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Computing Fekete and Lebesgue points: Simplex, square, disk☆

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ABSTRACT

We have computed point sets with maximal absolute value of the Vandermonde determinant (Fekete points) or minimal Lebesgue constant (Lebesgue points) on three basic bidimensional compact sets: the simplex, the square, and the disk. Using routines of the Matlab Optimization Toolbox, we have obtained some of the best bivariate interpolation sets known so far.

1. Introduction

An important and challenging topic of approximation theory is to provide, for a fixed degree n, *good* point sets ξ for polynomial interpolation over multivariate compact domains $K \subset \mathbb{R}^d$. Given a basis $\{\phi_j\}$ of the finite dimensional vector space $\mathbb{P}_n^d$ of d-variate polynomials of degree not greater than n, a first issue consists in finding $\xi = \{\xi_1, \dots, \xi_N\}$ that are *unisolvent*, i.e., the cardinality N of ξ is equal to the dimension of $\mathbb{P}_n^d$ and $\det(V_n(\xi)) \neq 0$, where

$$V_n(\xi) = [v_{ij}] = [\phi_j(\xi_i)], \quad 1 \le i,\ j \le N = \dim(\mathbb{P}_n^d) \tag{1}$$

is the Vandermonde matrix. It is well-known that if K is a bounded interval $[a, b]$, then any set of $N = n+1$ distinct points of K is unisolvent, but the problem is much more difficult for multivariate settings (cf., e.g., [1]). Moreover, as it has been clear since the discovery of the Runge phenomenon, unisolvence does not ensure that the set has good interpolation properties. From this point of view, one searches for unisolvent sets $\xi = \{\xi_i\}$ with low Lebesgue constant

$$\Lambda_n(\xi) = \max_{x \in K} \sum_{i=1}^{N} |\ell_i(x)| \tag{2}$$

where

$$\ell_i(x) = \frac{V_n(\xi_1, \dots, \xi_{i-1}, x, \xi_{i+1}, \dots, \xi_N)}{V_n(\xi_1, \dots, \xi_N)} \tag{3}$$

is the i-th Lagrange polynomial w.r.t. the points ξ. It is not difficult to see that for any continuous function f in the compact domain K

$$\|f - I_n f\|_\infty \le (1 + \Lambda_n(\xi))\|f - f_n^*\|_\infty \tag{4}$$

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where I_nf is the polynomial that interpolates f in ξ and $f_n^* \in \mathbb{P}_n$ is the polynomial of best approximation to f in the ∞-norm. The sets ξ with minimal $\Lambda_n(\xi)$ are known as *Lebesgue points*. From (4), it stems that low Lebesgue constants $\Lambda_n(\xi)$ provide sets whose interpolation error $\|f - I_nf\|_\infty$ is expected to be as close as possible to that of the best interpolant $\|f - f_n^*\|_\infty$. In general, it is not easy to find these sets theoretically (Lebesgue points are not known even for the interval [2]). On the other hand, the so called *Fekete points*, i.e., those sets ξ maximizing the absolute value of the Vandermonde determinant (w.r.t. any polynomial basis), possess Lebesgue constants growing at most as the dimension N of the polynomial space $\mathbb{P}_n$ since $\|\ell_i\|_\infty = 1$ by construction (but in practice can perform much better). They are analytically known only in few cases: the interval (Legendre–Gauss–Lobatto points) where $\Lambda_n = O(\log n)$, the complex circle (equispaced points) where again $\Lambda_n = O(\log n)$, and the cube (tensor-product of Legendre–Gauss–Lobatto points) for tensorial interpolation where $\Lambda_n = O(\log^d n)$; cf. [3,4].

Notice that, whereas the existence of Fekete points for a given compact set K is trivial, since $\det(V_n(\xi))$ is a polynomial in $\xi \in K^N$, the problem is more subtle concerning Lebesgue points. Indeed, the Lebesgue constant $\Lambda_n(\xi)$ is not continuous on the whole K^N, since the denominator of the Lagrange polynomials vanishes on a subset of K^N which is an algebraic variety. Nevertheless, if K is polynomial determining, that is polynomials vanishing there vanish everywhere (this is true for example whenever K has internal points), such is K^N and thus there are points in K^N where $\det(V_n(\xi))$ does not vanish. The Lebesgue constant is then positive, goes to infinity at the variety, and is continuous in the rest of K^N. By a suitable redefinition on the variety, the Lebesgue constant becomes lower-semicontinuous and thus has a global minimum on the compact K^N, that is taken at Lebesgue points (which, as Fekete points, are not unique, in general).

Computing Fekete and Lebesgue points requires solving a large-scale nonlinear optimization problem. Indeed, the number of variables (that are the coordinates of the optimal points) is $2N$, with $N = \dim(\mathbb{P}_n^d)$. In dimension $d = 2$, for example, we deal with $2 \times 66 = 122$ variables already at degree $n = 10$. In order to provide a cheap numerical approximation of Fekete points, recently *Approximate Fekete Points* and *Discrete Leja Points* have been introduced, cf. [5–7]. Though they are not optimal, the absolute values of their Vandermonde determinants are significantly high and the computation requires only basic linear algebra routines (*QR* and *LU* factorizations of Vandermonde matrices). Furthermore, they provide good interpolation sets on rather general compact domains, and can be used, as is done in the present work, as starting guess for more sophisticated optimization procedures.

