Thesis submitted for the degree of Doctor of Philosophy

Uncertainty in Sustainability and Economic Decisions

Yu Sun

Contents

Abstract

This dissertation analyses the uncertainties in economic decisions, and consists of six related chapters on sustainable growth and market competition.

The first chapter discusses the long-term policy under the threat of earthquakes. In specific, policy makers need tools to decide optimally on the economic trajectories that will maximize the society's welfare. Tools should be flexible and account for the consequences of earthquakes, incorporating the best estimate of their frequency and intensity. In this regard, this paper presents a modelling strategy that combines optimal control techniques and Bayesian learning: after an earthquake occurs, policy makers can improve their knowledge and adjust policies optimally. Some numerical examples illustrate the advantages of our modelling strategy along different dimensions. While Japan symbolizes the policy maker who has learned from earthquakes protecting the economy accordingly; Italy underlines the importance of prevention capital. China shows the hidden dangers of its extraordinary economic growth. Finally, the Chinese region of Yunan puts forward the roles of learning and of political independence.

The second chapter is an operational research and generalization of the model proposed in the first essay. We propose a predictive control model under great uncertainty. A multi-stage optimal control approach with predetermined switching time is presented. We apply this approach to the issue of global warming with an additional important environmental variable: catastrophes, which are correlated with global warming. The catastrophe learning process is described under both myopic agent and non-myopic agent assumptions. We prove the existence of the solution. In particular, the analytical solutions of parameters of Bayesian learning process are obtained.

In the third chapter, we analyse a stochastic endogenous growth model with an externality –through health degradation – from production-induced pollutants. Our model extends the one dimensional stochastic growth model of Bretschger and Vinogradova (2016) to two dimensions, where the correlated uncertainties in both capital accumulation and health regeneration are analysed. We propose an analytical model with flexible environmental policy not only at global level, but also with the advantage of being disaggregated by country. Our central results are two simple formulas for the optimal abatement policy (or, equivalently, the optimal carbon tax) and optimal growth rate. We demonstrate that the relationship between the abatement policy and growth rate is inverted-U shaped, when the correlated uncertainties include both capital and health regeneration processes. We also find that the optimal abatement policy is particularly sensitive to the assumptions regarding the elasticity of marginal utility to consumption and health. Our quantitative result suggests a higher carbon tax than the well-known estimates in literature.

The fourth chapter introduces the fat-tailed uncertainty to the stochastic capital and health demand model proposed in the third essay. In specific, we propose in this essay a stochastic growth model by introducing two kinds of catastrophic shocks: natural disasters and epidemics, driven by two independent Poisson processes. We provide the necessary and sufficient condition for the existence and uniqueness of the equilibrium. Our central results are simple closed-form formulas for the optimal abatement policy (or, equivalently, for the optimal carbon tax) and growth rate. The innovative message from our quantitative analysis is that optimal abatement policy reacts sensitively to the parameters of health and catastrophe intensity. Our quantitative result shows catastrophic shock leads to a higher carbon tax.

The fifth chapter proposes a analytical DSGE that allows the economy grows at a “healthy” rate when taking into account of two production externalities: carbon dioxide and pollutant. In this regard, we generalize Golosov et al. 2014 model with extension to a health demand framework. In contrast to the vast majority of analytical DSGE, our economic growth rate is endogenous and optimal carbon tax formula is very simple to derive and apply. The impact of parameters on economic variables are discussed. In particular, our model indicates that adjustment of pollution tax results in proportional adjustment of Carbon tax at a constant rate.

The sixth chapter discusses the uncertainty and heterogeneity in a life-cycle market competition, which includes both theoretical and empirical analysis.

Firms face uncertainty and the heterogeneity in the life-cycle of destination when they venture into foreign markets. We model firms with switching costs and market uncertainty in a two-period stochastic model. On one hand, switching costs motivate firms to lock consumers in period 1 by setting a price lower than the so-called conventional optimal price, which optimizes firms' current profit and ignore the future profitability (i.e. the "locked-in" effect). On the other hand, in potential high-growth market, the future profitability depends mainly on the market expansion, rather than the locked-in consumers. Thus, firms are less interested in locking current consumers at the cost of lower current profit. This is defined as "market development" effect, which, in contrast to the "locked-in" effect, motivates firms to set an equilibrium price close to the conventional optimal price. The weight of market potential in future profitably depends on the market foresight under uncertainty.

The empirical part is to analyse the strategy of family firms' internationalization using the heterogeneous market uncertainty model. Despite the conventional view on family firms who are slow to venturing into foreign markets, Hennart et al. (J Int Bus Stud, 2019) show that family firms in high-quality niche business are able to overcome the internationalization barriers. Eddleston et al. (J Int Bus Stud, 2019) argue that the effect only conditionally holds when external and internal contexts are considered, specified as pro-market development and professionalization practices respectively. We extend Hennart et al.'s (2019) and Eddleston et al.'s (2019) research by explicitly considering the heterogeneity in the life-cycle of destination markets. Specifically, we demonstrate that family firms, no matter selling niche or mass products, are encouraged to internationalize given satisfactory market growth potential – an external context, as defined in Eddlelston et al. (2019). We develop a two-period competition model with logistic market growth to assess the role of the life cycle of export markets on the decision to entry. Empirical evidence shows that family firms are more likely to enter markets with high growth potential in their early stages of development.

0. Abstract

1 Catastrophe, Bayesian learning and optimal control I

1.1 Introduction

Major natural catastrophes like tsunamis, volcano eruptions or earthquakes entail significant human and economic losses that modify national budget constraints, productive endowments and as a consequence, economic decisions. On the 26th December 2004, 10 countries along the Indian Ocean coast were hit by an earthquake of intensity 9.3 on the Richter scale and by a subsequent tsunami. There was no warning system to inform villagers that a tsunami was approaching and 270,000 people were killed, more than 500,000 were injured and 1,8 million people were left homeless (Davis, 2008). According to the Consultative Group on Indonesia (CGI, 2006), main infrastructures remained largely intact but small constructions like houses, clinics, schools were destroyed explaining the high death-toll. Damage is usually explained by the poor building techniques, unadapted to earthquake activity and the low quality of the materials. We believe that another key reason is the lack of tools to build optimal economic policies in time, which would take into account the arrival and consequences of earthquakes in a wide framework. For this reason, this paper aims to enable policy makers with a tool to maximize the society well-being while optimally preparing the economy with adequate investment. Our modeling framework encompasses features from classic models in economic growth, together with Bayesian and adaptive learning.

There is a relative recent and growing interest on the analysis of natural catastrophes and their effects on economic growth, both in the short and the long-term. The term natural catastrophe englobes different phenomena that vary in causes and in economic and human consequences. As shown in Loayza et al. (2012), earthquakes are the natural catastrophe generating the largest economic losses; whereas droughts affect the most people. Nevertheless, some general results have been established that apply to all natural catastrophes. Developing countries suffer on average more than developed countries (Noy, 2009), economic losses are lesser for countries with higher education and greater trade openness (Toya and Skidmore, 2007) and more intense disasters produce larger negative economic impact (Fornby et al., 2013; Hochrainer, 2009; Noy, 2009; Stephens, 2007). Nevertheless, results are not unanimous regarding long-term consequences. A first group of papers defend that natural catastrophes could enhance growth in the long-run à la Schumpeter. That is, natural catastrophes should foster capital replacement with new, more efficient capital, which stimulates growth. In Jaramillo (2009), low income countries enjoy a medium term increase in GDP growth, which fades away after some years, only if the event has low incidence. Nevertheless, most papers find short and medium term losses and that economies return to their growth path with time.[1]

There is also a growing number of theoretical models that highlight the role of natural catastrophes on economic growth. They allow to analyse the linkages between the natural catastrophe and the productive factors, technological progress, adaptation and mitigation policies, the role of the policy maker and public investment. There exist computable general equilibrium models, which have addressed these questions at a national scale like Shibusawa and Miyata (2011) for Japan, and at a regional level like Rose and Liao (2005). In this paper, we adopt a classical view in economic growth building a model à la Ramsey, focusing on the effects of earthquakes in the short and long terms. The closest to our approach is the NEDyM model built in the spirit of Solow (1956), developed and utilised in Hallegate and Ghil (2008) and Hallegate and Dumas (2009), for instance. Hallegate and Ghil (2008) show that natural catastrophes have a deeper effect on economies when they are booming since all resources are fully exploited. If on the contrary, the economy is at a recession, it can reallocate unexploited resources fast and in a efficient manner. Hallegate and Dumas (2009) explore whether the increase in investment that follows a natural catastrophe has an effect in the long-term. They show that since reconstruction investment comes at least partly from other national budgets, R&D for instance, economic growth could rather slow down.

[1] Among the papers supporting creative destruction à la Schumpeter, let us mention Albala-Bertrand (1993), Stewart and Fitzgerald (2001), Okuyama (2003) and Benson and Clay (2004). Then, Raddatz (2007), Anttila-Hughes and Hsiang (2013), Ströbl (2011), or Felbermayr and Gröschl (2014) find short and medium term losses, while Chhibber and Laajaj (2008) and Klomp and Valckx (2014) find that economies will return to their original growth path.

The issues of earthquake prevention and long-term optimal decision making have not been considered together so far in the literature. Nevertheless, as argued above, it seems urgent to search for models, which at the same time are flexible, comprehensive and easy to implement. Let us underline three of the main features of the present model. The main novelty and the key characteristic of our framework is the introduction of Bayesian adaptive learning. The policy maker has an estimation of the earthquake frequency and the associated damages, and she takes her optimal decisions accordingly. Standing policies are suspended upon each realisation of the earthquake and knowledge is updated. Then, the policy maker recomputes new optimal policies with revised, more accurate information on the expected frequency and intensity of earthquakes. Note that with time, the policy maker can learn the true distribution of the earthquake's frequency and intensity. Second, the original stochastic policy-making problem is transformed into a deterministic problem in which the policy maker expects earthquakes to arrive at the expected frequency and to hit the economy with the expected intensity. Although our model loses on instantaneous prediction precision, it does not lose in its capacity to build long-term policies. Moreover, our modelling strategy is simple to implement and it provides an accurate description of earthquakes with time. The third novelty is the modelization of prevention capital, which encompasses all accumulated resources that protect effectively the economy against earthquakes. Prevention capital protects the economy against earthquakes, diminishes instantaneous damage after an earthquake and shortens recovery times. Its dynamics are known to policy makers, who can invest to enhance prevention or neglect it; privileging consumption and the accumulation of physical capital. In the literature, the term adaptation capital is commonly used (Ewing et al., 2003; Palecki et al., 2001). As in Ewing et al. (2003), we understand that prevention capital includes social capital, as the procedures acquired by the population to face natural catastrophes, emergency plans (ex-ante warning alerts and ex-post evacuation and aid plans), secure sewage and lifelines, the adaptation of buildings, bridges, roads, and urban planning. Noteworthy, a parallel can be established between spending in defence and in earthquake prevention. Like spending in earthquake prevention, defence enhances security, promoting productivity and economic growth (Smith, 1776). This thesis has been widely tested and corroborated ever since, see, for instance, Aizenman and Glick (2006), Cuaresma and Reitschuler (2006) or Dunne and Perlo-Freeman (2003).

Differently from other papers in the field of natural catastrophes, our policy maker takes into consideration the future occurrence of earthquakes using an estimation of its frequency and intensity. These qualities permit the policy maker to elaborate plans considering earthquakes while using at every moment the most updated information possible. There is varied evidence about policy makers' learning. Kalkstein and Greene (1997) find that Southern States in the US are at less risk from heat waves than Northern States, due to their longer experience. According to Palacki et al. (2011), adaptation measures count with increased policy support when frequency increases. In particular, they focus on consecutive heat waves in St. Louis, Missouri and Chicago, finding that mortality was reduced between the heat waves of 1995 and 1999. From a theoretical perspective, different learning mechanisms can be considered ranging from the business-as-usual policy maker who does not include the natural catastrophe in his planning program, to the policy maker who uses the most updated information available averaging all registers. In the mid-point, there are myopic policy makers who neglect all past information and prudent policy makers, who incorporate new information but do not assign the same importance to new event as to old registered information (see Bréchet et al., 2011, and Bréchet et al., 2014, for two applications).

This paper is structured as follows. Section 1.2 makes a brief introduction to earthquakes. Section 1.3 presents a model for optimal inter temporal decision making for an economy frequently hit by earthquakes. Optimal decisions depend then on the evolution of physical and prevention capitals, both suffering from earthquakes. Beforehand, we devote a subsection to the choice of the damage function, which depends here on the catastrophe intensity and time elapsed since last occurrence, and it includes a recovery function. Both damage and recovery depend crucially on prevention. Section 1.4 closes this analysis in developing numerically some case studies.

1.2 A brief introduction to earthquakes

An earthquake is the detectable tremor of the Earth surface. The moving plates that cover the earth find opposition in the neighbouring plates, generating frictions, until one of the plates gives in. Then, the amassed energy is released producing an earthquake immediately.[2]

[2] Although here we only consider earthquakes having a natural origin, mine blasts and nuclear tests can also generate earthquakes.

Large earthquakes induce damages which include human life losses, infrastructure and building collapse, fires, soil liquefaction, landslides, avalanches, floods and tsunamies. According to its strength, earthquakes intensity is measured using two main scales: the Mercali and the Richter scales. In this paper, the Richter scale is privileged (see Table 1.5 in the Appendix for details). Earthquake intensity in the Richter scale ranges from 1 to 9, and for every unit increases in the scale, released energy increases roughly thirty-fold. According to the U.S. Geological Survey Earthquake Hazards Team, and as shown in Table 1.6 in the Appendix, there are between 15,000 and 32,000 earthquakes worldwide every year, all intensities together. Minor earthquakes that do not entail human nor economic losses, occur almost constantly in places like California, Alaska, El Salvador, Mexico, Turkey or Japan, but they can occur everywhere. Regarding high intensity earthquakes, there has been only one earthquake of magnitude larger than 8 from 2009 to 2011, and 2 in 2012.

Table 1.1 below lists the ten most intense earthquakes in history, showing the economic and human losses involved. The second largest earthquake hit Alaska in 1964, but despite its intensity it did not entail large economic losses, and only 131 people lost their lives. Several reasons can explain this somehow lucky outcome. First, the earthquake happened a Good Friday so that public buildings and schools were closed. Second, Alaska has a low population density. Indeed, according to Pipkin and Trent (1997), population density in Alaska in 1964 was less than one person per square kilometer. In the same line, the 1952 Kamchatka earthquake, the 5th most intense in history, did not take human lives and economic losses were relatively low. The Honshu earthquake in 2011 had the same intensity as the Kamchatka earthquake. In contrast to the Russian experience, 15,703 people died in Honshu and economic losses exceeded 300 billion U.S. dollars, and this despite the preparedness of the Japanese population and government.

It seems clear then that some countries would benefit from including earthquakes in their long-term economic decision making if they aim at maximising the country's welfare. This would require precise knowledge on earthquake arrival times, on generated instantaneous damage, recovery times but also on the dynamics of prevention. All these depend in turn

[3]Based on National Centers for Environmental Information of NOAA (U.S.). Retrieved from http://earthquake.usgs.gov/earthquakes/world/10_largest_world.php

No.	Location	Date	Magnitude	Economic Loss	Fatalities
1	Chile	22.05.1960	9.5	$550 million	1,655
2	Alaska	28.03.1964	9.2	$311 million	131
3	Northern Sumatra	26.12.2004	9.1	$10 billion	227,898
4	Honshu, Japan	11.03.2011	9.0	$309 billion	15,703
5	Kamchatka, Russia	04.11.1952	9.0	$800,000 - $1,000,000	0
6	Maule, Chile	27.02.2010	8.8	$30 billion	523
7	Ecuador-Colombia	31.01.1906	8.8	n.a.	500-1500
8	Rat Islands, Alaska	04.02.1965	8.7	$10,000	n.a.
9	Northern Sumatra	28.03.2005	8.6	n.a.	1,000
10	Assam, Tibet	15.08.1950	8.6	n.a.	780

Table 1.1 – *10 Largest Earthquakes in History*[3]

on the country's geographical situation, population and industrial distributions, on its administration as well as on its sensitivity to earthquakes. The following examples help us illustrate the idiosyncratic nature of earthquakes and their damage, which should be respected in this type of framework. Using data from the U.S. National Oceanographic and Atmospheric Administration (NOAA), Figure 1.1 displays the number of earthquakes by intensity and damage categories suffered in Japan, Italy, China and the Chinese region of Yunnan as from 1900.[4] Note that although Italy has suffered less earthquakes overall and of a lower intensity than Japan, damage is relatively higher.

Looking at Japan, Figure 1.2 highlights that most earthquakes are without effect, underlining that the country has been implementing effective prevention policies for long. In the numerical exercises of section 1.4, the roles of learning and earthquake sensitivity together with prevention capital and technology are examined for these two countries. Most earthquakes in China and Yunan are of intensity 7 and 8. Although the distribution of earthquake losses is similar for China and Yunan (Figure 1.2), their conse-

[4]The damage scale is produced by the NOAA and has 4 levels. An earthquake of level 0 has no damage. A level 1 earthquake has an economic damage of less than 1 million U.S. dollars. Level 2 earthquakes are labelled moderate and they induce a loss ranging from 1 to 5 million U.S. dollars. Level 3 earthquakes are severe and induce losses from 5 to 24 million U.S. dollars. Level 4 earthquakes are extreme and generated losses are larger than 25 million dollars.

quences for welfare and growth are quite different. In this regard, section 1.4 analyses the welfare impact of learning, of having an adequate level of adaptation and prevention capital, taking into account the attributes of each economy, including economic characteristics and population growth, among others.

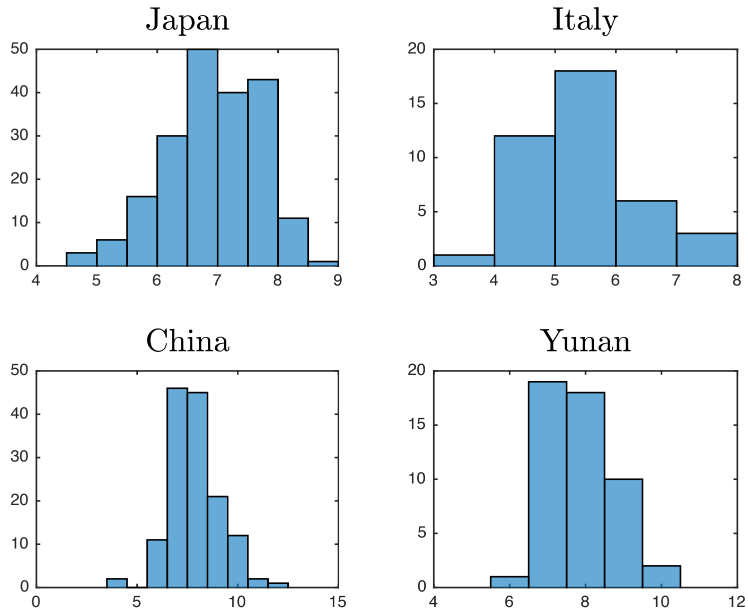


Figure 1.1 – Number of Earthquakes by intensity. Source: NOAA.

Knowing the significant damage effect of earthquakes, the next question is how to introduce earthquakes into a long-term decision making model. Hence, we need to make assumptions on the nature and behaviour of earthquakes. In this regard, let us highlight that although earthquakes have been analysed for more than 130 years, earthquake prediction has been fruitless (Mulargia et al., 2017), that is, it is difficult to forecast the occurrence of earthquakes in the medium or long-term. Nevertheless, there exist two well-established laws governing earthquakes. First, the Omari law (1894) forecasts the decay of earthquake aftermath. Second, the Gutenberg-Richter (1956) law establishes a relationship between the total number of earthquakes and the intensity threshold. According to this law, the more intense the earthquake, the less total numbers (and vice versa), while the relationship between each intensity and its associated number of earthquakes are not clear. Regarding the modelling of earthquake frequency, there are different approaches. Here, we follow Kagan and Jackson (1994) and Kagan (2002). Using the Harvard catalog on earthquakes, the authors estimate long-term worldwide earthquake probabilities assuming that the distribution of earthquakes in time follows a Poisson process. Like theirs, our estimates are not earthquake predictions to be used as warnings, but as guides for economic policy making. In contrast to earthquake frequency, earthquake intensity cannot be predicted nor estimated. As a consequence, in this paper, the policy maker will adapt to the probability distribution of earthquakes of the average of observed intensities.

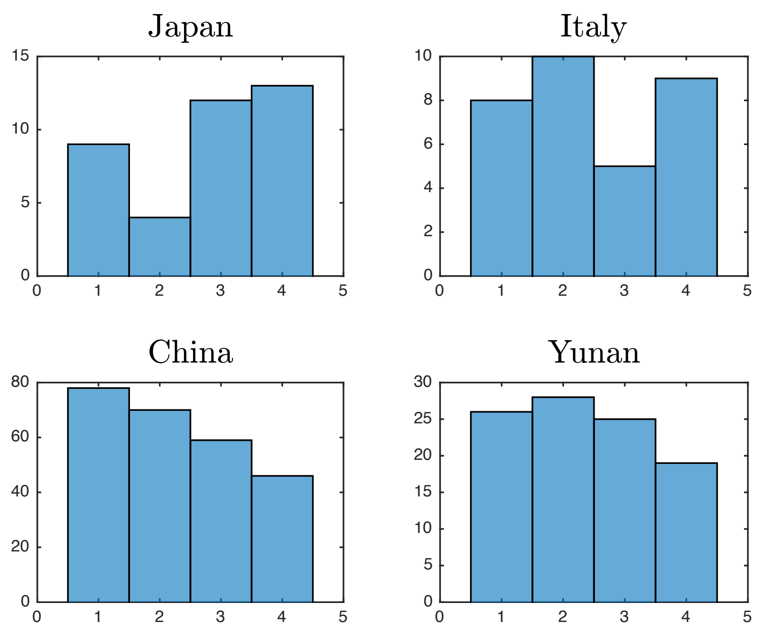


Figure 1.2 – Earthquake Intensity Histogram. Source: NOAA.

1.3 Inter temporal decision making under earthquake threat

We develop a theoretical model to address the problem of a policy maker who faces intense earthquakes frequently. Although the economy may have suffered from earthquakes for a long time, the policy maker knowledge or awareness may be insufficient to provide an accurate stochastic description of the earthquake frequency, its intensity and the associated economic damage.

The remaining of this section is structured as follows. After a brief introduction to damage functions, section 1.3.1 presents our choices for the instantaneous damage and recovery time functions. Then, section 1.3.2 introduces a simple version of Bayesian learning tailored to our framework. Finally, section 1.3.3 ensures the existence of an optimal solution to the policy maker's problem and provides a set of necessary optimal conditions.

1.3.1 The damage function

The literature on economic losses generated by disasters is vast and ranges from applications of Input-Output methodologies (see Cochrane, 1974, Kawashima and Kauch, 1990), econometrics (West and Lenze, 1994), to computable general equilibrium models (Rose et al., 1997). Other papers aim at producing accurate estimates of immediate economic and material losses. Focusing on earthquake damages, Brookshire et al. (1997) and Kircher et al. (2006) forecast the economic short-term impact of earthquakes depending on their intensity and the region's building inventory, infrastructure exposure, relative vulnerability of built environment to ground shaking and also socio-economic wealth associated to exposed assets. Chan et al. (1998), Dunbar et al. (2002) or Chen et al. (2001) predict economic losses depending on earthquake intensity. Regarding damage modelisation, this paper relies mainly on Jaiswal and Wald (2011), who provide a loss function to evaluate immediate economic losses after significant earthquakes worldwide.

Jaiswal and Wald (2011) design a loss ratio, r, defined as the ratio between direct economic loss to total economic exposure. One of the many advantages of this loss ratio is that the same functional form fits all sensitive regions. An earthquake of intensity s between 5.0 and 9.0 on the Richter scale generates an immediate loss of

$$r(s) = \phi\left[\frac{1}{\beta}\ln\left(\frac{s}{\chi}\right)\right]. \tag{1.1}$$

ϕ is the normal cumulative distribution function with free parameters χ and β. It suffices then to estimate the regional values of β and χ to compute the estimated loss ratio worldwide.[5]

In the current context of long-term decision making under the frequent arrival of earthquakes, the damage function also needs to account for delayed effects. Models forecasting damage generated by earthquakes do not usually consider recovery time simply because most models are static. However, recovery time itself is important because it determines overall economic losses. In as far as recovery depends at least partially on political decisions, it seems best suited a model with optimizing decision makers, who consider at the same time the economy and its structure, the earthquake random nature, its effects, recovery times and how these can be mitigated with adequate investment. There is debate on whether natural catastrophes induce long term gains or losses. The seminal paper of Albala-Bertrand (1993), predicted long-run gains after a natural catastrophe, led by an upgrade in capita *à* la Schumpeter. Old capital would be replaced by new, technologically more advanced capital. Other papers followed this line, like Stewart and Fitzgerald (2001), Okuyama (2003), Benson and Clay (2004), Jaramillo (2009) and Cuñado and Ferreira (2011). Note that this result is mainly driven by developing countries. Nevertheless, conclusions are nuanced when the origin of the solicited public investments is considered as in Hallegate and Dumas (2009). Indeed, the urgent investment in new capital may proceed from cuts in other budgets, like R&D. Underlining the effect of intense hurricanes on the US from 1970 to 2005, Ströbl (2011) proves that economic growth can be reduced by 0.93 percentage points on average. Regarding tropical storms, Antilla, Hughes and Hsiang (2011) also find persistent losses of 6.7% several years after the storm. Analysing natural catastrophes in general, Raddatz (2007) finds that climate disasters are responsible for an average loss of 2% of GDP, a year after the event. According to Noy (2009), sudden-set disasters can make lose a 9% of GDP in developing countries. An intertemporal damage function should then incorporate a recovery period leading to one of the following three possible scenarios: long-run losses, recovery and long-run gains as depicted in Figure 1.3. In the case studies developed in section 1.4, we consider that countries go back to their pre-catastrophe economic situation after a lapse of time.

Two novel additional features are added to the original loss function in

[5]Note that χ can be larger or smaller than s. If $\frac{s}{\chi} < 1$, then its logarithm is negative but it results in positive damage given that ϕ is the normal cumulative distribution and it is positive definite.

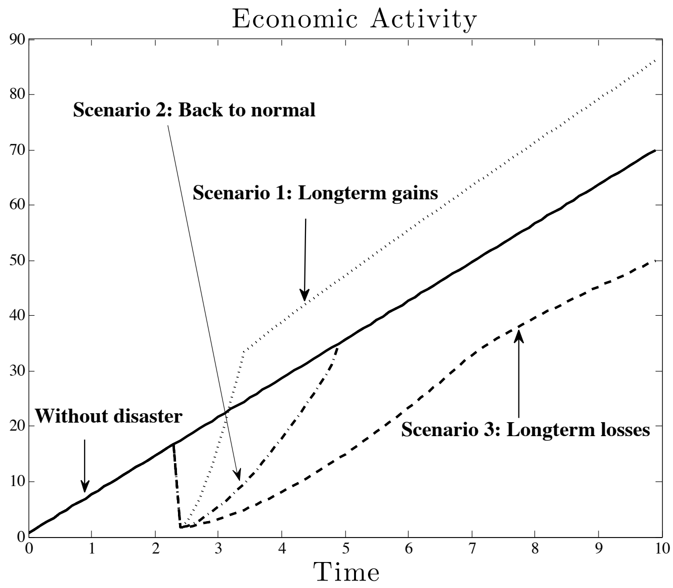


Figure 1.3 – Recovery scenarios

Jaiswal and Wald (2011), namely investment in prevention and recovery time. First, the policy maker can invest in prevention, to protect the economy, decrease short-run losses and shorten the recovery period. In this regard, the functions describing damage and recovery do depend on prevention capital. Second, the loss function includes a recovery function g, which depends on prevention capital and on time elapsed since last earthquake occurrence. If the last earthquake occurred at t', the loss function at time t is

$$\varphi(b, \mu, t, t') = g(b, t, t')r(\mu) = g(b, t - t')\phi \left[\frac{1}{\beta} \ln \left(\frac{\mu}{\chi} \right) \right]. \qquad (1.2)$$

Function g decreases in b so that the larger prevention capital, the lower the loss. In the limit, g converges asymptotically to zero when b tends to infinite. φ also depends time elapsed since last catastrophe and it could also encompass population density, and physical capital age.

1.3.2 Bayesian learning on earthquakes

Here earthquakes are considered as independent realizations of a random variable, with unknown probability distribution. At a given date, the policy maker solves the optimal control problem with the available information on the random variable frequency and intensity. This means that at time 0, the policy maker expects a realization of the variable every τ_0 years with expected intensity μ_0, and she incorporates the consequences to the problem. When the variable realizes, the policy maker actualises her expectations and solves again the problem. In this section we describe first Bayesian learning when the stochastic frequency of a variable follows a Poisson distribution. Then, we apply the obtained results to our particular problem. In particular, we underline the role of learning and information updating.

We assume that earthquake frequency follows a Poisson process for point processes as in Woo (1999) or Coles (2001) and as applied in Nogaj et al (2006). Indeed, as shown in Khintchine (1960), the superposition of independent processes, in which no process dominates, follows a Poisson probability distribution. The probability there will be q occurrences in a time period is

$$p(q \mid \gamma) = \frac{\gamma^q}{q!} e^{-\gamma},$$

when the average number of occurrences in the same period of time is γ.[6] Suppose that the average number of occurrences of a stochastic variable is unknown, that the time interval $[0, T]$ is divided in n subperiods and that the aim is to compute the probability that there will be q_1 occurrences in the first subperiod, q_2 in the second subperiod, until q_n in the last subperiod. Bayesian learning can be applied here since the probability that the average number of occurrences is γ, when there have been $\{q_1, q_2, ..., q_n\}$ occurrences in the last n subperiods, equals the product of the probability of observing γ occurrences on average over $[0, T]$, and the probability of observing q_i in subperiod i for all i, when the true average is γ, that is:

$$p(\gamma \mid q_1, \ldots, q_n) \propto p(q_1, \ldots, q_n \mid \gamma) \cdot p(\gamma). \tag{1.3}$$

$p(\gamma)$ and $p(\gamma \mid q_1, \ldots, q_n)$ are called respectively prior and posterior probability density function. Here it is assumed that $p(q_1, \ldots, q_n \mid \gamma)$ follows a

[6] We use here the term occurrence for a realisation of the random variable. In our context, an occurrence is the realisation of an earthquake.

Poisson distribution, which implies that its associated prior $p(\gamma)$ follows a Gamma distribution $\gamma \sim \Gamma(1,1)$, that is

$$p(\gamma) = e^{-\gamma}, \tag{1.4}$$

for any $\gamma > 0$. As a result, the expected value and variance of γ are $E(\gamma) = 1$ and $V(\gamma) = 1$. Hence, substituting (1.4) into (1.3)

$$p(\gamma \mid q_1, \ldots, q_n) \propto \gamma^{\sum_{i=1}^{n} q_i} e^{-(1+n)\gamma}. \tag{1.5}$$

Therefore, when $\{q_1, q_2, ..., q_n\}$ occurrences have been observed in the previous n subperiods, the number of occurrences follows a Gamma distribution $\Gamma(\alpha^*, \beta^*)$, with

$$\alpha^* = \sum_{i=1}^{n} q_i, \quad \text{and} \quad \beta^* = 1 + n.$$

The associated conditional mean is

$$\mathrm{E}(\gamma \mid q_1, \ldots, q_n) = \frac{1 + \sum_{i=1}^{n} q_i}{1+n},$$

and the corresponding conditional frequency is

$$\Upsilon = \frac{1}{\mathrm{E}(\gamma \mid q_1, \ldots, q_n)} = \frac{1+n}{1 + \sum_{i=1}^{n} q_i}.$$

Policy makers can be differentiated depending on their learning mechanism. Myopic decision makers actualise their beliefs on earthquake intensity using uniquely the most recent information. Others disregard current information using a conservative scheme, where they do not actualise their beliefs. These policy makers consider recent occurrences as non-significant accidents. Cautious decision makers actualize using a weighted average of past beliefs and current information. Only when all earthquake occurrences are equally weighted, the policy maker beliefs do converge to the true distribution in the long-run applying the law of large numbers.

Therefore, we shall assume that the policy maker approximates the true number of earthquake occurrences in a period of time by the average number of occurrences $\frac{\sum_{i=1}^{n} q_i}{n}$, with corresponding frequency is $n / \sum_{i=1}^{n} q_i$. Then, as time passes and the number of observations increases, the average of the number of occurrences during a given period converges to the expected value of the number of occurrences, and so estimated frequency also converges to expected frequency. We solve next the inter-temporal decision problem of a policy maker who faces earthquakes frequently and who learns about them. To underline the importance of learning, we consider some scenarios in section 1.4, where the policy maker does not update her beliefs on earthquakes' intensity nor frequency.

1.3.3 The policy maker's problem

Suppose a risk neutral policy maker who maximizes overall discounted welfare over a finite time period in a one-sector economy made of homogeneous individuals. For simplicity, we assume population grows at a constant rate $n \in \mathbb{R}$. On a given date, welfare depends on the amount of the unique final good consumed and it is measured by a standard utility function U, which is a positive, increasing and concave function:

$$U(\cdot) \geq 0, \;\; U'(\cdot) \geq 0, \;\; U''(\cdot) < 0.$$

Additionally, U satisfies the Inada conditions:

$$\lim_{C \longrightarrow 0} U'(C) = \infty, \;\; \text{and} \;\; \lim_{C \longrightarrow \infty} U'(C) = 0.$$

The objective of the policy maker is to choose the trajectories for consumption and investment in physical and prevention capital that will maximize aggregated welfare of a representative agent over the time interval $[0, T]$. In that regard, the policy maker solves

$$\max_{\{c,p\}} \int_0^T U(C(t))e^{-\rho t}dt,$$

where C stands for consumption per capital and c is the share of output devoted to consumption. In specific, we considered the general consumption together with public debt, which absorbs most of the shocks. p is the share of output invested in prevention capital. The policy maker discounts the future exponentially, so that parameter ρ is the time discount rate. Note that the arrival of earthquakes does not modify the policy maker preferences, nor the production function. Hence the problem solved by the policy maker does not change structurally with the occurrence of earthquakes.

Optimal decisions, production of the final good and earthquake damage depend on the economy's level of protection against earthquakes. To model this new investment possibility, we define prevention capital as follows:

Definition 1. Prevention capital is the aggregate of accumulated resources that protect the economy against earthquakes.

Prevention capital is then an aggregated stock made of human skills, adapted infrastructures, factories and housing (new or improved via retrofitting techniques); fire-breaks as well as national, regional and local emergency plans. Governments of all levels can invest in prevention capital, and like the Japanese government, they can even hedge insurance against catastrophes (OCDE, 2006).

We denote by k physical capital per capita and assume that production of the unique final good follows a Cobb-Douglas function with technological parameter A and physical capital share α. A share φ of output is bygone by earthquake damage. Hence, available output for consumption and investment in physical and prevention capital is the share $(1 - \varphi(b, \mu, t, \tau))Ak^{\alpha}$, which represents undamaged output.

Consumption per capita, $C(t)$, is a fraction c of total undamaged output and prevention investment a fraction p. Hence, consumption is $C(t) = c(t)\,[1 - \varphi(b, \mu, t, \tau)]\,Ak^{\alpha}(t)$ and $p(t)\,[1 - \varphi(b, \mu, t, \tau)]\,Ak^{\alpha}(t)$ is investment in prevention. Hence, the dynamic equation describing the evolution of physical capital per capita is

$$\dot{k}(t) = [1 - c(t) - p(t)]\,[1 - \varphi(b, \mu, t, \tau)]\,Ak^{\alpha}(t) - (\delta_k + n)k(t). \tag{1.6}$$

δ_k is the physical capital depreciation. b, accumulated prevention capital per capita, evolves according to the following ordinary differential equation:

$$\dot{b}(t) = \beta_1 \left\{p(t)\,[1 - \varphi(b, \mu, t, \tau)]\,Ak^{\alpha}(t)\right\}^{\beta_2} - (\delta_b + n)b,$$

that is, investment transforms into prevention capital via a power function, and it depreciates at a rate δ_b. β_1 and β_2 are efficiency parameters that depend on the economy earthquake prevention skills and on the overall technology.

Let us gather all elements and describe the model the policy maker utilises at $t = 0$. At the beginning of the planning horizon, the decision maker expects that an earthquake of intensity μ_0 will arrive every τ_0 years. We assume that the policy maker uses a CIES utility function with parameter σ, she solves the following problem:

$$\max_{\{c,p\}} \int_0^T \frac{\left\{c(t)[1 - \varphi(b, \mu_0, t, \tau_0)]Ak^{\alpha}(t)\right\}^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \tag{1.7}$$

subject to

$$
\begin{cases}
\dot{k}(t) = [1 - c(t) - p(t)][1 - \varphi(b, \mu_0, t, \tau_0)]Ak^{\alpha}(t) - (\delta_k + n)k(t), \\
\dot{b}(t) = \beta_1 \left\{p(t)[1 - \varphi(b, \mu_0, t, \tau_0)]Ak^{\alpha}(t)\right\}^{\beta_2} - (\delta_b + n)b, \quad \text{for all t} \in [0, T], \\
k(0) \text{ and } b(0) \text{ given.}
\end{cases}
\tag{1.8}
$$

The state equations of the policy maker exhibit (downward) jumps at fixed dates, which correspond to the expected arrival of an earthquake: τ_0, $2\tau_0$, $3\tau_0$,... Hence, $\dot{k}$ and $\dot{b}$ have a countable number of discontinuities and standard optimisation methods apply.

As mentioned, the optimal trajectory resulting from (1.7) and (1.8) will be applied from $t = 0$ until the first earthquake arrives at time t_1. After measuring the actual intensity, m_1, and actual frequency t_1, the policy maker actualises her beliefs and recomputes optimal trajectories from t_1.[7] In general, the policy maker actualises her beliefs and recomputes new optimal trajectories after each earthquake. We describe next how earthquake beliefs are actualised.

The j^{th} earthquake after $t = 0$ hits the economy at time $0 \leq t_j < T$ with intensity m_j. This j^{th} earthquake arrives then $t_j - t_{j-1}$ years after the last. Then, the policy maker updates first her beliefs on frequency and intensity as the average of past observations and the last:

$$
\mu_j = \frac{m_j + \sum_{i=0}^{j-1} \mu_i}{j} \quad \text{and} \quad \tau_j = \frac{\sum_{i=1}^{j-1} (t_{i+1} - t_i) + \tau_0}{j} = \frac{t_j + \tau_0}{j}. \tag{1.9}
$$

We know that if the policy maker updates her beliefs using the average of the observed values, as in (1.9), then beliefs converge towards their true values with time.

With the updated values for earthquake intensity and frequency, the policy maker solves the new problem, from t_j:

$$
\max_{\{c,p\}} \int_{t_j}^{T} \frac{\left\{c(t)[1 - \varphi(b, \mu_j, t, \tau_j)]Ak^{\alpha}(t)\right\}^{1-\sigma}}{1 - \sigma} e^{-\rho t} dt,
$$

[7] Since the earthquake happening at t_1 is the first in the planning period, t_1 is both (average) observed frequency and time elapsed since last occurrence.

subject to

$$\begin{cases} \dot{k}(t) = [1 - c(t) - p(t)][1 - \varphi(b, \mu_j, t, \tau_j)]Ak^\alpha(t) - (\delta_k + n)k(t), \\ \dot{b}(t) = \beta_1 \left\{p(t)[1 - \varphi(b, \mu_j, t, \tau_j)]Ak^\alpha(t)\right\}^{\beta_2} - \delta_b b, \quad \text{for all t} \in [t_n, T], \\ k(t_j) \text{ and } b(t_j) \text{ given.} \end{cases} \tag{1.10}$$

At t_j, the policy maker expects the arrival of an earthquake at every τ_j years with an intensity of μ_j. In other words, the state variables k and b are expected to receive shocks at fixed dates with known intensity at every τ_j years. Hence, at the moment the decision maker computes the optimal trajectory, the times of the expected shocks are $t_j + \tau_j, t_j + 2\tau_j, ..., t_j + N_j\tau_j$, where N_j is the expected number of earthquakes from t_j to T. N_j is computed as the integer part of N over τ_j, $N_j = \lfloor \frac{T}{\tau_j} \rfloor$.

Notably, the policy maker's problem at time t_j can be rewritten as

$$\max_{\{c,p\}} \sum_{i=0}^{N_j-1} \int_{t_j+i\tau_j}^{t_j+(i+1)\tau_j} \frac{\{c(t)[1 - \varphi(b, \mu, t, \tau)]Ak^\alpha(t)\}^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \tag{1.11}$$

$$+ \int_{t_j+N_j\tau_j}^{T} \frac{\{c(t)[1 - \varphi(b, \mu, t, \tau)]Ak^\alpha(t)\}^{1-\sigma}}{1-\sigma} e^{-\rho t} dt,$$

subject to

$$\begin{cases} \dot{k}(t) = [1 - c(t) - p(t)][1 - \varphi(b, \mu_j, t - i\tau_j, \tau_j)]Ak^\alpha(t) - (\delta_k + n)k(t), \\ \dot{b}(t) = \beta_1 \left\{p(t)[1 - \varphi(b, \mu_j, t - i\tau_j, \tau_j)]Ak^\alpha(t)\right\}^{\beta_2} - \delta_b b(t), \end{cases} \tag{1.12}$$

for $t \in [t_j + i\tau_j, t_j + (i+1)\tau_j]$ and $i = 0, ..., N_j$, with $k(t_j)$ and $b(t_j)$ known, $k(0)$ and $b(0)$ known. Abusing of notation, we identify $T = t_j + (N_j + 1)\tau_j$. Note how the damage function evolves with time, but only inasmuch as it defines time elapsed since last earthquake, that is, recovery time. We define the set of admissible controls to problem (1.11)-(1.12) as follows:

Definition 2. A pair of control variables $\{c, p\}$ is said to be τ_j-admissible for a given switching frequency $\tau_j \in [0, T]$ if

a) There exists a unique solution (k^*, p^*) to (1.12) with N_j shocks at times $t_j + \tau_j, t_j + 2\tau_j, ..., t_j + N_j\tau_j$, with initial condition $(k(t_j), b(t_j))$, for all j.

b) The pair of controls $\{c^*, p^*\}$ are measurable and bounded functions, with $0 \leq c^*(t) \leq 1$ and $0 \leq p^*(t) \leq 1$, for all $t \in [t_j, T]$.

Theorem 1 provides the set of necessary optimal conditions:

Theorem 1. Given t_j and the couple (τ_j, μ_j), let us define on each interval $[t_j + i\tau_j, t_j + (i+1)\tau_j]$, with $i = 0, ..., N_j$ and $(N_j+1)\tau_j = T$, the Hamiltonians $H_i(t, k, b, c, p, \lambda_1^i, \lambda_2^i)$ and the co-state variables λ_1^i and λ_2^i as:

$$H_i(t, k, b, c, p, \lambda_1^i, \lambda_2^i) = \frac{\{c(t)[1 - \varphi(b, \mu_j, t - i\tau_j, \tau_j))]Ak^\alpha(t)\}^{1-\sigma}}{1-\sigma} e^{-\rho t}$$

$$+\lambda_1^i \{[1 - c(t) - p(t)][1 - \varphi(b, \mu_j, t - i\tau_j, \tau_j)]Ak^\alpha(t) - (\delta_k + n)k(t)\}$$

$$+\lambda_2^i \left\{\beta_1 \{p(t)[1 - \varphi(b, \mu_j, t - i\tau_j, \tau_j)]Ak^\alpha(t)\}^{\beta_2} - (\delta_b + n)b(t)\right\}.$$

Assume there exists an optimal control set $\{c_j^*, p_j^*\}$ associated to τ_j defined on $[t_j, T]$. Let $\{c_{j,i}^*, p_{j,i}^*\}$ denote the restriction of $\{c_j^*, p_j^*\}$ to the time interval $[t_j + i\tau_j, t_j + (i+1)\tau_j]$ for all i. Then, it is necessary that there exists a couple $\{\lambda_1^i, \lambda_2^i\}$ such that $\{k^*, b^*\}$ satisfy a set of canonical equations at every $t \in [t_j + i\tau_j, t_j + (i+1)\tau_j]$:

$$\frac{\partial H_i}{\partial \lambda_1^i} \Rightarrow \dot{k}^* = (1 - c - p)[1 - \varphi(b^*)]Ak^{\alpha *} - (\delta_k + n)k^* \quad (1.13)$$

$$\frac{\partial H_i}{\partial \lambda_2^i} \Rightarrow \dot{b}^* = \beta_1 \{p[1 - \varphi(b^*)]Ak^{\alpha *}\}^{\beta_2} - (\delta_b + n)b^* \quad (1.14)$$

and

$$\dot{\lambda}_1^i(t) = -\frac{\partial H_i}{\partial k} = -\{c[1 - \varphi(b^*)]A\}^{1-\sigma}\alpha k^{\alpha(1-\sigma)-1}e^{-\rho t} - \lambda_1\{(1 - c - p)[1 - \varphi(b^*)]A\alpha k^{\alpha-1}$$
$$-(\delta + n)\} - \lambda_2\alpha\beta_1\beta_2\{p[1 - \varphi(b^*)]A\}^{\beta_2}k^{\alpha\beta_2-1}$$

$$\dot{\lambda}_2^i(t) = -\frac{\partial H_i}{\partial b} = (cAk^\alpha)^{1-\sigma}[1 - \varphi(b^*)]^{-\sigma}\varphi_b'(b^*)e^{-\rho t} + \lambda_1(1 - c - p)\varphi_b'(b^*)Ak^\alpha$$
$$+\lambda_2\{\beta_1\beta_2 [pAk^\alpha]^{\beta_2} \varphi_b'(b^*)(1 - \varphi(b^*))^{\beta_2-1} + \delta_b + n\} \quad (1.15)$$

with $\{k(t_j), b(t_j)\}$ known. We simply denote $\varphi(b^*, \mu_j, t - i\tau_j, \tau_j)$ as $\varphi(b^*)$. Calculating the partial derivatives with respect to control variables i.e. $\frac{\partial H_i}{\partial c_{j,i}^*} = 0$ and $\frac{\partial H_i}{\partial p_{j,i}^*} = 0$, we obtain the optimal values for consumption and prevention variables :

$$
\begin{aligned}
c^*_{j,i} &= \frac{\left(\lambda_1^i e^{\rho t}\right)^{-1/\sigma}}{(1-\varphi(b))Ak^\alpha}, \\
p^*_{j,i} &= \left(\frac{\lambda_2^i \beta_1 \beta_2}{\lambda_1^i}\right)^{\frac{1}{1-\beta_2}} \frac{1}{(1-\varphi(b))Ak^\alpha},
\end{aligned} \tag{1.16}
$$

for all i. Furthermore, the following matching conditions are satisfied:

$$
\begin{aligned}
\lambda_1^{i*}(t_j + (t+1)\tau_j)^- &= \lambda_1^{(i+1)*}(t_j + (t+1)\tau_j)^+, \\
\lambda_2^{i*}(t_j + (t+1)\tau_j)^- &= \lambda_2^{(i+1)*}(t_j + (t+1)\tau_j)^+,
\end{aligned} \tag{1.17}
$$

and

$$
\lim_{t \to (t_j + (t+1)\tau_j)^-} H_i(t, k, b, c, p, \lambda_1^i, \lambda_2^i) = \lim_{t \to (t_j + (t+1)\tau_j)^+} H_{i+1}(t, k, b, c, p, \lambda_1^{i+1}, \lambda_2^{i+1}). \tag{1.18}
$$

Theorem 1 provides a method to compute the optimal trajectories for the economy after each earthquake. Note that trajectories computed at t_j will be applied from t_j until t_{j+1}, the arrival time of next earthquake. We shall define the optimal trajectory for the time interval $[0, T]$ as the juxtaposition of the optimal solutions between two earthquakes. If there are $\mathcal{N}$ earthquakes during the time interval $[0, T]$, we denote by $t_{\mathcal{N}}$ the arrival time of the last earthquake and define $t_{\mathcal{N}+1} = T$. Hence, the policy maker re-computes $\mathcal{N}$ times the optimal trajectory over the periods $[t_j, T]$ to build the optimal policy for the entire period. The resulting optimal solution is the juxtaposition of the optimal solution between two earthquakes:

Definition 3. The optimal solution to problem (1.7)-(1.10) is a set $\{k^*, b^*, c^*, p^*\}$, constructed piece wisely:

$$
\begin{aligned}
k^*(t) &= k^*_j(t + t_j), \\
b^*(t) &= b^*_j(t + t_j), \\
c^*(t) &= c^*_j(t + t_j), \\
p^*(t) &= p^*_j(t + t_j),
\end{aligned} \tag{1.19}
$$

for $t \in [t_j, t_{j+1}]$, $j = 0, ..., \mathcal{N}$, and where $\{k^*_j, b^*_j, c^*_j, p^*_j\}$ is the optimal solution computed after the j^{th} earthquake.

1.4 Numerical case studies

In this section, the roles of learning and prevention capital are analysed numerically and their consequences are measured. We have chosen four economies that are afflicted by earthquakes frequently, namely Japan, Italy, China and the Chinese region of Yunnan. As shown in Figure 1.2 in section 1.2, these economies do not suffer equally from comparable earthquakes. This reflects differences in preparedness, which, in light of our model, stem from varying knowledge on earthquakes and their structural capacity to protect the economy. We have identified each country with an initial level of prevention capital, and run simulations under a number of scenarios that contrast the benchmark results. We start our analysis with the cases of Japan and Italy. On the one hand, Japan is the perfect example of learning and prevention since the country's behaviour reflects a deep knowledge of earthquake frequency and intensity. Indeed, the long history of enormous and necessary Japanese investment in prevention capital protects the economy and minimises losses. On the other hand, evidence suggests that Italy does not posses an effective level of prevention capital. We close this section with the numerical emulation of China and Yunnan. Our simulations underline the effect of the lack of learning in a context of tremendous economic growth. For Yunnan, we also bring to the forefront the effect on welfare of the lack of political independence.

Let us present first the calibration method and then the associated results for each country in separated subsections. Data on earthquakes' intensity, date and exact location are collected from the NOAA. Data on GDP and population are extracted from the World DataBank in the World Development Indicators Database. The learning period is fixed to 1990-2000, except for Italy, where it covers the period 1980-2000. All numerical exercises cover the years 2000 to 2015.

The model has two functional forms to specify and calibrate. The first of these functions is the production function, which we assume is of the Cobb-Douglas type. Although the benchmark model of section 1.3 does not consider any engine of exogenous technological progress, it is needed to reproduce the GDP trajectory of Japan, China and Yunnan. For simplicity, exogenous technological progress is assumed to be a linear function of time, so that the production function in per capita terms is:

$$f(k) = (A + \gamma t)k^{\alpha}. \tag{1.20}$$

Output is initialized using 2000 data by the World DataBank in the World Development Indicators Database. Hence, we need to calibrate A, γ and α so that $y(0) = (A + \gamma t)k(0)^\alpha$ coincides with the countries output, and that national output grows at the observed rate over the 15 years of the simulation exercise. Table 1.2 collects the calibrated values for the model parameters. Note that for Italy, China and Yunnan we only reproduce those parameters whose calibration differs from Japan's.

		Japan	Italy	China	Yunan
A	Scale parameter	0.6		1	1
α	Output elasticity	0.75		0.3	0.3
γ	Time trend	0.001	0	20	20
σ	Utility parameter	0.6			
ρ	Time discount rate	0.015			
δ_k	k depreciation rate	0.004			
n	Population growth rate	-0.015	0	0.05	0.05
β_1	Prevention skills production parameter	0.75			
β_2	Prevention skills production parameter	0.5			
δ_b	b depreciation rate	0.01			

TABLE 1.2 – *Calibration of the state equations*

Next, the final form of the damage function is

$$\varphi(b, \mu, t, t') = g(b, t-t')\phi\left[\frac{1}{\beta}\ln\left(\frac{\mu}{\chi}\right)\right] = \frac{z}{b(t)}e^{\Omega(\mu-m)^{\theta_1}}e^{-(t-t')^{\theta_2}}\phi\left[\frac{1}{\beta}\ln\left(\frac{\mu}{\chi}\right)\right].$$

The above specification for g encompasses all required features. That is, it decreases with $b(t)$ and time elapsed since last earthquake. Additionally, damage depends on the estimation error of the policy maker regarding intensity via the term $e^{\Omega(\mu-m)^{\theta_1}}$.

Under the calibration for Japan, whenever $b(0) = k(0)$, that is, if Japan had as much prevention capital as physical capital, then there would be almost no losses after an earthquake. In the benchmark scenario for Japan initial prevention capital is equivalent to 10% of its physical capital. With this initial prevention capital, Japan loses at most a 1.5% of its GDP upon an earthquake of intensity larger than 6, being this loss distributed in time. Italy's initial prevention capital is calibrated to be 0.05% of physical capital, which is a very low level. There are reasons to believe that most Italian infrastructures and public/private buildings, schools, etc. are not optimally adapted for earthquakes. In a 2006 study after the Potenza earthquake, all buildings in the region were analysed. Although the law enforces earthquake resistant buildings for new buildings after 1980, resistant buildings account only for 35% of the total, and mainly in urban areas. However, over the last 20 years, earthquakes have hit the countryside and the Italian mountains. The electronic database associated to Jaiswal and Wald (2011) provides the most recent estimations for the parameters in the loss ratio ϕ, as displayed in Table 1.3.[8] Using historical data, we have computed

		Japan	Italy	China	Yunnan
$b(0)$	Initial prevention capital	$0.1k(0)$	$0.0005k(0)$		
z	Recovery function parameter	10^5	10^6	10^5	10^3
Ω	Recovery function parameter	-0.1	5	-0.1	
θ_1	Recovery function parameter	0.005			
θ_2	Recovery function parameter	2			
β	Damage function parameter	0.1		0.15	0.15
χ	Damage function parameter	10.291	9.03	9.946	9.946

TABLE 1.3 – *Damage function calibration.*

the average of frequency and intensity related to earthquakes of intensity larger than 5 to act as initial condition for the simulation period, which covers the years 2000-2015. The resulting initial expected frequencies and intensities are displayed in Table 1.4.

		Japan	Italy	China	Yunnan
τ_0	Initial expected frequency	0.5	1.33	0.83	2.3
μ_0	Initial expected intensity	6.6	5.15	6	6.5

TABLE 1.4 – *Initial beliefs on the frequency and intensity of earthquake.*

1.4.1 Japan: the good learner

From 2000 to 2015 Japan has suffered 55 earthquakes. As a result of sustained investment in prevention, earthquakes of intensity lower than 8 hardly provoke any damage. As our numerical results show, Japan has been learning from earthquakes for so long that their knowledge is almost

[8]Note that in all exercises, we compare optimal trajectories under two different assumptions for $b(0)$: $b(0) = 0.1k(0)$ and $b(0) = 0.001k(0)$.

perfect. Consequently, their policies would follow the same lines even if they stopped learning today.

The numerical exercises for Japan start with the case in which there is no learning about the earthquake intensity nor about its frequency. Later the policy maker learns about earthquakes, assigning to expected frequency and intensity the average of the observed values. In each of these sets of exercises the role of prevention is underlined considering two initial levels of prevention capital, corresponding to a good and an insufficient level, respectively.

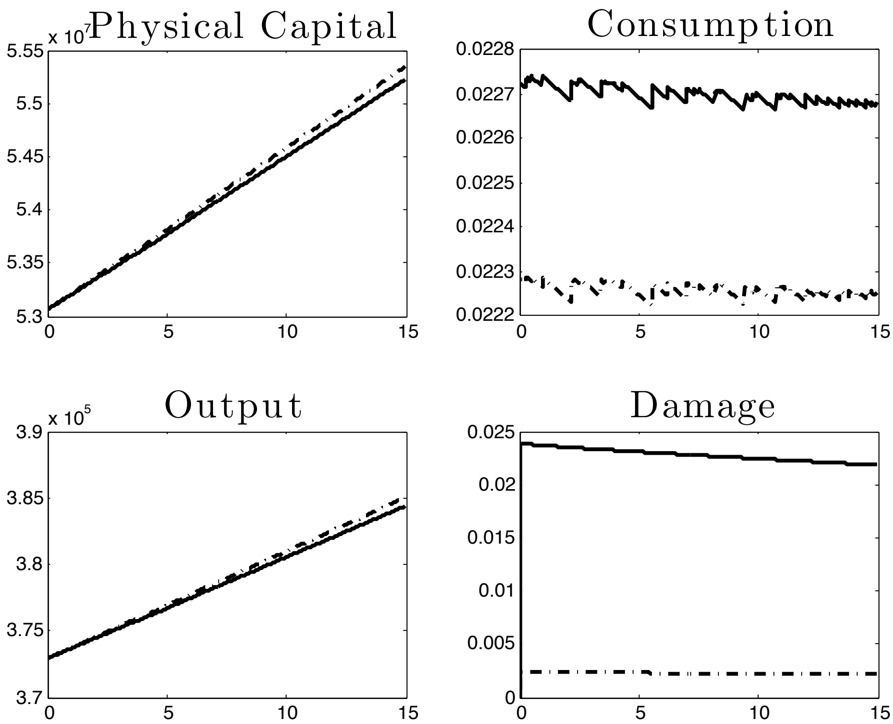


Figure 1.4 – Japan with no learning on earthquake. Solid lines: $b(0) = 0.01k(0)$*. Dash-dotted:* $b(0) = 0.1k(0)$*.*

Figure 1.4 shows the numerical results for the first exercises, with no learning. Time is represented on the horizontal axis of Figure 1.4, so that $t = 0$ corresponds to year 2000. As mentioned in the calibration section, when Japan starts with a prevention capital that is 10% of its physical capital stock (dashed lines), damage is low. Japan would lose less than 0.25% of its GDP on a continuous basis. Note the small figure for a country hit by 55 earthquakes in 15 years with an average intensity of 6.6. At the 15 year aggregate, an earthquake of intensity 6 generates at most a loss of 1.5% of Japanese GDP (if no other earthquake hits before). However, if Japan started with ten times less prevention capital (solid lines), then loses would amount to 2.5% of its production due to the accumulated effect of earthquakes. That is, Japan would lose 5 times more of its production if it was poorly protected against earthquakes. Note that damage decreases slightly with time because prevention capital increases in the 15 years of the simulation. Besides the role of prevention capital in growth, the Japanese economy recovers after an earthquake stimulated by the undamaged exogenous technological progress, as shown by the pictures of physical capital and output in Figure 1.4. Prevention capital also has an impact on GDP growth even in this case, where the damage induced by earthquakes is low. GDP overall growth rate during the period 2000-2015 is of 3.25% in the high prevention capital benchmark, and 3.08% in the case with ten times less prevention capital. Worth to underline, GDP and physical capital trajectories diverge with time, and their differences increase with time. Indeed, damage to GDP and capital accumulates.

As in the original Ramsey (1928) model, the policy maker choices result in a smooth trajectory for consumption with small downward jumps caused by earthquakes. The share of GDP devoted to consumption is larger when prevention capital is low. The average household consumes around 2.27% of annual GDP when prevention is poor compared to an average of 2.225% when better protected. Although this result may seem strange at first, it is not so if one thinks that investment in prevention also needs resources to be maintained and to eventually increase. Hence, the larger prevention capital, the more resources the economy needs to devote to prevention, even if it is just to maintain the level of prevention capital. Since earthquakes do not modify the policy maker's preferences, the consumption trajectories associated to the low and high prevention capital are qualitatively identical. We observe that both trajectories oscillate around an average value. Earthquakes shift consumption downwards showing that to reduce consumption is the policy maker's main tool to face economic losses. Indeed, among the three uses of production: consumption, investment in physical or prevention capital, it is consumption that absorbs most of the losses generated by earthquakes. This result deserves some comments. Consumption defines welfare, so one does not expect that a policy maker sacrifices consumption to privilege physical capital. Nevertheless, if everything else remains equal, sacrifices in consumption do not accumulate in time. That is, they do not have further negative consequences. On the contrary, reductions in investment in physical or prevention capital do have lasting effects on the future of production and consequently on future consumption. Upon an earthquake, the policy maker sacrifices today's consumption to preserve tomorrow's.

Then, earthquake intensity learning is introduced. Numerical results show little differences between the exercises with and without learning on earthquake intensity. For the sake of brevity we do not show here the numerical results of this exercise since they resemble results in Figure 1.4. This outcome teaches us one important lesson. When countries have acquired sufficient experience on earthquakes, devoting extra effort to learning is rewardless. And this is notably the case of Japan. Japan has been preparing for earthquakes for long, adapting infrastructures and designing mixed private-public insurance mechanisms (OCDE, 2006).

Our results for Japan can be briefly summarised in three main conclusions. First, prevention capital protects the economy and can generate important differences in GDP growth in the medium term. Second, when an earthquake hits the economy, the policy maker main tool to re-equilibrate the economy is to sacrifice the period's consumption. This allows the policy maker to maintain the level of the stocks of physical and prevention capital, which ensure future welfare. And third, when policy makers accumulate accurate knowledge on earthquakes, making further efforts does not transcend in higher welfare nor in larger GDP.

1.4.2 Italy: a historical country made of stone

Italy has suffered 25 earthquakes from 1980 to 2015, with an average intensity of 5.44 in the Richter scale. The largest earthquake during those 35 years happened in April 2009 in the region of L'Aquila in the center of the country.

In the first exercise, the benchmark, prevention capital is as low as 0.05% of physical capital and the policy maker does not update her beliefs on earthquake intensity nor frequency. In a second exercise, we consider that Italy mimics Japan, that is, that its initial prevention capital is 0.1% of

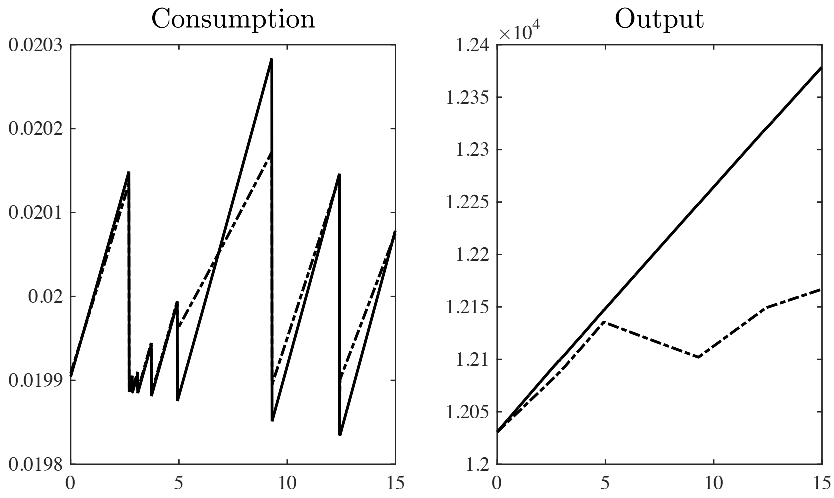


Figure 1.5 – Italy. Dashed lines, $b(0) = 0.05k(0)$*. Solid lines,* $b(0) = 0.1k(0)$*.*

the initial physical capital, and that the policy maker collects information after each earthquake to update her beliefs. Figure 1.5 shows the resulting trajectories for consumption and output. As observed in the previous exercise on Japan, the policy maker adapts to the shocks using consumption. Indeed, we observe relative large fluctuations in consumption that help preserving investment in future capital and hence future consumption. Next, we consider a second scenario in which Italy updates its beliefs on earthquake intensity and frequency, and in which it has the same relative initial stock of prevention capital as Japan. Results correspond to the solid lines in Figure 1.5. Broadly speaking, we can conclude that prevention capital enhances consumption. Take consumption after the November 2004 earthquake of Salo in Lombardy. Note how to maintain a high level of prevention capital, the well protected Italy decreases consumption further than the unaware Italy. Since the economy is better protected, the effects of the earthquake last shorter. As a result, production grows faster enhancing later consumption. This corresponds to the peak in consumption at $t = 10$ or year 2010.

