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Option pricing when correlations are stochastic: an analytical framework

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Abstract In this paper we develop a novel market model where asset variances–covariances evolve stochastically. In addition shocks on asset return dynamics are assumed to be linearly correlated with shocks driving the variance–covariance matrix. Analytical tractability is preserved since the model is linear-affine and the conditional characteristic function can be determined explicitly. Quite remarkably, the model provides prices for vanilla options consistent with observed smile and skew effects, while making it possible to detect and quantify the correlation risk in multiple-asset derivatives like basket options. In particular, it can reproduce and quantify the asymmetric conditional correlations observed on historical data for equity markets. As an illustrative example, we provide explicit pricing formulas for rainbow "Best-of" options.

Keywords Wishart processes · Best-of basket option · Stochastic correlation · FFT

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1 Introduction

This paper presents an analytically tractable model for capturing the joint behavior of prices of an underlying set of assets, options on the individual underlying assets, and derivatives on baskets of the underlying assets. We believe this is the first tractable model that allows for non-trivial stochastic volatility of asset returns and stochastic correlation of cross-sectional asset returns in a manner that is fully consistent with the sorts of smile and skew effects that are apparent in typical market pricing of plain vanilla option prices.

In this model, prices (conditional to volatilities) evolve according to a lognormal diffusion while the stochastic variance–covariance matrix follows a Wishart process as proposed by Gourieroux and Sufana (2004b). Originally introduced by Bru (1991), the Wishart process is the natural matrix counterpart of the Feller (1951) square root process and falls within the class of affine factor models as originally defined in Duffie and Kan (1996), in the extension discussed in Grasselli and Tebaldi (2008). Following this line of development, the present paper introduces the multi-asset extension of the Heston (1993) stochastic volatility model. In fact the present stochastic market model extends to the multiple asset setup the most appealing features of the Heston model: that is, its analytical tractability and the possibility to provide explicit parametric restrictions ensuring that the variance–covariance matrix remains positive definite, a key property when modeling stochastic dynamic covariances.

The first paper proposing the use of continuous-time Wishart processes in quantitative finance is by Gourieroux and Sufana (2004a). These authors highlighted the flexibility of the Wishart specification in modeling dynamic dependence within a linear affine framework. However, Gourieroux and Sufana (2004b) model for equity markets assumes that the noises driving asset returns and asset covariances are independent and therefore has limited ability in reproducing the observed skew effects of plain vanilla options.

Independently of our work, Buraschi et al. (2006) introduced a perfect dependence among these noises within a dynamic asset allocation framework: they highlighted that a correlation hedging demand arises when correlations become stochastic. In their model, asset returns are driven by a column of the volatilities' matrix noises, so that it can be viewed as the Wishart extension of the Heston and Nandi (2000) model, in which there is a perfect correlation between asset and volatility noises.

An alternative non-linear approach to the modeling of stochastic correlations has been proposed in Driessen et al. (2005), who chose to model the "market correlation risk" with a single risk factor capturing average market correlations and evolving like a Jacobi diffusion. By analyzing simultaneously prices of options on single name and on the market index, Driessen et al. (2005) show that correlation risk is priced in the options market and is a determinant of the value of dispersion trading strategies.

Note however that the affine specification is quite natural, since it allows to extend to the dynamic framework the linear decomposition of portfolio variance into common systematic factors and idiosyncratic ones as is customary in the (static) arbitrage pricing theory.

Within this line of development, our paper improves previous literature on correlation risk: first, we completely characterize all linear correlation structures between underlying asset returns and their volatilities, which are fully consistent with the smile and skew effects on vanilla options. Second, we quantify the effect of stochastic correlations on option prices by introducing the notion of implied correlation together with its smile and skew effects.

We provide a complete treatment in the case of a “Best-of” basket option, where the whole procedure can be carried out explicitly.

In this case the appearance of an implied correlation skew can be explained in terms of the asymmetric reaction of correlations with respect to positive or negative shocks on returns, in full analogy with the well known “leverage effect” explanation of the volatility skew effect in the Heston (1993) stochastic volatility model.

The paper is organized as follows: in the next section we introduce the Wishart affine stochastic correlation model (WASC) and we explain some properties of this process; in Sect. 3 we develop the analytical framework, while in Sect. 4 we focus on financial applications on both real and simulated data. A detailed analysis of the Best-of option contract is performed in order to explain some stylized facts about correlation products. Section 5 concludes the paper.

2 The Wishart affine stochastic correlation model

2.1 Modeling stochastic correlations in a linear affine factor model

Consider an equity market in which options on both single name and multiple names are traded. The single asset return dynamics $dY_t^i, i = 1, \ldots, n$, in traditional models like CAPM (see e.g., Sharpe 1964) and the APT of Ross (1976) are driven by a linear-affine combination of systematic and idiosyncratic risk factors:

$$dY_t^i = \alpha_t^i + \left(\beta_t^i\right)^T dY_t^* + dz_t^i.$$

A vector of systematic risk factors Y_t^* is identified by a vector of factor-mimicking portfolios. Correspondingly, the quadratic covariations of the traded risk factors are linear combinations of the quadratic covariation of the systematic risk factors:

$$d\left\langle Y^i, Y^j\right\rangle_t = \sum_{l,k=1}^{n} d\left\langle Y_k^*, Y_l^*\right\rangle_t \beta_t^{ik}\beta_t^{lj} + \varepsilon_i^2\delta_{i,j},$$

where we assumed that the non-systematic factors z^i are independent with a variance of $\varepsilon_i^2 << 1$.

In order to model dynamic stochastic dependence we also assume that quadratic covariations evolve as additional stochastic factors and we impose a factor-evolution linear affine with all relevant factors, i.e. in the vector of returns $Y_{i,t}^*$ and in the matrix of quadratic (co-)variations $\Sigma_{kl} = \left\langle Y_k^*, Y_l^*\right\rangle_t$.

Assumption 1 The continuous time-diffusive Factor Model is considered to be a linear-affine stochastic factor model w.r.t. returns Y^*_{it} and variance–covariance factors Σ_{kl}.

This assumption extends the traditional static APT linear factor decomposition to a dynamic framework with stochastic covariances. In fact, in this model (observed) returns and their variances can be explained as linear combinations of fundamental risk factors (including covariance risks) whose evolution is still a linear-affine process.

Note that this approach would be essentially unfeasible within the class of canonical admissible affine processes with domain $\mathbb{R}^m_+ \times \mathbb{R}^{n-m}$ analyzed in Dai and Singleton (2000) and Duffie et al. (2000). In fact, the parametric restrictions imposed in order to grant admissibility, i.e. the existence of the process at any future time, would impose severe limitations to the class of describable stochastic correlations. The following example illustrates the nature of these restrictions in a simple case.

Example 1 One can generate a linear-affine model with stochastic volatility and non-trivial correlation between the assets by introducing (at least) 3 CIR positive independent factors

$$
\begin{aligned}
dX^1_t &= \left(\alpha_1 X^1_t + \beta_1\right) dt + \sqrt{X^1_t} dW^1_t \\
dX^2_t &= \left(\alpha_2 X^2_t + \beta_2\right) dt + \sqrt{X^2_t} dW^2_t \\
dX^3_t &= \left(\alpha_3 X^3_t + \beta_3\right) dt + \sqrt{X^3_t} dW^3_t
\end{aligned}
$$

which drive the volatility of the assets in the following way:

$$
\begin{aligned}
dS^1_t &= S^1_t \left(r dt + \sqrt{X^1_t} dZ^1_t + \sqrt{X^3_t} dZ^3_t\right) \\
dS^2_t &= S^2_t \left(r dt + \sqrt{X^2_t} dZ^2_t + \sqrt{X^3_t} dZ^3_t\right),
\end{aligned}
$$

with

$$
\begin{aligned}
Z^1_t &= \rho_1 W^1_t + \sqrt{1-\rho_1^2} B^1_t \\
Z^2_t &= \rho_2 W^2_t + \sqrt{1-\rho_2^2} B^2_t \\
Z^3_t &= \rho_3 W^3_t + \sqrt{1-\rho_3^2} B^3_t
\end{aligned}
$$

where $\left(W^1, W^2, W^3, B^1, B^2, B^3\right)$ is a standard (6-dimensional) Brownian motion.

The model admits a linear affine infinitesimal generator and the (stochastic) volatilities are given by

$$\frac{1}{(S^1)^2} d\left\langle S^1, S^1 \right\rangle_t = \left(X_t^1 + X_t^3\right) dt$$
$$\frac{1}{(S^2)^2} d\left\langle S^2, S^2 \right\rangle_t = \left(X_t^2 + X_t^3\right) dt,$$

while the (stochastic) covariation between the assets is given by

$$\frac{1}{S^1 S^2} d\left\langle S^1, S^2 \right\rangle_t = X_t^3 dt > 0.$$

In conclusion, in this affine model (which in the terminology of Dai and Singleton 2000 could be called $A_3(5)$) the assets have a stochastic volatility allowing for smile and (even stochastic) skew effects, but *the covariance is constrained to remain positive*. This well-known limitation has already been pointed out in Dai and Singleton (2000, p. 1956).

Quite remarkably, the recent paper by Gourieroux and Sufana (2004a,b) shows that it is possible to describe the stochastic evolution of a generic stochastic positive definite variance–covariance matrix within a linear-affine model with a non linear (cone) state space, and Grasselli and Tebaldi (2008) establish sufficient parametric conditions in order to obtain an admissible process preserving the cone state space formed by the set of positive definite symmetric matrices. For this reason we retain the Gourieroux and Sufana (2004a,b) specification of the variance process:

Assumption 2 The stochastic covariance matrix follows a Wishart process.

The instantaneous variance–covariance of the risky assets is a matrix Σ_t which is assumed to satisfy the following dynamics:

$$d\Sigma_t = \left(\Omega\Omega^T + M\Sigma_t + \Sigma_t M^T\right) dt + \sqrt{\Sigma_t} dW_t Q + Q^T (dW_t)^T \sqrt{\Sigma_t}. \quad (1)$$

Equation 1 characterizes the Wishart process introduced by Bru (1991). Note that in terms of factor analysis, this model not only allows the stochastic evolution of principal components (eigenvalues of the covariance matrix) but also a stochastic evolution of the factor loadings (i.e. sensitivities) of observed factors with respect to latent factors.

