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PRICING DISCRETELY MONITORED BARRIER OPTIONS AND DEFAULTABLE BONDS IN LÉVY PROCESS MODELS: A FAST HILBERT TRANSFORM APPROACH

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This paper presents a novel method to price discretely monitored single- and double-barrier options in Lévy process-based models. The method involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the (Esscher transformed) Lévy process. A discrete approximation with exponentially decaying errors is developed based on the Whittaker cardinal series (Sinc expansion) in Hardy spaces of functions analytic in a strip. An efficient computational algorithm is developed based on the fast Hilbert transform that, in turn, relies on the FFT-based Toeplitz matrix–vector multiplication. Our method also provides a natural framework for credit risk applications, where the firm value follows an exponential Lévy process and default occurs at the first time the firm value is below the default barrier on one of a discrete set of monitoring dates.

KEY WORDS: Lévy processes, Esscher transform, discrete barrier options, first passage time problems, credit risk, defaultable bonds, Fourier transform, Hilbert transform, Whittaker cardinal series, Sinc expansion

1. INTRODUCTION

1.1. Background and Motivation

Pure jump and jump-diffusion asset pricing models based on Lévy processes have enjoyed remarkable popularity in recent years due to at least two reasons. First, non-Gaussian Lévy processes provide better fit to empirical time series behavior of underlying assets such as equities, currencies, and commodities, as well as better explain observed patterns in options prices across strikes such as volatility smile and skew effects. Popular Lévy process-based models include jump-diffusion models of Merton (1976) and Kou

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(2002), as well as infinite activity pure jump models such as the variance gamma (VG) model of Madan and Seneta (1990), Madan and Milne (1991) and Madan et al. (1998), the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), the generalized hyperbolic model of Eberlein et al. (1998), the Carr et al. (2002) model (CGMY), and the finite moment log-stable model of Carr and Wu (2003). Second, due to the celebrated Lévy–Khintchine formula, simple analytical expressions for characteristic functions are available for a wide range of non-Gaussian Lévy processes, making them highly tractable in practice. In particular, a fast and accurate Fast Fourier Transform (FFT)-based pricing method for European options has been proposed by Carr and Madan (1999) (see Lee [2004] for a detailed analysis of the method).

However, while the pricing of European options under non-Gaussian Lévy models is well understood, pricing path-dependent options remains a challenge. Discrete barrier options are one of the most important classes of path-dependent options. On one hand, in practice most barrier option contracts have discretely monitored barriers. On the other hand, in the area of credit risk default events are often modeled as barrier crossing events.

In their pioneering work, Black and Cox (1976) model the total market value of the assets of the firm by geometric Brownian motion. They interpret the (generally time dependent) lower barrier L as the minimum firm value required by the bond safety covenants for the firm to continue its operations. If the value of the firm falls below L, bondholders are entitled to enforce the safety covenant and force the firm into bankruptcy. The original Black and Cox (1976) first passage time model has since been extended in many directions by many authors (see Duffie and Singleton [2003] for a survey). Most of the first passage time literature consider continuous default barriers to attain analytical tractability. However, we argue that it makes sense to model default barriers as discretely monitored. The bondholders will only check the firm's compliance with the safety covenants at discrete time intervals, perhaps quarterly, monthly, weekly, or at most daily. The reason for the continuous default barrier assumption in this setting is the availability of a simple analytical solution for the first passage time distribution in the geometric Brownian motion model. However, note that the analytical tractability for continuous barriers does not extend to jump-diffusion or pure jump processes, such as Lévy processes (Kou's [2002] jump-diffusion model is the only exception; see Kou and Wang [2003, 2004] for analytical results for the first passage time problem in this model and Chen and Kou [2006] for credit risk applications). Thus, when one departs from the geometric Brownian motion assumption and moves on to consider more general jump-diffusion or pure jump firm value processes, there is no analytical advantage in maintaining a continuous default barrier assumption, and one can as well adopt a more realistic discrete barrier assumption.

In the Black–Scholes setting, Broadie, Glasserman, and Kou (1997) provide an explicit analytical formula for the barrier adjustment to be implemented in the continuous barrier option formula to correct for discrete observations and approximate the discrete barrier option with the continuous one with an appropriately adjusted barrier (see also Kou 2003; Howison and Steinberg 2007). While very convenient in practice, this analytical approximation is limited to single-barrier options and to the geometric Brownian motion process.

Eydeland (1994) pioneered FFT applications to the pricing of discrete path-dependent options. His algorithm is applicable to discretely monitored barrier options in the Gaussian framework. The approach is to sequentially compute convolutions of the value function of the option at the previous monitoring date with the transition density between the two dates. The Gaussian density is space-homogeneous (depends only on the difference $x - y$) and, hence, the expectation operator reduces to a convolution. The integral with the density is discretized, and the resulting matrix–vector multiplication approximating the convolution is efficiently computed using the Toeplitz matrix–vector multiplication algorithm via the FFT. The computational complexity of Eydeland's method is $O(NM\log_2 M)$, where N is the number of barrier monitoring dates and M is the number of sample points at each date. However, the discretization error due to numerical integration in the state space decreases slowly as h^2 for the trapezoidal rule (h is the discretization step size). A similar convolution approach to discrete barrier options has been proposed by Reiner (1998). While the convolution method can theoretically be extended to non-Gaussian Lévy processes, it is not practical in the general non-Gaussian setting, since for Lévy processes the transition density is usually not known in closed form.

Broadie and Yamamoto (2005) propose an efficient method for discrete barrier options based on combining the double-exponential quadrature rule and the fast Gauss transform early introduced to the options pricing literature in Broadie and Yamamoto (2003). Their method is remarkably fast and accurate for models where the return distribution is a mixture of independent Gaussians, such as in Merton's jump-diffusion model in addition to the Black–Scholes–Merton model. The computational complexity of their method is $O(N\ M)$, where N is the number of barrier monitoring dates and M is the number of sample points at each date needed to compute the fast Gauss transform, and the error decreases exponentially with increasing M. Broadie and Yamamoto's method is thus more efficient than Eydeland's and Reiner's convolution method, as it is linear in M and has exponentially rather than polynomially decaying errors. However, while Broadie and Yamamoto's method appears optimal for the Gaussian case, it is not applicable to non-Gaussian Lévy processes.

Petrella and Kou (2004) develop an interesting numerical method for discrete lookback and single-barrier options based on the celebrated Spitzer's identity for the maximum (minimum) of a random walk. Theoretically, their method is applicable to any Lévy process. However, while the method is reasonably fast for lookback options, for barrier options it is computationally very intensive, as it is quadratic in the number of monitoring dates and, furthermore, requires numerical two-dimensional Laplace transform inversion at each step, with the total computational effort $O(N^2 M)$, where N is the number of barrier monitoring dates and M is the number of sample points in the numerical two-dimensional Laplace transform inversion. Petrella and Kou (2004) provide a comprehensive study of discrete single-barrier options in Merton's and Kou's jump-diffusion models in this framework based on the Spitzer's identity. According to their results, it took about 180 seconds to price an up-and-out put option with 160 barrier monitoring dates to four significant digit accuracy in Merton's jump-diffusion model, and about 400 seconds for the same in Kou's jump-diffusion model. While no results are provided for infinite activity pure jump Lévy processes such as NIG or CGMY, apparently it would be even more challenging computationally. This method is also limited to single-barrier options.

1.2. The Contribution of the Present Paper: A Fast Hilbert Transform Method

The present paper develops a novel method to price discretely monitored single- and double-barrier options in Lévy process models. Our method is based on the key observation that multiplying a function with the indicator function in the state space (monitoring the barrier in the state space) corresponds to taking the Hilbert transform in the Fourier space. Our method thus involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the (Esscher transformed) Lévy process. A remarkably accurate discrete approximation with exponentially decaying errors is developed based on the Whittaker cardinal series (Sinc expansion) in Hardy spaces of functions analytic in a strip. The following are the main features and strengths of our method:

- The method is applicable to any Lévy process (jump-diffusion or infinite activity pure jump) and any terminal payoff function (subject to an integrability condition).
- The method is applicable to both single- and double-barrier options.
- The method is applicable to contracts with nonequally spaced barrier monitoring dates, nonconstant barriers, and valuation dates falling between the two barrier monitoring dates.
- The method exhibits discretization errors decaying exponentially in $1/h$, where h is the discretization step size used to compute the Hilbert transform, i.e., the discretization error is $O(\exp(-C_1/h))$. If the characteristic function of the Lévy process decays as $\exp(-tc|\xi|^\nu)$ as $|\xi| \to \infty$, we prove that the truncation error to truncate an infinite sum approximating the Hilbert transform after M terms is $O(h^{-1}(Mh)^{1-\nu}\exp(-\Delta c(Mh)^\nu))$ for single barrier options and $O((Mh)^{1-\nu}\exp(-\Delta c(Mh)^\nu))$ for double barrier options, where Δ is the barrier monitoring interval. Selecting $h = h(M)$ appropriately, the total error is $O(M^{1/(1+\nu)}\exp(-CM^{\frac{\nu}{1+\nu}}))$ for single barrier options and $O(\max(1, M^{(1-\nu)/(1+\nu)})\exp(-CM^{\frac{\nu}{1+\nu}}))$ for double barrier options and decays essentially exponentially as we increase the number of sample points M.
- The total computational effort is $O(NM\log_2 M)$, linear in the number of monitoring dates N and $M\log_2 M$ in the number of sample points M needed to compute the Hilbert transform. This linear dependence on the number of monitoring dates is particularly valuable for long-maturity credit risk applications, where the firm value process follows an exponential Lévy process and default occurs at the first time the firm value is below the default barrier on one of a discrete set of monitoring dates.
- Option delta and gamma are obtained with essentially no additional computational cost.

On one hand, our method extends the Carr and Madan (1999) FFT method for European options to discrete barrier options. On the other hand, our method extends the Eydeland (1994) FFT method for discrete barrier options in the Gaussian setting to general Lévy processes. Our method has features similar to Broadie and Yamamoto's (2005) algorithm (exponentially decaying errors and computational efficiency), but is applicable more broadly to general Lévy processes, and not just to mixtures of Gaussians as for Broadie and Yamamoto's method. To the best of our knowledge, our method is the first application of the Hilbert transform and Sinc expansions in mathematical finance.

1.3. An Illustrative Example

To give an idea of our method, we briefly describe the pricing of a discretely monitored down-and-out put with lower barrier $L > 0$, strike price $K > L$, barrier monitoring interval $\Delta > 0$, and maturity $T = N\Delta$. The asset price process is $S_t = Ke^{X_t}$, where X is a Lévy process started at $x = \ln(S/K)$, where S is the initial asset price at time zero. The price of this option is given by:

$$V(S) = e^{-rT}\mathbb{E}_x\big[\mathbf{1}_{(l,\infty)}(X_\Delta)\cdot\mathbf{1}_{(l,\infty)}(X_{2\Delta})\cdots\mathbf{1}_{(l,\infty)}(X_{N\Delta})\cdot K(1-e^{X_{N\Delta}})^+\big],$$

where $\mathbb{E}_x$ is the expectation with respect to the law of the Lévy process X started at $x = \ln(S/K)$, and $l = \ln(L/K)$ is the lower barrier for the Lévy process X. This can be computed recursively by the following backward induction in the state space:

$$\begin{aligned} v^N(x) &= K(1-e^x)^+\mathbf{1}_{(l,\infty)}(x), \\ v^{j-1}(x) &= \mathbf{1}_{(l,\infty)}(x)\cdot P_\Delta v^j(x), \quad j = N, N-1, \ldots, 2, \\ v^0(x) &= P_\Delta(x), \end{aligned}$$

where $P_t f(x) := \mathbb{E}_x[f(X_t)]$ is the expectation operator. The option price is recovered via $V(S) = e^{-rT}v^0(\ln(S/K))$.

We implement this backward induction in the Fourier space. Denote the Fourier transform of $v^j(x)$ by $\hat{v}^j(\xi)$, $\hat{v}^j(\xi) = (\mathcal{F}v^j)(\xi)$. First, we observe that $\mathcal{F}(P_t v^j)(\xi) = \phi_t(-\xi)\hat{v}^j(\xi)$, where $\phi_t(\xi)$ is the characteristic function of X_t. To compute $\hat{v}^{j-1}$, we note that the indicator function can be written as:

$$\mathbf{1}_{(0,\infty)}(x) = \frac{1}{2}(1+\operatorname{sgn}(x)),$$

where sgn (x) is the signum function. The following relationship from Fourier analysis is the key to our approach:

$$\mathcal{F}(\operatorname{sgn}\cdot g)(\xi) = i\mathcal{H}\hat{g}(\xi),$$

for any $g \in L^1(\mathbb{R})$ with $\hat{g} \in L^1(\mathbb{R}, \mathbb{C})$. Here $\mathcal{H}$ is the Hilbert transform defined by the following Cauchy principal value integral:

$$\mathcal{H}f(\xi) := \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(\eta)}{\xi - \eta}\, d\eta.$$

It follows that

$$\mathcal{F}(\mathbf{1}_{(l,\infty)}\cdot g)(\xi) = \frac{1}{2}\hat{g}(\xi) + \frac{i}{2}e^{i\xi l}\mathcal{H}(e^{-i\eta l}\hat{g}(\eta))(\xi).$$

That is, monitoring the barrier in the state space (multiplying with the indicator function in the state space) is transformed into taking the Hilbert transform in the Fourier space. The backward induction can therefore be implemented in the Fourier space:

$$\begin{aligned} \hat{v}^N(\xi) &= \frac{K(1-e^{i\xi l})}{i\xi} - \frac{K(1-e^{(1+i\xi)l})}{1+i\xi}, \\ \hat{v}^{j-1}(\xi) &= \frac{1}{2}\phi_\Delta(-\xi)\hat{v}^j(\xi) + \frac{i}{2}e^{i\xi l}\mathcal{H}\big(e^{-i\eta l}\phi_\Delta(-\eta)\hat{v}^j(\eta)\big)(\xi), \quad j = N, N-1, \ldots, 2, \\ v^0(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\phi_\Delta(-\xi)\hat{v}^1(\xi)\, d\xi. \end{aligned}$$

To compute $v^0(x)$, we need to compute $N-1$ Hilbert transforms and one Fourier transform inversion. Due to the powerful approximation theory in Hardy spaces of analytic functions, the Hilbert transform can be computed remarkably accurately with exponentially decaying error by the following simple trapezoidal-like quadrature rule. Fix discretization step size $h > 0$ and the truncation level M (a positive integer) to truncate the infinite sum. Then we have the following discrete approximation for the Hilbert transform of a function f:

$$\mathcal{H}_{h,M}f(\xi) = \sum_{m=-M}^{M} f(mh)\frac{1-\cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h}.$$

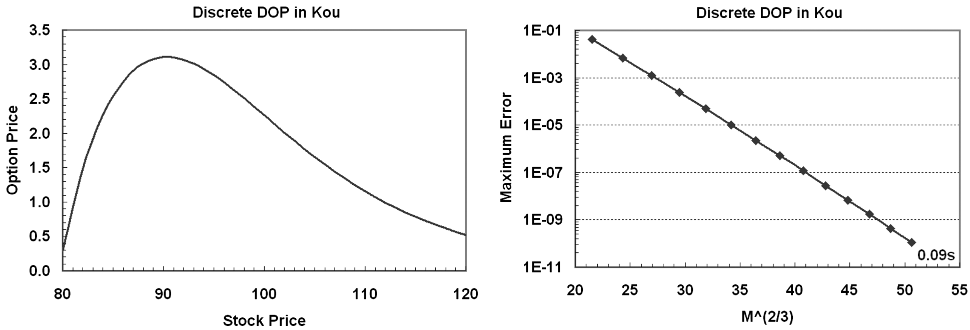

FIGURE 1.1. One year daily monitored down-and-out put option in Kou's double exponential jump-diffusion model.

For each time step in the backward induction, we need to evaluate sums of the following form (with ξ taking values on the discrete grid $\{kh, |k| \le M\}$):

$$\mathcal{H}_{h,M} f(kh) = \sum_{m=-M, m\neq k}^{M} f(mh) \frac{1-(-1)^{k-m}}{\pi(k-m)}, \quad k = -M, \dots, M.$$

This involves Toeplitz matrix–vector multiplication, where the matrix is of the Toeplitz form with entries depending only on $k-m$. Toeplitz matrix–vector multiplication can be accomplished in $O(M \log_2 M)$ operations by using the FFT (see Appendix B). Moreover, the Fourier inversion integral at the final step to compute $v^0(x)$ can also be accurately approximated using the simple trapezoidal rule and computed in $O(M)$ operations if the option price is desired for only one value of the initial asset price, or in $O(M \log_2 M)$ operations using the FFT if the option price is desired for a range of initial asset prices. Therefore, the total operation count for the algorithm is $O(NM \log_2 M)$ ($N-1$ Hilbert transforms and one Fourier inversion at the final step). This algorithm exhibits discretization errors decaying exponentially in $1/h$, i.e., the discretization error is $O(\exp(-C_1/h))$. If the characteristic function of the Lévy process decays as $\exp(-tc|\xi|^\nu)$ as $|\xi| \to \infty$, the truncation error to truncate an infinite sum approximating the Hilbert transform after M terms is $O(h^{-1}(Mh)^{1-\nu}\exp(-\Delta c(Mh)^\nu))$. Selecting $h = h(M)$ so that the discretization and truncation errors have about the same order (i.e., we let $h = h(M)$ be such that $\exp(-C_1/h) = \exp(-\Delta c(Mh)^\nu)$), the total error is $O(M^{1/(1+\nu)}\exp(-CM^{\frac{\nu}{1+\nu}}))$ and decays essentially exponentially as we increase M.

Consider a one-year daily monitored ($T = 1$, $N = 252$) down-and-out put option with lower barrier $L = 80$ and strike price $K = 100$. The risk free interest rate and the dividend yield are $r = 5\%$ and $q = 2\%$. Assume the asset price follows Kou's (2002) double exponential jump-diffusion process with parameters $\sigma = 0.1$ (diffusion component with volatility 0.1), $\lambda = 3$ (intensity of 3 jumps per year), $p = 0.3$ (the probability of a positive jump is 0.3), $\eta_1 = \eta_2 = 20$ (mean jump sizes of $\pm 5\%$ in the return process). Fixing M, the discretization step size h is selected according to $h = (\frac{\pi d}{\Delta c})^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}}$ for $d = 20$ (the width of a strip in the complex plane where the product of the Fourier transform $\hat{f}(\xi)$ of the option payoff and the characteristic function $\phi_\Delta(-\xi)$ of the jump-diffusion process X is analytic), $c = \sigma^2/2$, $\nu = 2$. For this selection of $h = h(M)$, the total pricing error of our method is $O(M^{1/3}\exp(-CM^{2/3}))$. The first graph in Figure 1.1 plots the option value as a function of the initial underlying asset price. The second graph plots the maximum pricing error, evaluated at the initial asset prices $S = 80, 81, \dots, 120$ in log-scale as a function of $M^{2/3}$, verifying experimentally the theoretical convergence rate. This plot shows that our Hilbert transform method is remarkably fast and accurate. It takes 0.09 seconds to price a one-year daily monitored down-and-out put with 10^{-10} accuracy in this example.

1.4. Organization of the Paper

The remainder of the paper is organized as follows. In Section 2, we collect some results on Lévy processes, their Markov semigroups, and their Fourier representations needed in the sequel. In Section 3, we briefly summarize some facts on geometric Lévy models. In Section 4, using the Esscher transform, we prove a theorem extending the semigroup of a Lévy process and its Fourier representation to the weighted spaces $L^1(\mathbb{R}, e^{\alpha x}\,dx)$ with such α that $\mathbb{E}[e^{-\alpha X_t}] < \infty$, and discuss the pricing of European options with payoffs in $L^1(\mathbb{R}, e^{\alpha x}\,dx)$. In Section 5, we formulate Theorem 5.1 that represents the value function of a discrete barrier option as a result of sequential evaluation of Hilbert transforms. In Section 6, after recalling the necessary results about the Wiener spaces of entire functions of exponential type and the Sinc approximation in Hardy spaces of functions analytic in a strip, we present a discrete approximation of our Hilbert transform-based method, derive error bounds, and propose an efficient computational implementation based on the fast Hilbert transform algorithm to efficiently evaluate the discrete Hilbert transform. In Section 7, we provide computational examples for several different single- and double-barrier contracts and jump-diffusions, as well as infinite activity pure jump Lévy process, that demonstrate the accuracy of our method both for option pricing applications and for credit risk applications. A comparison with an alternative FFT algorithm is provided to further illustrate the power of our method. Section 8 concludes the paper. Proofs are collected in Appendix A. Appendix B describes how to efficiently accomplish Toeplitz matrix–vector multiplication via the FFT needed to compute the fast Hilbert transform.

2. PRELIMINARIES ON LÉVY PROCESSES

In this section, we collect some results on Lévy processes that will be used in the subsequent development (see Applebaum 2004; Bertoin 1996; Sato 1999, for more details). Consider a real-valued stochastic process $\{X_t, t \geq 0\}$ starting at zero, $X_0 = 0$, and defined on a given complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual hypothesis. We assume that X is a Lévy process with respect to the filtration $\mathbb{F}$, i.e., X_t is adapted to $\mathcal{F}_t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s} for $0 \leq s < t$, X_t is continuous in probability, and sample paths of X are *cádlág* functions (right continuous with left limits). By the celebrated Lévy–Khintchine theorem (e.g., Bertoin 1996, p. 13), the characteristic function (CF) of X_t has the form

$$\phi_t(\xi) = \mathbb{E}[e^{i\xi X_t}] = e^{-t\Psi(\xi)}, \quad t \geq 0,$$

where the *characteristic exponent* $\Psi(\xi)$, $\xi \in \mathbb{R}$, admits a representation

$$\Psi(\xi) = \frac{1}{2}\sigma^2\xi^2 - i\mu\xi + \int_{\mathbb{R}} \left(1 - e^{i\xi x} + i\xi x \mathbf{1}_{\{|x|\leq 1\}}\right)\Pi(dx).$$

The Lévy process is specified by the volatility of its diffusion component $\sigma \geq 0$, its "drift" $\mu \in \mathbb{R}$, and its *Lévy measure* Π on $\mathbb{R}$ with $\Pi(\{0\}) = 0$ and $\int_{\{|x|\leq 1\}} x^2\Pi(dx) <$

∞, $\int_{\{|x|>1\}} \Pi(dx) < \infty$. The triplet (μ, σ^2, Π) is referred to as the *generating triplet* or *Lévy characteristics* of X.

The Lévy measure Π describes the arrival rates of jumps so that jumps of sizes in some set A (bounded away from zero) occur according to a Poisson process with intensity $\Pi(A)$. If $\Pi = 0$, the process is a Brownian motion with drift μ and volatility σ. If $\sigma = 0$, the process is a pure jump process. If $\int_{\mathbb{R}} \Pi(dx) < +\infty$, the jump component is of compound Poisson type with the Poisson arrival intensity $\lambda = \int_{\mathbb{R}} \Pi(dx)$ and jump size distribution $\lambda^{-1}\Pi$. If the integral $\int_{\mathbb{R}} \Pi(dx)$ is infinite, the jump process is of infinite activity. A pure jump Lévy process is of finite variation if and only if $\int_{[-1,1]} |x|\Pi(dx) < +\infty$.

