Inversion Homework #1

Professor H. Yao & H. Zhang

Jintao Li

SA20007037

E-mail: lijintao@mail.ustc.edu.cn

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Appendix A

Exercise 4

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \\ 4 & 6 & 7 & 11 \end{bmatrix} \tag{1}$$

Find bases for $\mathcal{N}(\mathbf{A})$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A}^T)$, and $\mathcal{R}(\mathbf{A}^T)$. What are the dimensions of the four subspaces?

Solution:

To find $\mathcal{N}(\mathbf{A})$, we solve the system of equations $\mathbf{Ax} = 0$,

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \\ 4 & 6 & 7 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Put the system of equations into an augmented matrix,

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 2 & 1 & 3 & 0 \\ 4 & 6 & 7 & 11 & 0 \end{array}\right] \tag{3}$$

and then find the reduced row echelon form (**RREF**),

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4}$$

We can find that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 + x_4 \\ -\frac{5}{2}x_3 - \frac{5}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} x_4 \tag{5}$$

So,

$$\mathcal{N}(\mathbf{A}) = \mathbf{space}\left(\begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} \right), \tag{6}$$

and the dimension of $\mathcal{N}(\mathbf{A})$ is **2**.

To find $\mathcal{R}(\mathbf{A})$, the equations becomes $\mathbf{Ax} = \mathbf{b}$,

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \\ 4 & 6 & 7 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{b} \tag{7}$$

Because the **RREF** of **A** is

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

We can write $\mathbf{b}$ as a linear combination of the first two columns of $\mathbf{A}$:

$$\mathbf{b} = x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \tag{9}$$

i.e.

$$\mathcal{R}(\mathbf{A}) = \textbf{space}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right), \tag{10}$$

and the dimension of $\mathcal{R}(\mathbf{A})$ is **2**.

To find $\mathcal{N}(\mathbf{A}^T)$ and $\mathcal{R}(\mathbf{A}^T)$, we first calculate the **RREF** of $\mathbf{A}^T$.

$$\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 6 \\ 3 & 1 & 7 \\ 4 & 3 & 11 \end{bmatrix}, \tag{11}$$

$$\textbf{RREF of A is} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

So,

$$\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} x_3, \tag{13}$$

i.e.

$$\begin{aligned} \mathcal{N}(\mathbf{A}^T) &= \textbf{space}\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \right), \\ \mathcal{R}(\mathbf{A}^T) &= \textbf{space}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \right). \end{aligned} \tag{14}$$

And the dimension of $\mathcal{N}(\mathbf{A}^T)$ is **1**, the dimension of $\mathcal{R}(\mathbf{A}^T)$ is **2**.

Exercise 10

Show that if $\mathbf{x} \perp \mathbf{y}$, then

$$\|\mathbf{x} + \mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2. \tag{15}$$

Solution:

Exercise 11

In this exercise, we will derive the formula (A.88) for the 1-norm of a matrix. Begin with the optimization problem

$$\|\mathbf{A}\|_1 = \max_{\|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1. \tag{16}$$

(a) Show that if $\|\mathbf{x}\|_1 = 1$, then

$$\|\mathbf{A}\mathbf{x}\|_1 \leq \max_j \sum_{i=1}^{m} |A_{i,j}| . \tag{17}$$

Solution:

(b) Find a vector $\mathbf{x}$ such that $\|\mathbf{x}\|_1 = 1$, and

$$\|\mathbf{A}\mathbf{x}\|_1 = \max_j \sum_{i=1}^{m} |A_{i,j}| . \tag{18}$$

Solution:

(c) Conclude that

$$\|\mathbf{A}\|_1 = \max_{\|\mathbf{x}\|_1=1} \|\mathbf{A}\mathbf{x}\|_1 = \max_j \sum_{i=1}^{m} |A_{i,j}| . \tag{19}$$

Solution:

Appendix B

Exercise 6

Suppose that $\mathbf{X} = (X_1, X_2)^T$ is a vector composed of two random variables with a multivariate normal distribution with expected value μ and covariance matrix $\mathbf{C}$, and that $\mathbf{A}$ is a 2 by 2 matrix. Use properties of expected value and covariance to show that $\mathbf{y} = \mathbf{A}\mathbf{x}$ has expected value $\mathbf{A}\mu$ and covariance $\mathbf{A}\mathbf{C}\mathbf{A}^T$

Solution:

Exercise 9

Using MATLAB, repeat the following experiment 1000 times. Generate five exponentially distributed random numbers from the exponential probability density function (B.10) with means $\mu = 1/\lambda = 10$. You may find the library function **exprand** to be useful here. Use (B.74) to calculate a 95% confidence interval for the 1000 mean determinations. How many times out of the 1000 experiments did the 95% confidence interval cover the expected value of 10? What happens if you instead generate 50 exponentially distributed random numbers at time? Discuss your results.

Solution:

Appendix C

Exercise 1

Let

$$f(\mathbf{x}) = x_1^2 x_2^2 - 2x_1 x_2^2 + x_2^2 - 3x_1^2 x_2 + 12x_1 x_2 - 12x_2 + 6. \tag{20}$$

Find the gradient, $\nabla f(\mathbf{x})$, and Hessian, $\mathbf{H}(f(\mathbf{x}))$. What are the critical points of f? Which of these are minima and maxima of f?

Solution:

Exercise 2

Find a Taylor's series approximation for $f(\mathbf{x} + \Delta\mathbf{x})$, where

$$f(\mathbf{x}) = e^{-(x_1+x_2)^2} \tag{21}$$

is near the point

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{22}$$

Solution: