A standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, the elements of $\vec{e}_i$ are all zero except for $i^{th}$ element, which is one.

$$\vec{m} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \cdots + \alpha_n \vec{e}_n$$

$$= \sum_{i=1}^{n} \alpha_i \vec{e}_i$$

$$G(\vec{m}) = G\left(\sum_{i=1}^{n} \alpha_i \vec{e}_i\right) = \sum_{i=1}^{n} \alpha_i G(\vec{e}_i)$$

$$= \left[ G(\vec{e}_1) \cdots G(\vec{e}_n) \right] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \left[ G(\vec{e}_1) \cdots G(\vec{e}_n) \right] \vec{m}$$

$$= \overline{\overline{\Gamma}} \, \vec{m}$$

So, $\overline{\overline{\Gamma}} = \left[ G(\vec{e}_1) \cdots G(\vec{e}_n) \right]$,

and its size is $m \times n$

$$t(z) = \int_0^\infty S(\xi) H(z-\xi)\, d\xi \qquad H = \begin{cases} 1 & 0 < \xi \leq z \\ 0 & \xi > z \end{cases}$$

$$\begin{aligned} t(z_i) &= \int_0^\infty H(z_i - \xi)\, S(\xi)\, d\xi \\ &= \int_0^{z_i} S(\xi)\, d\xi = \sum_{j=0}^{z_i} 1 \cdot S(z_j) \Delta z + \sum_{j=z_i}^{n} 0 \cdot S(z_j) \Delta z . \\ &= \sum_{j=0}^{z_i} S(z_j) \Delta z \end{aligned}$$

and $\Delta z = 0.2$

So. $G_{i,j} = \begin{cases} \Delta z & i \geq j \\ 0 & i < j \end{cases} = 0.2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \vdots \\ \vdots & 1 & \ddots & 0 \\ 1 & \cdots & & 1 \end{bmatrix}$

A system of linear equations is called inconsistent if it has no solutions. A system which has a solution is called consistent.

The system can be inconsistent even with only $m=3$. put the system of equations into an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & t_1 & -\frac{1}{2}t_1^2 & y_1 \\ 1 & t_2 & -\frac{1}{2}t_2^2 & y_2 \\ 1 & t_3 & -\frac{1}{2}t_3^2 & y_3 \end{array}\right] \xrightarrow{\text{find RREF}}$$

$$\left[\begin{array}{ccc|c} 1 & t_1 & -\frac{1}{2}t_1^2 & y_1 \\ 0 & t_2-t_1 & -\frac{1}{2}(t_2-t_1)(t_2+t_1) & y_2-y_1 \\ 0 & 0 & -\frac{1}{2}(t_2-t_1)(t_3-t_1)(t_3-t_2) & (y_3-y_1)(t_2-t_1)-(y_2-y_1)(t_3-t_1) \end{array}\right]$$

When $t_1=t_2$ or $t_3=t_1$ or $t_3=t_2$, but

$y_1 \neq y_2$ or $y_3 \neq y_1$ or $y_3 \neq y_2$,

the equation becomes $0=C$ $(C \neq 0)$.

So the system will be inconsistent.

This indicates that there are multiple observations at the same time by different people or machines, but the observations are different.

When the number of data point is 2, the system also can be inconsistent. This situation is the same as $m=3$.

$$\begin{bmatrix} 1 & t_1 & -\frac{1}{2}t_1^2 \\ 1 & t_2 & -\frac{1}{2}t_2^2 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

If $t_1 = t_2$, but $y_1 \neq y_2$, the system will be inconsistent.

$$S = \frac{1}{V} = \frac{1}{V_0 + kz} = \frac{1}{1000 + 40z}$$

Because $z_0 = 0$, $z_i = \frac{1}{2}\Delta z + (i-1)\Delta z = 0.2i - 0.1$

$$\vec{m}_{true} = \frac{1}{V_0 + k(\Delta z \cdot i - \frac{1}{2}\Delta z)} = \frac{1}{1000 + 4(2i-1)} \qquad i = 1, \ldots, n$$

$$d = \int_0^\infty s(\xi) H(z-\xi)\, d\xi = \int_0^z s(\xi)\, d\xi$$

$$= \int_0^z \frac{1}{1000 + 40z}\, d\xi = \frac{1}{40} \ln(1000 + 40z) \Big|_0^z$$

$$= \frac{1}{40} \ln\left(1 + \frac{z}{25}\right)$$

When N=100, and added some noise to d,
the m_pre became unstable compared to m_true.
There is a large gap between m_pre_add-noise and m_true

When $N=4$, the two type errors are close but the error of m-pre is large than the same error when $N=100$.

As same as (a). $G = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ 5 & 5 & 5 & 0 \\ 5 & 5 & 5 & 5 \end{bmatrix}$

The standard deviation of noise is $5\times10^{-5}$ s.

5 is larger than $5\times10^{-5}$ compared to 0.2 (when $N=100$). i.e., when $N=4$, the impact of noise is smaller than the condition when $N=100$.