Let X be a Lévy process with Lévy measure Π and $\mathcal{I}_X$ a set defined by

$$\mathcal{I}_X := \left\{\alpha \in \mathbb{R} : \int_{\{|x|>1\}} e^{-\alpha x}\Pi(dx) < \infty\right\}.$$

By theorem 25.17 of Sato (1999), $\alpha \in \mathcal{I}_X$ if and only if $\mathbb{E}[e^{-\alpha X_t}] < \infty$ for some $t > 0$ or, equivalently, for every $t > 0$. $\mathcal{I}_X$ is a (finite or infinite) interval containing the origin with endpoints λ_- and λ_+, $-\infty \le \lambda_- \le 0 \le \lambda_+ \le \infty$. Endpoints may or may not belong to $\mathcal{I}_X$, and it is possible that $\lambda_- = \lambda_+ = 0$, in which case $\mathcal{I}_X = \{0\}$. Suppose that at least one of the endpoints $\{\lambda_-, \lambda_+\}$ is not zero. Let $\mathcal{S}_X := \{z \in \mathbb{C} : \Im(z) \in (\lambda_-, \lambda_+)\}$. Then the characteristic exponent $\Psi(z)$, as a function of the complex variable z, is analytic in the strip $\mathcal{S}_X$ (throughout this paper ξ is a real variable and $z = \xi + iw$ is a complex variable). If we model a spot asset price $\{S_t, t \ge 0\}$ as the exponential of a Lévy process $\{X_t, t \ge 0\}$, then $\mathbb{E}[S_t^{-\alpha}] < \infty$ for every $\alpha \in \mathcal{I}_X$ and every $t > 0$. In particular, to insure that the asset itself is priced, we need to have $\mathbb{E}[e^{X_t}] < \infty$ for every $t > 0$, i.e., $\lambda_- \le -1$. Thus, in this paper we restrict our attention to Lévy processes for which $[-1, 0] \in \mathcal{I}_X$.

Consider a family of convolution operators indexed by $t \ge 0$:

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}} f(x+y) P_t(dy), \tag{2.1}$$

where $P_t(dy)$ is the transition measure of X_t starting at the origin. The family of operators $\{P_t, t \ge 0\}$ defines a Markov semigroup on any of the spaces $L^p(\mathbb{R})$ with $1 \le p \le \infty$ called an *L^p-Markov semigroup* (see Applebaum 2004, theorem 3.4.2, p. 149). In particular, consider a Markov semigroup $\{P_t, t \ge 0\}$ on $L^1(\mathbb{R})$. Define the Fourier transform of a function $f \in L^1(\mathbb{R})$ by

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\, dx, \quad \xi \in \mathbb{R}.$$

Then for every $f \in L^1(\mathbb{R})$, we have for every $\xi \in \mathbb{R}$ (Bertoin 1996, proposition 9, p. 23)

$$\mathcal{F}(P_t f)(\xi) = \phi_t(-\xi)\hat{f}(\xi), \quad t \ge 0. \tag{2.2}$$

Recall that, by the Riemann–Lebesgue theorem, the Fourier transform maps $L^1(\mathbb{R})$ into $C_0(\mathbb{R}, \mathbb{C})$, the space of continuous complex-valued functions on $\mathbb{R}$ vanishing at infinity (e.g., theorem IX.7 in Reed and Simon 1975, p. 10) and $\|\hat{f}\|_{L^\infty(\mathbb{R},\mathbb{C})} \le \|f\|_{L^1(\mathbb{R})}$. To invert the Fourier transform (2.2) for some $t > 0$, we need

$$\int_{\mathbb{R}} |\phi_t(-\xi)\hat{f}(\xi)|\, d\xi < \infty. \tag{2.3}$$

Since $\hat{f}$ is bounded, to invert the Fourier transform (2.2) for every $f \in L^1(\mathbb{R})$ and every $t > 0$, it is sufficient that

$$\int_{\mathbb{R}} |\phi_t(\xi)|\, d\xi = \int_{\mathbb{R}} e^{-t\,\Re\Psi(\xi)}\, d\xi < \infty \text{ for every } t > 0. \tag{2.4}$$

In particular, (2.4) holds if the characteristic function has an estimate with some $c, \kappa > 0$ and $\nu \in (0, 2]$:

$$|\phi_t(\xi)| = e^{-t\Re\Psi(\xi)} \leq \kappa e^{-tc|\xi|^{\nu}}. \tag{2.5}$$

Assuming (2.4) holds, we can invert the Fourier transform (2.2) for every $f \in L^1(\mathbb{R})$ and every $t > 0$ and write

$$P_t f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} \phi_t(-\xi) \hat{f}(\xi)\, d\xi. \tag{2.6}$$

If equation (2.4) holds, equation (2.6) gives the Fourier representation of the operator P_t on $L^1(\mathbb{R})$ for every $t > 0$. Equation (2.4) is obviously satisfied if the Lévy process has a diffusion component with $\sigma > 0$. For pure jump Lévy processes, the question of whether or not (2.4) is satisfied depends on the behavior of the Lévy measure in the neighborhood of the origin. Consider a pure jump Lévy process. Let Π satisfy

$$\liminf_{r \downarrow 0} \frac{\int_{[-r,r]} x^2 \Pi(dx)}{r^{2-\nu}} > 0 \tag{2.7}$$

for some $0 < \nu < 2$. Then for each $t > 0$ the transition measure $P_t(dy)$ has a density $p_t(y)$ of class $C^{\infty}(\mathbb{R})$ and all derivatives of the density tend to 0 as $|x| \to \infty$ (Sato 1999, theorem 28.3, p. 190). In this case the characteristic function has an estimate (2.5) and (2.4) is obviously satisfied. For Lévy processes for which (2.4) is not satisfied, the Fourier representation (2.6) holds for those $f \in L^1(\mathbb{R})$ and $t > 0$ for which (2.3) holds. In particular, it holds for every $t > 0$ for all $f \in L^1(\mathbb{R})$ for which $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$ (complex-valued L^1 functions on $\mathbb{R}$).

3. GEOMETRIC LÉVY MODELS OF ASSET PRICES

In this paper we take an equivalent martingale measure (EMM) as given and model the asset price dynamics as a process

$$S_t = Ke^{X_t}, \quad t \geq 0,$$

where $K > 0$ is a scale parameter (some given reference asset price level) and X is a Lévy process with $\lambda_- < -1$, $\lambda_+ > 0$, and starting at $X_0 = x = \ln(S/K) \in \mathbb{R}$ at time zero, where $S_0 = S > 0$ is the initial asset price at time zero. Typically one sets the scale parameter equal to the initial asset price value, $K = S_0$, but it is more convenient for us to keep the two separate (in options pricing applications it will be convenient to set K equal to the option strike).

To insure that under the EMM $\mathbb{E}[S_t] = e^{(r-q)t} S_0$, $t \geq 0$, where $r \geq 0$ is the risk-free interest rate and $q \geq 0$ is the dividend yield of the asset, we should have for the process $X\, \mathbb{E}[e^{X_t}] = e^{x+(r-q)t}$, $t \geq 0$, which fixes the "drift" parameter of the Lévy process

$$\mu = r - q - \frac{\sigma^2}{2} + w, \quad w = \Psi_{\Pi}(-i) = \int_{\mathbb{R}} \left(1 - e^x + x\mathbf{1}_{\{|x|\leq 1\}}\right)\Pi(dx), \tag{3.1}$$

TABLE 3.1
Examples of Lévy components

Process	Lévy density $\Pi(dx)/dx$	Characteristic exponent $\Psi(\xi)$	Parameters and $\mathcal{I}_X$
Diffusion component			
$\sigma B_t + \mu t$	$\Pi \equiv 0$	$\frac{1}{2}\sigma^2\xi^2 - i\mu\xi$	$\sigma > 0,\ \mu \in \mathbb{R}$, $\mathcal{I}_X = \mathbb{R}$
Finite-activity pure jump components			
Merton	$\lambda \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(x-m)^2}{2s^2}}$	$\lambda(1 - e^{im\xi - \frac{1}{2}s^2\xi^2})$	$\lambda > 0,\ m \in \mathbb{R},\ s > 0$,
Reference: Merton (1976)			$\mathcal{I}_X = \mathbb{R}$
Kou	$\lambda p\eta_1 e^{-\eta_1 x}\mathbf{1}_{\{x>0\}}$ $+\lambda(1-p)\eta_2 e^{-\eta_2\lvert x\rvert}\mathbf{1}_{\{x<0\}}$	$\lambda(1 - \frac{p\eta_1}{\eta_1 - i\xi} - \frac{(1-p)\eta_2}{\eta_2 + i\xi})$	$\lambda > 0,\ p \in [0, 1]$, $\eta_1 > 1,\ \eta_2 > 0$,
References: Kou (2002), Kou and Wang (2003)			$\mathcal{I}_X = (-\eta_1, \eta_2)$
Infinite-activity pure jump components			
NIG	$\frac{\delta\alpha}{\pi\lvert x\rvert} e^{\beta x} K_1(\alpha\lvert x\rvert)$	$\delta(\sqrt{\alpha^2 - (\beta + i\xi)^2}$ $-\sqrt{\alpha^2 - \beta^2})$	$\alpha, \delta > 0$, $\beta \in (-\alpha, \alpha - 1)$,
IV; (2.5) holds with $\nu = 1$			$\mathcal{I}_X = [\beta - \alpha, \beta + \alpha]$
Reference: Barndorff-Nielsen (1998)			
CGMY	$Cx^{-Y-1}e^{-Mx}\mathbf{1}_{\{x>0\}}$ $+C\lvert x\rvert^{-Y-1}e^{-G\lvert x\rvert}\mathbf{1}_{\{x<0\}}$	$C\Gamma(-Y)[M^Y - (M - i\xi)^Y$ $+G^Y - (G + i\xi)^Y]$,	$Y \in (0, 1) \cup (1, 2)$, $C, G > 0,\ M > 1$
$Y \in (0, 1)$: FV; $Y \in (1, 2)$: IV; (2.5) holds with $\nu = Y$			$\mathcal{I}_X = [-M, G]$
Reference: Carr et al. (2002)			
VG	$Cx^{-1}e^{-Mx}\mathbf{1}_{\{x>0\}}$ $+C\lvert x\rvert^{-1}e^{-G\lvert x\rvert}\mathbf{1}_{\{x<0\}}$	$\nu^{-1}\ln(1 - i\nu\theta\xi + \frac{1}{2}\nu s^2\xi^2)$, $C = 1/\nu$, $G = \sqrt{\frac{\theta^2}{s^4} + \frac{2}{s^2\nu}} + \frac{\theta}{s^2}$, $M = \sqrt{\frac{\theta^2}{s^4} + \frac{2}{s^2\nu}} - \frac{\theta}{s^2}$	$\nu, s > 0,\ \theta \in \mathbb{R}$ $C, G > 0,\ M > 1$ $\mathcal{I}_X = (-M, G)$
FV; (2.5) does not hold;			
Reference: Madan and Seneta (1990), Madan and Milne (1991), and Madan et al. (1998)			

where w is the value of the characteristic exponent of the jump component evaluated at $-i$. Equation (3.1) is called the *martingale condition* since this choice of μ insures that the discounted gains process (price changes plus dividends) is a martingale.

Table 3.1 gives some examples of Lévy processes. Further examples of Lévy processes used in finance, as well as further details on the processes summarized in Table 3.1, can be found in the monographs Boyarchenko and Levendorskii (2002), Cont and Tankov (2004), and Schoutens (2003) and the original papers cited therein. For each process, the table gives its Lévy measure, its characteristic exponent, parameter range, the interval $\mathcal{I}_X$, references, and, for infinite activity jump components, whether the process is of finite variation (FV) or infinite variation (IV), whether the estimate (2.5) holds, and, if yes, the exponent ν.

Equation (2.5) holds and, hence, the transition measure has a density of $C^\infty(\mathbb{R})$ class with all its derivatives vanishing at infinity for all infinite activity jump components in Table 3.1 except for the VG process. In the case of VG, the estimate (2.5) fails, and the condition (2.4) fails for all $t < \nu/2$ (it holds for $t > \nu/2$). Consequently, the transition density fails to be continuous for $t < \nu/2$. The transition density of the VG model was found explicitly by Madan et al. (1998). This density is continuous when $t > \nu/2$, but it becomes infinite for a particular value of x when $t < \nu/2$. Adding a diffusion with $\sigma > 0$ to the VG process restores the estimate (2.5) with $\nu = 2$ and, hence, restores the continuity of the density of the diffusion-extended VG for all $t > 0$.

The NIG density is known in closed form. To the best of our knowledge, the CGMY density is not known in closed form. For jump-diffusion processes with both a diffusion component and a jump component the densities are known in closed form only for Merton's and Kou's models.

All finite-activity jump components, if taken without a diffusion component, have bounded characteristic exponent, do not satisfy (2.4) for any $t > 0$, and their transition measures are not absolutely continuous with respect to the Lebesgue measure: they contain an atom at $x = 0$ for all $t > 0$ that corresponds to the positive probability of no jumps by time $t > 0$. Adding a diffusion component with $\sigma > 0$ restores the estimate (2.5) with $\nu = 2$ and the existence of a transition density of class C^∞.

4. PRICING EUROPEAN OPTIONS IN GEOMETRIC LÉVY MODELS

4.1. Extension of the Lévy Semigroup to $L^1(\mathbb{R}, e^{\alpha x}\,dx)$ via Esscher Transform

To value options, we need to consider payoff functions that are not in $L^1(\mathbb{R})$. For $\alpha \in \mathbb{R}$, let $L^1_\alpha(\mathbb{R}) := L^1(\mathbb{R}, e^{\alpha x}\,dx)$ and $\|f\|_{L^1_\alpha(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| e^{\alpha x}\,dx$. Option payoffs will typically lie in $L^1_\alpha(\mathbb{R})$ for some α. For call options, the payoff as a function of $x = \ln(S/K)$ is $f(x) = K(e^x - 1)^+ \in L^1_\alpha(\mathbb{R})$ for every $\alpha < -1$. For put options, $f(x) = K(1 - e^x)^+ \in L^1_\alpha(\mathbb{R})$ for every $\alpha > 0$. Generally, for a given payoff $f(x)$ define a set

$$\mathcal{I}_f := \{\alpha \in \mathbb{R} : f \in L^1_\alpha(\mathbb{R})\}.$$

For vanilla calls $\mathcal{I}_{\text{vanilla call}} = (-\infty, -1)$. For vanilla puts $\mathcal{I}_{\text{vanilla put}} = (0, \infty)$. The following theorem forms the foundation of the Fourier approach to pricing European options under Lévy processes. We give a proof based on the Esscher transform.

THEOREM 4.1. *Let X be a Lévy process such that at least one of the endpoints $\{\lambda_-, \lambda_+\}$ of $\mathcal{I}_X$ is not zero.*

(i) For every $\alpha \in (\lambda_-, \lambda_+)$, $\{P_t, t \geq 0\}$ is a strongly continuous semigroup on $L^1_\alpha(\mathbb{R})$, and $\{e^{t\Psi(i\alpha)} P_t, t \geq 0\}$ is a strongly continuous contraction semigroup on $L^1_\alpha(\mathbb{R})$.

(ii) If $f \in L^1_\alpha(\mathbb{R})$ with some $\alpha \in (\lambda_-, \lambda_+)$ is such that for some $t > 0$ (if the condition (4.1) is satisfied for some $t > 0$, it is also satisfied for all $s \geq t$)

$$\int_{\mathbb{R}} |\phi_t(-\xi + i\alpha) \hat{f}_\alpha(\xi)| d\xi < \infty, \tag{4.1}$$

where $\hat{f}_\alpha(\xi)$ is the Fourier transform of $f_\alpha(x) := e^{\alpha x} f(x) \in L^1(\mathbb{R})$, then the operator P_t has the Fourier representation:

$$P_t f(x) = \frac{1}{2\pi} e^{-\alpha x - t\Psi(i\alpha)} \int_{\mathbb{R}} e^{-i\xi x} \phi_t^{(\alpha)}(-\xi) \hat{f}_\alpha(\xi)\,d\xi, \tag{4.2}$$

where

$$(4.3)\quad \phi_t^{(\alpha)}(\xi) = \frac{\phi_t(\xi + i\alpha)}{\phi_t(i\alpha)} = e^{-t\Psi^{(\alpha)}(\xi)}, \quad \Psi^{(\alpha)}(\xi) = \Psi(\xi + i\alpha) - \Psi(i\alpha)$$

are the characteristic function and the characteristic exponent of the Esscher-transformed Lévy process $X^{(\alpha)}$ with the Lévy characteristics $(\mu^{(\alpha)}, \sigma^2, \Pi^{(\alpha)})$ *with*

$$\mu^{(\alpha)} = \mu - \sigma^2\alpha + \int_{[-1,1]} x(e^{-\alpha x} - 1)\Pi(dx), \quad \Pi^{(\alpha)}(dx) = e^{-\alpha x}\Pi(dx).$$

Proof.

(i) Since $\alpha \in (\lambda_-, \lambda_+)$, $\mathbb{E}[e^{-\alpha X_t}] = e^{-t\Psi(i\alpha)} = \phi_t(i\alpha) < \infty$ (recall that $\Psi(\xi)$ is analytic in the strip $\mathcal{S}_X$, and $\Psi(i\alpha)$ is real). Hence, the process $\{Z_t^{(\alpha)} := e^{-\alpha X_t + t\Psi(i\alpha)}, t \geq 0\}$ is an exponential martingale (Wald martingale). Introduce $\{P_t^{(\alpha)}, t \geq 0\}$ on $L^1(\mathbb{R})$ by

$$P_t^{(\alpha)} g(x) := \mathbb{E}_x\left[\frac{Z_t^{(\alpha)}}{Z_0^{(\alpha)}} g(X_t)\right] = e^{\alpha x + t\Psi(i\alpha)}\mathbb{E}_x\big[e^{-\alpha X_t} g(X_t)\big], \ t \geq 0, \ g \in L^1(\mathbb{R}).$$

This defines an equivalent probability measure $\mathbb{P}^{(\alpha)}$ (*Esscher transform* or *exponential tilting* of the original probability measure) such that under $\mathbb{P}^{(\alpha)}$ the process X is a Lévy process with the characteristic exponent (4.3) (e.g., Raible 2000, proposition 1.8; Jeanblanc et al. 2006). Thus, $\{P_t^{(\alpha)}, t \geq 0\}$ is a Markov semigroup on $L^1(\mathbb{R})$ of the Lévy process $X^{(\alpha)}$:

$$P_t^{(\alpha)} g(x) = \mathbb{E}_x^{(\alpha)}[g(X_t)], \quad t \geq 0, \quad g \in L^1(\mathbb{R}).$$

Then for $f \in L^1_\alpha(\mathbb{R})$ we can write

$$(4.4)\quad P_t f(x) = \mathbb{E}_x\big[e^{-\alpha X_t} f_\alpha(X_t)\big] = e^{-\alpha x - t\Psi(i\alpha)}\mathbb{E}_x^{(\alpha)}[f_\alpha(X_t)] = e^{-\alpha x - t\Psi(i\alpha)} P_t^{(\alpha)} f_\alpha(x).$$

The semigroup property of $\{P_t, t \geq 0\}$ on $L^1_\alpha(\mathbb{R})$ follows from the representation (4.4) and the semigroup property of $\{P_t^{(\alpha)}, t \geq 0\}$ on $L^1(\mathbb{R})$. Next, observe that

$$\|f\|_{L^1_\alpha(\mathbb{R})} = \|f_\alpha\|_{L^1(\mathbb{R})}, \quad \|P_t f\|_{L^1_\alpha(\mathbb{R})} = e^{-t\Psi(i\alpha)}\big\|P_t^{(\alpha)} f_\alpha\big\|_{L^1(\mathbb{R})}.$$

Then the strong continuity of $\{P_t, t \geq 0\}$ on $L^1_\alpha(\mathbb{R})$ follows from the strong continuity of $\{P_t^{(\alpha)}, t \geq 0\}$ on $L^1(\mathbb{R})$. The contractivity of $\{e^{t\Psi(i\alpha)} P_t, t \geq 0\}$ on $L^1_\alpha(\mathbb{R})$ follows from the contractivity of $\{P_t^{(\alpha)}, t \geq 0\}$ on $L^1(\mathbb{R})$:

$$(4.5)\quad \|P_t f\|_{L^1_\alpha(\mathbb{R})} \leq e^{-t\Psi(i\alpha)}\|f\|_{L^1_\alpha(\mathbb{R})}.$$

(ii) For $f_\alpha \in L^1(\mathbb{R})$, equation (2.2) gives the Fourier transform of $P_t^{(\alpha)} f_\alpha$ with the characteristic function $\phi_t^{(\alpha)}$. When the condition (4.1) is satisfied, we can invert the Fourier transform and obtain the Fourier representation of $P_t^{(\alpha)} f_\alpha$ in the form (2.6) with the characteristic function $\phi_t^{(\alpha)}$. Substituting this in (4.4), we arrive at equation (4.2) for $f \in L^1_\alpha(\mathbb{R})$. □

REMARK 4.1. In the proof of Theorem 4.1 we gave a probabilistic interpretation of the exponential damping of the option payoff as an Esscher transform or exponential tilting of the probability measure. Carr and Madan (1999) (see also Lee [2004] for a thorough investigation and extension of their approach) used exponential damping of the option payoff to obtain the Fourier transform in log-strike. When the transform is taken in strike, the probabilistic interpretation of the damping as the exponential tilting of the probability measure is not as apparent. Here we have taken the Fourier transform in the state variable (log-price), which made the Esscher transform apparent. Raible (2000), Lewis (2001), and Boyarchenko and Levendorskii (2002) take Fourier transform in the state variable similar to our approach in this paper, but interpret the result as a complex Fourier transform (also known as generalized Fourier transform or two-sided or bilateral Laplace transform in the literature), in contrast to the probabilistic Esscher transform interpretation given here. The complex Fourier transform representation of equation (4.2) is obtained by introducing a complex variable $z = \xi - i\alpha$ and re-writing (4.2)

$$P_t f(x) = \frac{1}{2\pi} \int_{\mathcal{C}_\alpha} e^{-izx} \phi_t(-z) \hat{f}(z)\, dz,$$

where $\hat{f}(z)$ is the complex Fourier transform of $f(x)$ with complex z and the integration contour in the complex plane is

$$\mathcal{C}_\alpha = \{z \in \mathbb{C} : \Im(z) = -\alpha\}.$$

In this interpretation, for a given payoff f, different choices of $\alpha \in \mathcal{I}_f \cap (\lambda_-, \lambda_+)$ correspond to different choices of the integration contour $\mathcal{C}_\alpha$, and the Cauchy theorem can be used to shift the contour of integration. In the probabilistic interpretation in Theorem 4.1, this corresponds to the Esscher transform of the probability measure.