The main purpose of this paper is to provide Fekete and Lebesgue points on three basic bidimensional compact sets, the simplex, the square, and the disk, by solving numerically the corresponding large-scale nonlinear optimization problems up to degree $n = 18$. Once such points have been computed in one reference set, they can be used on any triangle, parallelogram, and ellipse, by affine mapping. The results of our computational work reach and often improve those previously known. The interpolation sets and the Matlab codes are available at the webpage [8]. The codes can be easily extended to other domains, for instance simple polygons. In the case of the simplex, due to their relevance in developing spectral and high-order methods for PDEs we have also computed interpolation sets that have an assigned distribution on the sides (Legendre–Gauss–Lobatto side nodes), which appear to be better than those provided in [9,10]. Concerning the square, besides Fekete and Lebesgue points, we have computed some new sets that generalize the Padua points [11] and improve their already good quality. Very little seems to be known about Fekete and Lebesgue points for the disk (cf., e.g., [12]), and we hope that our computational results could put some insight into this topic.

2. Computational aspects

For computing almost optimal points, the Matlab Optimization Toolbox (cf. [13]) is particularly appealing since we can determine the desired point sets by methods that are considered state of the art. This numerical environment provides three built-in routines, `fmincon` for constrained minimization and `fminsearch`, `fminunc` for unconstrained optimization. Their usage is straightforward, one has only to provide the target function F to minimize, and a good starting guess. Each of these routines computes (approximately) the minimum of F. The optimization algorithms have default options to free the user from the burden of deciding some specific parameters as the size of the derivatives, the number of iterations, ... However, we experienced that these settings were not fully tailored to our purposes. For this reason, beyond `MaxIter` and `MatFunEvals` that determine the maximum number of iterations and of function evaluations, after several trials and numerical experiments, it has been important to put `RelLineSrchBnd` and `DiffMaxChange` equal to 10^{-3}. With such modifications, the methods that were erratic or too static achieved a better numerical behavior.

In the present context, for a given set of points $\xi \subset K$, we will consider as target functions the numerically evaluated Lebesgue constant and the absolute value of the determinant of $V_n(\xi)$, where the latter is the Vandermonde matrix of degree n w.r.t. a certain polynomial basis. We point out that the sets that we obtain are not the true Fekete or Lebesgue points, but that they share with them low Lebesgue constants and high (relative to the given polynomial basis) absolute values of $\det(V_n(\xi))$.

In order to compute the Lebesgue constant $\Lambda_n(\xi)$ of a particular point set $\xi = \{\xi_i\}$, usually one fixes a fine reference mesh $X \subset K$ and evaluates the Lebesgue function

$$\lambda_n(x;\xi) = \sum_{i=1}^{N} |\ell_i(x)|, \quad x \in X. \tag{5}$$

If we define as target function

$$F_1(\xi) = \max_{x \in X} \lambda_n(x; \xi) \tag{6}$$

and make the assumption that

$$\Lambda_n(\xi) = \max_{x \in K} \lambda_n(x; \xi) \approx \max_{x \in X} \lambda_n(x; \xi) = F_1(\xi) \tag{7}$$

the minimizers will compute sets with particularly low Lebesgue constant. In all our examples, we have chosen as reference discretization X an *admissible mesh* of K with approximately $250^2 = 62\,500$ points. Admissible meshes are good discretizations of a compact set, in the sense that they determine the ∞-norm of polynomials up to a constant, i.e., $\|p\|_K \le C\|p\|_X$ for all polynomials of a certain degree. They have been introduced recently by Calvi and Levenberg in [14], as a relevant tool for multivariate polynomial approximation. Admissible meshes for quadrangles, triangles and disks have been studied in [15], where it has been shown that there exists Chebyshev-like tensorial meshes for these compacts with approximately $(\mu\nu)^2$ points, such that the inequality above is satisfied by all polynomials of degree not greater than ν with $C = 1/(\cos 2\mu)^2$. For $\mu\nu = 250$, we get for example that the discrepancy between the maximum absolute value of a polynomial of degree $\nu = 50$ on the mesh and that on the whole compact is about 10%. Estimation of continuous functions that are not polynomials via admissible meshes, is more delicate and essentially related to the accuracy of their best polynomial approximation. In practice, however, we have found that admissible meshes with a large number of points provide very good estimates of the norms of projection operators on low degree polynomial subspaces, cf., e.g., [16]. Indeed, by these meshes we have recovered up to the second significant figure the Lebesgue constants of all the best interpolation sets known in the literature for the simplex and the square (see Section 3).

Concerning the computation of Fekete points, first of all we have to choose carefully a *well-conditioned* polynomial basis. One of the obvious reasons for such a decision is that if at degree n the Vandermonde determinant is approximated as `Inf` by the code, no more Fekete sets approximations can be computed. In the case of Fekete points, it is natural to consider as target function

$$F_2(\xi) = -|\det(V_n(\xi))| \tag{8}$$

(here the—sign depends from the fact that the Matlab optimization routines compute the minimum of a function). However, it has been important to scale this value to avoid possible overflows, i.e., if the initial set of points $\xi^{(\mathbf{0})}$ is *unisolvent*, i.e., $|\det V_n(\xi^{(\mathbf{0})})| > 0$, we took as target function

$$F_2(\xi) = -\left|\frac{\det(V_n(\xi))}{\det(V_n(\xi^{(0)}))}\right|. \tag{9}$$

We observe that in most of the previous papers about this subject, the authors have defined tailored algorithms, usually of unconstrained type. Since the domains K are bounded, it is not clear what to do when some points are not in K. In our numerical experiments, we used both, constrained and unconstrained methods. If at the k-th iteration the minimizer computed a set $\xi^{(k)}$ with some points not in K, we set the target function value equal to 10^{30} for the Lebesgue constant case and to `realmin` for the absolute value of Vandermonde determinant, so that the minimizer changes suitably to other directions. Observe also that, even though one can prove that there exists at least a point set with minimal Lebesgue constant, the function that associates to a point set its Lebesgue constant has singularities, hence derivative-free methods are preferable.