Finally, we consider the following extension of all previously known (affine) models:

Assumption 3 Brownian motions of the assets' returns and those driving their instantaneous covariance matrix are linearly correlated.

This extension is motivated by the well-known fact that within the Heston (1993) model it is possible to approximately reproduce observed skews only by introducing a non-zero correlation between the innovation of returns and the innovation-driving variance. In the same way, it is reasonable to expect that these correlations are needed in order to fully capture the effects of dynamic stochastic correlations on multiple asset options.

In formulas, we consider a n-dimensional risky asset S_t whose dynamics in a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are given by

$$dS_t = diag[S_t]\left[(r\mathbf{1} + \lambda_t(\omega))\, dt + \sqrt{\Sigma_t} dZ_t\right],$$

where $\mathbf{1} = (1, \ldots, 1)^T$ (^{T} denotes transposition), $Z_t \in \mathbb{R}^n$ is a vector Brownian motion and $\lambda_t(\omega) \in \mathbb{R}^n$ is the vector of risk premia. The general way in which we can correlate the Brownian motions Z and W consists in introducing n real matrices $R_k \in M_n$, $k = 1, \ldots, n$ such that:

$$dZ_t^k = \sqrt{1 - Tr\left[R_k R_k^T\right]} dB_t^k + Tr\left[R_k dW_t^T\right], \qquad k = 1, \ldots, n,$$

where the (vector) Brownian motion B is independent of W. That is, the element R_k^{ij} represents the correlation between the (scalar) Brownian motion Z^k and the (scalar) Brownian motion W^{ij}.

Note that for a generic choice of R_k the correlated model would not remain linear affine. Under the above assumptions it follows that the correlation matrix among the Brownian motions has a very special and parsimonious form.

Proposition 1 *Assumption 1 and Assumption 2 imply that for $k = 1, \ldots, n$, the correlation matrix R_k is given by:*

$$R_k = \begin{pmatrix} 0 & 0 & 0 \\ \rho_1 & \ldots & \rho_n \\ 0 & 0 & 0 \end{pmatrix} \longleftarrow k\text{-th row}, \tag{2}$$

where $\rho_i \in [-1, 1]$, $i = 1 \ldots n$ and $\rho^T \rho \leq 1$.

Proof Let us introduce the assets' returns $Y_t = \ln S_t$, whose dynamics are given by

$$dY_t = \left(r\mathbf{1} - \frac{1}{2} Vec\left[\Sigma_t^{ii}\right]\right) dt + \sqrt{\Sigma_t} dZ_t,$$

where

$$\begin{aligned} Vec\left[\Sigma_t^{ii}\right] &= \left(\Sigma_t^{11}, \ldots, \Sigma_t^{nn}\right)^T \\ &= Vec\left[Tr\left[e_{ii}\Sigma_t\right]\right], \end{aligned}$$

with $e_{ii} = \left(\delta_{ijk}\right)_{j,k=1,\ldots,n}$, that is the canonical basis of M_n (the set of square matrices).

We shall compute the infinitesimal covariation term between the asset return Y^h, $h = 1, \ldots, n$ and the volatility factor Σ^{ij}, i.e. $d < Y^h, \Sigma^{ij} >_t$, and we investigate under which condition this term is affine w.r.t. Σ. It is useful to denote by σ the square root of the Wishart matrix Σ, so that:

$$\begin{aligned} \sigma \quad &= \sqrt{\Sigma_t}, \\ \Sigma_t^{ij} &= \sigma^{il}\sigma^{jl}, \end{aligned} \tag{3}$$

where we used the Einstein convention that repeated indexes are summed.

By using the dynamics of the Wishart factors and the asset returns equation we obtain:

$$\begin{aligned} d < Y^h, \Sigma^{ij} >_t &= 2\left(\sigma^{hk} dZ^k\right) d\Sigma^{ij} \\ &= \sigma^{hk} Tr\left[R_k dW_t^T\right]\left(\sigma^{im} dW^{ml} Q^{lj} + \sigma^{jm} dW^{ml} Q^{li}\right) \\ &= \sigma^{hk} R_k^{pq} dW_t^{pq}\left(\sigma^{im} dW^{ml} Q^{lj} + \sigma^{jm} dW^{ml} Q^{li}\right), \end{aligned}$$

where summation is extended to the indexes: k, p, q, m, l.

Now, since W is a matrix Brownian motion (whose elements are independent scalar Brownian motions), we have

$$d < W_{\cdot}^{pq}, W_{\cdot}^{ml} >_t \; = \delta_{p,m}\delta_{q,l} dt,$$

so that

$$\begin{aligned} d < Y^h, \Sigma^{ij} >_t &= \sigma^{hk} R_k^{ml}\left(\sigma^{im} Q^{lj} + \sigma^{jm} Q^{li}\right) dt \\ &= \left(\sigma^{hk} R_k^{ml}\sigma^{im} Q^{lj} + \sigma^{hk} R_k^{ml}\sigma^{jm} Q^{li}\right) dt, \end{aligned}$$

where we see that the only possibility to recognize an element of Σ is to assume that for all indexes $k \neq m$ we have $R_k^{ml} = 0$ for all $l = 1, \ldots, n$. In other words, we find that the matrices R_k have zero entries except for the elements of the k-th row.

Moreover, in order to apply (3), it must be that the elements of such k-th row be the same for any k, in other words this row is the same for any matrix R_k as in (2).

Now we can apply (3) and find out the linearity w.r.t. Σ:

$$\begin{aligned} d < Y^h, \Sigma^{ij} >_t &= \left(\sigma^{hk}\sigma^{im} Q^{lj} + \sigma^{hk}\sigma^{jm} Q^{li}\right)\rho_l dt \\ &= \left(\Sigma^{hi} Q^{lj} + \Sigma^{hj} Q^{li}\right)\rho_l dt \\ &= \left(\Sigma^{hi} Tr\left[R_j Q\right] + \Sigma^{hj} Tr\left[R_i Q\right]\right) dt, \end{aligned}$$

which represents the affine covariation between the asset return and the volatility terms. □

As a by-product, we obtain an orthogonality condition on the matrices R_k:

$$Tr\left(R_i R_j^T\right) = 0, \qquad i \neq j.$$

The effect of imposing the expression (2) to R_k implies that the most general expression of the vector of return innovations is given by:

$$dZ_t = \sqrt{1 - \rho^T \rho} dB_t + dW_t \rho.$$

This expression is a direct generalization of the Heston (1993) stochastic volatility model. Note that in the multidimensional situation each coefficient ρ_i enters symmetrically in the definition of any Brownian noise Z_k, e.g. for $n = 2$:

$$\begin{aligned} dZ_t^1 &= \sqrt{1 - \left(\rho_1^2 + \rho_2^2\right)} dB_t^1 + \left(dW_t^{11}\rho_1 + dW_t^{12}\rho_2\right) \\ dZ_t^2 &= \sqrt{1 - \left(\rho_1^2 + \rho_2^2\right)} dB_t^2 + \left(dW_t^{21}\rho_1 + dW_t^{22}\rho_2\right), \end{aligned}$$

hence the determination of the vector of correlations ρ between returns and covariance innovations requires a global analysis of systematic risks driving the market and affecting all the stocks in the market.

2.2 Parametric conditions on the WASC process

We use the results of Grasselli and Tebaldi (2008) to state sufficient conditions for admissibility, thus providing the parametric conditions required to ensure that the Wishart process Σ_t remains inside the admissible domain, i.e. the cone of positive definite matrices. Remarkably these conditions are also sufficient to grant the solvability of the model, i.e. to obtain a closed-form expression of the characteristic function in terms of solutions to linear ODEs. Then we state the following.

Proposition 2 *Assume that* $\Omega, M, Q \in M_n$. *If*

- $\Omega\Omega^T = \beta Q^T Q, \quad \beta > n - 1$;
- Ω *is invertible;*

then the WASC model has a well-defined evolution for any $s \geq t$, *i.e. given an initial condition* Σ_t *in the cone* $Sym_n^+(\mathbb{R})$ *of* n *dimensional real-valued invertible symmetric positive definite matrices, then the Wishart evolved matrix* $\Sigma_s \in Sym_n^+(\mathbb{R})$, $\forall s \geq t$.

Proof See Sect. 4.2 in Grasselli and Tebaldi (2008). □

Note that the first condition requires a proportionality relation between $\Omega\Omega^T$ and $Q^T Q$ and provides a strong reduction on the number of free parameters in the model. The proportionality (Gindikin) parameter β is constrained to be larger than $n - 1$. When $n = 1$ we recover the standard condition granting the positivity of the process for the square-root diffusion model.

In full analogy with the square-root diffusion process, the term $\Omega\Omega^T$ is related to the expected long-term variance–covariance matrix Σ_∞ through the solution to the following linear (Liapunov) equation:

$$-\Omega\Omega^T = M\Sigma_\infty + \Sigma_\infty M^T.$$

The existence of a positive definite solution Σ_∞ is also a sufficient condition to grant the convergence of the expectation to its natural long-time limit:

$$\lim_{t\to\infty} E\left[\Sigma_t\right] = \Sigma_\infty,$$

in fact the expectation will obey the differential equation:

$$\frac{dE\left[\Sigma_t\right]}{dt} = \Omega\Omega^T + ME\left[\Sigma_t\right] + E\left[\Sigma_t\right]M^T,$$

for which Σ_∞ is an asymptotically stable equilibrium point.

In a one-dimensional model the matrix M would collapse to the mean reversion velocity; in the multivariate situation the matrix M is still responsible for the mean reversion effects, however in case of a non-diagonal M it will also generate an effective dynamic interaction between different asset volatilities. Similarly the matrix Q is the multivariate counterpart of the volatility of volatility parameter in the Heston model.