4.2. Pricing European Claims

Consider a European option with a nonnegative payoff $F(S_T)$ at time $T > 0$. Let $V(S, t)$ denote the price of the option at time t, $0 \le t \le T$, when the underlying asset price is $S_t = S > 0$. Then the pricing relationship under the EMM is $V(S, t) = e^{-r\tau}\mathbb{E}_{t,S}[F(S_T)]$, where $\tau = T - t$ is the time to maturity and $E_{t,S}[\cdot] = E[\cdot|S_t = S]$. Introduce $f(x) := F(Ke^x)$ and $v(x, t) := e^{r\tau} V(Ke^x, t)$ (the forward value function in the variable $x = \ln(S/K)$). We assume that the payoff is such that $\mathcal{I}_f \cap (\lambda_-, \lambda_+)$ is not empty. Then the pricing relationship is $v(x, t) = P_\tau f(x)$, where the expectation operator (2.1) has the Fourier representation (4.2) if the condition (4.1) is satisfied. The option pricing function is recovered via $V(S, t) = e^{-r\tau} v(\ln(S/K), t)$.

For vanilla call options, $F(S) = (S - K)^+$, $f(x) = K(e^x - 1)^+ \in L^1_\alpha(\mathbb{R})$ with $\alpha < -1$ ($\mathcal{I}_{\text{vanilla call}} = (-\infty, -1)$), $f_\alpha(x) = K(e^{(\alpha+1)x} - e^{\alpha x})^+ \in L^1(\mathbb{R})$, and

$$\hat{f}_\alpha(\xi) = -\frac{K}{(\xi - i\alpha)(\xi - i(\alpha + 1))}. \tag{4.6}$$

Moreover, $\hat{f}_\alpha(\xi) \in L^1(\mathbb{R}, \mathbb{C})$ for each $\alpha < -1$ and, hence, the condition (4.1) is automatically satisfied for all Lévy processes with $\lambda_- < -1$ and the Fourier representation (4.2) holds for the vanilla call payoff with $\alpha \in (\lambda_-, -1)$.

For put options, $F(S) = (K - S)^+$, $f(x) = K(1 - e^x)^+ \in L^1_\alpha(\mathbb{R})$ with $\alpha > 0$ ($\mathcal{I}_{\text{vanilla put}} = (0, \infty)$), $f_\alpha(x) = K(e^{\alpha x} - e^{(\alpha+1)x})^+$, and $\hat{f}_\alpha(\xi)$ is given by the same formula (4.6) as for the call, but now $\alpha > 0$. Again, $\hat{f}_\alpha(\xi) \in L^1(\mathbb{R}, \mathbb{C})$ for every $\alpha > 0$ and, hence, the condition (4.1) is automatically satisfied for all Lévy processes with $\lambda_+ > 0$ and the Fourier representation (4.2) holds for the vanilla put payoff with $\alpha \in (0, \lambda_+)$. To price both vanilla calls and puts via the Fourier representation (4.2), we need to require $\lambda_- < -1$ and $\lambda_+ > 0$. From now on we always assume that the Lévy process satisfies this assumption.

For any fixed $\alpha \in \mathbb{R}$, the function $\hat{f}_\alpha(z)$ of the complex variable z in (4.6) has two simple poles at $z = i\alpha$ and $z = i(\alpha + 1)$. Applying the Cauchy residue theorem, one can move the integration contour in (4.2) in the z-plane. If one moves the contour through the pole, one picks up the corresponding residue contribution. This leads to several alternative expressions for the European call and put pricing formulas in the literature (Lewis 2001; Boyarchenko and Levendorskii 2002, in the context of Fourier transform in log-price; and Lee 2004, in the context of Fourier transform in log-strike).

In addition to the vanilla call and put payoffs, for barrier options applications we are also interested in the *truncated call and put payoffs* $F(S) = (S - K)^+ \mathbf{1}_{\{S<U\}}$ and $F(S) = (K - S)^+ \mathbf{1}_{\{S>L\}}$, where $L < K < U$. In these cases the functions $f(x) = K(e^x - 1)^+ \mathbf{1}_{\{x<u\}}$, $u = \ln(U/K) > 0$, and $f(x) = K(1 - e^x)^+ \mathbf{1}_{\{x>l\}}$, $l = \ln(L/K) < 0$, are compactly supported on the intervals $[0, u]$ and $[l, 0]$, respectively, and, hence, $f \in L^1_\alpha(\mathbb{R})$ for any $\alpha \in \mathbb{R}$ ($\mathcal{I}_f = \mathbb{R}$). In particular, we could set $\alpha = 0$, but it will be convenient to keep dependence on α explicit to have the flexibility of selecting α to optimize convergence of the numerical algorithm. The Fourier transform of f_α is

$$\hat{f}_\alpha(\xi) = K\left(\frac{1 - e^{(i\xi+\alpha)b}}{i\xi + \alpha} - \frac{1 - e^{b(1+\alpha+i\xi)}}{1 + \alpha + i\xi}\right), \tag{4.7}$$

where $b = u > 0$ for the truncated call payoff and $b = l < 0$ for the truncated put payoff. Note that $\hat{f}_\alpha$ decays as $|\xi|^{-1}$ as $|\xi| \to \infty$ and $\hat{f}_\alpha(\xi) \notin L^1(\mathbb{R}, \mathbb{C})$. The Fourier representation (4.2) for the truncated calls and puts holds only for those Lévy processes and those times to maturity t for which the condition (4.1) is satisfied. In particular, it holds when the condition (2.4) is satisfied. Note that for the function $\hat{f}_\alpha(z)$ of the complex variable z in (4.7), $z = i\alpha$ and $z = i(1 + \alpha)$ are removable singularities, and $\hat{f}_\alpha(z)$ is an entire function of z.

For credit risk applications (pricing defaultable bonds), we are also interested in the *truncated bond payoff* $F(S) = \mathbf{1}_{\{S>L\}}$. In this case we set $K = L$, $f(x) = \mathbf{1}_{\{x>0\}}$, $x = \ln(S/L) > 0$, $f(x) \in L^1_\alpha(\mathbb{R})$ with $\alpha < 0$ ($\mathcal{I}_{\text{truncated bond}} = (-\infty, 0)$), $f_\alpha(x) = e^{\alpha x}\mathbf{1}_{\{x>0\}} \in L^1(\mathbb{R})$ for $\alpha < 0$, and

$$\hat{f}_\alpha(\xi) = \frac{i}{\xi - i\alpha}. \tag{4.8}$$

This function decays as $|\xi|^{-1}$ as $|\xi| \to \infty$ and $\hat{f}_\alpha(\xi) \notin L^1(\mathbb{R}, \mathbb{C})$. The Fourier representation (4.2) holds only for those Lévy processes and those times to maturity t for which the condition (4.1) is satisfied. In particular, it holds when the condition (2.4) is satisfied. The function $\hat{f}_\alpha(z)$ of the complex variable z in (4.8) has a simple pole at $z = i\alpha$.

5. DISCRETE BARRIER OPTIONS: A HILBERT TRANSFORM APPROACH

5.1. Backward Induction in State Space

We will consider discretely monitored down-and-out options, up-and-out options, and double-barrier options. A discretely monitored down-and-out option is specified by its expiration date $T > 0$, lower barrier $L > 0$, and terminal payoff $F(S) \geq 0$ for $S > L\,(F(S) \equiv 0$ for $S \leq L)$. For the down-and-out call (put), terminal payoff is the vanilla call payoff (truncated put payoff). For the down-and-out bond, the payoff is the truncated bond payoff. A discretely monitored up-and-out option is specified by its expiration $T > 0$, upper barrier $U > 0$, and terminal payoff $F(S) \geq 0$ for $S < U\,(F(S) \equiv 0$ for $S \geq U)$. For the up-and-out call (put), the payoff is the truncated call payoff (vanilla put payoff). A discretely monitored double barrier knock-out call (put) option is specified by its expiration $T > 0$, lower and upper barriers L and U, and the terminal payoff $F(S) \geq 0$ for $S \in (L, U)\,(F(S) \equiv 0$ for $S \notin (L, U))$. For the double-barrier call (put), the payoff is the truncated call payoff (truncated put payoff). For these payoffs the intervals $\mathcal{I}_f$ and explicit expressions for the Fourier transforms $\hat{f}_\alpha$ are given in Section 4.2.

In addition to the payoff, a discrete sequence of barrier monitoring dates $0 < t_1 < \ldots < t_N = T$ is specified. In general, time intervals between the monitoring dates $\{\Delta_j = t_{j+1} - t_j, j = 1, 2, \ldots, N-1\}$ do not have to be equal. To simplify notation and without loss of generality, in what follows we assume equally spaced monitoring dates with the monitoring interval $\Delta_j = \Delta = T/N$, so that the monitoring dates are $t_j = j\Delta$, $j = 1, 2, \ldots, N$. We assume that the asset price follows a process $S_t = Ke^{X_t}$, where X is a Lévy process with $\lambda_- < -1$, $\lambda_+ > 0$ and starting at $x = \ln(S/K)$. Here K is a scale parameter (that we set equal to the option strike K for calls and puts, and to the default barrier L for defaultable bonds) and S is the initial asset price. We use the same notation as in Section 4.2: $V(S, t)$ is the value function (price of the option) at time $t \in [0, T]$ and $v(x, t) = e^{r(T-t)}V(Ke^x, t)$ is the forward value function in the variable $x = \ln(S/K)$. We also denote $v^j(x) := v(x, t_j)$, $j = 0, 1, 2, \ldots, N$, the forward value functions at dates $t_j = j\Delta$, where $t_0 = 0$ is the option valuation date. Generally, the valuation date does not have to coincide with one of the monitoring dates, as one may be interested in valuing the option at some time t between the monitoring dates, and our method can handle arbitrary valuation dates. To lighten notation and without loss of generality, in what follows we assume that the valuation date is $t = t_0 = 0$. Then, the discrete barrier option can be valued as follows:

$$v^0(x) = \mathbb{E}_x\left[f(X_{N\Delta}) \prod_{j=1}^{N} \mathbf{1}_I(X_{j\Delta}) \right],$$

where $\mathbf{1}_I(x)$ is the indicator function of the interval $I = (l, \infty)$ for down-and-out options, $(-\infty, u)$ for up-and-out options, and (l, u) for double-barrier options. The option is knocked out (the option contract is canceled) if $X_t = \ln(S_t/K)$ is outside of I at any of the barrier monitoring dates t_j. Using the Markov property and time homogeneity of the Lévy process X, the discrete barrier option can be valued via the following backward induction:

$$\begin{aligned} v^N(x) &= f(x), \\ v^{j-1}(x) &= \mathbf{1}_I(x) \cdot P_\Delta v^j(x), \quad j = N, N-1, \ldots, 2, \\ v^0(x) &= P_\Delta v^1(x), \end{aligned}$$

and

$$V(S, 0) = e^{-rT} v^0(\ln(S/K)),$$

where $P_t f(x) = \mathbb{E}_x[f(X_t)]$ is the Markov semigroup of X.

Let X be a Lévy process with $\lambda_- < -1$ and $\lambda_+ > 0$ and fix some $\alpha \in (\lambda_-, \lambda_+)$ such that the option payoff $f(x) \in L^1_\alpha(\mathbb{R})$ (assume that $\mathcal{I}_f \cap (\lambda_-, \lambda_+)$ is not empty). We can perform the Esscher transform according to (4.4):

$$v_\alpha^N(x) = f_\alpha(x) = e^{\alpha x} f(x), \tag{5.1}$$

$$v_\alpha^{j-1}(x) = e^{-\Delta\Psi(i\alpha)} \mathbf{1}_I(x) \cdot P_\Delta^{(\alpha)} v_\alpha^j(x), \quad j = N, N-1, \ldots, 2, \tag{5.2}$$

$$v_\alpha^0(x) = e^{-\Delta\Psi(i\alpha)} P_\Delta^{(\alpha)} v_\alpha^1(x), \tag{5.3}$$

and

$$V(S,0) = e^{-rT} \left(\frac{S}{K}\right)^{-\alpha} v_\alpha^0(\ln(S/K)). \tag{5.4}$$

Note that, due to (4.5), $v_\alpha^j(x) = e^{\alpha x} v^j(x) \in L^1(\mathbb{R})$ and

$$\|v_\alpha^j\|_{L^1(\mathbb{R})} \leq e^{-\Delta(N-j)\Psi(i\alpha)} \|f\|_{L_\alpha^1(\mathbb{R})} \tag{5.5}$$

for each $j = 1, \ldots, N$. Hence, we can select the same value of α for all j. The backward induction (5.1–5.3) can be implemented in the Fourier space as well.

5.2. Hilbert Transform

To proceed further we need to recall some facts about the Hilbert transform. The Hilbert transform is well defined a.e. for any $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, by the following Cauchy principal value integral:

$$\mathcal{H}f(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

The following parity relation holds when $1 < p < \infty$:

$$g(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy, \quad f(x) = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{g(y)}{x-y} dy$$

and $g \in L^p(\mathbb{R})$. Furthermore, when $1 < p < \infty$ the Hilbert transform is a bounded operator on $L^p(\mathbb{R})$ with

$$\|\mathcal{H}f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}, \tag{5.6}$$

where $C_p = \tan(\pi/2p)$ if $1 < p \leq 2$ and $C_p = \cot(\pi/2p)$ if $2 \leq p < \infty$ (Pichorides's theorem; see Grafakos 2004, p. 255). In this paper we will consider Hilbert transforms of complex-valued functions $f \in L^p(\mathbb{R}, \mathbb{C})$. The previous discussion immediately carries over to the complex-valued case (the constant in (5.6) doubles to $2C_p$ for complex-valued $f \in L^p(\mathbb{R}, \mathbb{C})$). The theory of Hilbert transform was originally developed by E.C. Titchmarsh and G.H. Hardy. It was named after David Hilbert who first discovered the parity relation for this transform in his work on integral equations.

The Hilbert transform is closely related to the Fourier transform. Recall that for any $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, its Fourier transform $\hat{f} = \mathcal{F}f \in L^q(\mathbb{R}, \mathbb{C})$ with $1/p + 1/q = 1$. Furthermore, the following relationship holds for any $f \in L^p(\mathbb{R})$ with $1 < p < \infty$ or with $p = 1$ if in addition $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$:

$$\mathcal{F}(\text{sgn} \cdot f)(\xi) = i\mathcal{H}\hat{f}(\xi), \tag{5.7}$$

where $\mathrm{sgn}(x)$ is the signum function, and $\mathcal{H}\hat{f} \in L^q(\mathbb{R}, \mathbb{C})$, $1/p + 1/q = 1$ (see Champeney 1987, p. 79; Grafakos 2004, p. 251; Pinsky 2002, p. 186; Stenger 1993, p. 46). Writing

$$\mathbf{1}_{(0,\infty)}(x) = \frac{1}{2}(1 + \mathrm{sgn}(x)),$$

we obtain the following relationship:

$$\mathcal{F}(\mathbf{1}_{(0,\infty)} \cdot f)(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2}(\mathcal{H}\hat{f})(\xi).$$

Denote the translation operator by $\mathcal{T}_a$, $(\mathcal{T}_a f)(x) = f(x - a)$. Then

$$\mathbf{1}_{(l,\infty)} = \mathcal{T}_l \mathbf{1}_{(0,\infty)} = \frac{1}{2}(1 + \mathcal{T}_l \,\mathrm{sgn})$$

and

$$\mathbf{1}_{(l,\infty)} \cdot f = \frac{1}{2} f + \frac{1}{2} f \cdot \mathcal{T}_l \,\mathrm{sgn} = \frac{1}{2} f + \frac{1}{2}\mathcal{T}_l (\mathrm{sgn} \cdot \mathcal{T}_{-l} f).$$

Taking the Fourier transform of both sides and using equation (5.7), we obtain:

(5.8)
$$\mathcal{F}(\mathbf{1}_{(l,\infty)} \cdot f)(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{1}{2}\mathcal{F}(\mathcal{T}_l(\mathrm{sgn} \cdot \mathcal{T}_{-l} f))(\xi) = \frac{1}{2}\hat{f}(\xi) + \frac{i}{2}e^{i\xi l}\mathcal{H}(e^{-i\eta l}\hat{f}(\eta))(\xi).$$

Noting that $\mathbf{1}_{(-\infty,u)} = 1 - \mathbf{1}_{[u,\infty)}$, we also obtain the following:

$$\mathcal{F}(\mathbf{1}_{(-\infty,u)} \cdot f)(\xi) = \frac{1}{2}\hat{f}(\xi) - \frac{i}{2}e^{i\xi u}\mathcal{H}(e^{-i\eta u}\hat{f}(\eta))(\xi). \tag{5.9}$$

Finally, noting that $\mathbf{1}_{(l,u)} = \mathbf{1}_{(l,\infty)} - \mathbf{1}_{[u,\infty)}$, we have for the case of a finite interval:

$$\begin{aligned} \mathcal{F}(\mathbf{1}_{(l,u)} \cdot f)(\xi) &= \frac{i}{2}e^{i\xi l}\mathcal{H}(e^{-i\eta l}\hat{f}(\eta))(\xi) - \frac{i}{2}e^{i\xi u}\mathcal{H}(e^{-i\eta u}\hat{f}(\eta))(\xi) \\ &= \int_{-\infty}^{\infty} \hat{f}(\eta)\frac{e^{iu(\xi-\eta)} - e^{il(\xi-\eta)}}{2\pi i(\xi - \eta)} d\eta \\ &= \int_{-\infty}^{\infty} \hat{f}(\eta) e^{i(\xi-\eta)(l+u)/2} \frac{\sin((\xi - \eta)(u - l)/2)}{\pi(\xi - \eta)} d\eta, \end{aligned} \tag{5.10}$$

where the $P.V.$ can be dropped since $\lim_{x\to 0}(\sin x)/x = 1$ (equation (5.10) can alternatively be obtained by the convolution theorem).

5.3. Backward Induction in the Fourier Space: A Hilbert Transform Representation

We are now ready to formulate the backward induction in the Fourier space.

THEOREM 5.1. *Let $\Delta > 0$ be the barrier monitoring interval. Let X be a Lévy process with characteristic function $\phi_t(\xi)$ with $\lambda_- < -1$ and $\lambda_+ > 0$. Fix $\alpha \in (\lambda_-, \lambda_+)$ such that the option payoff $f \in L^1_\alpha(\mathbb{R})$ (assume $\mathcal{I}_f \cap (\lambda_-, \lambda_+)$ is not empty) and assume that the payoff is such that the condition (4.1) is satisfied for time $t = \Delta$. Assume also that there is some time $\chi > 0$ such that*

$$\|\phi_\chi(\cdot + i\alpha)\|_{L^1} \equiv \int_{\mathbb{R}} |\phi_\chi(\xi + i\alpha)| d\xi < \infty$$

(if this condition is satisfied for some $\chi > 0$, then it is also satisfied for all $t \geq \chi$). Let $\hat{f}_\alpha(\xi)$ be the Fourier transform of $e^{\alpha x} f(x)$. Let $\hat{v}^j_\alpha(\xi)$, $j = 1, \ldots, N$, be the Fourier transforms of

$v^j_\alpha(x) = e^{\alpha x} v^j(x)$. *Then the backward induction (5.1–5.3) takes the following form for the Fourier transforms* $\hat{v}^j_\alpha(\xi)$. *The first step is:* $\hat{v}^N_\alpha(\xi) = \hat{f}_\alpha(\xi)$. *The backward induction is: for each* $j = N, N-1, \dots, 2$, *we have:*

(i) Down-and-Out Options

(5.11)
$$\hat{v}^{j-1}_\alpha(\xi) = \frac{1}{2} e^{-\Delta\Psi(i\alpha)} \left[\phi^{(\alpha)}_\Delta(-\xi)\hat{v}^j_\alpha(\xi) + i e^{i\xi l} \mathcal{H}\left(e^{-i\eta l}\phi^{(\alpha)}_\Delta(-\eta)\hat{v}^j_\alpha(\eta)\right)(\xi)\right],$$

(ii) Up-and-Out Options

(5.12)
$$\hat{v}^{j-1}_\alpha(\xi) = \frac{1}{2} e^{-\Delta\Psi(i\alpha)} \left[\phi^{(\alpha)}_\Delta(-\xi)\hat{v}^j_\alpha(\xi) - i e^{i\xi u} \mathcal{H}\left(e^{-i\eta u}\phi^{(\alpha)}_\Delta(-\eta)\hat{v}^j_\alpha(\eta)\right)(\xi)\right],$$

(iii) Double-barrier Options

(5.13)
$$\hat{v}^{j-1}_\alpha(\xi) = e^{-\Delta\Psi(i\alpha)} \int_{\mathbb{R}} \phi^{(\alpha)}_\Delta(-\eta)\hat{v}^j_\alpha(\eta) e^{i(\xi-\eta)(u+l)/2} \frac{\sin((\xi-\eta)(u-l)/2)}{\pi(\xi-\eta)} d\eta.$$

At the final step (5.3), $v^0_\alpha(x)$ *is computed through the Fourier representation of* $P^{(\alpha)}_\Delta\, v^1_\alpha(x)$:

$$v^0_\alpha(x) = \frac{1}{2\pi} e^{-\Delta\Psi(i\alpha)} \int_{\mathbb{R}} e^{-i\xi x}\phi^{(\alpha)}_\Delta(-\xi)\hat{v}^1_\alpha(\xi) d\xi \tag{5.14}$$

from the function $\hat{v}^1_\alpha(\xi)$ *obtained at the penultimate step. The option pricing function is then recovered via equation (5.4).*

Proof. See Appendix A. □

For single-barrier options the backward inductions (5.11), (5.12) in the Fourier space reduce to the sequential evaluation of Hilbert transforms. For double-barrier options the backward induction (5.13) involves sequential evaluation of a convolution of $\phi^{(\alpha)}_\Delta(-\xi)\hat{v}^j_\alpha(\xi)$ with the function $e^{i\xi(u+l)/2}\sin(\xi(u-l)/2)/(\pi\xi)$.

For future reference we state the following useful inequality (an immediate consequence of the Riemann–Lebesgue theorem and equation (5.5)):

$$\|\hat{v}^j_\alpha\|_{L^\infty(\mathbb{R},\mathbb{C})} \le e^{-\Delta(N-j)\Psi(i\alpha)} \|f\|_{L^1_\alpha(\mathbb{R})} \text{ for every } j = 1, \dots, N. \tag{5.15}$$

REMARK 5.1 (Time-dependent barriers, nonequally spaced monitoring dates and arbitrary valuation dates). Note that time-dependent barriers L_k and/or U_k, as well as nonequally spaced monitoring dates with nonequal Δ_k can be easily handled in this framework. Moreover, the option can be priced at any time $t \in [0, T]$ that does not have to coincide with a barrier monitoring date. It is straightforward to modify the formulas to cover these cases.

