Inversion Homework #4

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November 25, 2020

Chapter 3

Exercise 2

Another resolution test commonly performed in tomography studies is a **checkerboard test**, which consists of using a test model composed of alternating positive and negative perturbations. Perform a checkerboard test on the tomography problem in Example 3.1 using the test model,

$$\mathbf{m}_{true} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \tag{1}$$

Evaluate the difference between the true (checkerboard) model and the recovered model in your test, and interpret the pattern of differences. Are any block values recovered exactly? If so, does this imply perfect resolution for these model parameters?

Solution:

The true model (Figure 1a, reshape result):

$$\mathbf{m}_{true} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \tag{2}$$

and

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \tag{3}$$

The generalized inverse solution (Figure 1b, reshape result):

$$\mathbf{m}_{\dagger} = \mathbf{G}^{\dagger}\mathbf{d}_{test} = \mathbf{G}^{\dagger}\mathbf{G}\mathbf{m}_{true} = \begin{bmatrix} -1.6667 \\ 1.0000 \\ -0.3333 \\ 1.0000 \\ -0.3333 \\ 0.3333 \\ -0.3333 \\ 0.3333 \\ -1.0000 \end{bmatrix} \tag{4}$$

As shown in the Figure 1, $m_{1,2}, m_{2,1}, m_{3,3}$ are recovered exactly. But it **doesn't mean perfect resolution** for these model parameters. The model resolution matrix is illustrated in Figure 2. This matrix shows **only** $m_{3,3}$ has perfect resolution.

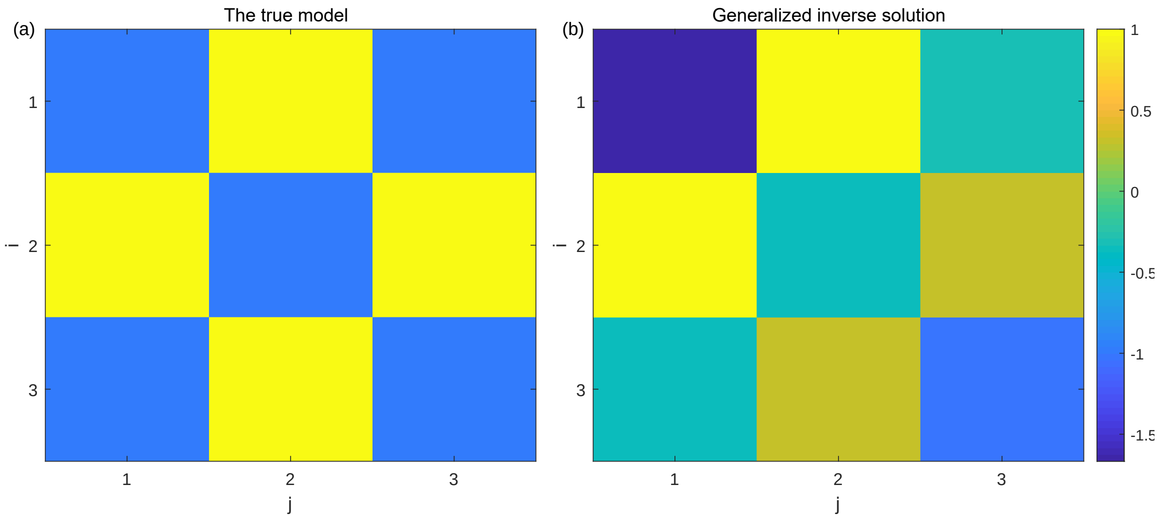

Figure 1: A comparison between the true model (a), and the generalized inverse solution (b)