In the three domains that we have studied, we started from already good sets of points, in terms of Lebesgue constants or absolute value of Vandermonde determinants. Then we applied some post-processing routine to get even better results. Since the approximation of these initial sets varies from domain to domain, we describe separately the details of these strategies for the simplex, the square, and the disk. All the experiments were made using Matlab 7.6 on a 2.13 GHz Intel Core 2 Duo, with the release 4.0 of the Matlab Optimization Toolbox.

3. Application to some bivariate domains

3.1. Simplex

In the case of the simplex, several good point sets are already known. We mention among the recent ones, without any sake of completeness, [10,17–20]. For a good description of what is available on the simplex, with a glimpse on spectral element approximation, consider also [21,22], and references therein. Notice that in all those papers the reference triangle may vary, but since there is an affine transformation between any two triangles, the points can be mapped preserving the quality of the sets. We will work on the unit simplex, i.e., the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

To our knowledge, the points with highest absolute value of the Vandermonde determinant were given in [10], while those with the best Lebesgue constants were obtained in [9]. In both the papers, the authors describe some unconstrained methods for computing the point sets, but the actual codes are not available to the users. Furthermore, the point sets are mentioned only for certain degrees. One of our main concern is to provide sets for all the degrees up to 18 as well as the relative Matlab codes so that the reader can adapt these routines to more general domains or check how we obtained our results. The codes and the sets ξ are available as Matlab files at [8].

At first, we have tested all the ξ, noticing their good behavior for low degrees. The acronyms correspond to: [10] (TWV), [9] (HEI—unsymmetric and HEI2—symmetric), [20] (WAR), [17] (BP), [18] (CB), [19] (HES), [5,6] (AFP and DLP). However, with the exception of TWV, HEI and CB, their performance seems sooner or later to deteriorate when n is increased. We cannot say much about what happens at higher degrees for CB, since the authors have determined point sets only up to $n = 13$.

Our intention was to compute new point sets trying to reach at least the results of TWV and HEI. These tests have been made evaluating the Lebesgue constant, that is maximizing the Lebesgue function, on a reference mesh consisting of more than 60 000 points, and computing the Vandermonde matrix w.r.t. the orthonormal Dubiner basis [23,24]. Denoting by

$$C_{m+1} = \{z_j^{m+1} = \cos((j-1)\pi/m),\ j = 1, \dots, m+1\} \tag{10}$$

the set of the $m + 1$ Chebyshev–Gauss–Lobatto points in $[-1, 1]$, our reference mesh is a Chebyshev-like grid

$$X = \sigma\,(C_{250} \times C_{250}) \tag{11}$$

σ being the (bilinear) *Duffy transform* from $[-1, 1] \times [-1, 1]$ to the simplex. This is an admissible mesh, as shown in [15] (see the discussion in the previous section after formula (7)).

As is well-known in this context (cf., e.g., [9,10]), a good choice of the starting guess is crucial for the effectiveness of the optimization process. We considered as initial set $\xi^{(0)}$ the Approximate Fekete Points of the simplex [5], since they are easily at hand for any degree and possess already good Lebesgue constants and rather high absolute values of Vandermonde determinants.

Since the evaluation of the target function $F_1(\xi)$ in (7) by the reference mesh (11) is still very costly, for the (quasi)-Lebesgue points we have found useful the following approach. We fix a sequence of positive integers $m_0 < m_1 < \cdots < m_k = 250$, and start the minimization process by evaluating the Lebesgue constant on a coarse mesh, namely $X_{m_0} = \sigma\left(C_{m_0} \times C_{m_0}\right)$, with a fixed number of iterations, say 50. When the approximate solution is at hand, in a multigrid fashion, we restart the process evaluating the Lebesgue constant on a finer mesh, in our case X_{m_j} with $m_j > m_{j-1}$, $j = 0, \dots, k$. After this first stage, we restart the algorithm from the initial m_0 and the point set, say $\xi^{(1)}$, just obtained, computing more stages. We repeat the process until there is no reasonable reduction between two subsequent stages $\xi^{(s)}, \xi^{(s+1)}$. Concerning the Matlab routines, we noticed the good performance of the *active-set* algorithm, that is called by `fmincon` when the preference `'Algorithm'` is put as `'active-set'` in the optimizer variable `option`. As for the post-processing, some improvements have been obtained performing additional stages with the Matlab built-in function `fminsearch`. A less sophisticated restart approach has been used also for computing the (quasi-)Fekete points by minimizing the target function $F_2(\xi)$ in (9). In both cases, as one can expect, depending from the degree n the cputime ranges from some minutes to several hours.