There is a very interesting phenomenon after 2005. From 2002 to 2005, there was a series of 5 earthquakes. The economy had not completely recovered from the previous earthquake when a new earthquake arrived. As a result, damage accumulated. Besides, the policy maker did not change her beliefs or her view on prevention capital, so that she was privileging consumption. As a consequence, output declines between 2005 and 2009. Then in 2009, output starts increasing. Indeed, following a period of more than 4 years of an austere consumption regime, the 2009 earthquake reduces further consumption. This time the policy maker privileges investment in physical capital and re-launches the final good sector. Indeed, after the 2009 earthquake of L'Aquila of intensity 6.3 the economy starts growing again, although at a lower rate than the well-protected economy. Three years later, another earthquake of intensity 6.1 arrives at Emilia-Romagna and destroys an important share of production. Consumption adjusts and output continues increasing, but at an even lower rate. Once again, the economy did not have time to recover, and it accumulates damage. In particular, had the economy been better protected, output would have increased a 2% more.

Three main results should be underlined. First, an economy which disregards the role of prevention capital does not diminish consumption as much as a better protected economy after an earthquake. Second, in poorly protected economies with no learning, earthquakes may have everlasting consequences on consumption, output and economic growth. Third, damage from a series of close earthquakes accumulates, being magnified upon each new event.

1.4.3 China: the growing giant

We analyse the Chinese exposure and adaptation to earthquakes at different levels. First, the entire country is considered as a unique and uniform decision unit. Then, the region of Yunnan is brought into focus. Yunnan is one of the Chinese regions the most exposed to earthquakes, it is a relatively poor region with a low population density and a large area. Yunnan optimal trajectories and welfare are first computed as if the region could take its own decisions. Then, in the last part of this section we consider that decisions regarding earthquake prevention in Yunnan stem from the Chinese central government.

As already noticed the Chinese policy maker uses consumption here as well to compensate for earthquake losses. Hence, when one earthquake hits, consumption per capita can decrease from a slight 0.1% to almost 3%

after an earthquake of intensity 6 in Yunnan. Independently of the initial level of prevention capital, damage is very close to zero and there are no significant differences with or without learning. Despite the instantaneous damage caused on production, technology grows so fast that it makes up for these losses. As a result, the Chinese government does not have any incentive to invest in prevention capital given that the overall economy does not feel the earthquake repercussions, which is hidden by Chinese economic growth.[9]

Yunnan as an independent regional decision maker

There have been 38 earthquakes in Yunnan from 2000 to 2015, with an average damage index of 2.55. That is, most of the earthquakes have implied serious economic losses.[10] In this section we run two exercises. In the first, there is no learning so that earthquake expected frequency and their intensity are fixed in time. Second, the same exercise is simulated allowing the policy maker of Yunnan to learn about earthquake frequency and intensity. In each exercise, we consider the cases of high and low initial prevention capital and we focus on consumption trajectories. The simulation results of this first exercise are shown in figure 1.6. Prevention capital increases slightly during the simulation period, independently of its initial level so that pre-existing differences are kept in time. When initial prevention capital is 10% of physical capital, damage is almost non-existent. But when prevention capital is rare, then an earthquake of intensity 6 destroys 30% of Yunnan's production instantaneously. Due to GDP growth, and despite the low investment in prevention capital, prevention capital increases and damage goes below 30% after 10 years. Although the trajectories of GDP and the stock of physical capital show a similar pattern in the cases of learning and no learning, physical capital is always larger when prevention capital is higher, that is, when physical capital is better protected.

When prevention capital is low, consumption declines with time. After 15 years, consumption decreases by 13% in 15 years (Figure 1.6). Note that consumption suffers downward shifts upon earthquake occurrences. Since the effects of earthquakes accumulate, consumption diminishes steadily.

[9]Results are not graphically displayed for the sake of brevity, but they are available upon request.

[10]As mentioned in Section 2, level 2 earthquakes imply losses up to 5 million U.S. dollars whereas level 3 earthquakes can entail losses from 5 to 24 million U.S. dollars.

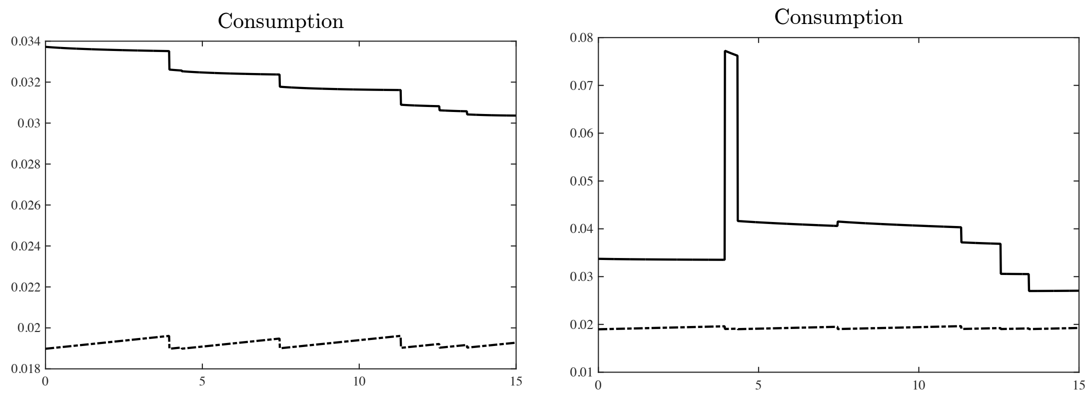


Figure 1.6 – Yunnan. Left panel: no learning on earthequake. Right panel: learning on earthquake. In both graphics, the solid line corresponds to $b(0) = 0.001k(0)$*, and the dashed line to* $b(0) = 0.1k(0)$*.*

In contrast, showing a complete different behavior consumption oscillates around 1.9% of GDP when prevention capital is high. As already noticed in other exercises, consumption is lower when prevention capital is high because maintenance of prevention capital requires resources that are shifted from consumption since the policy maker does not sacrifice investment in physical capital. Between earthquakes, the trajectory of consumption tends to increase. For instance, it increases almost a 4% from 2000 to 2004. Then, a first earthquake hits and consumption drops to almost the initial level, a drop of 3.25%. Contrary to the case of low prevention capital, shocks do not accumulate and the consumption trajectory rebounds after each earthquake. That is, the better protection against earthquakes allows the policy maker to continue accumulating both physical and prevention capital while increasing consumption.

Next, we allow the regional policy maker to learn about earthquake frequency and duration. Figure 1.6 evidences some remarkable differences between the two exercises. When prevention capital is low, the most noticeable difference are the tremendous jumps in consumption at the beginning of the simulation. Jumps are due to learning: at any time an earthquake hits, the policy maker has the opportunity to ameliorate policies and predictions. Additionally, Figure 1.6 shows that when the economy is sufficiently well endowed with prevention capital, learning is less important and optimal trajectories are stable, being only slightly modified upon earthquake shocks.

Let us compare now the exercises with and without learning using Figure 1.6. Information updates are crucial when prevention capital is low. When the economy is poorly endowed, learning increases welfare by 17% in the 15 years covered by the simulation. At $t = 0$, expected earthquake frequency is 2.314 years. Then, a first earthquake happens on the 14^{th} of January 2000 and expected frequency decreases to 1.35 years. As a result, the policy maker decides to shift resources from investment in physical capital to consumption and prevention capital. Hence, the trajectory of physical capital is slightly modified, results being noticeable only in the long-term.

On the contrary, if the economy is sufficiently well protected against earthquakes, learning increases welfare only by 0.07%. As noted in the example for Japan, a better prepared economy suffers less from earthquake occurrence and has less to learn and to improve. Trajectories are smooth, another characteristic usually sought by all policy makers. Hence, we can conclude that learning and prevention mitigate most of the shocks.

Yunan under Chinese rule

The aim of this subsection is to draw attention to the weight and consequences of political independence. In the mid-point between being under the Chinese national rule and being completely independent, we consider here that Yunnan is autonomous regarding consumption and investment in physical capital, but the region is tied to the national rule regarding earthquake prevention. We characterise a concerned and close policy maker who learns about earthquake frequency and intensity but who cannot invest in prevention. Hence, the region adapts to new earthquake expectations modifying consumption and investment in physical capital but still being unable to modify its prevention capital.

As in previous exercises, two levels of initial prevention capital are considered.[11] When prevention capital is high, the welfare loss induced by following the Chinese policy on earthquake prevention is equal to 19%.

[11] Simulation results are not graphically displayed since differences with respect to the previous section are not perceptible.

Otherwise stated, letting the regional government adapt to their needs can increase all individuals' welfare by one fifth. The welfare loss generated by following national rules is lower when the region is worse prepared, reaching a total loss of 14.3%. In this case, the economy is not sufficiently protected and the region suffers to a great extent from earthquakes whether independent or under the Chinese rule. Nevertheless, even in this case, the regional policy maker would, if she could, increase investment in prevention capital to protect the economy and secure future production and hence future consumption from subsequent earthquakes.

These last exercises have shed light on new issues. Chinese outstanding economic growth hides economic losses generated by earthquakes. As already underlined, Chinese authorities should take advantage of their current situation to protect the economy in the future. The exercises on Yunnan have pointed out at two aspects. Catastrophes hit harder on poorer economies, which recover with more difficultly. Finally, political independence seems key for a country facing earthquakes. Our results show that Yunnan would gain almost 20% of its welfare if it had the right to decide on investment in prevention.

It is worth to remark the proposed exercise can be understood as an argument for natural borders, where Yunnan could be understood as an autonomous territory with respect to the remaining of China. This requires further investigation in a framework of open economy modelling.

1.5 Conclusion

We have developed a modelling strategy to build optimal and flexible policies for economies frequently hit by earthquakes. Our framework allows the policy maker to learn and improve her knowledge on the stochastic nature of earthquakes as well as to invest in prevention to lessen the effects of earthquakes. Other novelties include a damage function which comprehends recovery time, the crucial time period after each earthquake.

Section 1.3 provides a practical description of the set of optimal necessary conditions that will be applied after each earthquake occurrence. When an earthquake hits the economy, the policy maker interrupts the on-going policies and obtains information about its frequency and intensity. Then, new policies are computed and applied. The overall optimal trajectory is then the juxtaposition of optimal policies between earthquake occurrences. Our numerical examples help us illustrate key elements in this decision problem: the initial level of prevention capital, the roles of technology and economic growth, of earthquake information and of awareness. Overall, our numerical results have shown how prevention capital and accurate earthquake information not only protect better the economy, but also increase consumption, long-term economic growth, enhancing welfare.

There are many extensions we consider for future work. First, the damage function would need further research to obtain a more accurate calibration of extreme earthquakes as well as of inter temporal damages. One of the most challenging and demanding projects is to allow the policy maker's preferences to vary with damage directly and not just via reductions in consumption. Then, in a second set of extensions, we would like to build a model considering together general natural catastrophes, global warming and economic growth as an extension of integrated assessment models *à* la Nordhaus (1991). Finally, in our learning approach, the heterogeneity results in the properties of different situations, but not the decision makers' parameters. To further analyse the impact of natural catastrophes on different countries and their learning patterns, we would like to extend the present model to an open economy with heterogenous decision makers.

1.6 Appendix: The Richter scale and earthquake world frequency

We provide in table 1.5 the Richter scale, indicating the effects of an earthquake depending on its magnitude as well as its frequency. Table 1.6 shows the frequency of each type of earthquake at the world scale.

Magnitude	Average earthquake effects	Average frequency (estimated)
<2.0	Microearthquakes, not felt or felt rarely.	Continual/several million per year
2.0-2.9	Felt slightly by some people. No damage to buildings.	Over one million per year
3.0-3.9	Often felt by people, but very rarely causes damage. Shaking of indoor objects can be noticeable.	Over 100,000 per year
4.0-4.9	Noticeable shaking of indoor objects and rattling noises. Felt by most people in the affected area. Slightly felt outside.	10,000 to 15,000 per year
5.0-5.9	Can cause damage of varying severity to poorly constructed buildings. Felt by everyone.	1,000 to 1,500 per year
6.0-6.9	Damage to a moderate number of well-built structures in populated areas. Strong to violent shaking in epicentral area.	100 to 150 per year
7.0-7.9	Causes damage to most buildings, some to partially or completely collapse or receive severe damage. Felt across great distances with major damage mostly limited to 250 km from epicenter.	10 to 20 per year
8.0-8.9	Major damage to buildings, structures likely to be destroyed. Damaging in large areas. Felt in extremely large regions.	One per year
≥ 9.0	At or near total destruction - severe damage or collapse to all buildings. Heavy damage and shaking extends to distant locations. Permanent changes in ground topography.	One per 10 to 50 years

TABLE 1.5 – *Richter Magnitude Scale from United States Geological Survey.*

Magnitude	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
8.0-9.9	1	1	0	1	2	1	2	4	0	1	1	1	2
7.0-7.9	14	15	13	14	14	10	9	14	12	16	23	19	11
6.0-6.9	146	121	127	140	141	140	142	178	168	144	150	185	96
5.0-5.9	1344	1224	1201	1203	1515	1693	1712	2074	1768	1896	2209	2276	1295
4.0-4.9	8008	7991	8541	8462	10888	13917	12838	12078	12291	6805	10164	13315	8710
3.0-3.9	4827	6266	7068	7624	7932	9191	9990	9889	11735	2905	4341	2791	2174
2.0-2.9	3765	4164	6419	7727	6316	4636	4027	3597	3860	3014	4626	3643	2721
1.0-1.9	1026	944	1137	2506	1344	26	18	42	21	26	39	47	34
0.1-0.9	5	1	10	134	103	0	2	2	0	1	0	1	0
0	3120	2807	2938	3608	2939	864	828	1807	1922	17	24	11	6
Total	22256	23534	27454	31419	31194	30478	29568	29685	31777	14825	21577	22289	15049
Estimated Deaths	231	21357	1685	33819	228802	88003	6605	712	88011	1790	320120	21953	629

Table 1.6 – *Number of Earthquakes Worldwide for 2000-2012. Data from United States Geological Survey.*

2 Catastrophe, Bayesian learning and optimal control II

2.1 Introduction

A predictive control model about the issue of global warming can be found in Bréchet, Camacho and Veliov (2012, 2014), while the latter paper is a practical study in a multi-country setting. The models can be seen as an extension of Nordhaus integrated assessment model (IAM, 1992), where the optimal policy is assumed to be homogeneous. The model presented here is a modification of the model of Brechet, Camacho and Veliov (2014) where two basic types of scenarios are considered, i.e. optional policy (OPT) and business-as-usual scenario (BAU). Besides the conventional environmental variables, i.e. GHGs, we consider an additional important environmental variable, i.e. a catastrophe correlated to the global warming.

Precisely, the decision maker has to decide on the optimal trajectories of consumption and abatement over time considering different types of variables: basically the deterministic or random variables. Among the random variables, the decision maker can collect information in a continuous way for some of them, such as temperature and rainfall. Nevertheless, others can only be observed on rare occasions, like the natural catastrophes, including tsunamis, earthquakes, volcano eruptions, and etc. Based on this construction, our proposed optimal control problem is defined on a piecewise trajectory.

Two-stage optimal control problem is studied by Tomiyama (1985), where necessary conditions for the optimal solutions are presented. Tomiyama and Rossana (1989) investigate the necessary conditions for the solutions of two-stage optimal problem where the switch point is a choice variable and appears as an argument of the integrands. Boucekkine, Saglam and Vallée (2004) use a two-stage optimal approach to study adoption problems under embodied technical change where switching time is adjustable. Boucekkine, Pommeret and Prieur (2011) extend this two-stage optimal approach into a multi-stage optimal control problem with application on Environmental Economics where both technology switching and ecology switching are considered. However, the switching time remains a choice variable. In this chapter, we consider a multi-stage optimal control problem with predetermined switching time based on the approach of Tomiyama (1985). The contributions are threefold.

Firstly, the predictive control model of Brechet, Camacho and Veliov (2014) is modified in order to allow a random catastrophe variable playing a fundamental role to better predict the right period and update the information.

Secondly, the catastrophe learning process is described under assumptions of both myopic agent and non-myopic agent. In particular, the analytical solutions of parameters of Bayesian learning process are obtained.

Last but not the least, a multi-stage optimal control approach with predetermined switching time is presented. We prove the existence of a solution, although its unicity cannot be granted (see Greiner, Grüner and Semmler, 2009).

2.2 The benchmark model: multiple stages optimal control

We postulate here that decision makers need to learn about the randomness of catastrophes, but they can only learn upon their realization. Suppose an earthquake happens in a certain region for the first time at time T. The local decision maker will think that another one will hit the region again in $2T$. If the second earthquake happens before $2T$, the decision maker will reconsider her beliefs. Here we introduce Bayesian learning in a predictive control setup where optimal decisions are recomputed on a fixed basis. We shall define a catastrophe as a major natural phenomenon that hits so seriously an economy. For example, the hurricane Katrina hit New Orleans in 2005, the tsunamis hit Indonesia in 2004 and Japan in 2011, the earthquake occurred in Nepal in 2015, fall into our definition.

Let us divide the interval $[0, \theta]$ with N nodes, and $N = \left\lfloor \frac{\theta}{h_k} \right\rfloor$, where the floor symbol $\lfloor \cdot \rfloor$ stands for the greatest integer less than or equal to $\cdot$, or simply called the integer part of $\cdot$. Furthermore, each node stands for a switch time. Thus, our dynamic control problem is a piece-wise optimal problem with t_j^σ switch time, where $t_j^\sigma = j\,h_k$, $j = 0, 1, \ldots, \left\lfloor \frac{\theta}{h_k} \right\rfloor$. It is worth noting that the integer $\left\lfloor \frac{\theta}{h_k} \right\rfloor$ should satisfy the following inequality condition:

$$\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \leq \theta < \left(\left\lfloor \frac{\theta}{h_k} \right\rfloor + 1 \right) h_k \tag{2.1}$$

Moreover, let us decompose θ in two independent parts: $\theta = \left\lfloor \frac{\theta}{h_k} \right\rfloor h_k + \left\{ \frac{\theta}{h_k} \right\} h_k$ where $\left\{ \frac{\theta}{h_k} \right\} = \frac{\theta}{h_k} - \left\lfloor \frac{\theta}{h_k} \right\rfloor$ denotes the the fractional part. The integer part can be further decomposed into N independent intervals as follows:

$$\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k = \sum_{j=0}^{\left\lfloor \frac{\theta}{h_k} \right\rfloor - 1} \left(t_{j+1}^\sigma - t_j^\sigma \right) = \sum_{j=0}^{\left\lfloor \frac{\theta}{h_k} \right\rfloor - 1} \left[(j+1)\, h_k - j h_k \right] , \quad j = 0, \ldots, \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1. \tag{2.2}$$

Thus, the time interval $[0, \theta]$ is decomposed into $N + 1$ independent intervals:

$$\theta = \sum_{j=0}^{N-1} (t_{j+1}^\sigma - t_j^\sigma) + (\theta - t_N^\sigma) \, , \, j = 0, 1, \ldots, N - 1 \, , \tag{2.3}$$

or in expanded form:,

$$\theta = \sum_{j=0}^{\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k - 1} \left[(j+1)\, h_k - j h_k \right] + \left(\theta - \left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \right) \, , \, j = 0, 1, \ldots, \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1 \, . \tag{2.4}$$

Let us define the piece-wise optimal control problem on $N+1$ independent intervals as follows:

$$\max \int_0^\theta L(t, v(t), x(t), y^\sigma(t), \kappa^\sigma(t)) dt \tag{2.5}$$

$$= \max \sum_{j=0}^{\left\lfloor \frac{\theta}{h_k} \right\rfloor - 1} \int_{j\,h_k}^{(j+1)\,h_k} L_j(t, v(t), x(t), y^\sigma(t), \kappa_j^\sigma(t)) dt$$

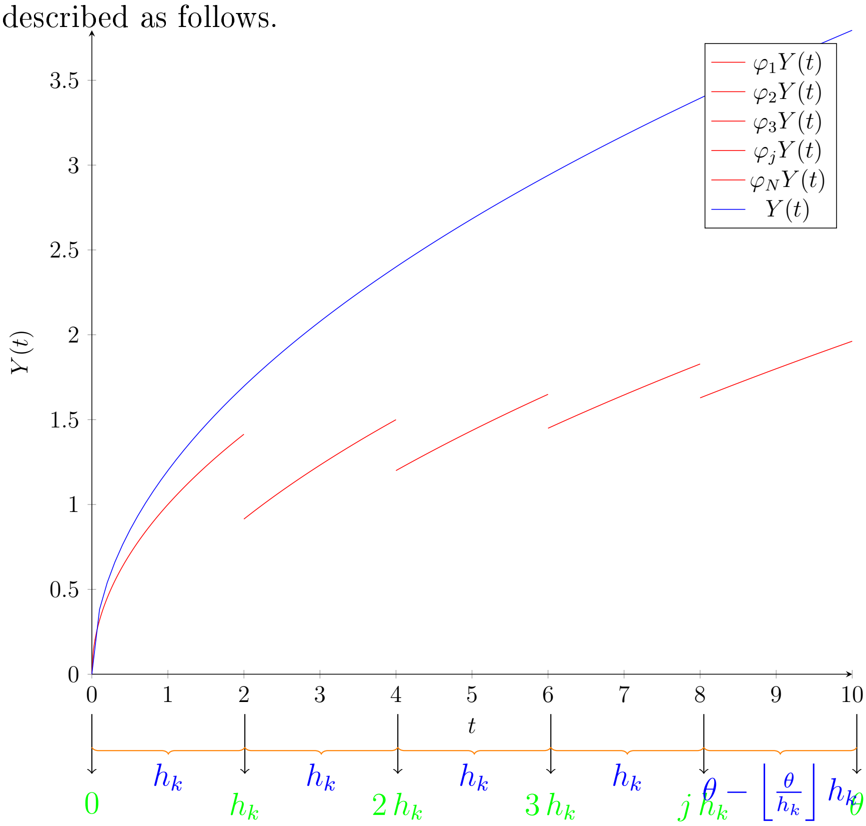
$$+ \int_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}^{\theta} L_j(t, v(t), x(t), y^\sigma(t), \kappa_j^\sigma(t)) dt \tag{2.6}$$

$$\text{where} \quad L_j(t, v(t), x(t), y^\sigma(t), \kappa_j^\sigma(t)) = \varphi_j(\kappa^\sigma, t) \cdot \bar{L}(t, v(t), x(t), y^\sigma(t)) \quad (2.7)$$

subject to

$$\begin{cases} \dot{x}(t) = f_j(t, x(t), v(t), y(t), \kappa_j^\sigma)\,, & \forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right], \quad t \in [j\,h_k, (j+1)h_k] \\ \qquad = \varphi_j(\kappa^\sigma, t) \cdot \bar{f}(t, x(t), v(t), y(t))\,, & \forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right], \quad t \in [j\,h_k, (j+1)h_k] \\ y(t) = \epsilon(y^\sigma_{[0,t_k]}, x^\sigma(t))(t), & \\ x(0) \in X_0\,, \quad x(\theta) \in X_\theta, & \\ v(t) \in \Omega_j\,, a.e \quad \text{on} \in [j\,h_k, (j+1)h_k]\,, \forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1\right] & \\ \kappa_j^\sigma := \kappa(t_j^\sigma) = l(\kappa(t_{j-1}^\sigma), y_t)\,, \quad \forall j = 0, 1, 2, \ldots & \end{cases} \tag{2.8}$$

X_0 and X_θ are non-empty closed set defined on n-dimensional Euclidean space $\mathbb{E}^n$. Ω_j is a non-empty compact set with $\Omega_j \in \mathbb{E}^m$. According to Rossana (1985) and Tomiyama (1985), we generalize the fixed switching time framework in a multi-stage optimal control problem. It is graphically described as follows.



Theorem 2. Considering the system of equations (2.6), (2.7) and (2.8) satisfying the following conditions:

- $T = [0, \theta]$, fixed control time,
- $z(t = 0) = z_0$, $z(\theta) < \infty$ fixed initial point, and free terminal point,
- $v(t)$, the control variables, is continuous everywhere, except (possibly) at each switching point $t_j^\sigma = j\, h_k$,
- $\forall j = 0, 1, 2, \dots, L_j$ and f_j are twice differentiable in x and v, and differentiable at t,

where $z(t) = (x(t), y(t))$, then, the Hamiltonian associated with above Pontryagin problem, i.e.

$$H_j(z, v, \kappa_j^\sigma, \lambda, t) = -L_j(z, v, t, \kappa_j^\sigma) + \lambda^T f_j(z, v, \kappa_j^\sigma, t) \tag{2.9}$$

where $\lambda(t)$ is n-dimensional adjoint vector (or vector of shadow price), exists optimal control v^* given the necessary condition that $\exists \lambda^*$ satisfying the following canonical equations:

$$\dot{z}^* = f_j\big(z^*, v_j^*, t, \kappa(t)\big) = \left[\frac{\partial H_j}{\partial \lambda}(z^*, v_j^*, t, \kappa_j^\sigma)\right]^T, \tag{2.10}$$

$$\dot{\lambda}^* = \left[\frac{\partial H_j}{\partial x}(z^*, v_j^*, t, \kappa_j^\sigma)\right]^T,$$

$$\text{where } t \in [j\, h_k, (j+1)h_k]\,, \forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1\right]. \tag{2.11}$$

with the boundary condition:

$$z^*(t_0 = 0) = z_0^*\,, \quad \lambda^*(\theta) = 0\,. \tag{2.12}$$

and the optimal control variable v_j^*, $\forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right]$ are the solutions of

$$\frac{\partial H_j}{\partial v}(z^*, v_j^*, t, \kappa_j^\sigma) = 0\,, \qquad \forall j \in [0, \left[\frac{\theta}{h_k}\right]], \quad t \in [j\, h_k, (j+1)h_k]\,. \tag{2.13}$$

At each switching point $t_j^\sigma = j\, h_k\,, \forall j \in \left[0, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right]$, a continuity condition should be satisfied, i.e.

$$\lambda^*(t_j^\sigma -) = \lambda^*(t_j^\sigma +)\,. \tag{2.14}$$

The proof of the theorem is straightforward (see Appendix). In a special case of $N = 1$ (or $\left[\frac{\theta}{h_k}\right] = 1$), the detail derivation can be found in Tomiyama (1985). Therefore, our multiple stage optimal control approach works in a backward recursion defined as follows:

Last (or N-th) Step: $t \in \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k, \theta\right]$

$$\max_v \int_{\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k}^{\theta} L_j(t, v, x, y^\sigma, \kappa_j^\sigma) dt \tag{2.15}$$

where

$$L_j(t, v, x, y^\sigma, \kappa_j^\sigma) = \varphi_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(\kappa^\sigma, t) \cdot \bar{L}(t, v, x, y^\sigma) \tag{2.16}$$

subject to

$$\begin{cases} \dot{x}(t) = f_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(t, x, v, y, \kappa^\sigma_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}) \,, \\ \qquad = \varphi_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(\kappa^\sigma, t) \cdot \bar{f}(t, x, v, y) \,, \qquad t \in \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \,, \theta\right] \,, \\ y(t) = \varepsilon\left(y^\sigma_{\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k\right)}, x^\sigma\right)(t) \text{ is fixed} \,, \quad t \in [\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \,, \theta] \,, \\ x\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k\right)) \text{ is fixed} \,, \quad x(\theta) \in X_\theta, \\ v(t) \in \Omega_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} \,, a.e \quad \text{on} \ \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k, \theta\right] \,, \\ \kappa^\sigma_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} := \kappa\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k\right) \,. \end{cases} \tag{2.17}$$

Hence, the associated Hamiltonian is as follows:

$$H_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}((x, y), v, \kappa_j^\sigma, \lambda, t) = -L_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}((x, y), v, t, \kappa_j^\sigma) + \lambda^T f_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}((x, y), v, \kappa_j^\sigma, t) \tag{2.18}$$

Specifically, let us describe the detailed dynamics in Eqs.(2.15) to (2.18) for an optimal (OPT) agent as follows:

$$\begin{aligned} &\max_v \int_{\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k}^{\theta} L_j(t, v, x, y^\sigma, \kappa_j^\sigma) dt \\ &= \max_v \left\{ V_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} := \int_{\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k}^{\theta} \varphi_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(\kappa^\sigma, t) \cdot \bar{L}(t, v, x, y^\sigma) \right\} \end{aligned} \tag{2.19}$$

subject to

$$\begin{cases} \dot{x} = f_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(t, v, x, y, \kappa^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}), \\ \qquad = \varphi_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(\kappa^{\sigma}, t) \cdot \bar{f}(t, x, v, y), \qquad t \in \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \,, \theta \right], \\ \dot{y} = g_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(t, e(t, v, x), y, \kappa^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}), \\ \qquad = \varphi_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(\kappa^{\sigma}, t) \cdot \bar{g}(t, e(t, v, x), y), \qquad t \in \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \,, \theta \right], \\ y\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \right) \text{ is fixed}, \quad y(\theta) \in Y_{\theta}, \\ x\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \right)) \text{ is fixed}, \quad x(\theta) \in X_{\theta}, \\ v(t) \in \Omega_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}, a.e \quad \text{on } \left[\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k, \theta \right], \\ \kappa^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} := \kappa\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor h_k \right). \end{cases} \tag{2.20}$$

We consider that $v = (u, a)$ stands for control variables, and $y = (m, \tau)$ stands for environmental variables. The associated Hamiltonian is defined as follows:

$$\begin{aligned} H_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(x, y, v, \kappa^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}, \lambda, t) = &-L_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(x, y, v, t, \kappa^{\sigma}_j) + \lambda_{\left(x, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right)} f_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(x, y, v, \kappa^{\sigma}_j, t) \\ &+ \lambda^T_{\left(y, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right)} g_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}(x, y, v, \kappa^{\sigma}_j, t), \end{aligned} \tag{2.21}$$

where $\lambda_{(z, n_1, n_2)}$ is the co-state variable associated with the state variable $z = (x, y)$ in both the economic regime N_1 and the environmental regime N_2. Obviously, in our case $N_1 = N_2 = \left\lfloor \frac{\theta}{h_k} \right\rfloor$. Thus, we could obtain optimal value function in the last step $V^*_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}\left(x^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor}, y^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} \right)$ with the following notations:

$$x^{\sigma}_j = x(t^{\sigma}_j), \quad y^{\sigma}_j = y(t^{\sigma}_j), \quad t^{\sigma}_j = j\, h_k. \tag{2.22}$$

Apparently, we choose $j = \left\lfloor \frac{\theta}{h_k} \right\rfloor$ in the last step.

Step N-1: $t \in \left[\left(\left\lfloor \frac{\theta}{h_k} \right\rfloor - 1 \right) h_k, \left[\frac{\theta}{h_k} \right] h_k \right] \left(\text{or } t \in \left[t^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor - 1}, t^{\sigma}_{\left\lfloor \frac{\theta}{h_k} \right\rfloor} \right] \right)$

In this time interval, we need to solve the following optimization problem:

$$\max_{v,x^\sigma_{j+1},y^\sigma_{j+1}} V_j := \int_{j\,h_k}^{(j+1)\,h_k} L_j(t,v,x,y^\sigma,\kappa^\sigma_j)dt + V^*_{j+1}\left(x^\sigma_{j+1},y^\sigma_{j+1}\right)\,, \tag{2.23}$$

$$\Leftrightarrow \max_{v,x^\sigma_{j+1},y^\sigma_{j+1}} \left\{ V_j := \int_{t^\sigma_j}^{t^\sigma_{j+1}} \varphi_{t^\sigma_j}(\kappa^\sigma,t)\cdot \bar{L}(t,v,x,y^\sigma) + V^*_{j+1}\left(x^\sigma_{j+1},y^\sigma_{j+1}\right) \right\}, \tag{2.24}$$

with $t^\sigma_j = j\,h_k$, and in this step $j = \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1$.

subject to the corresponding Pontryagin problem in Eq.(2.20). It is worth noting that the regime $N_1 = N_2 = j$, where $\left(t^\sigma_j, x^\sigma_j, y^\sigma_j\right)$, $j = \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1$ are given, and $x^\sigma_{j+1}, y^\sigma_{j+1}$ are free. Respectively, we can solve the associated Hamiltonian function and value function which are denoted by H_j and $V^*_j\left(x^\sigma_j, y^\sigma_j\right)$, $j = \left\lfloor \frac{\theta}{h_k} \right\rfloor - 1$.

Therefore, given the optimization solutions solved in **j-th Step**, we can recursively derive the solutions in **(j-1)-th Step**. The final step of this backward recursion problem is described as follows:

Step 1: $t \in [0, h_k]$ (or $[0, t^\sigma_k]$).
In this case, we focus on the optimization problem when $t \in [0, t^\sigma_k]$, i.e.

$$\max_{v,x^\sigma_1,y^\sigma_j} V_0 := \int_0^{h_k} L_0(t,v,x,y^\sigma)dt + V^*_1\left(x^\sigma_1, y^\sigma_1\right)\,, \tag{2.25}$$
$$\text{with } t^\sigma_0 = 0\,,\; t^\sigma_1 = h_k\,.$$

subject to the dynamics of associated regimes where $N_1 = N_2 = 0$, x_0, y_0 are given, and x^σ_1, y^σ_1 are free. Analogously, denoting the Hamiltonian with H_0 and we have $V^*_0 = V^*$. A necessary condition to guarantee solution existence is so called matching condition or continuity condition defined as follows (see, Rossana, 1985, and Tomiyama, 1985).

Proposition 2.2.1. ***(necessary condition)*** *At each switching point* t^σ_j, $\forall j \in \left[1, \left\lfloor \frac{\theta}{h_k} \right\rfloor\right]$, *i.e.* $0 < t^\sigma_1 < \ldots < t^\sigma_j < t^\sigma_{j+1} < \ldots < t^\sigma_{\lfloor \frac{\theta}{h_k} \rfloor} < \theta$, *the following equalities must hold:*

$$\lambda^*_{x,j}(t^\sigma_j) = \lambda^*_{x,j+1}(t^\sigma_j)\,, \quad \lambda^*_{y(i),j}(t^\sigma_j) = \lambda^*_{y(i),j+1}(t^\sigma_j)\,, i = 1,2\,. \tag{2.26}$$

2.3 An economy faces global warming and catastrophe

Let us consider the specific model of the Pontryagin problem with parameters:

$$\max_{u,a} \left\{ \int_0^{\infty} e^{-r\,t} \frac{[u(t)\,Y(t)]^{1-\alpha}}{1-\alpha}\, dt \right\} \tag{2.27}$$

subject to

$$Y(t) = p(t)\varphi(\tau,\kappa,t)k^{\gamma}\,, \qquad u(t), a(t) \in [0,1] \tag{2.28}$$

$$\dot{k} = -\delta\, k + [1 - u(t) - c(a(t))]\; Y(t)\,, \qquad k(0) = k^0 \tag{2.29}$$

$$\dot{\tau} = -\lambda(m)\tau + d(m)\,, \qquad \tau(0) = \tau^0 \tag{2.30}$$

$$\dot{m} = -\nu\, m + [(1 - a(t))e(t)Y(t)] + E(\tau)\,, \qquad m(0) = m^0 \tag{2.31}$$

$$\kappa_s(t) = \alpha\, \kappa_{s-1} + \beta\, \tau^{\theta_\kappa}(t) + \varepsilon_t\,, \varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma_\varepsilon^2\right)\,, \beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2) \tag{2.32}$$

State variable are vectors of x and y. Vector $x \in \mathbb{R}^n$ represents a stock of economic factors, and vector $y \in \mathbb{R}^m$ represents a stock of environmental factors. v is measurable functions which represents the admissible controls. Specifically, we define $x(t) = k(t)$, physical capital, $y = (m, \tau)$, a vector of CO_2 concentration and average temperature, and $v(t) = (u(t), a(t))$, a vector of control variables. $e(t, v, x)$ shows the impact of control and state variables, denoted by $v(t)$ and $x(t)$ respectively, on the environmental dynamic, i.e. $e(t, v, x) = (1 - a(t))e(t)p(t)\varphi(\tau)k^{\gamma}$. Referring to Bréchet, Camacho and Veliov (2011, 2014), we describe the explicit definition of each variable as follows:

$$\begin{aligned}
&k(t) : \text{Physical capital}\\
&m(t) : CO_2 \text{ concentration in atmosphere}\\
&\tau(t) : \text{average temperature}\\
&u(t) : \text{fraction of GDP for consumption}\\
&a(t) : \text{emission abatement rate}\\
&c(a(t)) : CO_2 \text{ abatement, i.e. the fraction of GDP used to reduce the}\\
&\qquad\qquad \text{emission intensity by } a(t).\\
&p(t) : \text{productivity level, exogenous.}\\
&\varphi(\tau(t), \kappa(t)) : \text{impact of climate on total factor of productivity.}\\
&e(t) : \text{emission for producing one unit of final good without abatement.}\\
&E(\tau) : \text{non-industrial emission at temperature}\, \tau.
\end{aligned} \tag{2.33}$$

It is worth noting that the technological progress $p(t)$ is a linear exogenous function defined as follows.

$$p(t) = \frac{0.25}{175}\, t + 1\,, \qquad \text{with}\, p(0) = 1\,,\; p(175) = 1.25 \tag{2.34}$$

The emission function for producing one unit of final good without abatement, $e(t)$, is also treated as another exogenous technology which reduces carbon emission intensity with time. It is assumed to be an exponential function. Especially, we assume an exogenous decrease in the emission output intensity by 25% in 75 years (Haurie, 2003).

$$e(t) = e(0)\, \cdot (0.75)^{(t/75)}\,, \qquad e(0) = 0.0427\,,\; e(75) = 0.75\, e(0) \tag{2.35}$$

$c(a)$, CO_2 abatement, is assumed to follow a linear function, and 1% of GDP is used for reducing the emission intensity by $a(t) = 50\%$, i.e.

$$c(a) = 0.01 \frac{a}{1-a}\,, \qquad \text{with } c(0) = 0 \text{ and}\,,\; \lim_{a \to 1} c(a) = \infty\,. \tag{2.36}$$

The effect of CO_2 concentration on the average temperature increase is captured by the standard function:

$$d(m) = \eta\, \ln \left(\frac{m}{m_0^*} \right) \tag{2.37}$$

A doubling of CO_2 concentration increases the average temperature by 0.41°C, where $m_0^* = 596.4$ GtC is the pre-industrial CO_2 concentration in the atmosphere.

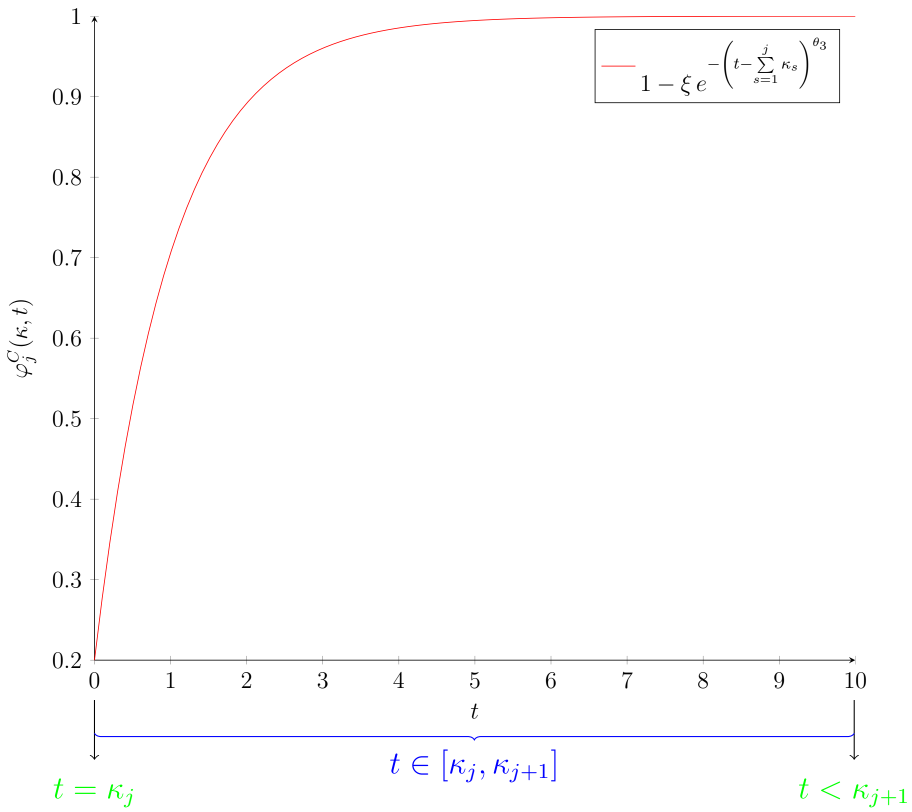
The damage function $\varphi(\tau(t), \kappa(t), t)$ is composed by two impacts of damages: global warming and catastrophe. Moreover, we assume $\varphi(\tau, \kappa, t) = \varphi(\tau)\, \varphi(\kappa, t)$. $\varphi(\tau)$ captures the impact of global warming on global productivity: a 2°C increase in the mean temperature reduces the global productivity by 3%. $\varphi(\kappa, t)$ describes the impact of catastrophe on global productivity.

$$\varphi(\tau, \kappa, t) = \varphi^{GHG}(\tau)\, \varphi^{C}(\kappa, t) \tag{2.38}$$

$$\varphi^{GHG}(\tau) = \frac{1}{1 + \theta_1 \tau^{\theta_2}} \tag{2.39}$$

$$\varphi^{C}(\kappa, t) = \prod_j \varphi_j^C(\kappa, t)\,, \qquad \forall j = 0, 1, 2, \dots, t \in [\kappa_j, \kappa_{j+1}] \tag{2.40}$$

$$\varphi_j^C(\kappa,t) = \begin{cases} 1 & : j = 0 \\ 1 - \xi\, e^{-\left(t - \sum\limits_{s=1}^{j} \kappa_s\right)^{\theta_3}} & : \forall j = 1,2,\ldots, t \in [\kappa_j, \kappa_{j+1}]\,, \xi \in [0,1] \end{cases}$$



where

$$1 - \xi\, e^{-\left(t - \sum\limits_{s=1}^{j} \kappa_s\right)^{\theta_3}} = \begin{cases} 1 - \xi : & \text{if a catastrophe happens}\,, \text{i.e.}\ t = \sum\limits_{s=1}^{j} \kappa_s \\ 1 - \omega\xi \in [1-\xi\,, 1] : & \text{if}\ t \in [\kappa_j, \kappa_{j+1}] \\ 1 : & \text{if no catastrophe for a sufficient long period}\,, \text{i.e.}\ t \gg \kappa_j\,, \kappa_{j+1} \end{cases}$$

$$\text{with}\ \forall j = 1,2,\ldots \left\lfloor \frac{\theta}{h_k} \right\rfloor .\ \text{and}\ \omega = e^{-\left(t - \sum\limits_{s=1}^{j} \kappa_s\right)^{\theta_3}} \in [0,1]$$

Let us define a piece-wise damage function for $j = 1, 2, \ldots, \left\lfloor \frac{\theta}{h_k} \right\rfloor$, $t \in [t_j^\sigma, t_{j+1}^\sigma]$, or $t \in [j\, h_k, (j+1)h_k]$. Moreover, we assume h_k is the catastrophe period (or $\Omega(\kappa_s, h_k) < \Delta$ from Eq.(2.64)), i.e for a given k, $t_j^\sigma = \sum\limits_s \kappa_s^\sigma$ (or

$j\,h_k = \sum_s \kappa_s^\sigma$), then we have:

$$\varphi(\tau, \kappa^\sigma, t) = \varphi^{GHG}(\tau)\,\varphi^{C}(\kappa^\sigma, t) = \varphi^{GHG}(\tau) \prod_j \varphi_j^C(\kappa^\sigma, t) \quad (2.41)$$

$$\varphi_j^C(\kappa^\sigma, t) = 1 - \xi\, e^{-(t-j\,h_k)^{\theta_3}}, \qquad \xi \in [0,1] \quad (2.42)$$

2.4 Model predictive control

The decision maker needs to compute the optimal trajectory for consumption and abatement over an infinite horizon, and since she is learning about the environment, she will compute trajectories step-wise.

At time $t = 0$, she has a prediction about the unknown but not random behaviour of the environment, represented by variable y, and the random variable related to catastrophe κ. Predictions are made based on past observations on the period $(-\phi, 0]$. Then, at times $t_j^\sigma = j\,h_k$, $\forall j = 1, 2, \ldots$ the decision maker re-evaluates the past of the environmental variables, where the distance of interval h_k stands for the (expected) catastrophe period. However, if a catastrophe occurs at t such that $t_j^\sigma < t < t_{j+1}^\sigma$, then she immediately re-evaluates the catastrophe interval h_k. Nevertheless, when a catastrophe happens, the decision maker re-evaluates both the past environmental behaviour and the probabilistic law governing natural catastrophes.

It is worth noting that, at each $t_j^\sigma = j\,h_k$ with $\forall k\,, h_k > 0$, the decision maker fixes her time horizon at $\theta >> h_k$, even if she knows she will re-evaluate her trajectory before. Indeed, motivated by altruism and to get a fair policy for generations to come, she optimizes over $[t_j^\sigma, t_j^\sigma + \theta]$. We shall define a recursive path to cover all the time periods as follows. We adopt here the standard notation in the literature. Denote $\sigma = (h_0, \ldots, h_k, \theta)$, we need to consider the vector of variables $(v^\sigma, x^\sigma, y^\sigma, \kappa^\sigma)$ where the upperscript reminds us about re-evaluation. v^σ is the vector of control variables. x^σ the vector of state variables with known initial value at every re-evaluation. y^σ is the vector of environmental variables, with known past upon re-evaluation, and κ^σ, the catastrophe variable, whose distribution is re-estimated upon each realization. The key to the model predictive control is that the decision maker makes predictions about the dynamics of a set of non-stochastic variables, based on a unknown and a stochastic variable about catastrophe.

At time t_j^σ, $y(t)$ is known over $(0, t_j^\sigma]$, and $x(t_j^\sigma) = x^\sigma(t)$ is known , thus the decision maker solves the following problem:

$$\max \int_{t_j^\sigma}^{t_j^\sigma+\theta} L(t, v(t), x(t), y(t), \kappa_s(t))dt \tag{2.43}$$

subject to

$$\begin{cases} \dot{x}(t) = f(t, x(t), v(t), y(t), \kappa(t)), \\ y(t) = \varepsilon(y \mid_{[0,t_k^\sigma]}, x(t))(t), \\ x(t_k^\sigma) = x^\sigma(t), \\ \kappa_s(t) = l(\kappa_{s-1}, y(t)) \,, \quad h_s = \mathrm{E}_{s-1}(\kappa_s) \end{cases} \tag{2.44}$$

$\varepsilon(\cdot)$ is the environment prediction function, which builds predictions for y relying on available information at time t_k^σ. The decision maker also estimates the average and the variance of the normal process describing the catastrophe occurrence. The choices for function $\varepsilon(\cdot)$ define different types of decision makers ranging from fully aware to denying any form of climate change. $\kappa_s(t)$ means s-th catastrophe period forecast at time t, with expectation h_s, which depends on historical observed catastrophe period κ_{s-1} and current environment status $y(t)$. At each t_j^σ, $\forall j = 1, 2, \ldots$ we update information about h_k based on our expectation on $\kappa_s(t)$. At $t \in [t_j^\sigma, t_{j+1}^\sigma]$, we set $\kappa_s^\sigma(t) = \kappa_s(t)$. Obviously, we choose v^σ to be equal to v at $[t_j^\sigma, t_{j+1}^\sigma]$ and extend continuously (x^σ, y^σ) taking the optimal solution to (2.43)-(2.44) for (x, y) over $[t_j^\sigma, t_{j+1}^\sigma]$. This way, we build $(v^\sigma, x^\sigma, y^\sigma, \kappa^\sigma)$ on $[0, \infty)$.

2.5 Environmental learning I

2.5.1 Learning about GHGs

We consider three types of policy makers: (1) fully aware of environmental change and make optimized decision (OPT), (2) a Business as usual (BAU) policy maker and (3) a free-rider. The OPT policy maker is assumed to know the real evolution of the environment at every t and integrates it into her optimization problem, that is

$$\dot{y}(t) = m(t, x(t), y(t), \tau(t)).$$

The BAU agent believe there is no change in the environment, or the change is not important enough (i.e. the damage is zero), or there is no link between her decision and the environment. They only notice changes upon the decision times, and evaluate the distance between the expected trajectory for the economic variables and the real trajectory as:

$$y(t) = y(t_j), \quad \forall t \in [t_j, t_{j+s}].$$

The free-rider observes the right evolution of the environment but does not take any costly decision to improve the environment, that is

$$\dot{y}(t) = m(t, x(t), y(t), \tau(t)).$$

2.5.2 Learning about catastrophe

As we discussed above, trajectories are revised at every h_k, except if a catastrophe happens at a time t^*, $t_j < t^* < t_{j+1}$. Here we define $\kappa_s^\sigma(t)$ as the unobservable s-th catastrophe period forecast at time $t \in [t_j^\sigma, t_{j+1}^\sigma]$ with $h_s = \mathrm{E}_{s-1}(\kappa_s(t))$, and update h_k with new h_s. Obviously, evidence shows that global warming will increase the frequency of catastrophe, or in other words, shorten the catastrophe occurrence interval.

Assumption of myopic agent

Here we suppose a catastrophe period, $\kappa(t)$ depends on the environmental variable, temperature $\tau(t)$ as well as the previous catastrophe period κ_{s-1}.

$$\kappa_s(t) = \alpha\, \kappa_{(s-1)} + \beta\, \tau^\eta(t) + \varepsilon_t \tag{2.45}$$

$\alpha\,, \beta\,, \eta$ are parameters, and ε_t follows a Gaussian white noise process with zero mean and finite unknown variance σ_ε^2. For simplicity, we assume $\alpha\,, \eta$ are constant. However, we do not know the true value of β, we suppose it follows a normal distribution $\mathcal{N}(\mu_\beta, \sigma_\beta^2)$ with mean and variance updated based on **Bayesian rule**.

$$f(\beta\,, \sigma_\varepsilon^2 \mid \kappa_s(t)) \quad = \frac{f(\kappa_s(t) \mid \beta\,, \sigma_\varepsilon^2)\, f(\beta\,, \sigma_\varepsilon^2)}{f(\kappa_s(t))} \tag{2.46}$$

where $f(\beta\,, \sigma_\varepsilon^2 \mid \kappa_s(t))$ is the posterior probability density function.

Moreover, for someone who does not believe that catastrophe period not depend on historical observation, we can simply assume $\alpha = 0$.

Proposition 1. Let us assume the priority probability density follows a normal distribution with finite mean and variance. i.e. $\beta \sim \mathcal{N}(\bar{\beta}, \bar{\sigma}^2)$. We choose a non informative prior (inverse gamma distribution) for σ_ε^2. Together with Eq.(2.45), we get

$$\mathrm{E}(\beta \mid \kappa_s(t)) = (\kappa_s(t) - \alpha\, \kappa_{s-1}^{\sigma})/\tau_t^{\eta} \tag{2.47}$$

$$\mathrm{E}(\sigma_\varepsilon^2 \mid \kappa_s(t)) = \frac{1}{n-1}\sum_{i=1}^{n}(\beta_i - \overline{\mu})^2 \tag{2.48}$$

Proof.

$$\begin{aligned} f(\beta\,, \sigma_\varepsilon^2 \mid \kappa_t) \quad &= \frac{f(\kappa_t \mid \beta\,, \sigma_\varepsilon^2)\, f(\beta\,, \sigma_\varepsilon^2)}{\int_0^\infty \int_{-\infty}^\infty f(\kappa_t \mid \beta\,, \sigma_\varepsilon^2) f(\beta\,, \sigma_\varepsilon^2) d\beta\, d(\sigma_\varepsilon^2)} \\ &\propto f(\kappa_t \mid \beta\,, \sigma_\varepsilon^2)\, f(\beta\,, \sigma_\varepsilon^2) \end{aligned} \tag{2.49}$$

where

$$\begin{aligned} &\kappa_t \mid \beta\,, \sigma_\varepsilon^2 \sim \mathcal{N}(\alpha\, \kappa^\sigma(t-1) + \beta\, \tau^\eta(t), \sigma_\varepsilon^2) \\ \text{or} \qquad &f(\kappa_t \mid \beta\,, \sigma_\varepsilon^2) = \frac{1}{\sqrt{2\pi\sigma_\varepsilon^2}} e^{-\frac{1}{2}\left(\frac{\kappa_t - (\alpha\, \kappa_{t-1}^\sigma + \beta\, \tau^\eta(t))}{\sigma_\varepsilon}\right)^2} \end{aligned} \tag{2.50}$$

Now, let us focus on exponential part of Eq.(2.50).

$$\begin{aligned} &-\frac{1}{2}\left\{\left(\frac{\kappa_t - (\alpha\, \kappa_{t-1}^\sigma + \beta\, \tau_t^\eta)}{\sigma_\varepsilon}\right)^2\right\} \\ &= -\frac{1}{2}\left\{\left(\frac{\beta - ((\kappa_t - \alpha\, \kappa_{t-1}^\sigma)/\tau_t^\eta)}{\sigma_\varepsilon/\tau_t^\eta}\right)^2\right\} \\ &= -\frac{1}{2}\left(\frac{\beta - \tilde{\beta}}{\tilde{\sigma}_\varepsilon}\right)^2 \end{aligned} \tag{2.51}$$

with notation $\tilde{\beta} = (\kappa_t - \alpha\, \kappa_{t-1}^\sigma)/\tau_t^\eta$, $\tilde{\sigma_\varepsilon} = \sigma_\varepsilon/\tau_t^\eta$.

Moreover, since ε_t is the white noise (systematic errors). We can assume β and σ_ε^2 are independent, and the prior distribution is

$$f(\beta\,,\sigma_\varepsilon^2) = f(\beta)\,f(\sigma_\varepsilon^2) \tag{2.52}$$

and

$$\begin{aligned} f(\beta\,,\sigma_\varepsilon^2 \mid \kappa_t) \quad &= f(\beta \mid \kappa_t)\,f(\sigma_\varepsilon^2 \mid \kappa_t) \\ &\propto f(\kappa_t \mid \beta\,,\sigma_\varepsilon^2)\,f(\beta\,,\sigma_\varepsilon^2) \end{aligned} \tag{2.53}$$

We choose a non informative prior (inverse gamma distribution) for σ_ε^2, i.e $\sigma_\varepsilon^2 \sim IG(a,b)$.

$$f(\sigma_\varepsilon^2 \mid a,b) \propto (\sigma_\varepsilon^2)^{-(a+1)}\,e^{-\frac{\beta}{\sigma_\varepsilon^2}} \tag{2.54}$$

We get

$$f(\beta,\sigma_\varepsilon^2) \propto \frac{1}{(\sigma_\varepsilon^2)^{n/2+1}}\,e^{-\dfrac{\sum_n(\beta_i-\beta)^2}{2\sigma_\varepsilon^2}} \tag{2.55}$$

So the product is

$$f(\beta\,,\sigma_\varepsilon^2 \mid \kappa_t) \propto \frac{1}{(\sigma_\varepsilon/\tau_t^\eta)^{1/2}}e^{-\frac{1}{2}\left(\dfrac{\beta-\tilde\beta}{\sigma_\varepsilon/\tau_t^\eta}\right)^2}\,\frac{1}{(\sigma_\varepsilon^2)^{n/2+1}}\,e^{-\dfrac{\sum_n(\beta_i-\beta)^2}{2\sigma_\varepsilon^2}} \tag{2.56}$$

Clearly, in this case, our updated $(\beta\,,\sigma_\varepsilon^2)$ follows an inverse gamma distribution with $a = \dfrac{n-1}{2}\,, b = \dfrac{(n-1)}{2}\mathrm{var}(\beta)$. Moreover, the learning process will give us the expectation as

$$\mathrm{E}(\beta \mid \kappa_t) = \tilde\beta = (\kappa_t - \alpha\,\kappa_{t-1}^\sigma)/\tau_t^\eta \tag{2.57}$$

$$\mathrm{E}(\sigma_\varepsilon^2 \mid \kappa_t) = \frac{1}{n-1}\sum_{i=1}^{n}(\beta_i - \overline{\mu})^2 \tag{2.58}$$

Therefore the proof is done.

Assumption of non-myopic agent

Here we suppose catastrophe period $\kappa(t)$ depends on the environmental variable, temperature $\tau(t)$ as well as all historical observed catastrophe period $\{\kappa_k\}_{k=1,\ldots,s-1}$.

$$\kappa_s(t) = \kappa_{s-1}^T \alpha + \beta\, \tau^\eta(t) + \varepsilon_t \tag{2.59}$$

$\beta\,, \eta$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1})^\intercal$ are parameters. Moreover, α can be rewritten as $\left(\sum_{k=1}^{s-1} \alpha_k\right) (\omega_1, \omega_2, \ldots, \omega_{s-1})^\intercal$, $\forall k = 1, 2, \ldots, s-1$, $\omega_k = \frac{\alpha_k}{\sum_{k=1}^{s-1} \alpha_k}$. Here $\omega = (\omega_1, \omega_2, \ldots, \omega_{s-1})^T$ represents a **weighted vector** for historical catastrophe observations. $\kappa_{s-1} = (\kappa_1, \kappa_2, \ldots, \kappa_{s-1})^\intercal$ is a vector of observed historical catastrophe periods. ε_t follows a Gaussian white noise process with zero mean and finite unknown variance σ_ε^2. For simplicity, we assume $\alpha\,, \eta$ are constant. We do not know the true value of β, and we suppose it follows a normal distribution $\mathcal{N}(\mu_\beta, \sigma_\beta^2)$ with mean and variance updated based on **Bayesian rule**.

2.6 Environmental learning II: the extension

Considering the co-existed agents OPT+BAU with a step-wise OPT-BAU process for index, we define the case of OPT ($i = 1$) and BAU ($i = 2$).

$$\max_{v_i \in V} \int_0^\theta L_i^n(v_i(t), x_i(t), y(t), \kappa(t)) dt \tag{2.60}$$

$$\dot{x}_i = f_i(t, v, x_i, y)\,, \quad t \in [0, \theta] \,\text{and}\, t_0 = 0\,, x_i(0) = x_0 \text{ given} \tag{2.61}$$

$$\dot{y} = g(t, \underbrace{e_1(t, v_1, x_1) + e_2(t, v_2, x_2)}_{\text{emission by OPT+BAU}}, y)\,, \; t \in [0, \theta]\,, \, t_0 = 0\,, y(0) = y_0 \,\text{given} \tag{2.62}$$

$$\dot{\kappa}(t) = l(y, \kappa)\,, \quad t_0 = 0\,, \text{and}\; \kappa(0) = \kappa(0)\,\text{given} \tag{2.63}$$

Now let us consider $\sigma = (h_0\,, \ldots\,, h_k, \theta)$ be a series of small $h_k \geq 0$, with $k = 0, 1, \ldots$, and large $\theta > 0$, and let $t_k^\sigma = \sum_k h_k$.

Step 1: We consider a piece-wise approximation of the original problem with revision time h_k, which depends on catastrophe period $\kappa(t_k^\sigma)$. Following Eq.(2.45) and (2.59), the catastrophe period updates following the Bayesian Rule. Thus, we can define the revision time h_{k+1} as the minimum between fixed revision Δ (due to GHGs) and the expectation of random revision $\Omega(\kappa_s, h_k)$ (due to catastrophes), i.e.

$$h_{k+1} = \min\{\Delta, \Omega(\kappa_s, h_k)\} \tag{2.64}$$

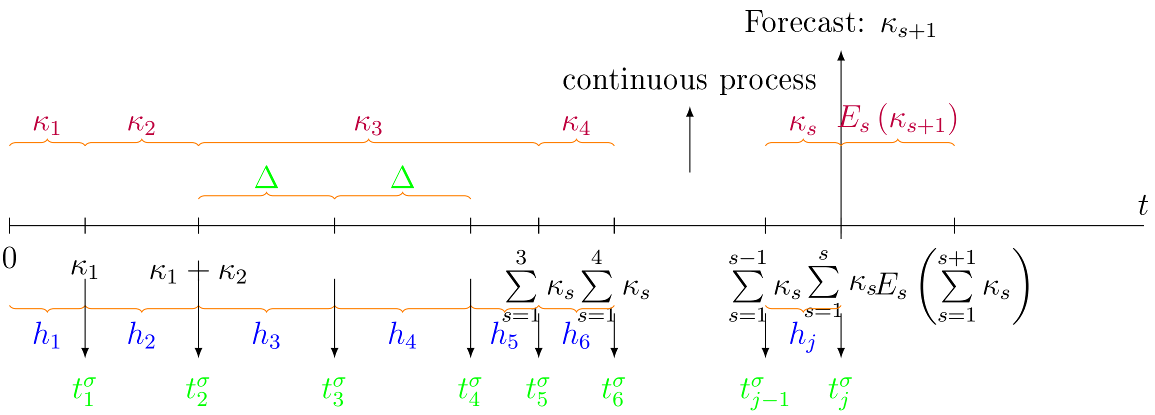
$$\text{where} \quad \Omega(\kappa_s, h_k) := \min_s \left(\mathrm{E}_s\left(\sum_{s+1} \kappa_{s+1}\right) - t_k^\sigma \right)_+ , \quad t_k^\sigma = \sum_k h_k \tag{2.65}$$

$$\mathrm{E}_s\left(\sum_{s+1} \kappa_{s+1}\right) = \sum_s \kappa_s + \mathrm{E}_s(\kappa_{s+1}) \tag{2.66}$$

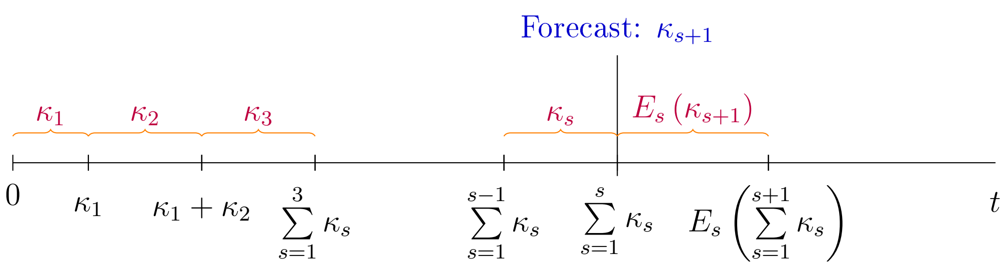
$$\mathrm{E}_s\left(\kappa_{s+1}\right) = \mathrm{E}\left(\kappa_{s+1} | \, \kappa_s, y(t_k^\sigma)\right) , \tag{2.67}$$

(or $\mathrm{E}_s\left(\kappa_{s+1}\right) = \mathrm{E}(\kappa_{s+1} | \, \kappa_s, \tau(t_k^\sigma))$, since only temperature matters)

Remark: Δ is an information updated period for GHGs. Here h_k also follows a learning process with initial condition $h_0 = 0$ given. When $k = 1$, we do not need to consider the catastrophe effect, so $h_1 = \Delta$ fixed. We can graphically explain about this as follows:



Let us consider an extreme case, when $\Delta >> \Omega$, i.e. $h_s = \kappa_s$, then we have



Step 2: The BAU agent ($i = 2$) firstly solves Eq.(2.60) and (2.61) on $[0, \theta]$

(where $\theta < \infty$) under prediction $y = y_0$.

$$\max_{v_2 \in V} \int_0^\theta L_2^n(v_2(t), x_2(t), y(t))dt \tag{2.68}$$

$$\dot{x_2} = f_2(t, v, x_2, y)\,, \quad t \in [0, \theta] \,\text{and}\, t_0 = 0\,, x_2(0) = x_0 \text{ given} \tag{2.69}$$

$$y(t) = y_0\,, \quad t \in [0, \theta] \,\text{and}\, t_0 = 0\,, y(0) = y_0 \,\text{given} \tag{2.70}$$

This way defines a hypothetical emission $\eta_2(t) = e_2(t, v_2, x_2)$ of a BAU agent on $[0, \theta]$. However, a BAU agent only implements her control on a short interval $[0, h_1]$.

