# Solving Homogeneous Linear Differential Equations in Terms of Second Order Linear Differential Equations

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# SOLVING HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS IN TERMS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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**1. Introduction.** Let $F$ be a differential field of characteristic zero and let $L(y) = 0$ be an $n^{\text{th}}$ order homogeneous linear differential equation with coefficients in $F$. In this paper we develop necessary and sufficient conditions for $L(y) = 0$ to be solvable in terms of second order linear differential equations. We use these conditions to show that, when $n = 3$ and $F$ is $k(x)$, the field of rational functions over a finitely generated extension of the rationals, we can decide if $L(y) = 0$ can be solved in terms of second order linear differential equations. We show, for example, that $y''' - xy = 0$ cannot be solved in these terms.

When $F = k(x)$, as above, the phrase "solved in terms of second order linear differential equations" intuitively means that the solutions of $L(y) = 0$ lie in the class $E$ of functions that can be gotten inductively from $F$ by successively adding functions that are algebraic over previously defined functions and by adding solutions of $ay'' + by' + cy = d$ where $a, b, c$ and $d$ are previously defined functions. A formal definition is given in Section 2. Note that $E$ contains the liouvillian functions (those defined in terms of integration and the exponential function) since $y' = d$ and $y' - cy = 0$ define $\int d$ and $\exp(\int c)$. $E$ also includes those special functions that are defined by homogeneous second order linear differential equations, such as Bessel functions, hypergeometric equations, etc. Even without a formal definition of solvability in terms of second order linear differential equations, one can get some insight into how solutions of second order linear equations arise in solutions of third order equations by considering the following examples:

*Example* 1.1.   Let

$$L(y) = y''' + \frac{2}{x} y'' - \frac{1}{4x^2} y' + \frac{1}{4x^3} y = 0$$

A basis for the solution space of this equation is $\{x, x^{1/2}, x^{-1/2}\}$, that is, all solutions are algebraic functions and are certainly in $E$.

*Example* 1.2.   Let

$$L(y) = y''' - xy'$$

This equation can be written as $L(y) = L_2(L_1(y))$ where $L_1(y) = y'$ and $L_2(y) = y'' - xy$. A basis for the solution space of this equation is therefore $\{1, \int y_1, \int y_2\}$ where $\{y_1, y_2\}$ is a basis for the solution space of $y'' - xy = 0$. Note that this last equation cannot be solved in terms of liouvillian functions ([Ka], p. 42).

*Example* 1.3.   Let

$$L(y) = y''' - xy' - y$$

This equation can be written as $L(y) = L_1(L_2(y))$ where $L_1(y)$ and $L_2(y)$ are as in Example 1.2. Hence, a basis for the solution space is $\{y_1, y_2, y_3\}$ where $\{y_1, y_2\}$ is a basis for the solution space of $L_2(y) = 0$ and $y_3$ satisfies $y_3'' - xy_3 = 1$.

*Example* 1.4.   Let $\{y_1, y_2\}$ be a basis for the solution space of $L_2(y) = y'' - xy = 0$. In this case, $\{y_1^2, y_1 y_2, y_2^2\}$ will be a basis for the solution space of

$$L(y) = y''' - 4xy' - 2y = 0$$

$L(y)$ has no algebraic solutions and cannot be factored in the manner of Examples 1.1 and 1.2 (this can be shown using the galois theory of differential equations). $L(y)$ is called the second symmetric power of $L_2(y)$. What we have just done can be repeated for any homogeneous second order linear operator $L_2$ to yield a homogeneous third order linear operator called the second symmetric power of $L_2(y)$ (c.f. Section 3. This new third order operator may or may not factor in general).