REMARK 5.2 (The Greeks). One of the advantages of our method is that the option delta and gamma can be obtained with essentially no additional computational effort. Indeed, we can differentiate equation (5.14) with respect to x under the integral sign. Thus, to compute delta and gamma, the last step (5.14) is modified as follows to compute the derivatives:

$$\frac{d^n v^0_\alpha}{dx^n}(x) = \frac{1}{2\pi} e^{-\Delta\Psi(i\alpha)} \int_{\mathbb{R}} (-i\xi)^n e^{-i\xi x}\phi^{(\alpha)}_\Delta(-\xi)\hat{v}^1_\alpha(\xi)\, d\xi.$$

To compute delta and gamma in addition to the option price, the first $N-1$ steps to obtain $\hat{v}^1_\alpha(\xi)$ are the same (no additional computational cost), and only at the last step one needs to compute one additional Fourier inversion each to obtain the first and second derivatives of $v^0_\alpha(x)$ with respect to x. Then, the first and second derivatives of the value function (5.4) with respect to S give the option delta and gamma.

6. DISCRETE APPROXIMATION: A DISCRETE HILBERT TRANSFORM REPRESENTATION

6.1. Preliminaries

The Hilbert transforms in (5.11) and (5.12), the convolution in (5.13), and the final Fourier inversion in (5.14) have to be evaluated numerically. Let X be a Lévy process with characteristic function ϕ_t and characteristic exponent Ψ. For $\Delta > 0$ and $l < 0 < u$, introduce the following linear operators:

$$\mathcal{P}^\Delta g(\xi) := \frac{1}{2}\phi_\Delta(-\xi)g(\xi) + \frac{1}{2} i\theta e^{i\xi b}\mathcal{H}\left(e^{-i\eta b}\phi_\Delta(-\eta)g(\eta)\right)(\xi), \tag{6.1}$$

$$\mathcal{Q}^\Delta g(\xi) := \int_{\mathbb{R}} \phi_\Delta(-\eta)g(\eta)e^{i(\xi-\eta)(u+l)/2}\frac{\sin((\xi-\eta)(u-l)/2)}{\pi(\xi-\eta)}d\eta, \tag{6.2}$$

$$\mathcal{R}^\Delta g(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}\phi_\Delta(-\xi)g(\xi)\,d\xi, \tag{6.3}$$

where one takes $b = l$ and $\theta = 1$ for down-and-out options and $b = u$ and $\theta = -1$ for up-and-out options in the definition of the operator $\mathcal{P}$.

To implement the backward induction in Theorem 5.1, we need to repeatedly compute operators $\mathcal{P}(\mathcal{Q})$ for single-barrier (double-barrier) options, and at the final stage compute the operator $\mathcal{R}$. In this section we will develop effective discrete approximations for these operators based on the approximation theory for analytic functions. In Section 6.2, we review some remarkable properties of entire functions of exponential type. In Section 6.3, we review some powerful results on the approximation in Hardy spaces of functions analytic in a strip containing the real axis that lead to an accurate discretization scheme for the Hilbert transform based on the trapezoidal rule. This approximation scheme has exponentially decaying errors. In Sections 6.4 and 6.5, we apply these results to develop an approximation scheme for the backward induction of Section 5.3 for discrete barrier options. Our main references for Sections 6.2 and 6.3 are McNamee et al. (1971), Stenger (1976, 1981), and in particular the monograph Stenger (1993).

6.2. Wiener Spaces $W(\pi/h)$ of Entire Functions of Exponential Type

Let $h > 0$, and let $W(\pi/h)$ denote the space of all entire functions (i.e., functions analytic in the whole complex plane), such that $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$, and such that for all $z \in \mathbb{C}$ the function has an exponential bound $|f(z)| \leq Ce^{\pi|z|/h}$ with some $C > 0$. Such functions are called *entire functions of exponential type* π/h, and $W(\pi/h)$ is called the *Wiener space of entire functions of exponential type*. The Paley–Wiener theorem states that for every $f \in W(\pi/h)$ there exists a function $g \in L^2(-\pi/h, \pi/h)$, such that for all $z \in \mathbb{C}$,

$$f(z) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-iz\xi}g(\xi)\,d\xi. \tag{6.4}$$

In other words, entire functions of exponential type have Fourier transforms that vanish outside of the finite interval $(-\pi/h, \pi/h)$ and are L^2 on this interval.

It is a remarkable property of entire functions of exponential type that they can be re-constructed exactly from the knowledge of their values on a discrete grid of points with the step size h, $\{f(kh), k \in \mathbb{Z}\}$.

THEOREM 6.1. *(Stenger 1993, theorem 1.10.1). For each $k \in \mathbb{Z}$, define the function*

$$S(k,h)(z) := \frac{\sin[\pi(z-kh)/h]}{\pi(z-kh)/h}. \tag{6.5}$$

The sequence $\left\{h^{-1/2}S(k,h)(z)\right\}_{k=-\infty}^{\infty}$ is a complete orthonormal sequence in $W(\pi/h)$ and every $f \in W(\pi/h)$ has the expansion

$$f(z) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(z) = h\sum_{k=-\infty}^{\infty} f(kh)\frac{\sin[\pi(z-kh)/h]}{\pi(z-kh)}. \tag{6.6}$$

Moreover, the function $g(\xi)$ in (6.4) is given by for $\xi \in \mathbb{R}$

$$g(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\,dx = h\sum_{k=-\infty}^{\infty} f(kh)e^{ikh\xi}, \quad \text{if} \quad |\xi| < \pi/h \tag{6.7}$$

and equals to zero if $|\xi| > \pi/h$, and $\int_{\mathbb{R}} |f(x)|^2 dx = h\sum_{k=-\infty}^{\infty} |f(kh)|^2$. In particular, the trapezoidal quadrature rule with step size h is exact in $W(\pi/h)$:

$$I(f) = \int_{\mathbb{R}} f(x)dx = h\sum_{k=-\infty}^{\infty} f(kh). \tag{6.8}$$

The proof of this theorem follows from the Paley–Wiener theorem by expanding the function $g \in L^2(-\pi/h, \pi/h)$ in the Fourier series on $(-\pi/h, \pi/h)$: $g(\xi) = \sum_{k=-\infty}^{\infty} c_k e^{ikh\xi}$, $\xi < (-\pi/h, \pi/h)$, with Fourier coefficients $c_k = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-ikh\xi} g(\xi)\,d\xi = hf(kh)$. This yields equation (6.7). Equation (6.8) follows by setting $\xi = 0$ in equation (6.7). Substituting (6.7) into (6.4) and integrating term-by-term yields (6.6) since $\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-iz\xi} he^{ikh\xi}\,d\xi = S(k,h)(z)$.

The resulting expansion (6.6) is called the *Whittaker cardinal series* or *Sinc representation of the entire function of exponential type*, the basis functions $S(k,h)(z)$ (6.5) are called the *Sinc functions*, and the function

$$C(f,h)(z) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(z) \tag{6.9}$$

is called the *Whittaker cardinal function* (the cardinal function was first discussed by Borel, but it was Whittaker [1915] who first recognized its significance for the theory of analytic functions; see historical account in Stenger [1993]). A version of this result is known as the *Shannon's sampling theorem for band-limited functions* (functions with Fourier transforms vanishing outside of a finite interval $(-\pi/h, \pi/h)$), where h is called the *Nyquist interval* (e.g., Pinsky 2002, section 4.2.3).

Equation (6.8) shows that the trapezoidal quadrature rule with step size h is *exact* for entire functions of exponential type. Similarly, the Hilbert transform can also be evaluated *exactly* in $W(\pi/h)$ by the trapezoidal rule (Stenger 1976, equation (2.19); Stenger 1981, equation (2.22)).

COROLLARY 6.1. *If $f \in W(\pi/h)$, then for $x \in \mathbb{R}$,*

$$\mathcal{H}f(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy = \sum_{k=-\infty}^{\infty} f(kh) \frac{1-\cos[\pi(x-kh)/h]}{\pi(x-kh)/h}. \tag{6.10}$$

The proof of this corollary follows by taking the Hilbert transform of the Sinc expansion (6.6) term-by-term and using the result for the Hilbert transform of the Sinc function

$$\mathcal{H}S(k,h)(x) = \frac{1-\cos[\pi(x-kh)/h]}{\pi(x-kh)/h}.$$

6.3. Sinc Approximation in Hardy Spaces of Functions Analytic in a Strip

While the cardinal series expansion is exact for entire functions of exponential type, it also forms the foundation for the powerful approximation theory for functions analytic in a strip including the real axis (and in more general domains in the complex plane that can be obtained from the strip by conformal mappings). Moreover, while equations (6.8) and (6.10) allow one to evaluate integrals and Hilbert transforms of entire functions of exponential type exactly, they also provide remarkably accurate approximations for integrals and Hilbert transforms of functions analytic in a strip.

For $d > 0$ and $\mathcal{D}_d = \{z \in \mathbb{C} : |\Im(z)| < d\}$, let $H^1(\mathcal{D}_d)$ denote the Hardy space[1] of functions analytic in the strip $\mathcal{D}_d$ containing the real axis and such that $\int_{-d}^{d} f(x+iy)\,dy \to 0$ as $x \to \pm\infty$ and the Hardy norm is finite

$$\|f\|_{H^1(\mathcal{D}_d)} := \lim_{y \to d-} \left\{ \int_{\mathbb{R}} |f(x+iy)|dx + \int_{\mathbb{R}} |f(x-iy)|dx \right\} < \infty.$$

For $f \in H^1(\mathcal{D}_d)$ and $h > 0$, let $C(f,h)(z)$ denote the Whittaker cardinal function defined by (6.9). We are interested in the error of the cardinal series (Sinc expansion) approximation for functions in $H^1(\mathcal{D}_d)$: $E_h(f)(z) = f(z) - C(f,h)(z)$ (recall that $E_h(f) \equiv 0$ for functions in $W(\pi/d)$).

THEOREM 6.2. *(Stenger 1993, theorem 3.1.3). Let $z = x + iy$ with $x, y \in \mathbb{R}$ and $|y| < d$ and $h > 0$. If $f \in H^1(\mathcal{D}_d)$, then*

$$\|E_h(f)(\cdot + iy)\|_{L^\infty} \le \frac{e^{-\pi(d-|y|)/h}(1+e^{-2\pi|y|/h})}{2\pi(d-|y|)(1-e^{-2\pi d/h})} \|f\|_{H^1(\mathcal{D}_d)}, \tag{6.11}$$

where $\|f(\cdot + iy)\|_{L^\infty} = \sup_{x \in \mathbb{R}} |f(x+iy)|$.

We observe that the bound on the uniform norm of the error of the Sinc approximation decays as $e^{-\pi(d-|y|)/h}$ when decreasing the approximation step size h (to obtain the bound on the uniform norm for the error on the real line $z = x$, set $y = 0$ in (6.11)). The proof of Theorem 6.2 is based on the application of the Cauchy residue theorem (Stenger 1993, pp. 132–135).

Integrals and Hilbert transforms of functions in $H^1(\mathcal{D}_d)$ can be approximated by equations (6.8) and (6.10). Fix $h > 0$ and define

$$I_h(f) := h \sum_{k=-\infty}^{\infty} f(kh) \quad \text{and} \quad E_h^I(f) := I(f) - I_h(f), \tag{6.12}$$

[1] The Hardy spaces H^p are commonly defined on the unit disk. They can then be defined on any domain in the complex plane that can be mapped into the unit disk by a conformal mapping.

and for $x \in \mathbb{R}$

(6.13)
$$\mathcal{H}_h f(x) := \sum_{k=-\infty}^{\infty} f(kh)\frac{1-\cos[\pi(x-kh)/h]}{\pi(x-kh)/h} \quad \text{and} \quad E_h^{\mathcal{H}}(f)(x) := \mathcal{H}f(x) - \mathcal{H}_h f(x).$$

THEOREM 6.3. *(Stenger 1993, theorems 3.2.1 and 3.4.4). If $f \in H^1(\mathcal{D}_d)$, then*

$$E_h^I(f) = \int_{\mathbb{R}} \left(\frac{e^{-2\pi d/h}}{e^{-2\pi d/h} - e^{2\pi i x/h}} f(x-id) + \frac{e^{-2\pi d/h}}{e^{-2\pi d/h} - e^{-2\pi i x/h}} f(x+id) \right) dx, \tag{6.14}$$

$$\begin{aligned} E_h^{\mathcal{H}}(f)(x) = \int_{\mathbb{R}} \Bigg(& \frac{(e^{-\pi d/h} e^{-i\pi y/h} - \cos(\pi x/h))e^{-\pi d/h}}{\pi(y-x-id)(e^{i\pi y/h} - e^{-i\pi y/h}e^{-2\pi d/h})} f(y-id) \\ & + \frac{(e^{-\pi d/h} e^{-i\pi y/h} - \cos(\pi x/h))e^{-\pi d/h}}{\pi(y-x+id)(e^{-i\pi y/h} - e^{i\pi y/h}e^{-2\pi d/h})} f(y+id) \Bigg) dy. \end{aligned} \tag{6.15}$$

Moreover,

$$\left| E_h^I(f) \right| \le \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} \| f \|_{H^1(\mathcal{D}_d)}, \tag{6.16}$$

$$\left\| E_h^{\mathcal{H}}(f) \right\|_{L^\infty} \le \frac{e^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})} \| f \|_{H^1(\mathcal{D}_d)}. \tag{6.17}$$

Estimates (6.16) and (6.17) show that the errors of the trapezoidal approximations (6.12) and (6.13) of the integrals and Hilbert transforms of functions in $H^1(\mathcal{D}_d)$ are $O(e^{-2\pi d/h})$ and $O(e^{-\pi d/h})$, respectively.

To evaluate Sinc expansions in practice, the infinite sums have to be truncated. Introduce

$$E_{h,M}(f)(z) := f(z) - \sum_{k=-M}^{M} f(kh)S(k,h)(z) = E_h(f)(z) + \sum_{|k|>M} f(kh)S(k,h)(z).$$

Then the total error is bounded by:

$$\| E_{h,M}(f)(\cdot + iy) \|_{L^\infty} \le \| E_h(f)(\cdot + iy) \|_{L^\infty} + \left\| \sum_{|k|>M} f(kh)S(k,h)(\cdot + iy) \right\|_{L^\infty}.$$

Since the Sinc function has the bound $\| S(k,h)(\cdot + iy) \|_{L^\infty} \le \cosh(\pi y/h)$ (Stenger 1993, equation (3.1.32)), the truncation error of the Sinc expansion can be bounded as follows

$$\left\| \sum_{|k|>M} f(kh)S(k,h)(\cdot + iy) \right\|_{L^\infty} \le \cosh(\pi y/h) \sum_{|k|>M} |f(kh)|.$$

Similar truncation bounds can be obtained for the Sinc approximations of integrals and Hilbert transforms. Introduce

$$I_{h,M}(f) := h \sum_{k=-M}^{M} f(kh) \text{ and } E_{h,M}^I(f) := I(f) - I_{h,M}(f),$$

and for $x \in \mathbb{R}$

$$\mathcal{H}_{h,M} f(x) := \sum_{k=-M}^{M} f(kh)\frac{1-\cos[\pi(x-kh)/h]}{\pi(x-kh)/h} \text{ and } E_{h,M}^{\mathcal{H}}(f)(x) := \mathcal{H}f(x) - \mathcal{H}_{h,M} f(x).$$

Then the total error bounds are

$$\left|E^I_{h,M}(f)\right| \le \left|E^I_h(f)\right| + h \sum_{|k|>M} |f(kh)| \tag{6.18}$$

and (using the bound $|(1-\cos x)/x| \le 1$)

$$\left\| E^{\mathcal{H}}_{h,M}(f)\right\|_{L^\infty} \le \left\| E^{\mathcal{H}}_h(f)\right\|_{L^\infty} + \sum_{|k|>M} |f(kh)|. \tag{6.19}$$

The sum $\sum_{|k|>M} |f(kh)|$ can be estimated from the behavior of $f(x)$ as $|x| \to \infty$.

6.4. Approximation of Operators $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$

6.4.1. Lévy Processes with the Estimate (2.5). We now apply the results of the previous section to develop remarkably accurate discrete approximations of the operators $\mathcal{P}, \mathcal{Q}, \mathcal{R}$. For fixed discretization step size $h > 0$ and truncation level $M > 0$, we define the following discretized and truncated operators:

$$\begin{aligned}
\mathcal{P}^\Delta_{h,M} g(\xi) &= \frac{1}{2}\phi_\Delta(-\xi)g(\xi) \\
&\quad + \frac{1}{2} i\theta e^{i\xi b} \sum_{m=-M}^{M} e^{-imhb}\phi_\Delta(-mh)g(mh)\frac{1-\cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h}, \quad \xi \in \mathbb{R}, \\
\mathcal{Q}^\Delta_{h,M} g(\xi) &= h \sum_{m=-M}^{M} e^{i(\xi - mh)(u+l)/2}\phi_\Delta(-mh)g(mh)\frac{\sin[(\xi - mh)(u-l)/2]}{\pi(\xi - mh)}, \quad \xi \in \mathbb{R}, \\
\mathcal{R}^\Delta_{h,M} g(x) &= \frac{1}{2\pi} h \sum_{m=-M}^{M} e^{-ixmh}\phi_\Delta(-mh)g(mh), \quad x \in \mathbb{R}.
\end{aligned}$$

Let $d > 0$ be such that $[-d, d] \subset (\lambda_-, \lambda_+)$, where $\lambda_\pm$ define the strip of analyticity $\mathcal{S}_X$ of the characteristic exponent Ψ of the Lévy process X. Denote $\mathcal{D}_d = \{z \in \mathbb{C} : \Im(z) \in [-d, d]\}$.

THEOREM 6.4. *Assume that the characteristic function $\phi_t(\xi) = e^{-t\Psi(\xi)}$ satisfies the estimate (2.5) for real ξ with some $\nu \in (0, 2]$, $c > 0$, and $\kappa > 0$, and $\|e^{-\Delta\Psi(\cdot \pm id)}\|_{L^1} < \infty$. Suppose $g(z)$ is analytic and bounded in $\mathcal{D}_d$. Then*

$$\left\|\mathcal{P}^\Delta g - \mathcal{P}^\Delta_{h,M} g\right\|_{L^\infty} \le A \frac{e^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})} + Bh^{-1}\Gamma(1/\nu, \Delta c(Mh)^\nu), \tag{6.20}$$

where $\Gamma(a, x) = \int_x^\infty e^{-y} y^{a-1} dy$ is the incomplete Gamma function, and A and B are independent of h and M ($\|g(\cdot + iy)\|_{L^\infty} := \sup_{x\in\mathbb{R}} |g(x + iy)|$ for any $y \in [-d, d]$) :

$$\begin{aligned}
A &= \frac{1}{2} e^{bd}\|g(\cdot + id)\|_{L^\infty}\|e^{-\Delta\Psi(\cdot - id)}\|_{L^1} \\
&\quad + \frac{1}{2} e^{-bd}\|g(\cdot - id)\|_{L^\infty}\|e^{-\Delta\Psi(\cdot + id)}\|_{L^1}, \quad B = \frac{\kappa\|g\|_{L^\infty}}{\nu(\Delta c)^{1/\nu}}.
\end{aligned}$$

In particular, let

$$h = \left(\frac{\pi d}{\Delta c}\right)^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}}. \tag{6.21}$$

Then there exists some $C > 0$ independent of M such that

$$\left\|\mathcal{P}^{\Delta} g - \mathcal{P}^{\Delta}_{h,M} g\right\|_{L^\infty} \le C M^{\frac{1}{1+\nu}} \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}}\right). \tag{6.22}$$

Proof. See Appendix A. □

THEOREM 6.5. *Under the same assumptions as in Theorem 6.4,*

$$\left\|\mathcal{Q}^{\Delta} g - \mathcal{Q}^{\Delta}_{h,M} g\right\|_{L^\infty} \le A \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} + B\Gamma(1/\nu, \Delta c(Mh)^{\nu}), \tag{6.23}$$

where A and B are independent of h and M:

$$A = \frac{e^{dl} + e^{du}}{2\pi d} \|e^{-\Delta\Psi(\cdot - id)}\|_{L^1} \|g(\cdot + id)\|_{L^\infty} + \frac{e^{-dl} + e^{-du}}{2\pi d} \|e^{-\Delta\Psi(\cdot + id)}\|_{L^1} \|g(\cdot - id)\|_{L^\infty}, \quad B = \frac{(u-l)\kappa \|g\|_{L^\infty}}{\pi \nu (\Delta c)^{1/\nu}}.$$

In particular, let

$$h = \left(\frac{2\pi d}{\Delta c}\right)^{\frac{1}{1+\nu}} M^{-\frac{\nu}{1+\nu}}. \tag{6.24}$$

Then there exists some $C > 0$ independent of M such that

$$\left\|\mathcal{Q}^{\Delta} g - \mathcal{Q}^{\Delta}_{h,M} g\right\|_{L^\infty} \le C \max(1, M^{\frac{1-\nu}{1+\nu}}) \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (2\pi d M)^{\frac{\nu}{1+\nu}}\right). \tag{6.25}$$

Proof. See Appendix A. □

THEOREM 6.6. *Under the same assumptions as in Theorem 6.4,*

$$\left|\mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x)\right| \le A \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} + B\,\Gamma(1/\nu, \Delta c(Mh)^{\nu}), \tag{6.26}$$

where A and B are independent of h and M:

$$A = \frac{1}{2\pi} e^{dx} \|e^{-\Delta\Psi(\cdot - id)}\|_{L^1} \|g(\cdot + id)\|_{L^\infty} + \frac{1}{2\pi} e^{-dx} \|e^{-\Delta\Psi(\cdot + id)}\|_{L^1} \|g(\cdot - id)\|_{L^\infty}, \quad B = \frac{\kappa \|g\|_{L^\infty}}{\pi \nu (\Delta c)^{1/\nu}}.$$

In particular, when $h = h(M)$ is taken according to (6.21), there exists some $C > 0$ independent of M such that

$$\left|\mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x)\right| \le C \max(1, M^{\frac{1-\nu}{1+\nu}}) \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}}\right). \tag{6.27}$$

When $h = h(M)$ is taken according to (6.24), there exists some $D > 0$ independent of M such that

$$\left|\mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x)\right| \le D \max(1, M^{\frac{1-\nu}{1+\nu}}) \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (2\pi d M)^{\frac{\nu}{1+\nu}}\right). \tag{6.28}$$

Proof. See Appendix A. □

REMARK 6.1. Selecting the step size h according to equation (6.21) (equation (6.24)) equalizes the discretization and truncation errors for the operator $\mathcal{P}(\mathcal{Q})$ in the leading order: $\exp(-\pi d/h) = \exp(-\Delta c(Mh)^{\nu})$ $(\exp(-2\pi d/h) = \exp(-\Delta c(Mh)^{\nu}))$. This results in the total error bound equation (6.22) (equation (6.25)).