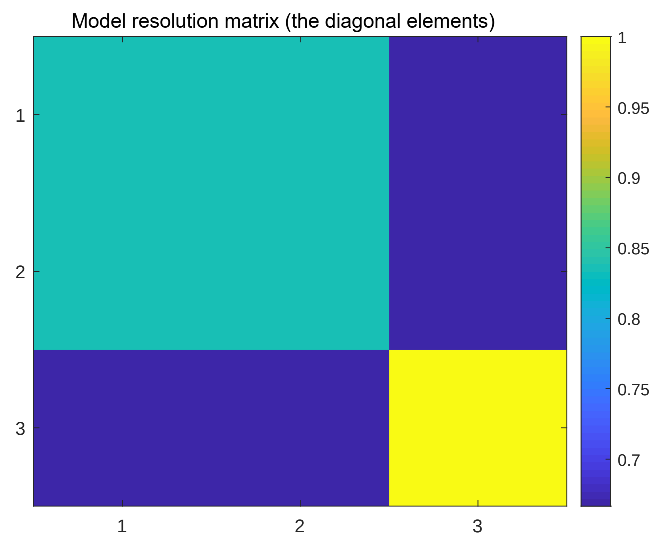

Figure 2: The model resolution matrix (only the diagonal elements, reshape)

matlab code

```
clear; clc; close all;

m_true = [-1, 1, -1, 1, -1, 1, -1, 1, -1]';
G = [1,0,0,1,0,0,1,0,0; 0,1,0,0,1,0,0,1,0; 0,0,1,0,0,1,0,0,1; ...
     1,1,1,0,0,0,0,0,0; 0,0,0,1,1,1,0,0,0; 0,0,0,0,0,0,1,1,1; ...
     sqrt(2),0,0,0,sqrt(2),0,0,0,sqrt(2); ...
     0,0,0,0,0,0,0,0,sqrt(2)];

% Forward, i.e. generate synthetic data
d = G * m_true;

```

```
% generalized inverse solution
diag_s = svd(G);
m = pinv(G, 0.001) * d;

% comparion between m and m_true
m = reshape(m, 3, 3)';
m_true = reshape(m_true, 3, 3)';

% model resolution matrix
R_m = pinv(G) * G;

% plot
figure(1)
imagesc(m)
xticks([1, 2, 3])
yticks([1, 2, 3])
colorbar;
temp1=caxis;
title("Generalized inverse solution")
xlabel("j")
ylabel("i")

figure(2)
imagesc(m_true)
xticks([1, 2, 3])
yticks([1, 2, 3])
caxis(temp1)
colorbar;
title("The true model")
xlabel("j")
ylabel("i")

figure(3)
imagesc(reshape(diag(R_m), 3,3))
xticks([1,2,3])
yticks([1,2,3])
colorbar;
title("Model resolution matrix (the diagonal elements)")
```

Exercise 4

A large north-south by east-west–oriented, nearly square plan view, sandstone quarry block (16 m by 16 m) with a bulk compressional wave seismic velocity of approximately 3000 m/s is suspected of harboring higher-velocity dinosaur remains. An ultrasonic tomography scan is performed in a horizontal plane bisecting the boulder, producing a data set consisting of 16 E→W, 16 N→S, 31 NE→SW, and 31 NW→SE travel times (see Figure 3). The travel time data (units of s) have statistically independent errors, and the travel time contribution for a uniform background model (with a velocity of 3000 m/s) has been subtracted from each travel time measurement.

The MATLAB data files that you will need to load containing the travel time data follow: **rowscan.mat**, **colscan.mat**, **diag1scan.mat**, and **diag2scan.mat**. The standard deviations of all data measurements are 1.5×10^{-5} s. Because the travel time contributions for a uniform background model (with a velocity of 3000 m/s) have been subtracted from each travel time measurement, you will be solving for slowness and velocity perturbations relative to a uniform slowness model of 1/3000 s/m. Use a row-by-row mapping between the slowness grid and the model vector (e.g., Example 1.12). The row format of each data file is $(x_1, \gamma_1, x_2, \gamma_2, t)$ where the starting point coordinate of each source is (x_1, γ_1), the end point coordinate is (x_2, γ_2), and the travel time along a ray path between the source and receiver points is a path integral (in seconds).