We also observe that in the literature, e.g., in spectral element approximation (cf., e.g., [22]), it is often required that the interpolation set has a fixed distribution on each side of the simplex (typically $n + 1$ Legendre–Gauss–Lobatto points, in view of a conjecture of Bos concerning Fekete points [25]). Hence we have also optimized the target function only on the $(n-1)\,(n-2)$ variables corresponding to the (x, y) coordinates of the internal points, forcing the Legendre–Gauss–Lobatto distribution on each side (see Fig. 1).

In Table 1 we compare the Lebesgue constants of all the best point sets known in the literature, with the new points we have computed. We have termed FEK our (quasi)-Fekete points, LEB our (quasi)-Lebesgue points, and LEBGL (Lebesgue Gauss–Lobatto) the (quasi)-Lebesgue points obtained forcing the Legendre–Gauss–Lobatto distribution along the sides. Observe that LEB and LEBGL give the smallest Lebesgue constants, improving the best results known in the literature. We think that the point set that we have computed for $n = 15$ could not be the actual minimum, though our methods are not able to get better results. In Table 2 we compare the absolute values of the corresponding Vandermonde determinants and in Table 3 the conditioning of the Vandermonde matrix (in the orthonormal Dubiner basis). Concerning the points computed in [9], only the unsymmetric HEI points are available, for $n = 6, 9, 12$. Finally, in Table 4 we display the Lebesgue constants of our (quasi)-Lebesgue and (quasi)-Fekete points for the whole range of degrees $n = 1, \dots, 18$. Though full symmetry is not imposed to the points (except for the sides), we guess that LEBGL points, which are available at [8] for all degrees up to $n = 18$, could become useful in the framework of bidimensional spectral and high-order element methods for PDEs discretized over triangulations.

3.2. Square

On the square $[-1, 1] \times [-1, 1]$, recently it has been discovered a family of interpolation sets that possess almost optimal properties, the so called *Padua points* [11,26–28]. Denoting by C_{n+1} the set of the $n + 1$ Chebyshev–Gauss–Lobatto points, cf. (10), and defining

$$C^E_{n+1} = \{z_j^{n+1} \in C_{n+1},\ j-1 \text{ even}\}$$

$$C^O_{n+1} = \{z_j^{n+1} \in C_{n+1},\ j-1 \text{ odd}\} \tag{12}$$

then

$$\mathrm{Pad}_n = (C^E_{n+1} \times C^O_{n+2}) \cup (C^O_{n+1} \times C^E_{n+2}) \subset C_{n+1} \times C_{n+2}. \tag{13}$$

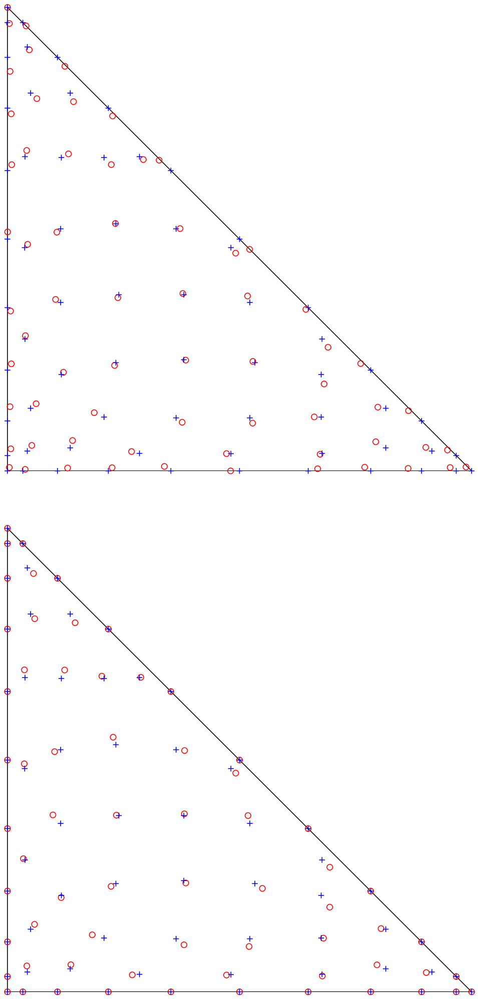

Fig. 1. Top: $N = 66$ (quasi-)Lebesgue points ($\circ$) and (quasi-)Fekete points ($+$) for $n = 10$. Bottom: forcing the Legendre–Gauss–Lobatto distribution on the sides.

Table 1
Lebesgue constants of interpolation sets in the simplex.

Deg	3	6	9	12	15	18
LEB	1.97	3.39	5.28	6.90	9.07	9.88
LEBGL	2.11	3.59	5.51	7.13	9.07	9.88
FEK	2.11	4.17	6.97	8.57	11.42	14.43
TWV	2.11	4.17	6.80	9.68	10.01	14.73
HEI	–	3.67	5.58	7.12	8.41	10.08
HEI2	–	3.87	5.59	7.51	9.25	11.86
WAR	2.11	3.70	5.73	9.36	17.64	36.76
BP	2.11	3.87	7.39	17.78	49.59	156.16
CB	2.11	3.79	5.91	10.08	–	–
HES	2.11	4.07	6.94	12.39	29.68	–
AFP	2.26	5.35	11.33	14.81	29.52	36.34
DLP	3.88	14.68	17.79	31.37	33.07	83.18

A first improvement is possible by using the Jacobi–Gauss–Lobatto points (cf. [29]), and minimizing the Lebesgue constant as a function of the parameters (α, β). Denoting the set of the $n+1$ Jacobi–Gauss–Lobatto points in $[-1, 1]$ with parameters (α, β) by

$$J_{n+1}^{\alpha,\beta} = \{z_j^{n+1,\alpha,\beta},\ j = 1, \ldots, n+1\} \tag{14}$$

Table 2
Absolute values of Vandermonde determinants of interpolation sets in the simplex (w.r.t. the orthonormal Dubiner basis).

