Inversion Homework #5

Professor H. Yao & H. Zhang

Jintao Li

SA20007037

E-mail: lijintao@mail.ustc.edu.cn

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Chapter 4 Tikhonov Regularization

Note:

Some code (the files in folder "lib/") is refered from github: https://github.com/brianborchers/PEIP

Exercise 2

Consider the integral equation and data set from Problem 3.5. You can find a copy of this data set in the file **ifk.mat**.

(a) Discretize the problem using simple collocation.

Solution:

$$\begin{aligned} \mathbf{d} &= \mathbf{G}\mathbf{m}, \\ \mathbf{G}_{ij} &= x_i e^{-x_i x_j} \Delta x, \\ \Rightarrow \mathbf{d}_i &= \sum_{j=1}^{n=20} \mathbf{G}_{i,j}\mathbf{m}_j \end{aligned} \tag{1}$$

(b) Using the data supplied, and assuming that the numbers are accurate to four significant figures, determine a reasonable bound δ for the misfit.

Solution:

Refered to page 98,

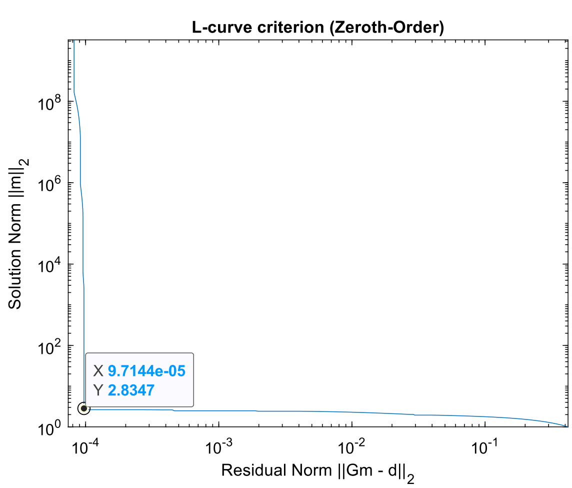
$$\delta = \sqrt{n} \times accurate = \sqrt{20} \times 10^{-4} = 4.472 \times 10^{-4}, \tag{2}$$

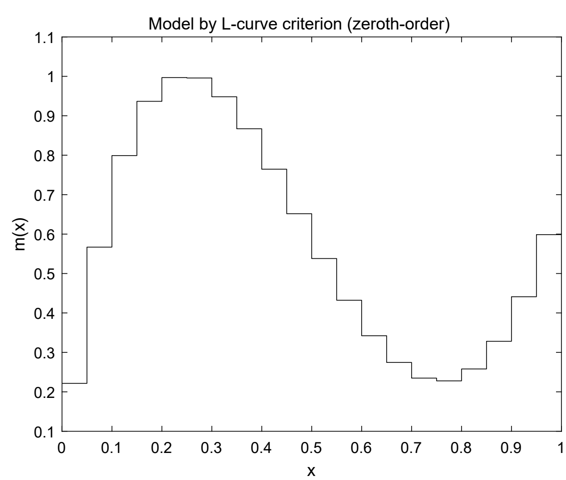
where n is the number of data points.

(c) Use zeroth-order Tikhonov regularization to solve the problem. Use GCV, the discrepancy principle, and the L-curve criterion to pick the regularization parameter.

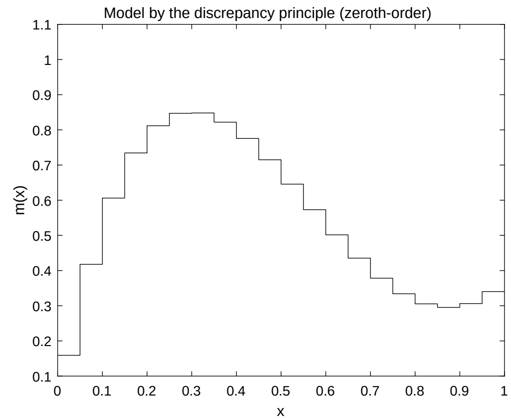
Solution:

The L-curve criterion reslut ($\alpha = 2.586 \times 10^{-5}$):