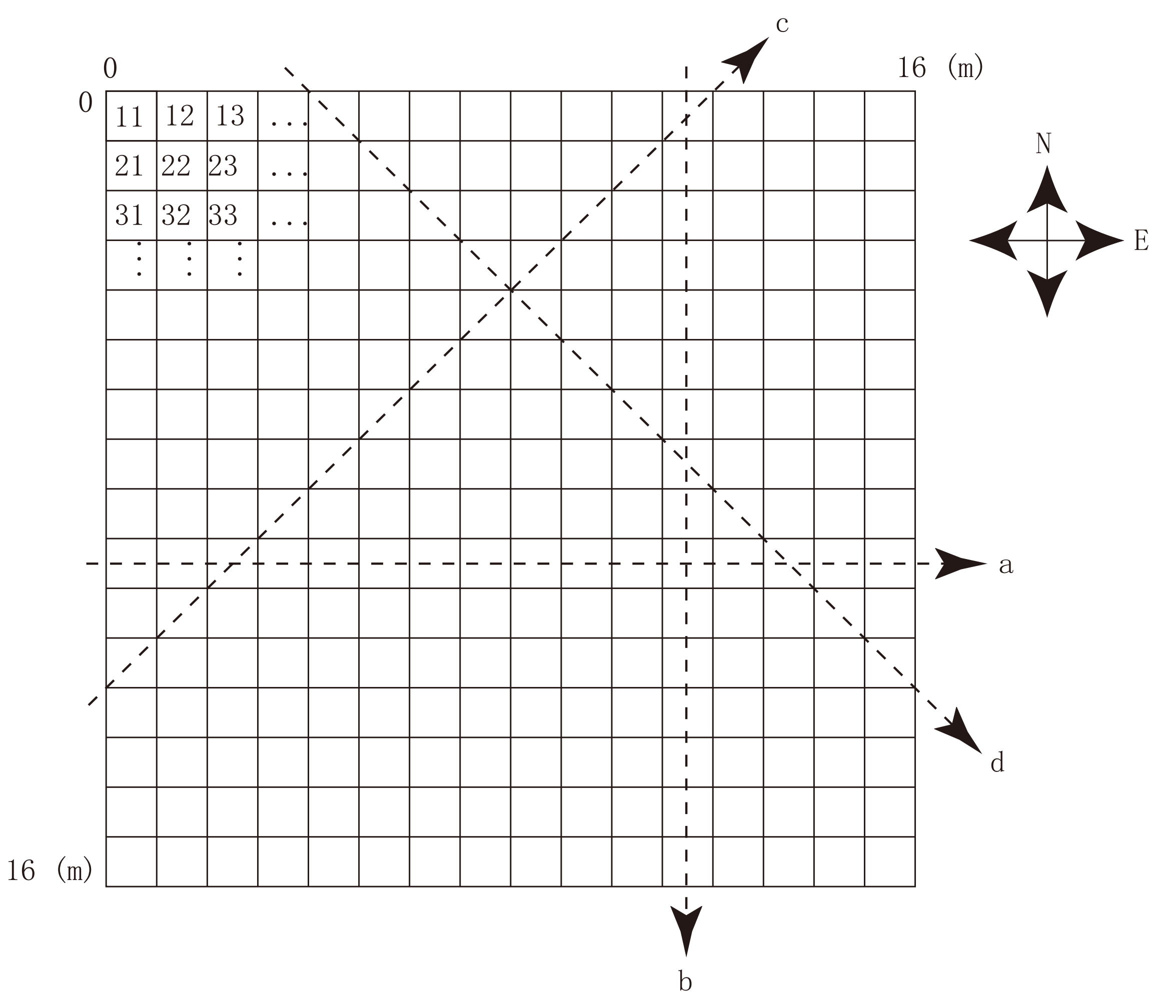

Figure 3: Tomography exercise, showing block discretization, block numbering convention, and representative ray paths going east-west (a), north-south (b), southwest-northeast (c), and northwestsoutheast (d).

Parameterize the slowness structure in the plane of the survey by dividing the boulder into a 16×16 grid of 256 1-m-square, north-by-east blocks and construct a linear system for the forward problem (Figure 3.29). Assume that the ray paths through each homogeneous block can be represented by straight lines, so that the travel time expression is

$$\begin{aligned} t &= \int_{\ell} s(\mathbf{x}) d\ell \\ &= \sum_{\text{blocks}} s_{\text{block}} \cdot \Delta l_{\text{block}} \end{aligned} \tag{5}$$

where Δl_{block} is 1 m for the row and column scans and $\sqrt{2}$ m for the diagonal scans.