*Example* 1.5.   Let $\{y_1, y_2\}$ be a basis for the solution space of $L_2(y) = y'' - xy = 0$ and let $v_1 = (y_1^2)'$, $v_2 = (y_1 y_2)'$ and $v_3 = (y_2^2)'$. $\{v_1, v_2, v_3\}$ is a basis for the solution space of

$$L(y) = y''' - 4xy' - 6y = 0$$

One can show (using Lemma 3.4(b) of Section 3) that this equation is not the second symmetric power of a second order linear operator, yet it clearly can be solved in terms of a linear second order equation. This example was concocted in the following manner: Let $F$ be a differential field and let $L_2(y) = 0$ be a homogeneous second order linear equation. Let $\{y_1, y_2\}$ be a basis for the solution space of $L_2(y) = 0$ and let $a_0, a_1, a_2$ be in $F$. Consider the elements

$$v_1 = a_0 y_1^2 + a_1 (y_1^2)' + a_2 (y_1^2)''$$

$$v_2 = a_0 y_1 y_2 + a_1 (y_1 y_2)' + a_2 (y_1 y_2)''$$

$$v_3 = a_0 y_2^2 + a_1 (y_2^2)' + a_2 (y_2^2)''$$

One can show that $\{v_1, v_2, v_3\}$ will, in general, be a basis for the solution space of a homogeneous third order linear differential equation. If $a_0 = 1$, $a_1 = a_2 = 0$, we get a second symmetric power, so Example 1.4 is a special case of this. In our example above, we let $a_0 = a_2 = 0$ and $a_1 = 1$.

One of the main results of this paper, Theorem 4.3, says that these are essentially the only ways a homogeneous third order linear differential equation can be solved in terms of second order linear equations. Furthermore we show how to decide when each of these cases occur.

The rest of the paper is organized as follows: Section 2 contains a formal definition of solvability in terms of second order linear differential equations and gives a necessary group theoretic condition for an $n^{\text{th}}$ order homogeneous linear differential equation to be solvable in this manner. Furthermore, in this section we show that for each $n \geq 3$, there exists an $n^{\text{th}}$ order homogeneous linear differential equation with coefficients in $C(x)$, $C$ the complex numbers, that cannot be solved in terms of second order linear equations. In Section 3, we develop the notion of a symmetric power of a linear differential operator. In Section 4, we present necessary and sufficient conditions for a homogeneous third order linear differential equation to be solvable in terms of second order linear differential equations and show that the necessary conditions developed in Section 2 are also sufficient. In Section 5, we show that $y''' - xy = 0$ cannot be solved in terms of second order linear equations. Section 6 contains a procedure to decide this question for an arbitrary homogeneous third order linear equation with coefficients in $k_0(x)$, $k_0$ a finitely generated extension of the rationals. Section 7 contains some final remarks. We assume that the reader

is familiar with the basic concepts in differential algebra and the galois theory of linear differential equations, for which the general references are [Ka] and [Ko 2]. We shall also need some facts from the theory of linear algebraic groups and representation theory for which the general references are [Hu 1] and [Hu 2]. All fields in this paper are assumed to be of characteristic zero and differential fields are assumed to be ordinary. We shall let $\mathbf{Z}$ denote the integers, $\mathbf{Q}$ the rationals and $\mathbf{C}$ the complex numbers.

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**2. Eulerian Extensions.**   In this section we shall give a formal definition of the notion of solvability in terms of second order linear equations and give a group theoretic characterization of those homogeneous linear differential equations that can be solved in terms of second order equations.

Let $F$ be a differential and let $f(y, y', \ldots, y^{(n)})$ be a differential polynomial with coefficients in $F$. We would like to give algebraic substance to the phrase "$f(y, y', \ldots, y^{(n)}) = 0$ has a solution that can be expressed in terms of solutions of second order linear differential equations." As a first attempt, we could say that this means that there exists a tower of fields $F = K_0 \subset \cdots \subset K_n$ such that $f(y, y', \ldots, y^{(n)}) = 0$ has a solution in $K_n$ and that each $K_i = K_{i-1}(u_i)$, $1 \leq i \leq n$, where each $u_i$ is either algebraic over $K_{i-1}$ or satisfies an equation of the form $a_i u_i'' + b_i u_i' + c_i u_i = d_i$ with $a_i, b_i, c_i, d_i$ lying in $K_{i-1}$ and not all zero. When one tries to prove statements using this definition, one ends up having to adjoin additional elements to $K_n$ (e.g. other solutions of the $a_i u'' + b_i u' + c_i u = d_i$). For this technical reason, we will use the following equivalent concept instead.