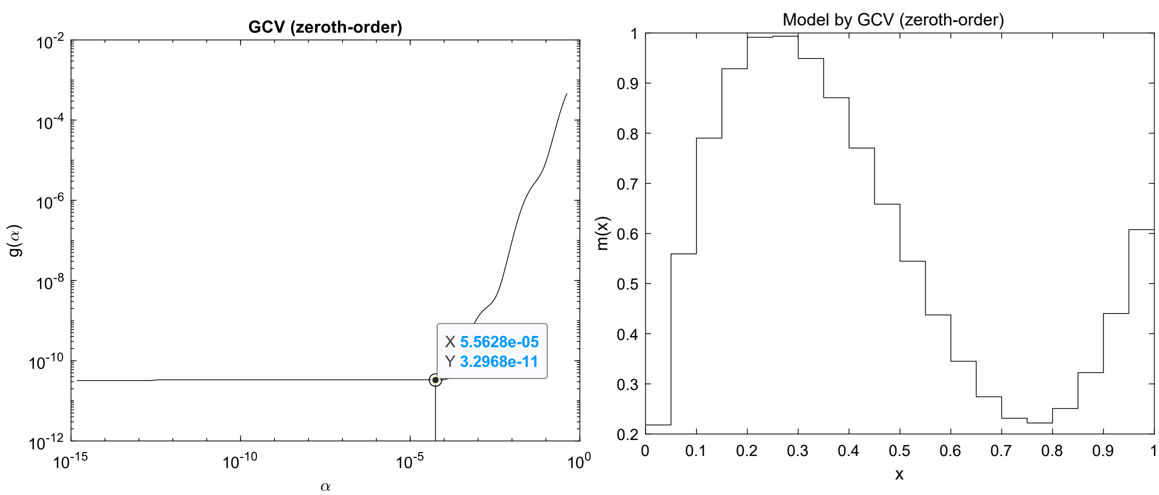




The discrepancy principle result ($\alpha = 6.7487 \times 10^{-4}$):



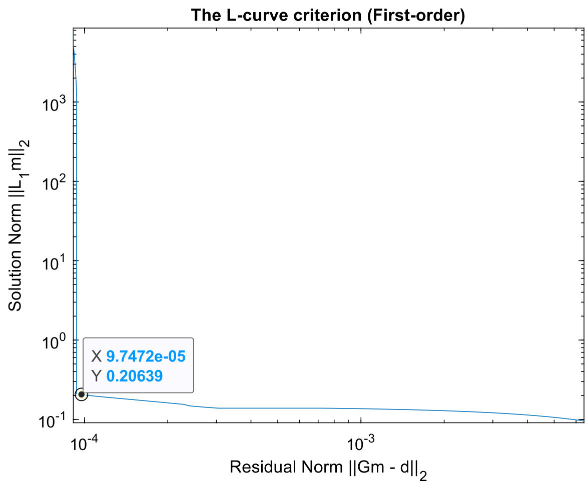
The GCV result($\alpha = 5.5628 \times 10^{-5}$):

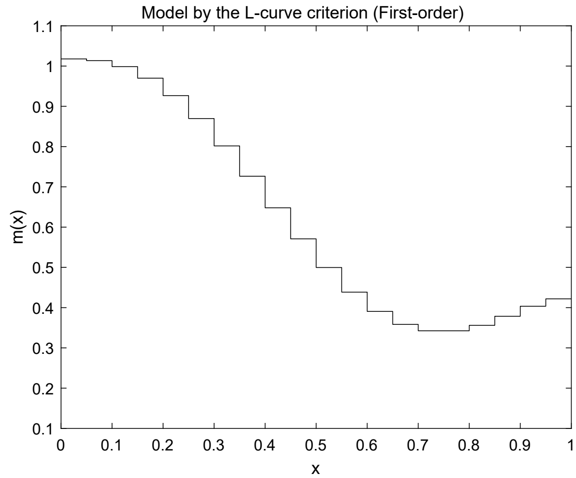


(d) Use first-order Tikhonov regularization to solve the problem. Use GCV, the discrepancy principle, and the L-curve criterion to pick the regularization parameter.

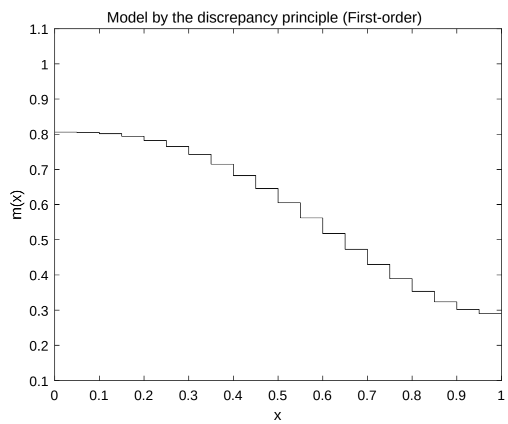
Solution:

The L-curve criterion reslut ($\alpha = 2.277 \times 10^{-4}$):

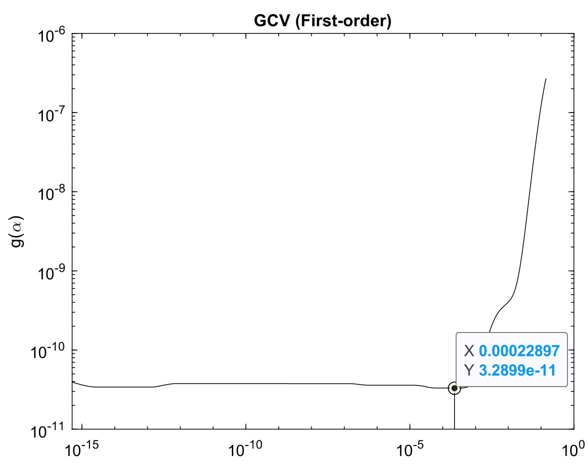


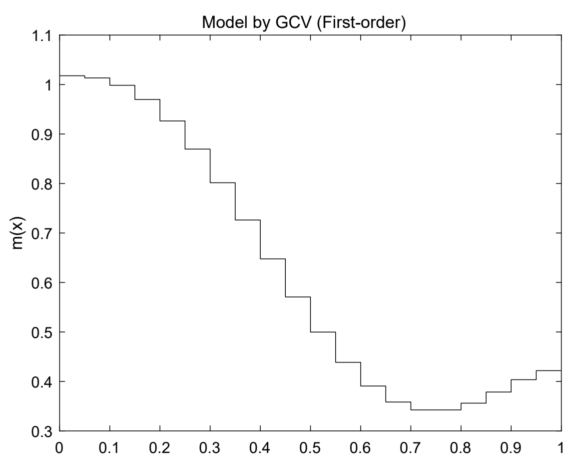


The discrepancy principle result ($\alpha = 1.68 \times 10^{-2}$):



The GCV result ($\alpha = 6.7487 \times 10^{-4}$):





Exercise 3

Consider the following problem in **cross-well tomography**. Two vertical wells are located 1600 m apart. A seismic source is inserted in one well at depths of 50, 150, ... , 1550 m. A string of receivers is inserted in the other well at depths of 50 m, 150 m, ... , 1550 m. See Figure 1. For each source-receiver pair, a travel time is recorded, with a measurement standard deviation of 0.5 ms. There are 256 ray paths and 256 corresponding data points. We wish to determine the velocity structure in the two-dimensional plane between the two wells.

Discretizing the problem into a 16 by 16 grid of 100 meter by 100 meter blocks gives 256 model parameters. The **G** matrix and noisy data, **d**, for this problem (assuming straight ray paths) are in the file **crosswell.mat**. The order of parameter indexing from the slowness grid to the model vector is row by row (e.g., Example 1.12).