Use the SVD to find a minimum-length/least squares solution, $\mathbf{m}_{\dagger}$, for the 256 block slowness perturbations that fit the data as exactly as possible. Perform two inversions in this manner:
1. Use the row and column scans only.
2. Use the complete data set.
For each inversion:

(a) Note the rank of your $\mathbf{G}$ matrix relating the data and model.

Solution:

If use the row and column scans only, $\mathbf{G}$ is a 32 by 256 matrix and its rank is 31.
If use the complete data set, $\mathbf{G}$ is a 94 by 256 matrix and its rank is 87.

(b) State and discuss the general solution and/or data fit significance of the elements and dimensions of the data and model null spaces. Plot and interpret an element of each space and contour or otherwise display a nonzero model that fits the trivial data set $\mathbf{Gm} = \mathbf{d} = \mathbf{0}$ exactly.

Solution:

For the two inversions, the dimensions of model null spaces are 225, 169, respectively, and the dimensions of data null spaces are 1, 7, respectively.
The dimensions of model/data null spaces equal to the dimensions of model/data spaces minus the number of nonzero singulars p. Thus, the smaller the dimensions of null spaces, the greater the number of nonzero singulars, the better the generalized inverse solution.
The Figure 4 shows all model and data null spaces, and each column represents a null space. And the Figure 5 shows one null space of the corresponding null spaces of Figure 4.
The sum of all elements of a model null space vector is 0.

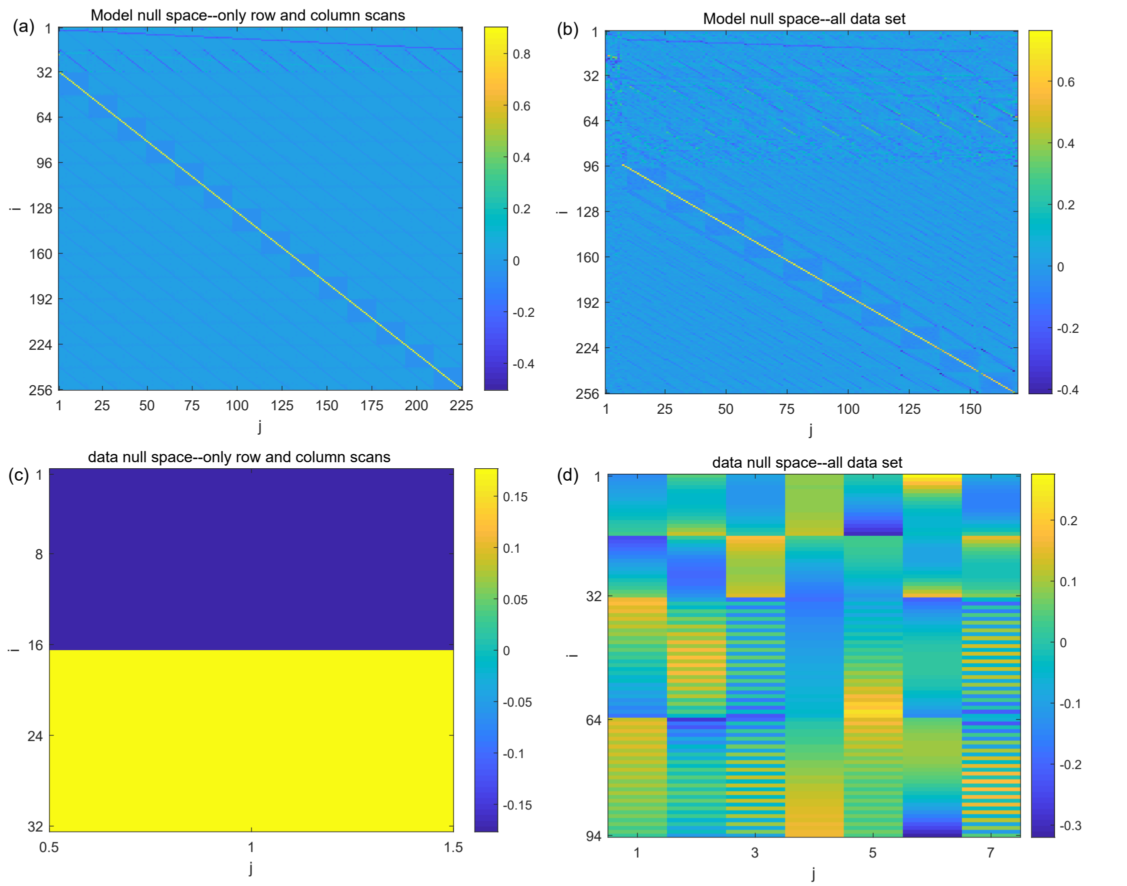

Figure 4: The model and data null spaces (a) model null spaces use row and column scans only, (b) model null spaces use the complete data set, (c) data null space use row and column scans only, (d) model null spaces use the complete data set.