Note that the average correlations among assets are fixed by the long term mean Σ_∞, so that the matrices M and Q cannot be chosen independently; in fact existence of a stationary state requires that:

$$-\beta Q^T Q = M\Sigma_\infty + \Sigma_\infty M^T \quad \beta > n - 1. \tag{4}$$

The implications of different choices for M and Q can be discussed analyzing their effect on derivative prices and in particular on the implied volatility surfaces for single assets. For example, the effects of a non-diagonal Q can have a static or a dynamic origin: since the matrix Q is related to the long-term covariance Σ_∞ and to M by Eq. 4, a non-diagonal Q can arise in two extreme (opposite) situations: if Σ_∞ is diagonal and M is non-diagonal the assets are statically linearly uncorrelated while their evolution introduces a dynamic interaction. On the contrary, if Σ_∞ is non-diagonal, static correlations are present while the evolution of the asset volatilities could eventually have independent drifts (M diagonal). In general, these dynamic and static effects will mix and will generate a non-diagonal Q which will affect the shape and level of the smile/skew.

The WASC model naturally accommodates a flexible parametrization of positive stochastic risk premia depending also on covariation levels. The potential role of these terms in explaining a large fraction of observed equity risk premia has been discussed more extensively in Buraschi et al. (2006). Since in this paper we focus on derivative pricing, we do not investigate here the (delicate) question of the most general expression for the risk premium, and for sake of simplicity we assume (as it is customary in the affine models) that the change of measure preserves the affine form of the model.

In the following proposition we simply state the expression of the stochastic discount factor extending the so called “completely affine parametrization” (in the terminology of Duffee 2002) to the WASC framework.

Proposition 3 *Consider a WASC model fulfilling the parametric conditions of Proposition 2. Let* $H_t = e^{-rt} E_t^{\mathbb{P}}\left[d\mathbb{Q}/d\mathbb{P}\right]$ *be the stochastic discount factor, where*

$d\mathbb{Q}/d\mathbb{P}$ is the Radon Nikodym derivative relating the historical measure $\mathbb{P}$ with the pricing measure $\mathbb{Q}$. Let the dynamics of H_t be defined by the SDE:

$$\frac{dH_t}{H_t} = -rdt + Tr\left[\Lambda^T\sqrt{\Sigma_t}dW_t\right] - l^T\sqrt{\Sigma_t}dB_t, \tag{5}$$

where $\Lambda \in M_{n\times n}(\mathbb{R})$, $l \in M_{1\times n}(\mathbb{R})$. Then the SDE describing the WASC process under the risk neutral measure $\mathbb{Q}$ is related to the original $\mathbb{P}$-process by the following reparametrization:

$$M^{\mathbb{Q}} = M^{\mathbb{P}} + Q^T\Lambda^T,$$

where the risk premium is given by $\lambda_t(\omega) = \Sigma_t\left(\sqrt{1-\rho^T\rho}l - \Lambda\rho\right)$.

Proof From (5), the market prices of risks associated to the Brownian motions W, B are as follows:

- for the matrix Brownian motion W there exists a matrix Λ such that

$$dW_t^{\mathbb{Q}} = dW_t - \sqrt{\Sigma_t}\Lambda dt, \tag{6}$$

  where $W_t^{\mathbb{Q}} \in M_n$ is a (matrix) Brownian motion under the risk-neutral measure $\mathbb{Q}$;
- for the Brownian motion B there exists a vector $l \in \mathbb{R}^n$ such that

$$dB_t^{\mathbb{Q}} = dB_t + \sqrt{\Sigma_t}ldt, \tag{7}$$

  where $B_t^{\mathbb{Q}}$ is a (vector) Brownian motion under $\mathbb{Q}$.

Given the functional form for the market prices of risks for all Brownian motions (6) and (7), it is easy to verify that the dynamics of the variance–covariance matrix are still Wishart with

$$M^{\mathbb{Q}} = M^{\mathbb{P}} + Q^T\Lambda^T,$$

so that the variance–covariance risk premium is given by the (affine) term $Q^T\Lambda^T\Sigma_t$, while the risk premium of the assets' returns is still an affine combination of the elements of Σ_t, in fact:

$$
\begin{aligned}
dS_t &= diag[S_t]\left((r\mathbf{1}+\lambda_t)\,dt+\sqrt{\Sigma_t}dZ_t\right)\\
&= diag[S_t]\left((r\mathbf{1}+\lambda_t)\,dt+\sqrt{\Sigma_t}\sqrt{1-\rho^T\rho}\left(dB_t^{\mathbb{Q}}-\sqrt{\Sigma_t}ldt\right)\right)\\
&\quad + diag[S_t]\sqrt{\Sigma_t}\left(dW_t^{\mathbb{Q}}+\sqrt{\Sigma_t}\Lambda dt\right)\rho\\
&= diag[S_t]\left((r\mathbf{1}+\lambda_t)\,dt-\Sigma_t\left(\sqrt{1-\rho^T\rho}l-\Lambda\rho\right)dt\right)\\
&\quad + \sqrt{\Sigma_t}\left(\sqrt{1-\rho^T\rho}dB_t^{\mathbb{Q}}+dW_t^{\mathbb{Q}}\rho\right)\\
&= diag[S_t]\left(r\mathbf{1}dt+\sqrt{\Sigma_t}dZ_t^{\mathbb{Q}}\right).
\end{aligned}
$$

Hence:

$$
\lambda_t = \Sigma_t\left(\sqrt{1-\rho^T\rho}l-\Lambda_t\rho\right).
$$

□

3 The analytical framework

3.1 The general pricing problem

In this section we show that using the WASC model, the general solution scheme to the pricing problem proposed in Bakshi and Madan (2000) extends to the pricing of a generic (European) derivative contract written on multiple underlyings. Following Bakshi and Madan (2000) we define the characteristic function of the state price density:

$$
\begin{aligned}
\Psi_{Y_t,\Sigma_t}(\gamma,\Gamma,t,\tau) &= \mathbb{E}_t^{\mathbb{P}}\left[\frac{H_{t+\tau}}{H_t}e^{\langle\gamma,Y_{t+\tau}\rangle+Tr[\Gamma\Sigma_{t+\tau}]}\right]\\
&= e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\langle\gamma,Y_{t+\tau}\rangle+Tr[\Gamma\Sigma_{t+\tau}]}\right], \qquad (8)\\
\gamma &\in i\mathbb{R}^n,\ \Gamma\in iM_{n\times n}\left(\mathbb{R}^n\right) \qquad (9)
\end{aligned}
$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the conditional expected value with respect to the risk-neutral measure, and asset returns $Y_{t+\tau}$ are defined by the equation $dY_t^i := dS_t^i/S_t^i$.

As remarked by Bakshi and Madan (2000), the central result of Fourier analysis states that the set of securities with payoff $e^{\langle\gamma,Y_{t+\tau}\rangle+Tr[\Gamma\Sigma_{t+\tau}]}$ $\gamma\in i\mathbb{R}^n$, $\Gamma\in iM_{n\times n}(\mathbb{R}^n)$ (also called "characteristic function claims") are a set of spanning securities for the set of contingent claims with integrable payoff depending on the factors $(Y_{t+\tau},\Sigma_{t+\tau})$. Consistently, the value of any security can be reconstructed given the value of characteristic function claims.

Let us consider the problem of pricing a contingent claim whose payoff is a function of final returns Y_T, say $F(Y_T, T)$. From the usual risk-neutral argument, the price P_0 of such options can be written as risk neutral expected value:

$$P_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[F(Y_T, T)],$$

and by applying standard arguments (see e.g. Bakshi and Madan 2000; Duffie et al. 2000; Carr and Madan 1999) it can be expressed in terms of the inverse Fourier transform of the payoff function and the characteristic function of the asset returns.

In fact,

$$\begin{aligned} P_0 &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[F(Y_T, T)] \\ &= e^{-rT}\frac{1}{(2\pi)^n}\mathbb{E}^{\mathbb{Q}}\int_{\mathcal{Z}} e^{-i\langle z,Y\rangle}\left(\int_{\mathbb{R}^n} e^{i\langle z,y\rangle}F(y,T)dy\right)dz \\ &= e^{-rT}\frac{1}{(2\pi)^n}\int_{\mathcal{Z}} \mathbb{E}^{\mathbb{Q}}\left[e^{-i\langle z,Y\rangle}\right]\left(\int_{\mathbb{R}^n} e^{i\langle z,y\rangle}F(y,T)dy\right)dz \\ &= \frac{1}{(2\pi)^n}\int_{\mathcal{Z}} \Psi_{Y_0,\Sigma_0}(-iz,0,0,T)\widehat{F}(z)dz, \end{aligned} \tag{10}$$

where $\langle .,.\rangle$ denotes the scalar product in $\mathbb{R}^n$ and

$$\widehat{F}(z) = \int_{\mathbb{R}^n} e^{i\langle z,Y\rangle}F(Y_T,T)dY_T$$

is the Fourier transform of the payoff function.

Therefore, in order to analytically solve the pricing problem, we have to:

1. compute the characteristic function of the assets' returns;
2. compute the Fourier transform of the specific payoff together with its admissible domain, i.e. the convergence set $\mathcal{Z} \subset \mathbb{C}^n$ in which the Fubini theorem implicitly used in (10) does hold;
3. compute (numerically) the remaining integral (e.g. through the FFT method).

In the following subsection we show that within the WASC model the computation of the first step (i.e. the characteristic function) can be performed explicitly, while in Sect. 4 we will provide an example of the whole methodology for the specific Best-of basket option.