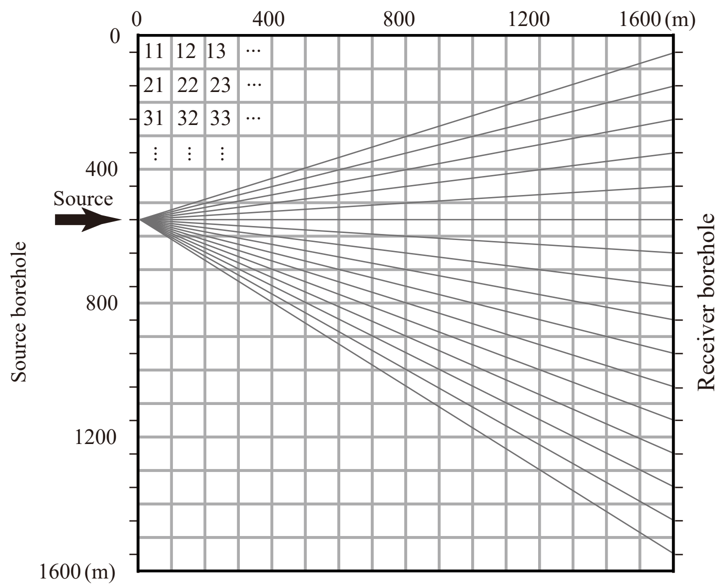
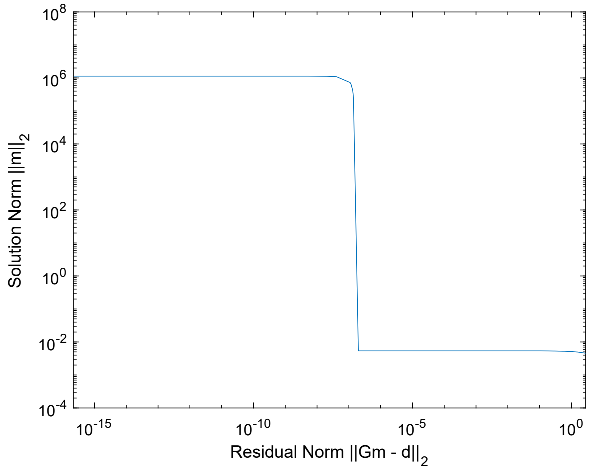
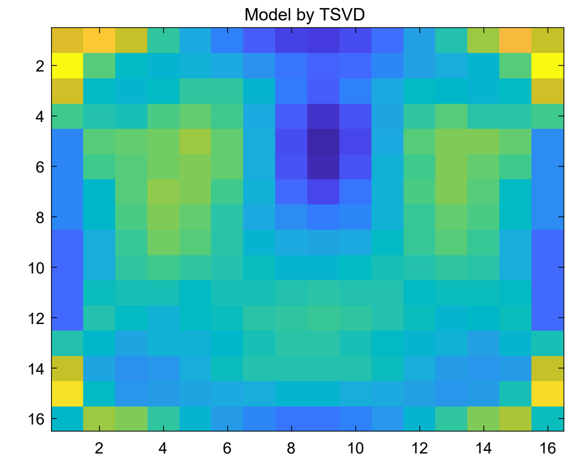


Figure 1: Cross-well tomography problem, showing block discretization, block numbering convention, and one set of straight source-receiver ray paths.

(a) Use the TSVD to solve this inverse problem using an L-curve. Plot the result.

Solution:

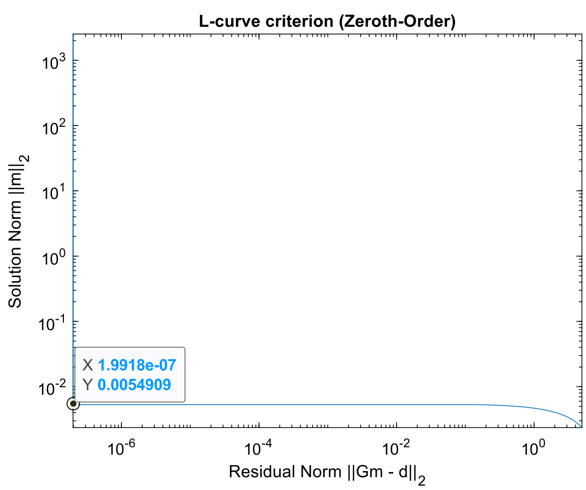


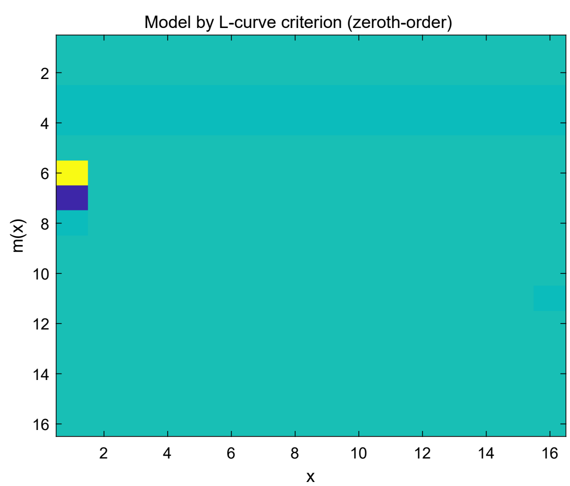


(b) Use zeroth-order Tikhonov regularization to solve this problem and plot your solution. Explain why it is difficult to use the discrepancy principle to select the regularization parameter. Use the L-curve criterion to select your regularization parameter. Plot the L-curve as well as your solution.

Solution:

Because $\delta = \sqrt{256} \times 5 \times 10^{-4} = 0.008$, it is too large to case a poor result by the discrepancy principle. The L-curve criterion's results are shown blow.