Step 3: The OPT agent takes η_2 as "prediction" for emission of BAU on $[0, \theta]$, and solves

$$\max_{v_1 \in V} \int_0^\theta L_1^n(v_1(t), x_1(t), y(t))dt \tag{2.71}$$

$$\dot{x_1} = f_1(t, v, x_1, y)\,, \quad t \in [0, \theta] \,\text{and}\, t_0 = 0\,, x_1(0) = x_0 \text{ given} \tag{2.72}$$

$$\dot{y} = g(t, \underbrace{e_1(t, v_1, x_1) + \eta_2(t)}_{\text{emission by OPT+BaU}}, y)\,, t \in [0, \theta] \,\text{and}\, t_0 = 0\,, y(0) = y_0 \,\text{given} \tag{2.73}$$

An OPT agent implements the control v_1 on $[0, h_1]$. Therefore, the controls of both OPT+BAU chosen on $[0, h_1]$ determine the evolution of overall system according Eq.(2.61) and (2.62).

The same procedure is repeated on $[h_1, \theta]$ to determine the $[h_1, h_2]$. i.e. for BAU we need to solve

$$\max_{v_2 \in V} \int_{h_1}^\theta L_2^n(v_2(t), x_2(t), y(t))dt \tag{2.74}$$

$$\dot{x_2} = f_2(t, v, x_2, y)\,, \quad t \in [h, \theta] \,\text{and}\, , x_2(h_1) = x_2^* \,\text{solved} \tag{2.75}$$

$$y(h_1) = y^*(h_1)\,, \text{solved from Eq.(2.72)} \quad t \in [h, \theta] \tag{2.76}$$

where $x_2^*(h_1)$ and $y^*(h_1)$ are solved in **Step 2** and **Step 3**. This way defines a hypothetical emission $\eta_2(t) = e_2(t, v_2, x_2)$ of a BAU agent on $[h_1, \theta]$. However, as a BAU agent only implements her control on a short interval $[h_1, h_2]$, an OPT agent takes η_2 as 'prediction' for emission of BAU on $[h_1, \theta]$, and determine her control, and so on so forth.

Actually, we can introduce the following general notations for the above optimal control of the problem

$$\max_{v_2 \in V} \int_s^\theta L_2^n(v_2(t), x_2(t), y^s)dt \tag{2.77}$$

$$\dot{x_2} = f_2(t, v, x_2, y^s)\,, \quad t \in [s, \theta] \,\text{and}\,, x_2(s) = x_2^s \,\text{solved} \tag{2.78}$$

We get the optimal control $\hat{v}_2[s, \theta, x_2^s; y^s](t)$ and optimal trajectory $\hat{x}_2[s, \theta, x_2^s; y^s]$.

For the OPT agent ($i = 1$), let us denote $\hat{v}_1[s, \theta, x_1^s, y^s; \eta_2(\cdot)](t)$, a solution of problem of Eq.(2.72) and (2.73) is on interval $[s, \theta]$ (instead of $[0, \theta]$) and with initial data $x_1(s) = x_1^s$, $y(s) = y^s$ given, and thus the function η_2 exogenously given.

Thus, we could define a step-wise OPT-BAU solution $(v_1^\sigma, v_2^\sigma, x_1^\sigma, x_2^\sigma, y^\sigma)$ recurrently as the previous process. Let us now define a parameter $\rho > 1$, and complete the recurrent definition of step-wise OPT-BAU solution:

$$\tilde{v}_2^\sigma(\tau) = \hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y_k^\sigma](\tau)\,, \quad \tau \in [t_k^\sigma, t_k^\sigma + \theta]\,, \tag{2.79}$$

$$\tilde{x}_2^\sigma(\tau) = \hat{x}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y_k^\sigma](\tau)\,, \quad \tau \in [t_k^\sigma, t_k^\sigma + \theta]\,, \tag{2.80}$$

$$\tilde{\eta}_2(\tau) = e_2(\tau, \tilde{v}_2^\sigma(\tau), \tilde{x}_2^\sigma(\tau))\,, \quad \tau \in [t_k^\sigma, t_k^\sigma + \theta]\,, \tag{2.81}$$

$$v_1^\sigma(\tau) = \hat{v}_1[t_k^\sigma, \rho\theta, x_1^\sigma(t_k^\sigma); y_k^\sigma](\tau)\,, v_2^\sigma(t) = \tilde{v}_2^\sigma(t)\,, \quad \tau \in [t_k^\sigma, t_{k+1}^\sigma] \tag{2.82}$$

Remark: $(x_1^\sigma(t), x_2^\sigma(t), y^\sigma(t))$ is the solution of Eq.(2.61) and (2.62) on $[t_k^\sigma, t_{k+1}^\sigma]$ with defined controls v_1^σ and v_2^σ in Eq.(2.82), and the initial value $(x_1^\sigma(t_k^\sigma), x_2^\sigma(t_k^\sigma), y^\sigma(t_k^\sigma))$.

2.6.1 Existence, uniqueness and convergence

To prove the existence, uniqueness and convergence of the solution, let us provide the following assumption, definitions and Proposition, which are analogous to Bréchet, Camacho and Veliov (2011), but here we consider a multi-stage optimal control problem with predetermined switching time based on the approach of Tomiyama (1985) and we update the information of switching time by Bayesian learning. **Definition 2.** We define (v_1^σ, v_2^σ), the limit point of a sequence defined above with $\forall k\,, h_k \to 0$ and $\theta \to \infty$ in $L_1^{loc}(0, \infty)$, as the OPT-BAU solution of the optimization system for Eq.(2.60),(2.61) and (2.62)

Let us formulate the conditions for existence of the solution. In addition, we will prove that the OPT-BAU solution and the choice of the parameter $\rho > 1$ are independent.

Assumption A.1

(i) We assume $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and $V \subset \mathbb{R}^r$ are given open set. We assume the set E contains all vectors $e_1(t, v_1, x_1) + e_2(t, v_2, x_2)$, $t \geq 0$, $v_i \in V$, $x_i \in X$. We also assume functions $f_i(t, v, x, y)$, $g(t, e, y)$, $L_i(t, v, x, y)$, $e_i(t, v, x)$ are bounded, continuous in t and *Lipschitz continuous* in the rest of the variables, $v \in V$, $x \in X$, $y \in Y$ and $e \in E$, uniformly with respect to t. In addition, $x_i^0 \in X$ and $y^0 \in Y$.

(ii) We assume the admissible control variables v_1 and v_2, $s \geq 0$, $x_i^s \in X$ and $y^s \in Y$ the solutions of Eq.(2.61) and (2.62) with initial condition $x_i(s) = x_i^s$, $y(s) = y^s$, exist in $X \times X \times Y$ and $[s, \infty]$.

Assumption A.2

(i) We assume each $s \geq 0$, $\theta \in (0, \infty)$, $x^s \in X$, $y^s \in Y$, and continuous $\eta_2 : [s, s+\theta] \mapsto E$, the optimal controls $\hat{v}_1[s, \theta, x^s; \eta_2(\cdot)](\cdot)$ and $\hat{v}_2[s, \theta, x^s; y^s](\cdot)$ exist and are uniquely exist in $L_1(s, s+\theta)$.

(ii) We assume $\exists\ \hat{l}$ and a continuous function α , δ : $[0, \infty) \mapsto [0, \infty)$ with $\delta(0) = 0$, s.t. $\forall t \geq s \geq 0$, $\tilde{\theta} \geq \theta \geq t - s$, $x, \tilde{x} \in X$, $y, \tilde{y} \in Y$, the inequality holds for continuous functions η_2 , $\tilde{\eta}_2 : [0, \infty) \mapsto E$ as follows.

$$\left|\hat{v}_1[s,\theta,x,y;\eta_2](t) - \hat{v}_1[s,\theta,x,y;\eta_2](t) - \hat{v}_1[s,\tilde{\theta},\tilde{x},\tilde{y};\tilde{\eta}_2](t)\right| \tag{2.83}$$

$$\leq \hat{}\cdot (\mid x - \tilde{x} \mid + \mid y - \tilde{y} \mid) + \sigma\left(\mid t - s \mid + \frac{1}{\theta}\right)$$

$$+ \underbrace{\int_t^{s+\theta} \alpha(\tau - s) \mid \eta_2(\tau) - \tilde{\eta}_2(\tau) \mid d\tau}_{\underbrace{\mathrm{E}^{N-1} \int_t^{s+t_N}}_{\int_t^{s+t_N^E}} \alpha(\tau - s)|\eta_2(\tau) - \tilde{\eta}_2(\tau)|d\tau} \tag{2.84}$$

(iii) $\exists\ \delta, \beta : [0, \infty) \mapsto [0, \infty)$ with $\delta(0) = 0$ are continuous functions, and $\int_0^\infty \alpha(\tau)\beta(\tau)d\tau < \infty$, s.t. $\forall t \geq s \geq 0$, $\tilde{\theta} \geq \theta \geq 0$, $\tau \in [t, s+\theta)$,

$x, \tilde{x} \in X$, $y, \tilde{y} \in Y$, we have the inequality as follows:

$$\left| \hat{v}_2[s, \theta, x; y](\tau) - \hat{v}_2[s, \tilde{\theta}, \tilde{x}; \tilde{y}](\tau) \right| + \left| \hat{x}_2[s, \theta, x; y](\tau) - \hat{x}_2[s, \tilde{\theta}, \tilde{x}; \tilde{y}](\tau) \right|$$
$$\leq \beta(\tau - s) \cdot \left[| \ x - \tilde{x} \ | + | \ y - \tilde{y} \ | + \delta \left(| \ t - s \ | + \frac{1}{s + \theta - \tau} \right) \right] \quad (2.85)$$

Assumption A.2 shows the relative requirement of *Lipschitz dependence* of the optimal solution on the length of the horizon, including the initial data. The following lemma shows that the dependence of the optimal solution is significant only close the end of the horizon, while melts out in case of infinite horizon.

Lemma 1. $\forall s \geq 0$, $(x, y) \in X \times Y$ and continuous $\eta_2 : [0, \infty) \mapsto E$, there exit the limits in $C(s, \infty)$ for the second and the third ones.

$$\hat{v}_1[s, \infty, x, y; \eta_2](s) := \lim_{\theta \to \infty} \hat{v}_1[s, \theta, x, y; \eta_2](s) \, , \quad (2.86)$$
$$\hat{v}_2[s, \infty, x; y](\cdot) := \lim_{\theta \to \infty} \hat{v}_2[s, \theta, x; y](\cdot) \, , \quad (2.87)$$
$$\hat{x}_2[s, \infty, x; y](\cdot) := \lim_{\theta \to \infty} \hat{x}_2[s, \theta, x; y](\cdot) \quad (2.88)$$

From Assumption A.2 (ii), we understand the first claim the following is applied for $t = s, \tilde{x} = x, \tilde{y} = y, \eta = \tilde{\eta}$. The last two equations use Assumption A.2 (iii) as well as the completeness of space $C(s, T)$.

We abbreviate the notations as follows.

$$\eta_2[x_2, y](t, \cdot) = e_2(\cdot, \hat{v}_2[t, \infty, x; y](\cdot), \hat{x}_2[t, \infty, x; y](\cdot)) \, , \quad (2.89)$$
$$v_1[x_1, x_2, y](t) = \hat{v}_1[t, \infty, x_1, y; \eta_2[x_2, y](t, \cdot)](t) \, , \quad (2.90)$$
$$v_2[x_2, y](t) = \hat{v}_2[t, \infty, x_2; y](t) \, , \quad (2.91)$$
$$\eta_1[x_1, x_2, y](t) = e_1(t, v_1[x_1, x_2, y](t), x_1) \quad (2.92)$$

Proposition 2. According to Assumptions A.1 and A.2, there uniquely exits a combined OPT-BAU optimal solution.

$$\dot{x_1} = f_1(t, v_1[x_1, x_2, y](t), x_1, y) \, , \quad x_1(0) = x_1^0 \text{ given} \quad (2.93)$$
$$\dot{x_2} = f_2(t, v_2[x_2, y], x_2, y) \, , \quad x_2(0) = x_2^0 \text{ given} \quad (2.94)$$
$$\dot{y} = g(t, \underbrace{\eta_1[x_1, x_2, y](t) + \eta_2[x_2, y](t, t)}_{\text{emission by OPT+BaU}}, y) \, , \quad y(0) = y_0 \, \text{given} (2.95)$$

with t of $x_1(t)$, $x_2(t)$ and $y(t)$ suppressed. Note here $\eta_2[x_2, y](t,t)$ in Eq.(2.95) satisfy the following equality:

$$\eta_2[x_2, y](t,t) = e_2(t, \hat{v}_2[t, \infty, x_2; y](t), x_2) \tag{2.96}$$

Proof of Proposition 2

(i) **Existence and uniqueness**
From **Assumption A.1** ,*Lipschitz continuous* $\eta_i(t) = e_i(t, v, x)$ w.r.t (v, x). Specifically, from Eq.(2.96), we know $\eta_2[x_2, y](t,t) = e_2(t, \hat{v}_2[t, \infty, x; y](t), x_2)$ and in L_1 space, we have

$$\begin{aligned} \mid \eta_2(z) - \eta_2(\tilde{z}) \mid &\leq l \cdot \mid\mid z - \tilde{z} \mid\mid_1 \\ &= l \mid |v - \tilde{v}| + |x - \tilde{x}| \mid \end{aligned} \tag{2.97}$$

where l is the *Lipschitz constant* of η_2, $|| \cdot ||_1$ is the norm in L_1 space, and $z = (v, x)^{\mathrm{T}}$. We substitute $\eta(z)$ with $\eta_2[x_2, y](t,t)$ and Eq.(2.96)

$$\begin{aligned} |\eta_2[x_2, y](t,\tau) - \eta_2[\tilde{x}, \tilde{y}](t,\tau)| &= |e_2(t, \hat{v}_2[t, \infty, x; y](t), x_2) - e_2(t, \hat{v}_2[t, \infty, \tilde{x}; \tilde{y}](t), \tilde{x}_2)| \\ &\leq l \mid \mid \hat{v}_2[t, \infty, x_2; y](\tau) - \hat{v}_2[t, \infty, \tilde{x}_2; \tilde{y}](\tau) \mid + \mid \hat{x}_2[t, \infty, x; y] - \hat{x}_2[t, \infty, \tilde{x}; \tilde{y}] \mid \mid \\ &\underbrace{\leq}_{\text{Assumption A.2 (iii)}} l\beta(\tau - t) \cdot \left[\mid x - \tilde{x} \mid + \mid y - \tilde{y} \mid + \sigma \left(\underbrace{\mid t - t \mid}_{=0} + \frac{1}{\underbrace{t + \infty - \tau}_{=\infty}} \right) \right] \\ &= l\beta(\tau - t) \cdot [\mid x - \tilde{x} \mid + \mid y - \tilde{y} \mid] \end{aligned} \tag{2.98}$$

where l is a constant or Lipschitz constant of η_2(or e_2).

From Assumption A.2 (ii), with shorten notation Eq.(2.89), we obtain

$$
\begin{aligned}
&|v_1[x_1, x_2, y](t) - v_1[\tilde{x}_1, \tilde{x}_2, \tilde{y}](t)| \\
&= |\hat{v}_1[t, \infty, x_1, y; \eta_2[x_2, y](t, \cdot)](t) - \hat{v}_1[t, \infty, \tilde{x}_1, \tilde{y}; \eta_2[\tilde{x}_2, \tilde{y}](t, \cdot)](t)| \\
&\leq \hat{} \cdot (| x_1 - \tilde{x}_1 | + | y - \tilde{y} |) + \int_t^{t+\infty} \alpha(\tau - t) \mid \eta_2[x_2, y](t, \tau) - \eta_2[\tilde{x}_2, \tilde{y}](t, \tau) \mid d\tau \\
&\leq \hat{} \cdot (\underbrace{| x_1 - \tilde{x}_1 | + | x_2 - \tilde{x}_2 |}_{=|x - \tilde{x}|\,, x = (x_1, x_2)^T} + | y - \tilde{y} |) + \int_t^{\infty} \alpha(\tau - t) \underbrace{\mid \eta_2[x_2, y](t, \tau) - \eta_2[\tilde{x}_2, \tilde{y}](t, \tau) \mid}_{\text{take eq.(2.98)}} d\tau \\
&= \hat{} \cdot (| x - \tilde{x} | + | y - \tilde{y} |) + \int_t^{\infty} \alpha(\tau - t) l \beta(\tau - t) \cdot [\, | x - \tilde{x} | + | y - \tilde{y} | \,] \, d\tau \\
&= \hat{} \cdot (| x - \tilde{x} | + | y - \tilde{y} |) \left(1 + \frac{l}{\hat{l}} \underbrace{\int_t^{\infty} \alpha(\tau - t) l \beta(\tau - t) d\tau}_{<\infty\,, \quad \text{Assumption A.2 (iii)}} \right) \\
&< C_{v1} \cdot (| x - \tilde{x} | + | y - \tilde{y} |)
\end{aligned} \tag{2.99}
$$

where C_{v1} is finite number, and $\hat{l}$ is a Lipschitz constant of v_1. Therefore,
(1) $v_1[x_1, x_2, y](t)$ in **Proposition 2** Eq.(2.93),(2.94) and (2.95) depends in a Lipschitz way on x_1, x_2, y (from Eq.(2.99)).
(2) $v_2[x_2, y](t)$ depends depends in a Lipschitz way on x_2, y.(from Assumption A.2(iii), Eq.(2.85)).
(3) $\eta_1[x_1, x_2, y](t)$ depends in a Lipschitz way on x_1, x_2, y(from Assumption A.1).
(4) $\eta_2[x_2, y](t)$ depends in a Lipschitz way on x_2, y(from Assumption A.1 or Eq.(2.98)).

From (1)-(4), the right hand of system Eq.(2.93)-(2.95) is Lipschitz continuous w.r.t (x_1, x_2, y), so a unique (x_i^*, y^*) exists at least locally. Obviously, this also solves Eq.(2.60)(2.61) and (2.62) with

$$v_1^*(t) = \hat{v}_1[t, \infty, x_1^*(t), y^*(t); \eta_2(t, \cdot)](t)\,, \tag{2.100}$$

$$v_2^*(t) = \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](t)\,, \tag{2.101}$$

$$\eta_2^*(t, \cdot) = e_2(\cdot, \hat{v}_2[t, \infty, x_2^*; y^*](\cdot), x_2^*(\cdot)) = \eta_2[x_2^*, y^*](t, \cdot)\,, \tag{2.102}$$

$$\eta_1^*(t) = e_1(t, v_1[x_1^*, x_2^*, y^*](t), x_1^*) = \eta_1^*[x_1^*, x_2^*, y^*](t) \tag{2.103}$$

In particular, **Assumption A.1**(ii) implies that (x_i^*, y^*) is extendible to $[0, \infty)$.

(ii) **OPT-BAU solution convergence/coincides**
Here we consider an arbitrage pair $\sigma = (h, \theta)$, with $h = \max_k h_k$, $k = 1, \ldots$ with $\forall k\,, h_k > 0, (\rho - 1)\theta > h_k$. Let us define $\rho > 1$, later shall set $h \to 0$ and $\theta \to \infty$. Here we compare (x_i^*, y^*) with step-wise OPT-BAU solution $(v_i^\sigma, x_i^\sigma, y^\sigma)$ corresponding to σ. Denote $t_k^\sigma = \sum_k h_k$ and

$$\varepsilon_k^\sigma = \max_{t\in[t_{k-1}^\sigma, t_k^\sigma]} (\,|x_1^\sigma(t) - x_1^*(t)| + |x_2^\sigma(t) - x_2^*(t)| + |y^\sigma(t) - y^*(t)|\,) \quad (2.104)$$

for $k = 0, 1, 2, \ldots$, and $t_{-1}^\sigma = 0$.

($\bullet$) For $k = 0$, from $t_{kh}^\sigma = \sum_k h_k$, we know $t_0^\sigma = 0$. Since $t \in [\,\underbrace{t_{k-1}^\sigma, t_k^\sigma}_{t_{k-1}^\sigma = t_k^\sigma = 0}\,] = 0$

$$\begin{aligned}
\varepsilon_0^\sigma &= \max_{t=0}(\quad \underbrace{\underbrace{|x_1^\sigma(0) - x_1^*(0)|}_{=0}}_{x_1^\sigma(0)=x_1^*(0)=x_1(0),\text{initial point}} \quad + |x_2^\sigma(0) - x_2^*(0)| + |y^\sigma(0) - y^*(0)|\,) \\
&= 0 \qquad (2.105)
\end{aligned}$$

($\bullet$) Recurrently estimate ε_{k+1}^σ with ε_k^σ
Lemma 2: fromm **Assumption A.1**(i), we know $f_i(t, v, x, y)$ and $g(t, e, y)$ are bounded. So the dynamic equations ($\dot{x}_i$, and $\dot{y}$) are bounded. i.e.

$$|\dot{x}_1^*(t)| + |\dot{x}_2^*(t)| + |\dot{y}^*| < M\,, \quad \text{every}\,, t \geq 0 \quad (2.106)$$

where $M < \infty$ is a constant. Thus for $t \in [t_k^\sigma, t_{k+1}^\sigma]$, we have

$$|x^\sigma(t_k^\sigma) - x^*(t)| + |y^\sigma(t_k^\sigma) - y^*(t)| \leq \varepsilon_k^\sigma + hM \quad (2.107)$$

Proof.

$$|x^\sigma(t_k^\sigma) - x^*(t)| = \sum_{i=1}^{2} |x_i^\sigma(t_k^\sigma) - x_i^*(t)| \quad (2.108)$$

$$\begin{aligned}
|x_i^\sigma(t_k^\sigma) - x_i^*(t)| &= |x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma) + x_i^*(t_k^\sigma) - x_i^*(t)| \\
&\leq |x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma)| + |x_i^*(t_k^\sigma) - x_i^*(t)| \quad (2.109)
\end{aligned}$$

The first part of right inequality is obviously bounded.

$$\begin{aligned}
|x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma) &\leq \max_{t=t_k^\sigma}(\,|x_1^\sigma(t_k^\sigma) - x_1^*(t_k^\sigma)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t_k^\sigma)| + |y^\sigma(t_k^\sigma) - y^*(t_k^\sigma)|\,) \\
&\leq \max_{t\in[t_{k-1}^\sigma, t_k^\sigma]}(\,|x_1^\sigma(t) - x_1^*(t)| + |x_2^\sigma(t) - x_2^*(t)| + |y^\sigma(t) - y^*(t)|\,) \\
&= \varepsilon_k^\sigma \qquad (2.110)
\end{aligned}$$

The second part of the right inequality satisfies

$$\left|\frac{x_i^*(t) - x_i^*(t_k^\sigma)}{t - t_k^\sigma}\right| = \left|\frac{\Delta x_i^*(t)}{\Delta t}\right| \le \max_{t\in[t_{k-1}^\sigma, t_k^\sigma]} |\dot{x}_1^*(t)| \le M \tag{2.111}$$

where M is defined in Eq.(2.106). Thus

$$|x_i^*(t) - x_i^*(t_k^\sigma)| \le |t - t_k^\sigma| \cdot M \le hM \tag{2.112}$$

Combine Eq.(2.110) and (2.112) with Eq.(2.109) we get

$$|x_i^\sigma(t_k^\sigma) - x_i^*(t)| \le \varepsilon_k^\sigma + hM \tag{2.113}$$

Furthermore, let us consider Eq.(2.107)

$$\begin{aligned}
& |x^\sigma(t_k^\sigma) - x^*(t)| + |y^\sigma(t_k^\sigma) - y^*(t)| = |x_1^\sigma(t_k^\sigma) - x_1^*(t)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t)| + |y^\sigma(t_k^\sigma) - y^*(t)| \\
= \; & |x_1^\sigma(t_k^\sigma) - x_1^*(t_k^\sigma) + x_1^*(t_k^\sigma) - x_1^*(t)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t_k^\sigma) + x_2^*(t_k^\sigma) - x_2^*(t)| \\
& + |y^\sigma(t_k^\sigma) - y^*(t_k^\sigma) + y^*(t_k^\sigma) - y^*(t)| \\
\le \; & |x_1^\sigma(t_k^\sigma) - x_1^*(t_k^\sigma)| + |x_1^*(t_k^\sigma) - x_1^*(t)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t_k^\sigma)| + |x_2^*(t_k^\sigma) - x_2^*(t)| \\
& + |y^\sigma(t_k^\sigma) - y^*(t_k^\sigma)| + |y^*(t_k^\sigma) - y^*(t)| \\
= \; & \underbrace{|x_1^\sigma(t_k^\sigma) - x_1^*(t_k^\sigma)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t_k^\sigma)| + |y^\sigma(t_k^\sigma) - y^*(t_k^\sigma)|}_{\text{Part1}} \\
& + \underbrace{|x_1^*(t_k^\sigma) - x_1^*(t)| + |x_2^*(t_k^\sigma) - x_2^*(t)| + |y^*(t_k^\sigma) - y^*(t)|}_{\text{Part2}}
\end{aligned} \tag{2.114}$$

The **Part 1** in Eq.(2.114) is bounded.

$$\begin{aligned}
& |x_1^\sigma(t_k^\sigma) - x_1^*(t_k^\sigma)| + |x_2^\sigma(t_k^\sigma) - x_2^*(t_k^\sigma)| + |y^\sigma(t_k^\sigma) - y^*(t_k^\sigma)| \\
\le \; & \max_{t\in[t_{k-1}^\sigma, t_k^\sigma]} \left(|x_1^\sigma(t) - x_1^*(t)| + |x_2^\sigma(t) - x_2^*(t)| + |y^\sigma(t) - y^*(t)| \right) \\
= \; & \varepsilon_k^\sigma
\end{aligned} \tag{2.115}$$

We can generalize this inequality Eq.(2.111) as follows:

$$\begin{aligned}
& \left|\frac{x_1^*(t) - x_1^*(t_k^\sigma)}{t - t_k^\sigma}\right| + \left|\frac{x_2^*(t) - x_2^*(t_k^\sigma)}{t - t_k^\sigma}\right| + \left|\frac{y^*(t) - y^*(t_k^\sigma)}{t - t_k^\sigma}\right| = \left|\frac{\Delta x_1^*(t)}{\Delta t}\right| + \left|\frac{\Delta x_2^*(t)}{\Delta t}\right| + \left|\frac{\Delta y^*(t)}{\Delta t}\right| \\
\le \; & \max_{t\in[t_{k-1}^\sigma, t_k^\sigma]} |\dot{x}_1^*(t)| + |\dot{x}_2^*(t)| + |\dot{y}^*(t)| = M
\end{aligned} \tag{2.116}$$

where M is defined in Eq.(2.106). Hence

$$|x_1^*(t) - x_1^*(t_k^\sigma)| + |x_2^*(t) - x_2^*(t_k^\sigma)| + |y^*(t) - y^*(t_k^\sigma)| \le |t - t_k^\sigma| \cdot M \le hM \tag{2.117}$$

Combine Eq.(2.115) and (2.117) with Eq.(2.114), we get Eq.(2.107).

$$|x^\sigma(t_k^\sigma) - x^*(t)| + |y^\sigma(t_k^\sigma) - y^*(t)| \le \varepsilon_k^\sigma + hM$$

By definition, we have

$$|v_2^\sigma(t) - v_2^*(t)| = |\hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](t) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](t)| \quad (2.118)$$

From **Assumption A.2**(iii) and Eq.(2.85), we have

$$\begin{aligned}
&|\hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](t) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](t)| \\
&+ |\hat{x}_2[t_k^\sigma, \rho\theta, x^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](t) - \hat{x}_2[t, \infty, x^*(t); y^*(t)](t)| \\
\le\ & \beta(t - t_k^\sigma) \cdot \left[|\ x^\sigma - x^*\ | + |\ y^\sigma - y^*\ | + \delta \left(|\ t - t_k^\sigma\ | + \frac{1}{t_k^\sigma + \rho\theta - t} \right) \right] \\
\le\ & \beta(t - t_k^\sigma) \cdot \left[\varepsilon_k^\sigma + hM + \delta \left(h + \frac{1}{\rho\theta} \right) \right] \\
\le\ & C_1 \cdot \left[\varepsilon_k^\sigma + hM + \delta \left(h + \frac{1}{\theta} \right) \right] \qquad (2.119)
\end{aligned}$$

where $C_1 = \beta(t - t_k^\sigma)$ is independent of $\sigma = (h, \theta)$. Obviously one can assume without any restriction that δ and **Assumption A.2** (ii) and (iii)is monotone increasing. Combining Eq.(2.118) with (2.119), we get

$$|v_2^\sigma(t) - v_2^*(t)| \le C_1 \cdot \left[\varepsilon_k^\sigma + hM + \delta \left(h + \frac{1}{\theta} \right) \right] \qquad (2.120)$$

Moreover, for $\tau \in [t_k^\sigma, t_k^\sigma + \theta]$, according to **Assumption A.2** (iii), we obtain

$$\begin{aligned}
&|\tilde{v}_2^\sigma(\tau) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau)| + |\tilde{x}_2^\sigma(\tau) - \hat{x}_2[t, \infty, x^*(t); y^*(t)](\tau)| \\
=\ & |\hat{v}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](\tau) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau)| \\
&+ |\hat{x}_2[t_k^\sigma, \rho\theta, x_2^\sigma(t_k^\sigma); y^\sigma(t_k^\sigma)](\tau) - \hat{x}_2[t, \infty, x^*(t); y^*(t)](\tau)| \\
\le\ & \beta(\tau - t_k^\sigma) \cdot \left[|\ x^\sigma - x^*\ | + |\ y^\sigma - y^*\ | + \delta \left(|\ \tau - t_k^\sigma\ | + \frac{1}{t_k^\sigma + \rho\theta - \tau} \right) \right] \\
\le\ & \beta(\tau - t_k^\sigma) \cdot \left[\varepsilon_k^\sigma + hM + \delta \left(h + \frac{1}{(\rho - 1)\theta} \right) \right] \\
\le\ & \tilde{C}_1 \cdot \left[\varepsilon_k^\sigma + hM + \delta \left(h + \frac{1}{(\rho - 1)\theta} \right) \right] \qquad (2.121)
\end{aligned}$$

where $\tilde{C}_1 = \beta(\tau - t_k^\sigma)$ is independent of $\sigma = (h, \theta)$. The same estimate holds for both $|\tilde{v}_2^\sigma(\tau) - \hat{v}_2[t, \infty, x_2^*(t); y^*(t)](\tau)|$ and $|\tilde{x}_2^\sigma(\tau) - \hat{x}_2[t, \infty, x^*(t); y^*(t)](\tau)|$.

Then we have

$$\begin{aligned}
& |\tilde{\eta}_2^\sigma(\tau) - \hat{\eta}_2(t,\tau)| \\
= \; & |e_2(\tau, \tilde{v}_2^\sigma(\tau), \tilde{x}_2^\sigma(\tau)) - e_2(\tau, \hat{v}_2[t,\infty, x_2^*(t); y^*(t)](\tau), \hat{x}_2[t,\infty, x^*(t); y^*(t)](\tau))| \\
& \leq l\beta(\tau - t_k^\sigma) \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{(\rho-1)\theta}\right)\right] \qquad (2.122)
\end{aligned}$$

where l is the Lipschitz constant of e_2. Hence for $t \in [t_k^\sigma, t_{k+1}^\sigma]$, according to assumption **Assumption A.2** (ii), Eq.(2.99) and (2.120), we obtain successively:

$$
\begin{aligned}
&|v_1^\sigma(t) - v_1^*(t)| \\
&= |v_1[x_1^\sigma, x_2^\sigma, y^\sigma](t) - v_1[x_1^*, x_2^*, y^*](t)| \\
&= |\hat{v}_1[t_k^\sigma, \theta, x_1^\sigma(t_k^\sigma), y^\sigma(t_k^\sigma); \tilde{\eta}_2^\sigma(\tau)](t) - \hat{v}_1[t, \infty, x_1^*(t), y^*(t); \eta_2^*(t,\tau)](t)| \\
&\leq \hat{l} \cdot (|\ x^\sigma - x^*\ | + |\ y^\sigma - y^*\ |) + \delta(|t_k^\sigma - t| + \frac{1}{t_k^\sigma + \theta - t}) \\
&\quad + \int_t^{t_k^\sigma + \theta} \alpha(\tau - t_k^\sigma)\ |\ \tilde{\eta}_2(\tau) - \eta_2^*(t,\tau)\ |\ d\tau \\
&\leq \hat{l} \cdot (\varepsilon_k^\sigma + hM) + \delta(h + \frac{1}{\theta}) \\
&\quad + l \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{(\rho - 1)\theta}\right)\right] \int_t^{t_k^\sigma + \theta} \alpha(\tau - t_k^\sigma)\beta(\tau - t_k^\sigma) d\tau \\
&= (\hat{l} - 1) \cdot (\varepsilon_k^\sigma + hM) + (\varepsilon_k^\sigma + hM) + \delta(h + \frac{1}{\theta}) \\
&\quad + \left(l \underbrace{\int_{t - t_k^\sigma}^{\theta} \alpha(z)\beta(z) dz}_{z = \tau - t_k^\sigma} \right) \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{(\rho - 1)\theta}\right)\right] \\
&\leq (\hat{l} - 1) \cdot (\varepsilon_k^\sigma + hM) + \left[(\varepsilon_k^\sigma + hM) + \delta(h + \frac{1}{\theta})\right] \\
&\quad + \left(l \underbrace{\int_0^\infty \alpha(z)\beta(z) dz}_{\alpha(z)\,,\beta(z) \geq 0} \right) \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{(\rho - 1)\theta}\right)\right] \\
&= (\hat{l} - 1) \cdot (\varepsilon_k^\sigma + hM) + \left[1 + \left(l \underbrace{\int_0^\infty \alpha(z)\beta(z) dz}_{< \infty, \quad \text{Assump A.2 (iii)}} \right)\right] \\
&\quad \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{\gamma\theta}\right)\right], \quad \gamma = \min\{1, \rho - 1\}
\end{aligned}
$$

$$
\begin{aligned}
&\le (\hat{l}-1)\cdot(\varepsilon_k^\sigma + hM + \underbrace{\delta\left(h+\frac{1}{\gamma\theta}\right)}_{\delta(\cdot)\ge 0})\\
&\quad + \left[1+\left(l\underbrace{\int_0^\infty \alpha(z)\beta(z)dz}_{<\infty,\quad \text{Assump A.2 (iii)}}\right)\right]\left[\varepsilon_k^\sigma + hM + \delta\left(h+\frac{1}{\gamma\theta}\right)\right]\\
&\le \left[(\hat{l}-1)\mathbb{1}_{[\hat{l}-1\ge 0]} + 1 + \left(l\underbrace{\int_0^\infty \alpha(z)\beta(z)dz}_{<\infty,\quad \text{Assump A.2 (iii)}}\right)\right]\left[\varepsilon_k^\sigma + hM + \delta\left(h+\frac{1}{\gamma\theta}\right)\right]\\
&:= C_2\cdot\left[\varepsilon_k^\sigma + hM + \delta\left(h+\frac{1}{\gamma\theta}\right)\right]
\end{aligned}
\tag{2.123}
$$

where $C_2 = (\hat{l}-1)\mathbb{1}_{[\hat{l}-1\ge 0]} + 1 + \left(l\underbrace{\int_0^\infty \alpha(z)\beta(z)dz}_{<\infty,\quad \text{Assump A.2 (iii)}}\right)$ is finite number and independent of σ. $\hat{l}$ is a Lipschitz constant of v_1, and $\gamma = \min\{1, \rho-1\}$.

Further from Eq.(2.61) and (2.62), we know that for $t \in [t_k^\sigma, t_{k+1}^\sigma]$, $\sigma = (h,\theta)$ and $t_k^\sigma = kh$

$$\dot{x}_i^\sigma(t) = f_i(t, v_i(t), x_i(t), y(t)) \tag{2.124}$$
$$\dot{x}_i^*(t) = f_i(t, v_i(t), x_i^*(t), y(t)) \tag{2.125}$$

Thus

$$
\begin{aligned}
\frac{d|x_i^\sigma - x_i^*|}{dt} \le |\dot{x}_i^\sigma(t) - \dot{x}_i^*(t)| &= |f_i(t, v_i(t), x_i(t), y(t)) - f_i(t, v_i(t), x_i^*(t), y(t))\\
&\le \lambda |x_i^\sigma(t) - x_i^*(t)|
\end{aligned}
\tag{2.126}
$$

Here we use the property that $f_i(t, v_i(t), x_i(t), y(t))$ is *Lipschitz* continuous in (x_i, y, v_i). With **Gronwall Inequality**, we obtain

$$
\begin{aligned}
|x_i^\sigma(t) - x_i^*(t)| &\le |x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma)|e^{\lambda(t-t_k^\sigma)}\\
&\le |x_i^\sigma(t_k^\sigma) - x_i^*(t_k^\sigma)|e^{\lambda h}
\end{aligned}
\tag{2.127}
$$

where $t \leq t_{k+1}^\sigma = (k+1)h$, and $t - t_k^\sigma \leq h$. From Eq.(2.107) and **Assumption**(iii), for $t \in [t_k^\sigma, t_{k+1}^\sigma]$,

$$
\begin{aligned}
|\hat{v}_2^\sigma(t) - \hat{v}_2^*(t)| + |x_2^\sigma(t) - x_2^*(t)| &\leq \beta(t - t_k^\sigma)(\underbrace{|x^\sigma - x^*| + |y^\sigma - y^*|}_{\text{take in eq.(2.107)}} + \delta(h + \frac{1}{\theta})) \\
&\leq C_3 \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{\gamma\theta}\right)\right] \qquad (2.128)
\end{aligned}
$$

So we have for $t \in [t_k^\sigma, t_{k+1}^\sigma]$

$$
\begin{aligned}
|x_2^\sigma(t) - x_2^*(t)| &\leq \beta(t - t_k^\sigma)(\underbrace{|x^\sigma - x^*| + |y^\sigma - y^*|}_{\text{take in eq.(2.107)}} + \delta(h + \frac{1}{\theta})) \\
&\leq C_3 \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{\gamma\theta}\right)\right] \qquad (2.129)
\end{aligned}
$$

From Eq.(2.104) and (2.106) we have for $t \in [t_k^\sigma, t_{k+1}^\sigma]$

$$
|x^\sigma(t) - x^*(t)| + |y^\sigma(t) - y^*(t)| \leq \varepsilon_k^\sigma \qquad (2.130)
$$

$$
\begin{aligned}
\varepsilon_{k+1}^\sigma &= \max_{t \in [t_k^\sigma, t_{k+1}^\sigma]} (\,|x_1^\sigma(t) - x_1^*(t)| + |x_2^\sigma(t) - x_2^*(t)| + |y^\sigma(t) - y^*(t)|\,) \\
&\leq e^{\lambda h}[\varepsilon_k^\sigma + C_3 \cdot \left[\varepsilon_k^\sigma + hM + \delta\left(h + \frac{1}{\gamma\theta}\right)\right]] \qquad (2.131)
\end{aligned}
$$

This clearly implies the existence of a number C4 independent of σ (see, e.g. Lemma 2.2 in Wolenski 1990).

Lemma 2.1 If $(\varepsilon_k^\sigma)_{k=1,2,\ldots,N}$ follows

$$
\varepsilon_{k+1}^\sigma \leq \beta\varepsilon_k^\sigma + \alpha\,, \quad \varepsilon_0^\sigma = 0 \qquad (2.132)
$$

Then exists $C_4 < \infty$ such that $\varepsilon_N^\sigma \leq C_4\,\alpha$.
Proof

$$
\begin{aligned}
\varepsilon_N^\sigma &\leq \beta\varepsilon_{N-1}^\sigma + \alpha \\
&\leq \beta^2\varepsilon_{N-1}^\sigma + \beta\alpha + \alpha \\
&\leq \beta^N \underbrace{\varepsilon_0^\sigma}_{=0} + (\beta^{N-1} + \beta^{N-2} + \ldots + \beta + 1)\alpha \\
&= \frac{1 - \beta^N}{1 - \beta}\alpha \qquad (2.133)
\end{aligned}
$$

Hence $C_4 = \dfrac{1-\beta^N}{1-\beta}$. Especially, when $\beta < 1$ and $N \to \infty$, we have $C_4 = \dfrac{1}{1-\beta}$ In our case, we have

$$\beta = e^{\lambda h}(1 + C_3 h) \tag{2.134}$$
$$\alpha = C_3 h[h + \delta(h + \frac{1}{\gamma\theta})] \tag{2.135}$$

We then can see

$$\varepsilon_k^\sigma \leq C_4 e^{\lambda k h}[h + \delta(h + \frac{1}{\gamma\theta})] \tag{2.136}$$

From the estimations for $|v_i^\sigma(t) - v_i^*(t)|$ obtained above, we conclude that for every finite interval [0,T], one can estimate

$$|v_i^\sigma(t) - v_i^*(t)| \leq \bar{C}_T \left[\eta + \delta\left(\eta + \frac{1}{\gamma\theta}\right)\right] \tag{2.137}$$
$$\begin{aligned} \mathrm{E}|v_i^\sigma(t) - v_i^*(t)| &\leq \bar{C}_T \left[\mathrm{E}(\eta) + \delta\left(\mathrm{E}(\eta) + \frac{1}{\gamma\theta}\right)\right] \\ &= \bar{C}_T \left[\min\{h, \mu\} + \delta\left(\min\{h, \mu\} + \frac{1}{\gamma\theta}\right)\right] \end{aligned} \tag{2.138}$$

Here, $\bar{C}_T$ and δ are independent. Hence, $\mathrm{E}(v_i^\sigma(t)) \to v_i^*(t)$ when $h \to 0, or\, \mu \to 0,\ \theta \to \infty$. So we prove the convergence which implies the claim of the proposition.

2.7 Conclusion

We propose a predictive control model under great uncertainty. A multi-stage optimal control approach with predetermined switching time is presented. We apply this approach to the issue of global warming with an additional important environmental variable: catastrophes, which is correlated with the global warming. The catastrophe learning process is described under both myopic agent and non-myopic agent assumptions. We prove the existence of the solution.

3 Sustainable growth I: environment and health uncertainty

3.1 Introduction

The detrimental impact of production-induced environmental degradation, such as pollution, global warming and associated catastrophes, leads to significant economic and health losses that are likely to change population's consumption and saving decisions, and as a consequence, economic growth. A growing number of empirical evidence shows that the influence of environmental pollution, e.g. haze, water pollution and food contamination, entails severe human health degradation, such as the increasing morbidity of chronic diseases and the associated mortality rate. As pollution-induced diseases have a major detrimental impact on the working-age population, the negative effect on labour productivity can significantly influence the overall economic activity . The World Health Organization (WHO, 2004) reports that 56% population suffering from chronic diseases are aged between 15 to 59 in high-income countries. Moreover, an overall 3 million deaths are caused by outdoor air pollution in global cities and rural areas in 2014, and the vast majority of those forms of deaths (70%) occurs in low- and middle-income countries.[1] Moreover, healthier employees are more productive and less likely to call in sick or reduce presenteeism.[2] The associated economic losses are striking. In the U.S., an amount of 277 billion U.S. dollar annual expenditure is spent on treatment to seven of the most common chronic diseases, and the productivity loss is equal to 1.1 trillion U.S. dollar per year (Devol and Bedroussian, 2007). In Europe, air pollution causes approximate 6 million premature deaths, and the associated economic loss is no less than 1.6 trillion U.S. dollar a year (WHO and OECD 2015). The situation is even worse in emerging economies. Taking

[1] WHO news, March 25, 2014. Available online: http://www.who.int/mediacentre/news/releases/2014/air-pollution/en/

[2] CDC resource, December 4, 2015. Available online: https://www.cdc.gov/workplacehealthpromotion/model/control-costs/benefits/productivity.html

China as an example, the World Bank (2007) estimates that the health costs due to air and water pollution is equal to 4.3% of its annual GDP.

Reducing production emissions has become an urgent and primary issue of environmental and health policies worldwide. Epidemiological research has discovered consistent evidence of the damaging effects of pollution on human health (Brunekreef and Holgate, 2002), and documented the long- and short-term effect of air pollution on respiratory and cardiovascular illness and the associated morbidity and mortality effects (Daniels et al., 2000; Dockeryet et al., 1993; Pope et al., 1995, 2002; Samet et al., 2000).

From the economic perspective, pollution-induced health problems have stimulated a wide number of research streams. Bovenberg and Smulders (1995), Pautrel (2008) and Mariani et al. (2010) have studied the impact of the negative effect of pollution on life expectancy, whereas Goenka et al. (2012) and Palivos and Varvarigos (2011) have analysed the correlation between environmental quality and longevity. Howitt (2005) adds the health status in a standard Solow-Swan model of growth and discusses different channels through which the state of health affects the growth path, whereas Wang et al. (2015) and Gutierrez (2008) analyse the link between pollution and growth. Bretscher and Vinogradova (2014, 2016) have extended this line of analysis to an economy which is endogenously growing over an infinite time horizon and is subject to recurring random shocks to the health status of the population. They derive a closed-form analytical solution for a growing and polluting economy which is subject to emission induced macroeconomic health shocks and characterized by a deterministic process of capital accumulation.

In this paper, we present an endogenous growth model that extends one dimensional model of Bretschger and Vinogradova (2016) into two dimensions by making both capital and health regeneration processes stochastic. This extension is motivated by the necessity to explicitly consider the influence of health status on production. Bretschger and Vinogradova (2016) consider that the uncertainty only exists in the health regeneration process. Given the basic AK-type economy where TFP parameter A is constant, health status (which can be viewed as human capital) influences neither the production function, nor the capital accumulation dynamics. In this scenario, society's health level does not effectively affect production, as the output can always be enlarged by investing in capital, even if the pollution generated by a larger amount of output negatively affects the working-age population and labour productivity.

To overcome this limitation, we introduce correlated uncertainties within the capital and health regeneration processes. This extension, that indirectly entails the effect of labours' health on production, leads to our central result which consists of two formulas for optimal abatement policy (or, equivalently, the optimal carbon tax) and optimal growth rate. We propose a stochastic dynamic system that allows the economy to grow at a "healthy" rate while considering production externalities. In this regard, we generalize a benchmark inter-temporal decision-making model à la Ramsey with extension to a stochastic optimal control problem. Inspired by Martin and Pindyck (2015), we consider an endogenous growing economy where the associated uncertainties are correlated and caused by various factors, such as global warming, earthquakes and uncontrolled viral epidemics. Using correlated uncertainties in capital and health regeneration processes, we indirectly entail the effect of labours' health on production. In our framework, the sign of correlation coefficient can be either positive or negative. On one hand, positive fluctuations on health increase labour productivity, and hence accelerate growth (i.e. "productivity effect"). On the other hand, positive fluctuations on production generates pollutants, which in turn degrade health (i.e. "environmental effect"). More generally, health regeneration and productivity can be positively (or negatively) correlated, depending on whether "productivity effect" dominates (or is dominated by) the "environmental effect". In a green economy with less pollutant industries where "productivity effect" dominates "environmental effect", we observe a positive correlation between production and health. On the contrary, in a "dirty" economy where more pollutant per unit of output is emitted, the "productivity effect" is dominated by the "environmental effect", thus we observe a negative correlation. If the economy is extremely dirty, a consumer can not compensate for her health-induced utility loss through more consumption on any other goods. In other words, unlimitedly production at the expense of health degradation is not desirable in our model.

The model offers several theoretical results. Firstly, we find that the economic growth rate is negatively affected by health fluctuation when the abatement policy, defined as a fraction of output for abating emission, is explicitly considered. Secondly, we show that the associated correlation between capital and health uncertainties may strengthen or offset this detrimental effect, depending on the sign of correlation coefficient. Thirdly, the relationship between abatement policy and growth rate is inverted-U shaped when both capital and health uncertainties are considered. Specifically, we show that, the greater the relative risk aversion, the more likely the environmental policy will stimulate economic growth. Even if compatible with the analysis of the optimal carbon tax at global level, the model permits the assessment of the environmental policy at country level, an important insight presented in Section 5 using numerical simulation. Finally, we demonstrate that the optimal abatement policy reacts significantly not only to economic parameters, like emission intensity, efficiency of abatement technology and TFP, but also to health and uncertainty parameters. The abatement policy is also sensitive to the sign of correlation coefficient, and this sensitivity is intensified if the measure of relative risk averse decreases. Specifically, we find that the optimal abatement-output ratio is 0.46%, equivalently as 33 U.S. dollar per ton of coal for carbon tax in the benchmark model, and 0.69% when the development of abatement technology is viewed in a less optimistic way, implying a relatively higher carbon tax equal to 50 U.S. dollar per ton of coal. The former result is higher but comparable to 0.42% and 30 U.S. dollar found in Nordhaus (2008). Our quantitative results suggest a higher carbon tax when the two-dimensional uncertainties are considered.

The remainder of the paper is organized as follows. Section 2 presents a short literature review on environmental and health uncertainties with economic growth. Section 3 develops our model which describes the uncertainties associated with capital accumulation and health regeneration. Section 4 provides an analytical illustration for optimal growth rate in different scenarios. Section 5 presents the model calibration in comparison with benchmark parameters and estimates. Section 6 concludes.

3.2 Literature review

3.2.1 Environment and health

There is a large amount of literature studying the detrimental impact of environmental externalities on human health and economic growth. The focus of these research is to analyse the linkages between production, pollutants, health degradation, mitigation policy and public investment. A number of overlapping generations (OLG) models have studied these effects (see, e.g., Gutiérrez, 2008; Balestra and Dottori, 2012; Wang et al., 2015). In these models, consumers are living for two periods. The young generation is always healthy, while the old generation faces the risk of mortality or unhealthy state, depending on the impact of pollutions. Specially, Gutiérrez (2008) finds that pollution raises health costs, and consequently, stimulates growth by fostering precautionary savings and capital accumulation. Other papers find welfare losses and growth slows down. A three period OLG model can be found in Mariani et al. (2010), where agents may invest in environmental care, which affects their life expectancy. An interesting scenario of multiple equilibria is discussed in this paper. This illustrates the so-called low-life-expectancy/low-environmental-quality trap caught by some developing countries. In the model of Mariani et al. (2010), the survival probability depends on the inherited environmental quality and is assumed to be constant in equilibrium states.

In order to overcome the dynamic inefficiency in OLG framework á la Diamond (1965), some papers (see, e.g., Davide and Dottori, 2012; Wang et al., 2015) discuss the second-best equilibrium using health insurance as an instrument. In particular, Wang et al. (2015) highlight that precautionary saving acts as a substitute for lacking health insurance. In contrast to Mariani et al. (2010), Davide and Dottori (2012) assume that the life expectancy in the second period depends on current environmental conditions. Hence, young generation can invest to improve environmental quality and receive returns when they are old. Political effect of population ageing is discussed as well.

One common feature of the models mentioned above is that human health does not enter the utility function. Introducing the health status in the welfare function á la Grossman (1972a), Pautrel (2012) extends the OLG model by considering the impact of pollution as endogenous depreciation on health dynamics. In Pautrel's model, working-age agents make trade-off between consumption and investment in health enhancing activities in order to maximize their welfare. Pautrel (2012) shows the relationship between environmental taxation and economic activities is inverted-U shaped. This inverted-U shape also holds between environmental taxation and lifetime welfare.

3.2.2 Health uncertainty

In health economics literature, the Grossman model of demand for health (1972a) is canonic to add health into consumers' welfare function. Theoretical extensions and some competing economic models can be found in Grossman (1972b, 2000), Wagstaff (1987), Zweifel and Breyer (1997), and Galama (2011), where health is viewed as a capital stock depreciating overtime but can be enhanced by investment in medical care to produce healthy time that benefit individual welfare and promote labour productivity. Some empirical studies on the model of demand for health can be found in Wagstaff (1986) who estimates Grossman model (1972a) using the 1976 Danish Welfare Survey. Van Doorslaer (1987) and Wagstaff (1993) extend the empirical study using longitudinal data. Uncertainty is introduced in the model of demand for health by a number of papers. Cropper (1977) assumes that illness will occur once the stock of health falling below a critical sickness level, which is random. This model indicates that an agent with higher income or wealth level will maintain higher stock of health than a poorer agent. Selden (1993) and Chang (1996) show that risk averse people enlarge their investments in health due to the effect of uncertainty. Compared to the model with perfect certainty, the expected value of the stock of health, optimal investment and health input are all larger when the effect of uncertainty is taken into account. Laporte and Ferguso (2007) extend the Grossman model by considering of health as a stochastic variable in order to characterize the case of permanent reduction in individual's stock of health when a serious illness happens.

3.2.3 Environmental uncertainty

An increasing amount of theoretical models highlights the role of environmental uncertainty on economic growth. Weitzman and Läfgren (1997) show the effect of environmental catastrophe on economic growth using national accounting and welfare measures, where the probability of catastrophe occurrence is driven by anthropogenic activities. Tsur and Withagen (2011) study a dynamic model on abrupt climate change, where a certain kind of capital is used to adapt the catastrophe. The catastrophe is caused by climate change, and the occurrence date is stochastic, whose distribution depends on atmospheric GHGs concentration. Bretschger and Vinogradova (2014) present a stochastic model of a growing economy where natural disasters occur randomly, and the damages are caused by polluting activity. Besides climate catastrophe, Martin and Pindyck (2015) analyse different types of catastrophes. They focus on the social cost of each catastrophe and develop a rule to determine which kind of catastrophe should be averted. While health and environment literature are explicit about health production by households and production-induced environmental degradation respectively, the correlation between health and environment are largely disregarded.

3.3 The model

We consider an endogenous growing economy with frequently occurred fluctuations caused by various factors, such as global warming damage, pollution and uncontrolled viral epidemics. Let us describe the uncertainties associated with capital accumulation and health regeneration through a two-dimensional Wiener process.[3]

3.3.1 Economy and climate: a general specification

We consider a stochastic version of the neoclassical growth model. Suppose a risk neutral policy maker maximizes overall expected discounted welfare over an infinite time period in an one-sector economy made of homogeneous individuals. There is a representative agent with the utility function

$$\mathbb{E}_0 \left\{ \int_0^\infty U(c_t, h_t)\, e^{-\rho\, t}\, dt \right\} \tag{3.1}$$

where c is consumption, h is health demand, and $\rho \in (0,1)$ is time discount factor. $U(\cdot)$ is a positive, increasing and concave function in both of its arguments:

$$U(\cdot) \geq 0, \;\; U'(\cdot) \geq 0, \;\; U''(\cdot) < 0.$$

Additionally, U satisfies the Inada conditions:

$$\lim_{c \to 0} U_c(c,h) = \infty, \;\; \lim_{h \to 0} U_h(c,h) = \infty, \;\; \text{and} \;\; \lim_{c \to \infty} U_c(c,h) = 0, \;\; \lim_{h \to \infty} U_h(c,h) = 0.$$

In contrast to the capital dynamic in a deterministic Keynes-Ramsey model, we assume that production is subject to random shocks which may come

[3]According to Steger (2005), the type of stochastic process (i.e. Wiener vs. Poisson) is less important once the expected value and variance of uncertainties are fixed

from consumers' health shock. More precisely, inspired by the stochastic economic growth model of Steger (2005), we define the law of motion of capital accumulation as follows:

$$dK_t = \big[Y_t - M_t - C_t\big]dt + \sigma K_t dZ_{k,t}\,, \quad K_t\,,\, M_t\,,\, C_t\,,\, t \geq 0\,,\ K(0) = K_0 \text{ given} \tag{3.2}$$

where M_t is the aggregate form of abatement to mitigate emissions. $Z_{k,t}$ is a standard Wiener process, and $\sigma \geq 0$ is defined as sensitivity of the capital to fluctuations, which measures the amplitude of randomness caused by pollution shock entering the process of capital accumulation. Y_t is output, and firms produce a composite consumption good using broadly defined capital stock, denoted K_t as input. We assume the production function is constant returns to scale, and, for simplicity, the population is constant and unitized to 1.

$$Y_t = F(K_t) = A_t K_t \tag{3.3}$$

where total factor productivity A_t is assumed to be constant for simplicity: $A_t \equiv A$. It is worth to remark that, as a broadly accepted model in endogenous growth literature, the input K_t in AK model is interpreted as a broad measure of capital in the economy, such as physical capital, human capital, knowledge, etc. Suggested by Grossman (1972a, 1972b) and Cropper (1981), we consider a variety of pollution degrades health status. Yet, the exact negative impact of pollution on health is non-deterministic, and therefore is assumed to be driven by a stochastic process:

$$dh_t = R(q_t) h_t dZ_{h,t}\,, \quad t\,, q_t \geq 0\,, R(q_t) \geq 0\,. \tag{3.4}$$

where q_t is emission concentrations per unit of capital, and $R(q_t)$ represents the impact of emissions on health. $Z_{h,t}$ is a standard Wiener process. The health regeneration process proposed here is inspired by Bretschger and Vinogradova (2016). In addition, we assume capital and health uncertainties are correlated:

$$\mathbb{E}(dZ_{k,t}\, dZ_{h,t}) = \rho_{k,h}\, dt\,, \quad \rho_{k,h} \in [-1, 1]\,,\, t \geq 0\,. \tag{3.5}$$

where $\mathbb{E}(\cdot)$ denotes expected values of $(\cdot)$ and $\rho_{k,h}$ is correlation coefficient, indicating the mutual influence between capital and health fluctuations.

3.3.2 Specializing some assumptions

Our optimal growth problem is characterized under three assumptions which are discussed below.

Preferences

We consider a standard constant relative risk aversion (CRRA) utility function

$$\textit{Assumption 1:} \qquad U(c_t, h_t) = \frac{c^{1-\epsilon}}{1-\epsilon} h^{\beta} \tag{3.6}$$

where the constant parameters $\epsilon\,, 1-\beta \in (0,1)$ represent Arrow-Pratt measure of relative risk aversion and the elasticity of marginal utility to consumption and health respectively[4]. The parameters of consumption and health are discussed in detail in the quantitative section of this paper.

Emissions

The aggregate form of mitigation (or abatement) M_t is provided to mitigate the emissions, financed by a fraction of output, denoted by $m_t \in [0,1)$. The remaining share of output $1 - m_t$ is divided between consumption and capital accumulation. So we define m_t as abatement-output ratio:

$$m_t = \frac{M_t}{Y_t}, \qquad \text{where}\, Y_t > 0\,, M_t \geq 0. \tag{3.7}$$

The total emission is an increasing function of the output and a decreasing function of abatement spendings. The stock of aggregate emissions, denoted by Q_t, can be represented as:

$$Q_t = \phi Y_t - \mu M_t\,, \quad \phi\,, \mu\,, t \geq 0\,. \tag{3.8}$$

Parameter $\phi \geq 0$ is so-called the emission intensity of output, which indicates the emissions per unite of output. $\mu \geq 0$ is the efficiency of abatement. By definition, we have $q_t = Q_t/K_t$, where q_t is the emission concentrations per unit of capital defined in Eq.(5.35). We can represent Eq.(4.11) in terms of q_t as follows:

$$\textit{Assumption 2:} \qquad q_t = (\phi - \mu m_t)A\,, \quad \phi\,, \mu\,, t \geq 0\,. \tag{3.9}$$

Apparently, the total emission needs to be non-negative: $Q_t \geq 0$. Hence, the abatement-output ratio m_t should be bounded from above: $m_t \leq \phi/\mu$,

[4]In order to guarantee the concavity of $U(\cdot)$ i.e. $U'_h(\cdot) > 0$, we require that $\epsilon \in (0,1)$. It is worth nothing that when $\epsilon = 1$, using L'hospital's rule, we have the logarithm utility function: $U(c_t, h_t) = \ln(c)\, h^{\beta}$.

if $\phi/\mu \in (0,1)$. Pollution is caused by production-released detrimental emissions. Therefore, we assume the pollution-induced health degradation is proportional to emission per capital.

$$\textit{Assumption 3:} \qquad R(q) = \delta q \tag{3.10}$$

where $\delta \geq 0$ denotes the sensitivity of health status to pollution.

3.3.3 The planning problem

We now substitute our key assumptions into the general formulation and state the planning problem. Since we consider the AK-type growth model with constant population size unitized to 1, capital per labour and output per labour are identical to overall capital and output respectively, i.e.

$$k_t \equiv \frac{K_t}{L_t} = K_t \,, \quad y_t \equiv \frac{Y_t}{L_t} = Y_t \tag{3.11}$$

Substituting m_t defined in *Assumption 2* into Eq.(3.2), we have

$$dk_t = \big[(1-m_t)y_t - c_t\big]dt + \sigma k_t dZ_{k,t} \,, \quad k_t \,,\, c_t \,,\, t \geq 0 \,,\; k(0) = k_0 \text{ given} \tag{3.12}$$

Moreover, combing Eqs.(5.35), (3.5) and (3.12), the policy maker solves the following problem

$$\max_{c,m} \mathbb{E}_0 \left\{ \int_0^\infty U(c_t, h_t)\, e^{-\rho t}\, dt \right\} \tag{3.13}$$

subject to

$$\begin{aligned}
dk_t &= \big[(1-m_t)y_t - c_t\big]dt + \sigma k_t dZ_{k,t} \,, \quad k_t \,,\, c_t \,,\, t \geq 0 \,,\; k(0) = k_0 \,\text{given} \,, \\
dh_t &= R(q_t) h_t dZ_{h,t} \,, \qquad q_t \,,\, h_t \,,\, t \geq 0 \,, \\
\mathbb{E}(dZ_{k,t} dZ_{h,t}) &= \rho_{k,h} dt \,, \qquad \rho_{k,h} \in [0,1] \,,\, t \geq 0 \,.
\end{aligned}$$

From now on, we omit the time subscripts for control and state variables for convenience. Eq.(3.12) can be further simplified if the saving rate is assumed to be constant, i.e. we assume $c = xk$, where x is a constant. Then capital dynamic can be expressed in a standard Geometric Brownian Motion (GBM):

$$dk = \mu k dt + \sigma k dZ_{k,t} \,, \quad k \,,\, t \geq 0 \,,\; k(0) = k_0 \text{ given} \,. \tag{3.14}$$

where drift parameter $\mu = (1-m)A - x$ is constant in optimal. The abatement-output ratio formula in our main proposition below relies on saving-rate constancy. In Appendix, we derive the analytical solution for x as function of the model's parameters by using the guess and verify method. In the data, moreover, saving rates do not tend to vary so much over time, and long-run growth models are often specified so that c_t/y_t is constant (see, e.g. Acemoglu, 2009).