3.2 The characteristic function of the WASC model

Let us recall for convenience the (discounted) characteristic function of the WASC model (8):

$$\Psi_{Y_t,\Sigma_t}(\gamma,\Gamma,t,\tau) = e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[e^{\langle\gamma,Y_{t+\tau}\rangle + Tr[\Gamma\Sigma_{t+\tau}]}\right] \tag{11}$$

By applying the Feynman-Kac argument, we obtain

$$\frac{\partial \Psi_{Y_t,\Sigma_t}(\gamma,\Gamma,t,\tau)}{\partial \tau} = \left(\mathcal{L}_{Y,\Sigma} - r\right) \Psi_{Y_t,\Sigma_t}(\gamma,\Gamma,t,\tau), \tag{12}$$

where $\mathcal{L}_{Y,\Sigma}$ denotes the infinitesimal generator of the couple (Y_t, Σ_t), which is given in the following:

Proposition 4

$$\begin{aligned}\mathcal{L}_{Y,\Sigma} = Tr&\left[\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right) D + 2\Sigma D Q^T Q D\right]\\ &+ \nabla_Y \left(r\mathbf{1} - \frac{1}{2} Vec\left[\Sigma^{ii}\right]\right) + \frac{1}{2}\nabla_Y \Sigma \nabla_Y^T\\ &+ 2Tr\left[D Q^T \rho \nabla_Y \Sigma\right],\end{aligned}$$

where D is a matrix differential operator with elements

$$D_{i,j} = \left(\frac{\partial}{\partial \Sigma^{ij}}\right).$$

Proof We can decompose the operator $\mathcal{L}_{Y,\Sigma}$ into three components:

$$\mathcal{L}_{Y,\Sigma} = \mathcal{L}_{\Sigma} + \mathcal{L}_{Y} + \mathcal{L}_{<Y,\Sigma>}, \tag{13}$$

where:

- $\mathcal{L}_{\Sigma}$ represents the infinitesimal generator for the Wishart process Σ_t, which has been computed by (Bru, 1991, p. 746) formula (5.12):

$$\mathcal{L}_{\Sigma} = Tr\left[\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right) D + 2\Sigma D Q^T Q D\right];$$

- $\mathcal{L}_Y$ denoted the infinitesimal generator of the assets' returns Y_t and is given by:

$$\begin{aligned}\mathcal{L}_Y &= \nabla_Y \left(r\mathbf{1} - \frac{1}{2} Vec\left[\Sigma^{ii}\right]\right) + \frac{1}{2}\nabla_Y \Sigma \nabla_Y^T\\ &= \sum_{i=1}^n \left(r - \frac{1}{2}\Sigma^{ii}\right)\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i,j=1}^n \Sigma^{ij}\frac{\partial^2}{\partial y_i \partial y_j},\end{aligned}$$

  where ∇_Y denotes the gradient operator, and finally
- $\mathcal{L}_{<Y,\Sigma>}$ denotes the cross infinitesimal generator, which involves the covariation among the assets' returns and the volatility noises. Since $< Y^i, \Sigma^{jk} >_t$ is associated to the term D_{jk} in the infinitesimal generator, from Proposition 1 we obtain

$$\begin{aligned}\mathcal{L}_{\langle Y,\Sigma\rangle} &= 2\sum_{i=1}^{n} Tr\left[\Sigma R_i Q D\right]\frac{\partial}{\partial y_i}\\ &= 2Tr\left[\left(\sum_{i=1}^{n} R_i \frac{\partial}{\partial y_i}\right) Q D \Sigma\right]\\ &= 2Tr\left[\begin{pmatrix}\rho_1\frac{\partial}{\partial y_1} & \cdots & \rho_n\frac{\partial}{\partial y_1}\\ \vdots & \ddots & \vdots\\ \rho_1\frac{\partial}{\partial y_n} & \cdots & \rho_n\frac{\partial}{\partial y_n}\end{pmatrix} Q D\Sigma\right]\\ &= 2Tr\left[\Sigma\nabla_Y^T \rho^T Q D\right],\end{aligned}$$

where $\rho = (\rho_1, \ldots, \rho_n)^T$. □

With a standard argument (see also Da Fonseca et al. 2005) we obtain the main result of this subsection:

Proposition 5 *The (discounted) characteristic function of the WASC model is given by*

$$\Psi_{Y_t,\Sigma_t}(\gamma, \Gamma, t, \tau) = e^{-r\tau}\exp\left\{Tr\left[A(\tau)\Sigma_t\right] + \gamma^T Y_t + c(\tau)\right\}, \tag{14}$$

where the deterministic function $A(t) \in M_n$ is as follows:

$$A(\tau) = \left(\Gamma A_2^1(\tau) + A_2^2(\tau)\right)^{-1}\left(\Gamma A_1^1(\tau) + A_1^2(\tau)\right), \tag{15}$$

with

$$\begin{pmatrix} A_1^1(\tau) & A_2^1(\tau)\\ A_1^2(\tau) & A_2^2(\tau)\end{pmatrix} = \exp\tau\begin{pmatrix} M + Q^T\rho\gamma^T & -2Q^T Q\\ \frac{1}{2}\left(\gamma\gamma^T - \sum_{i=1}^{n}\gamma_i e_{ii}\right) & -\left(M^T + \gamma\rho^T Q\right)\end{pmatrix}, \tag{16}$$

and where $c(\tau)$ can be obtained by direct integration, thus giving:

$$c(\tau) = -\frac{\beta}{2}Tr\left[\log\left(\Gamma A_2^1(\tau) + A_2^2(\tau)\right) + \tau M^T + \tau\gamma(\rho^T Q)\right] + \tau r\left(\gamma^T\mathbf{1} - 1\right). \tag{17}$$

with boundary conditions

$$\begin{aligned} A(0) &= \Gamma,\\ b(0) &= \gamma,\\ c(0) &= 0.\end{aligned}$$

Proof Since the characteristic function of the Wishart processes is exponentially affine (see e.g., Bru 1991), we look for three deterministic functions $A(t) \in M_n$; $b(t) \in \mathbb{R}^n$; $c(t) \in \mathbb{R}$ such that:

$$\Psi_{Y_t,\Sigma_t}(\gamma,\Gamma,t,\tau)$$
$$= e^{-r\tau}\exp\left\{Tr\left[A(\tau)\Sigma_t\right] + b^T(\tau)Y_t + c(\tau)\right\}.$$

By (12) we get,

$$\frac{\partial\Psi_{Y_t,\Sigma_t}}{\partial\tau} = Tr\left[\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)D + 2\Sigma D Q^T Q D\right]\Psi_{Y_t,\Sigma_t}$$
$$+ \nabla_Y\left(r\mathbf{1} - \frac{1}{2}Vec\left[Tr\left[e_{ii}\Sigma\right]\right]\right)\Psi_{Y_t,\Sigma_t} + \frac{1}{2}\nabla_Y\Sigma\nabla_Y^T\Psi_{Y_t,\Sigma_t}$$
$$+ 2Tr\left[DQ^T\rho\nabla_Y\Sigma\right]\Psi_{Y_t,\Sigma_t} - r\Psi_{Y_t,\Sigma_t},$$

and replacing the exponential affine expression for Ψ_{Y_t,Σ_t} we obtain

$$0 = -Tr\left[\frac{\partial}{\partial\tau}A(\tau)\Sigma\right] - \frac{\partial}{\partial\tau}b^T(\tau)Y - \frac{\partial}{\partial\tau}c(\tau) - r$$
$$+ b^T(\tau)\left(r\mathbf{1} - \frac{1}{2}Vec\left[Tr\left[e_{ii}\Sigma\right]\right]\right) + \frac{1}{2}b^T(\tau)\Sigma b(\tau)$$
$$+ Tr\left[\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)A(\tau) + 2\Sigma A(\tau)Q^T Q A(\tau)\right]$$
$$+ 2Tr\left[A(\tau)Q^T\rho b^T(\tau)\Sigma\right],$$

that is

$$0 = -Tr\left[\frac{\partial}{\partial\tau}A(\tau)\Sigma + \frac{\partial}{\partial\tau}b(\tau)Y^T\right] - \frac{\partial}{\partial\tau}c(\tau) - r$$
$$+ Tr\left[r\mathbf{1}b^T(\tau) - \frac{1}{2}\sum_{i=1}^{n}b_i(\tau)e_{ii}\Sigma + \frac{1}{2}b(\tau)b^T(\tau)\Sigma\right]$$
$$+ Tr\left[\left(\Omega\Omega^T + M\Sigma + \Sigma M^T\right)A(\tau) + 2\Sigma A(\tau)Q^T Q A(\tau) + 2A(\tau)Q^T\rho b^T(\tau)\Sigma\right]$$

with boundary conditions

$$A(0) = \Gamma,$$
$$b(0) = \gamma,$$
$$c(0) = 0.$$

By identifying the coefficients of Y we deduce

$$\frac{\partial}{\partial\tau}b(\tau) = 0,$$

hence $b(\tau) = \gamma$, for all τ.

By identifying the coefficients of Σ we obtain the Matrix Riccati ODE satisfied by $A(\tau)$:

$$\begin{aligned}
\frac{\partial}{\partial\tau}A(\tau) &= A(\tau)M + M^T A(\tau) - \frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii} + 2A(\tau)Q^T QA(\tau) + \frac{1}{2}\gamma\gamma^T \\
&\quad + A(\tau)Q^T\rho\gamma^T + \gamma\rho^T QA(\tau) \\
&= A(\tau)\left(M + Q^T\rho\gamma^T\right) + \left(M^T + \gamma\rho^T Q\right)A(\tau) + 2A(\tau)Q^T QA(\tau) \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\gamma_i e_{ii} + \frac{1}{2}\gamma\gamma^T \\
A(0) &= \Gamma.
\end{aligned}$$

We recall that a Matrix Riccati ODE can be linearized by doubling the dimension of the problem (see Grasselli and Tebaldi 2008 for a proof of the necessary and sufficient conditions for the linearizability over the entire time interval, which in turn grants the invertibility of the following matrices), thus obtaining

$$A(\tau) = \left(A(0)A_2^1(\tau) + A_2^2(\tau)\right)^{-1}\left(A(0)A_1^1(\tau) + A_1^2(\tau)\right),$$

which leads to the result. Finally, as usual the function $c(\tau)$ can be obtained by direct integration:

$$\begin{aligned}
\frac{\partial}{\partial\tau}c(\tau) &= Tr\left[r\mathbf{1}\gamma^T + \Omega\Omega^T A(\tau)\right] - r, \\
c(0) &= 0.
\end{aligned} \tag{18}$$

□

We emphasize that (16) and (17) are truly closed-form solution and do not involve any non-trivial integration, contrarily to the solution proposed by Gourieroux and Sufana (2004a,b) based on the variation of constants.

