Inversion Homework #2

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October 27, 2020

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Chapter 1

Exercise 1

Consider a mathematical model of the form $G(\mathbf{m} = \mathbf{d})$, where $\mathbf{m}$ is a vector of length n, and $\mathbf{m}$ is a vector of length m. Suppose that the model obeys the superposition and scaling laws and is thus linear. Show that $G(\mathbf{m})$ can be written in the form

$$G(\mathbf{m}) = \mathbf{\Gamma m} \tag{1}$$

where $\mathbf{\Gamma}$ is an m by n matrix. What are the elements of $\mathbf{\Gamma}$? Hint: Consider the standard basis, and write $\mathbf{m}$ as a linear combination of the vectors in the standard basis. Apply the superposition and scaling laws. Finally, recall the definition of matrix-vector multiplication.

Solution:

A standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, the elements of $\vec{e}_i$ are all zero except for i^{th} element, which is one.

$$\vec{m} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \cdots + \alpha_n\vec{e}_n = \sum_{i=1}^{n} \alpha_i \vec{e}_i$$

$$G(\vec{m}) = G\left(\sum_{i=1}^{n} \alpha_i \vec{e}_i\right) = \sum_{i=1}^{n} \alpha_i G(\vec{e}_i)$$

$$= [G(\vec{e}_1) \cdots G(\vec{e}_n)] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= [G(\vec{e}_1) \cdots G(\vec{e}_n)]\, \vec{m}$$

$$= \overline{\overline{\Gamma}}\, \vec{m}$$

So, $\overline{\overline{\Gamma}} = [G(\vec{e}_1) \cdots G(\vec{e}_n)]$,

and its size is $m \times n$

Exercise 2

Can (1.14) be inconsistent, even with only $m = 3$ data points? How about just $m = 2$ data points? If the system can be inconsistent, give an example. If not, explain why.

Solution:

A system of linear equations is called inconsistent if it has no solutions. A system which has a solution is called consistent.

The system can be inconsistent even with only $m=3$. put the system of equations into an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & t_1 & -\frac{1}{2}t_1^2 & y_1 \\ 1 & t_2 & -\frac{1}{2}t_2^2 & y_2 \\ 1 & t_3 & -\frac{1}{2}t_3^2 & y_3 \end{array}\right] \xrightarrow{\text{find RREF}}$$

$$\left[\begin{array}{ccc|c} 1 & t_1 & -\frac{1}{2}t_1^2 & y_1 \\ 0 & t_2-t_1 & -\frac{1}{2}(t_2-t_1)(t_2+t_1) & y_2-y_1 \\ 0 & 0 & -\frac{1}{2}(t_2-t_1)(t_3-t_1)(t_3-t_2) & (y_3-y_1)(t_2-t_1)-(y_2-y_1)(t_3-t_1) \end{array}\right]$$

When $t_1 = t_2$ or $t_3 = t_1$ or $t_3 = t_2$, but

$y_1 \neq y_2$ or $y_3 \neq y_1$ or $y_3 \neq y_2$,

the equation becomes $0 = C$ $(C \neq 0)$.

So the system will be inconsistent.

This indicates that there are multiple observations at the same time by different people or machines, but the observations are different.

When the number of data point is 2, the system also can be inconsistent. This situation is the same as $m=3$.

$$\begin{bmatrix} 1 & t_1 & -\frac{1}{2}t_1^2 \\ 1 & t_2 & -\frac{1}{2}t_2^2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

If $t_1 = t_2$, but $y_1 \neq y_2$, the system will be inconsistent.

Exercise 3

Consider the borehole vertical seismic profile problem of Examples 1.3 and 1.9 for $n = 100$ equally spaced seismic sensors located at depths of $z = 0.2, 0.4, ..., 20$ m, and for a model $\mathbf{m}$ describing n corresponding equal-length seismic slowness values for 0.2 m intervals having midpoints at $z - 0.1m$.

(a) Calculate the appropriate system matrix, $\mathbf{G}$, for discretizing the integral equation (1.21) using the midpoint rule.

Solution:

$$t(z) = \int_0^\infty s(\xi) H(z-\xi)\, d\xi \qquad H = \begin{cases} 1 & 0 < \xi \le z \\ 0 & \xi > z \end{cases}$$

$$\begin{aligned} t(z_i) &= \int_0^\infty H(z_i - \xi)\, s(\xi)\, d\xi \\ &= \int_0^{z_i} s(\xi)\, d\xi = \sum_{j=0}^{z_i} 1 \cdot s(z_j)\Delta z + \sum_{j=z_i}^{n} 0 \cdot s(z_j)\Delta z \\ &= \sum_{j=0}^{z_i} s(z_j)\Delta z \end{aligned}$$

and $\Delta z = 0.2$

So. $$G_{i,j} = \begin{cases} \Delta z & i \geq j \\ 0 & i < j \end{cases} = 0.2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \vdots \\ \vdots & 1 & \ddots & 0 \\ 1 & \cdots & & 1 \end{bmatrix}$$

(b) For a seismic velocity model having a linear depth gradient specified by

$$v = v_0 + kz, \tag{2}$$

where the velocity at $z = 0$ is $v_0 = 1$ km/s and the gradient is $k = 40$ m/s per m, calculate the true slowness values, $\mathbf{m}_{true}$, at the midpoints of the n intervals. Integrate the corresponding slowness function for (1.61) using (1.21) to calculate a noiseless synthetic data vector, $\mathbf{d}$, of predicted seismic travel times at the sensor depths.

Solution:

$$S = \frac{1}{v} = \frac{1}{v_0 + kz} = \frac{1}{1000 + 40z}$$

Because $z_0 = 0$. , $z_i = \frac{1}{2}\Delta z + (i-1)\Delta z = 0.2i - 0.1$

$$\vec{m}_{true} = \frac{1}{v_0 + k(\Delta z \cdot i - \frac{1}{2}\Delta z)} = \frac{1}{1000 + 4(2i-1)} \qquad i = 1, \ldots n$$

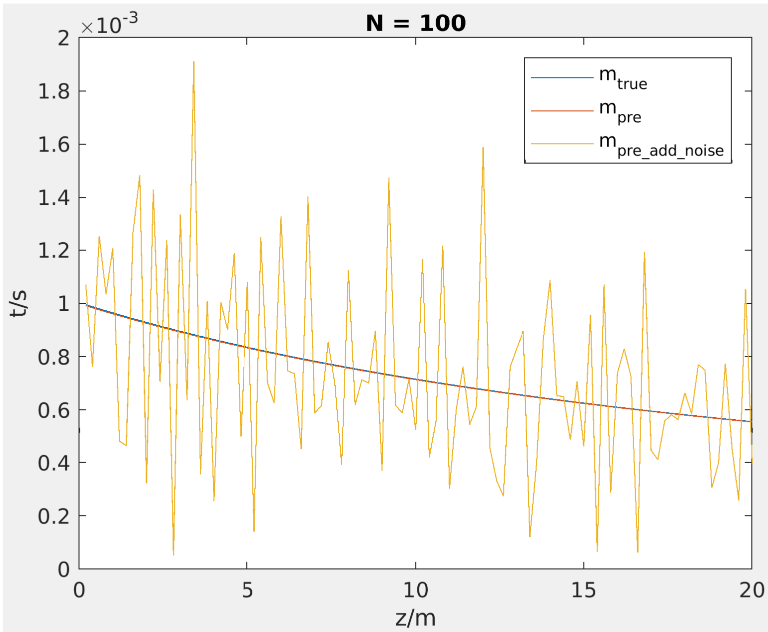
$$d = \int_0^\infty s(\xi) H(z-\xi) d\xi = \int_0^z s(\xi)\, d\xi$$

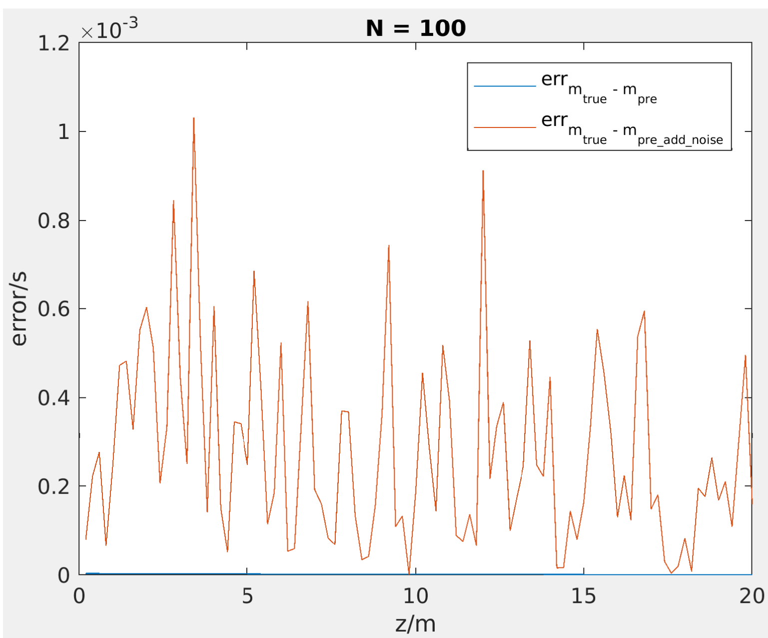
$$= \int_0^z \frac{1}{1000 + 40z} d\xi = \frac{1}{40} \ln(1000 + 40z)\Big|_0^z$$