3.3.4 Competitive equilibrium

In this subsection, we characterize the solutions to the planner's problem in the set-up described above. Following the approach of Dixit and Pindyck (1994), and Steger (2005), we write down the Hamilton-Jacobi-Bellman (HJB) equation for Eqs.(3.5) to (3.13) as follows:

$$\rho V(k,h) = \max_{c,m} \left\{ U(c,h) + \frac{1}{dt}\mathbb{E}\big[dV(k,h)\big] \right\} \tag{3.15}$$

where $V(k,h) = \max\limits_{c,m} U(c,h)$ is the value function associated with the optimal control problem. Applying Ito's lemma, we obtain HJB in the following expression:

$$\rho V(k,h) = \max_{c,m} \Big\{ U(c,h) + V_k(k,h)\big[(1-m)y - c\big] + \frac{1}{2}\big[V_{kk}(k,h)\sigma^2 k^2 + V_{hh}(k,h)R^2(q)h^2 + 2\rho_{k,h}V_{kh}(k,h)\sigma k R(q)h\big] \Big\} \tag{3.16}$$

Calculating the first-order conditions (FOCs) for control variables $\{c, m\}$, we have:

$$\text{For } c: \quad U_c = V_k \tag{3.17}$$

$$\text{For } m: \quad V_k\, y_t = (-\mu y_k')\Big(V_{hh}R(q)R'(q)h^2 + V_{kh}\sigma R'(q)\rho_{kh}kh\Big) \tag{3.18}$$

Applying envelope theorem to state variables $\{k\,, h\}$, we have

$$\text{For } k: \quad \rho V_k = V_{kk}[(1-m)y - c] + V_k(1-m)A + \frac{1}{2}\Big[V_{kkk}\sigma^2 k^2 + 2\sigma^2 V_{kk}k + V_{hhk}R^2(q)h^2 + 2\rho_{kh}\sigma V_{khk}R(q)hk + 2\sigma\rho_{kh}V_{kh}R(q)h\Big] \tag{3.19}$$

$$\text{For } h: \quad \rho V_h = U_h + V_{kh}[(1-m)y - c] + \frac{1}{2}\Big[V_{kkh}\sigma^2 k^2 + V_{hhh}R^2(q)h^2 + 2V_{hh}R^2(q) + 2\sigma\rho_{kh}V_{khh}R(q)kh + 2\sigma\rho_{kh}V_{kh}R(q)k\Big] \tag{3.20}$$

Solving the above system of differential equations, we also obtain the optimal abatement-output ratio:

$$m^* = \frac{\phi}{\mu} - \frac{(1-\epsilon)\big[1+\beta\rho_{kh}\mu\sigma\delta\big]}{A\beta(1-\beta)(\mu\delta)^2}\,, \qquad (3.21)$$
$$\text{with } A\,,\delta\,,\sigma\,,\rho\,,\mu \geq 0\,,\epsilon\,,\beta \in (0,1)\,,\rho_{kh} \in [-1,1].$$

In order to ensure $m^* \in [0,1)$, the parameters should satisfy the following inequality: $\frac{\phi}{\mu} - 1 < \frac{(1-\epsilon)\big[1+\beta\rho_{kh}\mu\sigma\delta\big]}{A\beta(1-\beta)(\mu\delta)^2} \leq \frac{\phi}{\mu}$. Substituting Eq.(3.21) into *Assumption 2*, we obtain the optimal emission concentration:

$$q^* = \frac{(1-\epsilon)\big[1+\beta\sigma\mu\rho_{kh}\delta\big]}{\beta(1-\beta)\mu\delta^2}\,, \qquad (3.22)$$
$$\text{with } \quad A\,,\delta\,,\sigma\,,\rho\,,\mu \geq 0\,,\epsilon\,,\beta \in (0,1)\,,\rho_{kh} \in [-1,1].$$

We require $-1 \leq \beta\rho_{kh}\sigma\mu\delta \leq 0$ to ensure $q^* \geq 0$.[5] Applying Ito's lemma and substituting Eq.(3.19) into Eq.(3.17), we obtain the expected growth rate of consumption, denoted g as follows:

$$g = \mathbb{E}\left(\frac{\dot{c}}{c}\right) = \frac{1}{\epsilon}\Big\{(1-m)A - \rho + \Delta\Big\}\,, \qquad (3.23)$$

where Δ represents the effect of uncertainty on the expected growth rate. Moreover, Δ can be decomposed into the following three items: $\Delta = \Delta_h + \Delta_k + \Delta_\rho$, with

$$\Delta_h = -\frac{1}{2}\beta(1-\beta)(\delta A)^2(\phi-\mu m)^2 \leq 0\,, \qquad (3.24)$$
$$\Delta_k = -\frac{1}{2}\sigma^2\epsilon(1-\epsilon) \leq 0\,, \qquad (3.25)$$
$$\Delta_\rho = \rho_{kh}(1-\epsilon)\beta\sigma\delta A(\phi-\mu m)\,. \qquad (3.26)$$

where $\Delta_h\,,\Delta_k\,,\Delta_\rho$ represent respectively the uncertainty effect of health, capital and their correlation. $\Delta_h\,,\Delta_k$ are non-positive. Δ could be positive or negative, depending on the sign of correlation parameter $\rho_{k,h}$. For instance, $\rho_{k,h} < 0$ implies $\Delta_\rho < 0$, and hence Δ is negative. So the uncertainties dampen the growth rate. Substituting optimal abatement-output ratio m^* into Eq.(3.23), we obtain expected optimal growth rate g^* in expression of model's parameters:

$$g^* = \frac{A-\rho}{\epsilon} + \frac{1}{\epsilon}\left(I_g - \frac{\phi}{\mu}A\right) \qquad (3.27)$$

[5] In practice, $-1 \leq \beta\rho_{kh}\sigma\mu\delta \leq 0$ holds, since $\beta\,,\mu\,,\delta\,,\sigma \in (0,1)$ and $\rho_{kh} \in [-1,1]$

where I_g represents the total uncertainty effect.

$$I_g = \frac{\left[1+(1-\epsilon)\beta\sigma\delta\mu\rho_{kh}\right]^2}{2\beta(1-\beta)(\mu\delta)^2} - \frac{\epsilon^2}{2\beta(1-\beta)(\mu\delta)^2} - \frac{1}{2}\epsilon(1-\epsilon)\sigma^2 \qquad (3.28)$$

In following section, we further discuss economic properties in the set-up described above.

3.4 Analytical illustration

3.4.1 Proposition of equilibrium

The expected growth rate represents the "trend" of the stochastic growth path. Pollution-induced health degradation slows down this trend. As a consequence, we expect a higher growth rate by enlarging the abatement-output ratio m. Meanwhile, consumers invest a smaller fraction of output when $1-m$ decreases, and hence dampens the growth rate. In particular, we have the following property.

Proposition 3.4.1. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to social planner's problem implies that c_t/y_t is constant in all states and at all times, then the expected growth rate has the following properties:*

(i) The uncertainties dampen the expected economic growth.

(ii) When $\rho_{k,h} < 0\,(or\,\rho_{k,h} > 0)$, a higher correlation between uncertainties worsens (or offsets) the uncertainty-induced growth loss.

(iii) The relationship between the abatement-output ratio and expected growth rate has an inverted-U shape. Specifically, when $m < m_g^*$ (or $m > m_g^*$), a greater mitigation fraction will raise (or lower) the expected growth rate. m_g is a threshold value, defined as follows:

$$m_g^* = \frac{\phi}{\mu} - \frac{1+(1-\epsilon)\beta\rho_{kh}\mu\sigma\delta}{A\beta(1-\beta)(\mu\delta)^2}, \qquad (3.29)$$
$$\textit{with} \quad \delta \geq 0\,,\ \beta\,,\epsilon \in (0,1]\,,\sigma\,,\delta \in [0,1]\,,\rho_{k,h} \in [-1,1]$$

Proof. Provided in the Appendix. □

The basic mechanism underlying Proposition (3.4.1) is that: the abatement policy influences the growth rate through two channels. Firstly, abatement spendings, occupying a share of output, reduces consumption and saving. In other words, when m increases, $1-m$ decreases, and hence growth rate slows down. Secondly, abatement can dampen uncertainties, thus reducing risk premium and stimulating economic growth. In Eq.(3.23), the first term in the RHS represents the negative impact of abatement spendings on growth rate (i.e. the "slow-down effect"), and the last term represents the positive impact of abatement spendings on growth rate (i.e. the "uncertainty effect"). When the abatement-output-ratio reaches the threshold value (i.e. $m = m_g^*$), the "slow-down effect" and the "uncertainty effect" exactly offset each other.

Indeed, abatement represents a transmission between capital and health fluctuations and production activities, which affects the consumption sector as well.

Given *Assumption 3* is satisfied and the health regeneration process defined in Eq.(5.35), emission increases the uncertainty of consumers' health. The abatement spendings reduce the stock of emissions, and thus diminishes this uncertainty. Consumers are better off. To understand how uncertainties' correlation affects the economy, we consider the following three scenarios:

Scenario 1: *Suppose the capital and health uncertainties are positively correlated (i.e.* $\rho_{kh} > 0$*).*
Then the abatement spendings also hedge the capital's uncertainty, and hence pro-growth. For a standard CRRA preference defined in *Assumption 1*, the impact of uncertainty reduces the expected economic growth rate at a deterministic level. The economic implication can be illustrated by employing the well-known concept of certainty equivalent return on saving (Weil, 1990).[6]

Scenario 2: *Suppose the capital and health uncertainties are negatively correlated (i.e.* $\rho_{kh} < 0$*).*
Then the uncertainties in capital and health offset each other: a positive fluctuation on capital responds to a negative fluctuation on health.

[6]It is worth to remark that we are not interested in the case of $\epsilon = 1$. Because the stochastic impact of capital accumulation on growth rate will disappear then.

The economic implication straightforward: a boom in production enhances emissions, and thus degrades health. In addition, a smaller ρ_{kh} implies a higher threshold value of abatement-output ratio (i.e. m_g^*), and hence the domain of abatement spendings to promote economic activity enlarges.

Scenario 3: *Suppose the capital and health uncertainties are uncorrelated (i.e. $\rho_{kh} = 0$).*
Then the emission-induced fluctuation on health does not transit into fluctuation on capital, and vice versa. Hence, superposition effect of uncertainty disappears, and the expected growth rate is higher. Considering an extreme case when $\sigma, \delta = 0$, both uncertainties vanishes. Thus, only the "slow-down effect" remains. In this scenario, the detrimental emissions do not affect health regeneration and capital accumulation. Therefore, the optimal abatement spendings are zero, and our growth rate is identical to the one in perfect certainty case. In the literature, moreover, environmental uncertainties are correlated and should not be considered in isolation (see, e.g. Martin and Pindyck, 2015).

An appealing policy implication of property (iii) in Proposition (3.4.1) is: when abatement spendings are relatively small and satisfying $m < m_g^*$, the "slow-down effect" is dominated by "uncertainty effect". In this way, a tighter mitigation policy reduces the economic uncertainty, leading to a higher growth rate. In this scenario, the environmental policy promotes economic growth. This scenario particularly provides insights to some emerging economies where we observe serious pollutions but less efficient abatement policies. In addition, it is worth to note that a larger measure of relative of risk aversion, ϵ, will decrease (or increase) m_g^* when $\rho_{k,h}$ is negative (or positive). To understand this, recalling that

$$\frac{\partial m_g^*}{\partial \epsilon} = \frac{\beta \rho_{kh} \mu \sigma \delta}{A\beta(1-\beta)(\mu\delta)^2} \begin{cases} < 0 & \text{when } \rho_{k,h} < 0 \\ > 0 & \text{when } \rho_{k,h} > 0 \end{cases} \tag{3.30}$$

Hence in ***Scenario 2***, a declining risk aversion will enlarge m_g^*. So the abatement spendings are more likely to promote the economic growth. Unsurprisingly, we get the opposite conclusion in ***Scenario 1***. To better illustrate this inverted-U shape, we provide a graphical interpretation in Fig.(3.1).

Recalling the expected optimal growth rate g^* in Eq.(3.27) is a polynomial of model's parameters. The first term of right hand side of Eq.(3.27)

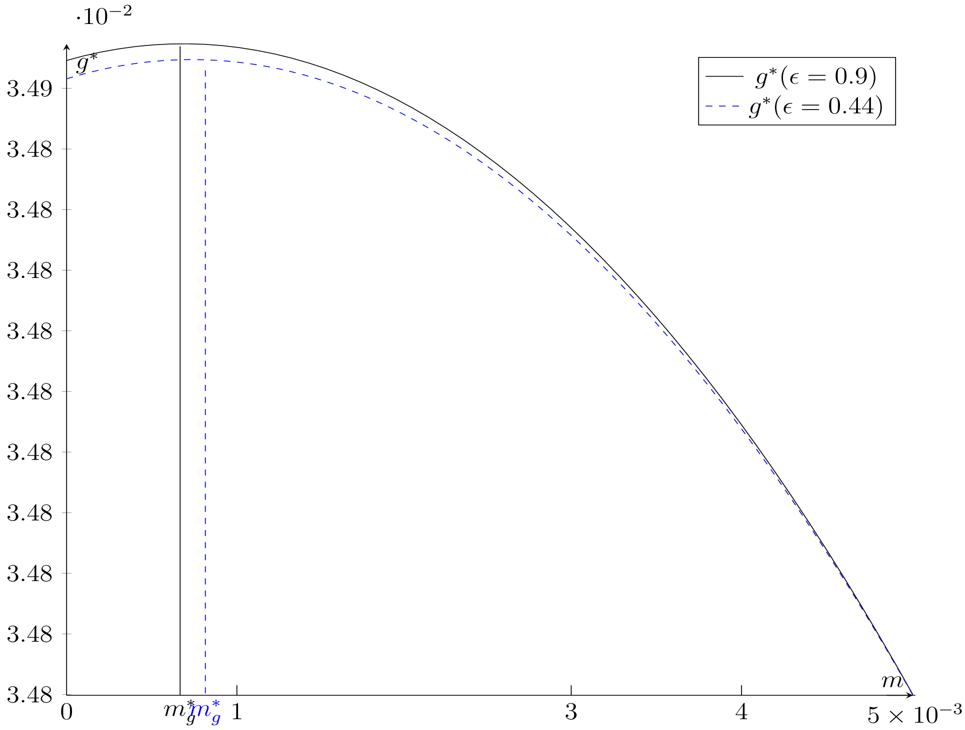


Figure 3.1 – The optimal growth rate g^* for different values of mitigation policy m.

reflects the standard Keynes-Ramsey rule of optimal consumption. That is the risk-free interest rate[7] less the preference rate, divided by the constant measure of relative risk aversion. The last term implies the implicit risk premium, adjusted by the elasticity of intertemporal substitution for consumption, à la Dorfman (1969). Hence, in our model, the real interest rate is composed by risk-free rate less the risk premium. It represents one of the main channels through which uncertainty affects economic growth (De Hek, 1999). Specifically, this can be illustrated by environmental abatement effect and uncertainty effect. The abatement effect is defined by the ratio of the emission intensity of output to the abatement efficiency, i.e ϕ/μ. The abatement has a negative impact on the real interest rate because production externalities have an unambiguously negative effect on marginal product of capital (MPK). This negative effect can be reduced by either reducing the pollution intensity or improving the abatement efficiency.

The overall uncertainty effect, represented by I_g in Eq(3.27), can be decomposed into health uncertainty effect, capital uncertainty effect, and superposition effect, denoted by $I_{g,h}$, $I_{g,k}$ and $I_{g,\rho}$ respectively:

$$I_g = \underbrace{\frac{\left[1 + (1-\epsilon)\beta\sigma\delta\mu\rho_{kh}\right]^2}{2\beta(1-\beta)(\mu\delta)^2}}_{I_{g,\rho}} - \underbrace{\frac{\epsilon^2}{2\beta(1-\beta)(\mu\delta)^2}}_{I_{g,h}} - \underbrace{\frac{1}{2}\epsilon(1-\epsilon)\sigma^2}_{I_{g,k}} \tag{3.31}$$

Specifically, the difference between the deterministic growth rate g^d and the stochastic growth rate g^* is given by $g^d - g^* = \frac{1}{\epsilon}\left(\frac{\phi}{\mu}A - I_g\right)$. Therefore, the uncertainties could either stimulate or dampen the economic growth depending on the sign of $\frac{\phi}{\mu}A - I_g$. The economic reason is best described by employing the concept of certainty equivalent return on savings (see Weil, 1990; Steger, 2005). For example, under risk aversion preference (i.e. $\epsilon > 0$), the certainty equivalent return on saving is smaller than the expected rate of return due to the presence of risk. As a result, there are two possible cases, depending on the well-known (intertemporal) "income effect" and (contemporaneous) "substitution effect".

Case 1: *"Income effect" dominates "substitution effect", then pro-growth.* The intertemporal income effect can depress contemporaneous consumption, which in turn induces more savings and pro-growth. The analysis is

[7] The risk-free interest rate equals to marginal product of capital (MPK), i.e. $r = A$.

analogous to the literature on economy's propensity to save (PTS) under uncertainty, which can be found in Wälde (1999), Toche (2001), Steger (2005), Bretschger and Vinogradova (2014). In our model, the gross savings are endogenously split between two purposes: capital investment and emission abatement. Obviously, abatement reduces emissions, and hence diminishes both capital and health uncertainties. On one hand, more capital investment leads to more output, associated with more emissions. So capital booming accelerates health degradation. On the other hand, increased health uncertainty motives precautionary savings and hence capital accumulation. This implies a negative relationship between capital and health (i.e. $\rho_{kh} < 0$). In literature, moreover, an intensified positive link between pollution and economic growth can be found when health effect is included (see, e.g. Gutierrez, 2008).

Case 2: *"Substitution effect" dominates "income effect", then anti-growth.* The substitution effect induces a rise in contemporaneous consumption, and in turn, fosters precautionary dissaving and dampens growth. Similar analysis can be found in Müller-Fürstenberger and Schumacher (2015), Bretschger and Vinogradova (2016). In our model, this case further implies a positive link between pollution and economic growth (i.e. $\rho_{kh} > 0$).

In conclusion, we analyse the uncertainties' structure and the effect of abatement spendings on economic growth. The optimal expected growth rate decreases when the number of the uncertainties increases. The correlation between uncertainties may amplify or offset this uncertainty impact, depending on the sign and model's parameter. A detailed illustration of some attractive properties is presented in the following subsection.

3.4.2 Proposition of parameters

To understand the properties of model's parameters on the equilibriums, recalling the optimal abatement-output ratio in Eq.(3.21), we have the following proposition:

Proposition 3.4.2. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to social planner's problem implies that c_t/y_t is constant in all states and at all times. The optimal abatement-output ratio m^* is constant over time, and a function of model's parameters with following properties:*

(i) m^* is a monotonic increasing function of the total factor productivity

(TPF) A, emission intensity of output ϕ, and the sensitivity of the health to pollution δ.

(ii) m^* is a decreasing function of the uncertainties' correlation coefficient $\rho \in [-1, 1]$.

(iii) m^* is an increasing (or decreasing) function of the abatement efficiency μ, if $\mu < \mu^*$ (or $\mu > \mu^*$). μ^* is the threshold value of abatement efficiency.

(iv) m^* is an increasing (or decreasing) function of the elasticity of marginal utility to health β, if $\beta < \beta^*$ (or $\beta > \beta^*$). β^* is the threshold value of health parameter.

(v) m^* is an increasing function of the elasticity of marginal utility to consumption ϵ.

Proof. Provided in the Appendix. □

The economic implication behind property (i) of Proposition (3.4.2) is straightforward. A higher level of TFP implies more production, and thus more emissions and pollutants. Higher emission intensity of output leads more emission per unit of output. A greater sensitivity of health to pollutants enlarges health fluctuation and degradation. Therefore, larger abatement spendings are required to relieve the pollution-induced deteriorating health.

Property (ii) reveals that the sign of the correlation may amplify or offset the detrimental impact of pollutions. Specifically, we have *a negative correlation*: an increasing positive fluctuation on capital leads more production, and hence more emissions. This accelerates health degradation, and thus more abatement spendings are required (i.e. "production effect"). Meanwhile, we have *a positive correlation*: labours' health degradation reduces labour productivity, and hence slows down production (i.e. "health effect"). Thus eventually we observe a negative (or positive) correlation, depending on whether the "production effect" dominates (or is dominated by) the "health effect". More precisely, it depends on the trade-off between precautionary saving and consumption under uncertainty. The detailed economic explanation behind is analogous to the previous subsection.

Property (iii) reveals the ambiguous relationship between abatement spendings and its efficiency. Recalling m^* defined in Eq.(3.21), the first term of m^* is positive, and tends to decrease when abatement efficiency μ increases. This implies a negative relationship between abatement spendings and abatement efficiency. Meanwhile, the second term is negative, and it's absolute value also tends to decrease when μ increases. An increased abatement efficiency offsets the health's fluctuations. So the sign of increment of m^* as μ increases is undetermined. Our calibration shows a positive increment of m^* only when the abatement efficiency is relatively poor. An alternative illustration could be straightforward but more intuitive: when efficiency of abatement technology is low, both abatement efficiency and abatement policy need to be strengthened in order to dampen health degradation until a certain level of abatement efficiency is reached. Afterwards, less abatement spendings are needed when the efficiency of abatement technology further improves.

Property (iv) shows an inverted-U shape between the health parameter β and the optimal abatement policy. Recalling the welfare function defined in Eq.(5.13), an increased health parameter β implies a larger weight of health relative to consumption in utility function (i.e. "weight effect"). Hence, a strengthened abatement policy is preferred in order to offset health fluctuation. Meanwhile, $1-\beta$, a measurement of risk aversion with respect to health, will decrease when β increases (i.e. "risk effect"). Hence, consumers become less risk averse towards health. This implies consumers are less sensitive to health fluctuation, and thus in favour of lessened abatement policy. There is a trade-off between the two effects. Specifically, the "weight effect" dominates (or is dominated by) the "risk effect" when β is smaller (or larger). Property (v) is obvious: a larger measure of risk aversion will lead a tight mitigation policy.

3.5 Quantitative analysis

3.5.1 Parameter selection

In our benchmark calibration, we choose the TFP A, emission intensity ϕ, and abatement efficiency σ, equal to 5%, 0.4‰ and 8% respectively[8]. It is worth to remark that CO_2 emission is 0.4‰ tons (0.4 kg) per unit of GDP in U.S. dollar for the world average during 2011-2015, and hence we choose $\phi = 0.4‰$. $\mu = 8\%$ corresponds to 12.5 U.S. dollar per ton coal burned (or CO_2 emission). For comparative study, we consider a larger TFP (e.g. $A = 7\%$) from an optimistic point of view, and a smaller TFP ($A = 2.1\%$) for some relatively slower growth economy. We adopt a higher emission intensity, $\phi = 2.1‰$ for high CO_2 emission countries (e.g. China). Also, $\mu = 5\%$ is considered in case of the technology development slows down. We set the health parameter $\beta = 0.5$, according to Pautrel (2012), for the benchmark calibration. We choose benchmark value of $\epsilon = 0.9$ (e.g. India), and $\epsilon = 0.44$ (e.g. Japan) as comparative analysis, according to Gandelman and Hernández-Murillo (2014). In addition, we choose time discount rate equal to 1.5% per year according to Nordhaus (2008). Our model highlights the effect of uncertainties in economic decisions, which include three key parameters σ , δ and ρ_{kh}. Parameters σ and δ measure the sensitivity of the capital and health to fluctuations. Parameter ρ_{kh} measure the correlation between uncertainties. These parameters are chosen intuitively inspired from various strands of literature. In order to accurately study the effect of uncertainties on optimal policy, we set different values for these parameters and comparatively study their effect. We provide both table and graphical presenting. The benchmark values of the parameters are summarized in Table 3.1.

3.5.2 Quantitative results

The emission concentration per unit of capital q_t is defined in Eq.(4.12), implying $0 \leq q_t \leq \phi\, A = 0.4 \cdot 10^{-3} \times 0.05 = 2 \cdot 10^{-5}$. Given the parameter defined in Table 3.1, we present the relationship between the Arrow-Pett measure of relative risk aversion and the optimal emission concentration per unit of capital q^*, given in in Table 3.2. Table 3.3 and Table 3.4 present the effects of model parameters on abatement policy and growth rate. We

[8]based on the World Bank series 2016 "CO_2 emissions per GDP" and empirical studies by Hood (2011), McKinsey (2009), and Bretschger and Vinogradova (2014).

A	the total factor productivity	5%
ϕ	output emission intensity	0.04%
μ	abatement efficiency	0.08
ρ	time discount rate	0.015
δ	health sensitivity to fluctuation	$1.7 \cdot 10^3$
σ	capital sensitivity to fluctuation	0.001
β	health parameter	0.5
ϵ	elasticity of marginal utility	0.9

Table 3.1 – *Values of parameters for the numerical example.*

CRRA parameter ϵ	Optimal Emissions q^* (10^{-5})	Optimal abatement m^* (10^{-3})
1	0	5
0.9	0.167	4.582
0.5	0.836	2.911
0.1	1.504	1.240

Table 3.2 – *The effect of CRRA measurement ϵ on optimal emission concentration per unit of capital q^* and fraction of output for abatment m^*.*

choose different correlation coefficient, and comparatively study its effect. The effect of the parameters on optimal abatement and growth rate are shown in the following figures. Our quantitative results coincide with the properties in **Proposition 3.4.2** and **Proposition 3.4.1** (when $\partial g/\partial m < 0$). In addition, we find that the optimal abatement policy reacts sensitively to the parameters of uncertainties. This implies the necessity of using multi-dimensional stochastic growth model to calculate optimal abatement policy (or equivalently, for the optimal carbon tax). Table 3.5 presents a specific case when the measure of relative risk aversion is relatively small, $\epsilon = 0.44$ (e.g. Japan). In this case, we observe that policy and economic activity are more sensitive to the parameters of diffusion terms who characterize the uncertainty effect.

Moreover, we graphically present the effect of model's parameters on the optimal mitigation policy and growth rate in Fig.(3.2) and Fig.(4.3).

The above calibrations suggest the necessity to consider the correlation structure between uncertainties. Moreover, our numerical results on abatement policy is comparable to the empirical study of ENERDATA (2014). ENERDATA reports that, to maintain global temperature rising below 2°C, a GHGs target of 50% domestic reduction in CO_2 emission by 2030

Parameters	Optimal Mitigation Policy and Growth Rate		
	$q^*(10^{-6})$	$m^*(‰)$	$g^*(\%)$
$A^H = 7\%$	1.671	4.702	6.0744
$A^L = 5\%$	1.671	4.582	3.8633
$\phi^H = 2.1‰$	1.671	25.832	3.7453
$\phi^L = 0.4‰$	1.671	4.582	3.8633
$\delta^H = 1.7 \cdot 10^4$	0.0114	4.997	3.8611
$\delta^L = 1.7 \cdot 10^3$	1.671	4.582	3.8633
$\sigma^H = 1\%$	1.142	4.715	3.8621
$\sigma^L = 0.1\%$	1.671	4.582	3.8633
$\rho_{kh}^H = 1$	1.848	4.538	3.8636
$\rho_{kh}^{NH} = 0.5$	1.789	4.553	3.8635
$\rho_{kh}^{NH} = 0$	1.730	4.567	3.8634
$\rho_{kh}^L = -0.5$	1.671	4.582	3.8633
$\rho_{kh}^{NL} = -1$	1.612	4.597	3.8632
$\beta^H = 0.9$	4.512	3.872	3.8679
$\beta^M = 0.5$	1.671	4.582	3.8635
$\beta^L = 0.1$	4.773	3.807	3.8675
$\mu^H = 0.08$	1.671	4.582	3.8633
$\mu^L = 0.05$	2.709	6.916	3.8502

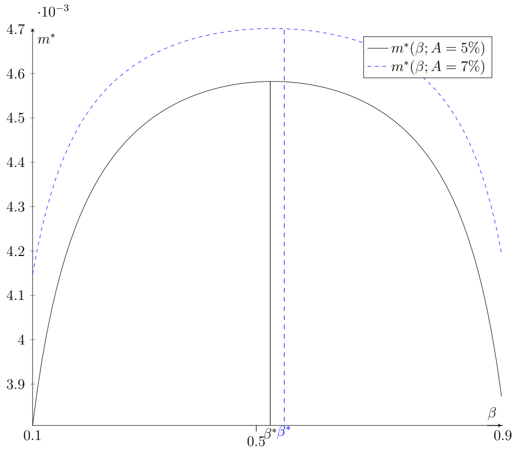
TABLE 3.3 – *The optimal emission concentration, mitigation fraction, and grwoth rate when* $\epsilon = 0.9\,, \rho_{kh} < 0$.

Parameters	Optimal Mitigation Policy and Growth Rate		
	$q^*(10^{-6})$	m^*‰	$g^*\%$
$A^H = 7\%$	1.789	4.768	6.0746
$A^L = 5\%$	1.789	4.553	3.8635
$\phi^H = 2.1$‰	1.789	25.925	3.7454
$\phi^L = 0.4$‰	1.789	4.553	3.8635
$\delta^H = 2.0 \cdot 10^3$	1.30	4.675	3.8628
$\delta^L = 1.7 \cdot 10^3$	1.789	4.553	3.8635
$\sigma^H = 1\%$	2.31	4.420	3.8637
$\sigma^L = 0.1\%$	1.789	4.553	3.8635
$\rho_{kh}^H = 1$	1.848	4.538	3.8636
$\rho_{kh}^L = 0.5$	1.789	4.553	3.8635
$\rho_{kh}^{NH} = 0$	1.730	4.567	3.8634
$\rho_{kh}^{NH} = -0.5$	1.671	4.582	3.8633
$\rho_{kh}^{NL} = -1$	1.612	4.597	3.8632
$\beta^H = 0.9$	4.02	3.725	3.8679
$\beta^M = 0.5$	1.848	4.553	3.8635
$\beta^L = 0.1$	2.90	3.790	3.8675
$\mu^H = 0.08$	1.848	4.553	3.8635
$\mu^L = 0.05$	2.827	6.869	3.8504

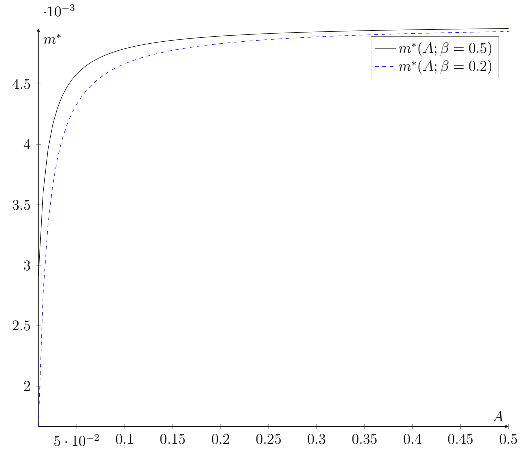
Table 3.4 – *The optimal emission concentration, mitigation fraction, and grwoth rate when* $\epsilon = 0.9\,, \rho_{kh} > 0$.

Parameters	Optimal Mitigation Policy and Growth Rate		
	$q^*(10^{-5})$	m^*‰	$g^*\%$
$\delta^H = 1.7 \cdot 10^4$	0.0102	5.016	1.33986
$\delta^L = 1.7 \cdot 10^3$	$\underline{1.517}$	$\underline{4.343}$	$\underline{1.35864}$
$\sigma^H = 1\%$	1.023	4.551	1.34832
$\sigma^L = 0.1\%$	$\underline{1.497}$	$\underline{4.343}$	$\underline{1.35864}$
$\rho_{kh}^H = 1$	1.656	4.274	1.36147
$\rho_{kh}^{NH} = 0.5$	1.603	4.297	1.36051
$\rho_{kh}^{NH} = 0$	1.550	4.567	1.35959
$\rho_{kh}^L = -0.5$	$\underline{1.497}$	$\underline{4.343}$	$\underline{1.35864}$
$\rho_{kh}^{NL} = -1$	1.445	4.367	1.35773
$\beta^H = 0.9$	4.043	3.228	1.39019
$\beta^M = 0.5$	$\underline{1.497}$	$\underline{4.343}$	$\underline{1.35864}$
$\beta^L = 0.1$	4.277	3.125	1.39427

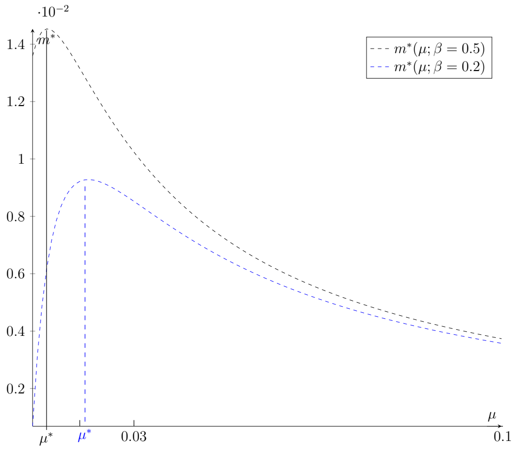
TABLE 3.5 – *The optimal emission concentration, mitigation fraction, and grwoth rate when* $\epsilon = 0.44, A = 2.1\%, \phi = 0.04\%, \delta = 1700, \mu = 0.08, \delta = 0.001, \rho_{kh} = -0.5$.



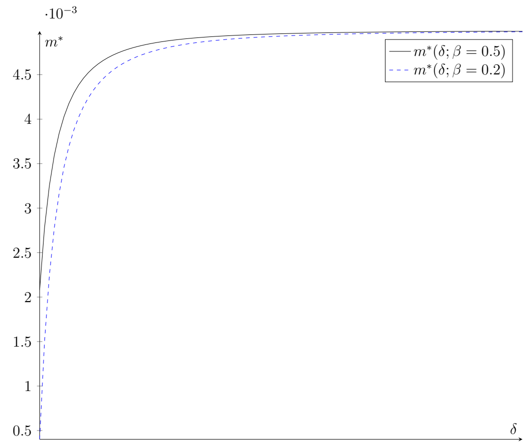
(a) The optimal mitigation policy m^* for different values of health parameter β.



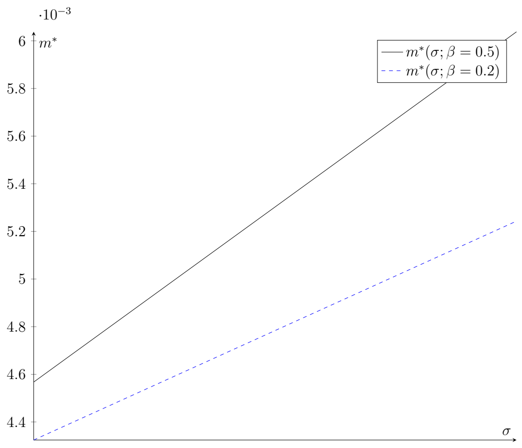
(b) The optimal mitigation policy m^* for different values of TFP, A.



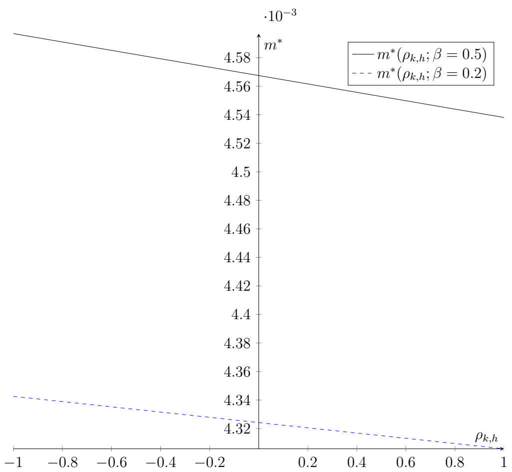
(c) The optimal mitigation policy m^* for different values of the efficiency of abatement μ.



(d) The optimal mitigation policy m^* for different values of the sensitivity of the health status to pollution δ.

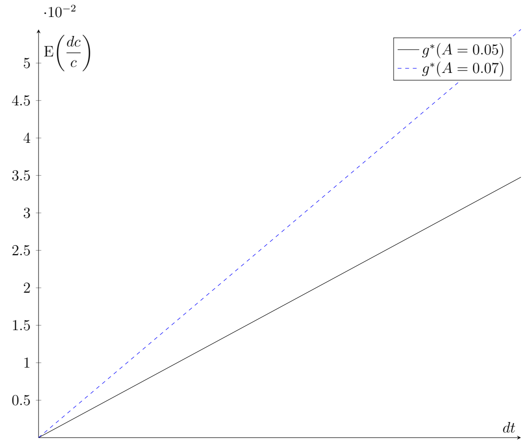


(e) The optimal mitigation policy m^* for different values of the (percentage) voatility parameter of capital σ.

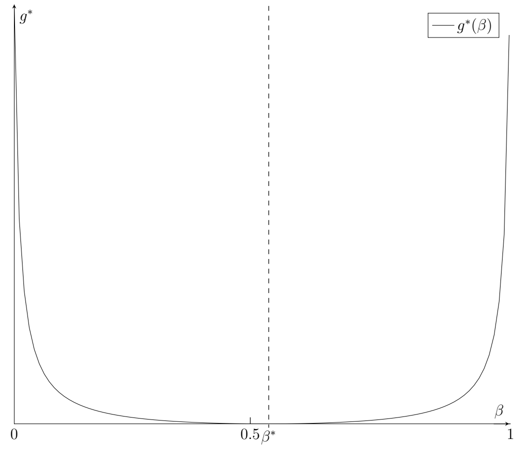


(f) The optimal mitigation policy m^* for different values of the correlation coefficient ρ_{kh}.

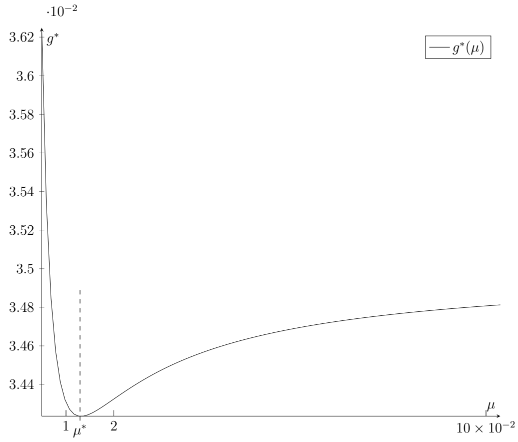
Figure 3.2 – The effect of model's parameters on the optimal mitigation policy m^*.



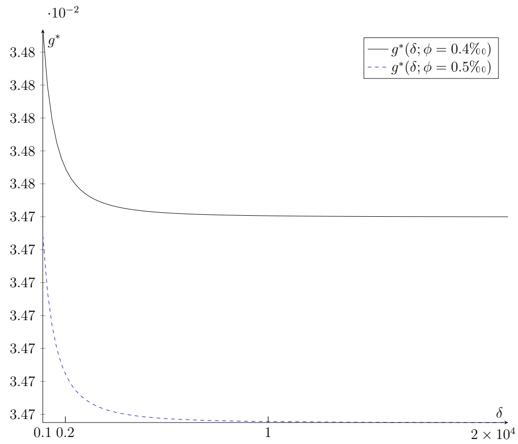
(a) The optimal growth rate g^* for different values of TFP, A over time.



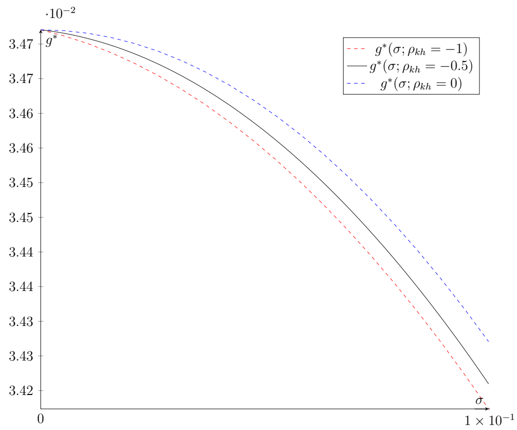
(b) The optimal growth rate g^* for different values of health parameter β.



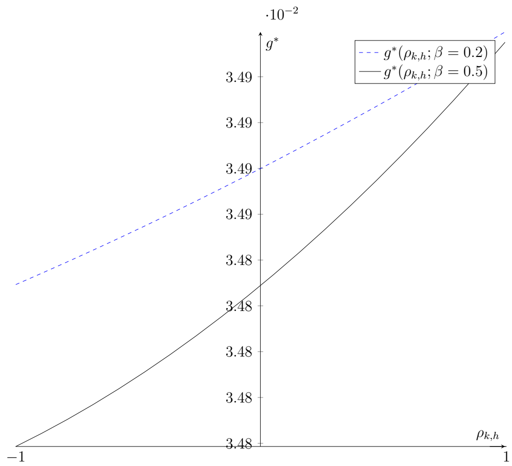
(c) The optimal growth rate g^* for different values of the efficiency of abatement μ.



(d) The optimal growth rate g^* for different values of the sensitivity parameter of health stautus to pollution δ.



(e) The optimal growth rate g^* for different values of the volatility parameter of captial σ and correlation coefficient ρ_{kh}.



(f) The optimal growth rate g^* for different values of correlation coefficient ρ_{kh}.

Figure 3.3 – The effect of model's parameters on economic growth rate g^*.

costs EU at 0.6% of GDP in 2030. Furthermore, it is worth to highlight that the GHGs emissions reduction target will reduce reliance on imported energy sources (e.g. fossil fuels), which could cut down EU health costs on respiratory illness by an average of 0.1% of GDP in 2030 under the mean value of health impacts. The average world GDP growth rate in th last 50 years (1961-2015) is 3.8% [9].

Under the benchmark calibration described above, we obtain the optimal abatement-out ratio m^* equal to 0.46%, 33 U.S. dollar per ton of coal for carbon tax, and the economic growth rate equal to 3.86%. Our value of m^* is slightly higher than 0.42% in Nordhaus (2008), who calculated the carbon tax as 30 U.S. dollar per ton of coal at a yearly global output of 70 trillion U.S. dollar in 2010. The difference is due to the factor of health fluctuation.

3.6 Conclusion

This paper investigates a sustainable growth economy, where the production induced environmental degradation and its detrimental impact on health are taken into account. Specifically, our model enables policy maker to maximize the society well-being while optimally balance the environmental health and GDP growth under the threat of uncertainties. We present a stochastic endogenous growth model. This model can be seen as a generalization of Bretschger and Vinogradova (2016). Yet in their model, the capital accumulation process is deterministic. Our model can also be viewed as an extension of Steger's one dimensional stochastic growth model (2005). We contribute to the literature by introducing a two dimensional stochastic endogenous growth model. The capital and health regeneration processes are driven by two correlated Winer processes. Roughly speaking, the proposed dynamic structure can be interpreted as a multi-factor Geometric Brownian motion. Our central results are two simple formulas for optimal abatement policy and growth rate. We particularly focus on the abatement spendings which, acting as the first-best policy, ensure the economy grows at a “healthy” rate.

We demonstrate that the relationship between abatement policy and eco-

[9]Based on Source: Global Growth Tracker the World Economy: 50 Years of Near Continuous Growth, editor: Dariana Tani, March 2016.

nomic growth rate is an inverted-U shaped. The fact is that larger abatement spendings decrease both consumption and investment, and thus dampens growth (i.e. "slow-down effect"). Meanwhile, reducing emissions, the abatement policy benefits economic activity through dampening labours' health degradation and improving human capital (i.e. "uncertainty effect"). Thus, a tighter mitigation policy is more likely to boom economic growth when health is highly pollution-sensitive and the efficiency of abatement technology is high. This is detailedly discussed in **Propositions (3.1)**. Moreover, the correlation structure between the uncertainties changes the optimal policy. The economic implication on environmental and health parameters are discussed in details in **Propositions (3.2)**. In quantitative analysis, we plot the impact of parameters on optimal policy and economic growth.

An important insight of our model is the country-based environmental policy. Even though, the model is compatible to analyse the optimal carbon tax at global level. The emission defined in our model includes not only the GHGs, but also other pollutants harmful to human health. Our numerical result suggests the abatement-output ratio equal to 0.46%, indicating 33 U.S. dollar per ton of coal when the elasticity of marginal utility is 0.9. This number is slightly higher but comparable to 0.42% and 30 U.S. dollar in Nordhaus (2008). The difference is due to the uncertainties in capital and health regeneration. The detrimental impact of emission on health will lead to tight environmental policy and thus higher abatement technology cost. It is worth noting that the abatement cost is very sensitive to fuel efficiency (Ekins et al., 2011), when we set a less optimistic value of abatement technology improvement, $\mu = 0.05$. The optimal abatement-output ratio dramatically increases to 0.69%, implying a relatively higher carbon price equal to 50 U.S. dollar per ton of coal, which is close to 56 U.S. dollar calculated by Golosov et al. (2014) using a DSGE framework.

3.7 Appendix: Optimization and proofs

3.7.1 Analytical treatment of the model

$V(k,h)$ is the value function defined by $V(k,h) = \max_{c,m} U(c,h)$. Applying Ito's formula to the value function, we have:[10]

$$
\begin{aligned}
dV(k,h) = &\Big\{V_k\big[(1-m)y-c\big] + \frac{1}{2}\big[V_{kk}\sigma^2 k^2 + V_{hh}R^2(q)h^2 + 2V_{kh}\sigma k R(q) h \rho_{k,h}\big]\Big\}dt \\
&+ V_k \sigma k dZ_{k,t} + V_h R(q) h dZ_{h,t}
\end{aligned} \tag{3.32}
$$

and thus $\frac{1}{dt}\mathbb{E}\big[dV(k,h)\big]$ is determined as follows.

$$
\frac{1}{dt}\mathbb{E}\big[dV(k,h)\big] = V_k\big[(1-m)y-c\big] + \frac{1}{2}\big[V_{kk}\sigma^2 k^2 + V_{hh}R^2(q)h^2 + 2V_{kh}\sigma k R(q) h \rho_{k,h}\big] \tag{3.33}
$$

Substituting Eq.(3.33) into the Hamilton-Jacobi-Bellman (HJB) equation associated with our problem:

$$
\rho V(k,h) = \max_{c,m}\Big\{U(c,h) + \frac{1}{dt}\mathrm{E}\big[dV(k,h)\big]\Big\} \tag{3.34}
$$

we obtain:

$$
\begin{aligned}
\rho V(k,h) = \max_{c,m}\Big\{&U(c,h) + V_k(k,h)\big[(1-m)y-c\big] + \frac{1}{2}\big[V_{kk}(k,h)\sigma^2 k^2 + \\
&V_{hh}(k,h)R^2(q)h^{2a} + 2V_{kh}(k,h)\sigma k R(q) h \rho_{k,h}\big]\Big\}
\end{aligned} \tag{3.35}
$$

Calculating the first-order conditions (FOCs) for control variables $\{c, m\}$, we have:

$$
\text{For } c: \qquad U_c = V_k \tag{3.36}
$$

$$
\text{For } m: \quad V_k\, y_t = (-\mu y_k')\Big(V_{hh}R(q)R'(q)h^2 + V_{kh}\sigma R'(q)\rho_{kh} k h\Big) \tag{3.37}
$$

[10] For convenience, we omit the time subscripts and denote the following abbreviations: $V_k \equiv \partial V(k,h)/\partial k$, $V_h \equiv \partial V(k,h)/\partial h$, $V_{kk} \equiv \partial V^2(k,h)/\partial k^2$, $V_{hh} \equiv \partial V^2(k,h)/\partial h^2$, $V_{kh} \equiv \partial V^2(k,h)/\partial k \partial h$.

Applying envelope theorem to state variables $\{k\,,h\}$, we have

$$\text{For } k: \quad \rho V_k = V_{kk}[(1-m)y-c] + V_k(1-m)A + \frac{1}{2}\Big[V_{kkk}\sigma^2 k^2 + 2\sigma^2 V_{kk}k + V_{hhk}R^2(q)h^2 + 2\rho_{kh}\sigma V_{khk}R(q)hk + 2\sigma\rho_{kh}V_{kh}R(q)h\Big] \tag{3.38}$$

$$\text{For } h: \quad \rho V_h = U_h + V_{kh}[(1-m)y-c] + \frac{1}{2}\Big[V_{kkh}\sigma^2 k^2 + V_{hhh}R^2(q)h^2 + 2V_{hh}R^2(q) + 2\sigma\rho_{kh}V_{khh}R(q)kh + 2\sigma\rho_{kh}V_{kh}R(q)k\Big] \tag{3.39}$$

From FOC for control variable c, we have:

$$U_c = V_k \quad \Rightarrow \quad c^* = c(k) \tag{3.40}$$

and let us define $V_k := f(k,h)$, and apply Ito's lemma to V_k:

$$dV_k = f_k dk + f_h dh + \frac{1}{2}\Big[f_{kk}(dk)^2 + f_{hh}(dh)^2 + 2f_{kh}dkdh\Big] \tag{3.41}$$

Substituting capital accumulation dynamic dk and health regeneration process dh defined in Section 3.3, we obtain:

$$dV_k = \Big\{V_{kk}[(1-m)y-c] + \frac{1}{2}\Big[V_{kkk}\sigma^2 k^2 + V_{khh}R^2(q)h^2 + 2\sigma\rho_{kh}V_{kkh}R(q)kh\Big]\Big\}dt + V_{kk}\sigma k dZ_k + V_{kh}R(q)hdZ_h \tag{3.42}$$

Calculating the difference between (3.42) and (3.38)$\cdot dt$, we obtain:

$$dV_k = V_k\Big\{\rho - (1-m)A - \Big[\sigma^2\frac{V_{kk}}{V_k}k + \sigma\rho_{kh}\frac{V_{kh}}{V_k}R(q)h\Big]\Big\}dt + \sigma V_{kk}kdZ_k + V_{kh}R(q)hdZ_h\,. \tag{3.43}$$

It is worth nothing that, from *Assumption 1*, we obtain $U_c(c,h) = c^{-\epsilon}h^{\beta}$. Therefore, consumption can be expressed as an inverse function of the marginal utility as follows:

$$c := f^c(U_c, h) = U_c^{-\frac{1}{\epsilon}}h^{\frac{\beta}{\epsilon}} \tag{3.44}$$

Applying Ito's lemma to c, and calculating its expected value, we have:

$$\frac{\mathbb{E}(dc)}{c\,dt} = -\frac{1}{\epsilon}\Big\{\rho - (1-m)A - \sigma\Big[-\epsilon\sigma\frac{c_k}{c}k + \rho_{kh}\beta R(q)\Big]\Big\} + \frac{1}{2}\Big\{$$
$$\frac{1}{\epsilon}(1+\frac{1}{\epsilon})\Big[\sigma^2\epsilon^2\frac{c_k^2}{c^2}k^2 + \beta^2 R^2(q) - 2\epsilon\sigma\rho_{kh}\beta\frac{c_k}{c}R(q)k\Big] + \frac{\beta}{\epsilon}\Big(\frac{\beta}{\epsilon}-1\Big)\,R^2(q)$$
$$-2\frac{\beta}{\epsilon^2}\Big[-\epsilon\sigma\rho_{kh}\frac{c_k}{c}kR(q) + \beta R^2(q)\Big]\Big\} \tag{3.45}$$

3.7.2 The optimal abatement

Lemma 1. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to a social planner's problem implies that c_t/y_t is constant in all states and at all times. The optimal abatement-output ratio m exists and is constant along the balanced growth path.*

Proof. Recall the linear capital accumulation function defined in Section 2.2. In balanced growth path, endogenous variables growth at the same rate, i.e.

$$g := \frac{\dot{C}}{C} = \frac{\dot{M}}{M} = \frac{\dot{Y}}{Y} \tag{3.46}$$

Recall that $M = mY$ and thus

$$\frac{\dot{M}}{M} = \frac{\dot{m}Y + m\dot{Y}}{mY} = \frac{\dot{m}}{m} + \frac{\dot{Y}}{Y} \tag{3.47}$$

Since Eq.(3.46) tells us $\dot{M}/M = \dot{Y}/Y$, we prove m is constant overtime, i.e.

$$\frac{\dot{m}}{m} = 0 \tag{3.48}$$

We prove the existence of m by calculating its analytical solution. The constant ratio of c/y can be expressed as $c = xk$, where x is constant. Recall that our preference in *Assumption 1* thus can be expressed as:

$$V(k,h) = \frac{(x\,k)^{1-\epsilon}}{1-\epsilon}h^{\beta}\,, \quad \epsilon\,,\beta \in (0,1)\,. \tag{3.49}$$

Substituting Eq.(3.49) into Eq.(3.37), we obtain the following equalities:

$$0 = -1 + \frac{\rho(\beta-1)}{1-\epsilon}\delta^2(\phi - \mu m)A(-\mu) + \beta\sigma\delta(-\mu)\rho_{kh} \tag{3.50}$$

and hence we obtain the optimal abatement-output ratio m is constant overtime and a function of the model's parameters:

$$m^* = \underbrace{\frac{\phi}{\mu} - \frac{1-\epsilon}{A\beta(1-\beta)(\mu\delta)^2}}_{\text{mitigation overall}} - \underbrace{\frac{(1-\epsilon)\beta\sigma\delta\mu\rho_{kh}}{A\beta(1-\beta)(\mu\delta)^2}}_{\text{“shock” as mitigation}} \tag{3.51}$$

Thus the existence of m is proved, given the parameters defined in our paper. In our quantitative analysis, we empirically show different m according to the change of the model's parameters. □

Recall the linear relationship between q_t and m_t in *Assumption 2*, the optimal emission concentration q^* is thus constant, and composed by the model's parameters.

$$q^* = \frac{(1-\epsilon)(1+\beta\sigma\delta\mu\rho_{kh})}{\beta(1-\beta)\mu\delta^2} \tag{3.52}$$

Corollary 1. *Suppose assumptions 1, 2 and 3 are satisfied, then the solution to a social planner's problem implies that c_t/y_t is constant in all states and at all times (along the balanced growth path).*

Proof. We prove this lemma by the "guess and verify" method. We assume c/y is constant. More precisely, $c = xk$, where x is a constant. Then straightforward, we have

$$c_k = x \quad \text{and} \quad \frac{c_k}{c} = k^{-1} \tag{3.53}$$

Furthermore, we have following equality:

$$\frac{\mathbb{E}\big(dc\big)}{c\,dt} = \frac{\mathbb{E}\big(dk\big)}{dt}\frac{1}{k} = \frac{(1-m)y-c}{k} = (1-m)A - x \tag{3.54}$$

We can derive x by substituting Eqs.(3.53) and (3.54) into the RHS and LHS of Eq.(3.45).

$$x = \frac{1}{\epsilon}\Big\{\rho - (1-\epsilon)(1-m)A + I_x\Big\}, \qquad \rho\,,\, \epsilon \in (0,1)\,. \tag{3.55}$$

$$\text{where} \quad I_x = \underbrace{\frac{1}{2}\beta(1-\beta)\delta^2\big[(\phi-\mu m)A\big]^2}_{I_{x_h}} + \underbrace{(1-\epsilon)\beta\sigma(-\rho_{kh})\delta\big[(\phi-\mu m)A\big]}_{I_{x_{h,k}}} + \underbrace{\frac{1}{2}\epsilon(1-\epsilon)\sigma^2}_{I_{x_k}}$$

with $\phi - \mu m \geq 0\,, A\,, \delta\,, \sigma \geq 0\,, \epsilon\,, \beta \in (0,1)\,, \rho_{kh} \in [-1,1]\,, m \in [0,1)$

$I_x \geq 0$ represents the effect of environmental uncertainty. Especially, it is composed by the single effect on health, single effect on capital and the joint effect on both, denoted by $I_{x_h} \geq 0$, $I_{x_k} \geq 0$ and $I_{x_\rho} \gtreqless 0 \Leftrightarrow \rho_{kh} \gtreqless 0$ respectively. Since m is constant at optimal state, we prove x is a constant polynomial composed by the model's parameters. □

3.7.3 The expected growth rate

Lemma 2. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to a social planner's problem implies that c_t/y_t is constant in all states and at all times. Then the optimal growth rate g is given by*

$$g^* = \frac{A-\rho}{\epsilon} + \frac{1}{\epsilon}\left(I_g - \frac{\phi}{\mu}A\right) \tag{3.56}$$

where I_g represents the total uncertainty effect.

$$I_g = \frac{\big[1+(1-\epsilon)\beta\sigma\delta\mu\rho_{kh}\big]^2}{2\beta(1-\beta)(\mu\delta)^2} - \frac{\epsilon^2}{2\beta(1-\beta)(\mu\delta)^2} - \frac{1}{2}\epsilon(1-\epsilon)\sigma^2 \tag{3.57}$$

Proof. Substituting Eq.(3.53) into Eq.(3.45), we have the expected growth rate, i.e. $\mathrm{E}\big(\dot{c}/c\big)$ with $\dot{c} = dc/dt$ as follows.

$$\mathbb{E}\left(\frac{\dot{c}}{c}\right) = \frac{1}{\epsilon}\left\{(1-m)A - \rho - \frac{1}{2}\beta(1-\beta)(\delta q)^2 - \frac{1}{2}\sigma^2\epsilon(1-\epsilon) + \rho_{kh}\big(1-\epsilon\big)\beta\sigma\delta q\right\} \tag{3.58}$$

Substituting Eqs.(3.51) and (3.52) into Eq.(3.58), we get the closed form of growth rate g as a constant function of the model's parameters.

$$\begin{aligned} g^* = \mathbb{E}\left(\frac{\dot{c}}{c}\right) = \frac{1}{\epsilon}\Bigg\{\left(1-\frac{\phi}{\mu}\right)A - \rho + \frac{(1-\epsilon)(1+\beta\sigma\delta\mu\rho_{kh})}{\beta(1-\beta)(\mu\delta)^2}\Bigg[\frac{1}{2}(1+\epsilon) \\ +\left(1-\frac{1}{2}(1+\epsilon)\beta\right)\sigma\delta\mu\rho_{kh}\Bigg] - \frac{1}{2}\epsilon(1-\epsilon)\sigma^2\Bigg\} \end{aligned} \tag{3.59}$$

The above polynomial is identical to Eq.(3.58) by rearranging the combination of parameters. Therefore the lemma is approved. □

3.7.4 Proof of Propositions

Proof of Proposition 3.1

Proof. The property (i) can be proved from Eqs.(3.24) and (3.25), where $\partial\Delta_h/\partial\delta = -\beta(1-\beta)\delta A^2 q^2 \leq 0$ and $\partial\Delta_k/\partial\sigma = -\sigma\epsilon(1-\epsilon) \leq 0$. The equalities hold when $\delta = \sigma = 0$, indicating there are no uncertainties. Without considering the correlation, each uncertainty dampens the expected growth rate. The property (ii) can be proved from Eq.(3.26), where $\partial\Delta/\partial\rho_{h,k} = \partial\Delta_\rho/\partial\rho_{h,k} = (1-\epsilon)\beta\sigma\delta q \geq 0$. Hence, when $\rho_{h,k} < 0$, the impact of uncertainty on growth rate Δ decreases as the correlation strengthens, i.e. $\rho_{h,k} \to -1$. When $\rho_{h,k} > 0$, Δ increases as $\rho_{h,k}$ increases. The equality holds when $\rho_{h,k} = 0$, implying that the shocks on health and capital are independent. To prove property (iii), recall that more abatement booms growth if $\dfrac{\partial g}{\partial m} > 0$. This implies:

$$m < \underbrace{\frac{\phi}{\mu} - \frac{1+(1-\epsilon)\beta\rho_{kh}\mu\sigma\delta}{A\beta(1-\beta)(\mu\delta)^2}}_{:=m_g^*} = \frac{\phi}{\mu} - \frac{(1-\epsilon)\big[1+\beta\rho_{kh}\mu\sigma\delta\big]+\epsilon}{A\beta(1-\beta)(\mu\delta)^2}\,, \tag{3.60}$$
$$\delta \geq 0\,,\ \beta\,,\epsilon \in (0,1]\,,\sigma\,,\delta \in [0,1]\,,\rho_{k,h} \in [-1,1].$$

Vice verse, $m > m_g^*$ implies $\dfrac{\partial g}{\partial m} < 0$. □

Proof of Proposition 3.2

Proof. This proposition can be proved by calculating the partial derivatives of the optimal abatement policy m^* in Eq.(3.51) respect to the related parameters in Eqs.(3.61) to (3.70). In specific, Eqs.(3.61) to (3.63) are the proof of property (i), and Eqs.(3.64) to (3.68) are corresponding to the proof of properties (ii) to (v), given the threshold values of abatement efficiency and health preference in Eqs.(3.69) and (3.70).

$$\frac{\partial m^*}{\partial A} = \frac{(1-\epsilon)(1+\beta\sigma\delta\mu\rho_{kh})}{A^2\beta(1-\beta)(\mu\delta)^2} > 0\,, \tag{3.61}$$

$$\frac{\partial m^*}{\partial \phi} = \frac{1}{\mu} > 0\,, \tag{3.62}$$

$$\frac{\partial m^*}{\partial \delta} = \frac{2(1-\epsilon)\Big(1+\frac{1}{2}\rho_{kh}\beta\mu\sigma\delta\Big)}{A\beta(1-\beta)\mu^2\delta^3} > 0\,, \tag{3.63}$$

$$\frac{\partial m^*}{\partial \sigma} = -\frac{(1-\epsilon)\big(\rho_{kh}\beta\mu\delta\big)}{A\beta(1-\beta)(\mu\delta)^2}\begin{cases} \geq 0 & \text{when } \rho_{kh} \leq 0\,, \\ < 0 & \text{when } \rho_{kh} > 0\,. \end{cases} \tag{3.64}$$

$$\frac{\partial m^*}{\partial \rho_{k,h}} = -\frac{(1-\epsilon)\big(\sigma\beta\mu\delta\big)}{A\beta(1-\beta)(\mu\delta)^2} < 0 \tag{3.65}$$

$$\frac{\partial m^*}{\partial \mu}\begin{cases} \geq 0 & \text{when } 0 < \mu \leq \mu^*\,, \\ < 0 & \text{when } \mu > \mu^*\,. \end{cases} \tag{3.66}$$

$$\frac{\partial m^*}{\partial \beta}\begin{cases} \geq 0 & \text{when } 0 < \beta \leq \beta^*\,, \\ < 0 & \text{when } \beta^* < \beta < 1\,. \end{cases} \tag{3.67}$$

$$\frac{\partial m^*}{\partial \epsilon} = \frac{1+\beta\sigma\delta\mu\rho_{kh}}{A\beta(1-\beta)(\mu\delta)^2} > 0\,, \tag{3.68}$$

$$\text{with}\, \mu^* = \frac{2(1-\epsilon)}{\phi A\beta(1-\beta)\delta^2 - (1-\epsilon)\rho_{kh}\beta\sigma\delta}\,,\ \text{when}\, \phi > \frac{(1-\epsilon)\rho_{kh}\sigma}{A(1-\beta)\delta}\,, \tag{3.69}$$

$$\text{and}\ \beta^* = \frac{1}{1+\sqrt{1-a}}\,,\ \text{when}\, a := -\rho_{kh}\mu\sigma\delta \in (-\infty, 1)\,. \tag{3.70}$$

Note that when $\phi \leq \frac{(1-\epsilon)\rho_{kh}\sigma}{A(1-\beta)\delta} \Leftrightarrow \mu^* \leq 0$. Hence only the case $\frac{\partial m^*}{\partial \mu} < 0$ in Eq.(3.66) holds. Moreover, when $a \geq 1$, only the case $\frac{\partial m^*}{\partial \beta} \geq 0$ in Eq.(3.67) holds. $\square$

4 Sustainable growth II: fat-tailed uncertainty

4.1 Introduction

Production-induced CO_2 and other greenhouse gases (GHGs) emissions entail unfavourable climate change worldwide: global warming, higher sea level, and intensified occurrence frequency of nature disasters, which significantly harms human life and causes tremendous economic losses. On 26 December, 2004, Indian Ocean earthquake and tsunami affected 10 countries, resulting in more than 2.2 million deaths and enormous economic losses. Take Indonesia for example, economic losses account 4,451 million U.S. dollar[1], which equals to 1.73% of its overall annual GDP and 30.35% of GDP growth rate. For Thailand, economic losses are 2,198 million U.S. dollar[2], accounting for 1.36% of its overall annual GDP and 29.56% GDP growth rate. On 27 February, 2010, the Chile earthquake affected 82% of the country's population. A preliminary report[3] estimates the value of total infrastructure losses at 30 billion U.S. dollar. Based on our observations of numerous destructive earthquakes, we estimate that the losses may double once the building contents, commercial and industrial business interruptions are taken into account.[4] The Global Catastrophe Recap (2015) reports that the expected economic losses in 2015 Nepal earthquake reaches and possibly exceeds 5 billion U.S. dollar[5], which is a quarter of the country's GDP.