REMARK 6.2. In the truncation error estimate in (6.20) for the operator $\mathcal{P}$, h^{-1} appears in front of the incomplete gamma function. If one takes the limit $h \to 0$ and $M \to \infty$ keeping Mh fixed, then the truncation error bound in (6.20) increases unbounded. Selecting $h = h(M)$ according to (6.21), M goes to infinity faster than h goes to zero, and the truncation error estimate goes to zero as $M^{\frac{1}{\nu+1}} \exp(-CM^{\frac{\nu}{\nu+1}})$ as M increases. We have observed in our numerical experiments that apparently the truncation error estimate in (6.20) can be improved to get rid of the factor h^{-1}. In numerical experiments, when we increase M and decrease h keeping Mh fixed, the error remains bounded above by a constant, rather than increases as h^{-1} as suggested by the truncation error estimate (6.20). Therefore, when we select $h = h(M)$ according to (6.21), the total error goes to zero as $\max(1, M^{\frac{1-\nu}{1+\nu}}) \exp(-CM^{\frac{\nu}{\nu+1}})$ rather than as $M^{\frac{1}{\nu+1}} \exp(-CM^{\frac{\nu}{\nu+1}})$ as M increases, and the estimate (6.22) could also be improved. However, we have not been able to prove this tighter bound for the operator $\mathcal{P}$. According to our numerical experiments, the error estimates (6.23)–(6.25) for the operator $\mathcal{Q}$ and (6.26)–(6.28) for the operator $\mathcal{R}$ appear tight.

6.4.2. Pure Jump Variance Gamma Model. The condition (2.5) does not hold for the pure jump VG model. Instead, the VG characteristic function has a polynomial estimate $|\phi_t(\xi)| \leq \kappa |\xi|^{-2t/\nu}$. The Proofs of Theorems 6.4, 6.5, and 6.6 can be extended to show that the discretization errors in approximating the operators $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ still have estimates $O(\exp(-c/h))$. However, the truncation errors decay only polynomialy due to the polynomial decay of the characteristic function. For $\Delta > \nu/2$, we have for the VG characteristic function:

$$h \sum_{|m|>M} |\phi_\Delta(-mh)| \leq 2\kappa h \sum_{m>M} (mh)^{-2\Delta/\nu}$$
$$\leq 2\kappa \int_{Mh}^{\infty} \xi^{-2\Delta/\nu} d\xi = \frac{2\kappa \nu}{2\Delta - \nu} (Mh)^{-(2\Delta/\nu - 1)}.$$

Therefore, we have the following error estimates in the pure jump VG model when $\Delta > \nu/2$. When $\Delta \leq \nu/2$, the characteristic function decays so slowly, that it is not integrable (equation (2.4) is not satisfied for $t \leq \nu/2$), and we are not generally able to bound the truncation error.

THEOREM 6.7. *Suppose $\Delta > \nu/2$ in the pure jump VG model. Then there exist $A_{\mathcal{P}}, B_{\mathcal{P}}, A_{\mathcal{Q}}, B_{\mathcal{Q}}, A_{\mathcal{R}}, B_{\mathcal{R}} > 0$ independent of h and M such that*

$$\left\| \mathcal{P}^{\Delta} g - \mathcal{P}^{\Delta}_{h,M} g \right\|_{L^\infty} \leq A_{\mathcal{P}} \frac{e^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})} + B_{\mathcal{P}}\, h^{-1} (Mh)^{-(2\Delta/\nu - 1)},$$

$$\left\| \mathcal{Q}^{\Delta} g - \mathcal{Q}^{\Delta}_{h,M} g \right\|_{L^\infty} \leq A_{\mathcal{Q}} \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} + B_{\mathcal{Q}}\, (Mh)^{-(2\Delta/\nu - 1)},$$

$$\left| \mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x) \right| \leq A_{\mathcal{R}} \frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} + B_{\mathcal{R}}\, (Mh)^{-(2\Delta/\nu - 1)}.$$

6.4.3. The Black–Scholes–Merton Model. In the special case of Brownian motion with drift and no jumps, the characteristic function $\phi_t(\xi) = \exp(-\sigma^2 t\xi^2/2 + i\mu t\xi)$ is analytic in the whole complex plane. For double barrier knock-out options, down-and-out puts, and up-and-out calls, the Fourier transforms of the truncated call and put payoffs (4.7) are also analytic in the whole complex plane, and we can set $\alpha = 0$ in this case. As a result, the parameter $d > 0$ defining the width of the strip of analyticity $\mathcal{D}_d$ appearing in the error estimates in theorems 6.4, 6.5, and 6.6 can be selected arbitrarily in this special case. In fact, we can select $d = c/h$ for some $c > 0$ so that d increases as the discretization step h is decreased. In the following theorem, we show that by selecting $d = c/h$ with an appropriate c, in this special case we obtain a discretization error estimate $O(\exp(-C/h^2))$, in contrast to our general error estimate $O(\exp(-C/h))$. Note that the constants A in the discretization error estimates in (6.20), (6.23), and (6.26) depend on the norms $\|g(\cdot \pm id)\|_{L^\infty}$. Due to equations (5.15) and (4.7), we can show that the following estimate holds for the functions $g(\xi - iw) = \hat{v}^j(\xi - iw) = \hat{v}^j_w(\xi)$ for all $1 \le j \le N$ and all $w \in \mathbb{R}$ (here T is the option maturity):

$$\|g(\cdot - iw)\|_{L^\infty} \le K e^{\frac{1}{2}(T-\Delta)\sigma^2 w^2 + (T-\Delta)|\mu w|}\left(\frac{1-e^{bw}}{w} - \frac{1-e^{b(1+w)}}{1+w}\right), \tag{6.29}$$

where $b = l$ for the truncated put payoff, and $b = u$ for the truncated call payoff.

THEOREM 6.8. *Let X be a Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion $\sigma > 0$. Suppose $g(z)$ is analytic in the whole complex plane and satisfies the estimate (6.29). Then we have the following error estimates:*

$$\begin{aligned}
\left\|\mathcal{P}^\Delta g - \mathcal{P}^\Delta_{h,M} g\right\|_{L^\infty} &\le \frac{A_{\mathcal{P}} h}{1 - e^{-\pi^2/(\sigma^2 T h^2)}} \exp\left(-\frac{\pi^2}{2\sigma^2 T h^2} + \frac{\pi \beta_{\mathcal{P}}}{\sigma^2 T h}\right) \\
&\quad + B_{\mathcal{P}} h^{-1} \Gamma\left(\frac{1}{2}, \frac{1}{2}\sigma^2 \Delta (Mh)^2\right) \\
\text{with}\quad A_{\mathcal{P}} &= \frac{2\sigma T K |b|(1 + e^{b+|b|})}{\pi\sqrt{2\pi\Delta}}, \quad B_{\mathcal{P}} = \frac{K}{\sigma\sqrt{2\Delta}}(e^b - b - 1), \\
&\quad \beta_{\mathcal{P}} = |b| + |\mu| T; \\
\left\|\mathcal{Q}^\Delta g - \mathcal{Q}^\Delta_{h,M} g\right\|_{L^\infty} &\le \frac{A_{\mathcal{Q}} h}{1 - e^{-4\pi^2/(\sigma^2 T h^2)}} \exp\left(-\frac{2\pi^2}{\sigma^2 T h^2} + \frac{2\pi \beta_{\mathcal{Q}}}{\sigma^2 T h}\right) \\
&\quad + B_{\mathcal{Q}} \Gamma\left(\frac{1}{2}, \frac{1}{2}\sigma^2 \Delta (Mh)^2\right) \\
\text{with}\quad A_{\mathcal{Q}} &= \frac{2\sigma T K |b|(1 + e^{|b|})}{\pi\sqrt{2\pi\Delta}}, \; B_{\mathcal{Q}} = \frac{K(u-l)}{\pi\sigma\sqrt{2\Delta}}(e^b - b - 1), \\
&\quad \beta_{\mathcal{Q}} = |b| + \max(u, -l) + |\mu| T; \\
|\mathcal{R}^\Delta g(x) - \mathcal{R}^\Delta_{h,M} g(x)| &\le \frac{A_{\mathcal{R}}}{1 - e^{-4\pi^2/(\sigma^2 T h^2)}} \exp\left(-\frac{2\pi^2}{\sigma^2 T h^2} + \frac{2\pi \beta_{\mathcal{R}}}{\sigma^2 T h}\right) \\
&\quad + B_{\mathcal{R}} \Gamma\left(\frac{1}{2}, \frac{1}{2}\sigma^2 \Delta (Mh)^2\right) \\
\text{with}\quad A_{\mathcal{R}} &= \frac{2K|b|(1 + e^{|b|})}{\sigma\sqrt{2\pi\Delta}}, \quad B_{\mathcal{R}} = \frac{K}{\pi\sigma\sqrt{2\Delta}}(e^b - b - 1), \\
&\quad \beta_{\mathcal{R}} = |b| + |\mu| T + |x|.
\end{aligned}$$

If $h = h(M)$ is taken according to

$$h = \left(\frac{\pi^2}{\Delta\sigma^4 T}\right)^{1/4} M^{-1/2}, \tag{6.30}$$

then there exist $C_{\mathcal{P}} > 0$ and $C_{\mathcal{R}} > 0$ independent of M such that

$$\left\|\mathcal{P}^{\Delta} g - \mathcal{P}^{\Delta}_{h,M} g\right\|_{L^\infty} \le C_{\mathcal{P}} \exp\left(-\sqrt{\frac{\pi^2\Delta}{4T}}M + \left(\frac{\pi^2\Delta}{\sigma^4 T^3}\right)^{1/4} \beta_{\mathcal{P}} M^{1/2}\right), \tag{6.31}$$

$$\left|\mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x)\right| \le C_{\mathcal{R}} \exp\left(-\sqrt{\frac{\pi^2\Delta}{4T}}M + 2\left(\frac{\pi^2\Delta}{\sigma^4 T^3}\right)^{1/4} \beta_{\mathcal{R}} M^{1/2}\right). \tag{6.32}$$

If $h = h(M)$ is taken according to

$$h = \left(\frac{4\pi^2}{\Delta\sigma^4 T}\right)^{1/4} M^{-1/2}, \tag{6.33}$$

then there exist some $C_{\mathcal{Q}} > 0$ and $C_{\mathcal{R}} > 0$ independent of M such that

$$\left\|\mathcal{Q}^{\Delta} g - \mathcal{Q}^{\Delta}_{h,M} g\right\|_{L^\infty} \le C_{\mathcal{Q}} \exp\left(-\sqrt{\frac{\pi^2\Delta}{T}}M + \left(\frac{4\pi^2\Delta}{\sigma^4 T^3}\right)^{1/4} \beta_{\mathcal{Q}} M^{1/2}\right), \tag{6.34}$$

$$\left|\mathcal{R}^{\Delta} g(x) - \mathcal{R}^{\Delta}_{h,M} g(x)\right| \le C_{\mathcal{R}} \exp\left(-\sqrt{\frac{\pi^2\Delta}{T}}M + \left(\frac{4\pi^2\Delta}{\sigma^4 T^3}\right)^{1/4} \beta_{\mathcal{R}} M^{1/2}\right). \tag{6.35}$$

Proof. See Appendix A. □

REMARK 6.3. This theorem shows that in the Black–Scholes–Merton model if $h = h(M)$ is taken according to (6.33) for double barrier options or (6.30) for down-and-out puts and up-and-out calls the total approximation errors for operators $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are $O(\exp(-CM))$, in contrast to the $O(\exp(-CM^{2/3}))$ error decay that is predicted by the general results in Theorems 6.4–6.6.

REMARK 6.4. In Merton's jump-diffusion model with Gaussian jumps the characteristic function $\phi_t(\xi) = \exp(-\frac{1}{2}\sigma^2 t\xi^2 + i\mu t\xi - \lambda t(1 - e^{im\xi - \frac{1}{2}s^2\xi^2}))$ is analytic in the whole complex plane. For options with truncated call and put payoffs, the Fourier transform of the payoff is also analytic in the whole complex plane. Hence, the width of the strip of analyticity d can also be selected arbitrarily. However, due to the term $-\lambda t(1 - e^{im\xi - \frac{1}{2}s^2\xi^2})$ in the characteristic function, there will be an extra factor $O(\exp(T\lambda e^{|m|d + \frac{1}{2}s^2 d^2}))$ in the estimates (A.3), (A.4), and (A.5) of the discretization error which grows rapidly as d increases. This prohibits us from taking $d = c/h$ as we did in the Black–Scholes–Merton model (see also Crouch and Spiegelman 1990). In this case, d should be selected so that $T\lambda e^{|m|d + \frac{1}{2}s^2 d^2}$ is well controlled. After d is selected, the discretization step $h = h(M)$ should be selected according to (6.21) or (6.24).

6.5. Computation: A Fast Hilbert Transform Algorithm

To implement the backward induction in the Fourier space (5.11)–(5.14), at each step we need to compute the operators (6.1) for single barrier options or (6.2) for double-barrier options, as well as the operator (6.3) at the final step. Here the underlying Lévy process is the Esscher-transformed process $X^{(\alpha)}$ with an appropriately selected $\alpha \in (\lambda_-, \lambda_+)$ such that the option payoff $f \in L^1_\alpha(\mathbb{R})$. Recall that $\phi_t(z)$ is analytic in the strip $\{z \in \mathbb{C} : \Im(z) \in (\lambda_-, \lambda_+)\}$ containing the real axis. Then $\phi_t^{(\alpha)}(-z) = \phi_t(-z + i\alpha)e^{t\Psi(i\alpha)}$ with $\alpha \in (\lambda_-, \lambda_+)$ is analytic in the strip $\{z \in \mathbb{C} : \Im(z) \in (\alpha - \lambda_+, \alpha - \lambda_-)\}$ containing the real axis. The Fourier transforms $\hat{f}_\alpha(z)$ of the payoffs considered in Section 4.2 are also analytic in some strips containing the real axis. Thus, the function $\phi_\Delta^{(\alpha)}(-z)\hat{f}_\alpha(z)$ is also analytic in some strip containing the real axis. Specifically, for the vanilla call payoff ($\alpha \in (\lambda_-, -1)$), vanilla put payoff ($\alpha \in (0, \lambda_+)$), truncated call and put payoffs ($\alpha \in \mathbb{R}$), and the truncated bond payoff ($\alpha \in (\lambda_-, 0)$), the strips of analyticity around the real axis for the function $\phi_\Delta^{(\alpha)}(-z)\hat{f}_\alpha(z)$ are:

$$\begin{aligned}
\mathcal{S}_{\text{vanilla call}} &= \{z \in \mathbb{C} : \Im(z) \in (\alpha + 1, \alpha - \lambda_-)\},\\
\mathcal{S}_{\text{vanilla put}} &= \{z \in \mathbb{C} : \Im(z) \in (\alpha - \lambda_+, \alpha)\},\\
\mathcal{S}_{\text{truncated call}} = \mathcal{S}_{\text{truncated put}} &= \{z \in \mathbb{C} : \Im(z) \in (\alpha - \lambda_+, \alpha - \lambda_-)\},\\
\mathcal{S}_{\text{truncated bond}} &= \{z \in \mathbb{C} : \Im(z) \in (\alpha, \alpha - \lambda_-)\}.
\end{aligned}$$

Generally, given a Lévy process X, a payoff f, and $\alpha \in (\lambda_-, \lambda_+)$ such that $f \in L^1_\alpha(\mathbb{R})$, we denote the corresponding strip of analyticity of $\phi_\Delta^{(\alpha)}(-z)\hat{f}_\alpha(z)$ by $\mathcal{S}^{(\alpha)}_{X,f} = \{z \in \mathbb{C} : \Im(z) \in (d_-, d_+)\}$. We select $\alpha \in (\lambda_-, \lambda_+)$ so that the strip $\mathcal{S}^{(\alpha)}_{X,f}$ is symmetric around the real axis, i.e., $-d_- = d_+ = d$. This gives $\alpha = (\lambda_- - 1)/2$ and $d = -(\lambda_- + 1)/2$ for the vanilla call, $\alpha = \lambda_+/2$ and $d = \lambda_+/2$ for the vanilla put, $\alpha = (\lambda_+ + \lambda_-)/2$ and $d = (\lambda_+ - \lambda_-)/2$ for the truncated call and put payoffs, and $\alpha = \lambda_-/2$ and $d = -\lambda_-/2$ for the truncated bond payoff.

Then the function $\phi_\Delta^{(\alpha)}(-z)\hat{f}_\alpha(z)$ is analytic in $\mathcal{D}_d := \{z \in \mathbb{C} : |\Im(z)| \le d\}$. Furthermore, if $g(z)$ is analytic in $\mathcal{D}_d$, then the Hilbert transform $(\mathcal{H}g)(\xi)$ may be analytically continued from the real line to the strip $\mathcal{D}_d$ (Stenger 1993, p. 152). Hence, all $\hat{v}^j_\alpha(z)$ are also analytic in $\mathcal{D}_d$. Moreover, all $\hat{v}^j_\alpha(z)$ are bounded in $\mathcal{D}_d$. Indeed, since $\hat{v}^j_\alpha(\xi + iw) = \hat{v}^j_{\alpha-w}(\xi) = \hat{v}^j(\xi + i(w - \alpha))$, from equation (5.15) we have the following inequality for any $|w| \le d$:

$$\left\| \hat{v}^j_\alpha(\cdot + iw)\right\|_{L^\infty(\mathbb{R},\mathbb{C})} \le e^{-\Delta(N-j)\Psi(i(\alpha-w))}\|f\|_{L^1_{\alpha-w}(\mathbb{R})}.$$

The results of Section 6.4 can thus be applied. We fix an integer $M > 0$ and approximate the operators $\mathcal{P}^\Delta$, $\mathcal{Q}^\Delta$, and $\mathcal{R}^\Delta$ with their discretized and truncated versions $\mathcal{P}^\Delta_{h,M}$, $\mathcal{Q}^\Delta_{h,M}$, and $\mathcal{R}^\Delta_{h,M}$, where the discretization step size $h = h(M)$ is selected according to Theorems 6.4, 6.5, or 6.8. Then for single-barrier options we obtain the discretization:

$$\hat{v}^N_{\alpha,M}(kh) = \hat{f}_\alpha(kh) \quad \text{for} \quad k = -M, \ldots, M, \tag{6.36}$$

$$\begin{aligned}
\hat{v}^{j-1}_{\alpha,M}(kh) &= \frac{1}{2}e^{-\Delta\Psi(i\alpha)}\phi^{(\alpha)}_\Delta(-kh)\hat{v}^j_{\alpha,M}(kh)\\
&+\frac{i\theta}{2\pi}e^{-\Delta\Psi(i\alpha)}e^{ikhb}\sum_{m=-M,m\neq k}^{M} e^{-imhb}\phi^{(\alpha)}_\Delta(-mh)\hat{v}^j_{\alpha,M}(mh)\frac{1-(-1)^{k-m}}{k-m}, \quad k=-M,\ldots,M
\end{aligned}\tag{6.37}$$

for $j = N, N-1, \ldots, 2$ ($\theta = 1$ and $b = l$ for down-and-out options, and $\theta = -1$ and $b = u$ for up-and-out options), and

$$v^0_{\alpha,M}(x) = \frac{1}{2\pi}e^{-\Delta\Psi(i\alpha)}\sum_{m=-M}^{M} e^{-imhx}\phi^{(\alpha)}_\Delta(-mh)\hat{v}^1_{\alpha,M}(mh)h. \tag{6.38}$$

The approximation $V_M(S, 0)$ to the option price $V(S, 0)$ at time zero when the stock price is S is finally obtained by equation (5.4) from $v^0_{\alpha,M}(x)$ with $x = \ln(S/K)$.

To obtain the *Discrete Hilbert Transform* from the (truncated) Sinc approximation (6.13)

$$\mathcal{H}_{h,M} f(kh) = \frac{1}{\pi} \sum_{m=-M, m\neq k}^{M} f(mh) \frac{1-(-1)^{k-m}}{k-m}, \tag{6.39}$$

note that

$$\frac{1-\cos(\pi(kh-mh)/h)}{\pi(kh-mh)/h} = \begin{cases} \dfrac{1-(-1)^{k-m}}{\pi(k-m)}, & m \neq k \\ 0, & m = k \end{cases}.$$

For double-barrier options, we replace the backward induction equation (6.37) with:

(6.40)

$$\begin{aligned} &\hat{v}^{j-1}_{\alpha,M}(kh) \\ &= e^{-\Delta\Psi(i\alpha)} e^{ikh(u+l)/2} \sum_{m=-M, m\neq k}^{M} e^{-imh(u+l)/2} \phi^{(\alpha)}_{\Delta}(-mh) \hat{v}^{j}_{\alpha,M}(mh) \frac{\sin(h(k-m)(u-l)/2)}{\pi(k-m)} \\ &\quad + e^{-\Delta\Psi(i\alpha)} \frac{(u-l)}{2\pi} \phi^{(\alpha)}_{\Delta}(-kh) \hat{v}^{j}_{\alpha,M}(kh) h, \quad \text{for} \quad k = -M, \ldots, M, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and} \quad j = N, N-1, \ldots, 2. \end{aligned}$$

To price single-barrier options, at each step in the backward induction we need to evaluate the discrete Hilbert transform (6.37) of a complex vector. To price double-barrier options, at each step in the backward induction we need to evaluate the discrete convolution (6.40). In both cases, the computation involves a Toeplitz matrix–vector multiplication. The corresponding matrices depend only on the difference $k - m$ and, hence, have the Toeplitz structure:

$$T_{k,m} = \begin{cases} \dfrac{1-(-1)^{k-m}}{\pi(k-m)}, & m \neq k \\ 0, & m = k \end{cases}$$

for single-barrier options, and

$$T_{k,m} = \begin{cases} \dfrac{\sin(h(k-m)(u-l)/2)}{\pi(k-m)}, & m \neq k \\ \dfrac{(u-l)h}{2\pi}, & m = k \end{cases}$$

for double-barrier options, respectively. At each step of the backward induction, the Toeplitz matrix–vector multiplication can be accomplished in $O(M \log_2 M)$ operations using the FFT (see Appendix B). We refer to the corresponding algorithm of computing the discrete Hilbert transform via the FFT as the *Fast Hilbert Transform*.

The computation of (6.38) for some given value $x = \ln(S/K)$ is done in $O(M)$ operations. If one wishes to obtain option prices for a large number of underlying asset price values, one may also use the FFT and accomplish the final step in $O(M \log_2 M)$ operations to recover option prices at multiple underlying prices. Hence, the total operation count for our algorithm is $O(NM\log_2 M)$. In summary, the computational algorithm is as follows.

1. (**Preparation**): For a given payoff $f(x)$ ($x = \ln(S/K)$ with some scale parameter K, such as the strike) and a given Lévy process X, select $\alpha \in (\lambda_-, \lambda_+)$ such that $f \in L^1_\alpha(\mathbb{R})$ and $-d_- = d+ = d$ and compute the Fourier transform $\hat{f}_\alpha(\xi) = (\mathcal{F} f_\alpha)(\xi)$. Fix an integer $M > 0$ and select an appropriate discretization step size $h = h(M)$ based on Theorems 6.4–6.8. Compute and store $\{\phi^{(\alpha)}_\Delta(-mh)\}_{m=-M}^{M}$ and $\{\hat{v}^N_\alpha(mh) = \hat{f}_\alpha(mh)\}_{m=-M}^{M}$.
2. (**Backward Induction**): For each $j = N, N-1, \ldots, 2$, for single-barrier options compute the vector $\{\hat{v}^{j-1}_{\alpha,M}(mh)\}_{m=-M}^{M}$ from $\{\hat{v}^{j}_{\alpha,M}(mh)\}_{m=-M}^{M}$ by equation (6.37) by computing the fast Hilbert transform via the Toeplitz matrix–vector multiplication algorithm (for double-barrier options, compute equation (6.40) by computing the discrete convolution via the Toeplitz matrix–vector multiplication algorithm).
3. (**Final Fourier Inversion**): Compute the final Fourier inversion (6.38).