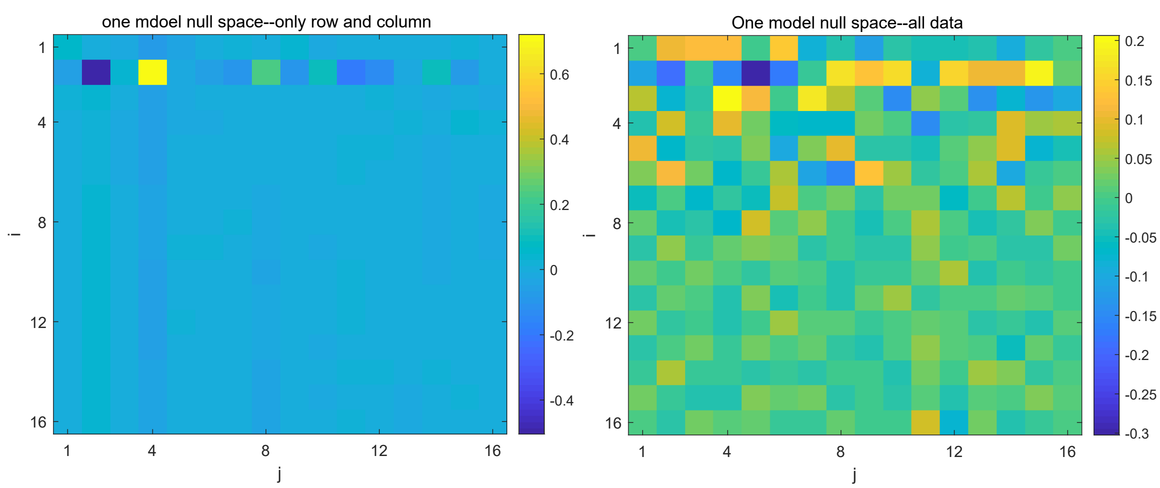

Figure 5: One model null space (reshape result) (a) one model null spaces use row and column scans only, (b) one model null spaces use the complete data set.

(c) Note whether there are any model parameters that have perfect resolution.

Solution:

The Figure 6 shows the model resolution matrix (only diagonal elements).

1. use the row and column scans only

As shown in Figure 6a, there is no model parameter that has perfect resolution.

2. use the complete data set

As shown in Figure 6b, The model parameters $m_{1,1}$, $m_{1,16}$, $m_{16,1}$, $m_{16,16}$ have perfect resolution.