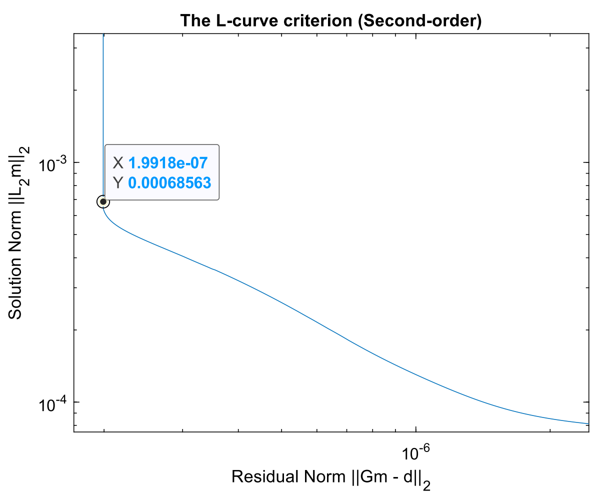
(c) Use second-order Tikhonov regularization to solve this problem and plot your solution. Because this is a two-dimensional problem, you will need to implement a finite-difference approximation to the Laplacian (second derivative in the horizontal direction plus the second derivative in the vertical direction) in the roughening matrix. The **L** matrix can be generated using the following **MATLAB** code:

```
L=zeros(14*14,256);
k=1;
for i=2:15,
    for j=2:15,
        M=zeros(16,16);
```

```
        M(i,j)=-4;
        M(i,j+1)=1;
        M(i,j-1)=1;
        M(i+1,j)=1;
        M(i-1,j)=1;
        L(k,:)=reshape(M,256,1)';
        k=k+1;
    end
end
```

What, if any, problems did you have in using the L-curve criterion on this problem? Plot the L-curve as well as your solution.

Solution:



(d) Discuss your results. If vertical bands appeared in some of your solutions, can you explain why?

Solution:

It might be decided by resolution matrix.

Exercise 5

In some situations it is appropriate to bias the regularized solution toward a particular model $\mathbf{m}_0$. In this case, we would solve

$$\min \|\mathbf{Gm} - \mathbf{d}\|_2^2 + \alpha^2 \left\|\mathbf{L}\left(\mathbf{m} - \mathbf{m}_0\right)\right\|_2^2 \tag{3}$$

Write this as an ordinary linear least squares problem. What are the normal equations? Can you find a solution for this problem using the GSVD?

Solution:

Refered to the page 95, we can get:

$$\begin{aligned}
&\min \| \begin{bmatrix} \mathbf{G} \\ \alpha\mathbf{L} \end{bmatrix} \mathbf{m} - \begin{bmatrix} \mathbf{d} \\ \alpha\mathbf{L}\mathbf{m}_0 \end{bmatrix} \|_2^2 \\
&\text{normal equation} \Rightarrow \\
&\begin{bmatrix} \mathbf{G}^T & \alpha\mathbf{L}^T \end{bmatrix} \begin{bmatrix} \mathbf{G} \\ \alpha\mathbf{L} \end{bmatrix} \mathbf{m} = \begin{bmatrix} \mathbf{G}^T & \alpha\mathbf{L}^T \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \alpha\mathbf{L}\mathbf{m}_0 \end{bmatrix} \\
\Rightarrow &(\mathbf{G}^T\mathbf{G} + \alpha^2\mathbf{L}^T\mathbf{L})(\mathbf{m} - \mathbf{m_0}) = \mathbf{G}^T(\mathbf{d} - \mathbf{G}\mathbf{m}_0)
\end{aligned} \tag{4}$$

Since, we assume $[UVXAB] = gsvd(G, L)$, so:

$$\begin{aligned}
(\mathbf{A}^T\mathbf{A} + \alpha^2\mathbf{B}^T\mathbf{B})\mathbf{X} = \mathbf{A}^T\mathbf{B}^T(\mathbf{d} - \mathbf{G}\mathbf{m}_0) \\
x_i = \frac{\gamma_i^2}{\gamma_i^2 + \alpha^2} = \frac{\mathbf{U}_{.i+k}^T(\mathbf{d} - \mathbf{G}\mathbf{m}_0)}{\lambda_i}.
\end{aligned} \tag{5}$$

If we set $\mathbf{KX} = \mathbf{m} - \mathbf{m}_0$, we will obtain:

$$\mathbf{m} = (\sum_{i=1}^{m} \frac{\gamma_i^2}{\gamma_i^2 + \alpha^2} \frac{\mathbf{U}_{.i+k}^T(\mathbf{d} - \mathbf{G}\mathbf{m}_0)}{\lambda_i} \mathbf{K}_{.i}) + \mathbf{m}_0 \tag{6}$$