4 Financial issues

In this section we show that the description of financial market stochastic correlations, obtained applying the WASC model, nicely reproduces many financial stylized facts and provides a consistent pricing framework for both vanilla (single asset) and multiple asset options such as rainbow basket options.

We perform this illustration providing some numerical results in the simplest case where only two assets are traded in the market. This choice has been made to simplify the exposition but we stress that our analysis remains valid for an arbitrary number of assets. In addition, the linear-affine assumption implies that linear factor analysis can be applied even in the presence of stochastic correlations. In particular, the possibility that a systematic market effect can be identified when the number of assets increases seems to be a reasonable and interesting possibility which deserves more empirical investigation.

4.1 Pricing plain vanilla options: a numerical illustration

In its construction, the WASC model is consistent with the single-asset Heston stochastic volatility model and can reproduce the stylized effects of the implied volatility surface, like the celebrated smile and skew effects. This ability comes directly from the fact that in the WASC model, the single assets evolve according to Heston-like dynamics: the price follows a conditionally log-normal process and the stochastic volatility process is driven by a Brownian noise which is partially correlated with the noise-driving assets' returns. However, due to the presence of a stochastic correlation between asset prices, a novel effect arises: the skew on each single asset's vanilla could possibly depend from cross-correlation effects as we explain below.

As shown in Lewis (2000), in the Heston model the main contribution to the short term skew is given by the volatility of volatility multiplied by the (scalar) correlation between the asset noise and the volatility noise. If for example we compute the correlation between the first asset return Y_1 and its variance Σ^{11} in the WASC model we get:

$$\begin{aligned} Corr_t\left(Noise(Y^1), Noise(Vol(S^1))\right) &= \frac{\left\langle Y^1, \Sigma^{11}\right\rangle_t}{\sqrt{\Sigma_t^{11}}\sqrt{\left\langle\Sigma^{11}\right\rangle t}} \\ &= \frac{2\Sigma_t^{11} Tr\left[R_1 Q\right]}{\sqrt{\Sigma_t^{11}}\sqrt{4\Sigma_t^{11}\left(Q_{11}^2+Q_{21}^2\right)}} \\ &= \frac{Q_{11}\rho_1 + Q_{21}\rho_2}{\sqrt{Q_{11}^2+Q_{21}^2}}, \end{aligned} \tag{19}$$

where, from Proposition 1, we can evaluate $\left\langle Y^1, \Sigma^{11}\right\rangle_t$ and also

$$d < \Sigma^{11}, \Sigma^{11} >_t = 4\Sigma_t^{11}(Q_{11}^2 + Q_{21}^2)dt. \tag{20}$$

Notice that, consistently with the Heston model, the correlations between the noise driving asset's returns and their volatilities are still deterministic.

Remark 1 Da Fonseca et al. (2005) chose the following dynamics for the single-asset model:

$$\frac{dS_t}{S_t} = rdt + Tr\left[\sqrt{\Sigma_t}dZ_t\right],$$

where as usual Σ_t follows (1). This leads to a richer structure for the volatility surface which permits to obtain a stochastic correlation between the asset's noise and its volatility.

The usual Heston correlations between returns and volatilities are recovered by considering a diagonal choice of the matrix Q

$$\begin{aligned}
Corr_t\left(Noise(Y^1), Noise(Vol(S^1))\right) &= \rho_1 \\
Corr_t\left(Noise(Y^2), Noise(Vol(S^2))\right) &= \rho_2.
\end{aligned}$$

On the other hand, if we relax the independence hypothesis and consider the more realistic assumption that assets are correlated, the WASC model highlights a new and remarkable effect: the correlation among returns and volatilities (19) of each asset involves all the coefficients ρ_i through the off-diagonal parameters of the matrix Q. In financial terms this implies that the model is flexible enough to take into account a situation where skews on single-asset implied volatilities are influenced by cross effects with other assets.

Remark 2 Gourieroux and Sufana (2004a,b) chose the trivial correlation structure corresponding to (in the 2-dim case)

$$\rho_1 = \rho_2 = 0,$$

and as a consequence their model is not consistent with the smile and skew effects on vanilla options. In an independent work, Buraschi et al. (2006) chose a special correlation structure corresponding to

$$\rho_1 = 1, \quad \rho_2 = 0.$$

In order to visualize the above effects, we consider a numerical experiment: the parameters used in our computations are given by:

$$\begin{aligned}
&M = \begin{pmatrix} -2.5 & -1.5 \\ -1.5 & -2.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.21 & -0.14 \\ 0.14 & 0.21 \end{pmatrix} \\
&\rho_1 = \rho_2 = -0.6.
\end{aligned}$$

The Gindikin coefficient is $\beta = 7.14286$, the interest rate $r = 0$ and the initial volatility matrix is given by

$$\Sigma_0 = \begin{pmatrix} 0.09 & -0.036 \\ -0.036 & 0.09 \end{pmatrix}.$$

Observe that the dynamics of the two assets differ only for the off-diagonal term in Q while $\sqrt{Q_{12}^2 + Q_{22}^2} = \sqrt{Q_{11}^2 + Q_{21}^2}$ are kept constant. With this choice the differences between the implied volatilities of asset 1 and asset 2 quantify the impact that the off-diagonal terms in the matrix Q have on the implied volatility surfaces, which are shown in Fig. 1.

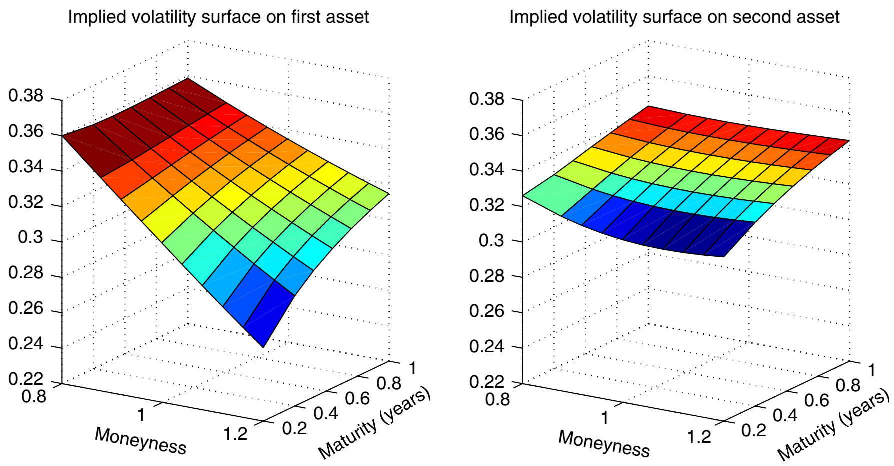

Fig. 1 Implied volatility surfaces

They show the typical smile-skew effect consistent with the Heston model. Since the (Heston) correlations between returns and volatility are given by (19):

$$Corr_t\left(Noise(Y^1), Noise(Vol(Y^1))\right) = -0.832$$
$$Corr_t\left(Noise(Y^2), Noise(Vol(Y^2))\right) = \frac{Q_{12}\rho_1 + Q_{22}\rho_2}{\sqrt{Q_{12}^2 + Q_{22}^2}} = -0.166,$$

in line with the Lewis analysis, a higher (absolute) level of the correlation implies a more pronounced skew effect of the first asset with respect to the second one.

4.2 WASC model calibrated on historical data

An estimation procedure of the WASC model based on the empirical characteristic function has been developed in Da Fonseca et al. (2007). The model has been estimated under the historical measure using a data set consisting of the time series for S&P500 and Dax daily quotes starting on January 1990 and ending on June 2007. The estimated parameters have been:

$$M = \begin{pmatrix} -3,635 & 1,21 \\ 0,679 & -2,809 \end{pmatrix}, \quad Q = \begin{pmatrix} -0,017 & -0,087 \\ -0,074 & -0,019 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0,454 \\ 0,406 \end{pmatrix},$$

with the Gindikin coefficient being equal to $\beta = 10{,}639$. Observe that the matrix M is definite negative, thus insuring a mean reverting behavior of the Wishart process. To analyze the model parameters it is worth to compute the single asset Heston correlation (19) and volatility of volatility (20), which are reported in Table 1.

Consistently with known results we found a negative correlation between the asset returns and their volatilities. The values for the volatility of volatility and correlation

Table 1 Volatility of volatility and skew for the S&P and DAX

	SP/DAX
Vol vol asset 1	0,152
Vol vol asset 2	0,178
Skew asset 1	−0,49
Skew asset 2	−0,53

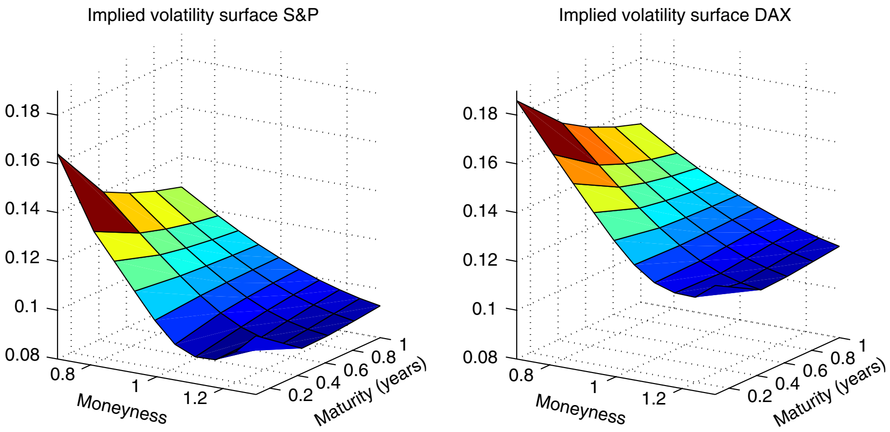

Fig. 2 Implied volatility surfaces on S&P and Dax vanilla options

are close to the ones obtained by Eraker et al. (2007) on the S&P500, thus aggregated values compare well with the calibrated single asset Heston model. Although the estimation has been done under the historical probability measure (on the spots) and not on vanilla options, for illustrative purpose in Fig. 2 we plot the implied volatility surface obtained from the model calibrated under the historical measure. As could be expected based on Table 3 and 4 of Eraker (2004), the smiles and skews obtained in the illustration are in qualitative agreement with those expected from the calibration to observed option prices. A complete calibration under the pricing measure will be performed in a future work.