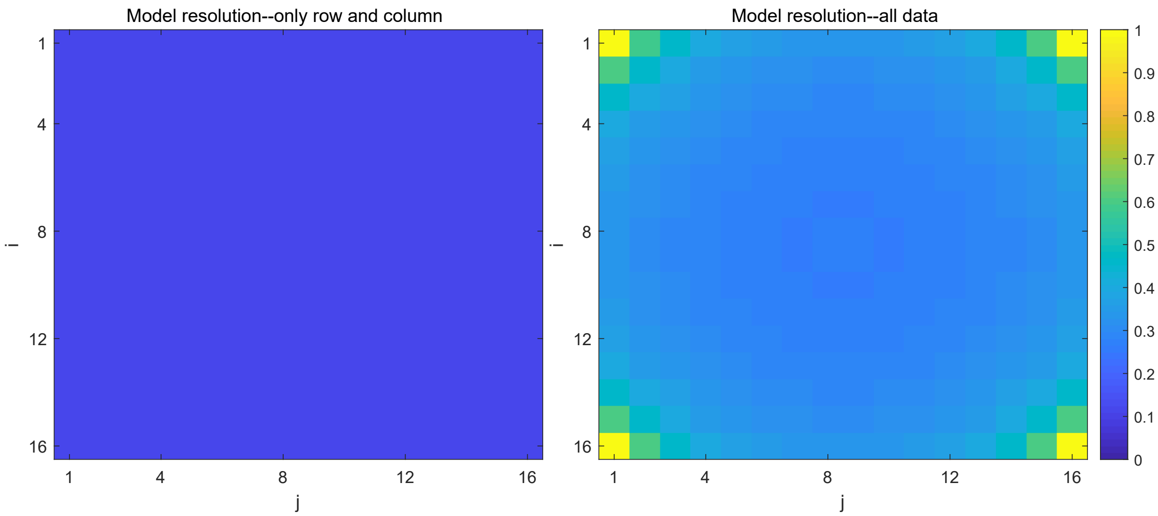

Figure 6: Two model resolution matrix (only diagonal elements and reshape them to 16 by 16), (a) use row and column scans only, (b) use the complete data set.

(d) Produce a 16 by 16 element contour or other plot of your slowness perturbation model, displaying the maximum and minimum slowness perturbations in the title of each plot. Interpret any internal structures geometrically and in terms of seismic velocity (in m/s).

Solution:

The results are illustrated in Figure 7.

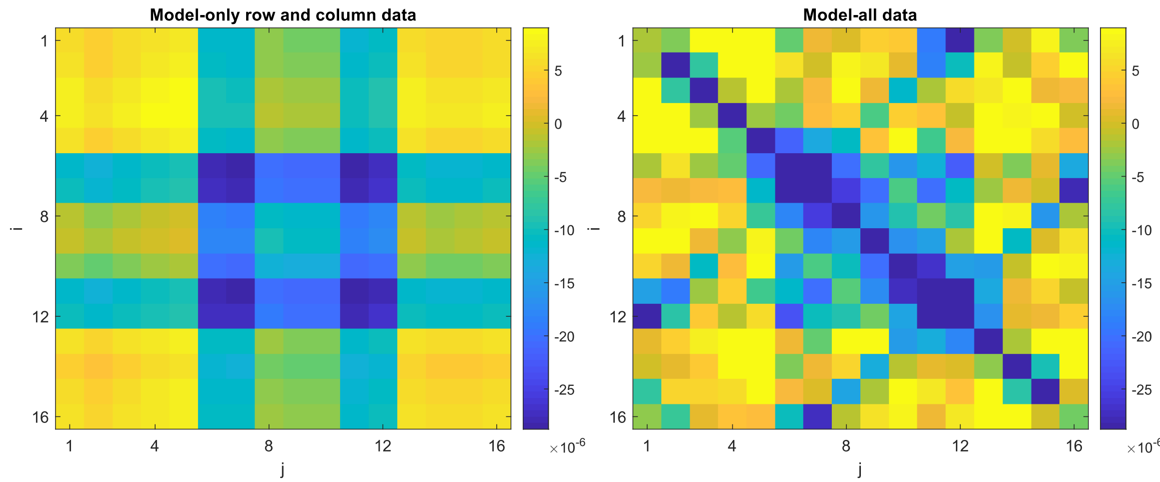

Figure 7: The generalized inverse solution. (a) use the row and column scans only, (b) use the complete data set.

(e) Show the model resolution by contouring or otherwise displaying the 256 diagonal elements of the model resolution matrix, reshaped into an appropriate 16 by 16 grid.

Solution:

The result is illustrated in Figure 6.

(f) Describe how one could use solutions to $\mathbf{Gm} = \mathbf{d} = \mathbf{0}$ to demonstrate that very rough models exist that will fit any data set just as well as a generalized inverse model. Show one such wild model.

Solution:

A basis of model null spaces is satisfied it. We set such basis model as m_0, and:

$$
\begin{aligned}
&\mathbf{G}(\mathbf{m} + \mathbf{m_0}) = d_2, \\
\Rightarrow & d_2 - d = 0, \\
\Rightarrow & \mathbf{Gm} = 0.
\end{aligned} \tag{6}
$$

One example: the models shown in the Figure 5.
see code (%% the 4.f problem)