*Definition.*   Let $F \subset E$ be differential fields. We say that $E$ is an *eulerian extension* of $F$ is there exists a tower of fields $F = F_0 \subset \cdots \subset F_n = E$ such that either

(i) $F_i = F_{i-1}(u_i)$ where $u_i$ is algebraic over $F_{i-1}$ or
(ii) $F_i = F_{i-1}(u_i)$ where $u_i' \in F_{i-1}$ or
(iii) $F_i = F_{i-1}(u_i)$ where $u_i \neq 0$ and $u_i'/u_i \in F_{i-1}$ or
(iv) $F_i = F_{i-1}(u_i, v_i)$ where $u_i$ and $v_i$ are linearly independent (over the constants of $F_i$) solutions of an equation of the form $y'' + a_i y = 0$ with $a_i$ in $F_{i-1}$.

When we say that $f(y, \ldots, y^{(n)}) = 0$ has a solution expressible in terms of solutions of second order linear equations, we shall mean that this equation has a solution in an eulerian extension of $F$. Clearly, the elements used to build up the tower in the definition of eulerian extensions are either algebraic or satisfy second order linear differential equations. Conversely, using variation of parameters and the fact that the substitution $y = wz$, where $2w' + bw = 0$, removes the $by'$ term from $y'' + by' + cy = 0$, we see that the tower used in our first attempt at a rigorous definition can be enlarged to give an eulerian extension.

In our definition of eulerian extension, we pay no attention to the possibility of new constants being introduced in the tower. This is because of the following result:

PROPOSITION 2.1. *Let $F$ be a differential field with algebraically closed field of constants $C$. Let $\Sigma \subset F\{y_1, \ldots, y_n\}$ be a set of differential polynomials and let $J \in F\{y_1, \ldots, y_n\}$ with $J \notin \{\Sigma\}$, the smallest radical differential ideal containing $\Sigma$. If $\Sigma$ has a solution $\zeta_1, \ldots, \zeta_n$ for which $J \neq 0$ in some eulerian extension $E$ of $K$, then $\Sigma$ has a solution $\eta_1, \ldots, \eta_n$ for which $J \neq 0$ in an eulerian extension $E_1$ of $K$ whose subfield of constants is $C$.*

The proof of this proposition follows closely (with the obvious modifications) the proof of Theorem 3 in [Ko 1] and will be omitted.

The above proposition implies that if $L(y) = 0$ is a homogeneous linear differential equation with coefficients in $F$ and if $L(y) = 0$ has a fundamental set of solutions in some eulerian extension of $F$, then it has a fundamental set of solutions in an eulerian extension of $F$ with no new constants. In particular, using the uniqueness of the Picard-Vessiot extension, this means that the Picard-Vessiot extension $M$ of $K$ associated with $L(y) = 0$ will lie in an eulerian extension of $K$ having the same constants as $K$. We will now see what this means in terms of the galois group of $M$ over $K$. First we need the following definition:

*Definition.* Let $C$ be an algebraically closed field and $G$ a linear algebraic group defined over $C$. We say that $G$ is *eulerian* if there exists a tower of subgroups $G = G_n \supset \cdots \supset G_0 = \{e\}$ such that each $G_i$ is a linear algebraic subgroup of $G$, defined over $C$, normal in $G_{i+1}$ and such that $G_{i+1}/G_i$ is isomorphic (as an algebraic group) to one of the following:

  (i) a finite group
  (ii) the additive group of $C$, $\mathbf{G}_a$

(iii) the multiplicative subgroup of nonzero elements of $C$, $\mathbf{G}_m$

(iv) the group of $2 \times 2$ matrices of determinant 1, $\mathbf{SL}(2, C)$

(v) $\mathbf{PSL}(2, C) = \mathbf{SL}(2, C)/\{\pm e\}$, $e$ the identity matrix.

The following lemma contains the facts that we shall need about eulerian groups.