7. NUMERICAL EXAMPLES

7.1. Discrete Barrier Options under the Pure-Jump NIG Process

In this section, we consider a pure-Jump NIG model with the following parameters: $\alpha_{\mathrm{NIG}} = 15$, $\beta_{\mathrm{NIG}} = -5$ and $\delta_{\mathrm{NIG}} = 0.5$. The risk-free interest rate and the dividend yield are $r = 0.05$ (5%) and $q = 0.02$ (2%). The characteristic exponent is given in Table 3.1, $\lambda_- = \beta_{\mathrm{NIG}} - \alpha_{\mathrm{NIG}} = -20$, and $\lambda_+ = \beta_{\mathrm{NIG}} + \alpha_{\mathrm{NIG}} = 10$. The characteristic function $\phi^{(\alpha)}_t$ satisfies the estimate (2.5) with $\nu = 1$ and $c = \delta_{\mathrm{NIG}}$ for any $\alpha \in (\lambda_-, \lambda_+)$, and the martingale condition (3.1) fixes the drift parameter $\mu = r - q + \delta_{\mathrm{NIG}}(\sqrt{\alpha^2_{\mathrm{NIG}} - (\beta_{\mathrm{NIG}}+1)^2} - \sqrt{\alpha^2_{\mathrm{NIG}} - \beta^2_{\mathrm{NIG}}}) = 0.1873$. The mean and standard deviation of the NIG process X_1 at one-year time horizon are 0.01 (1%) and 0.2 (20%), respectively.

We consider the following three option contracts: down-and-out puts (DOP) with $K = 100$ and $L = 80$, down-and-out calls (DOC) with $K = 100$ and $L = 80$, and double-barrier knock-out puts (DBP) with $K = 100$, $L = 80$, and $U = 120$. All options have one year to expiration ($T = 1$). The DOC has the same terminal payoff as the vanilla call. The DOP and DBP have the truncated put payoff. In our examples we consider daily ($\Delta = 1/252$) barrier monitoring (we assume 252 trading days per year). The parameter α is selected so that the strip of analyticity $\mathcal{S}^{(\alpha)}_{X,f}$ is symmetric about the real axis and, hence, the width parameter d of the symmetric strip $\mathcal{D}_d \subset \mathcal{S}^{(\alpha)}_{X,f}$ can be maximized. This leads to the fastest asymptotic error decay according to our error bounds in Section 6.4. For the DOP and DBP, we select $\alpha = -5$ with $d = 15$. For the DOC, we select $\alpha = -10.5$ with $d = 9.5$. Note that it may be advantageous to select nonzero α even for double-barrier options to maximize the width of the strip of analyticity. The discretization step size $h = h(M)$ is selected according to equation (6.24) for the DBP and equation (6.21) for the DOP and DOC, where $c = \delta_{\mathrm{NIG}} = 0.5$ and $\nu = 1$ in this case. For each of the three option contracts, in Figure 7.1 we plot the value function as a function of the underlying asset price S and the maximum error in log-scale (evaluated at underlying asset prices $S = 80, 81, \ldots, 120$) as a function of $M^{1/2}$. For the DBP we also provide the plots for the option delta and its maximum error. Computation times to attain 10^{-8} accuracy are indicated in seconds

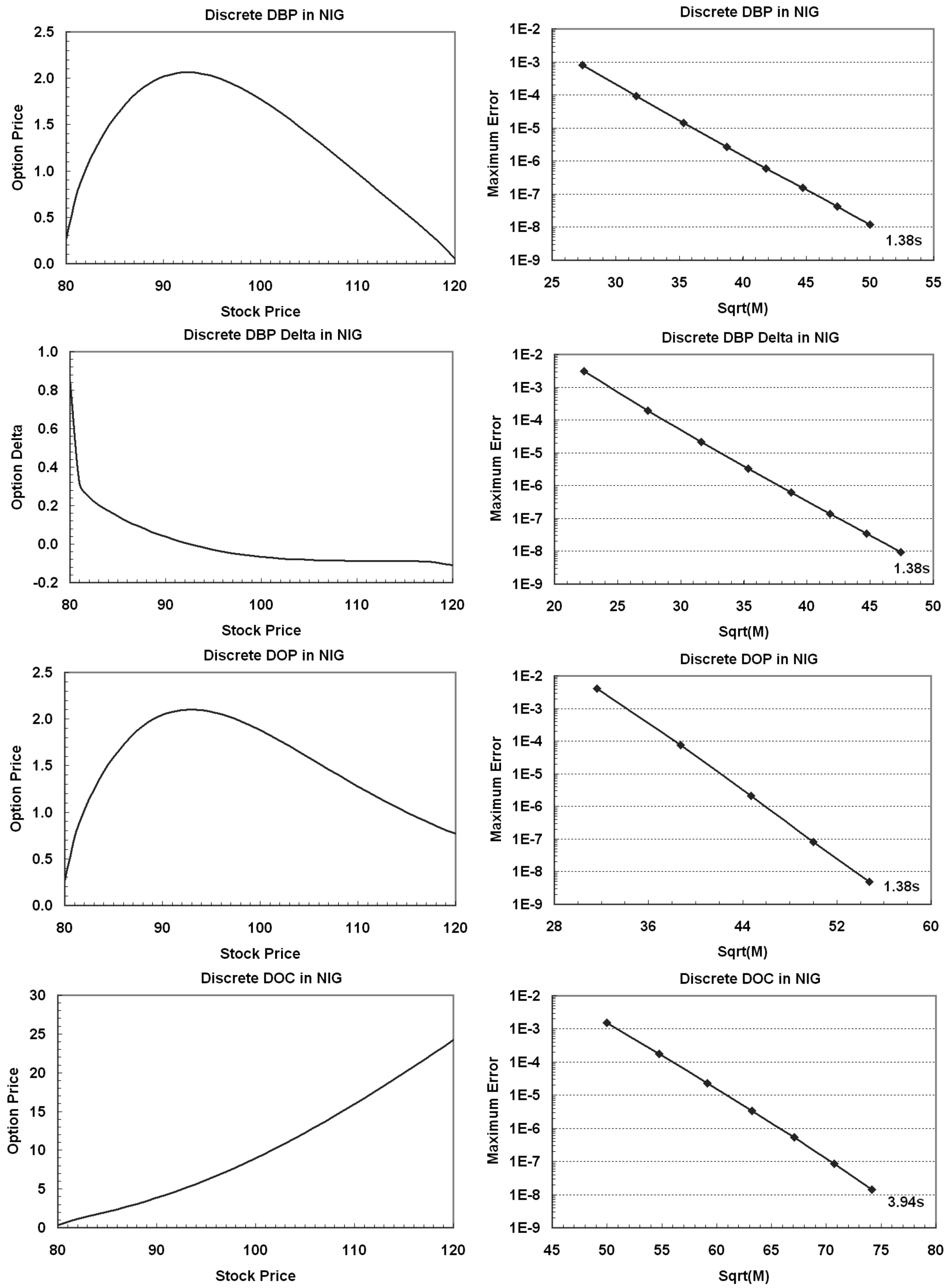

FIGURE 7.1. One-year daily-monitored discrete barrier options in the NIG model.

(all computations in this paper were performed on a Dell desktop with Xeon 3.06 GHz CPU). For all contracts considered, we attain accuracy of 10^{-8} for the one-year daily monitored single- or double-barrier option with 252 monitoring dates in the pure jump NIG model in computation times between 1.38 and 3.94 seconds. Moreover, the option's delta is computed at almost no additional computational cost. To obtain the option delta in addition to the price, only the final step of the algorithm needs to be repeated once. The error plots experimentally verify the $O(\exp(-CM^{1/2}))$ error decay as predicted by theory.

7.2. Discretely Monitored Defaultable Bonds in the Pure-jump NIG Model

In this section, we consider pricing discretely monitored defaultable bonds with $L = 15$, $S = 50$, and recoveries $R = 0$ or $R = 0.5$ paid at maturity in the NIG model with parameters $\alpha_{\mathrm{NIG}} = 5$, $\beta_{\mathrm{NIG}} = -1$, and $\delta_{\mathrm{NIG}} = 0.75$. The standard deviation of the NIG process X_1 at one-year time horizon is 0.4 (40%) for this set of parameters. The risk free interest rate and dividend yield are the same as in the previous section. We price weekly monitored defaultable bonds with maturity T ranging from 1 week to 30 years ($\alpha = -3$ is selected to make the strip of analyticity $\mathcal{S}^{(\alpha)}_{X,f}$ symmetric, and $h = h(M)$ is selected according to equation (6.21)). The relationship between the default probability p (i.e., the probability that the asset price falls below the default barrier L at one of the monitoring dates) and the price B_R of the defaultable bond with unit face value and recovery $R \in [0, 1]$ is given by

$$B_R = e^{-rT}(1 - p + Rp).$$

The first two plots in Figure 7.2 give the term structures of credit spreads and default probabilities in the NIG model. The third plot shows the error in log-scale as a function of $M^{1/2}$ for a 5-year weekly monitored defaultable bond. The error plot experimentally

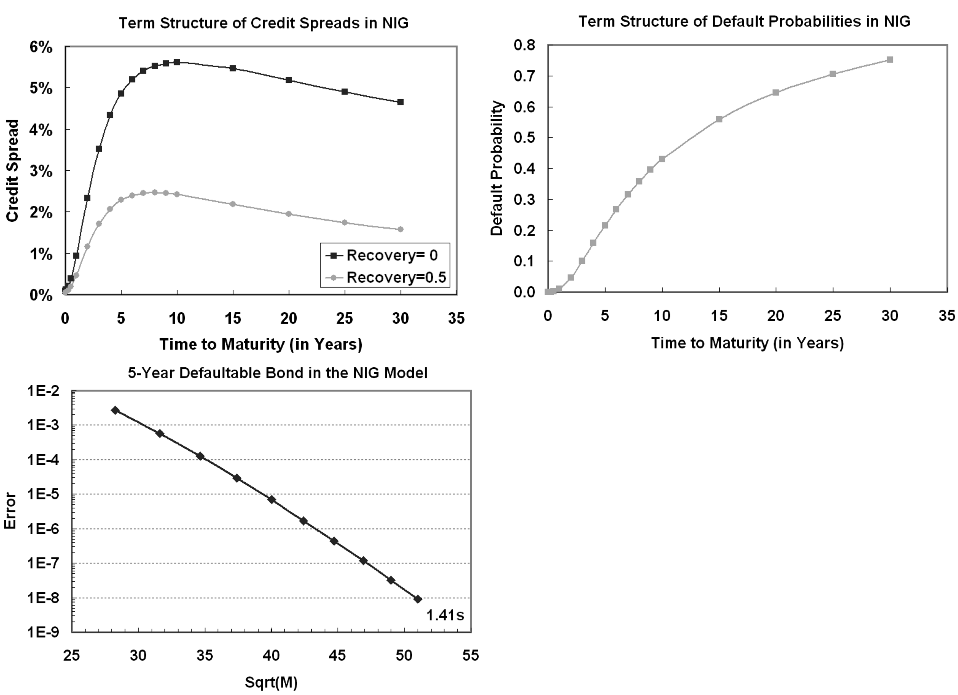

FIGURE 7.2. Weekly monitored defaultable bonds in the NIG model.

verifies the theoretical $O(\exp(-CM^{1/2}))$ error decay. It takes about 1.41 seconds to attain 10^{-8} accuracy in this case.

7.3. Other Lévy Processes

In this section, we present some example computations for other Lévy processes. To save space, we consider only the double-barrier knock-out put option with $K = 100$, $L = 80$, and $U = 120$. The interest rate and dividend yield are the same as in the NIG case. We consider the following models: Black–Scholes–Merton model with $\sigma = 0.2$, Merton's jump-diffusion model with $\sigma = 0.1$, $\lambda = 3$ (intensity of three jumps per year), $m = -0.05$ and $s = 0.086$ (mean jump size of -5% with the standard deviation of 8.6%), Kou's jump-diffusion model with $\sigma = 0.1$, $\lambda = 3$, $p = 0.3$, $\eta_1 = 40$, and $\eta_2 = 12$, the pure-jump CGMY model with $C = 4$, $G = 50$, $M = 60$, $Y = 0.7$, and the pure-jump VG model with $C = 4$, $G = 12$, and $M = 18$. With these parameters, all the processes are such that the distributions of X_1 at one year time horizon have the same mean and standard deviation of 1% and 20%, respectively. See Table 3.1 for characteristic exponents and other details about these processes. All contracts have one year to maturity and daily monitoring ($N = 252$) except for the quarterly monitored pure-jump VG model ($N = 4$). The discretization step size $h = h(M)$ is selected according to equation (6.33) for the BSM model, and equation (6.24) for Kou's model ($\alpha = -14, c = \sigma^2/2,\ \nu = 2,\ d = 26$), Merton's model ($\alpha = 0, c = \sigma^2/2,\ \nu = 2,\ d = 20$), and the CGMY model ($\alpha = -5, c = 2C|\Gamma(-Y)\cos(\pi Y/2)|,\ \nu = Y,\ d = 55$).

For each of the five processes, in Figure 7.3 we plot the maximum error evaluated at $S = 80, 81, \ldots, 120$. For the Black–Scholes model, the error is seen to decay as $O(\exp(-CM))$ as predicted by Theorem 6.8. We attain accuracy of 10^{-10} in just 0.016 seconds. This remarkable numerical performance of our method applied to the Black–Scholes model is comparable to the performance of the double exponential fast Gauss transform method of Broadie and Yamamoto (2005), which is specifically designed for Gaussian models. The advantage of our method is that it is applicable to general non-Gaussian Lévy processes. For Kou's, Merton's, and CGMY models we experimentally verify the $O(\exp(-CM^{\nu/(\nu+1)}))$ error decay as predicted by our theory. For the VG model, we separately plot the discretization error as a function of h, keeping the truncation level fixed so that $Mh = 30{,}000$, and the truncation error as a function of M, keeping the discretization step size fixed $h = 2$. The plots experimentally verify the $O(e^{-C/h})$ discretization error decay and the polynomial truncation error decay. The truncation error decays only polynomially and dominates the exponentially decaying discretization error. Hence, the total error decay is polynomial. Moreover, according to Theorem 6.7, the smaller the barrier monitoring interval Δ, the slower the truncation error decay (due to slow decay of the pure jump VG characteristic function as $|\xi| \to \infty$), slowing down the computation for smaller Δ. Adding a diffusion component to the VG model restores the estimate (2.5). Alternatively, one can consider a pure jump CGMY model with $Y > 0$, which again restores the estimate (2.5).

Table 7.1 presents sample prices for one-year vanilla call and put options ($N = 1$) and daily monitored knock-out barrier options ($N = 252$) for different processes. Parameters are the same as in the previous examples and the options are at-the-money $S = K = 100$ (for the diffusion extended variance gamma model (DEVG), $\sigma = 0.1, s = 0.16,\ \nu = 0.1,\ \theta = -0.2$). The intervals (λ_-, λ_+) are also included in the table. The prices are given with 10^{-8} accuracy.

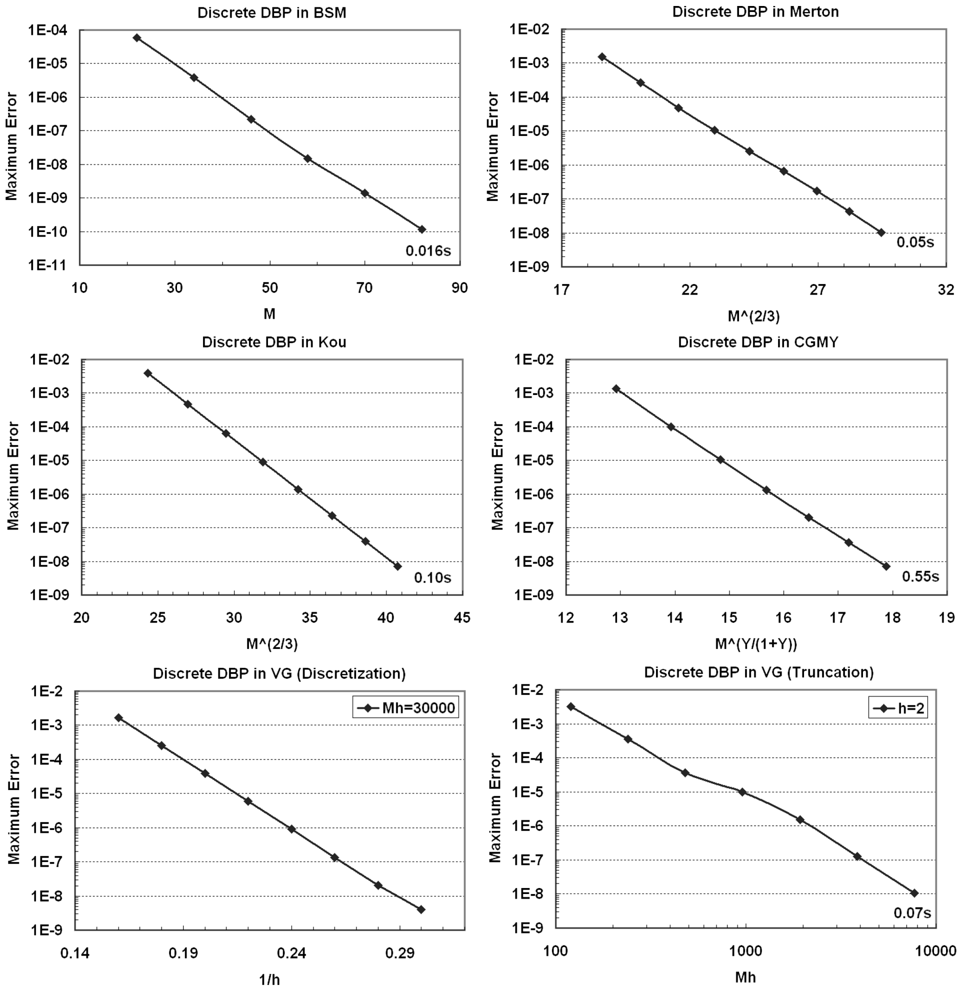

FIGURE 7.3. Pricing errors for one-year double-barrier knock-out put options in Lévy models. Quarterly monitoring ($N = 4$) for the VG model. Daily monitoring ($N = 252$) for all other models.

7.4. Comparison with an Alternative Algorithm

In this section, we compare our Hilbert transform method with an alternative approach. The backward induction (5.1–5.3) can be alternatively implemented as follows. The first step is the same as in Theorem 5.1: compute the Fourier transform $\hat{v}_\alpha^N(\xi) = \hat{f}_\alpha(\xi)$. Then the backward induction is:

$$v_\alpha^{j-1}(x) = \mathbf{1}_I(x) \cdot \frac{1}{2\pi} e^{-\Delta\Psi(i\alpha)} \int_{\mathbb{R}} e^{-i\xi x} \phi_\Delta^{(\alpha)}(-\xi) \hat{v}_\alpha^j(\xi)\, d\xi, \tag{7.1}$$

$$\hat{v}_\alpha^{j-1}(\xi) = \int_I e^{i\xi x} v_\alpha^{j-1}(x)\, dx, \quad j = N, N-1, \ldots, 2. \tag{7.2}$$

The final step is the same as in Theorem 5.1, equation (5.14). Here $I = (l, u)$ for double barrier options, $I = (-\infty, u)$ for up-and-out options, and $I = (l, \infty)$ for down-and-out options. For $j = N, N-1, \ldots, 2$, at each step this involves one Fourier transform

TABLE 7.1
One-year dailymonitored barrier option prices in Lévy models

Option/ model (λ_-, λ_+)	BS $(-\infty, \infty)$	Merton $(-\infty, \infty)$	Kou $(-40,12)$	DEVG $(-37,21)$	NIG $(-20,10)$	CGMY $(-60,50)$
DBP	1.72868009	1.60065569	1.43836344	1.72199580	1.77396718	1.77036472
DBC	1.22420234	2.07502090	2.49384291	1.59045177	1.90734010	1.30878441
DOP	1.87811268	1.71568710	1.53986638	1.85089232	1.88148753	1.91099247
DOC	9.15141382	8.97945779	8.86025111	9.04914284	8.96705248	9.11932528
UOP	6.13865136	5.93687139	5.77759181	6.01743589	5.93391783	6.10938803
UOC	1.27524635	2.10377673	2.50891679	1.62859597	1.93661373	1.35600461
V.PUT	6.33008063	6.12038666	5.98007999	6.20460772	6.11090222	6.29127501
V.CALL	9.22700551	9.01731154	8.87700487	9.10153260	9.00782710	9.18819989

inversion to compute $v_\alpha^{j-1}(x)$ from $\hat{v}_\alpha^j(\xi)$ and one Fourier transform to compute $\hat{v}_\alpha^{j-1}(\xi)$ from $v_\alpha^{j-1}(x)$. The final step involves one Fourier transform inversion to compute $v_\alpha^0(x)$ from $\hat{v}_\alpha^1(\xi)$. In contrast to our Hilbert transform method, this alternative approach does not use the key relationship (5.7) and, instead of computing $N-1$ Hilbert transforms in (5.11) or (5.12) for single-barrier options or $N-1$ convolutions (5.13) for double-barrier options, it computes $N-1$ Fourier transform inversions (7.1) and $N-1$ Fourier transforms (7.2).

To compute the Fourier integrals (7.2) numerically, for down-and-out options we need to truncate the infinite interval $I = (l, \infty)$ by choosing a large enough $u > 0$ and work on the truncated finite interval (l, u). For up-and-out options, we choose some $l < 0$ and work on the truncated interval (l, u). That is, we approximate the single-barrier option by a double-barrier option with an artificial second barrier placed far enough from the current asset price level so that the approximation error is well controlled. Thus, in what follows we only consider the pricing of a double-barrier option with the finite interval $I = (l, u)$.