Deg	3	6	9	12	15	18
LEB	2.00e08	3.73e28	2.76e62	3.42e111	1.88e182	1.44e268
LEBGL	3.45e08	1.87e29	2.23e64	1.69e115	1.89e182	1.44e268
FEK	3.45e08	2.29e29	3.79e64	7.22e115	9.66e183	4.10e269
TWV	3.44e08	2.29e29	2.40e64	6.15e115	4.60e183	1.04e269
HEI	–	1.84e29	1.43e64	2.59e115	–	–
WAR	3.44e08	2.14e29	1.84e64	2.28e114	1.54e180	3.30e262
BP	3.45e08	2.00e29	8.72e63	1.27e113	1.23e177	9.76e255
CB	3.45e08	2.21e29	1.93e64	1.84e114	–	–
HES	3.27e08	1.72e29	9.73e63	6.70e113	2.08e179	–
AFP	3.01e08	1.00e29	5.17e63	1.91e114	1.72e181	1.18e265
DLP	1.04e08	1.39e28	8.29e61	9.77e110	6.16e176	6.16e259

Table 3
Conditioning of the Vandermonde matrices of interpolation sets in the simplex (w.r.t. the orthonormal Dubiner basis).

Deg	3	6	9	12	15	18
LEB	1.95e01	6.10e01	1.39e02	1.73e02	3.73e02	5.05e02
LEBGL	2.04e01	6.80e01	1.36e02	2.44e02	3.63e02	4.90e02
FEK	2.04e01	7.00e01	1.55e02	2.40e02	3.52e02	4.65e02
TWV	1.44e01	7.01e01	1.41e02	2.35e02	3.28e02	4.24e02
HEI	–	7.40e01	1.45e02	2.46e02	–	–
WAR	2.04e01	5.92e01	1.38e02	1.97e02	3.92e02	1.15e03
BP	2.04e01	7.10e01	1.46e02	2.24e02	6.35e02	1.85e03
CB	2.03e01	6.93e01	1.43e02	2.32e02	–	–
HES	2.08e01	7.01e01	1.45e02	3.63e02	1.38e03	–
AFP	2.17e01	7.23e01	1.48e02	2.77e02	4.71e02	7.40e02
DLP	2.66e01	8.47e01	1.68e02	4.50e02	6.07e02	7.86e02

Table 4
Lebesgue constants of (quasi)-Lebesgue in the simplex.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
LEB	1.00	1.49	1.97	2.42	2.90	3.39	3.94	4.55	5.28	5.63	6.45	6.90	7.59	8.31	9.07	8.58	9.12	9.88
LEBGL	1.00	1.67	2.11	2.59	3.08	3.59	4.14	5.21	5.51	5.93	6.56	7.13	7.74	8.31	9.07	8.58	9.12	9.88

and defining

$$J_{n+1}^{\alpha,\beta,E} = \{z_j^{n+1,\alpha,\beta},\ j-1 \text{ even}\}$$
$$J_{n+1}^{\alpha,\beta,O} = \{z_j^{n+1,\alpha,\beta},\ j-1 \text{ odd}\} \tag{15}$$

then we can define the "Padua–Jacobi points"

$$\mathrm{Pad}_n^{(\alpha,\beta)} = (J_{n+1}^{\alpha,\beta,E} \times J_{n+2}^{\alpha,\beta,O}) \cup (J_{n+1}^{\alpha,\beta,E} \times J_{n+2}^{\alpha,\beta,O}) \subset J_{n+1}^{\alpha,\beta} \times J_{n+2}^{\alpha,\beta} \tag{16}$$

(clearly $\mathrm{Pad}_n = \mathrm{Pad}_n^{(-1/2,-1/2)}$), and compute the parameters which minimize the Lebesgue constant, with the constraint $\alpha, \beta > -1$, by the Matlab routine `fmincon`. Such optimal parameters, which turn out to be negative, are reported in Table 5 (rounded to three significant figures). Notice that for $n = 1, 3$ and for all even $n \geq 6$ they seem to correspond to Gegenbauer-Gauss–Lobatto points ($\alpha = \beta$). The results for the corresponding interpolation sets are reported as PdJ in Tables 6 and 7.

Moreover, using the Padua points as starting guess and following the same procedure used for the unit simplex, we have obtained (quasi)-Fekete points FEK and (quasi)-Lebesgue points LEB (for the Vandermonde matrices we used the product Chebyshev polynomial basis [24]), up to degree $n = 20$ (see Fig. 2); codes and point sets are available at [8]. The reference mesh to evaluate the Lebesgue constant is the product Chebyshev–Lobatto grid $X = C_{250} \times C_{250}$, which is an admissible mesh for the square (cf. [15]). From the Tables one immediately notices that for $n \geq 11$ the (quasi-)optimal results so obtained are almost indistinguishable from those of PdJ, confirming the very good quality of these Padua-type sets. The numerical results suggest to deepen the theoretical study of the Padua–Jacobi interpolation sets.