---

[1] Asian Disaster Preparedness Center 2004. Available online: http://cmsdata.iucn.org/downloads/social_and_economic_impact_of_december_2004_tsunami_apdc.pdf

[2] Asian Disaster Preparedness Center 2004. Available online: http://cmsdata.iucn.org/downloads/social_and_economic_impact_of_december_2004_tsunami_apdc.pdf

[3] Based on the MercoPress, 18 March 2010. Retrieved from http://en.mercopress.com/2010/03/18/chile-s-quake-death-toll-700-and-economic-damage-18-of-gdp

[4] Based on World Bank report, 27 February 2010. Retrieved from http://documents.worldbank.org/curated/en/750511468217448787/pdf/701380ESW0P12100WB0Report0final01EN.pdf

[5] Aon Benfield Analytics - Impact Forecasting, April 2015. Retrieved from http://thoughtleadership.aonbenfield.com/Documents/20150507-if-april-global-recap.pdf

Given the unfavourable climate change and intensified occurrence of catastrophes, how should an economy appropriately balance its production, investment, and abatement? How should optimal policy adjusts in an uncertainty environment? Does a tightened environmental policy improve the social welfare?

In this paper, we consider a generalized version of a stochastic dynamic model illustrated in previous chapter. This model is composed of a two dimensional stochastic processes: capital accumulation and health regeneration, explained by two types of randomness, i.e. small-scale continuous fluctuations occurring more frequently, and large-scale catastrophic shocks less frequently occurred. The small-scale capital and health diffusions are driven by two correlated Wiener processes. While the catastrophic shock are driven by Poisson process. Here we extend our model by analysing the expected growth rate under catastrophic shocks. We provide close-form solutions of the optimal mitigation policy and expected growth rate. The impact of parameters on economic variables are discussed in details with clear-cut implication for environmental policy.

Furthermore, an increasing amount of theoretical models study the effect of catastrophes on economic growth. Weitzman and Läfgren (1997) show the effect of environmental catastrophe on economic growth using national accounting and welfare measures, where the probability of catastrophe occurrence is driven by anthropogenic activities. Tsur and Withagen (2011) study a dynamic model on abrupt climate change, where a certain kind of capital is used to adapt the catastrophe. The catastrophe is caused by climate change, and the occurrence date is stochastic, whose distribution depends on atmospheric GHGs concentration. Bretschger and Vinogradova (2014) present a stochastic model of a growing economy where natural disasters occur randomly, and the damages are caused by polluting activity. Besides climate catastrophe, Martin and Pindyck (2015) analyse different types of catastrophes. They focus on the social cost of each catastrophe and develop a rule to determine which kind of catastrophe should be averted. While health and environment literature are explicit about health production by households and production-induced environmental degradation respectively, the correlation between health and environment are largely disregarded.

4.1.1 Outline of the results

Our objective is to propose a dynamic system that allows social planner to make decisions by considering economic growth, pollutants, GHGs emissions, extreme events and their detrimental impact on human health. The particularity of our model is the introduction of correlations between uncertainties in capital and health dynamics. The economy is affected by two types of uncertainties: small fluctuation and catastrophic shocks which are assumes to follow Wiener and Poisson processes respectively. We provide clear-cut analytically tractable solutions of optimal growth rate and the appealing first-best mitigation policy.

In a numerical experiment, we show that the optimal abatement policy reacts sensitively not only to the variations in conventional economic parameters, like emission intensity, efficiency of abatement technology, and TFP, but also to the health parameter and uncertainty parameters. When catastrophic shock is taken into account, the optimal abatement fraction will increase, i.e. $m^* = 1.4\%$, indicating a carbon taxation of 103 U.S. dollar per ton of coal in benchmark model, and $m^* = 0.8\%$ if the development of the abatement technology is viewed in an optimistic way. The latter number is identical to Golosov et al. (2014). Our calculation suggests a higher carbon tax when the catastrophic shocks in health and capital are taken into account.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the analytical treatment and derives the closed-form optimal solutions. Section 4 is an analytical discussion of appealing economic properties. Section 5 presents the numerical calibration to show the effects of model parameters on mitigation policy and growth rate. Section 6 is the conclusion.

4.2 The model

Besides pollution, production emissions cause deterioration of the natural environment and severe global warming, leading to a random occurrence of catastrophic disaster.

4.2.1 The economy and catastrophe: a general specification

We consider a stochastic version of the neoclassical growth model. Suppose a risk neutral policy maker maximizes overall expected discounted welfare over an infinite time period in an one-sector economy made of homogeneous individuals. There is a representative agent with the utility function

$$\mathbb{E}_0\left\{\int_0^\infty U(c_t, h_t)\, e^{-\rho t}\, dt\right\} \tag{4.1}$$

where c is consumption, h is health demand, and $\rho \in (0,1)$ is time discount factor. $U(\cdot)$ is a positive, increasing and concave function in both of its arguments:

$$U(\cdot) \geq 0, \;\; U'(\cdot) \geq 0, \;\; U''(\cdot) < 0.$$

Additionally, U satisfies the Inada conditions:

$$\lim_{c\to 0} U_c(c,h) = \infty, \;\; \lim_{h\to 0} U_h(c,h) = \infty, \;\; \text{and} \;\; \lim_{c\to\infty} U_c(c,h) = 0, \;\; \lim_{h\to\infty} U_h(c,h) = 0.$$

In contrast to the capital dynamic in a deterministic Keynes-Ramsey model, we assume that production is subject to random shocks which may due to consumer's health shock. More precisely, we extend the model of Brechet, Sun and Zhu (2017) to include the possibility of two kinds of sudden shocks to the health and capital. The first shock is a natural disaster, such as a hurricane, an earthquake or a nuclear leak, which seriously damages both production and human health. The second shock is an epidemic, can be caused by negative production externalities, which harms consumers' health regeneration. We define the law of motion of capital accumulation as follows:

$$dK_t = \big[Y_t - M_t - C_t\big]dt + \sigma K_t dZ_{k,t} - \xi k dN_{c,t}\,, K_t\,,\; M_t\,,\; C_t\,,\; t \geq 0\,,\; K(0) = K_0\,\text{given}\,. \tag{4.2}$$

where M_t is the aggregate form of abatement to mitigate emissions. $Z_{k,t}$ is a standard Wiener process, and $\sigma \geq 0$ is defined as sensitivity of the capital to fluctuations, which measures the amplitude of randomness caused by pollution shock entering the process of capital accumulation. The natural disaster follows a Poisson process, with increment $dN_{c,t}$ and intensity λ_c. Parameter ξ measures the percentage loss of capital due to catastrophe shocks. Y_t is the output, and firms produce a composite consumption good using broadly defined capital stock, denoted K_t as input. We assume the production function is constant returns to scale, and, for simplicity, the population is constant and unitized to 1.

$$Y_t = F(K_t) = A_t K_t \tag{4.3}$$

where total factor productivity A_t is assumed to be constant for simplicity: $A_t \equiv A$. It is worth remarking that, as a broadly accepted model in endogenous growth literature, the input K_t in AK model is interpreted as a broad measure of capital in the economy, such as physical capital, human capital, knowledge, etc. Suggested by Grossman (1972a, 1972b) and Cropper (1981), we consider a variety of pollution that degrades health status. Yet, the exact negative impact of pollution on health is non-deterministic, and therefore is assumed to be driven by a stochastic process:

$$dh(t) = R(q_t)h_t dZ_{h,t} - D_c(q_t)h_t dN_{c,t} - D_e(q_t)h_t dN_{e,t} \,. \tag{4.4}$$

where q_t is emission concentrations per unit of capital, and $R(q_t)$ represents the impact of emissions on health. $Z_{h,t}$ is a standard Wiener process. The epidemic follows a Poisson process, with increment $dN_{e,t}$ and intensity λ_e. $D_c(q_t)$ and $D_e(q_t)$ measure the percentage loss of health caused by nature disasters and epidemic shocks, respectively. The health regeneration process proposed here is inspired by Bretschger and Vinogradova (2016). In addition, we assume the correlation structure of capital and health uncertainties as follows.

$$\mathbb{E}(dZ_{k,t}\, dZ_{h,t}) = \rho_{k,h}\, dt\,, \qquad \rho_{k,h} \in [-1, 1]\,,\, t \geq 0\,. \tag{4.5}$$

$$\mathbb{E}(dN_{c,t} dN_{e,t}) = 0\,. \tag{4.6}$$

where $\mathbb{E}(\cdot)$ denotes expected values of $(\cdot)$. $dZ_{k,t}$ and $dZ_{h,t}$ are correlated standard Wiener processes with correlation coefficient $\rho_{k,h}$, indicating the mutual influence between capital and health fluctuations. $dN_{c,t}$ and $dN_{e,t}$ are independent Poisson processes with

$$dN_{i,t} = \begin{cases} 1: & \text{probability} \lambda_i dt \\ 0: & \text{probability} 1 - \lambda_i dt \end{cases}, \quad i = \{c, e\}$$

Therefore, we have two following remarkable results: $\mathrm{E}\left(dN_{c,t}\right) = \lambda_c dt$ and $\mathrm{E}\left(dN_{e,t}\right) = \lambda_e dt$. D_c and D_e represent the capital destruction and health damage (or morality rate) during catastrophic shocks, where the subscript "c" and "e" stand for "natural disaster" and "epidemic" respectively.

4.2.2 Specializing some assumptions

In this subsection, we discuss the optimal growth problem which is characterized under three assumptions.

Preferences

We consider a standard constant relative risk aversion (CRRA) utility function

$$U(c_t, h_t) = \frac{c^{1-\epsilon}}{1-\epsilon} h^{\beta} \tag{4.7}$$

where the constant parameters $\epsilon\,, 1-\beta \in (0,1)$ represent Arrow-Pratt measure of relative risk aversion and the elasticity of marginal utility to consumption and health respectively[6]. Considering that fact that epidemic, caused by contagious diseases, lowers an average health status $\bar{h}$ of all individuals, we adopt the utility function in Bretschger and Vinogradova (2016) and generalize the previously defined utility with an average health term as follows.

$$U(c, h, \tilde{h}) = \frac{c^{1-\epsilon}}{1-\epsilon} h^{\beta} \bar{h}^{\gamma}\,, \quad \beta\,, \gamma\,, \epsilon \in [0,1]\,. \tag{4.8}$$

Let us now consider an average health status $\bar{h}$ is proportional to h (i.e. $\bar{h} \sim h$), based on the fact that rising h of the representative household will increase $\bar{h}$ proportionally, and vice versa. Without loss of generality, we consider the utility function in the following form:[7]

$$\textit{Assumption 1:} \quad U(c, h) = \frac{c^{1-\epsilon}}{1-\epsilon} h^{\tilde{\beta}}\,, \quad \tilde{\beta} = \beta + \gamma \,\text{and}\, \tilde{\beta}, \beta, \gamma, \epsilon \in [0,1]. \tag{4.9}$$

The parameters of consumption and health are discussed in detail in the quantitative section of this paper.

Emissions

The aggregate form of mitigation (or abatement) M_t is provided to mitigate the emissions, financed by a fraction of output, denoted by $m_t \in [0,1)$. The remaining share of output $1 - m_t$ is divided between consumption and capital accumulation. So we define m_t as abatement-output ratio:

$$m_t = \frac{M_t}{Y_t}, \qquad \text{where}\, Y_t > 0\,, M_t \geq 0. \tag{4.10}$$

[6] In order to guarantee the concavity of $U(\cdot)$ i.e. $U_h'(\cdot) > 0$, we require that $\epsilon \in (0,1)$. It is worth nothing that when $\epsilon = 1$, using L'hospital's rule, we have the logarithm utility function: $U(c_t, h_t) = \ln(c)\, h^{\beta}$.

[7] This utility function is designed under the inspiration of Lucas (1988).

The total emission is an increasing function of the output and a decreasing function of abatement spendings. The stock of aggregate emissions, denoted by Q_t, can be represented as:

$$Q_t = \phi Y_t - \mu M_t \,, \quad \phi\,, \mu\,, t \geq 0\,. \tag{4.11}$$

Parameter $\phi \geq 0$ is so-called the emission intensity of output, which indicates the emissions per unite of output. $\mu \geq 0$ is the efficiency of abatement. By definition, we have $q_t = Q_t/K_t$, where q_t is the emission concentrations per unit of capital defined in Eq.(5.35). We can represent Eq.(4.11) in terms of q_t as follows:

$$\textit{Assumption 2:} \qquad q_t = (\phi - \mu m_t)A\,, \quad \phi\,, \mu\,, t \geq 0\,. \tag{4.12}$$

Apparently, the total emission needs to be non-negative: $Q_t \geq 0$. Hence, the abatement-output ratio m_t should be bounded from above: $m_t \leq \phi/\mu$, if $\phi/\mu \in (0, 1)$. Pollution is caused by production-released detrimental emissions. Therefore, we assume the pollution-induced health degradation is proportional to emission per capital.

$$\textit{Assumption 3:} \qquad R(q) = \delta q_t\,, \quad D_c(q) = \eta_c q_t\,, \quad D_e(q) = \eta_e q_t \tag{4.13}$$

where $\delta\,, \eta_c\,, \eta_c \geq 0$ denotes the sensitivity of the health to pollution, natural disaster and epidemic respectively.

4.2.3 The planning problem

We now substitute our key assumptions into the general formulation and discuss the planning problem. Since we consider the AK-type growth model with constant population size unitized to 1, capital per labour and output per labour are identical to overall capital and output respectively, i.e.

$$k_t \equiv \frac{K_t}{L_t} = K_t\,, \quad y_t \equiv \frac{Y_t}{L_t} = Y_t \tag{4.14}$$

Substituting m_t defined in *Assumption 2* into Eq.(3.2), we have

$$dk_t = \big[(1 - m_t)y_t - c_t\big]dt + \sigma k_t dZ_{k,t} - \xi k dN_{c,t}\,, \quad k_t\,, c_t\,, t \geq 0\,,\ k(0) = k_0 \text{ given} \tag{4.15}$$

Moreover, combing Eqs.(4.4), (4.5), (4.6) and (4.15), the policy maker solves the following problem

$$\max_{c,m} \mathbb{E}_0 \left\{ \int_0^\infty U(c_t, h_t)\, e^{-\rho t}\, dt \right\} \tag{4.16}$$

subject to

$$dk_t = \big[(1-m_t)y_t - c_t\big]dt + \sigma k_t dZ_{k,t} - \xi k dN_{c,t}\,,\ \ k(0) = k_0 \ \text{given} \tag{4.17}$$
$$dh_t = R(q_t)h_t dZ_{h,t} - D_c(q_t)h_t dN_{c,t} - D_e(q_t)h_t dN_{e,t}\,. \tag{4.18}$$
$$\mathbb{E}(dZ_{k,t}\, dZ_{h,t}) = \rho_{k,h}\, dt\,, \quad \rho_{k,h} \in [-1,1]\,,\ t \geq 0\,. \tag{4.19}$$
$$E(dN_{c,t}dN_{e,t}) = 0\,. \tag{4.20}$$

From now on, we omit the time subscripts for control and state variables for convenience. Eq.(4.15) can be further simplified if the saving rate is assumed to be constant, i.e. $c = xk$, where x is a constant. Then capital dynamic can be expressed in a simple jump-diffusion model (i.e. Geometric Brownian Motion with compound Poisson jumps):

$$dk = \mu k dt + \sigma k dZ_{k,t} - \xi k dN_{c,t}\,, \quad k\,,\, t \geq 0\,,\ k(0) = k_0 \ \text{given}\,. \tag{4.21}$$

where drift parameter $\mu = (1-m)A - x$ is constant in optimal. The abatement-output ratio formula in our main proposition below relies on saving-rate constancy. In Appendix, we derive the analytical solution for x as function of the model's parameters by using the guess and verify method. In the data, moreover, saving rates do not tend to vary so much over time, and long-run growth models are often specified so that c_t/y_t is constant(see, e.g., Acemoglu (2009)).

4.2.4 Competitive equilibriums

In this subsection, we characterize the solutions to the planner's problem in the set-up described above. Following the approach of Dixit and Pindyck (1994), and Steger (2005), we write down the Hamilton-Jacobi-Bellman (HJB) equation for the planning problem as follows:

$$\rho V(k,h) = \max_{c,m} \left\{ U(c,h) + \frac{1}{dt}\mathrm{E}\big[dV(k,h)\big] \right\} \tag{4.22}$$

where $V(k,h) = \max\limits_{c,m} U(c,h)$ is the value function associated with the optimal control problem. Applying Ito's lemma with Jumps, we obtain the

Hamilton-Jacobi-Bellman (HJB) equation as follows:[8]

$$\rho V(k,h) = \max_{c,m}\Big\{U(c,h) + V_k\big[(1-m)y - c\big] + \frac{1}{2}\Big[V_{kk}\sigma^2 k^2 + V_{hh}R^2(q)h^2 + 2\rho_{k,h}V_{kh}\sigma k R(q)h\Big] + \lambda_c\Big[\Big(V^{\tilde{k}} - V\Big) + \Big(V^{\tilde{h}} - V\Big)\Big] + \lambda_e\Big(V^{\tilde{\tilde{h}}} - V\Big)\Big\}. \quad (4.23)$$

Detailed analytical treatment is presented in the Appendix. Calculating the first-order conditions (FOCs) for control variables $\{c, m\}$, we have:[9]

$$\text{For } c: \qquad U_c = V_k \quad (4.24)$$

$$\text{For } m: \quad \frac{1}{\mu A}V_k\, y_t = -V_{hh}R(q)R'(q)h^2 - V_{kh}\sigma R'(q)kh\rho_{kh} + \Big(\lambda_c\, V^{\tilde{h}}_{\tilde{h}}D'_c(q) + \lambda_e V^{\tilde{\tilde{h}}}_{\tilde{\tilde{h}}}D'_e(q)\Big)h \quad (4.25)$$

Applying envelope theorem to state variables $\{k\,, h\}$, we have:

$$\text{For } k: \ \rho V_k = V_{kk}\big[(1-m)y - c\big] + V_k(1-m)A + \frac{1}{2}\Big[V_{kkk}\sigma^2k^2 + 2\sigma^2V_{kk}k + V_{hhk}R^2(q)h^2 + 2\rho_{kh}\sigma V_{khk}R(q)hk + 2\sigma\rho_{kh}V_{kh}R(q)h\Big] + \lambda_c\Big[\big(V^{\tilde{k}}_k - V_k\big) + \big(V^{\tilde{h}}_k - V_k\big)\Big] + \lambda_e\big(V^{\tilde{\tilde{h}}}_k - V_k\big) \quad (4.26)$$

$$\text{For } h: \ \rho V_h = U_h + V_{kh}[(1-m)y - c] + \frac{1}{2}\Big[V_{kkh}\sigma^2k^2 + V_{hhh}R^2(q)h^{2\alpha} + 2V_{hh}R^2(q)h^2 + 2\sigma V_{khh}R(q)kh\rho_{kh} + 2\sigma V_{kh}R(q)k\rho_{kh}\Big] + \lambda_c\Big[\big(V^{\tilde{k}}_h - V_h\big) + \big(V^{\tilde{h}}_h - V_h\big)\Big] + \lambda_e\big(V^{\tilde{\tilde{h}}}_h - V_h\big) \quad (4.27)$$

Before solving the above system of differential equations, we firstly prove the existence and uniqueness of the equilibrium.

Proposition 4.2.1. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to a social planner's problem implies that c_t/y_t is constant in all states and at all times. The implicit solution for the equilibrium emission concentration $\tilde{q}^*$ satisfies the following equation:*

$$\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big) = \tilde{\beta}\mu\Big\{\Big[\lambda_c\eta_c\big(1-\eta_c\tilde{q}^*\big)^{\tilde{\beta}-1} + \lambda_e\eta_e\big(1-\eta_e\tilde{q}^*\big)^{\tilde{\beta}-1}\Big] + \big(1-\tilde{\beta}\big)\delta^2\tilde{q}^*\Big\} \quad (4.28)$$

Moreover, Eq.(4.28) exists unique solution $\tilde{q}^*$ if and only if

$$\lambda_c\eta_c + \lambda_e\eta_e \le \frac{\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big)}{\tilde{\beta}\mu} \le \lambda_c\hat{\eta}_c + \lambda_e\hat{\eta}_e + \big(1-\tilde{\beta}\big)\delta^2\phi A\,, \quad \frac{\phi}{\mu} \in (0,1) \quad (4.29)$$

---

[8] We define the following notations for simplify: $V^{\tilde{k}} = V(\tilde{k}, h)$, $V^{\tilde{h}} = V(k, \tilde{h})$ and $V^{\tilde{\tilde{h}}} = V(k, \tilde{\tilde{h}})$, where $\tilde{k} = k(1-\xi)$, $\tilde{h} = h\big[1 - D_c(q)\big]$ and $\tilde{\tilde{h}} = h\big[1 - D_e(q)\big]$.

[9] We define: $V^{\tilde{h}}_{\tilde{h}} := \dfrac{\partial V^{\tilde{h}}}{\partial \tilde{h}} = \dfrac{\partial V(k,\tilde{h})}{\partial \tilde{h}} = V_h(k,h)\mid_{h=\tilde{h}}$ and $\tilde{V}^{\tilde{\tilde{h}}}_{\tilde{\tilde{h}}} := \dfrac{\partial V(k,\tilde{\tilde{h}})}{\partial \tilde{\tilde{h}}} = V_h(k,h)\mid_{h=\tilde{\tilde{h}}}$.

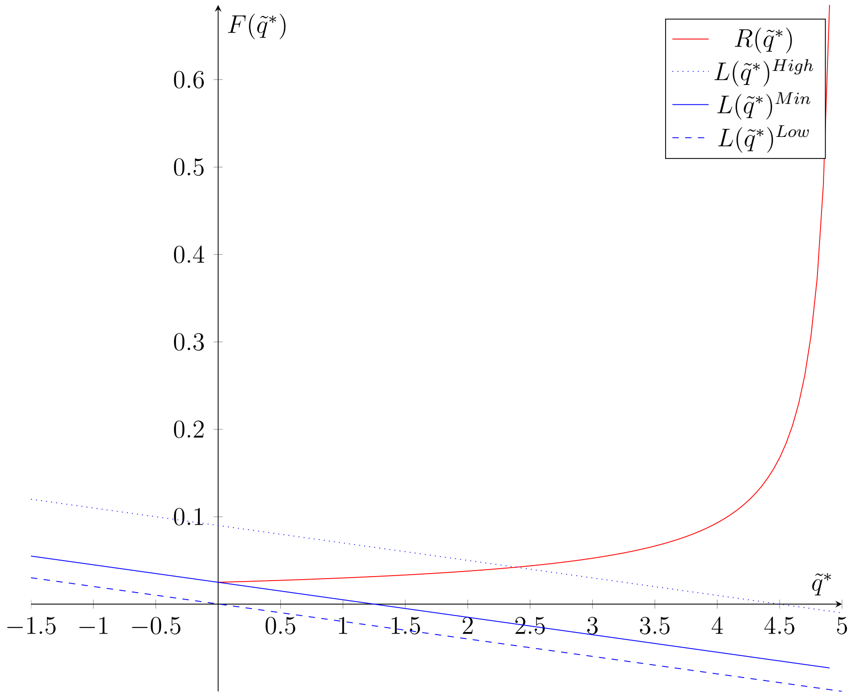


Figure 4.1 – The condition for solution existence and uniqueness.

where $\hat{\eta}_c = \eta_c\big(1-\eta_c\phi A\big)^{\tilde{\beta}-1}$ *and* $\hat{\eta}_e = \eta_e\big(1-\eta_e\phi A\big)^{\tilde{\beta}-1}$.

Proof. Detailed algebra proof is provided in the Appendix. Instead, we consider a geometrical illustration of the proposition. Firstly, we rearrange polynomial in Eq.(4.28) for $\forall\,\tilde{\beta},\ \mu \neq 0$ as follows.

$$\begin{aligned}&\frac{\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big)}{\tilde{\beta}\mu} - \big(1-\tilde{\beta}\big)\delta^2\tilde{q}^* \\ &= \lambda_c\eta_c\big(1-\eta_c\tilde{q}^*\big)^{\tilde{\beta}-1} + \lambda_e\eta_e\big(1-\eta_e\tilde{q}^*\big)^{\tilde{\beta}-1}\end{aligned} \tag{4.30}$$

Let us define LHS and RHS of Eq.(4.30) as

$$L(\tilde{q}^*) = \frac{\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big)}{\tilde{\beta}\mu} - \big(1-\tilde{\beta}\big)\delta^2\tilde{q}^*$$

and

$$R(\tilde{q}^*) = \lambda_c\eta_c\big(1-\eta_c\tilde{q}^*\big)^{\tilde{\beta}-1} + \lambda_e\eta_e\big(1-\eta_e\tilde{q}^*\big)^{\tilde{\beta}-1}$$

We plot $L(\tilde{q}^*)$ and $R(\tilde{q}^*)$ in Figure 1. Obviously, $L(\tilde{q}^*)$ is a linear monotonic decreasing function of $\tilde{q}^*$, and $R(\tilde{q}^*)$ is a power monotonic increasing function of $\tilde{q}^*$. The monotonicity guarantees the uniqueness once $L(\tilde{q}^*)^{Min} \leq L(\tilde{q}^*) \leq L(\tilde{q}^*)^{Max}$, which implies Eq.(4.29) holds. $\square$

Analogously, we can prove the existence and uniqueness of optimal abatement-output ratio m, using the linear relationship between m and q defined in *Assumption 2*. Although we prove the existence of the optimal emission concentration $\tilde{q}^*$, the closed-form solution of Eq.(4.28) is not obvious. We overcome this difficulty by linearising $\tilde{q}^*$ around original points using Taylor expansion. Therefore, for both natural disasters and epidemic shocks, we approximate $(1-\eta\,\tilde{q}^*)^{\tilde{\beta}-1}$, $\eta = \{\eta_c\,, \eta_e\}$ as follows.

$$\begin{aligned}(1-\eta\,\tilde{q}^*)^{\tilde{\beta}-1} &= 1 + (-\eta)(\tilde{\beta}-1)(1-\eta\,\tilde{q}^*)^{\tilde{\beta}-1}\,|_{\tilde{q}^*=0} + o(\tilde{q}^*) \\ &\doteq 1 + \eta(1-\tilde{\beta})\tilde{q}^*\,, \quad \eta = \{\eta_c\,, \eta_e\}\,. \end{aligned} \tag{4.31}$$

Substituting Eq.(4.31) into (4.28), we obtain optimal emission concentration:

$$\tilde{q}^* = \frac{\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big) - \tilde{\beta}\mu\big(\lambda_c\eta_c + \lambda_e\eta_e\big)}{\tilde{\beta}(1-\tilde{\beta})\mu\Big[\delta^2 + \big(\lambda_c\eta_c + \lambda_e\eta_e\big)\Big]} \tag{4.32}$$

Substituting Eq.(4.32) into Eq.(4.12), we obtain the optimal abatement-output ratio as follows:

$$\tilde{m}^* = \frac{\phi}{\mu} - \frac{\big(1-\epsilon\big)\big(1+\tilde{\beta}\mu\sigma\delta\rho_{k\,h}\big) - \tilde{\beta}\mu\big(\lambda_c\eta_c + \lambda_e\eta_e\big)}{\tilde{\beta}(1-\tilde{\beta})\mu^2 A\Big[\delta^2 + \big(\lambda_c\eta_c + \lambda_e\eta_e\big)\Big]} \tag{4.33}$$

The existence and uniqueness condition in Eq.(4.29) guarantees the interior solutions of $\tilde{q}^*$ and $\tilde{m}^*$. Apparently, $\tilde{q}^*$ is smaller than $\tilde{q}^*$, the optimal emission excluding catastrophe shocks (i.e. $\eta_c = \eta_e = 0$, see previous chapter). Because the denominator and nominator of $\tilde{q}^*$ are, respectively, larger and smaller than q^*. This implies a larger optimal abatement spendings (i.e. $\tilde{m}^* < m^*$, where m^*, the optimal abatement-output ratio in previous chapter), where the catastrophic environmental degradations are not included.

Proposition 4.2.2. *Suppose Assumptions 1, 2 and 3 are satisfied, and the solution to a social planner's problem implies that c_t/y_t is constant in all states and at all times. The optimal abatement-output ratio $\tilde{m}^*$ is positively correlated with both the frequencies and the intensities of catastrophes i.e.*

$$\frac{\partial\tilde{m}^*}{\partial\lambda_c} > 0\,,\ \frac{\partial\tilde{m}^*}{\partial\eta_c} > 0\,,\ \frac{\partial\tilde{m}^*}{\partial\lambda_e} > 0\,,\ \frac{\partial\tilde{m}^*}{\partial\eta_e} > 0$$

.

Proof. Provided in Appendix. □

Applying Ito's lemma with Poisson jump to the consumption function $c(U_c, h)$, we obtain the expected optimal growth path of consumption as follows:

$$\tilde{g}^* := \mathrm{E}\left(\frac{\dot{c}}{c}\right) = \frac{A-\rho}{\epsilon} + \frac{1}{\epsilon}\left(I_g + \Gamma - \frac{\phi}{\mu}A\right) \tag{4.34}$$

The first term of right hand side of Eq.(4.34) reflects the standard Keynes-Ramsey rule of optimal consumption. That is the risk-free interest rate[10] less the preference rate, divided by the constant measure of relative risk aversion.[11] The last term implies the implicit risk premium, adjusted by the elasticity of intertemporal substitution for consumption, à la Dorfman (1969). Hence, in our model, the real interest rate is composed by risk-free rate less the risk premium. It represents one of the main channels through which uncertainty affects economic growth (De Hek, 1999). Specifically, this can be illustrated by environmental abatement effect and uncertainty effect. The abatement effect is defined by the ratio of the emission intensity of output to the abatement efficiency, i.e. ϕ/μ. The abatement has a negative impact on the real interest rate because production externalities have an unambiguously negative effect on marginal product of capital (MPK). This negative effect can be reduced by either reducing the pollution intensity or improving the abatement efficiency. The overall uncertainty effect is composed by small fluctuations I_g and catastrophic shocks Γ, driven by Wiener and Poisson processes respectively. Moreover, we can write Eq(4.34) as:

$$\tilde{g}^* = g^* + \frac{1}{\epsilon}\Gamma \tag{4.35}$$

where $g^* = \dfrac{A-\rho}{\epsilon} + \dfrac{1}{\epsilon}\left(I_g - \dfrac{\phi}{\mu}A\right)$ and $I_g = \dfrac{\left[1+(1-\epsilon)\beta\sigma\delta\mu\rho_{kh}\right]^2}{2\beta(1-\beta)(\mu\delta)^2} - \dfrac{\epsilon^2}{2\beta(1-\beta)(\mu\delta)^2} - \dfrac{1}{2}\epsilon(1-\epsilon)\sigma^2$ are expected growth rate and uncertainty

[10] The risk-free interest rate equals to a marginal product of capital (MPK), i.e. $r = A$.

[11] The Keynes-Ramsey rule in isoelastic utility (or CRRA utility) function has the following form: $\dfrac{\dot{c}}{c} = \dfrac{r_t - \rho}{\epsilon}$ with $\epsilon = -\frac{cU_c''}{U_c'}$ constant

effect excluding catastrophe shocks (see previous chapter). The effect of catastrophic shocks Γ can be decomposed into the effect of natural disaster and epidemic shocks, driven by two independent Poisson processes as follows.

$$\Gamma = \lambda_c \Gamma_c(q) + \lambda_e \Gamma_e(q)$$

where

$$\Gamma_c(q) = \left(\left[\left(1-\xi\right)^{1-\epsilon} - 1\right] + \left[\left(1 - D_c(q)\right)^{\tilde{\beta}} - 1\right]\right) + \epsilon\left[\left(1 + \left[\left(1-\xi\right)^{-\epsilon} - 1\right] + \left[\left(1 - D_c(q)\right)^{\tilde{\beta}} - 1\right]\right)^{-\frac{1}{\epsilon}} - 1\right] + \epsilon\left[\left(1 - D_c(q)\right)^{\frac{\tilde{\beta}}{\epsilon}} - 1\right] \tag{4.36}$$

$$\Gamma_e(q) = \left[\left(1 - D_e(q)\right)^{\tilde{\beta}} - 1\right] + \epsilon\left[\left(1 - D_e(q)\right)^{-\frac{\tilde{\beta}}{\epsilon}} - 1 + \left(1 - D_e(q)\right)^{\frac{\tilde{\beta}}{\epsilon}} - 1\right] \tag{4.37}$$

In practice, emission concentration per capital is close to zero (around 10^{-5}). So we focus on the sign of Eqs.(4.36) and (4.37) for a small $q \ll 1$. A convenient approach to show this point is to use linearisation, though we give an analytic proof in the Appendix. We linearise Γ_c and Γ_e using a first-order Taylor expansion around $q = 0$ and $\xi = 0$ as follows.

$$\left(1 - D_c(q)\right)^{\tilde{\beta}} \doteq 1 - \tilde{\beta}\eta_c q\,, \quad \text{where}\, D_c(q) = \eta_c q \tag{4.38}$$

$$\left(1 - D_e(q)\right)^{\tilde{\beta}} \doteq 1 - \tilde{\beta}\eta_e q\,, \quad \text{where}\, D_e(q) = \eta_e q\,. \tag{4.39}$$

$$\left(1 - \xi\right)^{1-\epsilon} \doteq 1 - (1-\epsilon)\xi\,. \tag{4.40}$$

Substituting Eqs.(4.38)-(4.40) into Eqs.(4.36) and (4.37) , we obtain both $\Gamma_c(q)$ and $\Gamma_e(q)$ are negative when $q \ll 1$.

$$\Gamma_c(q) = -\xi - \tilde{\beta}\eta_c q < 0 \tag{4.41}$$

$$\Gamma_e(q) = -\tilde{\beta}\eta_e q < 0 \tag{4.42}$$

Thus, the effect of catastrophic shocks are negative.

$$\begin{aligned}\Gamma &= \lambda_c \Gamma_c(q) + \lambda_e(q)\Gamma_e \\ &= -\xi\lambda_c - \left(\lambda_c\eta_c + \lambda_e\eta_e\right)\tilde{\beta}q < 0\end{aligned} \tag{4.43}$$

Substituting Eqs.(4.32) and (4.43) into (4.35), we obtain the expected growth rate as a function of model's parameters, where

$$\Gamma = -\xi\lambda_c - \left(\lambda_c\eta_c + \lambda_e\eta_e\right)\frac{(1-\epsilon)\left(1 + \tilde{\beta}\mu\sigma\delta\rho_{kh}\right) - \tilde{\beta}\mu\left(\lambda_c\eta_c + \lambda_e\eta_e\right)}{(1-\tilde{\beta})\mu\left[\delta^2 + \left(\lambda_c\eta_c + \lambda_e\eta_e\right)\right]} \tag{4.44}$$

4.3 Analytical illustration

To understand the effect of catastrophic shocks on equilibriums, recalling the optimal growth rate in Eq.(4.35) and the associated parameters defined in previous section, we have the following proposition:

Proposition 4.3.1. *The effect of catastrophic shocks slows down the expected economic growth rate. However, this show-down effect could be accelerated or dampened (even recovered), depending on the threshold values of the intensity and severity of catastrophic shocks. Indeed, the relationship between the effect of catastrophic shocks and expected growth rate has an inverted-U shape. The quantitative description is as follows.*

$$\frac{\partial \tilde{g}^*}{\partial \lambda_c} \begin{cases} < 0 & \textit{when} \begin{cases} z \geq 0 & \textit{if } 0 < \tilde{\beta} \leq \xi/(\xi+\eta_c)\,, \\ 0 \leq z < \tilde{z}^* & \textit{if } \xi/(\xi+\eta_c) < \tilde{\beta} < 1 \end{cases} \\ \geq 0 & \textit{when } z \geq \tilde{z}^*\,, \textit{if } \xi/(\xi+\eta_c) < \tilde{\beta} < 1\,. \end{cases}, \tag{4.45}$$

$$\frac{\partial \tilde{g}^*}{\partial \eta_c} \begin{cases} < 0 & \textit{when } 0 \leq z < z^*\,, \\ \geq 0 & \textit{when } z \geq z^*\,. \end{cases} \tag{4.46}$$

$$\frac{\partial \tilde{g}^*}{\partial \lambda_e} \begin{cases} < 0 & \textit{when } 0 \leq z < z^*\,, \\ \geq 0 & \textit{when } z \geq z^*\,. \end{cases} \tag{4.47}$$

$$\frac{\partial \tilde{g}^*}{\partial \eta_e} \begin{cases} < 0 & \textit{when } 0 \leq z < z^*\,, \\ \geq 0 & \textit{when } z \geq z^*\,. \end{cases} \tag{4.48}$$

$$\textit{with } z^* = \left(\sqrt{1+(1-\epsilon)(1+\tilde{\beta}\mu\sigma\delta\rho_{kh})/(\tilde{\beta}\mu\delta^2)} - 1\right)\delta^2 \tag{4.49}$$

$$\textit{and } \tilde{z}^* = \left(\sqrt{\frac{1+(1-\epsilon)(1+\tilde{\beta}\mu\sigma\delta\rho_{kh})/(\tilde{\beta}\mu\delta^2)}{1-\xi(1-\tilde{\beta})/(\eta_c\tilde{\beta})}} - 1\right)\delta^2 \tag{4.50}$$

Proof. Provided in the Appendix. □

Recall that the expression of expected growth rate in Eq.(4.34) gives a similar Keynes-Ramsey rule albeit the last polynomial of right-hand side (RHS) of the equation, which represents the uncertainty effect, driven by Wiener and Poisson processes respectively. In addition, Γ on RHS of Eq.(4.34) accounts for the effect of catastrophic shocks, including both natural disasters and epidemics, driven by two independent Poisson processes. We are particularly interested in the sign of this item. When Γ is negative i.e. $0 \leq q \leq \bar{q}$, with $\bar{q} = \min\left\{ \frac{1}{\eta_c}\left[1-\left(2-\left(1-\xi\right)^{-\epsilon}\right)^{\frac{1}{\beta}}\right], \frac{1}{\eta_e}\left[1-\left(1+\frac{1}{\epsilon}\right)^{-\frac{\epsilon}{\beta}}\right]\right\}$. The catastrophic shocks dampen the economic growth. The presence of risky leads to a smaller certainty equivalent return on saving than the expected rate of return (Steger, 2005). The economic implication can be described as a precautionary dissaving motive (Muller-Furstenberger and Schumacher 2015; Bretschger and Vinogradova, 2016). Recall that (i) Γ is composed by two kinds of catastrophic shocks: nature disaster $\Gamma_c(q)$ and epidemic $\Gamma_e(q)$; (ii) $d\tilde{k}/dk$, $d\tilde{h}/dh$ and $d\tilde{\tilde{h}}/dh$ represent the change of the post-shock capital and health to the pre-shock capital and health when natural disaster and epidemic happen respectively. It is worth noting that the shocks on consumption include both a "direct effect" on capital and health (i.e. $\tilde{k}$, $\tilde{h}$, and $\tilde{\tilde{h}}$), and an "indirect effect" on the marginal utility (i.e. $dV_{\tilde{k}}^{\tilde{k}}/dV_k$, $dV_{\tilde{k}}^{\tilde{h}}/dV_k$ and $dV_{\tilde{k}}^{\tilde{\tilde{h}}}/dV_k$). This "indirect effect" is indeed a shock reaction of the marginal utility to the jump in capital and health, for a given multiplier ϵ. Whether the "direct effect" dominates the "indirect effect" depends on the magnitude of ϵ. Recall that ϵ, defined in our context, is also equal to the Arrow-Pratt measure of relative risk aversion and also the elasticity of marginal utility to consumption, which is required to be strictly larger than zero and smaller than one. Empirical literature suggests $q^* \ll 1$. The "indirect effect" is unambiguously dominated by the "direct effect". Therefore, the overall effect of catastrophic shocks slows down the economy.

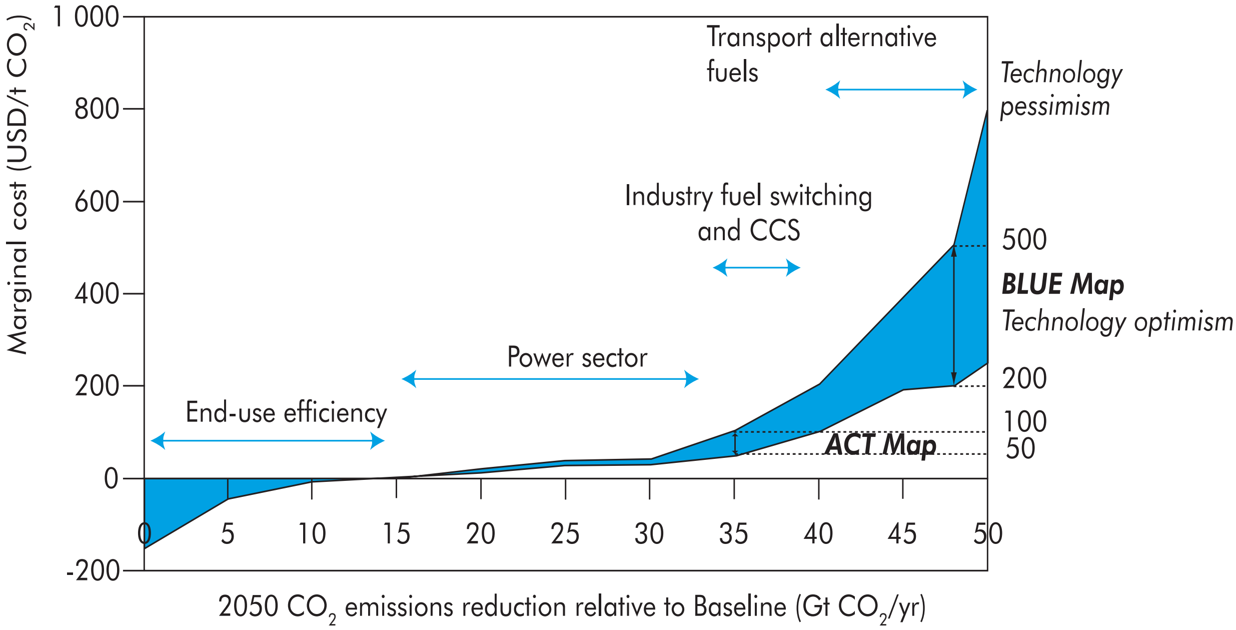
4.4 Quantitative analysis

4.4.1 Parameter selection

Firstly, let us focus on the abatement efficiency μ. Energy Technology Perspectives (2008)[12] reports that the ACT and BLUE scenarios in Figure.(4.2) represent a set of optimal pathways to reduce energy-related GHGs emissions. The family of ACT and BLUE scenarios describe least-cost pathways to return CO_2 emissions back to 2005 level by 2050, and reduce 50% of emission to the level of 2005 by 2050, respectively. The ACT scenario requires options with a marginal cost up to 50 U.S. dollar per ton CO_2 emission, while BLUE scenario needs 200 U.S. dollars per ton CO_2 emission, when significant technology cost reductions is viewed in an optimistic

[12]ETP, publication of The International Energy Agency (IEA).

way. Under pessimistic assumption about abatement technology development, the associated two scenarios will require 100 U.S. dollar and 500 U.S. dollar per ton respectively. The difference is due to the cost uncertainty for the abatement technology improvement. Nevertheless, technology improvement is observed by time. In IEA 2011, Hood summaries that significant level of emissions abatement could be achieved with existing technology, at carbon price less than 50 U.S. dollar per ton of CO_2 emission. However, further emission reduction for achieving a 2°C target will require new technology which associates with higher and more uncertain cost, such as carbon capture and storage (CCS) in industry, and alternative transport fuels. According to ETP (2010), up to 175 U.S. dollar per ton of CO_2 emission is needed to achieve the BLUE Map scenario.



Marginal costs increase significantly between ACT Map and BLUE Map, and the cost uncertainty increases.

Figure 4.2 – Marginal emissions reduction costs for the global energy system, 2050.[14]

[14]Based on source: Energy Technology Perspectives (ETP) of IEA, June, 2008.

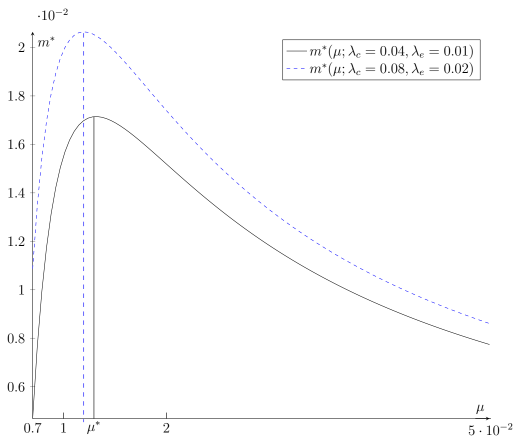
The emission defined in our model includes not only the GHGs but also other pollutants harmful to human health. The detrimental impact of emission on health will lead to tight mitigation policy, and consequently a higher abatement technology cost. In addition, uncertainty effect of CO_2 emission is underestimated. Ekins et al. (2011) argue that the abatement cost is very sensitive to many factors, like discounting rate, variations in investment cost and fuel efficiency, etc. Hereby, we compare various abatement measures and aim to include potential abatement technology development as well as uncertainty cost by setting emission price at 50 U.S. dollar per ton coal (implying abatement efficiency $\mu = 0.02$) as benchmark scenario, and 20 U.S. dollar per ton coal (implying $\mu = 0.05$), and 100 U.S. dollar per ton of coal (implying $\mu = 0.01$) respectively, for both optimistic and pessimistic assumptions of technology development.

Secondly, we set the TFP A, emission intensity ϕ, and abatement efficiency σ, equal to 5%, 0.4‰ and 8% respectively[15]. It is worth to remark that CO_2 emission is 0.4‰ tons (0.4 kg) per unit of GDP in U.S. dollar for the world average during 2011-2015, and hence we choose $\phi = 0.4$‰. We set the health parameter $\beta = 0.5$, according to Pautrel (2012), for the benchmark calibration. We choose benchmark value of $\epsilon = 0.9$ (e.g. India) according to Gandelman and Hernández-Murillo (2014). In addition, we choose time discount rate equal to 1.5% per year according to Nordhaus (2008).

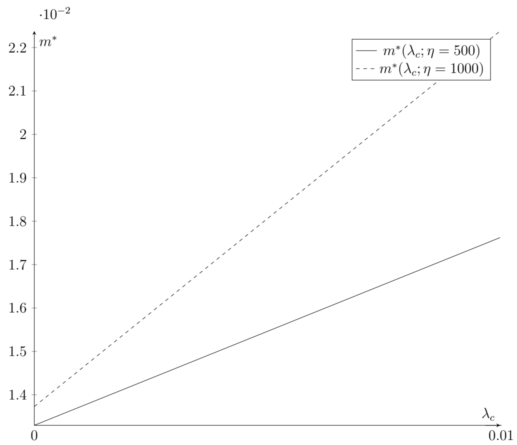
Last but not least, our model highlights the effect of catastrophic shocks in economic decisions, which include five key parameters λ_c, λ_e and ξ, η_c, η_e. Parameters ξ and η_c, η_e measure the sensitivity of the capital and health to catastrophic shocks. Parameter λ_c and λ_e measure the intensities of natural disasters and epidemics. These parameters are chosen intuitively inspired from various strands of literature. In order to accurately study the effect of catastrophic shocks on optimal policy, we set different values for these parameters and comparatively study their effect. The benchmark values of the parameters are summarized in **Table 1**. Our quantitative results in displayed in **Table 2**.

In **Figure 3**, we graphically show the effect of abatement efficiency and catastrophic shocks on the optimal abatement policy and growth rate.

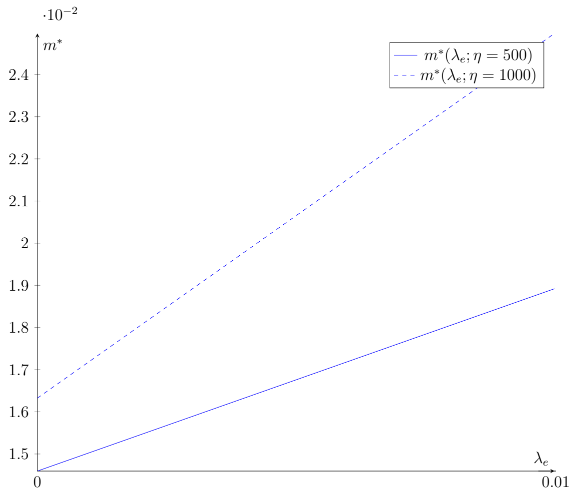
[15]based on the World Bank series 2016 "CO_2 emissions per GDP" and empirical studies by Hood (2011), McKinsey (2009), and Bretschger and Vinogradova (2014).



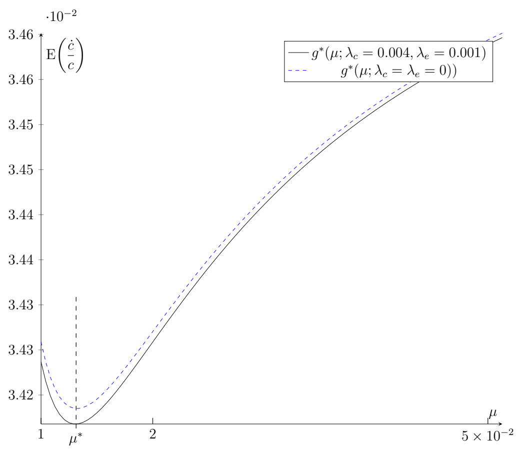
(a) The optimal mitigation policy m^* for different values of the efficiency of abatement μ.



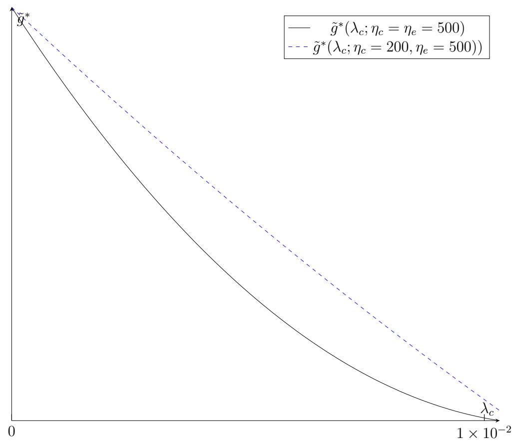
(b) The optimal mitigation policy m^* for different values of catastrophe intensity λ_c.



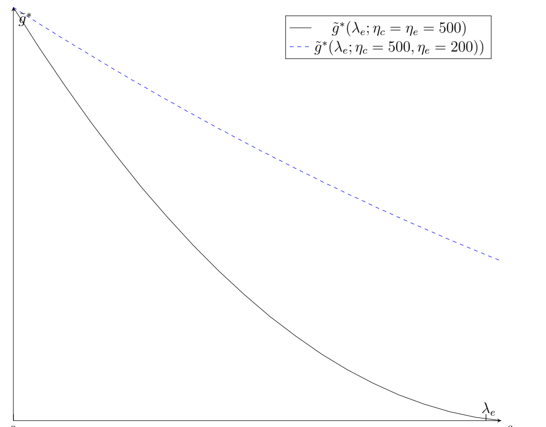
(c) The optimal mitigation policy m^* for different values of epidemic intensity λ_e.



(d) The optimal growth rate g^* for different values of the efficiency of abatement μ.



(e) The optimal growth rate g^* for different values of nature disaster intensity λ_c.



(f) The optimal growth rate g^* for different values of epidemic intensity λ_e.

Figure 4.3 – The effect of model's parameters on the optimal abatement policy m^* and growth rate g^*.

A	the total factor productivity	5%
ϕ	output emission intensity	0.04%
μ	abatement efficiency	0.08
ρ	time discount rate	0.015
δ	health sensitivity to fluctuation	$1.7 \cdot 10^3$
σ	capital sensitivity to fluctuation	0.001
β	health parameter	0.5
ϵ	elasticity of marginal utility	0.9
λ_c	natural catastrophe intensity	0.004
ρ_{kh}	correlation coefficient of Wiener uncertainties	-0.05
ξ	capital destruction during a natural disaster	0.001
λ_e	epidemic intensity	0.004
η_c	health sensitivity to nature disaster shock	500
η_e	health sensitivity to epidemic shock	500

TABLE 4.1 – *Values of parameters for numerical example*

4.4.2 Quantitative results

Our calibration shows that the abatement spendings dramatically increase when a greater risk is taken into account, i.e. the economy is under the threat of catastrophes. Calibrating the model under the benchmark parameters described above, we obtain the optimal abatement-output ratio $m^* = 1.4\%$, indicating carbon taxation up to 103 U.S. dollar per ton of coal in our benchmark setting, and $m^* = 0.8\%$, indicating 56.9 U.S. dollar per ton of coal, when abatement technology development is viewed in an optimistic way. Coincidentally, the latter number is the same with the calculation by Golosov et al. (2014) using a dynamic stochastic general-equilibrium (DSGE) model. Their suggested optimal carbon tax is equal to 56.9 U.S. dollar per ton of coal (or, equivalently, the optimal abatement-output ratio equals to 0.8% of world output), when the subjective discount is 1.5% per year, and the total world emissions is 9.7 billion tons of carbon in 2010. Our calculation suggests a higher carbon tax when the catastrophic shocks are taken into account. Under the threat of catastrophes, we have reason to view the abatement efficiency in a less optimistic way. Therefore, the real social cost of carbon should be higher because the extreme events (or so-called "fat tail") are neglected in conventional approaches (Weitzman 2011, 2014).

Parameters	Optimal Mitigation Policy and Growth Rate		
	$q^*(10^{-6})$	$m^*\%$	$g^*\%$
$\lambda_c = 0, \lambda_e = 0$	1.497	1.185	3.8155
$\mu = 0.01$	12.181	1.564	3.8157
$\mu = 0.02$	4.973	1.432	3.8142
$\mu = 0.05$	0.647	0.774	3.8494
$\eta^H = 1 \cdot 10^3$	2.810	1.679	3.8140
$\eta^L = 5 \cdot 10^2$	4.973	1.432	3.8142
$\lambda_c^H = 0.01$	2.377	1.728	3.8135
$\lambda_c^L = 0.004$	4.973	1.432	3.8142
$\lambda_e^H = 0.004$	3.675	1.58	3.8140
$\lambda_e^L = 0.001$	4.973	1.432	3.8142

TABLE 4.2 – *The optimal emission concentration, mitigation fraction and grwoth rate in banchmark setting when catastrophie happens,* $\epsilon = 0.9, A = 5\%, \phi = 0.04\%, \delta = 1700, \mu = 0.08, \delta = 0.001, \rho_{kh} = -0.5$.

4.5 Conclusion

This paper investigates a sustainable growth economy in an uncertainty environment, where environmental degradation (i.e. pollution, global warming, catastrophes) and its detrimental impact on health are taken into account. Specifically, our model enables policy maker to maximize the society well-being while optimally balance the environmental health and GDP growth under the threat of catastrophes. We present a stochastic endogenous growth model. This model is a generalization of the previous chapter in which the extreme events are not specified since the overall uncertainties are explained by one type of randomness: Wiener process In this paper, we highlight the great environmental uncertainties by introducing two kinds of catastrophes: natural disasters and epidemics. In particular, the capital and health regeneration processes are driven by two correlated Wiener processes and two independent Poisson processes. The proposed dynamic structure can be interpreted as a multi-factor jump-diffusion model (i.e. Geometric Brownian motion with composed jumps). We prove the existence and uniqueness of the equilibriums under the necessary and sufficient condition. Our central results are two closed-from formulas for optimal abatement policy and growth rate respectively. The abatement spendings, acting as the first-best policy, ensure the economy grows at a "healthy" rate.

It demonstrates that the relationship between the effect of catastrophic shocks and expected growth rate is an inverted-U shape. The catastrophic shocks could stimulate the growth rate, depending on whether the effect of shocks on capital and health regeneration (i.e. the "direct effect") is dominated by the effect of shocks on marginal utility (i.e. "indirect effect"). The economic implications are discussed in details in **Propositions (3.1)**. In the quantitative analysis, we plot the impact of catastrophes on optimal abatement policy and economic growth.

Even if our model aims to provide the country-based environmental policy, it is compatible to analyse optimal carbon tax at global level. The emission defined in our model includes not only the GHGs, but also other pollutants harmful to human health. Our numerical study suggests the abatement-output ratio $m^* = 1.4\%$, indicating 103 U.S. dollar per ton of coal taxation, in our benchmark setting, and $m^* = 0.8\%$, indicating 56.9 U.S. dollar per ton of coal respectively, if the abatement technology development is viewed in an optimistic way. Coincidentally, the latter number is the same as Golosov et al. (2014) using a DSGE framework. Apparently in face of catastrophic shocks, it is reasonable to have less optimistic view on the abatement efficiency. Therefore, our model suggests a tighten climate policy and higher social cost of carbon, since the extreme events (or so called "fat tail") are neglected in conventional approaches (Weitzman 2011, 2014).

5 Sustainable growth III: climate change and pollution

5.1 Introduction

5.1.1 Purpose and stylized facts

The detrimental impact of production-induced environmental degradation, such as pollution, global warming and the associated catastrophes, leads significant economic and health losses that change population's consumption and saving decisions, and as a consequence, economic growth. First of all, a growing number of empirical evidence shows that the influence of environmental pollution, like haze, water pollution and food contamination, entails severe human health degradation, for instance, the increasing morbidity of chronic diseases and the associated mortality rate. Specially, pollution-induced disease has a major detrimental impact on working-age population, and hence, reduces economic activity in terms of lacking labour productivity. The World Health Organization (WHO, 2004) reports that 56% population suffering from chronic diseases are aged between 15 to 59 in high-income countries. Moreover, an overall 3 million deaths are caused by outdoor air pollution in global cities and rural in 2014, and the vast majority of those form of deaths (70%) occurs in low- and middle-income countries.[1] The associated economic losses are striking. In U.S., an amount of 277 billion U.S. dollar annual expenditure is spent on treatment to seven most common chronic diseases, and the productivity loss is equal to 1.1 trillion U.S. dollar per year (Devol and Bedroussian, 2007). In Europe, air pollution causes approximate 6 million premature deaths, and the associated economic loss is no less than 1.6 trillion U.S. dollar a year (WHO and OECD 2015). The situation is even worse in emerging economies. Taking China as an example, World Bank (2007) estimates that the health costs due to air and water pollution is equal to 4.3% of its annual GDP.

[1] WHO news, March, 25, 2014. Available online: http://www.who.int/mediacentre/news/releases/ 2014/air-pollution/en/

Consequently, reducing production emissions has become an urgent and primary issue of environmental and health policies worldwide. This goal can be achieved by tightening mitigation policy and enlarging investment in abatement technologies, such as low carbon transport system and renewable fuels. From economic point of view, we motivate our analysis with a number of stylized facts of pollution-induced health degradation, matched by the corresponding properties of our theoretical model, as follows:

(i) Both CO2 emission and pollutant, as two externalities of production, causes global warming and degrades consumers' health. As a consequence, the classical free market optimal allocations are no longer efficient without taking into account of environmental policies.

(ii) Both CO2 emission and pollutant are proportional to economic activity, yet only the pollution has detrimental impact on human health.

(iii) As a fundamental component of labour productivity, consumer's health status thus constitutes an imperative part of sustainable economic growth.

5.1.2 Summary of contributions

The aim of this paper is to study the production-induced environment degradation and its detrimental impact on long-term growth. Specifically, our model enables policy maker to maximize the society well-being while optimally balance between emissions, health and growth in an uncertainty environment. We present an analytical DSGE model with two externalities: CO2 emissions and pollutant. This model can be seen as a generalization of Golosov et al. 2014. Yet in their model, the pollutant and health degradation is not considered. We contribute to the literature by introducing a two dimensional stochastic endogenous growth model. Our central result are three closed-form formulas for optimal health investment, growth rate and carbon tax. The impact of parameters on economic variables are detailedly discussed.

5.1.3 Literature review

An increasing amount of analytical dynamic stochastic general equilibrium (DSGE) models of climate change highlights the advantage of simple models with clear climate policy and expert opinions. (see, e.g. Rezai and van der Ploeg 2014, Golosov et al. 2014, Bretschger and Vinogradova 2014, Pindyck 2015). The DSGE models have the advantage of incorporating environmental uncertainty, tractability and provide policy makers with straightforward results. For instance, Golosov et al. (2014) provides closed-form solution to the optimal carbon tax, composed by the model's parameters. But the model only accounts the CO2 emission, not the pollutant. As the recent literature shows, industrial production emits both carbon dioxide and pollution. The latter is crucial for understanding the links between sustainable growth and health degradation, and hence the carbon tax and mitigation policies.

We define health as a human capital, inspired by the demand for health model à la Grossman (1972). Theoretical extensions and some competing economic models can be found in Grossman (1972b, 2000), Wagstaff (1987), Zweifel and Breyer (1997), and Galama (2011), where health is viewed as a capital stock depreciating overtime but can be enhanced by investment in medical care to produce healthy time that benefiting individual welfare and promoting labour productivity. Some empirical studies on the model of demand for health can be found in Wagstaff (1986) who estimates Grossman model (1972a) using 1976 Danish Welfare Survey. Van Doorslaer (1987) and Wagstaff (1993) extend the empirical study using longitudinal data.

There is a series of overlapping generations (OLG) models studying these effects (see, e.g., Gutiérrez (2008), Balestra and Dottori (2012), and Wang et al. (2015)). In these models, consumers are living for two periods. The young generation is always healthy, while the old generation faces the risk of mortality or unhealthy state, depending on the impact of pollutions. Specially, Gutiérrez (2008) finds that pollution raises health costs, and as a consequence, stimulates growth by fostering precautionary savings and capital accumulation. Yes, other papers find welfare losses and growth slows down. A three period OLG model can be found in Mariani et al. (2010), where agents may invest in environmental care, which affects their life expectancy. An interesting scenario of multiple equilibria is discussed. This illustrates the so-called low-life-expectancy/low-environmental-quality trap caught by some developing countries. In Mariani et al. model, the survival probability depends on the inherited environmental quality and is assumed to be constant in equilibrium states. In order to overcome the dynamic inefficiency in OLG framework á la Diamond (1965), some papers (see, e.g., Davide and Dottori (2012), and Wang et al. (2015)) discuss the second-best equilibrium using health insurance as an instrument. In particular, Wang et al. (2015) highlight that precautionary saving acts as a substitute for lacking health insurance. In contrast to Mariani et al. (2010), Davide and Dottori (2012) assume that the life expectancy in second period depends on current environmental condition. Hence, young generation can invest to improve environmental quality and benefit when they are old. Political effect of population ageing is discussed as well. Bretschger and Vinogradova (2014) present a stochastic model of a growing economy where natural disasters occur randomly, and the damages are caused by polluting activity.

5.1.4 Outline of the results

Our objective is to propose a analytical DSGE that allows the economy grows at a "healthy" rate when taking into account of two production externalities: carbon dioxide and pollutant. In this regard, we generalize Golosov et al. 2014 model with extension to a health demand framework. In contrast to the vast majority of analytical DSGE, our economic growth rate is endogenous and optimal carbon tax formula is very simple to derive and apply. An important insight of our model is the country-based environmental policy. However, the model is compatible to analyse optimal carbon tax at global level. We aim to accurately describe the impact of uncertainties by considering two stochasticities in damage functions on production and health, respectively.