We observe that the functions $e^{-i\xi x}\phi_\Delta^{(\alpha)}(-\xi)\hat{v}_\alpha^j(\xi)$ have odd imaginary part and even real part. Hence, the Fourier inversion integrals (7.1) can be written in the form:

$$v_\alpha^{j-1}(x) = \mathbf{1}_I(x) \cdot \frac{1}{\pi} e^{-\Delta\Psi(i\alpha)} \Re\left[\int_0^\infty e^{-i\xi x}\phi_\Delta^{(\alpha)}(-\xi)\hat{v}_\alpha^j(\xi)\, d\xi\right].$$

We discretize the integral with some step size h_ξ and truncate the infinite sum after M terms:

$$v_\alpha^{j-1}(x) = \mathbf{1}_I(x) \cdot \frac{1}{\pi} e^{-\Delta\Psi(i\alpha)} \Re\left[\sum_{m=0}^{M-1} e^{-imh_\xi x}\phi_\Delta^{(\alpha)}(-mh_\xi)\hat{v}_\alpha^j(mh_\xi)a_m h_\xi\right],$$

where $\{a_m\}$ are the weights of the quadrature rule. For the trapezoidal rule, $a_0 = 1/2$ and $a_m = 1$ for $1 \le m \le M-1$. The Fourier integrals (7.2) are discretized with some step size h_x as follows:

$$\hat{v}_\alpha^{j-1}(\xi) = \sum_{k=0}^{M-1} e^{i\xi(l+kh_x)} v_\alpha^{j-1}(l+kh_x) b_k h_x,$$

where $\{b_k\}$ are the weights of the quadrature rule used to approximate the Fourier integral. For the trapezoidal rule, $b_0 = b_{M-1} = 1/2$, $b_k = 1$, $1 \le k \le M-2$. For the Simpson rule, $M-1$ should be even and $(b_0, b_1, \ldots, b_{M-1}) = (1/3, 4/3, 2/3, \ldots, 2/3, 4/3, 1/3)$.

To implement the algorithm, we need to compute the values $\hat{v}_\alpha^{j-1}(\xi)$ for $\xi = mh_\xi$, $0 \le m \le M-1$, and $v_\alpha^{j-1}(x)$ for $x = l + kh_x$, $0 \le k \le M-1$ (where $l + (M-1)h_x = u$, so that $h_x = (u-l)/(M-1)$):

$$v_\alpha^{j-1}(l + kh_x) = \frac{1}{\pi} e^{-\Delta\Psi(i\alpha)} \Re\left[\sum_{m=0}^{M-1} e^{-imkh_\xi h_x} e^{-imh_\xi l}\, \phi_\Delta^{(\alpha)}(-mh_\xi)\hat{v}_\alpha^j(mh_\xi) a_m h_\xi\right],$$

$$\hat{v}_\alpha^{j-1}(mh_\xi) = e^{imh_\xi l} \sum_{k=0}^{M-1} e^{imkh_\xi h_x} v_\alpha^{j-1}(l + kh_x) b_k h_x.$$

To use the FFT algorithm to compute the above sums, we need $h_x h_\xi = 2\pi/M$. For a fixed M, the integration step size in the state space is fixed, $h_x = (u-l)/(M-1)$. Then the relationship $h_x h_\xi = 2\pi/M$ also fixes the integration step size in the Fourier space $h_\xi = 2\pi(M-1)/((u-l)M)$. This introduces an undesirable restriction. Alternatively, we allow $h_x h_\xi = 2\pi\theta$ for arbitrary $\theta > 0$. However, the standard FFT may not be applied directly unless $\theta = 1/M$. Instead, we use the so-called *fractional fast Fourier transform* (FrFFT) described in Appendix B. Since we are now free to select h_ξ, we select $h_\xi = h_\xi(M)$ according to (6.24) or (6.33). For double-barrier options, M is the only free parameter controlling the approximation, as both h_x and h_ξ are selected in relation to M. For single-barrier options, we also need to select a second artificial barrier to truncate the infinite integration domain to a finite one.

On the surface of things, at least in the case of double-barrier options, it appears that the alternative method requires twice as many computations, as we need to compute both the Fourier transform and the inverse Fourier transform at each step in the backward induction using the FrFFT, but is otherwise comparable to the Hilbert transform method. At closer examination, the alternative algorithm turns out to be dramatically inferior to the Hilbert transform algorithm. The essence of the Hilbert transform method is that it allows us to stay in the Fourier space until the last Fourier inversion at the final step. Fourier transforms are analytic in a strip, which insures the exponential error decay of the trapezoidal quadrature rule used in evaluating Hilbert transforms and convolutions. In contrast, the alternative method goes back and forth between the state space and the Fourier space. The trapezoidal quadrature used in computing the Fourier inversions (7.1) has exponentially decaying errors due to the analyticity of the integrand in the appropriate strip in the complex plane. However, computing the Fourier integrals (7.2) in the state space with the trapezoidal quadrature is only *second order* accurate in the discretization step size h_x, since we do not have the special analyticity properties in the state space! Thus, the numerical integration error in the state space is only $O(h_x^2)$ or $O(CM^{-2})$, in contrast to the $O(\exp(-CM^{\nu/(1+\nu)})$ or $O(\exp(-CM))$ integration error in the Fourier space. Thus, to achieve the same error, the alternative method requires orders of magnitude larger computation size M in contrast to the Hilbert transform algorithm that does not leave the Fourier space until the final step and does not require one to numerically compute any integrals in the state space. Using a higher-order quadrature rule such as Simpson's in the alternative method improves the error estimate (e.g., $O(CM^{-4})$ for Simpson's), but still the error decay is only polynomial in contrast to the exponential error decay of the Hilbert transform method.

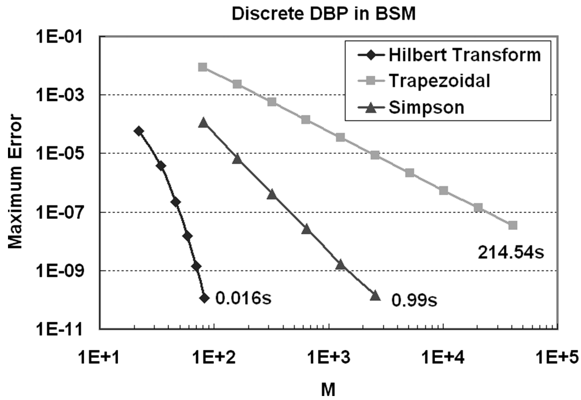

FIGURE 7.4. The Hilbert transform method vs. the alternative method for a one-year, daily monitored double-barrier put in the Black–Scholes–Merton model.

Furthermore, the Hilbert transform method prices single-barrier options directly without first approximating them by double-barrier options with an artificial additional barrier. In contrast, the alternative method requires one to truncate the infinite integration interval in the state space by approximating the single-barrier problem with a double-barrier one. Consider down-and-out options to be specific. We have to place an artificial upper barrier u high enough in order to control the truncation error. But taking a large u forces us to select large M since $M-1=(u-l)/h_x$ for a given discretization step h_x. Thus, the alternative method would require even larger computation size M for single-barrier options to control the additional truncation error that is not present in the double-barrier case (and in the Hilbert transform method).

We now compare numerical performance of the alternative method and the Hilbert transform method for a one-year daily monitored ($T=1$, $N=252$) double-barrier knock-out put option in the Black–Scholes–Merton model with $\sigma=0.2$, $K=100$, $L=80$, $U=120$, $r=0.05$, and $q=0.02$. The maximum pricing error (evaluated at $S=80, 81, \ldots, 120$) is shown in Figure 7.4 in log-log-scale as a function of M for the Hilbert transform method and for the alternative methods with the trapezoidal rule and the Simpson rule. The computational times are shown in seconds. The plot experimentally verifies the polynomial convergence of the alternative method, in contrast to the exponential convergence of the Hilbert transform method.

8. CONCLUSIONS

This paper presented a novel method to price discretely monitored single- and double-barrier options in Lévy process-based models. Our method is based on the key observation that multiplying a function with the indicator function in the state space (monitoring the barrier in the state space) corresponds to taking the Hilbert transform in the Fourier space. Our method thus involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the (Esscher transformed) Lévy process. A remarkably accurate discrete approximation with exponentially decaying errors is developed based on the Whittaker cardinal expansion (Sinc expansion) in Hardy spaces of functions analytic in a strip in the complex plane containing the real axis.

On one hand, our method extends the Carr and Madan (1999) FFT method for European options to discrete single- and double-barrier options. On the other hand, our method extends the Eydeland (1994) FFT method for discrete barrier options in the Gaussian setting to Lévy processes. Our method also provides a natural framework for credit risk applications, where the firm value follows an exponential Lévy process and default occurs at the first time the firm value falls below the default barrier on one of a discrete set of monitoring dates. In addition to barrier option pricing and credit risk applications, our method can also be applied to other problems where one needs to accurately compute distributions of the discretely monitored maximum or minimum of a Lévy process. In particular, our method can be extended to the pricing of discretely monitored lookback options. An extension of our method applicable to lookback options will be reported in a subsequent publication.

A. PROOFS OF THEOREMS

Proof of Theorem 5.1. First consider down-and-out options and take the Fourier transform of equation (5.2) (by Theorem 4.1, $P_\Delta^{(\alpha)} v_\alpha^j \in L^1(\mathbb{R})$ and by equation (2.2) $\mathcal{F}(P_\Delta^{(\alpha)} v_\alpha^j)(\xi) = \phi_\Delta^{(\alpha)}(-\xi)\hat{v}_\alpha^j(\xi)$):

$$\begin{aligned}\hat{v}_\alpha^{j-1}(\xi) &= e^{-\Delta\Psi(i\alpha)}\mathcal{F}(\mathbf{1}_{(l,\infty)} \cdot P_\Delta^{(\alpha)} v_\alpha^j)(\xi)\\ &= \frac{1}{2}e^{-\Delta\Psi(i\alpha)}\left[\phi_\Delta^{(\alpha)}(-\xi)\hat{v}_\alpha^j(\xi) + e^{i\xi l}\mathcal{H}(e^{-i\eta l}\phi_\Delta^{(\alpha)}(-\eta)\hat{v}_\alpha^j(\eta))(\xi)\right],\end{aligned}$$

where we used equation (5.8). This proves equation (5.11). Equation (5.12) for up-and-out options is similarly shown using equation (5.9). Equation (5.13) for double-barrier options is similarly shown using equation (5.10). The final step (5.14) follows from (4.2).

In the statement of the theorem, we assumed that the payoff is such that the condition (4.1) is satisfied for time $t = \Delta$. In order for the Hilbert transforms to be well defined at each step j and in order to be able to invert the Fourier transform at the final step according to equation (4.2), we need $\int_{\mathbb{R}} |\phi_\Delta(-\xi + i\alpha)\hat{v}_\alpha^j(\xi)|d\xi < \infty$ for each $j = 1, 2, \ldots, N$. We can show using Hölder's inequality that if the Fourier transform of the payoff $\hat{v}_\alpha^N(\xi) = \hat{f}_\alpha(\xi)$ satisfies this condition and $\int_{\mathbb{R}} |\phi_\chi(\xi + i\alpha)|d\xi < \infty$ for some $\chi > 0$, then $\hat{v}_\alpha^j(\xi)$ for all $j = 1, \ldots, N$ satisfy this condition due to the boundedness of the Hilbert transform in L^p for all $p > 1$ due to Pichorides' theorem equation (5.6) (note that here we are working with complex-valued functions and the constant in the estimate (5.6) doubles to $2C_p$; see remark below equation (5.6)). Indeed, let $p, q > 1$ be such that $p\Delta \geq \chi$ and $1/p + 1/q = 1$. Then $(\phi_\Delta(-\xi + i\alpha))^p = \phi_{p\Delta}(-\xi + i\alpha) \in L^1$. That is, $\phi_\Delta(-\xi + i\alpha) \in L^p$. Then, using Hölder's inequality, for $j = N - 1$ we have

$$\begin{aligned}&\int_{\mathbb{R}} |\phi_\Delta(-\xi + i\alpha)\hat{v}_\alpha^{N-1}(\xi)|d\xi\\ &\leq \frac{1}{2}\int_{\mathbb{R}} |(\phi_\Delta(-\xi + i\alpha))^2\hat{f}_\alpha(\xi)|d\xi + \frac{1}{2}\int_{\mathbb{R}} |\phi_\Delta(-\xi + i\alpha)\mathcal{H}(e^{-i\eta b}\phi_\Delta(-\eta + i\alpha)\hat{f}_\alpha(\eta))(\xi)|d\xi\\ &\leq \frac{1}{2}\int_{\mathbb{R}} |\phi_{2\Delta}(-\xi + i\alpha)\hat{f}_\alpha(\xi)|d\xi + \frac{1}{2}\|\phi_\Delta(-\cdot + i\alpha)\|_{L^p}\|\mathcal{H}(e^{-ib\cdot}\phi_\Delta(-\cdot + i\alpha)\hat{f}_\alpha(\cdot))\|_{L^q}\\ &\leq \frac{1}{2}\int_{\mathbb{R}} |\phi_{2\Delta}(-\xi + i\alpha)\hat{f}_\alpha(\xi)|d\xi + C_q\|\phi_\Delta(-\cdot + i\alpha)\|_{L^p}\|\phi_\Delta(-\cdot + i\alpha)\hat{f}_\alpha(\cdot)\|_{L^q} < \infty.\end{aligned}$$

We showed that if the Fourier transform of the payoff $\hat{v}_\alpha^N = \hat{f}_\alpha$ satisfies the condition (4.1), then so does $\hat{v}_\alpha^{N-1}$. Hence, by induction, all $\hat{v}_\alpha^j,\ j = 1, \dots, N$, satisfy (4.1), the Hilbert transforms are well defined at each step j, and we can invert the Fourier transform at the final step. □

Proof of Theorem 6.4. The discretization error of replacing $\mathcal{P}^\Delta g$ by $\mathcal{P}^\Delta_{h,\infty} g$ is bounded by

$$\|\mathcal{P}^\Delta g - \mathcal{P}^\Delta_{h,\infty} g\|_{L^\infty} \le \frac{e^{-\pi d/h}}{\pi d(1 - e^{-\pi d/h})} \|\frac{1}{2} e^{-ib\cdot} \phi_\Delta(-\cdot) g(\cdot)\|_{H^1(\mathcal{D}_d)}.$$

The Hardy norm is estimated as follows:

$$\begin{aligned}
&\left\| \frac{1}{2} e^{-ib\cdot} \phi_\Delta(-\cdot) g(\cdot) \right\|_{H^1(\mathcal{D}_d)} \\
&= \frac{1}{2} e^{bd} \int_{\mathbb{R}} |e^{-\Delta\Psi(-\eta - id)} g(\eta + id)| d\eta + \frac{1}{2} e^{-bd} \int_{\mathbb{R}} |e^{-\Delta\Psi(-\eta + id)} g(\eta - id)| d\eta \\
&\le \frac{1}{2} e^{bd} \|g(\cdot + id)\|_{L^\infty} \|e^{-\Delta\Psi(\cdot - id)}\|_{L^1} + \frac{1}{2} e^{-bd} \|g(\cdot - id)\|_{L^\infty} \|e^{-\Delta\Psi(\cdot + id)}\|_{L^1} = A.
\end{aligned}$$

The truncation error of replacing $\mathcal{P}^\Delta_{h,\infty} g(\xi)$ by $\mathcal{P}^\Delta_{h,M} g(\xi)$ is bounded by the following:

$$\begin{aligned}
&\left|\mathcal{P}^\Delta_{h,\infty} g(\xi) - \mathcal{P}^\Delta_{h,M} g(\xi)\right| \\
&\le \sum_{|m|>M} \left| \frac{1}{2} e^{-imhb} \phi_\Delta(-mh) g(mh) \frac{1 - \cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h} \right| \\
&\le \frac{1}{2} \|g\|_{L^\infty} \sum_{|m|>M} |\phi_\Delta(-mh)| \le \frac{\kappa}{\nu(\Delta c)^{1/\nu}} \|g\|_{L^\infty} \quad h^{-1} \Gamma(1/\nu, \Delta c (Mh)^\nu),
\end{aligned}$$

where we have used $|(1 - \cos(x))/x| \le 1$ and

$$\begin{aligned}
h \sum_{|m|>M} |\phi_\Delta(-mh)| &\le 2\kappa h \sum_{m>M} \exp(-\Delta c (mh)^\nu) \\
&\le 2\kappa \int_{Mh}^\infty \exp(-\Delta c \xi^\nu)\, d\xi = \frac{2\kappa}{\nu(\Delta c)^{1/\nu}} \Gamma(1/\nu, \Delta c (Mh)^\nu).
\end{aligned} \tag{A.1}$$

Then (6.20) follows by noticing that the above estimate does not depend on ξ.

Suppose $h = h(M)$ is selected according to (6.21). Recall that $\Gamma(a, x) \sim x^{a-1} e^{-x}$ for large $x > 0$. In particular, for any $a > 0,\ \Gamma(a, x)$ can be bounded by a multiple of $x^{a-1} e^{-x}$ for all positive x bounded away from 0. Since $M \ge 1$, we have that $h \le (\frac{\pi d}{\Delta c})^{\frac{1}{1+\nu}}$ and $Mh = (\frac{\pi d}{\Delta c})^{\frac{1}{1+\nu}} M^{\frac{1}{1+\nu}} \ge (\frac{\pi d}{\Delta c})^{\frac{1}{1+\nu}}$. Therefore, there exist $C_0, C_1 > 0$ independent of M such that

$$\begin{aligned}
\frac{e^{-\pi d/h}}{1 - e^{-\pi d/h}} &\le C_0 e^{-\pi d/h} = C_0 \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}}\right), \\
h^{-1} \Gamma(1/\nu, \Delta c (Mh)^\nu) &\le C_1 h^{-1} (Mh)^{1-\nu} \exp(-\Delta c (Mh)^\nu) \\
&= C_1 \left(\frac{\pi d}{\Delta c}\right)^{-\frac{\nu}{1+\nu}} M^{\frac{1}{1+\nu}} \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}}\right).
\end{aligned}$$

Therefore, there exists some $C > 0$ independent of M such that (6.22) holds. □

Proof of Theorem 6.5. The discretization error of replacing $\mathcal{Q}^{\Delta,\alpha}g$ by $\mathcal{Q}^{\Delta,\alpha}_{h,\infty}g$ is bounded by

$$\left|\mathcal{Q}^{\Delta,\alpha}g(\xi)-\mathcal{Q}^{\Delta,\alpha}_{h,\infty}g(\xi)\right| \le \frac{e^{-2\pi d/h}}{1-e^{-2\pi d/h}}\|\phi_\Delta(-\cdot)g(\cdot)\frac{e^{iu(\xi-\cdot)}-e^{il(\xi-\cdot)}}{2\pi i(\xi-\cdot)}\|_{H^1(\mathcal{D}_d)}.$$

Note that

$$\left|\frac{e^{iu(\xi-(\eta\pm id))}-e^{il(\xi-(\eta\pm id))}}{2\pi(\xi-(\eta\pm id))}\right| \le \frac{e^{\pm dl}+e^{\pm du}}{2\pi d}. \tag{A.2}$$

The Hardy norm is estimated as follows:

$$\begin{aligned}
&\|\phi_\Delta(-\cdot)g(\cdot)\frac{e^{iu(\xi-\cdot)}-e^{il(\xi-\cdot)}}{2\pi i(\xi-\cdot)}\|_{H^1(\mathcal{D}_d)}\\
&\le \frac{e^{dl}+e^{du}}{2\pi d}\int_{\mathbb{R}}|e^{-\Delta\Psi(-\eta-id)}g(\eta+id)|d\eta+\frac{e^{-dl}+e^{-du}}{2\pi d}\int_{\mathbb{R}}|e^{-\Delta\Psi(-\eta+id)}g(\eta-id)|d\eta\\
&\le \frac{e^{dl}+e^{du}}{2\pi d}\|e^{-\Delta\Psi(\cdot-id)}\|_{L^1}\|g(\cdot+id)\|_{L^\infty}+\frac{e^{-dl}+e^{-du}}{2\pi d}\|e^{-\Delta\Psi(\cdot+id)}\|_{L^1}\|g(\cdot-id)\|_{L^\infty}=A.
\end{aligned}$$

The truncation error of replacing $\mathcal{Q}^{\Delta}_{h,\infty}g$ by $\mathcal{Q}^{\Delta}_{h,M}g$ is bounded by:

$$\begin{aligned}
\left|\mathcal{Q}^{\Delta}_{h,\infty}g(\xi)-\mathcal{Q}^{\Delta}_{h,M}g(\xi)\right| &\le h\sum_{|m|>M}\left|\phi_\Delta(-mh)g(mh)\frac{\sin[(\xi-mh)(u-l)/2]}{\pi(\xi-mh)}\right|\\
&\le \frac{u-l}{2\pi}\|g\|_{L^\infty}h\sum_{|m|>M}|\phi_\Delta(-mh)|\le\frac{(u-l)\kappa}{\pi\nu(\Delta c)^{1/\nu}}\|g\|_{L^\infty}\Gamma(1/\nu,\Delta c(Mh)^\nu),
\end{aligned}$$

where we have used $|\sin(x)/x|\le 1$ and (A.1). Then (6.23) follows by noticing that the above estimates do not depend on ξ.

If $h=h(M)$ is selected according to (6.24), then there exist $C_0, C_1>0$ such that

$$\begin{aligned}
\frac{e^{-2\pi d/h}}{1-e^{-2\pi d/h}} &\le C_0e^{-2\pi d/h}=C_0\exp\left(-(\Delta c)^{\frac{1}{1+\nu}}(2\pi dM)^{\frac{\nu}{1+\nu}}\right),\\
\Gamma(1/\nu,\Delta c(Mh)^\nu) &\le C_1(Mh)^{1-\nu}\exp(-\Delta c(Mh)^\nu)\\
&=C_1\left(\frac{2\pi d}{\Delta c}\right)^{\frac{1-\nu}{1+\nu}}M^{\frac{1-\nu}{1+\nu}}\exp\left(-(\Delta c)^{\frac{1}{1+\nu}}(2\pi dM)^{\frac{\nu}{1+\nu}}\right).
\end{aligned}$$

Therefore, there exists some $C>0$ independent of M such that 6.25) holds. □

Proof of Theorem 6.6. The discretization error of replacing $\mathcal{R}^{\Delta}g$ by $\mathcal{R}^{\Delta}_{h,\infty}g$ is bounded by

$$\left|\mathcal{R}^{\Delta,\alpha}g(x)-\mathcal{R}^{\Delta,\alpha}_{h,\infty}g(x)\right| \le \frac{e^{-2\pi d/h}}{1-e^{-2\pi d/h}}\left\|\frac{1}{2\pi}e^{-ix\cdot}\phi_\Delta(-\cdot)g(\cdot)\right\|_{H^1(\mathcal{D}_d)}.$$

The Hardy norm is estimated as follows:

$$\begin{aligned}
&\left\| \frac{1}{2\pi} e^{-ix\cdot} \phi_\Delta(-\cdot) g(\cdot) \right\|_{H^1(\mathcal{D}_d)} \\
&= \frac{1}{2\pi} e^{dx} \int_{\mathbb{R}} |e^{-\Delta\Psi(-\xi - id)} g(\xi + id)| d\xi + \frac{1}{2\pi} e^{-dx} \int_{\mathbb{R}} |e^{-\Delta\Psi(-\xi + id)} g(\xi - id)| d\xi \\
&\le \frac{1}{2\pi} e^{dx} \|e^{-\Delta\Psi(\cdot - id)}\|_{L^1} \|g(\cdot + id)\|_{L^\infty} + \frac{1}{2\pi} e^{-dx} \|e^{-\Delta\Psi(\cdot + id)}\|_{L^1} \|g(\cdot - id)\|_{L^\infty} = A.
\end{aligned}$$

The truncation error of replacing $\mathcal{R}^\Delta_{h,\infty} g$ by $\mathcal{R}^\Delta_{h,M} g$ is bounded by:

$$\begin{aligned}
&\left| \mathcal{R}^\Delta_{h,\infty} g(x) - \mathcal{R}^\Delta_{h,M} g(x) \right| \\
&\le h \sum_{|m|>M} \left| \frac{1}{2\pi} e^{-ixmh} \phi_\Delta(-mh) g(mh) \right| \\
&\le \frac{1}{2\pi} \|g\|_{L^\infty} h \sum_{|m|>M} |\phi_\Delta(-mh)| \le \frac{\kappa}{\pi\nu(\Delta c)^{1/\nu}} \|g\|_{L^\infty} \Gamma(1/\nu, \Delta c(Mh)^\nu),
\end{aligned}$$

where we have used (A.1).