$$= \frac{1}{40} \ln\left(1 + \frac{z}{25}\right)$$

(c) Solve for the slowness, $\mathbf{m}$, as a function of depth using your $\mathbf{G}$ matrix and analytically calculated noiseless travel times using the MATLAB backslash operator. Compare your result graphically with $\mathbf{m}_{true}$.

Solution:





(d) Generate a noisy travel time vector where independent normally distributed noise with a standard deviation of 0.05 ms is added to the elements of $\mathbf{d}$. Resolve the system for $\mathbf{m}$ and again compare your result graphically with $\mathbf{m}_{true}$. How has the model changed?

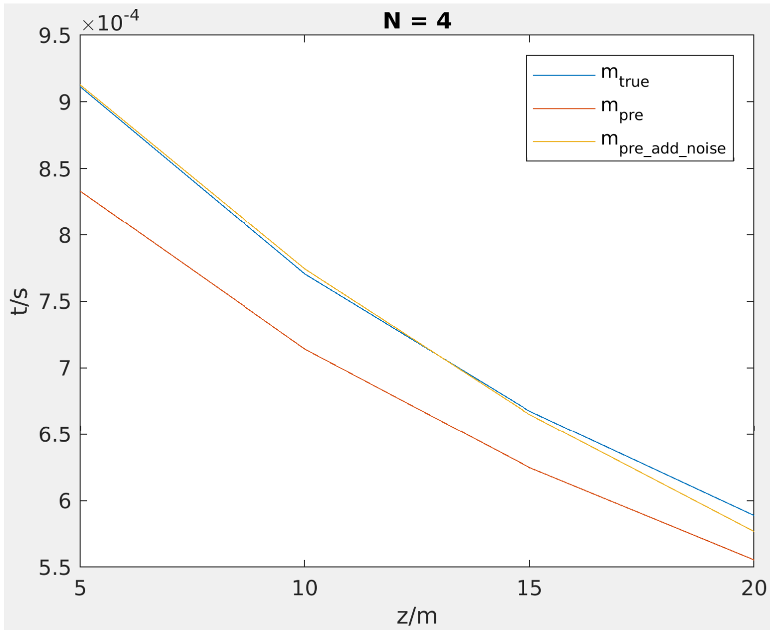
Solution:

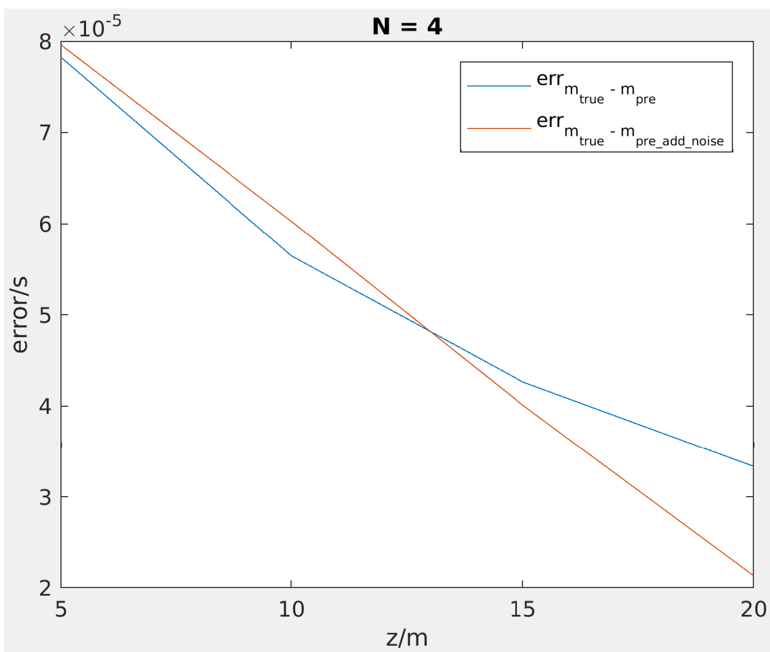
The figures are seen in Exercise 3.c (above)