Our model stands on the shoulder of Golosov et al. 2014, but with extension to health dynamics and pollutant. The rest of the paper is organized as follows. Section 2 describes the model with specialized assumptions. Section 3 presents analytical treatment and fully solves the dynamic model.

5.2 The standard model

We consider a human-capital-based endogenous growth model a la Uzawa (1965) and Lucas (1988). Suppose a policy maker maximizes overall discounted welfare over an infinite time period in an one-sector economy made of homogeneous individuals. There is a representative agent with the utility function

$$\max_{C,u} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{5.1}$$

where C is consumption and $\rho \in (0,1)$ is time discount factor. $U(\cdot)$ is a positive, increasing and concave function in both of its arguments:

$$U(\cdot) \geq 0, \;\; U'(\cdot) \geq 0, \;\; U''(\cdot) < 0.$$

Additionally, U satisfies the Inada conditions:

$$\lim_{C\to 0} U_C(\cdot) = \infty, \;\; \text{and} \;\; \lim_{C\to\infty} U_C(\cdot) = 0\,.$$

Analogous to Golosov et.al 2014, we consider a multi-sector neoclassical growth model, which as $I+1$ sectors. When $i=0$ presents the final-good sector and for every I presents intermediate-good sector producing energy inputs E_i, $i \in \{1,\ldots,I\}$ for all sectors. We consider the output in the final-goods sector as the following aggregate production function $F_{0,t}$:

$$Y_t = F_{0,t}(K_{0,t}, u_t, H_{0,t}, \underline{E_{0,t}}, P_t, S_t) \tag{5.2}$$

$$H_{0,t} = h_t N_{0,t} \tag{5.3}$$

The arguments of $F_{0,t}$ include the standard inputs: capital $K_{0,t}$ and health capital $H_{0,t}$ used in this sector, along with $\underline{E_{0,t}} = (E_{0,1,t}, E_{0,2,t}, \ldots E_{0,I,t})$, denoting a vector of energy inputs used in this final sector at t. u_t is the fraction of human capital allocated to the production of final sector, and thus $1-u_t$ is the corresponding fraction of human capital used in health regeneration process. We can think health as human capital, $H_{0,t}$, equal to the number of workers in final sector, $N_{0,t}$, multiplied by the human capital of the typical worker, h, which are perfect substitutes in production in the sense that only the product $N_{0,t}h_t$ matters for output. We assume that the raw labor $\dot{N}_{0,t}$ is assumed to grow at an exogenous rate:

$$\frac{\dot{N}_{0,t}}{N_{0,t}} = n \tag{5.4}$$

Hence, despite of population growth, H_t grows only because of improvements in the healty quality, h_t. We omit any technological progress (that is, we assume that A is constant in our standard model). The feasibility constraint in the final-goods sector is:

$$C_t + K_{t+1} = F_{0,t}(K_{0,t}, u_t, H_{0,t}, \underline{E_{0,t}}, P_t, S_t) + (1 - \delta_k)K_t \tag{5.5}$$

We allow climate variables to affect output through two channels: the stock of GHGs and the flow of pollutant, denoted by S_t and P_t respectively. S_t is to be read as the total amount of CO2 emissions in the atmosphere at time t. We assume that S_t affects production directly through an externality: GHGs, which refers to Nordhaus's DICE and RICE treatments. Another externality, pollutant P_t, affects production indirectly through it's detrimental effect on labour's health H. Moreover, we assume the two externalities are caused by emission through the production of energy services in intermediate sectors.

Referring to Grossman (1972) and Cropper (1981), the health is assumed to depreciate over time and proportional to the pollution level (denoted $\tilde{P}_t$). The health-enhancing investment $\eta(1-u)H_t$ dampens or mitigates against this deterioration. Therefore, the equation of motion for aggregate health capital $H_t = h_t N_t$ is:

$$H_{t+1} = \eta(1-u)H_t + \left[\left(1 - D^h(P_t)\right) + n\right]H_t \tag{5.6}$$

or

$$h_{t+1} = \eta(1-u)h_t + \left(1 - D^h(P_t)\right)h_t \tag{5.7}$$

with $\eta > 0$ being the productivity scalar for health-enhancing activities. $D^h(P_t)$ is the damage function of pollutant on health.

The production of energy services includes both inputs and outputs. Let us define each element of energy $\underline{E_{0,t}}$ as $E_{0,i,t}$, which is self-produced by technology, capital, labor, and energy inputs. In addition, for some resources in finite supple, we define $R_{i,t}$ denote its stock at the beginning of period t, and define $E_{i,t}$ the total extracted amount at t. Thus, the decumulation dynamic of the exhaustible resource i is

$$R_{i,t+1} = R_{i,t} - E_{i,t} \geq 0 \tag{5.8}$$

We assume the technology for producing resource i is

$$E_{i,t} = F_{i,t}(K_{i,t}, H_{i,t}, \underline{E_{i,t}}, R_{i,t}) \geq 0 \tag{5.9}$$

where

$$H_{i,t} = h_t N_{i,t} \tag{5.10}$$

Raw labor is assumed to grow at an exogenous rate:

$$\frac{\dot{N}_{i,t}}{N_{i,t}} = n\,, \quad i = 0, 1, \ldots, I \tag{5.11}$$

Analogous to Golosove et.al 2014, we consider three sectors, i.e. (1) $i = 1, \ldots, I_g - 1$ "dirty" sectors which emits fossil carbon to the atmosphere. (2)'green" sectors with no environmental externalities, when $i = I_g, \ldots, I$. For simplicity, we assume each E_i for $i = 1, \ldots, I_g - 1$ produces 1 unit of carbon emission. In each time t, the physical capital and human capital are assumed to allocate freely across all sectors:

$$\sum_{i=0}^{I} K_{i,t} = K_t\,, \quad \sum_{i=0}^{I} H_{i,t} = H_t \quad \text{and} \quad E_{i,t} = \sum_{j=0}^{I} E_{j,i,t} \tag{5.12}$$

5.2.1 Specializing some assumptions

In the subsection followed, our optimal growth problem is characterized under three assumptions discussed in this subsection.

Preferences

In order to derive a closed-form solution we need to assume that an iso-elastic utility function. Let the instantaneous utility from goods consumption C be given by

Assumption 1:

$$U(C_t) = \frac{C_t^{1-\sigma} - 1}{1 - \sigma} \tag{5.13}$$

where the constant parameters $\sigma \in [0, \infty)$ represent Arrow-Pratt measure of relative risk aversion and the elasticity of marginal utility to consumption. Without loss of generality, we specify $\sigma = 1$ for simplicity. By virtue of L'hospital's rule, we have the logarithm preference

Assumption 1':

$$U(C) = \ln C \tag{5.14}$$

Production function

We consider that the production damages are caused by two externalities: GHGs and pollutant. In specific, the total amount of emissions in the atmosphere at time t, are composed into the cumulative emission of GHGs and the flow of pollutant, denoted by S_t and P_t respectively. Following Nordhaus, we assume that damage of S_t on production is multiplicative: through a direct damage function $\Omega^F(T_t)$. Moreover, pollutant P_t affects production indirectly through it's detrimental effect on labour's health $H_{0,t}(P_t)$. Therefore, the production function is defined as:

$$\textit{Assumption 4:} \quad F_{0,t}(K_{0,t}, u_t, H_{0,t}, \underline{E_{0,t}}, P_t, S_t) = \Omega^F(T_t)\tilde{F}_{0,t}(K_{0,t}, u_t H_{0,t}(P_t), \underline{E_{0,t}}) \tag{5.15}$$

Furthermore, $\tilde{F}_{0,t}(K_{0,t}, u_t H_{0,t}(P_t), \underline{E_{0,t}})$ is assumed to be a Cobb-Douglas form of its arguments:

$$\textit{Assumption 4.1:} \quad \tilde{F}_{0,t}(K_{0,t}, u_t H_{0,t}(P_t), \underline{E_{0,t}}) = AK_i^{\alpha_k}(uH_{0,t})^{\alpha_h}\tilde{H}^{\tilde{\alpha}_h}\left[\underline{E_{0,t}}\right]^{\alpha_e} \tag{5.16}$$
$$\alpha_k, \alpha_h, \tilde{\alpha}_h, \alpha_e \in [0,1], \quad \alpha_k + \alpha_h + \alpha_e = 1.$$

where total factor productivity A is assumed to be constant. Raw labor is assumed to be constant. u is a fraction of labour allocated to production of physical capital. We take into account that air pollution causing diseases (e.g. epidemics) are either contagious or affecting all the population. Thus, let us consider the wellbeing of individual positively related to the average health $\tilde{H}$. Referring to modelling the spillovers from average human capital à la Lucas (1988), we assume the average stock of health $\tilde{H}$ taken as exogenous. The social planner would consider $\tilde{H}$ as being endogenous, while by noting that, in equilibrium, $H = \tilde{H}$.

Emissions

Figure (5.1) shows that the combined emissions of greenhouse gases (GHGs), short-lived climate pollutants (SLCPs) and air pollutants have a range of impacts on climate change, human health and agriculture productivity.
For simplicity, we consider a general defined pollutants which includes both the air pollutants and the SLCPs. Thus, we define the aggregate emissions as follows.

$$E^M = E^M(GHGs, Po), \quad \text{where } Po \text{ stands for Pollutants} \tag{5.17}$$

We consider the GHGs emissions are composed by CO2 and other GHGs, denoted by $\tilde{GHGs}$. We also consider the Pollutants are composed by the intersection part of Pollutants and GHGs, denoted by PoG, together with other Pollutants $\tilde{Po}$.

$$GHGs = CO_2 + \tilde{GHGs} \tag{5.18}$$

$$Po = PoG + \tilde{Po} \tag{5.19}$$

Thus, the aggregate emission is composed by

$$E^M = GHGs + Po - PoG = CO_2 + \tilde{GHGs} + Po \tag{5.20}$$

Let us assume that we know the proportion of pollutants and GHGs in the aggregate emissions, denoted by θ^P and θ^G respectively. We also assume the weight of CO2 in the GHGs is known, denoted by θ^{CG}, i.e.

$$Po = \theta^P E^M \quad \Rightarrow \quad \frac{\partial Po}{\partial E^M} = \theta^P \tag{5.21}$$

$$GHGs = \theta^G E^M \quad \Rightarrow \quad \frac{\partial GHGs}{\partial E^M} = \theta^G \tag{5.22}$$

$$CO_2 = \theta^{CG} GHGs \quad \Rightarrow \quad \frac{\partial CO_2}{\partial GHGs} = \theta^{CG} \tag{5.23}$$

Therefore, we obtain

$$\frac{Po}{CO_2} = \theta^{PC} \quad \Rightarrow \quad \frac{\partial CO_2}{\partial GHGs} = \theta^{PC} \ , \quad \text{with } \theta^{PC} := \frac{\theta^P}{\theta^G \theta^{CG}} \tag{5.24}$$

The warming process

Global warming is approximately linearly proportional to cumulative CO2 emissions (Matthews et al., 2009; Allen et al., 2009; Zickfeld et al., 2009, 2013; Gillett et al., 2013; Collins et al., 2013). Thus, we can interpret it as

$$\textit{Assumption 2:} \qquad T_t = \zeta S_t \tag{5.25}$$

where ζ is a time-invariant parameter, which has been defined by IPCC as the Transient Climate Response to Cumulative Carbon Emissions (TCRE:

[3]Based on source: https://www.in-en.com/article/html/energy-2300089.shtml.

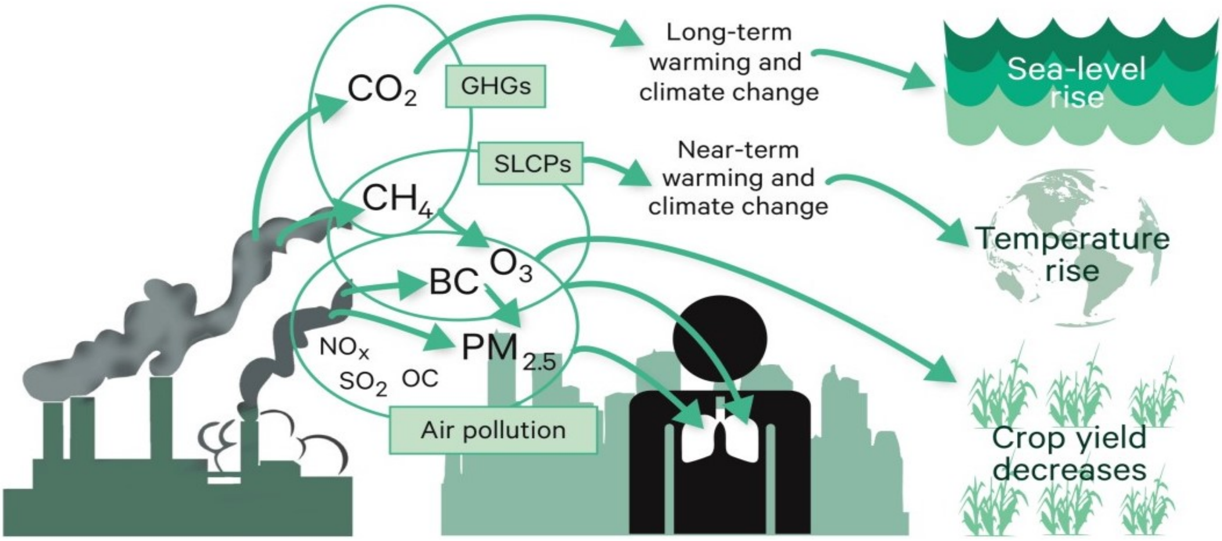


Figure 5.1 – Emissions from combustion include greenhouse gases (GHGs), short-lived climate pollutants (SLCPs) and air pollutants.[3]

Collins et al., 2013). T_t is the warming at time t, and S_t is current cumulative emissions of CO2, which depends on its historical anthropogenic emissions since pre-industrial time, defined as $t = 0$:

$$\textit{Assumption 3:} \qquad S_t\big(E_0^f, \ldots, E_t^f\big) = \sum_{s=0}^{t} E_s^f \,, \quad \forall s \geq 0\,. \tag{5.26}$$

with $E_s^f = \sum\limits_{i=1}^{I_g-1} E_{i,s}$ is instantaneous fossil emission at time $s = 1, \ldots, t$. We recall that instantaneous flow of emissions $E_{i,s}$ is measured of carbon emission in intermediate-good sector i.

Damages on production

We consider exponential-form damage function, which is widely used in Integrated Assessment Model(IAM) to calculate the closed-form formula of carbon pricing.(e.g. Golosov et.al 2014, Rezai and Van der Ploeg 2014, Traeger, 2015). The output scaling factor due to damages from climate change is defined as:

$$\textit{Assumption 5:} \qquad \Omega^F(T_t) = e^{-\gamma_t^F T_t}\,, \ \gamma_t^F \geq 0 \tag{5.27}$$

where T_t is the relative warming with respect to the pre-industry. The damage parameter $\{\gamma_t^F\}$ is stochastic with law of motion:

$$\gamma_t^F = \phi_f \gamma_{t-1}^F + \sigma_f \varepsilon_{f,t}\,, \qquad \varepsilon_{f,t} \overset{i.i.d.}{\sim} \mathcal{N}\big(0,1\big)\,, \quad t = 1, 2, \ldots \tag{5.28}$$

We firstly consider the case $\phi_f = 1$. This simplicity indicates the sequence of damage parameters $\{\gamma_t^F\,,\ t = 1, 2, \ldots\}$ is a discrete-time martingale, i.e. $\mathbb{E}_t\left(\gamma_{t+j}^F\right) = \gamma_t^F\,, \forall j > 0$.

Damages on health

We assume that health depreciates over time as a function of the air pollutions i.e.

$$\textit{Assumption 6:} \qquad D^H(P_t) = \delta_H(1 + \gamma_t^H P_t) \tag{5.29}$$

δ_H is the conventional health depreciation rate. The damage parameter to health γ_t^H is stochastic with law of motion:

$$\gamma_t^H = \phi_h \gamma_{t-1}^H + \sigma_h \varepsilon_{h,t}\,, \qquad \varepsilon_{h,t} \overset{i.i.d.}{\sim} \mathcal{N}\big(0,1\big)\,, \quad t = 1, 2, \ldots \tag{5.30}$$

We firstly consider the case $\phi_h = 1$. This simplicity indicates the sequence of damage parameters $\{\gamma_t^H\,,\ t = 1, 2, \ldots\}$ is a discrete-time martingale, i.e. $\mathbb{E}_t\left(\gamma_{t+j}^H\right) = \gamma_t^H\,, \forall j > 0$.

5.2.2 The planning problem

We now substitute our key assumptions in Section 1.2 into the general formulation in Section 1.1 and state the planning problem. Moreover, substituting Eq.(5.13) into (5.1) and combing Eqs.(5.5) (5.8) (5.9) (5.6), the policy maker solves the following problem:

$$\max_{C,u} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1 - \sigma} \right) \tag{5.31}$$

subject to

$$C_t + K_{t+1} = \Omega^F(T_t) A K_0^{\alpha_k} \Big(u_t H_{0,t}\Big)^{\alpha_h} \tilde{H}_{0,t}^{\tilde{\alpha}_h} \Big[\underline{E_{0,t}}\Big]^{\alpha_e} + (1-\delta_k) K_t\,, \tag{5.32}$$

$$E_{i,t} = A K_i^{\alpha_k} \Big(H_{i,t}\Big)^{\alpha_h} \tilde{H}_{i,t}^{\tilde{\alpha}_h} \Big[\underline{E_{i,t}}\Big]^{\alpha_e} \mathbb{1}_{\{R_{i,t}-E_{i,t}\geq 0\}}\,, \quad i = 1, \ldots, I \tag{5.33}$$

$$R_{i,t+1} = R_{i,t} - E_{i,t} \geq 0\,,\,, \quad i = 1, \ldots, I \tag{5.34}$$

$$h_{t+1} = \eta(1-u_t) h_t + \Big[1 - D^h(P_t)\Big] h_t\,, \tag{5.35}$$

$$K_i(0) = K_{i,0} \geq 0\,,\, E_i(0) = E_{i,0} \geq 0\,,\, N_i(0) = N_{i,0} \geq 0\,,\, i = 0, \ldots, I \text{ are given.}$$

$$H_i(0) = N_i(0)\, h_0 \geq 0\,,\, i = 0, \ldots, I\,, \text{ with } h(0) = h_0 \geq 0 \text{ given}\,,$$

$$A, \tilde{\alpha}_h > 0,\ 0 < \alpha_k, \alpha_h, \alpha_e < 1\,, \text{ and } \alpha_k + \alpha_h + \alpha_e = 1 \tag{5.36}$$

where the indicator function $\mathbb{1}_{\{R_{i,t}-E_{i,t}\geq 0\}}$ is defined as

$$\mathbb{1}_{\{R_{i,t}-E_{i,t}\geq 0\}} := \begin{cases} 1 & \text{if } R_{i,t} - E_{i,t} \geq 0, \\ 0 & \text{otherwise .} \end{cases}$$

together with our assumptions, defined in previous section, and non-negativity constraints. Since damages to the energy sectors are expected to be a small part of the overall economy, we do not include climate damages, nor pollution-induced health degradation in the energy sectors. This simplification seems quantitatively unimportant. In addition, this omission allows for comparison with Nordhaus and Golosove's treatment. Therefore, we have the simplified system of dynamics as follows:

$$C_t + K_{t+1} = \Omega^F(T_t) A K_0^{\alpha_k} \Big(u_t H_t\Big)^{\alpha_h} \tilde{H}_t^{\tilde{\alpha}_h} \Big(\underline{E_{0,t}}\Big)^{\alpha_e} + (1-\delta_k) K_t\,, \tag{5.37}$$

$$E_{i,t} = A K_i^{\alpha_k} N_i^{\alpha_h} \Big(\underline{E_{i,t}}\Big)^{\alpha_e} \mathbb{1}_{\{R_{i,t}-E_{i,t}\geq 0\}}\,, \quad i = 1, \ldots, I \tag{5.38}$$

$$R_{i,t+1} = R_{i,t} - E_{i,t} \geq 0\,,\,, \quad i = 1, \ldots, I \tag{5.39}$$

$$H_{t+1} = \eta(1-u_t) H_t + \Big[1 - D^h(P_t)\Big] H_t\,, \tag{5.40}$$

$$K_i(0) = K_{i,0} \geq 0\,,\, E_i(0) = E_{i,0} \geq 0\,,\, N_i(0) = N_{i,0} \geq 0\,, i = 0, \ldots, I\,, \text{ are given,} \tag{5.41}$$

$$H(0) = H_0 = N_0\, h_0 \geq 0 \text{ is given}\,.\ A, \tilde{\alpha}_h > 0,\ 0 < \alpha_k, \alpha_h, \alpha_e < 1\,, \text{ and } \alpha_k + \alpha_h + \alpha_e = 1$$

In each period, the production factors are allocated freely across sectors:

$$\sum_{i=0}^{I} K_{i,t} = K_t\,, \quad \sum_{i=0}^{I} N_{i,t} = N_t \quad \text{and} \quad E_{i,t} = \sum_{j=0}^{I} E_{j,i,t} \tag{5.42}$$

5.3 Competitive equilibrium

In this subsection, we characterize the solutions to the planner's problem in the setup described above. Given the system of dynamics defined in Eq.(5.31) to (5.36), we define the Lagrange $\mathcal{H}$ as follows:

$$\mathcal{H}(t; K_t, H_t, E_{i,t}; C_t, u_t; \lambda^F_{0,t}, \lambda^F_{i,t}, \lambda^H_t; \mu_t, \chi^e_{i,t}, \chi^k_t, \chi^h_t)$$

$$= \frac{C_t^{1-\sigma}}{1-\sigma}\beta^t + \sum_{s=t}^{\infty}\Big\{\lambda^F_{0,s}\Big[F_{0,s} - \delta_k K_{0,s} - C_s\Big]\Big\} + \sum_{s=t}^{t+1}\Big\{\lambda^H_s\Big[\eta(1-u_s)H_s$$

$$-D^h(P_s)H_s\Big]\Big\} + \sum_{i=1}^{I}\lambda^F_{i,t}\Big(F_{i,t} - E_{i,t}\Big) + \sum_{i=1}^{I}\mu^F_i\Big(-E_i\Big) + \sum_{i=1}^{I}\chi^e_{i,t}\Big(E_{i,t} - \sum_{j=0}^{I}E_{j,i,t}\Big)$$

$$+\chi^k_{i,t}\Big(K_t - \sum_{i=0}^{I}K_{i,t}\Big) \tag{5.43}$$

where $\lambda^F_{0,t}, \lambda^F_{i,t}$ are the co-state variables on the production constraint for sector $i \in \{0, \dots I\}$ at time t (w.r.t. Eqs.(5.32) and (5.33)). λ^H_t is the co-state variable on health sector (Eq.(5.35)). $\mu_{i,t}$ is the multiplier on the decumulation equation for exhaustible resource (Eq.(5.33)). Finally, $\chi^e_{i,t}, \chi^k_t$ are the co-state variables on the feasibility constraint for energy, physical capital of type i and human capital (i.e. Health). From now on, we omit the time subscripts t for control and state variables for convenience. The optimality conditions for control variable $\{C_t, u_t\}$ and the state variables $\{K_{0,t}, H_t\}$ are summarized as follows:

$$\text{FOC } C: \quad \frac{\partial \mathcal{H}}{\partial C} = 0 \quad \Rightarrow \quad C^{-\sigma}\beta^t = -\lambda^F_0. \tag{5.44}$$

$$\Rightarrow \text{Overlapping } C: \qquad \beta^j\left(\frac{C_t}{C_{t+j}}\right)^{\sigma} = \frac{\lambda^F_{0,t+j}}{\lambda^F_{0,t}}, \; j = 1, 2, \dots. \tag{5.45}$$

$$\Rightarrow \text{Dyanmic } C: \qquad \frac{\Delta C}{C} = -\frac{1}{\sigma}\left[\frac{\Delta\lambda^F}{\lambda^F} + \rho\right], \qquad \rho = 1/\beta - 1, \tag{5.46}$$

$$\text{FOC } u: \quad \frac{\partial \mathcal{H}}{\partial u_t} = 0 \quad \Rightarrow \qquad \lambda^F_0\alpha_h\frac{F_0}{u} = \lambda^H\eta H \tag{5.47}$$

$$\text{Co-state } \lambda^F_0: \; \lambda^F_{0,t+1} - \lambda^F_{0,t} = -\frac{\partial \mathcal{H}}{\partial K_{0,t}} \quad \Rightarrow \quad -\frac{\Delta\lambda^F_0}{\lambda^F_0} = \alpha_k\frac{F}{K_0} - \delta_k \tag{5.48}$$

$$\text{co-state } \lambda^H: \; \lambda^H_{t+1} - \lambda^H_t = -\frac{\partial \mathcal{H}}{\partial H_t} \quad \Rightarrow -\frac{\Delta\lambda^H}{\lambda^H} = \eta(1-u) - D^H(P) + \alpha_h\frac{F_0}{H}\frac{\lambda^F_0}{\lambda^H} \tag{5.49}$$

$$\text{Transversality condition:} \quad \lim_{t\to\infty}\lambda^F_{i,t}K_{i,t} = 0, \quad i = 0, 1, \dots. \quad \lim_{t\to\infty}\lambda^H_t H_t = 0 \tag{5.50}$$

5.3.1 The dynamics

We consider the intermediate sectors are produced by fossil energy which emits carbon to atmosphere. We focus on our competitive equilibrium around the balanced growth path (BGP) where the emissions approach zero. We omitted the climate and health damaged in the energy sectors, which only account a small part of the overall economy and is quantitatively negligible.

Balanced growth path

Let us calculate the growth rate of the physical capital and health, based on their dynamics in Eqs.(5.37) and (5.40) as follows:

$$g^K := \frac{\Delta K}{K} = \frac{F_0}{K} - \frac{C}{K} - \delta_k \,, \tag{5.51}$$

$$g^H := \frac{\Delta H}{H} = \eta(1-u) - D^H(P) \tag{5.52}$$

Moreover, the following *Corollaries* are remarkable for calculating the balanced growth path.

Corollary 2. *Given the Cobb-Douglas production function defined as $Y_t = K_t^\alpha L_t^\beta$, the discrete-time growth rate of the variables satisfies the following chain-rule:*

$$\frac{\Delta Y_t}{Y_t} = \alpha \frac{\Delta K_t}{K_t} + \beta \frac{\Delta L_t}{L_t} \tag{5.53}$$

$\Delta x = x_{t+1} - x_t$, $x = \{K, L, Y\}$ refers to the changes of x in sufficiently small time interval between $t+1$ and t. Moreover, Δx is supposed to be sufficiently small respect to x, i.e. $\Delta x/x \ll 1$.

Proof. Provided in the Appendix. □

Applying the discrete-time chain rule of the growth rate defined in *Corollary 1*, we obtain the growth rate of production in final sector, defined in *Assumption 4.1* as follows:

$$g^F := \frac{\Delta F}{F} = -\gamma^F g^T + \alpha_k g^K + (\alpha_h + \tilde{\alpha}_h) g^H + \alpha_e g^E \,,$$
$$\text{with } g^T := \Delta T/T \,, \text{ and } g^H := \Delta H/H = \Delta \tilde{H}/\tilde{H} \,. \tag{5.54}$$

where $\Delta H/H = \Delta\tilde{H}/\tilde{H}$ holds in equilibrium. Furthermore, we consider the carbon neutrality in the equilibrium, where the carbon emissions is offset by its removal to achieve net-zero carbon emissions. This implies $g^E = 0$. Thus, the growth in the long-run is

$$g = \frac{\alpha_h + \tilde{\alpha}_h}{1 - \alpha_k}\, g^H\,, \quad 1 - \alpha_k > 0 \tag{5.55}$$

In addition, we apply the discrete-time chain rule of growth rate defined in *Corollary 1* into Eq.(5.47), we have:

$$\frac{\Delta\lambda_0^F}{\lambda_0^F} + \frac{\Delta F_0}{F_0} - \frac{\Delta u}{u} = \frac{\Delta\lambda^H}{\lambda^H} + \frac{\Delta H}{H} \tag{5.56}$$

Let us substitute Eqs.(5.44), (5.49) and (5.47) into Eq.(5.56) for $\Delta\lambda_0^F/\lambda_0^F$, $\Delta\lambda^H/\lambda^H$ respectively:

$$-\rho - \sigma g^C + g^F - g^u = -\eta + D^H(P) + g^H \tag{5.57}$$

Let us focus on the steady-state, where the fraction of human capital allocated to the production of final sector is constant, i.e. $\Delta u = 0\,,\ g^u = 0$. Moreover, let us focus on the balanced growth path, where the growth rates of economy variables are identical, i.e. $g^F = g^K = g^C := g$.

$$\eta + (1-\sigma)g = \rho + g^H + D^H(P) \tag{5.58}$$

Optimal health investment

We calculate the steady state u^* in the following *Lemma 1*.

Lemma 3. *Suppose Assumptions 1-6 are satisfied, and the solution to social planner's problem implies that C_t/Y_t is constant in all states and at all times. Then the optimal fraction of human capital allocated to the production of final sector u^* is constant and a function of model's parameter:*

$$u^* = \frac{\rho + (\sigma - 1)g}{\eta} \geq 0 \tag{5.59}$$

where growth rate of economy g is a function of model's parameters and constant over time. Specifically, we have the following properties:

(i) For logarithm preference (i.e. $\sigma = 1$), $u^* = \rho/\eta$.

(ii) u^* is an increasing (or decreasing) function of the growth rate g, if the constant of relative risk aversion $\sigma > 1$ (or $0 < \sigma < 1$).

Proof. Provided in the Appendix. □

Optimal growth rate

Substituting Eqs.(5.55)into Eq.(5.58), we obtain the steady-state growth rate of the economy and the health, given by the following *Lemma 4*.

Lemma 4. *Suppose assumptions 1-6 are satisfied, and the solution to social planner's problem implies that C_t/Y_t is constant in all states and at all times, the steady-state growth rates of the production g and health g^H are constant over time, and functions of model's parameters given by*

$$g = \frac{(\alpha_h + \tilde{\alpha}_h)(\eta - \rho - \delta_h)}{\alpha_h} \tag{5.60}$$

$$g^H = \frac{(1 - \alpha_k)(\eta - \rho - \delta_h)}{\alpha_h} \tag{5.61}$$

with $\eta - \rho - \delta_h > 0$.

Proof. Provided in Appendix. □

Optimal carbon taxation

Lemma 5. *Suppose assumptions 1-6 are satisfied, and the solution to social planner's problem implies that C_t/Y_t is constant in all states and at all times. We obtain optimal Carbon tax Λ_t^C and Pollution tax Λ_t^P as follows*

$$\begin{aligned}
\Lambda^C/F_{0,t} &= \theta^C \zeta(-\gamma_t^F) \cdot \sum_{j=0}^{\infty} \big(\beta\big)^j \\
&= -\gamma_t^F \zeta \frac{1}{1-\beta} \theta^C, \qquad \textit{with } 0 < \beta < 1 \qquad (5.62) \\
\Lambda^P/F_{0,t} &= \frac{\alpha_h}{\eta} \frac{1}{u^*} \delta_H(-\gamma_t^H)\theta^P \\
&= -\gamma_t^H \frac{\alpha_h}{\rho} \delta_H \theta^P \qquad (5.63)
\end{aligned}$$

with $\mathbb{E}_t\big(\gamma_{t+j}^\kappa\big) = \gamma_t^\kappa$, $\kappa = \{F, H\}$, $j = 1, \dots ..$ Moreover, the ratio between Carbon tax and Pollution tax is

$$\frac{\Lambda^P}{\Lambda^C} = \frac{\gamma_t^H \alpha_h \delta_H \big(1-\beta\big)}{\gamma_t^F \zeta \rho} \theta^{PC} \tag{5.64}$$

Eq(5.64) indicates that adjustment of Pollution tax results in proportional adjustment of Carbon tax at a constant rate of $\big[\gamma^H \alpha_h \delta_H \big(1-\beta\big)\theta^{PC}\big]/\gamma^F \zeta \rho$. Here we consider the damage function parameters are time-invariant,i.e. $\gamma_t^\kappa = \gamma^\kappa$, , $\kappa = \{F, H\}$.

5.3.2 Transitional dynamics

Existence of equilibrium

We consider 3 energy producing sectors: oil sector, coal sector and green sector, denoted by $i = 1, 2, 3$ respectively.

(I) $E_{1,t} = R_{1,t} - R_{1,t+1}$, with $i = 1$ stands for oil sector, including fossil fuels.
We consider the oil sector is in finite supply and very cheap to extract (the extraction cost is zero). Oil is cheaper to convert to energy but rely on exhaustible resources in limited supply.

(II) $E_{2,t} = A_{2,t}N_{2,t}$, with $i = 2$ stands for coal sector.
We consider the coal sector is in finite supply but not be used up. We also consider coal is expensive to convert to energy and reply on exhaustible resource in larger supply.

(III) $E_{3,t} = A_{3,t}N_{3,t}$, with $i = 3$ stands for green sector.
We consider the green sector is expensive to convert into energy. We assume the technology growth of the economy is from the green sector, i.e. $A_{0,t} = A_{3,t}$.

We consider the energy sector $E_t(E_{1,t}, E_{2,t}, E_{3,t})$ has the following Cobb-Douglas form:

$$E_t = E_{1,t}^{\alpha_1^e} E_{2,t}^{\alpha_2^e} E_{3,t}^{\alpha_3^e} \tag{5.65}$$

It is worth to remark that the Cobb-Douglas function is a special cases of the constant elasticity of substitution (CES) function in Golosov et.al (2014): $E_t = \left(\alpha_1^e E_{1,t}^\rho + \alpha_2^e E_{2,t}^\rho + \alpha_3^e E_{3,t}^\rho\right)^{1/\rho}$, when ρ approaches to zero.

Theorem 3. Suppose *Assumptions 1-6* are satisfied, and the solution to social planner's problem implies that C_t/Y_t is constant in all states and at all times.

Proof. Let us consider the planner's problem:

$$\max_C \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{5.66}$$

subject to the budget constraint

$$K_{t+1} - K_t = Y_t - C_t - \delta_k K_t \tag{5.67}$$

From the budget constraint, we have

$$C_t = Y_t - K_{t+1} + (1-\delta_k)K_t \quad \Rightarrow \quad \frac{\partial K_{t+1}}{\partial C_t} = -1 \tag{5.68}$$

Substituting the Eq.(5.68) into Eq.(5.66), we obtain

$$\begin{aligned} &\max_C \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\big[Y_t - K_{t+1} + (1-\delta_k)K_t\big] \\ &= \mathbb{E}_0\Big\{ \ldots + U\big[\underbrace{Y_t - K_{t+1} + (1-\delta_k)K_t}_{C_t}\big] + U\big[\underbrace{Y_{t+1} - K_{t+2} + (1-\delta_k)K_{t+1}}_{C_{t+1}}\big] + \ldots \Big\} \end{aligned} \tag{5.69}$$

Calculating the 1st order condition with respect to K_{t+1} implies the Euler equation as follows.

$$U'(C_t) = \beta\, \mathbb{E}_t \left\{ U'(C_{t+1}) \left(\frac{\partial Y_{t+1}}{\partial K_{t+1}} + 1 - \delta_k \right) \right\} \tag{5.70}$$

The Cobb-Douglas production function implies

$$Y_t = F_0(K_t, \ldots) = \Omega^F(T_t) A K_0^{\alpha_k} \Big(u_t H_{0,t} \Big)^{\alpha_h} \tilde{H}_{0,t}^{\tilde{\alpha}_h} \Big[\underline{E_{0,t}} \Big]^{\alpha_e} \quad \Rightarrow \quad \frac{\partial Y_t}{\partial K_t} = \alpha_k \frac{Y_t}{K_t} \tag{5.71}$$

The preference function implies

$$U(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} \quad \Rightarrow \quad U'(C_t) = C^{-\sigma} \tag{5.72}$$

Substituting Eqs.(5.71) and(5.72) into Eq.(5.70), we obtain

$$\frac{Y_t - C_t - (1-\delta_k)K_t}{C_t^{\sigma}} = \beta \alpha_k\, \mathbb{E}_t \left\{ \frac{Y_{t+1} + \alpha_k^{-1}(1-\delta_k)K_{t+1}}{C_{t+1}^{\sigma}} \right\} \tag{5.73}$$

For simplicity, we further assume

- the logarithm utility $U(C_t) = \ln C_t$, implying $\sigma = 1$
- full depreciation of capital $\delta_k = 1$

The Eq.(5.73) is simplified as

$$\frac{Y_t}{C_t} - 1 = \beta\alpha_k \, \mathbb{E}_t\left(\frac{Y_{t+1}}{C_{t+1}}\right) \tag{5.74}$$

To calculate the steady state of Y_t/C_t, We will use 'guess and verify' method to solve this equation. Suppose that the consumption rate is constant in steady state, defined as $\mu_{y/c}$ (or $\mu_{c/y} = 1/\mu_{y/c}$).

$$\mathbb{E}_t\left(\frac{C_{t+1}}{Y_{t+1}}\right) = \frac{C_t}{Y_t} := \mu_{c/y} \qquad \text{or} \qquad \mathbb{E}_t\left(\frac{Y_{t+1}}{C_{t+1}}\right) = \frac{Y_t}{C_t} := \mu_{y/c}\,, \quad \forall t \geq 0 \tag{5.75}$$

Substituting Eq.(5.75) into Eq. (5.74), we obtain

$$\mu_{y/c} = 1 - \beta\alpha_k \tag{5.76}$$

$\square$

5.4 Conclusion

This essay is to study the production-induced environment degradation and its detrimental impact on long-term growth. Specifically, our model enables policy maker to maximize the society well-being while optimally balance between emissions, health and growth in an uncertainty environment. We present an analytical DSGE model with two externalities: CO2 emissions and pollutant. This model can be seen as a generalization of Golosov et al. 2014. Yet in their model, the pollutant and health degradation is not considered. We contribute to the literature by introducing a two dimensional stochastic endogenous growth model. Our central result are three closed-form formulas for optimal health investment, growth rate and carbon tax. The impact of parameters on economic variables are discussed. I n particular, our result indicates that adjustment of Pollution tax results in proportional adjustment of Carbon tax at a constant rate.

Appendix

2.4 Competitive Equilibrium

Proof of Lemma 1

Proof. We re-arrange the LHS and RHS of Eq.(5.52) to obtain

$$g^H + D^H(P) = \eta(1-u) \tag{5.77}$$

Substituting Eq.(5.77) into Eq.(5.58), we obtain: $u^* = \dfrac{\rho + (\sigma-1)g}{\eta} \geq 0$. □

Proof of Corollary 1

Proof.

$$\begin{aligned} Y_t = K_t^\alpha L_t^\beta\,,\ \forall t \geq 0 \quad &\Rightarrow \ln Y_t = \alpha \ln K_t + \beta \ln L_t \text{ and } \ln Y_{t+1} = \alpha \ln K_{t+1} + \beta \ln L_{t+1} \\ &\Rightarrow \Delta \ln Y_t = \alpha \Delta \ln K_t + \beta \Delta \ln L_t \end{aligned} \tag{5.78}$$

It is worth nothing that the natural logarithm has Maclaurin series as follows:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \tag{5.79}$$

When $|x| < 1$, the series are convergence, and $-\ln(1-x) := x + o(x)$, with $o(x)$ stands for an infinitesimal of higher order than x, i.e. $\lim\limits_{x\to 0} \frac{o(x)}{x} = 0$. Thus, the linearization i.e. $\ln(1+x) \doteq x$ will be a very good approximation as long as x is approximately zero or $|x| \ll 1$. This is satisfied, as the growth rate of the variables in our model are at the level of 10^{-2} (i.e. at most 2%).

$$\Delta \ln Y_t = \ln Y_{t+1} - \ln Y_t = \ln\left(1 + \frac{Y_{t+1} - Y_t}{Y_t}\right) \doteq \frac{Y_{t+1} - Y_t}{Y_t} = \frac{\Delta Y_t}{Y_t} \tag{5.80}$$

Analogously, we have the following linear approximation with respect to the growth rate of K, L:

$$\Delta \ln K_t = \Delta K_t / K_t\,, \text{ and } \Delta \ln L_t = \Delta L_t / L_t \tag{5.81}$$

Substituting Eqs.(5.80) and (5.80) into (5.78), we have discrete product rule of growth rate as follows:

$$\frac{\Delta Y_t}{Y_t} = \alpha \frac{\Delta K_t}{K_t} + \beta \frac{\Delta L_t}{L_t} \tag{5.82}$$

$\square$

6 Uncertainty and heterogeneity in market competition

6.1 Introduction

The extant literature has shown how the existence and relevance of switching cost makes investing in future market share relevant. If consumers do not switch, then setting low price today helps firms to increase their future market share and its future profits (Klemperer, 1985). However, this mechanism can be affected by the life stage of the market, i.e. its relative stability or potential growth. If future market shares are valuable (as in Klemperer), firms may adopt a different pricing policy according to the condition of the market, i.e. growing and mature market. Investing in market share may be a rational strategy in growing markets, in expectation of later larger profits. But when the market potential is limited, as in mature markets, investing in future market shares can be less relevant, as the consumer lock-in can be limited. Therefore, the value of investing in market share may be affected by market potential, and by product and country.

There is a growing consensus that firms' corporate governance influences their ability to export and internationalize. Prior research has emphasized both positive and negative aspects of the influence of family ownership on the ability of the company to expand abroad (Banalieva and Eddleston, 2011; Arregle et al., 2012; Graves and Shan, 2014; Hennart et al., 2019; Kano and Verbeke, 2018; Stadler et al., 2018). Scholars have argued that the successful experience of family firms in international markets can be related to their ability to develop resources and attitudinal commitment toward international involvement (Shi et al., 2019), to a more flexible and fast management teams (Gallo and Pont, 1996; Tsand, 2001; Fernàndez and Nieto, 2006; Kontinen and Ojala, 2015), or even to their long-termism, that is, their ability to internalize the long-run benefits of expanding abroad (The Economist: 2012, 2013, 2015). However, this positive view is not universally shared. Family businesses have often been described as reluctant to abandon their initial geographical niche and prefer to stay culturally "closer to home" so as to minimize risks (Gomez-Mejia, 2010: 244), or reduce the geographical scope of activity and choose what Rugman and Verbeke (2008) describe as a "home-region focus" (Banalieva and Eddleston, 2011; Baschieri et al., 2017; Zahra, 2003).

Hennart et al. (2019) show that family-managed SMEs selling high-quality products in global niches are able to overcome internationalization limits. The global niche business model leverages the strengths of family governance and minimizes family firms' limitations on recruiting specialized managers. It also relaxes constraints on financial resource requirement which can imply to dilute control to obtain the necessary finance to expand abroad. Eddleston et al. (2019) add a counterpoint by showing that a high-quality niche strategy only conditionally supports family firms' internationalization, as it depends on external and internal contexts, specified as country of origin pro-market development and professionalization practices respectively (Eddleston et al., 2019).[1] Their research papers stress the role of heterogeneity in the competitive framework as a driver for the success of family firms to expand abroad. The heterogeneity can be related to some structural feature of the destination market, i.e. the existence of market segments (mass vs. niche markets) as in the Hennart et al. (2019), or to other exogenous contexts, either external and internal, like the perception of home country by consumers and professionalization practices (Eddleston et al., 2019).

We contribute to this debate by showing that another source of heterogeneity in export markets, specifically the life-cycle of the market, can influence the entry decisions even in mass product markets, where the pro-competitive features might play a minor role. We sketch a simple model showing that the entry of family firms in a new export market depends on the market perspective growth, with family firms more likely to enter into markets at their growth stage. In this simple framework, which is well suited for mass products where competition is mainly driven by a price setting behaviour, high quality market niches turn out as a specific case of inelastic demand. The paper extends Hennart et al.'s (2019) research by showing that family firms are likely to sell even mass-market

[1] Eddleston et al. (2019) show that a global high-quality niche approach only works when family firms are from countries with strong pro-market development, as consumers' perceptions of a country influence their comparative evaluation of products from different countries (Klein, 2002; Maheswaran and Chen, 2009).

products abroad, in addition to niche products, but only when market potential growth is significant. The rationale behind this intuition is that pricing conditions prevailing before entry can be relaxed by potential market growth, thus making entry profitable even for cost-inefficient firms. If family firms are less efficient than other firms, which could be a direct effect of late entry, future market potential emerges as a remarkable driver for family firms to enter a new export market even in mass products. In addition, by helping entry, potential market growth comes out as a substitute for the (favourable) perception of customers, i.e. the pro-market development external context (Eddleston et al., 2019), when coupled with a replication strategy.

6.1.1 Theoretical background

When family firms decide to enter foreign markets, they show a peculiar behaviour compared to non-family business (Pukall and Calabrò , 2013; Claver et al., 2007; Graves and Thomas, 2006). Family firms are usually indicated as late starters in the internationalization process and generally lightly committed to international markets because mainly domestic-oriented and run by managers with no or scarce international experience (Claver et al., 2007). Their internationalization process is often biased towards countries with less psychological distance and to proceed slowly in order to accumulate knowledge and experience on foreign markets (Pedersen and Sharer, 2011). Limitations also arise from the relevance of risk avoidance in decision making, that worsen the contribution of unfamiliarity with the international scene, unavailability of relevant information, complexity resulting from the high number of alternatives and irreversibility of the investment (Cumberland 2006). Besides, the foreign expansion decision is difficult to change ex-post without considerable loss of time and money (Choo and Mazzarol, 2001). The relevance and complexity of this decision derives not only from the associated risk, but also from the number of factors that managers may have to evaluate, such as the number of foreign markets to enter or the economic distance between foreign and home markets (Reid, 1981). Export propensity is also lower if executives have no previous foreign experience and when strategic decision making is not decentralised. On the positive side, research suggests that family firms that invest in entrepreneurship have greater potential for good performance, stimulate corporate entrepreneurship and renewal, especially in later generation family firms (Cucculelli et al., 2016).

However, it is still not fully clear which features of the family governance mainly affect their internationalization process. As regard the antecedents of internationalization, in particular, scholars agree that family involvement in the firm and, more precisely, in the ownership influences the foreign expansion, but have not a common view whether this effect is positive or negative. Building on a sample of 409 U.S. manufacturing firms, Zahra (2003) demonstrates a positive influence of the family ownership on the scale and scope of international trade. Moreover, the relation is strengthened when family members actively participate in the company management. By contrast, Fernàndez and Nieto (2005) found evidence in the opposite direction. Barba Navaretti et al. (2012) also show that the presence of family members in executive management drives the negative relationship between family ownership and export performance. However, once this factor is controlled for, family ownership per se is no longer significant. Finally, in the attempt to find a plausible explanation for prior contrasting results, Sciascia et al. (2012) propose a non-linear relationship between family ownership and international entrepreneurship: they provide evidence for a U-shape relationship according to export intensity which is supposed to peak when ownership stays at moderate levels.

6.1.2 Core tenets of the model

Despite the several studies in the literature that have addressed the issue, to date no study has attempted to examine systematically the extent to which the life-cycle of the destination market, i.e. an external influence to the corporate decision (Boehe, 2016), plays a role in the international involvement of the company. This is an important gap in the literature, as the match between the resources of the firm and the requirements of the destination market may depend substantially on the life-cycle stage of the foreign market. Indeed, the current theory has highlighted differences in the behaviour of firms across phases of the industry life-cycle (Klepper, 1996, 1997), but without explicitly considering the influence of the ownership.

In mature markets, competition can be tougher and mark-ups lower, and the competitive environment induces a strict selection on productivity for newcomers (Mayer et al., 2013). This may end up as a major limitation for family firms if they are less efficient than non-family firms (Pérez-González, 2006; Bloom and van Reenen, 2006; Bertrand and Schoar, 2007), or just late entrants. Moreover, despite their long-term orientation (Randerson et al., 2015), family firms may be less prone to undertake entry initiatives in mature markets with significant financial and managerial requirements, or substantial risk of exiting by shakeout. By contrast, competition in growing markets may be less harsh and markups higher, thus making the productivity constraint less binding. Opportunities may be more frequent and easily seized by the family firms' ability to recognize and exploit them; small family boards and flexible management teams may turn out to be an advantage in terms of alertness and quick reaction to market opportunities (Kontinen and Ojala, 2010; Gallo and Pont, 1996; Tsang, 2001). This could make growing markets very attractive for family firms and a viable solution for their internationalisation process.

To date, very few studies have dealt with this issue or provided significant empirical evidence on it. We explore this gap and investigate whether growing export markets may emerge as a viable solution for family firms when individual cost conditions or the timing of entry can make difficult for these firms to enter new export markets.

On the cost side, the preference for growing markets may come at a cost if these markets are more information-demanding about their future evolution, which implies significant sunk costs of entry. In this framework, firms wishing to enter international markets may bear additional information costs, that could be larger the newer the destination market. Therefore, we explore and empirically test the intuition by Winter and Szulanski (2001) who showed that the adoption of a replication strategy, i.e. a strategy that allows the company to replicate abroad the market model followed in domestic markets , helps firms to control their cost disadvantage. Unfortunately, in addition to the scarcity of research that explicitly addresses the replication approach in international markets (Jonsson and Foss, 2011), there is an almost complete dearth of research studies on the role of family ownership behind the replication strategy. We contribute to this literature by showing that a replication strategy can offset some negative features of family governance that hinder their ability to expand abroad, as the weak managerial practices pointed out by Eddleston et al. 2019.

6.1.3 Analysis

To test these intuitions, we developed an index of market potential growth that summarizes the market stage of development and its future perspectives. The index – developed purposely for this test - is computed as the ratio between the actual market size and its saturation level, i.e. the estimated maximum level the market will reach in the future. Saturation levels have been computed by fitting a logistic equation for all the pairs exporter-destination market of first-time exporters in 2008. As a proxy for future business opportunities, the index provides a snapshot of the market potential expansion estimated from historical data. We use the index to identify "where" (family firms' preference for growing markets) and "how" (a replication vs. a new-product approach) family firms internationalize. The index is explicitly included as an explanatory variable in a model where the entry decision depends on the measure of potential market growth (k).

To identify the influence of potential market growth on the decision to enter a new market, we matched individual entry decisions by first time exporters with exogenous characteristics of foreign markets, as summarized by the index of market development. The index has been computed using trade data from 1998 up to 2008, and provides an estimate of the potential market growth from 2008 onwards (3 to 10 years). Therefore, it is the best market forecast available in 2008 to the group of first-time exporters, i.e. companies that decided to enter a new market in 2008.[2] This permits to evaluate the decision to go abroad in 2008 as only dependent on the information set related to the specific market when the decision has been made, without any other confounding factors from previous foreign activities. Being exogenous to individual firm's decisions, future market perspectives can be used to identify how ownership correlates to structural market characteristics. In particular, it permits to test if and to what extent the future market potential affects firm decisions differently in family and non-family firms.

As in Hennart et al. (2019), we investigate the behaviour of firms entering international markets using a large representative sample of European companies (EFIGE dataset, see Altomonte and Aquilante, 2012) which includes

[2]The decision to use data up to 2008 was due to simulate the post crisis evolution of destination markets not using data already affected by the structural break experienced by the global economy in the years of the great financial crisis (2008-2010).

financial and survey data from a sample of 14,759 European firms. From the full EFIGE dataset, we selected the sub-sample of firms that entered a new foreign market in 2008. This group, labelled as first-time exporters, was the focus of our analysis. Following Pedersen and Sharer (2011) we hypothesize that international expansion will be manifested by an initial long time period before a firm first expands overseas, followed by an initial "big step". In our analysis, the "big step" is the first entry in a foreign market. This entry decision is a major step in the strategic profile of a company, as it forces firms to develop processes and routines for managing foreign operations in the future, according to the economic environment existing in the selected destination country and the perspective evolution of the sectoral demand.

6.1.4 Core findings

The main findings of the study confirm the peculiar approach of family firms to international markets. Firstly, family firms and family CEOs do have a preference for growing markets: family ownership positively affects the probability that a firm enters foreign markets in their early stage of growth, instead of highly competitive, low profitability markets. Secondly, this preference for growing markets is associated to the adoption of a replication strategy that consists of selling abroad the product which is the most competitive in the domestic market. This evidence is consistent with an approach to international markets that exploits foreign opportunities while keeping the venturing risk well below the safety threshold mandated by the family ownership.

Empirical results provide further implications for research and practice. As for the research, the paper extends the literature on the internationalisation of family firms by stressing the connection between the firm's preference for growing markets and its ownership structure. Following Hennart et al. (2019), the paper shows that family firms are able to enter an export destination market when the market has a significant potential of growth, no matter if the firm is selling niche or mass products. It also highlights the value of a replication strategy for firms whose decision-making process is characterized by flexibility and long term orientation. Provided sufficient conditions on the side of potential market growth, even cost-inefficient firms can expand abroad by using a replication strategy which overcomes the limitations of internal context factors related to the managerial structure

(Eddleston et al., 2019). Finally, the paper provides empirical evidence based on a large data set of European family and non-family firms that extends the literature on replication which has been exclusively based on case studies (Winter and Szulanski, 2001; Rui et al., 2016; Jonsson and Foss, 2011) or pure theoretical approach (D'Adderio, 2016).

As for the practice, the paper recommends family firm owners to look closely to (fast) growing markets as elective markets to explore and penetrate. The increasing presence of small and medium size family firms in large and fast-growing markets like BRICS (The Economist, 2015) provides an excellent ex-post validation of this strategic fit. Moreover, family firms have the option to adopt a replication strategy to enter international markets in order to offset the scarce managerial resources that often characterise the family boards.

6.2 Heterogeneity in the life-cycle of destination market

6.2.1 Family firms' preference for growing markets

From an international business perspective, firm's preferences about alternative foreign markets may be affected by their stage of development because of the existence of different market-specific competitive conditions. In mature markets, competition is tougher and markups lower. Tougher competition induces exporters to reduce the set of exported products to only those whose efficiency levels are above the average productivity threshold set by incumbents (Mayer et al., 2013). This may result in a major limitation for family firms, at least for those firms that do not position themselves in the top deciles of the productivity distribution (Barba Navaretti et al., 2008; Pérez-González, 2006; Bloom and van Reenen, 2006; Bertrand and Schoar, 2007). Furthermore, from a resource dependent perspective, mature and competitive markets may require specific managerial abilities that, when scarce or absent, may negatively affect the probability to enter mature and more competitive markets (Naldi and Nordquist, 2008).

Conversely, in growing markets competition may be less severe and markups higher, thus making the productivity constraint less binding. Growing markets may provide more opportunities, as demand is expanding and consumer preferences are not already locked to specific characteristics of the product. In these contexts, family firms' ability to recognize and exploit opportunities may end up as a definite advantage, especially when these are seized by small and flexible family boards.

The literature on family firms and internationalisation provides further reasons that make family firms less suitable to engage competition in mature markets. As these markets imply tough competition, low margins and large sunk investments, they may generate classical hold up problems because of the risk profile of asset specificity (Coase, 1937; Grossman and Hart, 1986; Hart and Moore, 1990; Klein and Leffler, 1981; Williamson, 1979). This peculiar feature of mature markets makes the market-idiosyncratic risk a key variable for understanding the decision of family firms to enter foreign markets.

Compared to other owners' categories, family owners present peculiar family-related priorities and risk preferences which may bias strategic decisions for internationalization towards a low-risk approach (Miller et al., 2010, Gomez-Mejia et al., 2010). Their decisions, besides performance considerations, may be driven by the importance of preserving the family's socio-economic endowment (Gomez-Mejia et al., 2007; 2011) and see potential gains or losses in this endowment as their primary frame of reference in making strategic decisions (Chrisman et al., 2015). Internationalization can be perceived as a threat for socio-emotional wealth preservation, as it may require more external funding and managerial talents which dilutes family holdings and transfer control and decision-making power to other actors (Gomez-Mejia et al., 2011). Besides, the internationalization process could stimulate the firm to change its objectives, culture and structure, which are supposed to be unwelcome outcomes for the family (Gallo and Sveen, 1991). Family firms may deliberately limit their growth objectives (Donckels and Fröhlich, 1991; Claver, Rienda, and Quer, 2008), and become unwilling to borrow from external sources to facilitate international expansion (Graves and Thomas, 2006). Also, due to the founder's personal involvement, they may be unwilling to change strategy (Goodstein and Boeker, 1991; Brunninge et al., 2007) and to stick on niche strategies related to flexibility or product differentiation.

On a different perspective, family firms have also many peculiar traits that make them particularly suited to engage competition in fast-growing, early-stage foreign markets, where hold-up can be a minor issue. Firstly, the decision to internationalize can be considered as a choice towards a long-term investment, an activity which should be better suited to the long term horizon of family owners. Agency theory stresses that the extent of involvement in risky activities is likely to be influenced by the ownership of the firm (Fama, 1980). Thomsen and Pedersen (2000) stress that even profit-maximizing owners may disagree about corporate strategy because of different preferences regarding risk and the time profile of expected cash flow. Therefore, the decision to internationalise may result from a balanced combination between risky ventures and different risk attitude of the decision-maker. Seizing market opportunities in foreign markets typically involves risk-taking, and requires a particular long term disposition of the decision makers. Whereas defining actions aimed at reducing inefficiency and costs (i.e. cost restructuring) is primarily a matter of managerial skills, entering a new market requires entrepreneurship and the aptitude to bear the risk of the venture in the long term. "Discovering an area of a firm's comparative advantage calls for much more innovation and involves much greater uncertainty than eliminating inefficiencies, especially when the firm faces new environments in highly competitive export markets" (Frydman et al., 1998; Grosfeld and Roland, 1996).

Although family-owners may be characterized by a general sense of cautiousness regarding the entry process in foreign markets (Kontinen and Ojala, 2010a, 2010b), their long term orientation can have beneficial effects on the adoption of a strategy based on gaining market shares. The long-term nature of family ownership allows families to invest in innovation and risk-taking, thereby empowering them with the potential and ability to innovate and pursue entrepreneurial activities (Salvato, 2004). A large and recent family business literature has stressed the role of family entrepreneurial orientation as a major determinant of the firm's long term performance (Sharma, Chrisman and Chua 1997; Kellermans et al., 2008; Zellweger and Sieger, 2012; Casillas et al., 2010; Casillas and Moreno, 2010), together with the owner's inclination toward growth (Kirchoff, 1994; Baum and Locke, 2004; Avlonitis and Salavou, 2007).

In this framework, the entrepreneurial orientation of family firms may lead to a more sustained internationalization process, especially in the case of multigenerational family firms where the need to find employment opportunities for new generations may be a leading motivation for international expansion. Nordqvist et al. (2008), Lumpkin et al. (2010) and Zellweger et al. (2012) suggested that when family firms combine the attributes of family and business in their process of long-term value creation, the family entrepreneurial factors may positively affect the process of international expansion by shaping the way in which market opportunities are identified, defined, and exploited. All these subjective factors can help to explain the process of market selection by family firms, i.e. "where" family firms want to enter in the process of market selection.

Finally, if new and growing markets are endowed with business opportunities that must be quickly identified and exploited, family firms may have inherent comparative advantage. Kontinen and Ojala (2010) find that due to the flexibility of the management team, family firms were able to react quickly to new international opportunities. In their interviews, most of the entrepreneurs recognized that the flexibility and small management teams of family firms enabled them to be alert and reactive to international opportunities. In this sense, small management teams provide a distinct advantage in relation to the alertness of family SMEs: they allow decision processes to be quick and flexible (Gallo and Pont, 1996; Tsang, 2001). Hence, they can proactively seize emerging opportunities, which may be particularly present in growing markets.

To sum up, family firms' specific traits and the competitive characteristics of foreign destination markets make family firms more likely to engage competition in new and growing markets. We sketch a simple theoretical model of firm entry in the next section.

6.3 The general model

In this section, we generalize a two-period switching costs model à la Klemperer (1995) to the scenario where the 2nd period profit is not perfectly predictable. A representative firm's total profit is defined as a value function, taken into consideration of the expected 2nd period profit as follows

$$V^{F_j}(p_1, p_2) = \pi_1^{F_j}(p_1) + \delta\, \mathbb{E}_1\Big[\pi_2^{F_j}\big(\sigma_1^{F_j}(p_1); p_2\big)\Big] \tag{6.1}$$

$$p_i = p_i\big(\sigma_i^{F_j}\big)\,, \quad p_i(\cdot) \text{ is a decreasing function, } i = 1, 2. \tag{6.2}$$

$$\Big(\text{or } \sigma_i^{F_j} = \sigma_i^{F_j}\big(p_i\big)\,, \quad \sigma_i(\cdot) \text{ is a decreasing function, } i = 1, 2.\Big)$$

where V^{F_j} is total discounted value of a firm F_j, $j = 1, \ldots, J$. π_1^F is the firm j's current (or first-period) profit. $\mathbb{E}_1\Big(\pi_2^{F_j}\Big)$ is the expected future profit estimated at period 1, where $\pi_2^{F_j}$ is future (or second-period) profit which is uncertainty. $\sigma_1^{F_j}$ is the firm j's current market occupation, depending on its current price. If future market shares are valuable (because of the lock-in effect on consumers), competition for market share is expected to be fierce in the current period. As a consequence, firms aim to maximize their expected total discounted overall profits, which also depends their forecast on the future market occupations. δ is the time discounting coefficient.

It is worth noting that the monotonic (decreasing) relationship between p_i and $\sigma_i^{F_j}$ in Eq.(6.2) indicates the two functions are invertible. Therefore, it is substantially the same to replace the argument of p_i with $\sigma_i^{F_j}$ in the value function, and the Eq.(6.1) can be rephrased as follows

$$\begin{aligned} &V^{F_j}\left(\sigma_1^{F_j}, \sigma_2^{F_j}\right) = \pi_1^{F_j}\left(\sigma_1^{F_j}\right) + \delta\, \mathbb{E}_1\left[\pi_2^{F_j}\left(\sigma_1^{F_j}; \sigma_2^{F_j}\right)\right] \\ &\sigma_i = \sigma_i\big(p_i\big)\,, \quad \sigma_i(\cdot) \text{ is a decreasing function, } i = 1, 2. \end{aligned} \tag{6.3}$$

The uncertainty of the future profits $\pi_2^{F_j}(\sigma_1^{F_j}, \sigma_2^{F_j})$ comes from the unobservable future market occupation $\sigma_2^{F_j}$. Thus, the expected future profits $\mathbb{E}_1\left(\pi_2^{F_j}\right)$ is determined by the expected future market $\mathbb{E}_1\left(\sigma_2^{F_j}\right)$, depending on the potential market growth. In specific, $\mathbb{E}_1\left(\pi_2^{F_j}\right)$ is calculated as a decreasing function of its future price P_2, conditional on the observed information filter $\mathcal{F}_1$, i.e. $\mathbb{E}_1\left(\sigma_2^{F_j}\right) = \mathbb{E}_1\left(\sigma_2^{F_j} \middle| \mathcal{F}_1\right) := \hat{\sigma}_2^{F_j}\left(p_2 \middle| \mathcal{F}_1\right)$. It is worth noting that first equality holds as the probability distribution of a random variable is independent form the deterministic observations. Moreover, let us specify the filter as a market development index k, and we have the following form:

Assumption 1.1
$$\mathbb{E}_1\left(\sigma_2^{F_j}\right) = \mathbb{E}_1\left(\sigma_2^{F_j} \middle| k\right) = \hat{\sigma}_2^{F_j}(p_2, k) \tag{6.4}$$

Analogously, $\mathbb{E}_1\left(\pi_2^{F_j}\right)$ also depends on the potential market growth, i.e.

$$\mathbb{E}_1\left[\pi_2^{F_j}\left(\sigma_1^{F_j}, \sigma_2^{F_j}\right)\right] = \hat{\pi}_2^{F_j}\left(\sigma_1^{F_j}, p_2, k\right) \tag{6.5}$$

where k is the index of market development, defined as a ratio between the current market N_1 and the market saturation level N^*, i.e.

$$k = \frac{N_1}{N^*}, \quad N_1 \in [0, N^*]\,,\, k \in [0, 1] \tag{6.6}$$

Empirically, we use this index to describe the potential market evolution in a 3 to 10-year time frame. $\sigma_2^{F_j}\big(p_2, k\big)$ includes the firm's foresight of its future gains from potential market development. Moreover, we set the aggregate market is shared by each firm j to clear the market in every period, i.e. $N_1 = \sum_{j=1}^{J} \sigma_1^{F_j}$, $j = 1, 2, \ldots, J$.