3.3. Disk

The last domain we consider is the unit disk K with center in the origin and radius equal to 1. We started the numerical optimization process from the Approximate Fekete points of the disk (cf. [5]) and then we applied the process previously

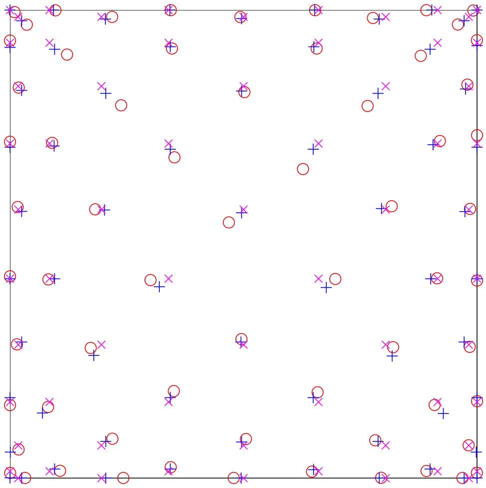

Fig. 2. $N = 66$ (quasi-)Lebesgue points ($\circ$), (quasi)-Fekete points ($+$), and (near-)optimal Padua–Jacobi points ($\times$), for $n = 10$.

Table 5
Values of the parameters that minimize the Lebesgue constant of the Padua–Jacobi points (16), rounded to three significant figures.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$-\alpha$	$0.359 \cdot 10^{-3}$	0.352	0.614	0.868	0.999	0.956	0.966	0.977	0.958	0.907	0.872	0.855	0.832	0.806	0.775	0.770	0.766	0.741	0.707	0.732
$-\beta$	$0.359 \cdot 10^{-3}$	0.999	0.614	0.999	0.982	0.956	0.999	0.977	0.940	0.907	0.883	0.855	0.821	0.806	0.779	0.770	0.761	0.741	0.712	0.732

Table 6
Lebesgue constants of some interpolation sets in the square.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
FEK	2.00	2.93	3.72	4.00	4.73	6.07	5.48	5.96	6.21	6.65	6.88	7.37	7.55	7.95	8.05	8.33	8.43	8.68	8.84	8.99
LEB	2.00	2.39	2.73	3.24	3.59	4.00	4.34	4.90	5.18	5.32	6.18	6.44	6.70	6.91	7.11	7.35	7.58	7.76	7.89	8.14
PdJ	2.00	2.65	3.71	3.74	4.14	4.58	4.94	5.30	5.60	5.92	6.19	6.45	6.70	6.93	7.16	7.36	7.58	7.76	7.99	8.14
Pd	2.00	3.00	3.78	4.41	4.95	5.42	5.84	6.21	6.66	6.88	7.17	7.45	7.71	7.95	8.19	8.41	8.62	8.82	9.01	9.20

Table 7
Absolute values of Vandermonde determinants of some interpolation sets in the square (w.r.t. the product Chebyshev basis).

Deg	3	6	9	12	15	18
FEK	6.27e03	9.23e14	2.38e35	9.46e66	2.95e110	5.92e166
PdJ	3.83e03	2.52e14	7.94e34	3.33e66	1.33e110	5.84e166
Pd	3.89e03	3.96e14	1.69e35	6.67e66	2.39e110	4.78e166

discussed to obtain (quasi)-Fekete and (quasi)-Lebesgue points. The Vandermonde matrices correspond to the Koornwinder type II polynomial basis [30]. To evaluate the Lebesgue constant we adopted the symmetric Chebyshev polar grid $X = \{(\rho_h \cos\theta_k, \rho_h \sin\theta_k)\}$, where $\{\rho_h\} = C_{250}$ (cf. (10)), and $\{\theta_k\}$ are 250 equally spaced angles in $[0, \pi]$, which is an admissible mesh for the disk, as it has been shown in [15]. We have been able to obtain reasonable results only up to degree $n = 16$, beyond the point sets deteriorate showing too large Lebesgue constants. The numerical results are shown in Table 8.

Very little seems to be known in the literature about Fekete and Lebesgue points of the disk (cf., e.g., [12]), even though some symmetry is expected due to the complete symmetry of the domain. In our computational results, some patterns seem to arise, at least in the considered degree range. The points appear to lie approximately on concentric circles (with radii distributed approximately as positive Legendre–Gauss–Lobatto points, see Fig. 3), and to be equispaced there in the angles, forming (suitably rotated) regular polygons. Moreover, there are $2n + 1$ points on (or very close to) the boundary of the disk, for n even the center is an interpolation point and 5 points lie on the most internal circle, whereas for n odd the center is not an interpolation point and 3 points lie on the most internal circle. The cardinality distribution on the concentric circles for n even follows initially the progression $1, 5, 9, 13, \ldots$, i.e., $4k+1$, $k = 0, \ldots, n/2$, but this breaks down at degree $n = 10$ for the (quasi)-Fekete points and $n = 12$ for the (quasi)-Lebesgue points. The situation is even more complicated with odd degrees. Such distributions up to degree $n = 13$ are listed in Tables 9 and 10 (beyond degree 13 it becomes hard to approximately locate on circles the points near the boundary). Much further computational and theoretical study will be needed to understand the actual distribution of the circles radii, of the number of points on the circles, and of the rotations of the regular polygons. As with the other domains, the Matlab codes and the point sets are available at [8].