LEMMA 2.2.    *Let $G$ be a linear algebraic group defined over an algebraically closed field $C$ and let $G^0$ be the connected component of the identity in $G$.*

(a) *If $G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = \{e\}$ is a normal tower of linear algebraic groups with each $G_i/G_{i-1}$ eulerian, then $G$ is eulerian.*

(b) *If $G$ is eulerian, then any homomorphic image of $G$ is eulerian.*

(c) *If $G^0$ is solvable, then $G$ is eulerian.*

(d) *If $G$ is eulerian and $H$ is a closed subgroup of $G$, then $H$ is eulerian.*

(e) *$G$ is eulerian if and only if $G^0$ is eulerian.*

(f) *If $G$ is connected and semisimple, then there exists a finite product $G_1 \times \cdots \times G_n$ of simple groups and a homomorphism $\psi: G_1 \times \cdots \times G_n \to G$ with $\psi$ having finite kernel. If, in addition, $G$ is eulerian, we may take each $G_i$ to be $\mathbf{SL}(2, C)$.*

(g) *If $G$ is connected, then $G$ contains a closed normal subgroup $H$ such that $H$ is solvable and $G/H$ is the homomorphic image (with finite kernel) of a product $G_1 \times \cdots \times G_n$ of simple groups. If, in addition, $G$ is eulerian, we may take each $G_i$ to be $\mathbf{SL}(2, C)$.*

*Proof.*    (a) follows easily from the definition of eulerian.

(b) Since the only connected one dimensional linear groups are $\mathbf{G}_a$ and $\mathbf{G}_m$ ([Hu 1], p. 131), we see that the homomorphic image of either of these is either trivial or $\mathbf{G}_a$ or $\mathbf{G}_m$. Since the only nontrivial normal subgroup of $\mathbf{SL}(2, C)$ is $\{\pm e\}$, we see that the only homomorphic images of $\mathbf{SL}(2, C)$ or $\mathbf{PSL}(2, C)$ are the trivial groups, $\mathbf{SL}(2, C)$ or $\mathbf{PSL}(2, C)$. If $G$ is a group with normal tower $G = G_n \supset \cdots \supset G_0 = \{e\}$ satisfying the conditions of an eulerian group, then, from the preceding remarks, we see that $\psi(G) = \psi(G_n) \supset \cdots \supset \psi(G_0) = \{e\}$ is a normal tower also satisfying these conditions.

(c) If $G^0$ is solvable, then there exists a normal tower $G^0 = G_n \supset \cdots \supset G_0 = \{e\}$ such that for each $i$, $1 \le i \ n$, $G_i/G_{i-1}$ is isomorphic to either $\mathbf{G}_a$ or $\mathbf{G}_m$ ([Hu 1], Theorem 19.3, p. 123). Therefore, since $G/G^0$ is finite, $G$ is eulerian by (a).

(d) Let $G = G_n \supset \cdots \supset G_0 = \{e\}$ be an eulerian group with its defining tower and let $H$ be a closed subgroup. $H = H_n \supset H_{n-1} = H \cap G_{n-1} \supset \cdots \supset H_0 = \{e\}$ is a normal tower and for each $i$, $1 \le i \le n$, we have a canonical injection $\psi_i : H_i/H_{i-1} \to G_i/G_{i-1}$. If $G_i/G_{i-1}$ is finite, $\mathbf{G}_a$ or $\mathbf{G}_m$, then $H_i/H_{i-1}$ will also be one of these types. If $G_i/G_{i-1}$ is $\mathbf{SL}(2, C)$ or $\mathbf{PSL}(2, C)$, then either the image of $\psi_i$ will be one of these two groups or it will have dimension less than or equal to 2. Since all connected linear algebraic groups of dimension less than or equal to 2 are solvable ([Hu 1], p. 137, Example 4), and therefore eulerian by (c), we have that $H = H_n \supset \cdots \supset H_0 = \{e\}$ is a normal tower of groups where each $H_i/H_{i-1}$ is eulerian so by (a), $H$ is eulerian.

(e) follows from (a) and (d).

(f)   The first statement is just Theorem 27.5 on p. 167 of [Hu 1]. If $G$ is eulerian, then each $G_i$ must be eulerian. The only eulerian simple groups are $\mathbf{SL}(2, C)$ and $\mathbf{PSL}(2, C)$. Since $\mathbf{PSL}(2, C)$ is $\mathbf{SL}(2, C)/\{\pm e\}$ we may take each $G_i$ to be $\mathbf{SL}(2, C)$.