By the definition, the market development index k is exogenous to the second period price p_2, and thus, we assume the expectation of the future profit function can be decomposed as follows:

$$\textit{Assumption 1.2} \qquad \hat{\pi}_2^{F_j}\left(\sigma_1^{F_j}, p_2, k\right) = \pi_2^{F_j}\left(\sigma_1^{F_j}, p_2\right)\gamma(k) \tag{6.7}$$

$\gamma(k)$ is defined as market foresight function, which stands for the market development effect. Apparently, firms' future profits tend to increase when the market's potential growth is larger. For convenience, let us omit the firm subscripts j for a representative firm. Maximizing firms overall value with respect to its first period price, we have the first order condition as follows:

$$\begin{aligned} 0 &= \frac{\partial \pi_1^F}{\partial p_1^F} + \frac{\partial \gamma(k)}{\partial p_1^F}\pi_2^F(\sigma_1^F) + \frac{\partial \pi_2^F(\sigma_1^F)}{\partial p_1^F}\gamma(k) \\ &= \frac{\partial \pi_1^F}{\partial p_1^F} + \big(\mathbf{E}_{\gamma,\sigma_1^F} + \mathbf{E}_{\pi_2^F,\sigma_1^F}\big)\frac{\partial \sigma_1^F}{\partial p_1^F}\frac{1}{\sigma_1^F}\gamma(k)\pi_2^F(\sigma_1^F) \end{aligned} \tag{6.8}$$

where $\mathbf{E}_{\gamma,\sigma_1^F} := \dfrac{\partial \gamma}{\partial \sigma_1^F}\dfrac{\sigma_1^F}{\gamma}$ and $\mathbf{E}_{\pi_2^F,\sigma_1^F} := \dfrac{\partial \pi_2^F}{\partial \sigma_1^F}\dfrac{\sigma_1^F}{\pi_2^F}$ are current market elasticity of market foresight and current market elasticity of future profit. Moreover, we algebraically rewrite the second item in $\mathbf{E}_{\gamma,\sigma_1^F}$ as follows:

$$\mathbf{E}_{\gamma,\sigma_1^F} = \frac{\partial \gamma}{\partial \sigma_1^F}\frac{\sigma_1^F}{\gamma} = \left(\frac{\partial \gamma}{\partial k}\frac{\partial k}{\partial \sigma_1^F}\right)\left(\frac{\sigma_1^F}{k}\frac{k}{\gamma}\right) = \frac{\partial \gamma}{\partial k}\left(\frac{\partial k}{\partial \sigma_1^F}\frac{\sigma_1^F}{k}\right)\frac{k}{\gamma} = \left(\frac{\partial \gamma}{\partial k}\frac{k}{\gamma}\right)\cdot\left(\frac{\partial k}{\partial \sigma_1^F}\frac{\sigma_1^F}{k}\right) \tag{6.9}$$

or

$$\mathbf{E}_{\gamma,\sigma_1^F} = \mathbf{E}_{\gamma,k}\cdot\mathbf{E}_{k,\sigma_1^F} \tag{6.10}$$

where $\mathbf{E}_{\gamma,k}$ and $\mathbf{E}_{k,\sigma_1^F}$ are market development elasticity of market foresight, and current market elasticity of market development respectively, and defined as $\mathbf{E}_{\gamma,k} = \dfrac{\partial \gamma}{\partial k}\dfrac{k}{\gamma}$ and $\mathbf{E}_{k,\sigma_1^F} = \dfrac{\partial k}{\partial \sigma_1^F}\dfrac{\sigma_1^F}{k}$.

6.3.1 Assumptions and propositions

Market foresight function $\gamma(k)$

Due to the lock-in effect to market, the future (second period) profit depends on the competition on current (first period) market share, which becomes more fierce when the potential of market development decreases, i.e. k increases (or $N_1 \to N^*$). The market development will increase the weight of the future profit in the overall profit.

In an extreme case where $k = 1$ (or $N_1 = N^*$), the market reaches its saturation level in the first period. Hence the market do not have potential growth. In this scenario, the model reduces to Klemperer model, where $\gamma(1) = 1$. Moreover, we have $k = k_t$ in dynamic market growth model for further extension. It is worth to remark that $\gamma(k) = \bar{\gamma}$ is a scalar for constant market growth rate. On the contrary, $k = 0$ (or $k \ll 1$) could exist in some emerging economies with sufficient potential market growth for certain goods.

Proposition 6.3.1. *Suppose Assumptions 1.1-1.2 are satisfied and the index of market development is defined in Eq.(6.23), then we have the following properties:*

(i) The first-order derivative of $\gamma(k)$ is negative.

$$\frac{\partial \gamma}{\partial k} < 0\,, \quad \gamma(k) \geq 0\,, \forall k \in [0,1] \tag{6.11}$$

(ii) $\gamma(1) = 1$ and $1 < \gamma(0) < \infty$.

(iii) The market development elasticity of market foresight $\mathbf{E}_{\gamma,k}$ is negative.

$$\mathbf{E}_{\gamma,k} = \frac{\partial \gamma}{\partial k}\frac{k}{\gamma} \sim \frac{\partial \gamma}{\partial k} < 0 \tag{6.12}$$

Market structure – oligopoly

We consider an oligopoly market form, wherein the market is shared and dominated by a few number of firms (oligopolists).

Assumption 2: We consider an oligopolistic competition, where J firms F_j , $j = 1, \ldots, J$ dominate the market N_i in each period $i = 1, 2$ with their shares $\sigma^i_{F_j}$ and $N_i = \sum\limits_j^J \sigma_i^{F_j}$, $j = 1, 2, \ldots, J$.

Proposition 6.3.2. *Suppose Assumptions 2 is satisfied, which implies each firm is a market-maker (instead of market-taker), i.e.* $\partial N_1/\partial \sigma_1^F > 0$ *(instead of* $\partial N_1/\partial \sigma_1^F = 0$*) holds for all firms and in all periods. Then we have the following properties:*

(i) The first-order derivative of $k(\sigma_1^F)$ *is positive.*

$$\frac{\partial k}{\partial \sigma_1^F} = \frac{\partial k}{\partial N_1}\frac{\partial N_1}{\partial \sigma_1^F} = \frac{1}{N^*}\frac{\partial N_1}{\partial \sigma_1^F} > 0 \tag{6.13}$$

(ii) The current market elasticity of market development $\mathbf{E}_{k,\sigma_1^F}$ *is positive.*

$$\mathbf{E}_{k,\sigma_1^F} = \frac{\partial k}{\partial \sigma_1^F}\frac{\sigma_1^F}{k} \sim \frac{\partial k}{\partial \sigma_1^F} > 0 \tag{6.14}$$

Substituting Eqs. (6.12) and (6.14) into (6.10), we obtain $\mathbf{E}_{\gamma,\sigma_1^F} < 0$. Apparently, we also have $\mathbf{E}_{\pi_2^F,\sigma_1^F} > 0$ and $\dfrac{\partial \sigma_1^F}{\partial p_1^F} < 0$ due to the lock-in effect. Let us rewrite the Eq.(6.8) as follows:

$$0 = \underbrace{\frac{\partial \pi_1^F}{\partial p_1^F}}_{\leqq 0} \Longleftarrow + \underbrace{\Big(\underbrace{\mathbf{E}_{\gamma,\sigma_1^F}}_{<0} + \underbrace{\mathbf{E}_{\pi_2^F,\sigma_1^F}}_{>0} \Big)}_{\leqq 0\,;\quad :=\Delta} \underbrace{\frac{\partial \sigma_1^F}{\partial p_1^F}}_{<0} \frac{\gamma \pi_2^F}{\sigma_1^F} \tag{6.15}$$

where $\Delta := \mathbf{E}_{\gamma,\sigma_1^F} + \mathbf{E}_{\pi_2^F,\sigma_1^F}$, defined as the sum of two elasticities, could be larger or smaller than zero. The basic mechanism underlying Eq.(6.15) is that, the market share in current period σ_1^F influences the firm's value through two channels. First, market foresight, implied by the elasticity $\mathbf{E}_{\gamma,\sigma_1^F}$, represents the firm's potential gain in the future. In specific, from Eqs.(6.14) and (6.12) we have $\mathbf{E}_{\gamma,\sigma_1^F} \sim \mathbf{E}_{\gamma,k} \sim \partial\gamma/\partial k < 0$, indicating that firm tends to evaluate more the future market than the current market when market development index is small. In other words, given optimistic foresight of market development, firms prefer to focus on the potential gains from the future market, instead of enlarging the current market share. Second, future profit is, of course, positively affected by the current market share due to the lock-in effect (i.e. $\mathbf{E}_{\pi_2^F,\sigma_1^F} \sim \partial \pi_2^F/\partial \sigma_1^F > 0$). The reasoning is analogously to Klemperer (1987). Therefore, firm also has the intention to enlarge the current market share. For simplicity, let us denote them "market development" effect and "lock-in" effect respectively. The sign will depend on which effect dominate the other. Therefore, we have two scenarios.

- **The "lock-in" effect dominates the "market development" effect, i.e. when $0 < k \leq 1$.**
  An appealing example could be when a market is close to its saturation level $k \to 1$. Then firm's foresight of market gains from potential market development becomes relatively small, i.e. $\mathbf{E}_{\gamma,\sigma_1^F} \sim \mathbf{E}_{\gamma,k} \sim \partial\gamma/\partial k \to 0$. Hence we have $\Delta > 0$, and the "market development" effect is dominated by the "lock-in" effect in this scenario From the FOC in Eq.(6.15), we have

  $$\frac{\partial \pi_1^F}{\partial p_1^F} > 0 \tag{6.16}$$

  Thus, p_1^F is lower than the price at which $\partial\pi_1^F/\partial p_1^F = 0$. That is, price is lower than if the firm ignored the effect of market share on its second-period profits (Klemperer, 1987).
  Remark: in this scenario, the "market development" effect dampens the "lock-in" effect. Thus the price is higher than Klemperer' price, though it is smaller than the price when the firm ignored the effect of market share on its second-period profits, i.e. $p_{1,Kemperer}^F \leq p_{1,ours}^F \leq p_{1,optimal}^F$.

- **The "market development" effect offsets the "lock-in" effect, i.e. when $k = 0$.**
  The small market development index $k \ll 1$ could exist in some emerging economies, such as China, where there exists sufficient potential market for certain goods. Thus, we could have $\left|\mathbf{E}_{\gamma,\sigma_1^F}\right| = \mathbf{E}_{\pi_2^F,\sigma_1^F}$ and $\Delta = 0$, which indicates the "market development" effect completely offsets the "lock-in" effect. Thus, the FOC in Eq.(6.15) implies

  $$\frac{\partial \pi_1^F}{\partial p_1^F} = 0 \tag{6.17}$$

  In this scenario, p_1^F equals to the optimal price at which $\partial\pi_1^F/\partial p_1^F = 0$. That is equivalent to say the current market and future market are considered to be independent for the pricing strategies. In other words, the future market is so large that the number of previous "locked-in" consumers is negligible.

It is worth to remark that we can not have $\Delta < 0$, i.e. the "market development" effect dominates the "lock-in" effect. According to our model's assumptions, the firm's expected future profits is better off if the numbers of locked-in consumers become larger, i.e. $\partial\, \mathbb{E}_1\big(\pi_2^F\big)/\partial\sigma_1^F \geq 0$. As the inherited market, composed of the locked-in consumers in the previous period, will not affect the price strategy in future period. In fact, we adopted the basic assumptions of the Klemperer's model, which suggests strictly better off, i.e. $\partial\pi_2^F/\partial\sigma_1^F > 0$, when the uncertainty and market potential development are absent. Substituting the above inequality into Eq.(6.15), we could deduce the Klemperer's result: $\partial\pi_1^F/\partial p_1^F > 0$.

Our model extends the Klemperer's two-period switching costs model to the uncertainty scenario where the 2nd period profit is not perfectly predictable. The 'locked-in' effect is dampened or even off-setted by the 'market development' effect in case of sufficiently large future market, i.e. $\partial\pi_1^F/\partial p_1^F = 0$ when $k = 0$.

6.3.2 An explicit uncertain switching costs model with market development

In this section, we generalize a two-period switching costs model à la Klemperer (1995) by incorporating the uncertainty into the analysis. The value function is defined straightforwardly as follows:

$$V^{F_j} = \pi_1^{F_j}(\sigma_1^{F_j}) + \gamma\, \mathbb{E}_1\Big[\pi_2^{F_j}(\sigma_1^{F_j}, \sigma_2^{F_j})\Big] \tag{6.18}$$

where V^{F_j} is total discounted value of a firm F_j, $j = 1, \ldots, J$. $\gamma \geq 0$ indicates the time discounting effect. π_1^F is the firm j's current (or first-period) profit, $\pi_2^{F_j}$ is future (or second-period) profit and $\sigma_1^{F_j}$ is the firm j's current market occupation, depending on its current price. We assume there is uncertainty regarding the firm j's future (or second period) market occupation $\sigma_2^{F_j}$, which depends on its future price. Indeed, many structures are possible to incorporate the uncertainty, we make the most straightforward assumption. For convenience, let us omit the firm subscripts F^j for a representative firm and simplify the value function as follows:

$$V = \pi_1(\sigma_1) + \gamma\, \mathbb{E}_1\big[\pi_2(\sigma_1, \sigma_2)\big] \tag{6.19}$$

σ_2 can either be equal to $\bar{\sigma}_2$ or to $\underline{\sigma}_2$, with $\bar{\sigma}_2 > \underline{\sigma}_2$. The ex-ante probability of the high value is denoted q. Analogously, the second period profit takes the following form:

$$\pi(\sigma_1, \sigma_2) = \begin{cases} \bar{\pi} := \pi(\sigma_1, \bar{\sigma}_2), & \text{with probability } q \quad (6.20) \\ \underline{\pi} := \pi(\sigma_1, \underline{\sigma}_2), & \text{with probability } 1-q \quad (6.21) \end{cases}$$

Moreover, we have

$$\begin{aligned} \mathbb{E}_1\big[\pi_2(\sigma_1, \sigma_2)\big] &= \bar{\pi}q + \underline{\pi}(1-q) \\ &= (\bar{\pi} - \underline{\pi})q + \underline{\pi} \end{aligned} \tag{6.22}$$

Furthermore, we also assume the probability q depends on the market development index k, i.e. $q = q(k)$. The index of market development is defined as the ratio between the current market size N_1 and the market's saturation level N^* as follows:

$$k = \frac{N_1}{N^*}, \quad N_1 \in [0, N^*], \, k \in [0,1] \tag{6.23}$$

We use this index to describe the potential market evolution in a 3 to 10-year time frame. From market cleaning condition, we know that the aggregate market is shared by each firm j in every period, i.e. $N_1 = \sum_{j=1}^{J} \sigma_1^{F_j}$, $j = 1, 2, \ldots, J$.

When the first period market N_1 is much smaller than the market saturation level, the potential future market is expected to be larger. Thus, the probably of the ex-ante probability of the high σ_2 is larger, i.e. $N_1 \downarrow \Rightarrow q(k) \uparrow$, or $\partial q(k) \partial \sigma_1 < 0$ and $\partial q(k) \partial k < 0$.

Moreover, we set the following two assumptions in order to satisfy the stylized effect.

Assumption 3.1 Firm's second-period profit should not be better-off for a larger inherited market, given other conditions unchanged, i.e. $\partial \mathbb{E}_1\big[\pi_2\big] / \partial \sigma_1^F \geq 0$.

Assumption 3.2 $\bar{\pi}_2 \geq \underline{\pi}_2$.

Calculating the firm's expected profit in second period with respect to the locked-in consumers as follows

$$\frac{\partial \mathbb{E}_1\big(\pi_2\big)}{\partial \sigma_1} = \underbrace{\frac{\partial q}{\partial \sigma_1}}_{<0} \underbrace{\big(\bar{\pi}_2 - \underline{\pi}_2\big)}_{\geq 0} + q \underbrace{\frac{\partial \bar{\pi}_2}{\partial \sigma_1}}_{>0} + (1-q) \underbrace{\frac{\partial \underline{\pi}_2}{\partial \sigma_1}}_{>0} \tag{6.24}$$

Substituting Eq.(6.24) into the firm's optimization problem, where firms maximize their overall value with respect to its first period price as follows:

$$\begin{aligned} 0 &= \frac{\partial \pi_1}{\partial p_1^F} + \delta \frac{\partial \mathbb{E}_1\big(\pi_2\big)}{\partial \sigma_1}\frac{\partial \sigma_1}{\partial p_1} \\ &= \frac{\partial \pi_1}{\partial p_1^F} + \delta\, \frac{\partial \sigma_1}{\partial p_1}\left[\underbrace{\frac{\partial q}{\partial \sigma_1}}_{<0}\big(\bar{\pi}_2 - \underline{\pi}_2\big) + q\frac{\partial \bar{\pi}_2}{\partial \sigma_1} + (1-q)\frac{\partial \underline{\pi}_2}{\partial \sigma_1}\right] \end{aligned} \tag{6.25}$$

Therefore, the sign of $\partial \pi_1 / \partial p_1^F$ depends on the sign of Eq.(6.24), i.e.

$$\frac{\partial \mathbb{E}_1\big(\pi_2\big)}{\partial \sigma_1} \gtreqqless 0 \quad \Rightarrow \quad \frac{\partial \pi_1}{\partial p_1^F} \gtreqqless 0 \tag{6.26}$$

We are particularly interested in the case when the equality holds, i.e. $\dfrac{\partial \mathbb{E}_1\big(\pi_2\big)}{\partial \sigma_1} = 0 = \dfrac{\partial \pi_1}{\partial p_1^F}$. Substituting this equality to Eq.(6.24), we have:

$$\underbrace{\frac{\partial q}{\partial \sigma_1}}_{<0}\, \Delta^{\pi_2} + q\, \Delta^{\pi_2}_{\sigma_1} + \underline{\pi}_{2,\sigma_1} = 0 \tag{6.27}$$

$$\Delta^{\pi_2} = \bar{\pi}_2 - \underline{\pi}_2\,, \quad \Delta^{\pi_2}_{\sigma_1} = \frac{\partial\big(\bar{\pi}_2 - \underline{\pi}_2\big)}{\partial \sigma_1}\,, \quad \underline{\pi}_{2,\sigma_1} = \frac{\partial \underline{\pi}_2}{\partial \sigma_1} \tag{6.28}$$

$$q\big(\sigma_1 = 0\big) = 1\,, \quad \text{where } q \in [0,1]\,, \quad \sigma_1 \in [0, N^*] \tag{6.29}$$

Eq(6.27) is a first order linear non-homogeneous differential equation, and has the closed-form solution $q^*(\sigma_1)$ as follows:

$$q^*(\sigma_1) = \tilde{q}^{-1}\Big|_{\sigma_1=0}\tilde{q}(\sigma_1) \;+\; M(\sigma_1) - M\Big|_{\sigma_1=0} \tag{6.30}$$

$$\tilde{q}(\sigma_1) = \exp\left[-\int_0^{\sigma_1}\left(\frac{\Delta^{\pi_2}_{\sigma_1}}{\Delta^{\pi_2}}\right) \mathrm{d}\sigma_1\right], \quad M(\sigma_1) = \int_0^{\sigma_1} \underline{\pi}_{2,\sigma_1}\, \tilde{q}(\sigma_1)\, \mathrm{d}\sigma_1 \tag{6.31}$$

Therefore, for a given $\sigma_1 \in [0, N^*]$, we have a $q^*(\sigma_1)$, such that "locked-in" effect is off-setted by the 'market development' effect, and our optimal price p_1^* satisfies $\partial \pi_1^F / \partial p_1^* = 0$.

6.3.3 The simplified model for empirical analysis

Firms compete to capture buyers in their life-cycle, and these problems are more subtle than the mere fact that firms set optimal price strategies to maximize their yearly profits. Firms compete ex-ante for ex-post market power by using aggressive price strategy, such as penetration pricing, introductory offers, and price wars. Such "competition for the market" motivates firms to set a lower price in order to lock-in consumers (Farrell and Klemperer, 2007) and exploit this consumer base in the future with higher prices. To support this intuition, we borrow a two-period market model from Klemperer (1985, pag.521) and extend it by including an index of potential market growth, k. A firm aims to maximize its total discounted future profits that depend on second period market share that, in turn, depends on first period price p_1^F:

$$V^F = \pi_1^F + \beta\pi_2^F(\sigma_1^F, k) \tag{6.32}$$

where V^F is total discounted value, π_1^F is first-period profit, π_2^F second-period profit, β is a time discounting coefficient, and σ_1^F is the first-period market share that depends on first-period price, i.e. $\sigma_1^F(p_1^F)$. The Index k is the measure of market potential growth, defined as a ratio between the actual size of the market at time t, N_t, and the market saturation level N^* , i.e. $k_t = N_t/N^*$. The higher the index, the more the destination market is approaching a maturity stage; vice-versa, a lower k identifies a growing market in its early stage of development. We will discuss and compute the index in a next section. A firm chooses prices in period one. Maximizing with respect to first-period price, firm's first order condition for equilibrium is:

$$0 = \frac{\partial\pi_1^F}{\partial p_1^F} + \beta\frac{\partial\pi_2^F(\sigma_1^F, k)}{\partial\sigma_1^F}\frac{\partial\sigma_1^F}{\partial p_1^F} \tag{6.33}$$

As $\partial\sigma_1^F/\partial P_1 < 0$, and $\partial\pi_2^F(\sigma_1^F, k)/\partial\sigma_1^F > 0$, it turns out that $\partial\pi_1^F/\partial p_1 > 0$, i.e. firms wish to set a price p_1^F lower than the optimal price to gain future market shares. Importantly, from Eq.(6.33) we observe that the pricing policy depends on the market potential expansion k. If a new entrant is a price taker, as in the case of a producer of mass products or, more generally, of a cost-inefficient manufacturer that experiences inferior organizational and managerial ability or is just a late entrant, the index k of potential market growth becomes a crucial variable in the decision to enter the export market.

Specifying some assumptions

Let us assume that the market with higher growth potential also implies larger future gains, i.e.

$$\textit{Assumption 1} \qquad \partial \pi_2^F / \partial k \leq 0 \tag{6.34}$$

Let us also consider a destination market in which a number of non-family and family firms compete, where each firm's market share is proportional to the overall market size, i.e.

$$\textit{Assumption 2} \qquad \partial N_1 / \partial \sigma_1^F > 0 \tag{6.35}$$

We can finally assume that, because of their managerial structure, non-family firms are more prone to go abroad or early-entrants and, thus, are able to set the market price. This leads family firms to be price-takers in their initial entry decision, which makes relative efficiency relevant in modelling the entry decision. If family firms are supposed to be cost-inefficient compared to non-family firms, for example, because they use the same scale-intensive technology of non-family firms but are late entrants into the market, they can end up with a higher production cost than non-family firms, i.e.

$$\textit{Assumption 3} \qquad C^{FF} > C^{NF} \tag{6.36}$$

where upper index FF and NF indicate family firm and non-family firm respectively.

Results

Proposition 6.3.3. *Suppose Assumptions 1-3 are satisfied, and the market development index k is defined as in section 3.3.1, then we have the following propositions:*

- *Family firms selling mass-products are more likely to venture abroad if the destination market is growing.*
- *Family firms with niche-products are always better-off from selling abroad, regardless of the market saturation level.*

Proof. Provided in Appendix.
To venture into foreign markets, family firms selling mass-products (price-takers) face the competition from non-family firms that set the market price. In a life-cycle market competition context, the future market shares are valuable because of the lock-in effect on consumers: therefore, to capture the market share in the first period, non-family firms tend to set a low price strategy that could make family firm' profit too low (actually, negative) to enter the market. By contrast, if the market has large growth potential, a low price strategy by early entrants finalized to invest in future market share could be less relevant, thus pushing prices upwards and relaxing the exclusion price restriction for cost-inefficient entrants. This beneficial effect for cost-inefficient entrants is expected to be stronger with larger expected market growth, and prices are expected to be closer to the optimal ones, i.e. those that maximize non-family firm's current profit and ignore the lock-in effect, the less mature is the destination market. As a consequence, price competition for market share is expected to be less fierce in the first period which, in turn, allows late entrants or cost-inefficient firms to enter the market. Therefore, we put forward the following hypothesis:
Hyp. 1 - Family firms are more likely to expand into new foreign markets that are in the growing stage of evolution.

6.3.4 The adoption of a replication strategy

Business organizations may expand internationally by replicating a part of their value chain, such as a sales and marketing format, in other countries. However, little is known regarding how such "international replicators" build a format for replication, or how they can adjust it in order to adapt to local environments and under the impact of new learning. As a strategic means to leverage a firm's knowledge assets, the replication of a successful template (a working example, or a superior operational routine) provides a fundamental source of competitive advantage (Winter 1995; Teece et al., 1997; Argote and Ingram 2000; Szulanski and Jensen, 2006, 2008). However, replicating routines and the related web of coordinating relationships, across multiple locations is far from trivial and provides a significant strategic value to the company (Jonsson and Foss, 2011). It can also be combined with deliberate company intent to enter foreign markets as a learning mechanism to feed the accumulation of business information in the organisation.

Replication is a strategy that leverages existing strengths of the product (produced domestically), whereby organisations deliberately try to reproduce the success they have enjoyed in some limited setting (Winter and Szulanski, 2001). A company seeking to derive first mover advantage by moving rapidly into new market opportunities may adopt a replication strategy to facilitate this rapid entry. As firms make an effort to develop their best business model to enter international market, firms may skew towards their best performing products (Mayer et al. 2013). International management theory, and specifically internalization theory (Rugman and Verbeke, 2004), has long maintained that firms need proprietary or ownership advantages to offset the liabilities of foreignness when they enter foreign markets.

Winter and Szulanski (2001) state that although replicators are becoming one of the dominant organizational forms of our time, they have been neglected by scholars interested in organizations. As a result of this neglect, replication is typically conceptualized as little more than the exploitation of a simple business formula. Conversely, the formula is typically a complex set of information that is discovered, adjusted, and integrated with further detailed information. Specifically, the conditions of emerging markets result in different learning processes about information: not only what is learned differs, but also how this is learned (Rui et al., 2016). For this reason, "replication requires effort, and naturally takes time. Its value is eroded by delay, and urgency is a hallmark of a replication strategy" (Winter and Szulanski, 2001). This strategic approach fits the flexibility and speediness of the decision-making process of family boards, as well as the tacit knowledge that backs their stewardship approach.

The adoption of a replication strategy is also consistent with many other traits of the family governance. Family firms may have lower willingness to introduce product innovation than non-family firms (Chrisman et al., 2015): this may push family firms to rely on consolidated products when enter foreign markets. Family firms are more likely to engage in innovative activities that provide more reliable performance rather than higher performance, except when performance is below aspiration levels (Patel and Chrisman, 2014), or to adopt a replication strategy, instead of an innovation strategy, because of concerns about the risk intensity of the strategy (Hoskisson et al., 2002; Goranova et al., 2007; Datta et al., 2009; Danneels, 2006). From a resource dependence perspective (Naldi and Nordquist, 2008; Gallo and Garcia Pontes, 1996; Fernandez and Nieto, 2006; Graves and Thomas, 2006; Okoroafo, 1999), family firms may overcome their potential shortage of resources by exploiting the endowment of resources already existent in the company through a replication strategy.

The implications for the resource approach are also consistent with the more general results from the international business literature about the factors that explain the preference for a replication strategy by an entrant. From the Vernon's approach, showing that products in their initial steps of life-cycle can be managed more easily when entering international markets, to the Dunning's (1993) eclectic paradigm and Buckley and Casson's (1976) Internalisation theory, a large literature in international business has stressed the role of unique resources to sustain the international expansion. The link between resources and internationalisation is also at the core of the Uppsala model that explains how firms gradually intensify their activities in foreign markets by first gaining experience from the domestic market and then move to foreign markets – using exports - more culturally and/or geographically close (Johanson and Vahlne, 1977).

Additional support to the adoption of a strategy that replicates the exiting course of actions comes from the literature on the technological diversification of family firms. Given family owners' desire to preserve control over their firms, they tend to resist introducing products that diverge in industry and technological characteristics from the existing product portfolio, simply because those products will be more risky. That is, such products will take a firm beyond its normal level of technological and market competency and thus may yield negative returns. Conversely, in introducing related products family firms can leverage existing resources, reduce risk (Shamsie et al., 2009) and avoid discontinuities in technological and in marketing processes (Mc Nally et al., 2010). For example, they would need to change manufacturing facilities less, and exploit the same staff (Hamel and Prahalad, 1993). They may also leverage a positive reputation, and avoid risking reputation on an untried unrelated new product – a key consideration for family firms with long time horizons (James, 1999; Deephouse and Jaskiewicz, 2013).

Moreover, using a replication strategy may be less risky as the firm does not invest additional resources in developing new variant of the product or establish commercial networks. Even though entering a new market may be a way to edge domestic demand fluctuations, entering export markets is a risky choice, as it involves sunk costs, potentially higher volatility of revenues, organisational adaptation, new investment and increased competition. If sunk costs of entry are high, the risk of bankruptcy may be substantial and the potential impact on family control extremely relevant. Furthermore, for family firms, responding appropriately to the intense competition in mature market "requires building competencies they are ill-equipped to acquire" (Henderson, 2006), thus making the product adoption too risky when control matters.

Correlated to the issue of risk there is also an agency rationale. One of the main benefits of the family governance is its ability to overcome agency problems in management, mainly adopting a decentralised management that is more efficient in dealing with complex and high risky markets (Barba Navaretti et al., 2008). However, when enter international markets, family members are forced to delegate a part of their decisional power to external managers, thus making the agency problem emerge as a result of this decision, and the benefit of centralized decision-making procedure get diluted. Therefore, if family owners wish to enter international markets, they may want to avoid losing control and enter using the simplest approach available in order not to delegate, that is just replicate the domestic strategy and use the same product.

A further issue concerns the entrepreneurial orientation of family firms and the flexibility of their executive boards. Kontinen and Ojala (2010) find that due to the small size and the flexibility of the management team in family SMEs, family firms were able to react quickly to new international opportunities. Flexibility may help in adopting a replication strategy because one of the factors for failure in international markets is the intention to replicate without adaptation, of to "adhere dogmatically to the initial template" (Winter and Szulanski, 2001, pag.735). The structure of the boards comes up as a definite advantage for family firms as they allow small adaptations to external conditions in a quick and timely way. The cost of mistakes and the risk of failure in undertaking many simultaneous rollouts of an untested business model are enormous and are those that mostly drive firms to replicate existing course of action by using products already sold (Winter and Szulanski, 2001). Furthermore, adaptation through replications requires time and it is coherent with the long-term orientation of family members, who spend their entire life in one company and most of the time with one or few products. As the speed of replication is crucial in a competitive setting, the replicator cannot afford delays in the adoption of its model: therefore, rapid decisional processes in small family teams are a definite advantage to the development into (fast) growing markets.

Finally, the presence of fewer specialized resources in fast growing markets leads companies toward a more intensive use of integration, because firms are less able to rely on external knowledge providers. This may favour family firms as their organisational structure is oriented towards a more hierarchical structure, with low delegation, low external involvement and the use of less complex value networks. Similarly, higher market growth leads companies to undertake more repetition to benefit from the market opportunities, a task which is much easier to accomplish by small, flexible and quick-acting board teams. Therefore, we posit the following hypotheses:
Hyp. 2 - Family firms mainly adopt a replication strategy when enter growing foreign markets.

6.4 Data and method

6.4.1 Data and sample

To build our dataset we combine information from three main data sources: (1) the EU-EFIGE survey, (2) the BvD-Amadeus database, (3) the EU Comext dataset. The EU-EFIGE survey, carried out in 2010 on a sample of about 14,759 companies from seven European countries (Austria, France, Germany, Hungary, Italy, Spain and UK), provides detailed information about companies' ownership and management structure. This dataset includes the results of a representative survey carried among companies having at least 10 employees. The survey includes a rich variety of information on firms' organizational, R&D and innovative behaviour for the period of 2007-2009 (Altomonte and Aquilante, 2012).

For all the surveyed companies, balance-sheet data for the period 2007-2009 are recovered from the BvD-Amadeus database, the most comprehensive and widely used source of financial information for public and private enterprises in Europe. The company accounting statements are harmonized by BvD, making the cross-country comparison reliable. Data are given for unlisted firms. Due to national legislations, the coverage of financial variables varies across countries. This limits the number of countries included in the analysis. Given that the firms included in the Amadeus Top 1 million companies have at least 10 employees, this inclusion criterion makes the source biased against the smallest companies.

EU Comext dataset provide information on bilateral trade between 28 European reporting countries and more than a hundred partner countries. Comext includes statistics on the trade in goods between the Member States (intra-EU trade) and between Member States and non-EU countries (extra-EU trade) from 1995. The Community Legislation ensures the harmonisation of the concepts and definitions applied by the Member States when sending Community data to Eurostat. In addition, the coherence is strengthened by a harmonised approach implemented at Eurostat level for the data production and dissemination whatever the type of trade – intra or extra-EU trade – and the Member State.[3]

6.4.2 Family Firms

Despite the widespread literature on family businesses, there is not a clear consensus on how family firms should be defined. Theoretical and empirical studies ground on definitions based on ownership shares, family involvement in the business, and some combinations of the two criteria (La Porta et al., 1999; Anderson and Reeb, 2003, 2006; Banalieva and Eddleston, 2012).

In this study, in order to avoid getting distorted results due to the adoption of a subjective definition of family company, we employ firm self-reported information to distinguish family owned and managed businesses. In particular, by relying on questions A20 and A21 of the EFIGE survey[4], we define: (i) family owned firms those companies directly or indirectly controlled by an individual or a family-owned entity; (ii) family managed firms those companies run by the individual who owns or controls the firm or a

[3]Data record the yearly trade in terms of imports and exports at the most detailed level of several product nomenclatures: the Combined Nomenclature (CN) which corresponds to the Harmonised System (HS 2007) plus a further breakdown at 8-digit level, the SITC, the BEC, the Classification of Products by Activity (CPA) and the Standard Goods Classification for Transport Statistics/Revised (NST/R). The following dimensions are provided across databases: reporting country (MSs), trading partner (e.g. EU27, China, US, world,), reference period, trade flow (import, export), product (according to the nomenclatures), indicators (value, quantity, supplementary units).

[4]The corresponding questions are: A20. Is your firm directly or indirectly controlled by an individual or a family-owned entity? (i) yes; (ii) no. A21. Is the CEO of your firm...? (i) the individual who owns or controls the firm or a member of the family that owns/controls it; (ii) a manager recruited from outside the firm; (iii) a manager appointed within the firm.

member of the controlling family. As reported in Table 1.1-2, it is worth noting that family-owned firms represent more than 70 percent of the whole sample (10365/14759 $\approx$ 70.2 percent). This percentage further raises when we focus our attention on four of the major continental Europe countries: Austria, Germany, Italy and Spain (Panel A). Among family-owned firms, moreover, almost 88 percent of the surveyed companies promote family members as CEOs (9174/10365 $\approx$ 88.5 percent), while less than 20 percent decide to hire professional managers external to the controlling family.

Table 6.0 – *Table 1.1 - Distribution of the sample by country, ownership structure and management type*

Country	All firms Obs.	Non-family firms Obs.	Family firms Obs.	Family CEOs Obs.	Non-family CEOs Obs.
Austria	443	83	360	316	44
France	2976	1292	1681	1443	238
Germany	2935	513	2409	2139	270
Hungary	488	228	254	210	44
Italy	3021	777	2244	2118	126
Spain	2832	700	2132	1829	303
UK	2067	769	1285	1119	166
Total	14759	4362	10365	9174	1191

Table 6.0 – *Table 1.2 - Distribution of the sample by sector, ownership structure and management type*

Sector(NACE 2)	All firms Obs.	Non-family firms Obs.	Family firms Obs.	Family CEOs Obs.	Non-family CEOs Obs.
Manufacture of metal products	3430	959	2466	2219	247
Manufacture of food products and beverage	1520	383	1135	1008	127
Manufacture of rubber and plastic products	937	282	654	567	87
Manufacture of textiles and wearing apparel	1966	574	1385	1235	150
Manufacture of furniture	1038	262	773	685	88
Manufacture of chemicals and pharmaceutical products	563	220	343	273	70
Manufacture of motor vehicles and other transport equipment	424	171	253	209	44
Manufacture of wood and products of wood and fork	705	137	568	520	48
Manufacture of coke and refined petroleum products	21	9	12	11	1
Manufacture of computer, electrical products and electrical equipment	2353	824	1523	1355	168
Manufacture of machinery and equipment	1802	541	1253	1092	161
Total	14759	4362	10365	9174	1191

In order to investigate the differences that exist between family and non-family businesses, we start carrying out several differences of means tests for the main variables included in the analysis. Table 2.1-2 present the results of these univariate tests. The first two columns in Table 2.1, we simply differentiate between family and non-family owned businesses, and in Table 2.2 we go a step further by dividing the family firm sample in two different subgroups depending on the CEO type.

TABLE 6.0 – *Table 2.1 - Descriptive statistics and univariate tests on family firms and non-family firms*

	Family firms			Non-family firms			
	Mean	St.dev	Obs.	Mean	Std.dev	Obs.	t-statistics
Age	27.51	24.22	10365	24.14	17.83	4362	3.37***
Employees	57.46	114.38	8157	101.48	184.65	3721	-44.02***
Total assets	7222.7	20588.7	9660	15601.9	37176.9	4184	-8379.2***
Debit ratio	66.61	27.12	9645	65.25	28.93	4170	1.36***
Innovative products	0.39	0.41	78442	0.38	0.38	4258	-0.01**
TFP growth	-0.04	0.17	7381	0.057	0.22	4036	-0.10***
Export(%)	0.64	0.48	10356	0.66	0.47	4346	-0.02***

TABLE 6.0 – *Table 2.2 - Descriptive statistics and univariate tests on family CEOs and non-family CEOs*

	Family CEOs			Non-family CEOs			
	Mean	St.dev	Obs.	Mean	Std.dev	Obs.	t-statistics
Age	27.53	24.25	9174	27.35	24.01	1191	0.18
Employees	50.28	101.15	7159	108.04	174.01	1016	-57.7***
Total assets	6055.3	17446.2	8531	16043.7	35212.9	1129	-9988.4***
Debit ratio	66.73	26.91	8517	65.74	28.66	1128	0.99
Innovative products	0.39	0.40	8897	0.41	0.42	1097	-0.02***
TFP growth	-0.06	0.17	7320	0.095	0.21	17	-0.16***
Export(%)	0.63	0.48	9167	0.73	0.44	1189	-0.10***

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

The tables report univariate statistics. Accounting figures are expressed in thousands of Euros. Balance-sheet indicators refer to the period 2007-2009.

As the data show, family-owned companies differ significantly from their non-family counterparts in several aspects. Family firms in the sample are, on average, older, smaller (both in terms of the number of employees and total assets) and more indebted than non-family owned firms. Probably due to the greater risk aversion and regional focus, family firms are less likely to promote both innovation and internationalization activities. The following univariate tests on the subsamples of family and non-family run firms additionally show that family businesses are heterogeneous when management type is accounted for. In fact, the reported results support the idea that companies appointing family and non-family CEOs differ from each other in terms of size and export intensity: family run businesses are on average significantly smaller and less export-oriented than firms run by external CEOs.

6.4.3 Sample of ex-novo exporting firms

To focus our analysis on the influence of potential market growth on the decision to enter a market, we selected only firms that entered a new foreign market for the very first time in 2008. This allows us to evaluate the decision as only affected by future market perspective, without any confounding factors from previous – occasional or systematic – foreign activities. From the total sample of firms included in EFIGE (n= 14,759) we first selected the group of firms that exported in 2008 (n=8,009). Then, we split this group in three mutually exclusive sub-groups that include: i) Regular exporters in 2008 (n=5,866); ii) occasional exporters in 2008 (n=1,705); exporter in 2008 and never before (n=437). This last group, labelled as first-time exporters, will be the focus of our analysis.

We follow Pedersen and Sharer (2011) in using first time export by firms as the "big step" in their internationalisation process. Even if Pedersen and Sharer (2011) ground their analysis in theories of foreign direct investment, we are confident that a similar approach can be also found in the case of first-time exporting. Indeed, the starting point is similar, as foreign firms face disadvantages in foreign markets when competing with indigenous firms for a variety of reasons that include differences in the language, laws, business environment, consumer demand and the structure of suppliers and, finally, exchange rate risks. Firms that expand internationally have to create the architecture that enables them to handle the cross-border activities, like creating new routines and often changing the nature of their managers' mindsets. However, once the company has made these changes, it does not have to make the same investment should it starts exporting to other countries. That is, the big step has been done. In fact, "when the infrastructure of the firm (the architecture, systems, and mind-set) are adapted to support international operations in the first place, it is much easier to 'plug-in' and add more international activities into this infrastructure" (Pedersen and Sharer, 2011).

6.4.4 Variables and measures

Dependent variables

To test the hypothesis on the preference of family firms for growing market, we used as dependent variable $NewExport_{i,h,c}$ (firm i, sector h and country

c), which is a dummy variable equal to one if firm i entered a new market (sector h and country c) in 2008 and zero otherwise. The same variable has been used for both the test of the preference (hypothesis 1) and for the relevance of the replication approach (Hyp. 2).

Independent variable

The main variable in the estimates is given by the $PotentialMarketGrowth_{h,c}$, i.e. the estimated market potential for sector h and country c.

The index of market development

To describe the potential market evolution in a 3- to 10-year time frame, we first define the index of market development as given by the ratio

$$k_i = \frac{N_t}{N^*} \tag{6.37}$$

Where k_i is the index of market development, N_t is the actual size of the market and N^* is the market saturation level to be estimated. The market saturation level is obtained by estimating a logistic curve on historical data from a specific product market. If $\dot{N}$ is the growth rate of market size (number of customers, sales, etc), the logistic model implies that the expansion of the market depends on the size of potential market:

$$\dot{N} = \delta N_t (N^* - N_t) \tag{6.38}$$

The growth of the market is a function of the size of the existing market N_t and the size of potential market $(N^* - N_t)$, where δ is a constant growth coefficient. At the beginning stage of the market, the market size is small and consequently the growth rate is relatively small. This reveals that the product is still unknown, or that exists a moderate imitation effect in customers. When market reaches a higher level, new customers accelerate their buying decisions, thus significantly increasing the growth rate of market. At the end of the process, when the actual market size is very large with respect to the saturation level, market growth rate decreases progressively because of the reduction of the potential market. The solution of the differential Eq.(6.38) gives the expansion path of the market:

$$N_t = \frac{N^*}{1 - e^{\alpha - bt}} \tag{6.39}$$

where

$$\alpha = ln(\frac{N^* - N_0}{N_0}) \tag{6.40}$$

and

$$b = \delta N^* \tag{6.41}$$

A more realistic version of this function is given by using a variable growth coefficient α instead of δ. Under the hypothesis that α depends on per-capita income, we can set:

$$\alpha = \mu g \tag{6.42}$$

where g is the per-capita income growth rate and μ is a parameter that summarizes the income elasticity of the market. Substituting in Eq.(6.38) we obtain:

$$\dot{N} = \alpha N_t(N^* - N_t) = \mu g N_t(N^* - N_t) \tag{6.43}$$

Integrating we get:

$$N_t = \frac{N^*}{1 - e^{b - \mu g t}} \tag{6.44}$$

that connects the existing market N_t and the potential market size N^*.

Estimation procedure

In order to estimate the parameters of the growth function, a regression analysis has been employed. A linear regression can be employed in the logit transformation of the logistic function:

$$ln(\frac{N^*}{N_t} - 1) = \alpha - bt \tag{6.45}$$

that allows the estimates of parameters α and b, given the level of N^*. In order to estimate the saturation level N^*, the Hotelling procedure based on the following equation has been used:

$$\frac{dN}{dt}\frac{1}{N} = b - \frac{1}{N^*}N_t \tag{6.46}$$

It allows to estimate parameters b and N^* thanks to the linearity of the log function. Finally, by substituting N^* in the previous Eq.(6.43) we get an estimate of parameter α.

The following graphs provide an example of the use of the model: the size of the expected market growth is predicted through the logistic approximation for China and the UK for one 8-digit CN sector. By estimating and comparing absolute values, the model provides an accurate forecast of the expected size and growth of the destination market, that is larger in China than in the UK in this specific case. The magnitude of "market potential", as summarized by the variable K, will be used in the following analysis to identify the strategic behaviour of companies by ownership in international markets.

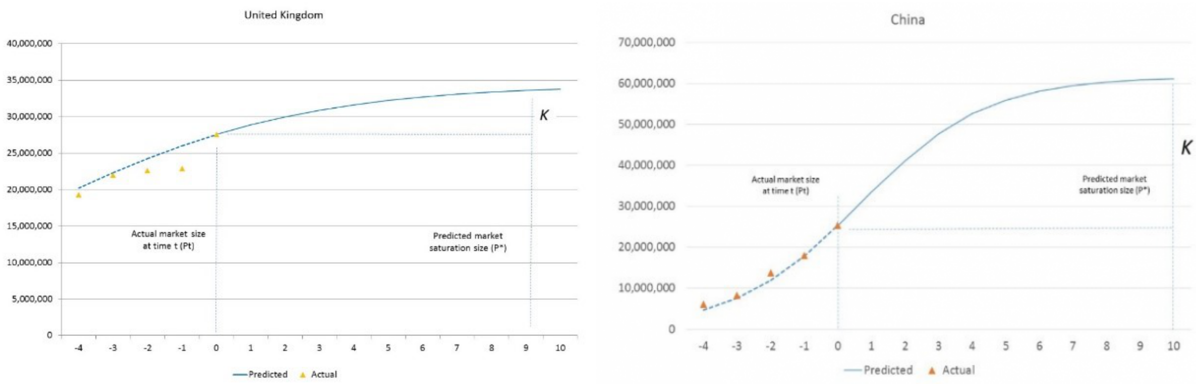


Figure 6.1 – Market potential in UK and China.

Preliminary evidence on the role of family governance on the external involvement of the company is provided in Table 3.1. Data reported in Table 3.1 are average values across exporting companies of the estimated market growth by pairs of countries-sectors. For example, the average 3-year future growth of all sectors in all countries where firms run by family CEO decided to export in 2008 is 9.0%, whereas it is 8.4% for the group of pairs country-sector markets entered by external CEOs. By using the expected growth rate by product/country as a measure of the potential market attractiveness, firms run by family CEOs have selected markets with higher potential growth (9.0%) than those selected by external CEOs (8.4%) and non-family firms (8.6%). The preference for high growth markets is even more evident when market potential is evaluated on a 5- and 10-years interval: firms run by family CEOs selected markets whose growth rate is above 20%, vs. only 12.7% in the case of non-family firms.

Table 6.0 – *Table 3.1 - Annual expected market growth in new markets*

Forecast interval	Non-family firms (a)	Family firms (b)	Diff, (a)-(b)	t-statistic	Family CEO (c)	External CEO (d)	Diff, (c)-(d)	t-statistic
3-year	0.086	0.088	-0.002	*	0.090	0.084	0.006	*
5-year	0.110	0.136	-0.026	***	0.145	0.115	0.030	***
10-year	0.127	0.198	-0/0711	***	0.201	0.192	0.009	**

Note:Market potential growth has been estimated using the procedure illustrated in Section 4. Coefficients have been obtained from Eq.(1).

Table 3.2 summarizes market data form the perspective of structural characteristics. Destination markets have been split into two groups, i.e. mature and growing markets, using the median score of the distribution of the index of market potential by product. For each group, aggregate descriptive statistics on company data have been reported. The main indication in Table 3.2 is that mature markets actually come out as more competitive and more difficult to entry than growing markets. Using profits per employees of companies operating in the market as a measure of competitive pressure, mature markets are significantly less profitable than growing markets. They also appear to be less risky, at least when market risk is measured using the standard deviation of profits per employee. Same indications can be drawn from a different measure of intensity of competition (Ebit on Sales), even if differences between markets are not statistically significant. Finally, both firm age and the financial status of the company are not a discriminant variable in explaining the decision to enter, even if liquidity seems to play a role in supporting entry in growing markets.

Table 6.0 – *Table 3.2 - Selected firms' indicators by type of foreign market: mature vs. growth*

Markets	Liquidity ratio	Firm age	Profit per employee		Ebit on sales	
			Mean	Std.dev	Mean	Std.dev
Mature	1.105	25.432	13.128	7.708	0.040	0.031
Growth	1.338	26.148	21.531	12.779	0.075	0.030
Total	1.219	25.790	17.256	9.964	0.057	0.030
Diff.	0.234	0.716	8.403**	5.701**	0.036	-0.001
T-statistic	-1.160	-0.278	-2.222	-1.900	-0.675	0.097
Pr(|T|>|t|)	0.248	0.782	0.042	0.049	0.501	0.923

Control variables

In order to correctly identify the impact of family ownership and management, we control for a large set of possible confounding effects. First of all, we consider a number of standard firm-specific characteristics and balance sheet indicators. In particular, we control for: i) firm age, measured by the number of years from its inception; ii) firm size in terms of employees; (iii) innovativeness, measured by the share of sale accounted for by innovative products in the three-year period covered by the survey; (iv) liquidity, given by the ratio of current assets over current liabilities; (v) efficiency, as indicated by the TFP growth in the period 2001-2007 available from the EFIGE dataset; (vi) profitability, proxied by the Return on Sales (ROS).

6.5 Empirical analysis

In the following analysis, Hyp. (1) and (2) will be tested in two sequential steps. In the first step, the role of the potential market growth will be investigated in order to explore its influence on the international activity of the firm by splitting the sample by ownership. In the second step, we will check whether a replication strategy is adopted by family firms. Finally, we will include into our framework a test of the Hennart et al's (2019) and Eddleston et al's (2019) results concerning i) the role of niche vs. mass markets and ii) the relevance of a replication strategy.

6.5.1 Methodology and estimation strategy I

To test Eq.(6.8) and investigate the role of the lock-in and market potential in the pricing decisions, we test the following Eq.(6.47):

$$p_{i,h,c} = \alpha_0 + \beta_1\,\mathrm{SC}_{h,c} + \beta_2\,k_{h,c} + \beta_3\big(k_{h,c} * \mathrm{SC}_{h,c}\big) + \beta_4\,\mathrm{PG}_h + \beta_5 X_i + \varepsilon_i \quad (6.47)$$

where the dependent variable $p_{i,h,c}$ represents price for firm i, sector h (Toys and games; professional coffee machines; household appliances; mechanotherapy and electro-physical-therapy devices) in different export destination countries c (see the list in Appendix A - Table A.1). $k_{h,c} = \big(N_1/N^*\big)_{h,c}$ is a measure of the export destination market potential development, and it is computed using the procedure explained in Section 2. $SC_{h,c}$ is an estimate (Shy, 2002) of the average switching cost by destination markets for each of the four sectors included in the analysis.

As for controls, past-growth variable, denoted by PG_h, is the growth pattern observed in the last 10 years, X_i is a set of firm specific covariates that include: i) firm age, measured by the number of years from its inception; ii) four dummies for firm size (employees); (iii) a financial liquidity indicator, measured as current assets over current liabilities and (iv) the firm profitability proxied by the Return on Assets (ROA) adjusted by the median sectoral ROA. ε_i are the error terms.

The relationship between firms' pricing policy, the market development effect and the lock-in effects has been examined in the simple case of exporters that compete in a destination (export) market. Following Armington (1969), all the firms in country i are treated as a single firm (country i exporters) that compete in the world market with all foreign "country-firms". This is possible by assuming that all products within an industry are differentiated according to the country of production, i.e. goods produced inside a source country are perfect substitute with each other, but imperfect substitute with the same products produced in another country. This simplified framework allows using detailed trade data (8-digit product export data) to estimate the relationship using export unit values as average prices by destination country. The same data source permits to compute country market shares using country exports in each destination market and the market development potential index.

In presence of of both "market development" and "lock-in" effects, the equilibrium price $p^F_{1,*}$ is lower than the conventional optimal price $p^F_{1,opt}$, but strictly higher than the "locked-in" price $p^F_{1,Kp}$. Moreover, the equilibrium price is expected to be closer to the conventional optimal price, if the "market development" effect counterbalances the "lock-in" effect. We test this intuition by interacting the development and the "lock-in" effect in the estimation model. Our variable of interest is the interaction between growth potential and the lock-in variable, i.e. $k_{h,c} * SC_{h,c}$. In particular, we expect a positive coefficient of the interaction dummy, as the larger the potential market (far from it's saturation level), the more the price increases and moves towards to the optimal price, indicating that the market development effect dampens the lock-in effect. In fact, we analyse the "locked-in" effect on price by calculating the partial derivative of $SC_{h,c}$ with respect to

$p_{i,h,c}$, defined in Eq.(6.47):

$$\frac{\partial p_{i,h,c}}{\partial \mathrm{SC}_{h,c}} = \beta_1 + \beta_3\, k_{h,c} \tag{6.48}$$

Since $\beta_2 < 0$, i.e. the locked-in effect has negative effect on price (Klemperer 1987) and the price ratio $k_{h,c}$ is positive, we expect $\beta_3 > 0$, indicating to the market development effect dampens the lock-in effect. Therefore, a balancing effect of market development on lock-in pricing is expected in a cross section analysis where prices are regressed on proxies for market development and switching cost.

The estimation of Eq.(6.8) using OLS has the problem of not considering the endogeneity of switching cost, giving rise to biased and inconsistent estimates. We use a two-stage least square (2SLS) approach, where the impact of switching costs on prices are estimated using the predicted values, instead of the actual values. Predicted values are estimated at the first stage of the model by regressing actual switching costs on the index of market development and other controls. The use of predicted values, which are correlated with actual values and uncorrelated with the error term, helps in mitigating the endogeneity problem in Eq.(6.47).

6.5.2 Does the market potential offset the swiching costs effect?

Table 1 summarizes the estimated results of Eq.(6.47). Firm size positively impacts on pricing in two out of four sectors, whereas firm age and past sectoral growth have weak explanatory power. Liquidity helps firms to support lower prices, whereas the positive impact of profitability may suggest some market power. For comparison, we did two tests. In Test 1 - without market development, the coefficients of the switching cost variables are all negative and significant for all sectors, thus supporting the hypothesis that markets with higher consumer lock-in also experience lower prices. To find out the role of market development, we did Test 2 by including a proxy for market development and its interaction with switching costs. A compensating/offsetting hypothesis would be supported if the interaction variable showed a positive and significant coefficient. Results from estimated show that the hypothesis cannot be rejected in all sectors. It is worth to remark that in Test 2, we obtain negative and significant coefficients for the sectors of Toys and Coffee machines, but not significant coefficients for the sectors of Household application and Therapy devices. The reason is the

Table 6.0 – *Table 1 – Model estimates. Dept variable: Price (export unit values) in single destination markets by exporting country - EUV: Averages 2012-2015*

	Switching costs				Switching costs + Mkt potential			
	Toys and games	Coffee machines	Household appliances	Therapy devices	Toys and games	Coffee machines	Household appliances	Therapy devices
Switching costs/SC	-0.061**	-0.147***	-0.070*	-0.155*	-0.199*	-0.153**	-0.102	-0.181
	(0.020)	(0.053)	(0.042)	(0.083)	(0.0070)	(0.074)	(0.052)	(0.107)
2.size_class	0.022	0.022	0.043**	0.028	0.020	0.018	0.039**	0.026
	(0.021)	(0.024)	(0.019)	(0.024)	(0.021)	(0.024)	(0.019)	(0.024)
3.size_class	0.033	0.028	0.116**	0.035	0.023	-0.017	0.103**	0.025
	(0.040)	(0.044)	(0.037)	(0.044)	(0.040)	(0.044)	(0.036)	(0.044)
4.size_class	0.122**	0.092	0.283**	0.000	0.111	0.093	0.274**	-0.002
	(0.073)	(0.073)	(0.068)	(0.076)	(0.072)	(0.074)	(0.068)	(0.078)
Firm age	-0.016	-0.005	0.006	-0.014	-0.017	0.004	0.004	-0.013
	(0.014)	(0.016)	(0.013)	(0.016)	(0.014)	(0.016)	(0.013)	(0.016)
Past growth	0.052	0.107	0.091	0.045	0.056	0.107*	0.097	0.048
	(0.069)	(0.066)	(0.070)	(0.071)	(0.069)	(0.064)	(0.069)	(0.071)
Liquidity ratio	-0.019**	-0.021**	-0.017**	-0.014	-0.017*	-0.020*	-0.014	-0.012
	(0.009)	(0.011)	(0.009)	(0.011)	(0.009)	(0.010)	(0.009)	(0.011)
Adjusted-ROS	0.386**	0.224	0.071	0.362*	0.395**	0.236	0.087	0.374**
	(0.173)	(0.187)	(0.151)	(0.187)	(0.170)	(0.183)	(0.149)	(0.185)
Mkt potential					0.052	-0.017	-0.109*	-0.035
					(0.064)	(0.064)	(0.060)	(0.067)
Mkt potential ∗ SC					0.060**	0.133***	0.055*	0.063*
					(0.023)	(0.052)	(0.028)	(0.034)
_cons	0.387***	0.715***	0.251***	0.464***	1.669***	2.032***	0.946***	1.894***
	(0.092)	(0.105)	(0.090)	(0.109)	(0.259)	(0.245)	(0.229)	(0.269)
N	2,2683	2,2683	2,2682	2,2683	2,2673	2,2673	2,2672	2,2673
R^2	0.19	0.15	0.13	0.17	0.24	0.31	0.13	0.28

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

fact that, in Test 2, the coefficient associated with Switching Costs indicates the net "locked-in" effect counteracted by the "market development" effect, i.e. $\beta_1 + \beta_3\, k_{h,c}$ (instead of β_1) defined in Eq.(6.48). Therefore, we find the "market development" effect completely offsets the "locked-in" effect in the sectors of Household application and Therapy devices. In general, results support the idea that, though switching costs are a key driver of the pricing policy, their influence varies with market life-cycle, as it declines with the potential and saturation level of the market.

6.5.3 Does the market potential influence the probability of exporting of family firms?

To investigate the role of the market potential growth in the decision to enter a new market, we test whether family firms (and firms appointing a family member as CEO) are more (less) likely to enter a new high potential market than a non-family counterpart (Hpy. 1). We estimate the following probability model:

$$\begin{aligned} Pr(NewExport_{i,h,c}) = \alpha(PotentialMarketGrowth_{h,c}) + \beta(PastGrowth_{h,c}) \\ +\gamma X_i + \delta Sector + \varepsilon_i \end{aligned} \tag{6.49}$$

where $NewExport_{i,h,c}$ represents the observed dependent variable (firm i, sector h and country c) and is a dummy variable equal to one if firm i entered a new market (sector h and country c) in 2008 or to zero otherwise. $PotentialMarketGrowth_{h,c}$ is the estimated market potential for sector h and country c. $PastGrowth_{h,c}$ is the growth pattern observed in the past years. X_i is a set of exogenous covariates. Sector include sectoral 3-digit NACE dummies. ε_i are the error terms, which are assumed to be i.i.d. Model (1) is estimated by splitting the total sample in three subsamples, i.e. non-family firms, family firms run by family CEOs and family firms run by external CEOs. In all models, the coefficient of interest is α, as it represents the influence of the potential market growth on the decision taken by the company to enter a selected market. Therefore, in line with Hyp.(1), we expect α>0 with a larger value for family firms and firms run by family CEOs.

6.5.4 Results

Table 4 summarizes the estimated results of Eq.(6.49). Firm size negatively impacts on the probability of entering a new market in the case of non-family firms, whereas the contribution is positive when family CEOs and external CEOS in family firms are considered. Older firms are more oriented toward the domestic market, and high profitability and liquidity prevent firms to look for new destination markets. Finally, innovation helps to enter new markets except in the case of non-family firms.

Table 6.0 – *Table 4 - The influence of market potential on the decision to export by governance types*

variables	**Non-family firms** (1)	**Fam firms & fam CEO** (2)	**Fam firms & ext CEO** (3)	**Fam firms & ext CEO** (4)
Market potential(5 years)	0.035*	0.042***	0.521***	0.296***
	(0.020)	(0.004)	(0.046)	(0.038)
Employees (log)	-0.021*	0.002*	0.025*	0.026***
	(0.011)	(0.001)	(0.011)	(0.008)
Age (log)	-0.012***	-0.029***	-0.026***	-0.028***
	(0.003)	(0.002)	(0.008)	(0.003)
Past growth	0.002*	0.004**	-0.005**	0.001**
	(0.001)	(0.002)	(0.002)	(0.000)
Liquidity ratio(%)	-0.002*	-0.002	-0.004	-0.007*
	(0.001)	(0.003)	(0.003)	(0.004)
ROS(%)	-0.047	-0.019**	-0.083**	-0.101***
	(0.041)	(0.007)	(0.025)	(0.003)
TFP growth	0.050***	0.091***	0.013**	0.014**
	(0.010)	(0.003)	(0.005)	(0.006)
Innovative products(%)	-0.003**	0.009***	0.004***	0.004***
	(0.001)	(0.000)	(0.001)	(0.001)
Family CEO				-0.0047
				(0.004)
Market potential*family CEO				-0.254***
				(0.039)
Intercept	0.161***	0.217***	-0.130***	-0.117***
	(0.020)	(0.014)	(0.031)	(0.050)
Firms	6750	6750	6750	6750
R-squared	0.151	0.227	0.465	0.478
F	10.49	10.35	46.57	41.40

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

When it comes to the market potential growth, the 5-year expected market growth significantly impacts the decision to enter a new market in all the sub-groups included in the analysis. The evidence is stronger for family firms, and even more significant for external CEOs in the group of family firms. The implication of these results is that the perspective evolution of market potential is a crucial variable that drives the decision to enter a new market in all types of firms, and especially in the case of family owned- and managed firms. As market potential provides an estimate of the expected growth profile after 2008, i.e. it is proxy for the information set available at the time the decision has to be taken, the evidence in Table 4 supports our intuition that family owned and managed firms have a preference for markets with a higher expected growth. This occurs when the decision to enter the market takes into account the stock of knowledge available up to the period before the decision.

Differently from models in columns (1) to (3), model in column (4) tests the role of family CEOs compared to external CEOs by using homogeneous controls across groups. So, while family CEOs and external CEOs do not differ statistically in the decision to enter a new market, market potential has a slightly lower influence in firms run by family CEOs than non-family ones (the interaction variable is negative and significant).

To recap, family owned- and managed- firms are affected by the potential growth of a destination market more than non-family firms. Their preference for growing markets is substantial even when a full set of factors potentially affecting the decision to go abroad are taken into account. Therefore, in accordance to Hypothesis 1, family firms seem to be more prone to invest in growing markets than non-family firms. Furthermore, the sensitivity is larger for external CEOs than family CEOs: therefore, family firms with external CEOs offer the best combination of ownership features, on the one hand, and professional abilities on the other hand, to fully exploit all the competitive advantages of growing markets.