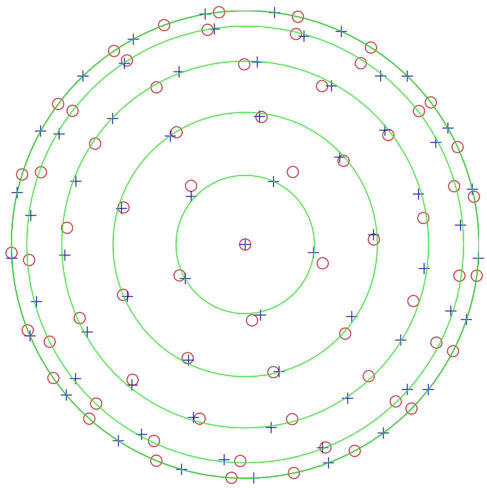

Fig. 3. $N = 66$ (quasi-)Lebesgue points ($\circ$) and (quasi-)Fekete points ($+$) for $n = 10$; the radii of the circles correspond to positive Legendre–Gauss–Lobatto points.

Table 8
Lebesgue constants of (quasi)-Fekete and (quasi)-Lebesgue points in the unit disk.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
FEK	1.67	1.99	2.63	3.19	3.95	4.38	5.58	6.45	7.02	7.84	8.92	14.85	17.02	18.51	21.44	28.91
LEB	1.67	1.99	2.47	2.97	3.50	4.30	5.08	5.43	6.73	7.62	8.48	9.65	11.98	12.39	14.13	17.44

Table 9
Cardinality distribution (number of vertices of the regular polygons) on concentric circles of (quasi-)Fekete points in the unit disk.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13
	3	1	3	1	3	1	3	1	3	1	3	1	3
		5	7	5	7	5	7	5	7	5	7	5	7
				9	11	9	11	9	13	9	11	11	13
						13	15	13	13	15	17	12	15
								17	19	15	17	18	19
										21	23	19	21
												25	27
N	3	6	10	15	21	28	36	45	55	66	78	91	105

Table 10
Cardinality distribution (number of vertices of the regular polygons) on concentric circles of (quasi-)Lebesgue points in the unit disk.

Deg	1	2	3	4	5	6	7	8	9	10	11	12	13
	3	1	3	1	3	1	3	1	3	1	3	1	3
		5	7	5	7	5	8	5	8	5	7	5	7
				9	11	9	10	9	12	9	11	9	15
						13	15	13	13	13	17	13	17
								17	19	17	17	19	18
										21	23	19	18
												25	27
N	3	6	10	15	21	28	36	45	55	66	78	91	105

4. Conclusions

We have computed, by the Matlab Optimization Toolbox, Fekete and Lebesgue points for polynomial interpolation on three basic bidimensional compacts: the simplex, the square, and the disk. These point sets can be affinely mapped on any triangle, parallelogram, and ellipse, respectively, and give for almost all degrees the best results known so far in terms of Lebesgue constant. For the simplex we have also computed constrained Lebesgue points, forcing the Legendre–Gauss–Lobatto distribution on the sides, in view of applications to spectral and high-order methods. Such constrained Lebesgue points show a near-optimal behavior. Moreover, for the square we have also experimented a new

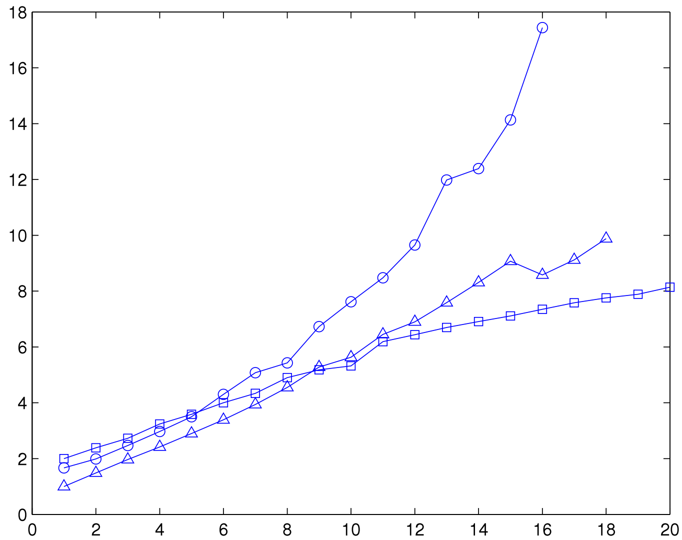

Fig. 4. Lebesgue constants of (quasi)-Lebesgue points of the simplex (△), square (□) and disk (○).

family of points, the "Padua–Jacobi" points, constructed as the Padua points [26], but on Jacobi–Gauss–Lobatto grids, minimizing the Lebesgue constant w.r.t. the parameters (α, β). The resulting Padua–Jacobi point sets turn out to be near-optimal and deserve further study.

In Fig. 4 we have plotted the Lebesgue constants of our (quasi)-Fekete points for the degree ranges available. It turns out that the Lebesgue constant is substantially linear for the simplex (as already observed in all previous computational studies), whereas it is sublinear for the square: indeed, it is smaller than the Lebesgue constant of the Padua points, that is theoretically known to be $\mathcal{O}(\log^2 n)$, cf. [26]. On the other hand, the growth appears superlinear for the disk, whereas there is a lower bound of order $\sqrt{n}$, cf. [31]. Though the results for the disk seem still to be far from optimality, especially at the highest degrees, some geometric patterns arise that deserve further consideration.