(g) Let $H$ be the radical of $G$ (p. 125, [Hu 1]). $G/H$ will be semisimple so that the result follows from (f).                                    ■

We shall now discuss the connection between eulerian extensions and eulerian groups.

PROPOSITION 2.3.   *Let $F$ be a differential field with algebraically closed field of constants $C$ and let $M$ be a Picard-Vessiot extension of $F$. If $M$ lies in an eulerian extension of $F$, then the galois group of $M$ is eulerian.*
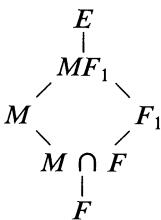
*Proof.*   By Proposition 2.1, we may assume that there exists an eulerian extension $E$ of $F$, having the same constants as $F$, such that $M \subset E$. We may write $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ where the $F_i$ are as in the definition of an eulerian extension. We will proceed by induction on $n$. Note that the compositum $MF_1$ is a Picard-Vessiot extension of $F_1$ (see Figure 1), which lies in an eulerian extension $E$ of $F_1$ with smaller $n$. By induction, the galois group $G(MF_1/F_1)$ of this extension is eulerian. Furthermore, $G(MF_1/F_1)$ is isomorphic to the galois group $G(M/M \cap F_1)$ of $M$ over $M \cap F_1$, so $G(M/M \cap F_1)$ is eulerian. If $F_1$ is algebraic over $F$, then $M \cap F_1$ is algebraic over $F$ so $G(M/M \cap F_1)$ is of finite index in $G(M/F)$, the galois group of $M$ over $F$. Therefore $G(M/F)$ is eulerian by Lemma 2.2(a).

Figure 1

If $F_1 = F(u)$ where $u' \in F$ or $u'/u \in F$, then $F_1$ is a Picard-Vessiot extension of $F$ with galois group $G(F_1/F)$ which is a subgroup of $\mathbf{G}_m$ or $\mathbf{G}_a$. In particular, $G(F_1/F)$ is abelian, so all subgroups are normal. Therefore $M \cap F_1$ is a normal extension of $F$ with galois group $G(M \cap F_1/F)$. This latter group is a homomorphic image of $\mathbf{G}_m$ of $\mathbf{G}_a$ and so must be finite, $\mathbf{G}_m$ or $\mathbf{G}_a$. Since this latter group is also isomorphic to $G(M/F)/G(M/M \cap F_1)$, we have that $G(M/F) \supset G(M/M \cap F_1) \supset \{e\}$ is a normal tower where each quotient is eulerian, so $G(M/F)$ is eulerian.

Now conider the case $F_1 = F(u, v)$, where $u$ and $v$ are linearly independent solutions of $y'' + ay = 0$, with $a \in F$. $F_1$ is a Picard-Vessiot extension of $F$ with galois group $G(F_1/F)$ which can be considered an algebraic subgroup of $\mathbf{SL}(2, C)$. Since both $F_1$ and $M$ are normal extensions of $F$, $M \cap F_1$ is a normal extension of $F$([Ko 2], Corollary 2, p. 402) so $G(M \cap F_1/F_1)$ is isomorphic to $G(F_1/F)/G(F_1/M \cap F_1)$. Any closed subgroup of $\mathbf{SL}(2, C)$ either has a solvable component of the identity or is $\mathbf{SL}(2, C)$. Therefore both $G(F_1/F)$ and $G(F_1/M \cap F_1)$ are eulerian, so $G(M \cap F_1/F)$ is eulerian. $G(M \cap F_1/F)$ is isomorphic to $G(M/F)/G(M/M \cap F_1)$ so we have that $G(M/F) \supset G(M/M \cap F_1) \supset \{e\}$ is a normal tower where each quotient is eulerian. This implies that $G(M/F)$ is eulerian. ∎

COROLLARY 2.4.   *For each integer $n \geq 3$, there exists a homogeneous linear differential equation of order $n$ with coefficients in $\mathbf{C}(x)$, having no nonzero eulerian solutions. Furthermore, this equation can be closer to have only Fuchsian singularities.*