4.2.1 Correlation risk analysis

Correlation risks are among the most relevant sources of incompleteness which arise when moving from single asset to multiple asset markets. In view of the above technical results, the WASC model provides a natural framework to value the impact of multivariate risks and dynamic correlations on prices and properly extends the framework of Bakshi and Madan (2000) where the pricing problem in multiple asset markets with stochastic volatility was solved under the hypothesis of constant instantaneous correlations.

We analyze first a stylized effect which appears to be a stylized effect highlighted in the time series analysis of asset returns, then we consider the even more dramatic effects of these new risk sources on the pricing of derivative contracts.

4.2.2 Asymmetric conditional correlation effect

A striking example of the WASC ability to reproduce stylized effects documented in time, series analysis of financial markets is given by the following computation which allows to describe analytically the asymmetric conditional correlation effect empirically documented by Roll (1988) and more recently by Ang and Chen (2002). These authors observed that looking at historical time series, an asymmetric response of correlations to positive and negative shocks on historical asset returns appears: a decrease in correlation will increase the dispersion of individual assets and thus the probability that any asset may reach high values.

This effect in the WASC model can be quantified computing the closed-form expression of the covariation between the asset returns Y^i and the stochastic cross correlation, ρ_t^{12}, between assets 1 and 2 defined by

$$\rho_t^{12} = \frac{\Sigma_t^{12}}{\sqrt{\Sigma_t^{11}\Sigma_t^{22}}}. \tag{21}$$

Proposition 6

$$d < Y^i, \rho^{12} >_t = \left(\sqrt{\frac{\Sigma_t^{ii}}{\Sigma_t^{jj}}}\left(1 - \left(\rho_t^{12}\right)^2\right) Tr\left[R_j Q\right]\right) dt \quad i, j = 1, 2.$$

Proof Differentiating $\rho_t^{12}\sqrt{\Sigma_t^{11}\Sigma_t^{22}} = \Sigma_t^{12}$ we obtain

$$\sqrt{\Sigma_t^{11}\Sigma_t^{22}} d\rho_t^{12} + \rho_t^{12} d\left(\sqrt{\Sigma_t^{11}\Sigma_t^{22}}\right) + (.)dt = d\Sigma_t^{12};$$

$$\sqrt{\Sigma_t^{11}\Sigma_t^{22}} d\rho_t^{12} + \rho_t^{12}\left(\frac{1}{2}\sqrt{\frac{\Sigma_t^{22}}{\Sigma_t^{11}}} d\Sigma_t^{11} + \frac{1}{2}\sqrt{\frac{\Sigma_t^{11}}{\Sigma_t^{22}}} d\Sigma_t^{22}\right) + (.)dt = d\Sigma_t^{12},$$

then

$$d\rho_t^{12} = \frac{1}{\sqrt{\Sigma_t^{11}\Sigma_t^{22}}} d\Sigma_t^{12} - \frac{1}{2}\rho_t^{12}\left(\frac{1}{\Sigma_t^{11}} d\Sigma_t^{11} + \frac{1}{\Sigma_t^{22}} d\Sigma_t^{22}\right) + (.)dt.$$

Now we take the covariation with the first asset returns and obtain

$$
\begin{aligned}
d < Y^1, \rho^{12} >_t &= \frac{1}{\sqrt{\Sigma_t^{11}\Sigma_t^{22}}} d < Y^1, \Sigma^{12} >_t \\
&\quad - \frac{1}{2}\rho_t^{12}\left(\frac{1}{\Sigma_t^{11}} d < Y^1, \Sigma^{11} >_t + \frac{1}{\Sigma_t^{22}} d < Y^1, \Sigma^{22} >_t\right) \\
&= \frac{1}{\sqrt{\Sigma_t^{11}\Sigma_t^{22}}}\left(\Sigma_t^{11} Tr\left[R_2 Q\right] + \Sigma_t^{12} Tr\left[R_1 Q\right]\right) \\
&\quad - \rho_t^{12}\left(Tr\left[R_1 Q\right] + \frac{\Sigma_t^{12}}{\Sigma_t^{22}} Tr\left[R_2 Q\right]\right) \\
&= Tr\left[R_2 Q\right]\left(\frac{\Sigma_t^{11}}{\sqrt{\Sigma_t^{11}\Sigma_t^{22}}} - \rho_t^{12}\frac{\Sigma_t^{12}}{\Sigma_t^{22}}\right),
\end{aligned}
$$

which gives the result. □

From the historical estimation of the WASC model we can evaluate the size and the direction of this asymmetry. Since $Tr\left[R_1 Q\right] < 0$ we get (as expected) a negative response of the correlation to return variations. In order to quantify the size of the effect we recover the asymptotic levels of the variance-covariance matrix Σ_∞ from Eq. 4 and we get:

$$
\Sigma_\infty = \begin{bmatrix} 0.01406 & 0.00857 \\ 0.00857 & 0.01284 \end{bmatrix}
$$

which corresponds to a level of the cross correlation $\rho_{12}^{\infty} = 0.6382$. Inserting the values in (21) and normalizing by the standard deviation of the return Y_1

$$
\frac{1}{\sqrt{\Sigma_{11}}} \frac{d < Y^1, \rho^{12} >_t}{dt} = \frac{1}{\sqrt{0.01406}}(-0.0234) = -0.1975
$$

thus we obtain that the expected variation of the correlation level determined by a positive return fluctuation of one standard deviation is about -0.2.

4.3 Valuation of the "Best-of" basket option

Correlation fluctuations are expected to play a major role in the valuation of multiple asset derivatives and in its deviations from the constant volatility B&S setup.

In order to exemplify the ability of the WASC process to quantify the effect of correlation risks on basket option prices, we continue the numerical illustration of Sect. 4.2 and we analyze the particular example of the Best-of two assets option contract. This traded basket option is well known to be a "correlation-dependent product", as well described e.g. in Fengler and Schwendner (2004), while maintaining a closed-form solution for the option price. The payoff of the Best-of rainbow option is given by:

$$\max\left(\max(S_T^1, S_T^2) - K, 0\right),$$

where K denotes the strike price and $S_0^1 = S_0^2 = S_0$ by normalization.

Proposition 7 *The price function of the Best-of contract is given by:*

$$P_0^{Best\text{-}of} = e^{-rT}\frac{1}{(2\pi)^2}\left(\int_{\mathcal{Z}_1}\Psi_{Y_0,\Sigma_0}(-iz,0,0,T)\widehat{F}_1(z)dz\right.$$
$$\left.+\int_{\mathcal{Z}_2}\Psi_{Y_0,\Sigma_0}(-iz,0,0,T)\widehat{F}_2(z)dz\right),$$

where $\widehat{F}_1(z)$, $\widehat{F}_2(z)$ *are given by*

$$\begin{aligned}\widehat{F}_1(z) &= -\frac{e^{(1+i(z_1+z_2))k}}{z_2(z_1+z_2)(1+i(z_1+z_2))}\\ \widehat{F}_2(z) &= -\frac{e^{(1+i(z_1+z_2))k}}{z_1(z_1+z_2)(1+i(z_1+z_2))},\end{aligned}$$

and whose integrability domains are given by

$$\begin{aligned}\mathcal{Z}_1 &= \left\{z \in \mathbb{C}^2 : \mathrm{Im}(z_2) \le 0 \text{ and } \mathrm{Im}(z_1) + \mathrm{Im}(z_2) \ge 1\right\}\\ \mathcal{Z}_2 &= \left\{z \in \mathbb{C}^2 : \mathrm{Im}(z_1) \le 0 \text{ and } \mathrm{Im}(z_1) + \mathrm{Im}(z_2) \ge 1\right\}.\end{aligned}$$

Proof See the Appendix □

The impact of the dynamic correlations on the Best-of price is confirmed by plotting the option price as a function of the long term correlation where a monotonic relation is found (see Fig. 3).

A more accurate analysis of the impact of correlations on the prices of basket options can be obtained extending the notion of implied volatilities to the analysis of correlations. We recall that the implied volatility for a standard option corresponds to the level of volatility that makes the price of the (observed on the market) option contract equal to the Black & Scholes price for a fixed level of the initial price S_0, of the strike K and of the time to maturity T; in full analogy we define an implied correlation level as the level of the correlation that makes the observed price of the Best-of contract equal to the B&S one; notice however that for the B&S basket option price we need to specify also the levels of volatilities for each underlying asset.

In order to be consistent with the market calibration of the model, we fix the single asset volatilities to be the implied levels obtained from the observed prices of plain vanilla options with the same strike and the same maturity of the Best-of basket option.