When $h = h(M)$ is taken according to (6.21), there exist $C_0, C_1 > 0$ such that

$$\begin{aligned}
\frac{e^{-2\pi d/h}}{1 - e^{-2\pi d/h}} &\le C_0 e^{-2\pi d/h} \le C_0 e^{-\pi d/h} = C_0 \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}} \right), \\
\Gamma(1/\nu, \Delta c(Mh)^\nu) &\le C_1 (Mh)^{1-\nu} \exp\left(-\Delta c(Mh)^\nu\right) \\
&= C_1 \left(\frac{\pi d}{\Delta c} \right)^{\frac{1-\nu}{1+\nu}} M^{\frac{1-\nu}{1+\nu}} \exp\left(-(\Delta c)^{\frac{1}{1+\nu}} (\pi d M)^{\frac{\nu}{1+\nu}} \right).
\end{aligned}$$

Therefore, (6.27) holds for some $C > 0$. The proof of (6.28) is the same as in Theorem (6.5). □

Proof of Theorem 6.8. The discretization error of replacing $\mathcal{P}^\Delta g$ by $\mathcal{P}^\Delta_{h,\infty} g$ is given by

$$\mathcal{P}^\Delta g(\xi) - \mathcal{P}^\Delta_{h,\infty} g(\xi) = \frac{1}{2} i\theta e^{i\xi b} E^{\mathcal{H}}_h (g_1)(\xi)$$

where $g_1(\eta) = e^{-i\eta b} \phi_\Delta(-\eta) g(\eta) = e^{-\frac{1}{2}\sigma^2\Delta\eta^2} e^{-i\eta(b+\mu\Delta)} g(\eta)$. Denote $g_1(\eta) = e^{-a\eta^2} g_0(\eta)$ with $a = \sigma^2\Delta/2$, $g_0(\eta) = e^{-i\eta(b+\mu\Delta)} g(\eta)$. For an arbitrary $d > 0$, recall the error estimate of the discrete Hilbert transform (6.15):

$$\begin{aligned}
E^{\mathcal{H}}_h g_1(\xi) = \int_{\mathbb{R}} \Bigg(&\frac{e^{ad^2}(e^{-\pi d/h} e^{-i\pi\eta/h} - \cos(\pi\xi/h)) e^{-\pi d/h}}{\pi(\eta - \xi - id)(e^{i\pi\eta/h} - e^{-i\pi\eta/h} e^{-2\pi d/h})} g_0(\eta - id) e^{-a\eta^2 + 2ia\eta d} \\
&+ \frac{e^{ad^2}(e^{-\pi d/h} e^{-i\pi\eta/h} - \cos(\pi\xi/h)) e^{-\pi d/h}}{\pi(\eta - \xi + id)(e^{-i\pi\eta/h} - e^{i\pi\eta/h} e^{-2\pi d/h})} g_0(\eta + id) e^{-a\eta^2 - 2ia\eta d} \Bigg) d\eta.
\end{aligned}$$

It follows that:

$$\|\mathcal{P}^\Delta g - \mathcal{P}^\Delta_{h,\infty} g\|_{L^\infty} \le \frac{e^{ad^2 - \pi d/h}}{2\pi d(1 - e^{-\pi d/h})} \int_{\mathbb{R}} (|g_0(\eta - id)| + |g_0(\eta + id)|) e^{-a\eta^2} d\eta.$$

Recalling (6.29) and noting that $|\frac{1-e^x}{x}| \le e^{|x|}$ (and hence $|\frac{1-e^{bx}}{x}| \le |b|e^{|bx|}$), we have:

$$\begin{aligned}
|g_0(\eta + id)| &= e^{d(b+\mu\Delta)}|g(\eta + id)| \\
&\le e^{db+\frac{1}{2}\sigma^2(T-\Delta)d^2+|\mu|Td} K\left(\frac{1-e^{-db}}{-d} - \frac{1-e^{(1-d)b}}{1-d}\right) \\
&= Ke^{\frac{1}{2}\sigma^2(T-\Delta)d^2+|\mu|Td}\left(\frac{e^{bd}-1}{-d} - \frac{e^{b(d-1)}-1}{1-d}e^b\right) \\
&\le K|b|(1+e^{b+|b|})\exp\left(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td + |b|d\right).
\end{aligned}$$

Similarly, $|g_0(\eta - id)| \le K|b|(1+e^{b+|b|})\exp(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td + |b|d)$. Recalling that $a = \sigma^2\Delta/2$ and noting that $\int_{\mathbb{R}} e^{-a\eta^2 d\eta} = \sqrt{\pi/a}$, we obtain the following bound:

$$\text{(A.3)}\quad \left\|\mathcal{P}^{\Delta}g - \mathcal{P}^{\Delta}_{h,\infty}g\right\|_{L^\infty} \le \frac{2K|b|(1+e^{b+|b|})}{\sigma d\sqrt{2\pi\Delta}(1-e^{-\pi d/h})}\exp\left(\frac{1}{2}\sigma^2Td^2 - \pi d/h + \beta_{\mathcal{P}}d\right)$$

with $\beta_{\mathcal{P}} := |\mu|T + |b|$. Taking $d = \pi/(\sigma^2 Th)$ leads to:

$$\left\|\mathcal{P}^{\Delta}g - \mathcal{P}^{\Delta}_{h,\infty}g\right\|_{L^\infty} \le \frac{2\sigma TK|b|(1+e^{b+|b|})h}{\pi\sqrt{2\pi\Delta}(1-e^{-\pi^2/(\sigma^2Th^2)})}\exp\left(-\frac{\pi^2}{2\sigma^2Th^2} + \frac{\pi\beta_{\mathcal{P}}}{\sigma^2Th}\right).$$

By Theorem 6.4 (with $c = \sigma^2/2$, $\nu = 2$, $\kappa = 1$) and (6.29), the truncation error of replacing $\mathcal{P}^{\Delta}_{h,\infty}g$ by $\mathcal{P}^{\Delta}_{h,M}g$ is bounded by

$$\left\|\mathcal{P}^{\Delta}_{h,\infty}g - \mathcal{P}^{\Delta}_{h,M}g\right\|_{L^\infty} \le \frac{K}{\sigma\sqrt{2\Delta}}(e^b - b - 1)h^{-1}\Gamma(1/2, \sigma^2\Delta(Mh)^2/2).$$

This proves the error estimate for $\mathcal{P}$.

The discretization error of replacing $\mathcal{Q}^{\Delta}g(\xi)$ by $\mathcal{Q}^{\Delta}_{h,\infty}g(\xi)$ is given by

$$\mathcal{Q}^{\Delta}g(\xi) - \mathcal{Q}^{\Delta}_{h,\infty}g(\xi) = E^I_h(g_1)(\xi),$$

where $g_1(\eta) = e^{-\frac{1}{2}\sigma^2\Delta\eta^2 - i\eta\mu\Delta}g(\eta)\frac{e^{iu(\xi-\eta)}-e^{il(\xi-\eta)}}{2\pi i(\xi-\eta)}$. Denote $g_1(\eta) = e^{-a\eta^2}g_0(\eta)$ with $a = \sigma^2\Delta/2$ and $g_0(\eta) = e^{-i\eta\mu\Delta}g(\eta)\frac{e^{iu(\xi-\eta)}-e^{il(\xi-\eta)}}{2\pi i(\xi-\eta)}$. Recall the error of the trapezoidal scheme (6.14),

$$E^I_h(g_1)(\xi) = \int_{\mathbb{R}}\left(\frac{e^{ad^2-2\pi d/h+2ia\eta d}}{e^{-2\pi d/h} - e^{2\pi i\eta/h}}g_0(\eta - id) + \frac{e^{ad^2-2\pi d/h-2ia\eta d}}{e^{-2\pi d/h} - e^{-2\pi i\eta/h}}g_0(\eta + id)\right)e^{-a\eta^2}d\eta.$$

It follows that:

$$\left|\mathcal{Q}^{\Delta}g(\xi) - \mathcal{Q}^{\Delta}_{h,\infty}g(\xi)\right| \le \frac{e^{ad^2-2\pi d/h}}{1-e^{-2\pi d/h}}\int_{\mathbb{R}}(|g_0(\eta - id)| + |g_0(\eta + id)|)e^{-a\eta^2}d\eta.$$

Recall (6.29) and note that $|\frac{1-e^{bx}}{x}| \le |b|e^{|bx|}$,

$$\begin{aligned}
|g_0(\eta + id)| &= e^{\Delta\mu d}|g(\eta + id)|\left|\frac{e^{iu(\xi-\eta)+du} - e^{il(\xi-\eta+dl)}}{2\pi i(\xi - \eta - id)}\right| \\
&\le \frac{e^{du}+e^{dl}}{2\pi d}e^{\frac{1}{2}\sigma^2(T-\Delta)d^2+|\mu|Td}K\left(\frac{1-e^{-db}}{-d} - \frac{1-e^{(1-d)b}}{1-d}\right) \\
&\le \frac{K|b|}{\pi d}(1+e^{|b|})\exp\left(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td + du + |b|d\right).
\end{aligned}$$

Similarly, $|g_0(\eta - id)| \le \frac{K|b|}{\pi d}(1+e^{|b|})\exp(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td - dl + |b|d)$. Recall that $a = \sigma^2\Delta/2$ and $\int_{\mathbb{R}} e^{-a\eta^2}d\eta = \sqrt{\pi/a}$. Note that the above bounds for $g_0(\eta \pm id)$ do not depend on ξ. We thus obtain the following bound:

$$\text{(A.4)} \quad \left\| \mathcal{Q}^\Delta g - \mathcal{Q}^\Delta_{h,\infty} g \right\|_{L^\infty} \le \frac{4K|b|}{\sigma\sqrt{2\pi\Delta}}(1+e^{|b|}) \frac{\exp\left(\frac{1}{2}\sigma^2 Td^2 - 2\pi d/h + \beta_{\mathcal{Q}} d\right)}{d(1-e^{-2\pi d/h})}$$

with $\beta_{\mathcal{Q}} := |b| + \max(u, -l) + |\mu|T$. In particular, taking $d = 2\pi/(\sigma^2 Th)$ leads to the following:

$$\|\mathcal{Q}^\Delta g - \mathcal{Q}^\Delta_{h,\infty} g\|_{L^\infty} \le \frac{2K|b|\sigma T(1+e^{|b|})h}{\pi\sqrt{2\pi\Delta}(1-e^{-4\pi^2/(\sigma^2Th^2)})} \exp\left(-\frac{2\pi^2}{\sigma^2 Th^2} + \frac{2\pi\beta_{\mathcal{Q}}}{\sigma^2 Th}\right).$$

Moreover, from Theorem 6.5 (with $c = \sigma^2/2$, $\nu = 2$, $\kappa = 1$) and (6.29), the truncation error of replacing $\mathcal{Q}^\Delta_{h,\infty} g$ by $\mathcal{Q}^\Delta_{h,M} g$ is bounded by

$$\left\| \mathcal{Q}^\Delta_{h,\infty} g - \mathcal{Q}^\Delta_{h,M} g \right\|_{L^\infty} \le \frac{K(u-l)}{\pi\sigma\sqrt{2\Delta}}(e^b - b - 1)\Gamma(1/2, \sigma^2\Delta(Mh)^2/2).$$

This proves the error estimate for $\mathcal{Q}$.

The discretization error of replacing $\mathcal{R}^\Delta g(x)$ by $\mathcal{R}^\Delta_{h,M} g(x)$ is given by

$$\mathcal{R}^\Delta g(x) - \mathcal{R}^\Delta_{h,M} g(x) = E^I_h g_1(x)$$

where $g_1(\xi) = \frac{1}{2\pi}e^{-i\xi x}\phi_\Delta(-\xi)g(\xi) = e^{-a\xi^2}g_0(\xi)$, with $a = \sigma^2\Delta/2$ and $g_0(\xi) = \frac{1}{2\pi}e^{-i\xi(x+\mu\Delta)}g(\xi)$. Therefore,

$$\left|\mathcal{R}^\Delta g(x) - \mathcal{R}^\Delta_{h,\infty} g(x)\right| \le \frac{e^{ad^2 - 2\pi d/h}}{1 - e^{-2\pi d/h}} \int_{\mathbb{R}} (|g_0(\xi - id)| + |g_0(\xi + id)|)e^{-a\xi^2} d\xi.$$

For $g_0(\xi + id)$ we have:

$$\begin{aligned} |g_0(\xi + id)| &= \frac{1}{2\pi} e^{d(x+\Delta\mu)} |g(\xi + id)| \\ &\le \frac{1}{2\pi} e^{\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td + xd} K\left(\frac{1 - e^{-db}}{-d} - \frac{1 - e^{(1-d)b}}{1-d}\right) \\ &\le \frac{K|b|}{2\pi}(1+e^{|b|})\exp\left(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td + xd + |b|d\right). \end{aligned}$$

Similarly, $|g_0(\xi - id)| \le \frac{K|b|}{2\pi}(1+e^{|b|})\exp(\frac{1}{2}\sigma^2(T-\Delta)d^2 + |\mu|Td - xd + |b|d)$. We thus obtain the following estimate:

$$\text{(A.5)} \quad \left|\mathcal{R}^\Delta g(x) - \mathcal{R}^\Delta_{h,\infty} g(x)\right| \le \frac{2K|b|}{\sigma\sqrt{2\pi\Delta}}(1+e^{|b|}) \frac{\exp\left(\frac{1}{2}\sigma^2 Td^2 - \frac{2\pi d}{h} + \beta_{\mathcal{R}} d\right)}{1 - e^{-2\pi d/h}},$$

where $\beta_{\mathcal{R}} = |b| + |\mu|T + |x|$. Taking $d = 2\pi/(\sigma^2 Th)$ leads to the following estimate:

$$\left|\mathcal{R}^\Delta g(x) - \mathcal{R}^\Delta_{h,\infty} g(x)\right| \le \frac{2K|b|(1+e^{|b|})}{\sigma\sqrt{2\pi\Delta}(1-e^{-4\pi^2/(\sigma^2Th^2)})} \exp\left(-\frac{2\pi^2}{\sigma^2 Th^2} + \frac{2\pi\beta_{\mathcal{R}}}{\sigma^2 Th}\right).$$

From Theorem 6.6 (with $c = \sigma^2/2$, $\nu = 2$, $\kappa = 1$) and (6.29), the truncation error of replacing $\mathcal{R}^\Delta_{h,\infty} g(x)$ by $\mathcal{R}^\Delta_{h,M} g(x)$ is bounded by

$$\left|\mathcal{R}^\Delta_{h,\infty} g(x) - \mathcal{R}^\Delta_{h,M} g(x)\right| \le \frac{K}{\pi\sigma\sqrt{2\Delta}}(e^b - b - 1)\Gamma(1/2, \sigma^2\Delta(Mh)^2/2).$$

This proves the error estimate for $\mathcal{R}$. Equations (6.31)–(6.32) (equations (6.34)–(6.35)) are obtained by direct substitutions of (6.30) (equation (6.33)). □

B. TOEPLITZ MATRIX–VECTOR MULTIPLICATION VIA THE FAST FOURIER TRANSFORM

The discrete Fourier transform (DFT) $(\hat{f}_n)_{n=0}^{N-1}$ of a (complex) vector $(f_n)_{n=0}^{N-1}$ is defined by:

$$\hat{f}_k = \sum_{n=0}^{N-1} e^{-2\pi i n k/N} f_n, \quad k = 0, 1, \dots, N-1.$$

Let w be the primitive N-th root of unity, $w = e^{-2\pi i/N}$, and introduce the DFT matrix $\mathbb{F}$:

$$\mathbb{F} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)(N-1)} \end{pmatrix}.$$

Then the DFT can be written in the matrix form: $\hat{f} = \mathbb{F} f$. The inverse discrete Fourier transform (IDFT) is

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i n k/N} \hat{f}_k, \; n = 0, 1, \dots, N-1,$$

or, in the matrix form, $f = \mathbb{F}^{-1} \hat{f}$, where the inverse of the DFT matrix is $\mathbb{F}^{-1} = \frac{1}{N}\mathbb{F}^H$, where $\mathbb{F}^H = \bar{\mathbb{F}}^\top$ is the Hermitian conjugate transpose of $\mathbb{F}$. The fast Fourier transform (FFT) is an efficient algorithm to compute the DFT (and IDFT) in $O(N \log_2 N)$ complex multiplications compared to N^2 complex multiplications for standard matrix–vector multiplication (e.g., Van Loan 1992).

An $N \times N$ matrix is called *circulant* if it has the following form (e.g., Davis 1994):

$$C = \begin{pmatrix} c_0 & c_{N-1} & \cdots & c_1 \\ c_1 & c_0 & \cdots & c_2 \\ \vdots & \vdots & \vdots & \vdots \\ c_{N-1} & c_{N-2} & \cdots & c_0 \end{pmatrix}.$$

It is completely specified by its first column $c = (c_0, \dots, c_{N-1})^\top$, and each subsequent column is obtained by doing a wrap-around downshift of the previous column. A circulant matrix is diagonalized by the DFT matrix (e.g., Davis 1994): $C = \mathbb{F}^{-1} \triangle \mathbb{F}$, where $\triangle$ is a diagonal matrix with the diagonal containing the eigenvalues of C: $\triangle = \mathrm{diag}(\mathbb{F}c)$. This factorization can be used to perform matrix–vector multiplication. Let $x = (x_n)_{n=0}^{N-1}$ be an N-dimensional vector and $C = (C_{n,m})_{n,m=0}^{N-1}$ an $N \times N$ circulant matrix. Then

$$Cx = \mathbb{F}^{-1} \triangle \mathbb{F} x = \mathbb{F}^{-1}(\mathbb{F}c \circ \mathbb{F}x),$$

where $\circ$ denotes the Hadamard element-wise vector multiplication. This can be computed efficiently using the FFT, by first computing the two DFTs $\mathbb{F}c$ and $\mathbb{F}x$ and then computing the IDFT $\mathbb{F}^{-1}(\mathbb{F}c \circ \mathbb{F}x)$. If the matrix–vector multiplication is performed repeatedly with the same circulant matrix and different vectors, the DFT $\mathbb{F}c$ needs to be computed only once.

An $M \times M$ matrix T is called *Toeplitz* if it has constant values along each (top-left to lower-right) diagonal. That is, a Toeplitz matrix has the form:

$$T = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-(M-1)} \\ t_1 & t_0 & \cdots & t_{-(M-2)} \\ \vdots & \vdots & \vdots & \vdots \\ t_{M-1} & t_{M-2} & \cdots & t_0 \end{pmatrix}.$$

It is completely specified by its first row and its first column. An $M \times M$ Toeplitz matrix T can be embedded into an $N \times N$ circulant matrix C with the first column $c = (t_0, \ldots, t_{M-1}, 0, \ldots, 0, t_{-(M-1)}, \ldots, t_{-1})^\top$. Here $N = 2^l$ is the smallest power of two such that $N \geq 2M - 1$. Note that $N - (2M - 1)$ zeros are padded into the vector c. Using this embedding, the Toeplitz matrix–vector multiplication Tx can be computed as follows:

$$(Tx)_k = (Cx^*)_k = (\mathbb{F}^{-1}(\mathbb{F}c \circ \mathbb{F}x^*))_k, \quad k = 0, 1, \ldots, M-1,$$

where the N-dimensional vector $x*$ is an extension of the original M-dimensional vector x by appending $N - M$ zeros to x. Now the problem is reduced to computing the circulant matrix–vector multiplication, which can be computed efficiently using the FFT as described previously (we chose N to be a power of two in order to use the FFT of radix 2). Applications in finance of the Toeplitz matrix–vector multiplication were pioneered by Eydeland (1994). See Feng and Linetsky (2006) for an application of the Toeplitz matrix-vector multiplication in numerically solving partial integro-differential equations for options pricing in jump diffusion models.

In order to implement the alternative method in Section 7.3, we need to compute the following:

$$\hat{f}_k = \sum_{m=0}^{M-1} e^{-2\pi i m k \theta} f_m, \quad k = 0, \ldots, M-1.$$

When $\theta = 1/M$, it is the usual discrete Fourier transform and can be computed efficiently within $O(M\log_2 M)$ floating point operations as discussed above. For an arbitrary real θ not equal to $1/M$, this is the so-called *fractional discrete Fourier transform*. In the following, we show that the fractional Fourier transform can still be computed in $O(M\log_2 M)$ operations for an arbitrary $\theta \in \mathbb{R}$. Note that

$$\hat{f}_k = e^{-i\pi k^2\theta} \sum_{m=0}^{M-1} e^{i\pi(k-m)^2\theta} e^{-i\pi m^2\theta} f_m, \quad k = 0, \ldots, M-1.$$

This corresponds to the Toeplitz matrix–vector multiplication (the matrix $T_{k,n} = e^{i\pi(k-n)^2\theta}$ only depends on the difference $k - n$). The Toeplitz matrix can be embedded into an $N \times N$ circulant matrix with the first column

$$c = \left(e^{i\pi 0^2\theta}, e^{i\pi 1^2\theta}, \ldots, e^{i\pi(M-1)^2\theta}, 0, \ldots, 0, e^{i\pi(M-1)^2\theta}, \ldots, e^{i\pi 1^2\theta}\right)^\top.$$

Here N is the first power of 2 such that $N \geq 2M - 1$. Note that $N - (2M - 1)$ zeros are padded into c. Denote $g_m = e^{-i\pi m^2\theta} f_m,\ m = 0, \cdots, M-1$. Let $g*$ be the N-dimensional extension of the M-dimensional vector $g = (g_0, \ldots, g_{M-1})^\top$ by appending $N - M$ zeros to g. Then

$$\hat{f}_k = e^{-i\pi k^2\theta}(\mathbb{F}^{-1}(\mathbb{F}c \cdot \mathbb{F}g^*))_k, \quad k = 0, \ldots, M-1,$$

which can be computed efficiently using the fast Fourier transform. The resulting algorithm is called the *fractional fast Fourier transform*. The fractional fast Fourier transform was originally studied in Bailey and Swarztrauber (1991, 1994). Its application to European option pricing in the context of the Carr–Madan method was recently considered by Chourdakis (2004).

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