This evidence is backed by some additional motivations. As a general consideration, literature has shown that hiring non-family managers has been identified as a common factor of success for family SMEs that invest overseas (Crick et al., 2006), whereas the lack of involvement of external expertise, such as non-family managers, has been recognized as one of the main limiting factors of family firms internationalization (Gallo and Sveen, 1991; Graves and Thomas, 2006; Kontinen and Ojala, 2010b). Therefore, the presence of an external manager in a family firm may counterbalance the "family effect" on internationalization decisions, which can be strongly affected by risk aversion, control motivations and lack of adequate expertise. Non-family manager might be more neutral decision-makers than family members. This does not mean to assume that family members managing a company are not as good as professional managers (Dekker et al., 2015), but that employing and delegating decision-making power to non-family managers may blur constraints related to family-based human asset specificity (Verbeke and Kano, 2012). Moreover, the presence of a non-family manager is a signal that the family firm has overcome the reluctance to allow non-family members to hold strategic positions and to share the decision-making process with them (Claver et al., 2009).

These findings provide evidence on the relevance of professional abilities in the decision to enter foreign markets. How to reconcile this evidence with the remarkable preference of family CEOs for external markets that we have perceived from the descriptive analysis? To find an answer, we have to look at the process of international expansion by considering the choice between different approaches when family CEOs decide to enter new foreign markets.

6.5.5 Replication strategy in foreign markets

Because of the typical features of family ownership, firms owned and managed by family members may reveal a significant interest towards growing markets, on the one hand, together with a supposed negative attitude towards risky investments on the other hand. This set of conditions may affect the product strategy to enter a new market. To investigate the role of the existing product portfolio in the decision to enter a foreign market, we exploit the following question from the EU-EFIGE survey:
(D16) The main product line you sell to foreign markets: 1. is also the main product line in your domestic market; 2. is also sold in your domestic market but it is not the main product line; 3. is not sold in your domestic market.

As we consider the sub-set of firms that decided to enter a new market for the very first time in 2008, question D16 can be used to identify three different – and non-overlapping – "strategic" approach to foreign markets. We recode answer n.1 as a "replication strategy" ("firms are entering a new market using an existing product"); answer n.2 as an "exploitation strategy" ("firms are using a product already sold in the domestic market, but not the most important one"); answer n.3 as a "new product strategy" ("a firm entering a new foreign market using a product not sold in the domestic market").

Table 5 summarizes the estimated results for the sub-samples of companies by ownership split into three groups according the type of product strategy. Non-family firms reveal a strong preference for a product strategy that involves the exploitation of (non-core) products already sold in domestic markets as main drivers to market entry. The coefficient is positive and significant, whereas the coefficient estimated for the replication strategy is negative and significant. Similar evidence comes out also for family firms run by external CEOs, as the use (exploitation) of existing products to enter new markets is substantial. However, it is worth noting that the coefficient for the new product approach is positive, albeit not significant, thus supporting the intuition that non-family CEOs do rely also on new products when decide to enter a foreign market.

Table 6.0 – *Table 5 - Product strategy in export markets by type of governance*

	Replication			New products			Exploitation		
	Non-family firms	Family CEOs	External CEOs	Non-family firms	Family CEOs	External CEOs	Non-family firms	Family CEOs	External CEOs
Variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Market potential growth	0.024*	0.081***	-0.007***	-1.101	-0.226***	0.013	0.118**	-0.046	0.171***
	(0.011)	(0.016)	(0.055)	(0.138)	(0.018)	(0.040)	(0.042)	(0.037)	(0.022)
Employees (log)	-0.016*	0.004	0.012	0.020*	-0.008*	0.006*	0.017*	0.010	0.080*
	(0.010)	(0.003)	(0.017)	(0.010)	(0.004)	(0.003)	(0.009)	(0.016)	(0.035)
Age (log)	-0.051***	-0.044***	-0.056***	-0.058***	-0.028***	-0.033***	-0.050***	-0.033***	-0.064***
	(0.007)	(0.009)	(0.011)	(0.013)	(0.005)	(0.006)	(0.011)	(0.07)	(0.012)
Past growth	0.087*	0.054**	0.055**	0.016**	0.012*	0.014**	0.035***	0.019***	0.031**
	(0.034)	(0.022)	(0.019)	(0.006)	(0.005)	(0.005)	(0.012)	(0.006)	(0.014)
Liquidity ratio(%)	-0.006**	-0.003	-0.005	-0.007	-0.002	-0.001	-0.005	-0.008	-0.007*
	(0.003)	(0.003)	(0.00)	(0.006)	(0.003)	(0.004)	(0.003)	(0.006)	(0.003)
ROS(%)	-0.055	-0.024**	-0.033***	-0.157***	-0.078	-0.089*	-0.061*	-0.111***	-0.094**
	(0.040)	(0.008)	(0.009)	(0.043)	(0.054)	(0.037)	(0.035)	(0.023)	(0.043)
TFP growth	0.013***	0.0140**	0.009***	0.011***	0.020*	0.011**	0.012***	0.014***	0.008**
	(0.002)	(0.007)	(0.003)	(0.002)	(0.010)	(0.005)	(0.003)	(0.004)	(0.004)
Innovative products(%)	-0.064**	0.039***	0.044***	0.024***	0.016**	0.019***	0.044***	0.049***	0.037***
	(0.021)	(0.009)	(0.007)	(0.003)	(0.007)	(0.000)	(0.012)	(0.008)	(0.010)
Intercept	0.116***	0.211***	-0.134***	-0.171***	0.144**	0.211***	-0.180***	-0.127***	-0.100***
	(0.010)	(0.021)	(0.019)	(0.031)	(0.060)	(0.004)	(0.021)	(0.033)	(0.020)
Firms	6750	6750	6750	6750	6750	6750	6750	6750	6750
R-squared	0.151	0.227	0.465	0.478	0.151	0.227	0.465	0.478	0.478
F	10.49	10.35	46.57	41.40	10.49	10.35	46.57	41.40	41.40

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

The evidence for family CEOs points towards a different direction, as the adoption of a replication approach appears to be the most likely strategy, together with a significant negative role of the introduction of new products. Therefore, firms run by family CEOs seem to use their principal domestic product to enter new foreign markets, and follow a replication approach targeted to specific market niches where the company is likely to have relevant technical and commercial competencies (Hyp. 2). Contrarily, non-family firms and firms run by an external CEO mainly adopt an exploitation strategy, based on non-core products sold in new fast-growing markets. Besides, external CEOs are also prone to use product innovation to enter new markets.

6.5.6 Testing Hennart et al.'s (2019) and Eddleston et al.'s (2019) results in the framework

The basic framework we have developed so far shows that exporters have a significant benefit from the growing stage of the destination market and that they also exploit replication to support entry in export markets. In this section we provide a test of the Hennart et al.'s (2019) and Eddleston et al.'s (2019) hypotheses in our framework.

As far as the niche v.s mass approach is concerned, we followed Hennart et al.'s (2019) methodology for the definition of niche markets and used the same data on the quality of products from the EFIGE questionnaire to rank the quality of products between 0 and 100. As we use the same dataset, we can borrow some results from the Hennart et al.'s (2019) analysis to support our intuition. In particular, they show that family firms are less efficient than non-family firms (in terms of labour productivity) and relatively more frequent in niche markets, at least for countries like Germany and Italy. Results in Table 6 extend these results to take into account the life-cycle of the destination market. In mature markets, the share of FF in niche markets is larger than non-family firms. By contrast, efficiency (TFP growth) has a negative drift for family firms, whereas it is positive for non-family firms, especially in niche markets. Also, the competitive structure of the export markets looks different when firms' individual characteristics are considered: family firms are smaller that non-family firms especially in mature markets, where size-related efficiency can play a crucial role in supporting performance.

Table 6.0 – *Table 6 - Structural characteristics of export markets by ownership, lifecycle and saturation level*

	Non-family firms			Family firms		
Share of firms	Mass	Niche	Total	Mass	Niche	Total
Growth markets	0.774	0.226	1.000	0.724	0.276	1.000
Mature markets	0.755	0.245	1.000	0.709	0.291	1.000
Total	0.758	0.242	1.000	0.711	0.289	1.000
TFP growth						
Growth markets	-0.045	0.197	0.002	-0.123	-0.153	-0.130
Mature markets	0.063	0.106	0.072	-0.029	-0.003	-0.023
Total	0.039	0.125	0.057	-0.049	-0.034	-0.046
Firm average size						
Growth markets	43,881	59,048	46,707	21,262	44,350	26,781
Mature markets	24,417	53,100	30,451	9,428	14,544	10,703
Total	28,437	54,188	33,730	11,737	20,104	13,807

Table 7 presents the estimated results of the basic model presented in Section 5.1, extended with a variable indicating the mass market (Mass market) and its interaction with market potential. Instead of the share of firms serving niche markets for the sub-set of four countries considered by Hennart et al.'s (2019), in our framework the Mass market variable measures the prevalence of mass exporters in a specific destination market in 2008. Therefore, it identifies each destination market as either a niche market or a mass market. Regressions in Table 7, Models (1) to (3), show that the direct effect of the mass market on the probability to start exporting is negative for all types of ownership and significant only in the case of family firms. However, when interacted with the variable summarizing market potential (Models 4 to 6), the influence of the Mass market turns positive, as the sum of the direct and interaction variables is always greater than zero, and highly significant for family firms. For example, if the influence of Mass market is negative for family firms run by family members (b=-0.031** in Model (2), it turns positive when the role of market potential is considered (-0.021 +2.006-1.185 = 0.8 in Model (5) when the market potential growth is considered at its mean value). Therefore, market potential plays a significant role in mass markets as it reverses their negative influence on the propensity to enter a new market into a positive one. This evidence thus supports the intuition that the negative view of family firms on mass markets can be significantly offset by the market potential growth.

When it comes to the role of replication in family-owned and -run firms, Models (7) to (12) present basic results of the test of the replication as a substitute for inferior managerial abilities hypothesis in our framework. Overall, we observe that estimated results are highly significant only for family firms run by family members, whereas tests run on other types or governance are not significant. Therefore, a replication appears only relevant for family-run firms: Model (8) shows that market potential is not significant when a new-product strategy is adopted, i.e. Replication is set to zero, but very significant when a replication strategy is at play. In this case, the total coefficient (direct effect plus interaction) turns positive when the average market potential growth is higher than a 5.4% threshold on a yearly basis (7.1% when only Mass markets are considered in Model (11).

Table 8 summarizes the average 5-year market growth computed for different ownership types, split by niche and mass market. The evidence for family firms run by family CEOs shows potential growth rates are substantially higher than the thresholds indicated above, in almost all internationalisation strategies. In sum, when coupled with a replication strategy,

TABLE 6.0 – *Table 7 - Test of the Hennart et al.'s (2019) and Eddleston et al.'s (2019) hypotheses*

	Non-family firms	Family firm & family CEOs	Family firm & ext CEOs	Non-family firms	Family firm & family CEOs	Family firm & ext CEOs	Non-family firms	Family firm & family CEOs	Family firm & ext CEOs	Non-family firms	Family firm & family CEOs	Family firm & ext CEOs
								Full sample			Mass markets only	
Variables	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
Mass market	-0.047	-0.031*	-0.138**	-0.060	-0.021	0.020						
	(0.033)	(0.016)	(0.070)	(0.086)	(0.065)	(0.210)						
Market potential (5 years)				-0.100	2.006***	4.908***	-0.812	-1.389	11.427	-1.548	-7.698**	10.525
				(0.607)	(0.393)	(1.326)	(2.362)	(1.760)	(9.399)	(2.698)	(3.073)	(9.080)
Market potential & mass market				0.260	-1.185***	-2.578*						
				(0.710)	(0.418)	(1.566)						
Replication							-0.256	-0.504**	1.823	-0.512	-0.899***	1.718
							(0.305)	(0.203)	(1.794)	(0.336)	(0.226)	(1.755)
Market potential & replication							0.315	3.331*	-10.249	1.152	9.157***	-12.262
							(2.641)	(1.887)	(9.628)	(2.941)	(3.075)	(9.648)
Intercept	-1.778***	-1.883***	-0.144***	-1.700***	-1.918***	-2.366***	-1.467***	-1.517***	-3.879**	-1.277***	-1.106***	-3.635**
	(0.071)	(0.048)	(0.016)	(0.183)	(0.131)	(0.426)	(0.283)	(0.214)	(1.774)	(0.304)	(0.306)	(1.717)
Firms	4394	9174	1191	2400	4479	669	1931	3557	546	1442	2563	379
Pseudo R-squared	0.12	0.10	0.12	0.12	0.25	0.23	0.14	0.28	0.22	0.31	0.31	0.23
LR chi2	10.3	10.2	10.8	13.4	18.7	14.9	22.6	19.6	22.8	25.8	29.2	22.1

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

the potential market growth helps family firms to venture abroad as in the pro-market development external context indicated by Eddleston et al. (2019).

TABLE 6.0 – *Table 8 - Structural characteristics of export markets growth potential by ownership, type of market and export strategy*

	Non-family firms		Family firm&family CEOs		Family firm&external CEOs		Total sample	
	New products	Replication	New products	Replication	New products	Replication	New products	Replication
Niche market annual potential growth in 5 years(# of firms)	12.3	14.4	23.2	18.5	13.8	12.1	18.4	14.5
	(56)	(433)	(114)	(880)	(17)	(150)	(187)	(1463)
Mass market annual potential growth in 5 years(# of firms)	11.9	13.3	13.8	23.2	12.3	13.3	12.2	19.4
	(159)	(1283)	(240)	(2323)	(47)	(332)	(446)	(3938)

Data in Table 8 are median values of the (5 years) potential growth of niche and mass destination markets respectively, by ownership. Number of firms in parentheses.

Table 1 summarizes the estimated results of Eq.(6.47). Firm size positively impacts on pricing in two out of four sectors, whereas firm age and past sectoral growth have weak explanatory power. Liquidity helps firms to support lower prices, whereas the positive impact of profitability may suggest some market power. For comparison, we did two tests. In Test 1 - without market development, the coefficients of the switching cost variables are all negative and significant for all sectors, thus supporting the hypothesis that markets with higher consumer lock-in also experience lower prices. To find out the role of market development, we did Test 2 by including a proxy for market development and its interaction with switching costs. A compensating/offsetting hypothesis would be supported if the interaction variable showed a positive and significant coefficient. Results from estimated show that the hypothesis cannot be rejected in all sectors. It is worth to remark that in Test 2, we obtain negative and significant coefficients for the sectors of Toys and Coffee machines, but not significant coefficients for the sectors of Household application and Therapy devices. The reason is the fact that, in Test 2, the coefficient associated with Switching Costs indicates the net "locked-in" effect counteracted by the "market development" effect, i.e. $\beta_1 + \beta_3\, k_{h,c}$ (instead of β_1) defined in Eq.(6.48). Therefore, we find the "market development" effect completely offsets the "locked-in" effect in the sectors of Household application and Therapy devices. In general, results support the idea that, though switching costs are a key driver of the pricing policy, their influence varies with market life-cycle, as it declines with the potential and saturation level of the market.

6.6 Conclusion

This article investigates the firms' pricing strategy in considering both the switching costs and potential market growth within a two-period model, where the uncertainty of future profit comes from the unobservable future market occupation. Our model extends Klemperer's two-period switching costs model to the uncertainty scenario where the 2nd period profit is not perfectly predictable. According to a sector life-cycle perspective, markets can be divided into different types on the basis of their stage of development. From a strategy point of view, different development stages of the market suggest dissimilar strategic approach to international activity. Mature markets usually imply price competition, which is mainly sustained by production efficiency and effective management practices. The intense competitive pressure inherent to mature markets can turn up to be a substantial threat for family firms when they are locked into irreversible investments and held up in idiosyncratic, non-diversifiable market risk. Conversely, new and growing markets can benefit from a long-term involvement, reputation and, in general, entrepreneurial orientation of family firms aimed at seizing market opportunities.

First, given the switching costs, firms tend to set a "lock-in" price in order to lock more current consumers, which thus benefits the future market share and profit. Apparently, the "lock-in" price is lower than the so-called conventional optimal price that optimizes firms current profit while ignore the effect of market share on its future profit. Indeed, firms reduce their current profit to exchange for the future gains in the two-period equilibrium.

Second, the potential of market expansion motivates firms to focus their gains from future consumers rather than current consumers. With optimistic foresight of market development, the future profitability depends mainly on the market expansion rather than the locked-in consumers. Thus, firms are less interested in locking current consumers at the cost of lower current profit. This is defined as "market development" effect, which, in contrast to the "locked-in" effect, motivates firms to set a price close to the conventional optimal price. It is worth to remark that the weight of market potential in future profitably depends on the market foresight under uncertainty.

In case both effects are present, the equilibrium price is lower than the conventional optimal price, but higher than the "locked-in" price from the Klemperer model where only the switching costs are presented. Moreover, if the future market is of great potential, the "market development" effect could completely offsets the "locked-in" effect. Thus the equilibrium price is identical to the conventional optimal price, indicating firms do not need to compromise their current profit by setting a lower price to lock consumers. We provide sufficient conditions for these scenarios, including expected profit function, market foresight and the current market elasticity with respect to market foresight and future profit respectively.

These differences fit into a wider framework in which strategic decisions concerning international expansion are strongly dependent on the combination of both firm characteristics and market attributes. Specifically, firms wishing to enter a new market have to carefully evaluate the stage/phase of development of the market and wisely match market attributes with traits of the company's governance structure. If potential entrants overlook the point, they run the risk of delaying the benefits from the entry process, or even failing the venture. In addition to stress the market structure/ firm governance connection, the paper also suggests to carefully assess the adoption of well-established approaches to foreign markets, such as the preference for geographically or culturally-close markets or the importance to hire experienced professional managers, and coordinate them with the recommendations drawn from the competitive dynamics of the destination market.

The empirical analysis has been carried out using data from four European sectors from 2012 to 2016. The dataset includes statistics on the trade in goods between the Member States (intra-EU trade) and between Member States and non-EU countries (extra-EU trade). We tested the model by including a proxy for market development as an interaction variable. Estimated results show that the hypothesis that potential market development counterbalances switching costs cannot be rejected in all sectors, with stronger evidence for Toys and Coffee machines. In general, the results support the idea that switching costs are a key driver of the pricing policy, but their influence varies with market life-cycle, as it declines with the saturation level of the market.

Overall, the paper provides evidence that decisions about international markets taken by family firms and family CEOs are highly influenced by the stage of development of the destination markets. High potential destination markets provide a better stage for family firms wishing to expand abroad and exploit their proprietary advantage in growing markets. In these markets, entrepreneurial orientation and flexibility can play a crucial role, whereas lower level of efficiency can be offset by higher potential market development. However, the significant sunk costs of entry involved by these international ventures can be a factor that prevents family owners to put the survival of the company at risk. As a consequence, firms run by family CEOs may be willing to exploit the benefit of a replication strategy that permits them to leverage the strength of core domestic products into new markets with high potential development. We believe that, in addition to the specific implications from modelling, this paper extends Hennart et al.'s (2019) and Eddelston et al.'s (2019) research by taking into consideration of heterogeneity in life-cycle market. Specifically, we demonstrate that family firms, no matter selling niche or mass products, are encouraged to internationalise, given satisfactory market growth potential – which operates as an external context (Eddelston et al. (2019) – and adopt a replication strategy as a substitute for professionalization practices when the ability of family CEOs may hinder the chance for family firms to venture abroad.

Appendix A: Proof of Proposition 1

[1] Mature market (k=1 is a constant)

In this case, Eq.(6.33) will be reduced to Eq.(6.50), as the market development effect k is absent.

$$0 = \frac{\partial \pi_1}{\partial p_1^F} + \beta \frac{\partial \pi_2^F(\sigma_1^F)}{\partial \sigma_1^F} \frac{\partial \sigma_1^F}{\partial p_1} \tag{6.50}$$

Provided the firm's first-period market share decreases with its first-period price, $\partial \sigma_1^F / \partial p_1^F < 0$, and the firm's second-period profits are increasing in its first-period market share, $\partial \pi_2^F / \partial \sigma_1^F > 0$, then the equilibrium price p_1^L is lower than the price p_1^H at which $\partial p_1^F / \partial p_1^F = 0$. That is, price is lower than that if the firm ignored the effect of market share on its second-period profits (Klemperer, 1987).

[2] Growing market (k is a variable and $k \in (0, 1)$)

Let us consider the market with some growth potential, implying larger future gains, i.e. $\partial pi_2^F / \partial k \leq 0$. Let us also consider an oligopoly market, where each firm's market share is proportional to the overall market size, i.e. $\partial N_1 / \partial \sigma_1^F > 0$. Thus, we apply chain rule to calculate the derivative of the composite function $k(N_1(\sigma_1^F)$ as follows:

$$\frac{\partial k}{\partial \sigma_1^F} = \frac{\partial k}{\partial N_1} \frac{\partial N_1}{\partial \sigma_1^F} = \frac{1}{N^*} \frac{\partial N_1}{\partial \sigma_1^F} < 0 \tag{6.51}$$

Substituting Eq.(6.50) into Eq.(6.51), we have:

$$0 = \frac{\partial \pi_1^F}{\partial p_1} + \beta (\frac{\partial \pi_2^F}{\partial \sigma_1^F} + \frac{\partial \pi_2^F}{\partial k} \frac{\partial k}{\partial \sigma_1^F}) \frac{\partial \sigma_1^F}{\partial p_1} \tag{6.52}$$

with

$$0 \leq \frac{\partial \pi_2^F}{\partial \sigma_1^F} + \frac{\partial \pi_2^F}{\partial k} \frac{\partial k}{\partial \sigma_1^F} \leq \frac{\partial \pi_2^F}{\partial \sigma_1^F} \tag{6.53}$$

Substituting the result from Eq.(6.53) into Eq.(6.52) and comparing with Eq.(6.50) , we find that the equilibrium price p_1^* in a growing market is higher than the price in the mature market, i.e.

$$p_1^L \leq p_1^* \leq p_1^H \tag{6.54}$$

This means that the growth potential of a market offsets the need for entrants to invest in market share by lowering prices. The higher the potential market growth, the lower the necessity for entrants to start a price competition.

Now, under the hypothesis that both FF and NF adopt a scale intensive technology, the late entry of FF in the market may keep their costs above the threshold set by competitive NF and prevent entry. However, this price limit strategy can be offset by the potential growth of the market, which makes the investment in market share through lower prices less relevant for future profitability. In the model, we assume non-family firms are price makers and have a lower production cost than family firms because of the scale effects, i.e. $c^N F < c^F F$, where upper index FF and NF indicates family firm and non-family firm, respectively. In specific, this model works well for family firms selling mass products, which can be modelled correctly through a competitive price setting framework. Without loss of generality, let us consider the case where $c^N F < p_1^L < c^F F < p_1^* < p_1^H$, and firms decide to enter a market as long as it is profitable. We have the following occurrences.

Scenario [1]: in a mature market, the fierce competition among non-family firms reduces the market price to p_1^L, and the profits for family firms and non-family firms are respectively:

$$\pi_1^{FF} = (p_1^L - c^{FF})\sigma_1^{FF} < 0$$
$$\pi_1^{NF} = (p_1^L - c^{NF})\sigma_1^{NF} < 0$$

Only non-family firms are profitable in a mature market, whereas the family firms selling mass products will not enter the market.

Scenario [2]: in a growing market, the equilibrium price goes up to p_1^*, and the profits for both family firm and non-family firms are respectively:

$$\pi_1^{FF} = (p_1^* - c^{FF})\sigma_1^{FF} > 0$$
$$\pi_1^{NF} = (p_1^* - c^{NF})\sigma_1^{NF} > 0$$

Thus, family firms selling mass products will enter a growing market as long as the potential market growth offsets the low-price strategy and allows a higher price p_1^* that is above the cost of FF.

Scenario [3]: the demand for niche is small but inelastic, i.e. $\partial\sigma_1^F/\partial p_1^F = 0$ in Eq.(6.50) and Eq.(6.52), which indicates family firms selling niche products can always set a monopolistic price $p_1^{niche} = p_1^H$. The profit is:

$$\pi_1^{FF} = (p_1^H - c^{FF})\sigma_1^{FF} = 0$$

Therefore, family firms selling niche products are always better-off when enter the market, regardless of the market structure.

Appendix B: Tables

Table 6.0 – *Table B.1 - Destination countries/regions for exports by sector*

Games and Toys	**Professional Coffee-maker machines**	**Household appliances**	**Mechano, electro and physical therapy devices**
Spain	USA	Poland	USA
Belgium	China	Russia	Germany
France	Germany	Turkey	Canada
Germany	South Korea	Czeck Rep.	France
UK	Japan	Romania	UK
Poland	France	Hungary	Russia
Portugal	Netherlands	Slovenia	Australia
Turkey	UAE	Slovakia	China
	Australia	Bulgaria	Spain
	Russia	Ukraine	India
	Norway	Lithuania	Poland
	Belgium	Latvia	Mexico
	Austria	Albania	Turkey
	Switzerland		Brazil
	Spain		Romania
	Taiwan		Chile
	Sweden		Kazakhstan
	Thailand		Morocco
	Denmark		Azerbaijan

Table 6.0 – *Table B.2 - Combined nomenclature(NC) 8 digit codes for sectors included in the sample*

Games and Toys	**Professional Coffee-maker machines**	**Household appliances**	**Mechano, electro and physical therapy devices**
9603.30.10	8419.81.20	8418.21.10	9018.90.10
9609.90.10	8419.90.85	8418.21.51	9018.90.20
9609.20.00	8438.80.10	8418.21.59	9018.90.30
9503.00.10	8516.71.00	8418.21.91	9018.90.40
9503.00.30		8418.21.99	9018.90.50
9503.00.55		8418.22.00	9018.90.60
9503.00.35		8418.29.00	9018.90.75
9503.00.41		8418.30.91	9018.90.84
9503.00.69		8418.30.99	9019.10.10
9503.00.87		8418.40.91	9019.10.20
9503.00.95		8418.40.99	9019.20.00
9503.00.99		8418.50.00	
		8422.11.00	
		8422.19.00	
		8422.90.00	
		8422.90.10	
		8450.11.10	
		8450.11.11	
		8450.11.19	
		8450.11.90	
		8450.12.00	
		8450.19.00	
		8516.60.10	
		8516.60.51	
		8516.60.59	
		8516.60.70	
		8516.60.80	

Bibliography

[1] Acemoglu, D. (2009). Introduction to modern economic growth. Princeton, NJ: Princeton University Press.

[2] Altomonte, C., Aquilante, T. (2012). The Eu-Efige/Bruegel-Unicredit Dataset. Bruegel Working Paper.

[3] Aizenman, J., Glick, R. (2006), Military expenditure, threats, and growth, Journal of International Trade and Economic Development, 15, 129-155.

[4] Albala-Bertrand, J. M. (1993), Natural disaster situations and growth: A macroeconomic model for sudden disaster impacts, World Development, 21(9), 1417-1434.

[5] Amit, R., Ding Y., Villalonga, B., Zhang, H. (2015). The role of institutional development in the prevalence and performance of entrepreneur and family-controlled firms. Journal of Corporate Finance, Volume 31, 284-305.

[6] Anderson, R., Reeb, D. (2003). Founding family owner and firm performance: Evidence from the S& P 500. Journal of Finance, 58, 1302-1328.

[7] Anttila-Hughes, J. K. and Hsiang, S. M. (2013), Destruction, disinvestment, and death: Economic and human losses following environmental disaster. SSRN 2220501.

[8] Argote, L., Ingram, P. (2000). Knowledge transfer: A basis for the competitive advantage of firms. Organ. Behav. Human Decision Processes, 82(1):150-169.

[9] Arregle, J.L., Naldi, L., Nordqvist, M., Hitt, M. (2012). Internationalization of family-controlled firms: A study of the effects of external involvement in governance. ETP, November, 1115-1143.

[10] Avlonitis, G., Salavou, H. (2007). Entrepreneurial orientation of SMEs, product innovativeness and performance. Journal of Business Research, 60.

[11] Balestra, C., Dottori, D. (2012). Ageing society, health, and the environment. Journal of Population Economics, 25, 1045-1076.

[12] Banalieva, E.R., Eddleston, K. (2011). Home-region focus and performance of family firms: The role of family vs. non-family leaders. Journal of International Business Studies, 42, 1060-1072.

[13] Barba Navaretti, G., Faini, R., Tucci, A. (2008). Does family control affect trade performance? Evidence from Italian firms. CEP Discussion Paper n.896, November.

[14] Barro, R.J. (1998). Determinants of economic growth: a cross-country empirical study. The MIT Press, edition 1, volume 1.

[15] Baschieri, G., Carosi, A., Mengoli, S. (2017). Family firm local involvement and the Local Home Bias Phenomenon. Long Range Planning, Volume 50, Issue 1, 93-107.

[16] Baum, J.R., Locke, E.A. (2004). The relationship of entrepreneurial traits, skill, and motivation to subsequent venture growth. Journal of Applied Psychology, 89(4), 587-598.

[17] Benson, C. and Clay, E.J. (2004), Understanding the economic and financial impacts of natural disasters, World Bank Publications.

[18] Bertrand, M., Schoar, A. (2006). The role of family in family firms. Journal of Economic Perspectives, 20(2), 73-96.

[19] Bloom, D.E., Canning, D., Sevilla, J. (2004). The effect of health on economic growth: a production function approach. World Development 32(1), 1-13.

[20] Bloom, N., Van Reenen, J. (2007). Measuring and explaining management practices across firms and countries. Quarterly Journal of Economics, 122(4), 1351-1408.

[21] Boehe, D.M. (2016). The internationalization of service firms from emerging economies: An internalization perspective. Long Range Planning, Volume 49, Issue 5, 59-569.

[22] Boucekkine, R.,Pommeret, A., Prieur, F. (2011). Technological vs ecological switch and the environmental Kuznets curve. <halshs-00633024>

[23] Boucekkine, R.,Saglam, C., Vallee, V. (2004). The Technology Adoption Under Embodiment: A Two-Stage Optimal Control Approach. Macroeconomic Dynamics, 8, 250-271.

[24] Bréchet, T., Camacho, C., Veliov, V.M. (2011). Model predictive control, the economy, and the issue of global warming. Annals of Operations Research, Volume 220, Issue 1, 25-48.

[25] Bréchet, T., Camacho, C., Veliov, V.M. (2014). Adaptive Model-Predictive Climate Policies in a Multi-Country Setting. The Oxford Handbook of the Macroeconomics of Global Warming, Oxford University Press.

[26] Bretschger, L., Vinogradova, A. (2014). Growth and mitigation policies with uncertain climate damage. OxCarre Research Paper No.145.

[27] Bretschger, L., Vinogradova, A. (2016). Human development at risk: economic growth with pollution-induced health shocks. Environmental and Resource Economics, 1-15.

[28] Brookshire, D.S., Chang, S.E., Cochrane, H., Olson, R.A. Rose, A. and Steenson, J. (1997). Direct and Indirect Economic Losses from Earthquake Damage, Earthquake Spectra, 13(4), 683-701.

[29] Brunninge O., Nordqvist M., Wiklund J. (2007). Corporate governance and strategic change in SMEs: The effects of ownership, board composition and top management teams. Small Business Economics, 29.

[30] Buckley, P. J., Casson, M. C. (1976). The future of the multinational enterprise, London: Macmillan.

[31] Casillas, J.C., Moreno, A.M., Barbero, J.L. (2010). A configurational approach of the relationship between entrepreneurial orientation and growth of family firms. Family Business Review, 23, 27-44.

[32] Casillas, J.C., Moreno, A.M. (2010). The relationship between entrepreneurial orientation and growth: The moderating role of family involvement. Entrepreneurship & Regional Development, Vol. 22, Iss. 3-4.

[33] Chan, L. S., Chen, Y., Chen, Q., Liu, J., Dong, W. and Shah, H. (1998), "Assessment of global seismic loss based on macroeconomic indicators", *Natural Hazards*, 17, 269-283.

[34] Chang, F. (1996). Uncertainty and investment in health. Journal of Health Economics, 15, 369-376.

[35] Chen, Y., Chen, Q. and Chen, L. (2001). Vulnerability analysis in earthquake loss estimate, *Natural Hazards*, 23, 349-364.

[36] Chhibber, A., Laajaj, R. (2008). Disasters, climate change and economic development in sub-Saharan Africa: lessons and directions, Journal of African Economies, 17.

[37] Chrisman, J. J., Chua J.H., De Massis A., Frattini F., Wright M. (2015). The ability and willingness paradox in family firm innovation. Journal of Product Innovation Management, 23(3), 310-318.

[38] Claver, E., Rienda, L., Quer, D. (2007).The internationalisation process in family firms: choice of market entry strategies. Journal of General Management, 33(1), 1-14.

[39] Claver, E., Rienda, L., Quer, D. (2008). Family firms' risk perception: Empirical evidence on the internationalization process. Journal of Small Business and Enterprise Development, Vol. 15, No. 3, 457 -471.

[40] Claver, E., Rienda, L. Quer, D. (2009). Family firms' international commitment, the influence of family-related factors. Family Business Review, 22(2), 125-135.

[41] Choo, S., Mazzarol, T.W. (2001). An impact on performance of foreign market entry choices by small and medium sized enterprises. Journal of Enterprising Culture, 9, 3, 291-312.

[42] Coase, R. (1937). The nature of the firm. Economica, 4, 386-405.

[43] Cochrane, H. C. (1974). Predicting the Economic Impact of Earthquakes, Institute of Behavioural Science, University of Colorado, Working Paper, No.15.

[44] Coles, S. (2001). An Introduction to Statistical Modeling of Extreme Values, Springer, New York.

[45] Consultative Group on Indonesia (CGI, 2006). The 15th Meeting of The Consultative Group on Indonesia, Jakarta, June 14, 2006. Preliminary Damage and Loss Assessment. Yogyakarta and Central Java Natural Disaster.

[46] Crick, D., Bradshaw, R., Chaudry, S. (2006). "Successful" internationalizing UK family and non-family-owned firms: A comparative study. Journal of Small Business and Enterprise Development, 13(4), 498-512.

[47] Cropper, M. (1977). Health, investment in health, and occupational choice. Journal of Political Economy, 85, 1273-1294.

[48] Cropper, M. (1981). Measuring the benefits from reduced morbidity. American Economic Review, 71, 235-240.

[49] Cuaresma, J.C., Reitschuler, G. (2006). Gun or better? Revisited. Robustness and nonlinearity issues in the defense-growth nexus, Scottish Journal of Political Economy, 4, 523-541.

[50] Cucculelli, M., Le Breton-Miller, I., Miller, D. (2016). Product innovation, firm renewal and family governance. Journal of Family Business Strategy, 7 (2), 90-104.

[51] Cumberland, F. (2006). Theory development within international market entry mode - an assessment. The Marketing Review, Volume 6, Number 4, 349-373(25).

[52] D'Adderio, L. (2016). The replication dilemma unravelled: How organizations enact multiple goals in routine transfer. Organization Science, Vol. 25, No. 5, 1325-1350.

[53] Danneels, E. (2006). Dialogue on the effects of disruptive technology on firms and industries. Journal of Product Innovation Management, 23, 2-4.

[54] Datta, D. K., Rajagopalan, N., Rasheed, A. M. A. (1991). Diversification and performance: Critical review and future directions. Journal of Management Studies, 28(5): 529-558.

[55] Davis, L. (2008), Natural Disasters, Facts on File, New York.

[56] Deephouse, D. L., Jaskiewicz, P. (2013). Do family firms have better reputations than non-family firms? An integration of socioemotional wealth and social identity theories. Journal of Management Studies, 50: 337-360.

[57] De Hek, P.A. (1999). On endogenous growth under uncertainty. International Economic Review, 40(3), 727-744.

[58] Dekker, J., Lybaert N., Steijvers T., Depaire, B. (2015). The effect of family business professionalization as a multidimensional construct on firm performance. Journal of Small Business Management, 53(2), 516-538.

[59] Devol, R., Bedroussian, A. (2007). An Unhealthy America: The Economic Impact of Chronic Disease. Milken Institute.

[60] Diamond, P.A. (1965). National debt in a neoclassical growth model. American Economic Review, 55, 1126-1150.

[61] Dixit, A.K., Pindyck, R.S. (1994). Investment under uncertainty. Princeton University Press.

[62] Donckels, R., Fröhlich, E. (1991). Are family businesses really different? European experiences from STRATOS. Family Business Review, 4, 149-160.

[63] Dorfman, R. (1969). An economic interpretation of optimal control theory. American Economic Review, 59(5), 817-831.

[64] Dunbar, P., Bilham, R. and Laituri, M. (2002), "Earthquake loss estimation for India based on macroeconomic indicators", in Beer, T., and Ismail-Zadeh, A., eds., *Risk science and sustainability? Science for reduction of risk and sustainable development of society*, Dordrecht, Kluwer Academic Publishers, 163-180.

[65] Dunne, P. and Perlo-Freeman, S. (2003), "The Demand for Military Spending in Developing Countries", *International Review of Applied Economics*, 17, 23-48.

[66] Dunning, J.H. (1993). Trade, location of economic activity and the multinational enterprise: A search for an eclectic approach. The Theory of Transnational Corporations, 1, 183-218.

[67] Eddleston, K., Sarathy, R., Banalieva, R. E. (2019). When a high-quality niche strategy is not enough to spur family-firm internationalization: The role of external and internal contexts. Journal of International Business Studies, 50, 783-808.

[68] Ekins, P., Kesicki, F., Smith, A.Z.P. (2011). Marginal abatement cost curves: a call for caution. UCL Energy Institute. A Report to Greenpeace UK.

[69] Ewing, B.T., Kruse, J.B., Thompson, M.A. (2003), "A comparison of employment growth and stability before and after the Fort Worth tornado", Environmental Hazards, 5, 83-91.

[70] Fama, E. (1980). Agency problem and the theory of the firm. Journal of Political Economy, 88.

[71] Farrell, J., Klemperer, P. (2007). Coordination and locked-in: competition with switching costs and network effects. Handbook of Industrial Organization, 1-129.

[72] Feenberg, D., Skinner, J. (1994). The risk and duration of catastrophic health care expenditures. Review of Economics and Statistics, 76(4), 633-647.

[73] Felbermayr, G. and Groschl, J. (2014), "Naturally negative: The growth effects of natural disasters", *Journal of development economics*, 111, 92-106.

[74] Fernandez, Z., Nieto, M. J. (2006). Internationalization strategy of small and medium sized family businesses: Some influential factors. Family Business Review, 18(1), 77-89.

[75] Fogel, R.W. (1994). Economic growth, population theory, and physiology: the bearing of long-term processes on the making of economic policy. American Economic Review, 84(3), 369-395.

[76] Fomby, T., Ikeda, Y., Loayza, N.V. (2013), The growth aftermath of natural disasters. Journal of Applied Econometrics, 28, no. 3, 412-434.

[77] Frydman, R., Hessel, M., & Rapaczynski, A. 1998. Why ownership matters? Columbia Law School Working paper n.172

[78] Galama, T. (2011). A contribution to health capital theory. RAND Corporation, Working Papers, No.831.

[79] Gallo, M., Pont, C. (1996). Important factors in family business internationalization. Family Business Review, Volume 9, issue 1, 45-59.

[80] Gallo, M., Sveen, J. (1991). Internationalizing the family business: Facilitating and restraining factors. Family Business Review, 4 (2), 181-190.

[81] Gandelman, N., Hernández-Murillo, R. (2014). Risk aversion at the country level. Federal Reserve Bank of St. Louis, Working Paper, No.2014-5.

[82] Golosov, M., Hassler, J., Krusell, P. (2014). Optimal taxes on fossil fuel in general equilibrium. Econometrica, 82(1), 41-88.

[83] Gomez-Mejia, L., Haynes, K.T., Nunez-Nickel, M., Jacobson, K., Moyano-Fuentes, J. (2007). Socio-emotional wealth and business risk in family-controlled firms: Evidence from Spanish olive market. Administrative Science Quarterly, 52, 106-137.

[84] Gomez-Mejia, L. R., Makri, M., Larraza-Kintana, M. (2010). Diversification decisions in family-controlled firms. Journal of Management Studies, 47(2): 223-252.

[85] Gomez-Mejia, L.R., Cruz, C., Berrone, P., De Castro, J. (2011). The bind that ties: Socioemotional wealth preservation in family firms. Academy of Management Annals, (5), 653-707.

[86] Goodstein, J.G., Boeker, W.B., (1991). Turbulence at the top: A new perspective on governance structure changes and strategic change. Academy of Management Journal, 34 (2).

[87] Goranova, M., Alessandri, T. M., Brandes, P., Dharwadkar, R. (2007). Managerial ownership and corporate diversification: A longitudinal view. Strategic Management Journal, 28(3): 211-225.

[88] Graves, C., Shan, Y.G. (2014). An empirical analysis of the effect of internationalization on the performance of unlisted family and non-family firms in Australia. Family Business Review, Vol. 27, Iss. 2, 142-160.

[89] Graves, C., Thomas, J. (2006). Internationalization of Australian family businesses: A managerial capabilities perspective. Family Business Review, Volume 19, issue 3, 207-224.

[90] Greiner, A., Grüne, L., Semmler, W. (2009). Growth and climate change: threshold and multiple equilibria. Dynamic Systems, Economic Growth, and the Environment. Series: Dynamic Modeling and Econometrics in Economics and Finance, 12, 63-78.

[91] Grosfeld, I., Roland, G., (1996). Defensive and strategic restructuring in Central European enterprises. Emergo, Journal of Transforming Economies and Societies, 3 (4).

[92] Grossman, M. (1972a). On the concept of health capital and the demand for health. Journal of Political Economy, 80, 223-255.

[93] Grossman, M. (1972b). The demand for health: a theoretical and empirical investigation. Columbia University Press for the National Bureau of Economic Research, New York.

[94] Grossman, S. Hart, O. (1986). The costs and benefits of ownership: A theory of lateral and vertical integration. Journal of Political Economy 94, 691-719.

[95] Gutenberg, B., Richter, C.F. (1956). Magnitude and energy of earthquakes. Annals of Geophysics, 9, 15.

[96] Gutierrez, M.J. (2008). Dynamic inefficiency in an overlapping generation economy with pollution and health costs. Journal of Public Economic Theory, 10(4), 563-594.

[97] Hallegatte, S., Ghil, M. (2008). Natural disasters impacting a macroeconomic model with endogenous dynamics. Ecological Economics, 68(1), 582-592.

[98] Hallegatte, S., Dumas, P. (2009). Can natural disasters have positive consequences? Investigating the role of embodied technical change. Ecological Economics, 68(3), 777-786.

[99] Hamel, G.H., Prahalad, C.K. (1993). Strategy as stretch and leverage. Harvard Business Review.

[100] Hart, O. (1995). Firms, contracts, and financial structure. Oxford: Clarendon Press.

[101] Hart, O., Moore, J. (1990). Property rights and the nature of the firm. Journal of Political Economy 98, 1119-58.

[102] Haurie, A. (2003). Integrated assessment modelling for global climate change: an infinite horizon optimization viewpoint. Environmental Modeling and Assessment, 8, 117-132.

[103] Henderson, R. (2006). The innovator's dilemma as a problem of organisational competence. Journal of Product Innovation Management, 23, 5-11.

[104] Hennart, J.F., Majocchi, A., Forlani, E. (2019). The myth of the stay-at-home family firm: How family-managed SMEs can overcome their internationalization limitations. Journal of International Business Studies, 50, 758-782.

[105] Hochrainer, S. (2009). Assessing the macroeconomic impacts of natural disasters: are there any? World Bank Policy Research Working Paper Series. Vol. 2009.

[106] Hood, C. (2011). Summing up the parts, combining policy instruments for least-cost climate mitigation strategies, International Energy Agency, Information Paper. Available online: https://www.iea.org/publications/freepublications/publication/Summing_Up.pdf

[107] Hoskisson, R. E., Hitt, M. A., Johnson, R. A., Grossman, W., (2002). Conflicting voices: The effects of institutional ownership heterogeneity and internal governance on corporate innovation strategies. Academy of Management Journal, 45(4): 697-716.

[108] Jaiswal, K.S., Wald, D.J. (2011), Rapid estimation of the economic consequences of global earthquakes. U.S. Geological Survey, 47, 2011-1116.

[109] Jaramillo, H. (2009). Do natural disasters have long-term effects on growth? Documentos CEDE.Universidad de los Andes, Bogotá, D. C., Colombia.

[110] James, H. S. (1999). Owner as manager, extended horizons and the family firm. International Journal of the Economics of Business, 6: 41-55.

[111] Jonsson, A., Foss, N. (2011). International expansion through flexible replication: Learning from the internationalization experience of IKEA. Journal of International Business Studies, 42, 1079-1102.

[112] Johanson, J., Vahlne, J. E. (1977). Internationalization process of the firm-A model of knowledge development and increasing foreign market commitments. Journal of International Business Studies, 8(1): 23-32.

[113] Kagan, Y.Y., Jackson, D.D. (1994). Long-term probabilistic forecasting of earthquake. Journal of Geophysical Research: Solid Earth, 99.B7, 13685-13700.

[114] Kagan, Y.Y. (2002). Seismic moment distribution revisited: I. Statistical results. Geophysical Journal International, 148.3, 520-541.

[115] Kalkstein, L. S., Greene, J.S. (1997). An evaluation of climate/mortality relationships in large US cities and the possible impacts of a climate change. *Environmental health perspectives*, 105(1), 84.

[116] Kano, L., Verbeke, A. (2018). Family firm internationalization: Heritage assets and the impact of bifurcation bias. Global Strategy Journal, Volume 8, Issue 1. Special Issue: Family Firms and Family Governed Multinationals in the Global Economy, 158-183.

[117] Kawashima, K. and Kanoh, T. (1990). Evaluation of Indirect Economic Effects Caused by the 1983 Nihonkai-Chubu Japan Earthquake. Earthquake Spectra, 6, 739-756.

[118] Kellermanns, F.W., Eddleston, K.A., Barnett, T., Pearson, A. (2008). An exploratory study of family member characteristics and involvement: Effects on entrepreneurial behaviour in the family firm. Family Business Review 21 (1), 1-14.

[119] Kirchoff, B.A. (1994). Entrepreneurship and dynamic capitalism: The economics of business firm formation and growth. Praeger Publishers, Westport CT, USA.

[120] Khintchine, A. Y. (1960). Mathematical Methods in the Theory of Queuing. Griffin, London.

[121] Kircher, C. A., Holmes, W.T. and Whitman, R.V. (2006). HAZUS earthquake loss estimation methods. *Earthquake Spectra*, 22, 297-339.

[122] Klein, J. G. (2002). Us versus them, or us versus everyone? Delineating consumer aversion to foreign goods. Journal of International Business Studies, 33(2): 345-363.

[123] Klein, B., Leffler, K. (1981). The role of market forces in assuring contractual performance. Journal of Political Economy 89, 615-41.

[124] Klemperer, P.D. (1987). Entry deterrence in market with consumer switching cost. Economic Journal, 97, 99-117.

[125] Klepper, S. (1996). Entry, exit, growth, and innovation over the product life cycle. American Economic Review, vol. 86(3), 562-583.

[126] Klepper, S. (1997). Industry life cycles. Industrial and Corporate Change. Oxford University Press, vol. 6(1), 145-181.

[127] Klomp, J., Valckx, K. (2014). Natural disasters and economic growth: A meta-analysis. Global Environmental Change, 26, 183-195.

[128] Kontinen, T., Ojala, A. (2010a). The internationalization of family businesses: A review of extant research. Journal of Family Business Strategy, 1(2), 97-107.

[129] Kontinen, T., Ojala, A. (2010b). Internationalization pathways of family SMEs: Psychic distance as a focal point. Journal of Small Business and Enterprise Development, 17(3), 437-454.

[130] La Porta, R., Lopez-De-Silanes, F., Shleifer, A. (1999). Corporate ownership around the world. Journal of Finance 54, 471-518.

[131] Laporte, A., Ferguson, B. (2007). Investment in health when health is stochastic. Journal of Population Economics, 20(2), 423-444.

[132] Loayza, N. V., Olaberria, E., Rigolini, J., Christiaensen, L (2012). Natural disasters and growth: Going beyond the averages. World Development, 40(7), 1317-1336.

[133] Lumpkin, G.T., Brigham, K. H., Moss, T. W. (2010). Long-term orientation: Implications for the entrepreneurial orientation and performance of family businesses. Entrepreneurship and Regional Development, Vol. 22, Iss. 3-4.

[134] Maheswaran, D., Chen, C. Y. (2009). Nation equity: Country-of-origin effects and globalization. In M. Kotabe & K. Kelsen (Eds.), Handbook of international marketing: 91-113. Thousand Oaks, CA: Sage.

[135] Mariani, F., Pérez-Barahona, A., Raffin, N. (2010). Life expectancy and the environment. Journal of Economic Dynamics and Control, 34, 798-815.

[136] Martin, I.W.R., Pindyck, R.S. (2015). Averting catastrophes: the strange economics of Scylla and Charybdis. American Economic Review, 105(10), 2947-2985.

[137] McKinsey&Company (2009). Pathways to a low carbon economy. Version 2 of the Global Greenhouse Gas Abatement Cost Curve. Available online: http://www.mckinsey.com/business-functions/sustainability-and-resource-productivity/our-insights/pathways-to-a-low-carbon-economy

[138] Mc Nally, R.C., Cavusgil, E., Calantone, R.J. (2010). Product innovativeness dimensions and their relationships with product advantage, product financial performance and project protocol. Journal of Product Innovation Management. Volume 27, 991-1006.

[139] Miller, D., Le Breton-Miller, I., Lester, R.H. (2010). Family ownership and acquisition behaviour in publicly-traded companies. Strategic Management Journal, 31(2), 201-223.

[140] Mulargia F., Stark, P.B. and Geller, R.J. (2017), "Why is Probabilistic Seismic Hazard Analysis (PSHA) still used?", *Physics of the Earth and Planetary Interiors*, Volume 264, 63-75.

[141] Müller-Furstenberger, G., Schumacher, I. (2015). Insurance and climate-driven extreme events. Journal of Economic Dynamics and Control, 54(C), 59-73.

[142] Naldi, L., Nordqvist, M. (2008). Family firms venturing into international markets: A resource dependence perspective. Frontiers of Entrepreneurship Research, 28(14), 1-18.

[143] Nogaj, M., Yiou, P., Parey, S., Malek, F. and Naveau, P. (2006), "Amplitude and frequency of temperature extremes over the North Atlantic region", *Geophysical Research Letters*, 33(10).

[144] Nordqvist, M., Habbershon, T.G., Melin, L. (2008). Transgenerational entrepreneurship: exploring EO in family firms. In H. Landstrom, H. Crijns and E. Laveren (Eds), Entrepreneurship, sustainable growth and performance: Frontiers in European entrepreneurship research, pp.93-116, Edward Elgar, Cheltenham, England.

[145] Nordhaus, W. D. (1991). To slow or not to slow: the economics of the greenhouse effect. The economic journal, 101(407), 920-937.

[146] Nordhaus, W.D. (2008). A question of balance: weighing the options on global warming policies. Yale University Press, New Haven.

[147] Nordhaus, W. D. (2011). The economics of tail events with an application to climate change. Review of Environmental Economics and Policy, 5.2, 240-257.

[148] Noy, I. (2009). The macroeconomic consequences of disasters. Journal of Development Economics, 88, 221-231.

[149] OCDE. (2006). OECD Studies in Risk Management. Japan Earthquakes.

[150] OCDE. (2008). OECD Territorial Reviews. Istanbul, Turkey.

[151] Okoroafo, S.C. (1999). Internationalization of Family Businesses: Evidence from Northwest Ohio, U.S.A. Family Business Review, Volume 12, Issue 2, 147-158.

[152] Okuyama, Y. (2003), Economics of natural disasters: a critical review, Research Paper 2003-12, Regional Research Institute, West Virginia University, USA.

[153] Omori, F. (1895). On after-shocks of earthquakes. Journal of the College of Science, Imperial University of Tokyo, 7, 111-200.

[154] Palecki, M.A., Changnon, S.A., Kunkel, K.E. (2001). The nature and impacts of the July 1999 heat wave in the Midwestern United States: learning from the lessons of 1995. Bulletin of the American Meteorological Society 82 (7), 1353-1367.

[155] Patel, P.C., Chrisman, J.J. (2014). Risk abatement as a strategy for family firms. Strategic Management Journal, Volume 35, Issue 4, 617-627.

[156] Pautrel, X. (2012). Pollution, private investment in healthcare, and environmental policy. Scandinavian Journal of Economics, 114, 334-357.

[157] Pedersen, T., Shaver, J. M. (2011). Internationalization revisited: The big step hypothesis. Global Strategy Journal, 1(3/4): 263-274.

[158] Pérez-González, F. (2006). Inherited Control and Firm Performance. American Economic Review. 96(5): 1559-1588.

[159] Picone, G., Uribe, M., Wilson, R.M. (1998). The effect of uncertainty on the demand for medical care, health capital and wealth. Journal of Health Economics, 17(2), 171-185.

[160] Pipkin, B., Trent, D. (1997), Geology and the Environment, West/Wadsworth, Belmont.

[161] Posner, R.A. (2006), Efficient responses to catastrophic risk, Chicago Journal of International Law, 6 (2), 511-525.

[162] Pukall, T.J., Calabrò, A. (2013). The Internationalization of family firms: A critical review and integrative model. Family Business Review, 20(10), 1-23.

[163] Raddatz, C. (2007). Are external shocks responsible for the instability of output in low income countries? Journal of Development Economics, 84, 155-187.

[164] Reid, S., (1981). The decision-maker and export entry and expansion. Journal of International Business Studies, Fall, 101-112.

[165] Rose, A., Benavides, J. Chang, S.E., Szczesniak, P., Lim, D. (1997). The Regional Economic Impact of an Earthquake: Direct and Indirect Effects of Electricity Lifeline Disruptions. Journal of Regional Science, 37(3), 437-458.

[166] Rose, A., Liao, S.Y. (2005). Modeling regional economic resilience to disasters: a computable general equilibrium analysis of water service disruptions. *Journal of Regional Science*, 45(1), 75-112.

[167] Rossana, R.J. (1985). Delivery lags and buffer stocks in the theory of investment by the firm. Journal of Economic Dynamics and Control, 9, 153-193.

[168] Rugman, A. M., Verbeke, A. (2008). The theory and practice of regional strategy: A response to Osegowitsch and Sammartino. Journal of International Business Studies, 39(2): 326-332.

[169] Rui, H., Cuervo-Cazurra, A., Un, C. A. (2016). Learning-by-doing in emerging market multinationals: Integration, trial and error, repetition, and extension. Journal of World Business, Volume 51, Issue 5, 686-699.

[170] Salvato, C. (2004). Predictors of entrepreneurship in family firms. Journal of Private Equity, 27(3), 68-76.

[171] Sciascia, S., Mazzola, P., Astrachan J.H., Pieper T.M. (2012). The role of family ownership in international entrepreneurship: exploring nonlinear effect. Small Business Economics, 38, 15-31.

[172] Selden, T.M. (1993). Uncertainty and health care spending by the poor: the human capital model revisited. Journal of Health Economics. 12: 109-115.

[173] Shamsie, J., Martin, X., Miller, D. (2009). In with the old, in with the new: Capabilities, strategies and performance among the Hollywood studios. Strategic Management Journal, 30, 1440-1452.

[174] Sharma, P., Chrisman, J.J., Chua, J.H. (1997). Strategic management of the family business: Past research and future challenges. Family Business Review, 10(1).

[175] Shi, H. X., Graves, C., Barbera, F. (2019). Intergenerational succession and internationalisation strategy of family SMEs: Evidence from China. Long Range Planning, Volume 52, Issue 4, 101838.

[176] Shibusawa, H., Miyata, Y. (2011), Evaluating the Dynamic and Spatial Economic Impacts of an Earthquake: A CGE Application to Japan, *Regional Science Inquiry*, 3(2), 13-25.

[177] Smith, A. (1776). The Wealth of Nations, Book V. London, Methuen & Co.

[178] Solow, R.M. (1956). A Contribution to the theory of economic growth. Quarterly Journal of Economics, 70, 65-94.

[179] Stadler, C., Mayer, M.C.J., Hautz, J., Matzler, K. (2018). International and product diversification: which strategy suits family managers? Global Strategy Journal, 8 (1), 184-207.

[180] Steger, T.M. (2005). Stochastic growth under Wiener and Poisson uncertainty. Economics Letters, 86, 311-316.

[181] Stephens, K.U. (2007). Testimony to the United States Senate, Committee on Homeland Security and Governmental Affairs, Ad Hoc Subcommittee on Disaster Recovery. Post-catastrophe Crisis: Addressing the Dramatic Need and Scant Availability of Mental Health Care in the Gulf Coast.

[182] Stewart, F., Fitzgerald, V. (2000). War and Underdevelopment: Country Experiences, Vol. 2, Oxford University Press, USA.

[183] Ströbl, E. (2011). The economic growth impact of hurricanes: evidence from US coastal counties. Review Economics and Statistics, 93 (2), 575-589.

[184] Szulanski, G., Jensen, R. J. (2006). Presumptive adaptation and the effectiveness of knowledge transfer. Strategic Management Journal, 27(10): 937-957.

[185] Szulanski, G., Jensen, R. J. (2008). Growing through copying: The negative consequences of innovation on franchise network growth. Research Policy, 37(10): 1732-1741.

[186] Teece, D.J., Pisano, G., Shuen, A. (1997). Dynamic capabilities and strategic management. Strategic Management Journal, 18(7):509-533.

[187] The Economist. (2012). The big engine that couldn't. 19 May.

[188] The Economist. (2013). Global niche players. 2 February.

[189] The Economist. (2015). Dynasties. 18 April.

[190] Thomas, J., Graves, C. (2005). Internationalization of the family firm: The contribution of an entrepreneurial orientation. Journal of Business and Entrepreneurship, 17(2), 91.

[191] Thomsen, S., Pedersen, T. (2000). Ownership structure and economic performance in the largest European companies. Strategic Management Journal, 1-2.

[192] Tomiyama, K. (1985). Two-stage optimal control problems and optimality conditions. Journal of Economic Dynamics and Control, 9, 317-337.

[193] Tomiyama, K., Rossana, R. J. (1989). Two-stage optimal control problems with an explicit switch point dependence: Optimality criteria and an example of delivery lags and investment. Journal of Economic Dynamics and Control, 13, 319-337.

[194] Toya, H., Skidmore, M. (2007). Economic development and the impacts of natural disasters. Economics Letters, 94(1), 20-25.

[195] Tsang, Eric W.k. (2001). Internationalizing the family firm: A Case study of a Chinese family business. Journal of Small Business Management, 39:1, 88-93.

[196] Tsur, Y., Withagen, C. (2011). Preparing for catastrophic climate change. Center for Agricultural Economic Research, Hebrew University of Jerusalem, Discussion Paper, No.5.11.

[197] United Nations Office of the High Commissioner for Human Rights. (2008). The right to health. Printed at United Nations, Geneva.

[198] United Nations. (2015). World population ageing 2015. New York: Department of Economic and Social Affairs, Population Division.

[199] Van Doorslaer, E.K.A. (1987). Health, knowledge and the demand for medical care. An Econometric Analysis, Van Gorcum, Maastricht and Wolfeboro, New Hampshire.

[200] Verbeke, A., Kano, L. (2012). The transaction cost economics theory of the family firm: Family-based human asset specificity and the bifurcation bias. Entrepreneurship Theory and Practice, 36(6), 1183-1205.

[201] Vernon, R. (1966). International Investment and International Trade in the Product Cycle. The Quarterly Journal of Economics, Volume 80, Issue 2, 190-207.

[202] Wagstaff, A. (1986). The demand for health: a simplified Grossman model [on the concept of health capital and the demand for health]. Bulletin of Economic Research, Wiley Blackwell, 38(1), 93-95.

[203] Wagstaff, A. (1993). The demand for health: an empirical reformulation of the Grossman model. Health Economics, 2(2), 189-198.

[204] Wälde, K. (1999). Optimal saving under Poisson uncertainty. Journal of Economic Theory, 87, 194-217.

[205] Wang, M., Zhao, J., Bhattacharya, J. (2015). Optimal health and environmental policies in a pollution-growth nexus. Journal of Environmental Economics and Management, 71, 160-179.

[206] Weil, P. (1990). Non-expected utility in macroeconomics. Quarterly Journal of Economics, 105(1), 29-42.

[207] Weitzman, M.L. (2011). Fat-tailed uncertainty in the economics of catastrophic climate change. Review of Environmental Economics and Policy, 5(2), 275-292.

[208] Weitzman, M.L. (2014). Fat tails and the social cost of carbon. American Economic Review, 104(5), 544-546.

[209] Weitzman, M.L., Lofgren, K.G. (1997). On the welfare significance of green accounting as taught by parable. Journal of Environmental Economics and Management, 32(2), 139-153.

[210] West, C. T., Lenze, D.G. (1994). Modelling the regional impact of natural disaster and recovery: a general framework and an application to hurricane Andrew. International Regional Science Review, 17, 121-150.

[211] Williamson, O. (1979). Transactions-cost economics: The governance of contractual relations. Journal of Law and Economics, 22, 233-62.

[212] Winter, S.G. (1995). Four Rs of profitability: Rents, resources, routines, and replication. In Montgomery, C., ed. Resource-Based and Evolutionary Theories of the Firm - Towards a Synthesis (Kluwer, Dordrecht, The Netherlands), 147-178.

[213] Winter, S., Szulanski, G. (2001). Replication as strategy. Organisation Science, Volume 12, Issue 6, 730-743.

[214] Woo G. (1999). The Mathematics of Natural Catastrophes, Imperial College Press, London.

[215] World Bank. (2011). Toward a healthy and harmonious life in China: stemming the rising tide of Non-communicable diseases. World Bank Report, No.62318-CN.

[216] World Bank. (2013). Health or wealth: which comes first? Africa Health Forum. Available online: http://siteresources.worldbank.org/INTAFRICA/Resources/AHF-health-or-wealth-which-comes-first.pdf

[217] World Health Organization. (2004). Clinical trials on treatment using a combination of Traditional Chinese medicine and Western medicine. WTO Report, available online: http://apps.who.int/medicinedocs/pdf/s6170e/s6170e.pdf

[218] World Health Organization. (2004). Global burden of disease report 2004. Available online: http://www.who.int/healthinfo/global _burden_disease/GBD_report_2004update_part4.pdf

[219] World Health Organization. (2008). China: Great strides for village doctors. WHO 60th anniversary commemorative volume, 86(12), 914-915.

[220] World Health Organization. (2014). Global Status Report on Non-communicable Diseases 2014. Available online: http://www.who.int/nmh/publications/ncd-status-report-2014/en/

[221] World Health Organization. (2016). Ambient air pollution: a global assessment of exposure and burden of disease. Available online: http://who.int/phe/publications/air-pollution-global-assessment/en/

[222] Zahra, S.A. (2003). International expansion of U.S. manufacturing family businesses: The effect of ownership and involvement. Journal of Business Venturing, 18(4), 495-512.

[223] Zellweger, T.M., Kellermanns, F.W., Eddleston, K.A., Memili, E. (2012). Building a family firm image: How family firms capitalize on their family ties. Journal of Family Business Strategy 3(4):239-250.

[224] Zellweger, T., Sieger, P. (2012). Entrepreneurial orientation in long-lived family firms. Small Business Economics 38 (1), 67-84.

[225] Zweifel, P., Breyer, F.(1997). Health Economics. Oxford University Press; New York.