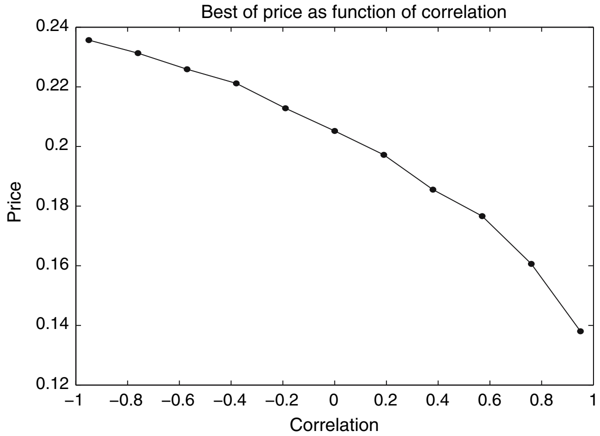

Fig. 3 Inverse relation between correlation level and Best-of price quoted at the money (maturity 1 year)

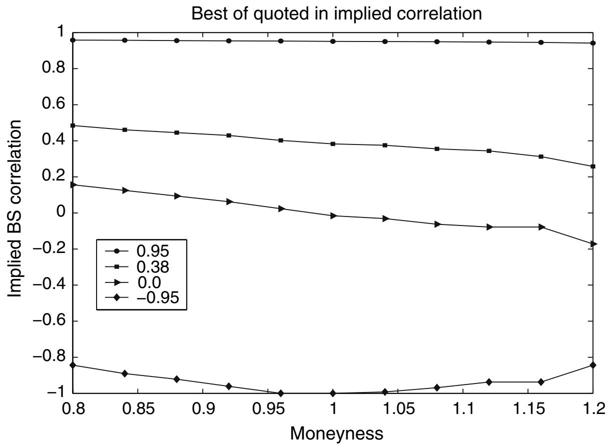

Fig. 4 Best-of option price quoted in implied correlation

In this way, the implied correlation remains indirectly defined by the relation:

$$P_{BS}^{Best\text{-}of}\left(\sigma_1^{imp}(K/S_0), \sigma_2^{imp}(K/S_0), \rho_{12}^{imp}(K/S_0)\right) = P_{WASC}^{Best\text{-}of}(K/S_0)$$

In the Fig. 4 we plot for different levels of ρ_{12}^{∞} the level of implied correlations $\rho_{12}^{imp}(K/S_0)$ against the moneyness K/S_0. We consider small deviations from the money options since these are essentially the only ones which are liquidly traded. A straight horizontal line will correspond to constant correlations, in fact, fixing the volatilities to their implied levels, the remaining deviations from a flat line of the implied correlation plot represents the effect induced by stochastic correlations on the basket price.

Quite remarkably we can identify three different regimes; when correlations are strongly negative (see Fig. 4) there's no systematic skew effect and a smile shape appears, on the contrary as ρ_{12}^{∞} increases and approaches $\rho_{12}^{\infty} = 0$ a pronounced downward sloping skew appears and finally, when ρ_{12}^{∞} approaches 1 we recover a straight line because perfect correlation implies that the Best-of option reduces to a single plain vanilla option.

The WASC model can describe and quantify this asymmetric effect and the induced "implied correlation skew" in the same way as the Heston model reproduces the so-called volatility skew thanks to the "volatility leverage effect" (a negative correlation between stock returns and volatility shocks).

Observe that skew of equity vanilla options are typically decreasing, implying that the terms $Tr\left[R_j Q\right]$ will typically be negative, as one can deduce from (19). Recalling that the indicator function has bounded variation and $\max(Y_T^1, Y_T^2) = Y_T^1 \mathbb{I}_{Y^1 > Y^2} + Y_T^2 \mathbb{I}_{Y^2 > Y^1}$, the Proposition 6 implies:

$$\begin{aligned} & d < \max(Y^1, Y^2), \rho^{12} >_t \\ & \quad = \left(1 - \left(\rho_t^{12}\right)^2\right) \left(\mathbb{I}_{Y^1 > Y^2} \sqrt{\frac{\Sigma_t^{11}}{\Sigma_t^{22}}} Tr\left[R_2 Q\right] + \mathbb{I}_{Y^2 > Y^1} \sqrt{\frac{\Sigma_t^{22}}{\Sigma_t^{11}}} Tr\left[R_1 Q\right] \right) dt \end{aligned} \tag{22}$$

hence a negative skew on single options implies a negative response of correlations to an increase of asset prices, then a "correlation leverage" effect on the price of the Best-of option.

In the Heston model the implied volatility skew is due to the covariation between returns and volatility. i.e. the volatility leverage. In full analogy it is reasonable to relate the skew of the implied correlation with the "correlation leverage", i.e. the quadratic covariation between returns and stochastic correlations. Equation 22 shows that the "correlation leverage" is proportional to the coefficient $1 - \left(\rho_{12}^{\infty}\right)^2$. In order to test whether the correlation skew also shows the same dependence with respect to ρ_{12}^{∞} we performed a regression of the skew slopes against the coefficient $1 - \left(\rho_{12}^{\infty}\right)^2$ finding an approximate linear relation $R^2 \simeq 0, 96$. These numerical results suggest that the implied correlation plot as defined above can be considered as a sensible indicator of the impact of stochastic correlations on market prices.

5 Conclusions

In this paper we introduced an analytical framework where correlations among assets evolve stochastically. We characterized all linear correlation structures between underlying assets and their volatilities which are fully consistent with the smile and skew effects on vanilla options. The model proposed is flexible enough to accommodate documented stylized facts like correlation leverage effects on equity markets. In the special case of a Best-of basket option we described and quantified the impact of the correlation risk on option prices through the notion of implied correlation. Future research will be devoted to the market calibration of the model on observed prices and to the estimation of the historical and risk-neutral joint dynamics of variance–covariances of asset returns. More particularly, we will investigate in a forthcoming paper the possibility of extending within this model the notions of risk factor decomposition and dynamic principal components analysis. Such extensions appear to be the necessary ingredients in order to provide an accurate description of correlation risk and to quantify its market value.

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Appendix

Proof of Proposition 7 Notice that the integrability set $\mathcal{Z}$ in (10) is empty. In fact, due to the particular payoff function, we must distinguish between the two possibilities $\max\left(Y_T^1, Y_T^2\right) = Y_T^1$ and $\max\left(Y_T^1, Y_T^2\right) = Y_T^2$, so let us write the payoff function as follows:

$$\begin{aligned} F(Y_T, T) &= \left(e^{Y_T^1} - e^k\right)_+ I_{\{Y_T^1 \geq Y_T^2\}} + \left(e^{Y_T^2} - e^k\right)_+ I_{\{Y_T^1 \leq Y_T^2\}} \\ &=: F_1(Y_T, T) + F_2(Y_T, T). \end{aligned}$$

Now we come back to the definition of the option price and we can obtain separate integrability conditions for the two functions defining the payoff:

$$\begin{aligned} price_0^{Best\text{-}of} &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{\max\left(Y_T^1, Y_T^2\right)} - e^k\right)_+\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[F_1(Y_T, T)\right] + e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[F_2(Y_T, T)\right] \\ &= e^{-rT}\frac{1}{(2\pi)^2}\mathbb{E}^{\mathbb{Q}}\int_{\mathcal{Z}_1} e^{-i\langle z, Y\rangle}\left(\int_{\mathbb{R}^2} e^{i\langle z, Y\rangle}F_1(Y_T, T)dY_T\right)dz \\ &\quad + e^{-rT}\frac{1}{(2\pi)^2}\mathbb{E}^{\mathbb{Q}}\int_{\mathcal{Z}_2} e^{-i\langle z, Y\rangle}\left(\int_{\mathbb{R}^2} e^{i\langle z, Y\rangle}F_2(Y_T, T)dY_T\right)dz \\ &= e^{-rT}\frac{1}{(2\pi)^2}\left(\int_{\mathcal{Z}_1}\Psi_Y(-iz)\widehat{F}_1(z)dz + \int_{\mathcal{Z}_2}\Psi_Y(-iz)\widehat{F}_2(z)dz\right), \end{aligned}$$

where

$$\begin{aligned} \widehat{F}_1(z) &= \int_{\mathbb{R}^2} e^{i\langle z, Y\rangle}F_1(Y_T, T)dY_T \\ &= \int_{\mathbb{R}^2} e^{i\langle z, Y\rangle}\left(\left(e^{Y_T^1} - e^k\right)_+ I_{\{Y_T^1 \geq Y_T^2\}}\right)dY_T, \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} e^{iz_1y_1+iz_2y_2}\left(\left(e^{y_1}-e^k\right)_+ I_{\{y_1\geq y_2\}}\right)dy_1dy_2\\
&= \int_{\mathbb{R}} e^{iz_1y_1}\left(e^{y_1}-e^k\right)_+ dy_1 \int_{-\infty}^{y_1} e^{iz_2y_2}dy_2\\
&= \int_{\mathbb{R}} e^{iz_1y_1}\left(e^{y_1}-e^k\right)_+ dy_1 \frac{e^{iz_2y_1}}{iz_2} \qquad \text{(provided that } \operatorname{Im}(z_2)\leq 0)\\
&= \int_{k}^{+\infty}\left(\frac{e^{(1+i(z_1+z_2))y_1}}{iz_2}-\frac{e^{k+i(z_1+z_2)y_1}}{iz_2}\right)dy_1\\
&= \left[\frac{e^{(1+i(z_1+z_2))y_1}}{iz_2\,(1+i\,(z_1+z_2))}-\frac{e^{k+i(z_1+z_2)y_1}}{i\,(z_1+z_2)\,iz_2}\right]_k^{+\infty}\\
&= -e^{(1+i(z_1+z_2))k}\left(\frac{1}{iz_2\,(1+i\,(z_1+z_2))}+\frac{1}{(z_1+z_2)\,z_2}\right)\\
&\quad \text{(provided that } \operatorname{Im}(z_1)+\operatorname{Im}(z_2)\geq 1)\\
&= -\frac{e^{(1+i(z_1+z_2))k}}{z_2\,(z_1+z_2)\,(1+i\,(z_1+z_2))},
\end{aligned}
$$

so that

$$
\mathcal{Z}_1 = \left\{z\in\mathbb{C}^2 : \operatorname{Im}(z_2)\leq 0 \text{ and } \operatorname{Im}(z_1)+\operatorname{Im}(z_2)\geq 1\right\}.
$$

