathrm{WIS}(\alpha, \beta, \gamma, p, s_0)$) where $\alpha\in\Delta_p$, $a\in\mathrm{GLP}(p)$, $b\in\tilde{\mathscr{S}}_p^-$ commutes with $\sqrt{a^Ta}$, $s_0\in\tilde{\mathscr{S}}_p^+$, $u\in\mathscr{S}_p^+$ (resp. $\beta\in\mathbb{R}_-$, $\gamma\in\mathbb{R}$), we have the following:
In the matrix case:

$$E_{s_0}[\exp\ -\operatorname{tr} uS_t] = (\det b^{-1}(b-ua^Ta+ua^Tae^{2bt}))^{-\alpha/2}$$
$$\cdot\exp\ -\operatorname{tr}[e^{bt}s_0e^{bt}b(b-ua^Ta+ua^Tae^{2bt})^{-1}u] \tag{5.22}$$

In the real case:

$$E_{s_0}[\exp\ -\operatorname{tr} uS_t] = \left(\det\left(\frac{\beta I-\gamma^2u+e^{2\beta t}\gamma^2u}{\beta}\right)\right)^{-\alpha/2}$$
$$\cdot\exp[-\beta e^{2\beta t}\operatorname{tr}(s_0(\beta I-\gamma^2u+e^{2\beta t}\gamma^2u)^{-1}u)] \tag{5.23}$$

Remark 14. With $\gamma=1$ and $\beta\to 0$, we find (3.8).

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