All the numerical codes and the point sets are available at the web site [8].

References

[1] M. Gasca, T. Sauer, Polynomial interpolation in several variables, Adv. Comput. Math. 12 (2000) 377–410.
[2] J.P. Boyd, A numerical comparison of seven grids for polynomial interpolation on the interval, Comput. Math. Appl. 38 (1999) 35–50.
[3] L. Bos, M.A. Taylor, B.A. Wingate, Tensor product Gauss–Lobatto points are Fekete points for the cube, Math. Comp. 70 (2001) 1543–1547.
[4] L. Fejér, Bestimmung derjenigen abszissen eines intervalles, für welche die quadratsumme der grundfunktionen der lagrangeschen interpolation im intervalle ein möglichst kleines maximum besitzt. (German), Ann. Sc. Norm. Super Pisa Cl. Sci. 1 (1932) 263–276.
[5] L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva, M. Vianello, Geometric weakly admissible meshes, discrete least squares approximation and approximate Fekete points, Math. Comp. 80 (2011) 1601–1621.
[6] L. Bos, S. De Marchi, A. Sommariva, M. Vianello, Computing multivariate Fekete and Leja points by numerical linear algebra, SIAM J. Numer. Anal. 48 (2010) 1984–1999.
[7] A. Sommariva, M. Vianello, Computing approximate Fekete points by QR factorizations of Vandermonde matrices, Comput. Math. Appl. 57 (2009) 1324–1336.
[8] M. Briani, A. Sommariva, M. Vianello, LEBFEK: Matlab codes and data sets for bivariate Lebesgue and Fekete interpolation points. Available at: http://www.math.unipd.it/~alvise/software.html.
[9] W. Heinrichs, Improved Lebesgue constants on the triangle, J. Comput. Phys. 207 (2005) 625–638.
[10] M.A. Taylor, B.A. Wingate, R.E. Vincent, An algorithm for computing Fekete points in the triangle, SIAM J. Numer. Anal. 38 (2000) 1707–1720.
[11] M. Caliari, S. De Marchi, A. Sommariva, M. Vianello, Padua2DM: fast interpolation and cubature at the Padua points in Matlab/Octave, Numer. Algorithms 56 (2011) 45–60.
[12] L. Bos, N. Levenberg, S. Waldron, On the spacing of Fekete points for a sphere, ball or simplex, Indag. Math. 19 (2008) 163–176.
[13] MathWorks, Matlab Optimization Toolbox, Version 4.0. Online description at: http://www.mathworks.com/products/optimization/.
[14] J.-P. Calvi, N. Levenberg, Uniform approximation by discrete least squares polynomials, J. Approx. Theory 152 (2008) 82–100.
[15] L. Bos, M. Vianello, Low cardinality admissible meshes for quadrangles, triangles and disks, Math. Inequal. Appl. 15 (2012) 229–235.
[16] L. Bos, A. Sommariva, M. Vianello, Least-squares polynomial approximation on weakly admissible meshes: disk and triangle, J. Comput. Appl. Math. 48 (2010) 1984–1999.
[17] M.G. Blyth, C. Pozrikidis, A Lobatto interpolation grid over the triangle, IMA J. Appl. Math. 71 (2006) 153–169.
[18] Q. Chen, Ivo Babuska, Approximate optimal points for polynomial interpolation of real functions in an interval and in a triangle, Comput. Methods Appl. Mech. Engrg. 128 (1995) 405–417.
[19] J.S. Hesthaven, From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex, SIAM J. Numer. Anal. 35 (1998) 655–676.
[20] T. Warburton, An explicit construction for interpolation nodes on the simplex, J. Eng. Math. 56 (2006) 247–262.
[21] R. Pasquetti, F. Rapetti, Spectral element methods on triangles and quadrilaterals: comparisons and applications, J. Comput. Phys. 198 (2010) 349–362.
[22] R. Pasquetti, F. Rapetti, Spectral element methods on unstructured meshes: which interpolation points? Numer. Algorithms 55 (2010) 349–366.
[23] M. Dubiner, Spectral methods on triangles and other domains, J. Sci. Comput. 6 (1991) 345–390.
[24] C.F. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, in: Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge University Press, Cambridge, 2001.
[25] L. Bos, On certain configurations of points in R^n which are uniresolvant for polynomial interpolation, J. Approx. Theory 64 (1991) 271–280.

[26] L. Bos, M. Caliari, S. De Marchi, M. Vianello, Y. Xu, Bivariate Lagrange interpolation at the Padua points: the generating curve approach, J. Approx. Theory 143 (2006) 15–25.
[27] L. Bos, S. De Marchi, M. Vianello, Y. Xu, Bivariate Lagrange interpolation at the Padua points: the ideal theory approach, Numer. Math. 108 (2007) 43–57.
[28] M. Caliari, S. De Marchi, M. Vianello, Bivariate polynomial interpolation on the square at new nodal sets, Appl. Math. Comput. 165 (2005) 261–274.
[29] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, in: Num. Math. and Scient. Comput., Oxford University Press, Oxford, 2004.
[30] T. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, Theory and Application of Special Functions, Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975, pp. 435–495.
[31] B. Sündermann, On projections constants of polynomial spaces on the unit ball in several variables, Math. Z. 188 (1984) 111–117.