By the symmetry of the problem we immediately obtain:

$$
\begin{aligned}
\widehat{F}_2(z) &= \int_{\mathbb{R}^2} e^{i\langle z,Y\rangle}F_2(Y_T,T)dY_T\\
&= \int_{\mathbb{R}^2} e^{i\langle z,Y\rangle}\left(\left(e^{Y_T^2}-e^k\right)_+ I_{\{Y_T^1\leq Y_T^2\}}\right)dY_T\\
&= \int_{\mathbb{R}^2} e^{iz_1y_1+iz_2y_2}\left(\left(e^{y_2}-e^k\right)_+ I_{\{y_1\leq y_2\}}\right)dy_1dy_2\\
&= -\frac{e^{(1+i(z_1+z_2))k}}{z_1\,(z_1+z_2)\,(1+i\,(z_1+z_2))},
\end{aligned}
$$

with

$$
\mathcal{Z}_2 = \left\{z\in\mathbb{C}^2 : \operatorname{Im}(z_1)\leq 0 \text{ and } \operatorname{Im}(z_1)+\operatorname{Im}(z_2)\geq 1\right\}.
$$

□

Notice that the above procedure can be easily extended in the case $n = 3$. In fact, in this case the payoff function is given by

$$F(Y_T, T) = \max\left(e^{\max\left(Y_T^1, Y_T^2, Y_T^3\right)} - e^k, 0\right),$$

and its Fourier transform is given by:

$$\widehat{F}(z) = \int_{\mathbb{R}^3} e^{i\langle z,Y\rangle} F(Y_T, T) dY_T.$$

In analogy with the 2 dimensional case, we can write

$$\begin{aligned} F(Y_T, T) \; &= \; \left(e^{Y_T^1} - e^k\right)_+ I_{\left\{Y_T^1 \geq \max\left(Y_T^2, Y_T^3\right)\right\}} + \left(e^{Y_T^2} - e^k\right)_+ I_{\left\{Y_T^2 \geq \max\left(Y_T^1, Y_T^3\right)\right\}} \\ &\quad + \left(e^{Y_T^3} - e^k\right)_+ I_{\left\{Y_T^3 \geq \max\left(Y_T^1, Y_T^2\right)\right\}} \\ &=: F_1(Y_T, T) + F_2(Y_T, T) + F_3(Y_T, T). \end{aligned}$$

Now

$$\begin{aligned} \widehat{F}_1(z) &= \int_{\mathbb{R}^3} e^{i\langle z,Y\rangle} F_1(Y_T, T) dY_T \\ &= \int_{\mathbb{R}} e^{iz_1y_1} \left(e^{y_1} - e^k\right)_+ dy_1 \int_{\mathbb{R}^2} \left(I_{\{y_1>y_2>y_3\}} + I_{\{y_1>y_3>y_2\}}\right) e^{iz_2y_2+iz_3y_3} dy_2 dy_3 \\ &= \int_{\mathbb{R}} e^{iz_1y_1} \left(e^{y_1} - e^k\right)_+ dy_1 \left[\int_{-\infty}^{y_1} \frac{e^{i(z_2+z_3)y_2}}{iz_3} dy_2 + \int_{-\infty}^{y_1} \frac{e^{i(z_2+z_3)y_3}}{iz_2} dy_3\right] \\ &= \int_{\mathbb{R}} e^{iz_1y_1} \left(e^{y_1} - e^k\right)_+ dy_1 e^{i(z_2+z_3)y_1} \left(-\frac{1}{z_3(z_2+z_3)} - \frac{1}{z_2(z_2+z_3)}\right) \\ &\quad \text{(provided that } \operatorname{Im}(z_2) \leq 0, \operatorname{Im}(z_3) \leq 0) \\ &= -\frac{1}{z_3z_2} \int_k^{+\infty} \left(e^{(1+i(z_1+z_2+z_3))y_1} - e^{k+i(z_1+z_2+z_3)y_1}\right) dy_1 \\ \widehat{F}_1(z) &= \frac{1}{z_3z_2} \left(\frac{e^{(1+i(z_1+z_2+z_3))k}}{1+i(z_1+z_2+z_3)} - \frac{e^{(1+i(z_1+z_2+z_3))k}}{i(z_1+z_2+z_3)}\right) \\ &\quad \text{(provided that } \operatorname{Im}(z_1) + \operatorname{Im}(z_2) + \operatorname{Im}(z_3) \geq 1) \\ &= -\frac{e^{(1+i(z_1+z_2+z_3))k}}{iz_3z_2\left(1+i(z_1+z_2+z_3)\right)(z_1+z_2+z_3)}, \end{aligned}$$

with

$$\mathcal{Z}_1 = \left\{ z \in \mathbb{C}^2 : \operatorname{Im}(z_2) \leq 0, \operatorname{Im}(z_3) \leq 0 \text{ and } \operatorname{Im}(z_1) + \operatorname{Im}(z_2) + \operatorname{Im}(z_3) \geq 1 \right\}.$$

Analogously,

$$\begin{aligned}\widehat{F}_2(z) &= \int_{\mathbb{R}^3} e^{i\langle z,Y\rangle} F_2(Y_T, T)dY_T \\ &= -\frac{e^{(1+i(z_1+z_2+z_3))k}}{iz_1z_3\,(1+i(z_1+z_2+z_3))\,(z_1+z_2+z_3)},\end{aligned}$$

with

$$\mathcal{Z}_2 = \left\{ z \in \mathbb{C}^2 : \operatorname{Im}(z_1) \leq 0, \operatorname{Im}(z_3) \leq 0 \text{ and } \operatorname{Im}(z_1) + \operatorname{Im}(z_2) + \operatorname{Im}(z_3) \geq 1 \right\},$$

and

$$\begin{aligned}\widehat{F}_3(z) &= \int_{\mathbb{R}^3} e^{i\langle z,Y\rangle} F_3(Y_T, T)dY_T \\ &= -\frac{e^{(1+i(z_1+z_2+z_3))k}}{iz_1z_2\,(1+i(z_1+z_2+z_3))\,(z_1+z_2+z_3)},\end{aligned}$$

with

$$\mathcal{Z}_3 = \left\{ z \in \mathbb{C}^2 : \operatorname{Im}(z_1) \leq 0, \operatorname{Im}(z_2) \leq 0 \text{ and } \operatorname{Im}(z_1) + \operatorname{Im}(z_2) + \operatorname{Im}(z_3) \geq 1 \right\}.$$

The general case can be similarly handled with some notational heaviness: for example, the first term can be computed as follows:

$$\begin{aligned}\widehat{F}_1(z) &= \int_{\mathbb{R}^n} e^{i\langle z,Y\rangle} F_1(Y_T, T)dY_T \\ &= \int_{\mathbb{R}} e^{iz_1y_1}\left(e^{y_1} - e^k\right)_+ dy_1 \int_{\mathbb{R}^{n-1}} \left(\sum_{\sigma(2,\ldots,n)} I_{\{y_1 > y_{\sigma(2)} > \cdots > y_{\sigma(n)}\}} \right) \\ &\quad \times e^{iz_2y_2 + iz_3y_3 + \cdots + iz_ny_n} dy_2dy_3 \ldots dy_n,\end{aligned}$$

where $\sigma(2, \ldots, n)$ denotes the permutations of the terms $\{2, \ldots, n\}$.

References

Ang, A., & Chen, J. (2002). Asymmetric correlations on equity portfolios. *Journal of Financial Economics, 63*, 443–494.

Bakshi, G. S., & Madan, D. B. (2000). Spanning and derivative-security valuation. *Journal of Financial Economics, 55*, 205–238.

Bru, M. F. (1991). Wishart processes. *Journal of Theoretical Probability, 4*, 725–743.
Buraschi, A., Porchia, P., & Trojani, F. (2006). Correlation hedging. Preprint. Now available at SSRN: http://www.ssrn.com/abstract=908664 as "Correlation Risk and Optimal Portfolio Choice".
Carr, P., & Madan, D. B. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance, 2*, 61–73.
Da Fonseca, J., Grasselli, M., & Ielpo, F. (2007). Estimating the Wishart Affine Stochastic Correlation model using the empirical characteristic function. Working paper ESILV, RR-35.
Da Fonseca, J., Grasselli, M., & Tebaldi, C. (2005). Wishart multi-dimensional stochastic volatility. Working paper ESILV, RR-31. To appear in *Quantitative Finance* as "A Multifactor Volatility Heston Model".
Dai, Q., & Singleton, K. (2000). Specification analysis of affine term structure models. *The Journal of Finance, 55*, 1943–1978.
Driessen, J., Maenhout, P. J., & Vilkov, G. (2005). Option implied correlations and the price of correlation risk. EFA 2005 Moscow Meetings. Available at SSRN: http://www.ssrn.com/abstract=673425.
Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *Journal of Finance, 57*, 405–443.
Duffie, D., Filipovic, D., & Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Probability, 13*, 984–1053.
Duffie, D., & Kan, R. (1996). A yield-factor model of interest rates. *Mathematical Finance, 6*, 379–406.
Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica, 68*, 1343–1376.
Eraker, B. (2004). Do stock prices and volatility jumps? Reconciling evidence from spot and option prices. *The Journal of Finance, 59*, 1367–1403.
Eraker, B., Johannes, M., & Polson, N. (2007). The impact of jumps in volatility and return. *The Journal of Finance, 58*, 1269–1300.
Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics, 54*, 173–182.
Fengler, M. R., & Schwendner, P. (2004). Quoting multiasset equity options in the presence of errors from estimating correlations. *Journal of Derivatives, Summer*, 43–54.
Gourieroux, C., & Sufana, R. (2004a). Wishart quadratic term structure models. CREF 03-10, HEC Montreal.
Gourieroux, C., & Sufana, R. (2004b). Derivative pricing with multivariate stochastic volatility: Application to credit risk. Working paper CREST.
Grasselli, M., & Tebaldi, C. (2008). Solvable affine term structure models. *Mathematical Finance, 18*, 135–153.
Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies, 6*, 327–343.
Heston, S. L., & Nandi, S. (2000). A closed-form GARCH option valuation model. *The Review of Financial Studies, 13*, 585–625.
Lewis, A. L. (2000). *Option valuation under stochastic volatility with mathematica code*. Finance Press.
Roll, R. (1988). The international crash of October, 1987. *Financial Analysts Journal, September–October*, 19–35.
Ross, S. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory, 13*, 341–360.
Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance, 19*, 425–442.