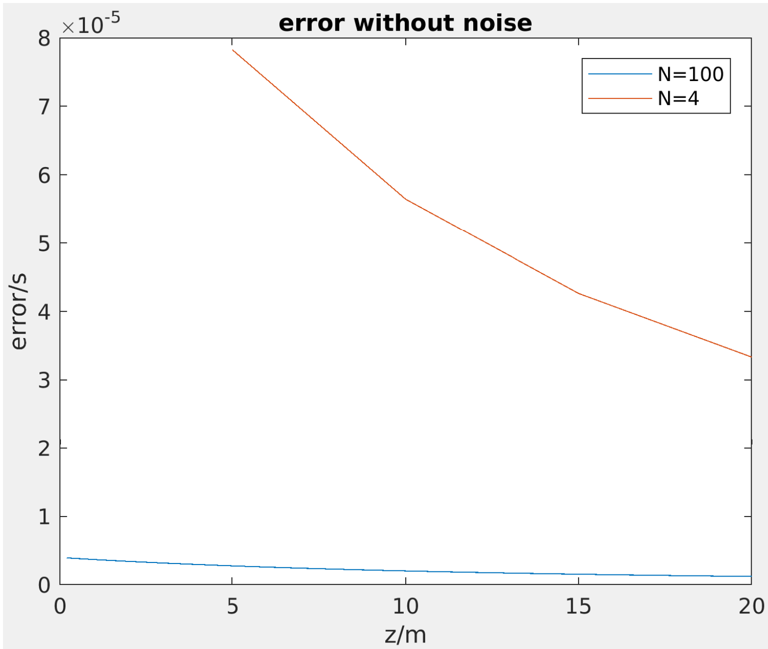
When N=100, and added some noise to d,
the m_pre became unstable compared to m_true.
There is a large gap between m_pre_add_noise and m_true

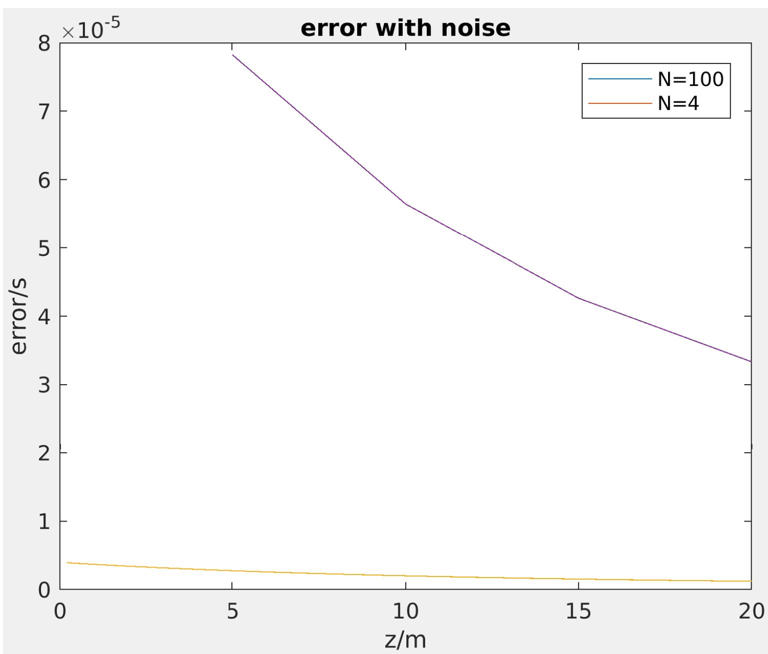
(e) Repeat the problem, but for just $n = 4$ sensor depths and corresponding equal length slowness intervals. Is the recovery of the true model improved? Explain in terms of the condition numbers of your $\mathbf{G}$ matrices.

Solution:






error without noise
×10^-5
error/s
z/m
N=100
N=4
0
1
2
3
4
5
6
7
8
0
5
10
15
20


error with noise
×10^-5
error/s
z/m
N=100
N=4
0
1
2
3
4
5
6
7
8
0
5
10
15
20

When $N=4$, the two type errors are close but the error of m-pre is large than the same error when $N=100$.

As same as (a), $G = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 5 \end{bmatrix}$

The standard deviation of noise is 5×10^{-5} s.

5 is larger than 5×10^{-5} compared to 0.2 (when $N=100$). i.e., when $N=4$, the impact of noise is smaller than the condition when $N=100$.