*Proof.*   In $[T - T]$, it was shown that, for any linear algebraic group $G \subset \mathbf{GL}(n, \mathbf{C})$, there exists an $n^{\text{th}}$ order linear operator $L(y)$ with coefficients in $\mathbf{C}(x)$ and having only Fuchsian singularities, such that the Picard-Vessiot extension of $\mathbf{C}(x)$ associated with $L(y) = 0$ has galois group $G$. Let $L_n(y)$ be a linear operator whose associated Picard-Vessiot extension $M$ of $\mathbf{C}(x)$ has galois group $\mathbf{SL}(n, \mathbf{C})$. Let $V$ be the vector space of solutions of $L_n(y) = 0$. Note that $V$ is an irreducible $\mathbf{SL}(n, \mathbf{C})$ module. Let $V_0$ be the vector space of eulerian solutions of $L_n(y) = 0$ in $M$. $V_0$ is left invariant by $\mathbf{SL}(n, \mathbf{C})$. Since $\mathbf{SL}(n, \mathbf{C})$ is not eulerian for $n \geq 3$, Proposition 2.3 tells us that $V_0 \neq V$. Therefore $V_0 = \{0\}$. ∎

The converse to Proposition 2.3 is also true. Its proof relies on some facts which we shall prove in Section 4, and we shall postpone the complete proof of this converse to that section. We will however prove the following result, which will be needed in that section.

PROPOSITION 2.5.  *Let F be a differential field with algebraically closed field of constants and let M be a Picard-Vessiot extension of F. There exist subfields $E_0$ and E of M such that*:

(i) *$E_0$ is algebraic over F*

(ii) *$E = E_1 \cdot \cdots \cdot E_n$ where each $E_i$ is a Picard-Vessiot extension of $E_0$ whose galois group is a simple algebraic group.*

(iii) *M is a liouillian extension of E.*

*Furthermore, if the galois group G of M over F is eulerian then the galois group of each $E_i$ over $E_0$ is isomorphic to* **SL**$(2, C)$ *or* **PSL**$(2, C)$.

*Proof.*  Let $G^0$ be the component of the identity of $G$ and let $E_0$ be the fixed field of $G^0$. Let $H$ and $\psi: G_1 \times \cdots \times G_n \rightarrow G^0/H$ be as in Lemma 2.2(g). If $E$ is the fixed field of $H$ then, since $H$ is solvable, $M$ is a liouvillian extension of $E$. The galois group of $M$ over $E_0$ is $G^0/H$. Let $H_i$ be the image of $G_1 \times \cdots \times \hat{G}_i \times \cdots \times G_n$ under $\psi$. $H_i$ is a normal subgroup of $G^0/H$, so its fixed field $E_i$ is a Picard-Vessiot extension of $E_0$. The galois group of $E_i$ over $E_0$ is a homomorphic image of $G_i$ (and so isomorphic to **SL**$(2, C)$ or **PSL**$(2, C)$ in the eulerian case). Since the compositum of the galois groups of the $E_i$ over $E_0$ is the galois group of $E$ over $E_0$, we have that $E = E_1 \cdot \cdots \cdot E_n$. ∎

## 3. Symmetric products of linear operators.

Let $F$ be a differential field with algebraically closed constants $C$. Let $L_1(y)$ and $L_2(y)$ be homogeneous linear differential polynomials with coefficients in $F$ and let $M_1$ and $M_2$ be the Picard-Vessiot extensions associated with $L_1(y)$ and $L_2(y)$. Recall that $M_1$ and $M_2$ are unique (up to $F$-isomorphism ([Ko 2], Proposition 13, p. 412)).

LEMMA 3.1.  (a) *Let $F, L_1(y), L_2(y), M_1$, and $M_2$ be as above. Then there exists a Picard-Vessiot extension $M_3$ of F containing copies of $M_1$ and $M_2$.*

(b) *Let M be a Picard-Vessiot extension of F containing copies of $M_1$ and $M_2$ and let V be the C-vector space spanned by $\{v_1 v_2 | L_1(v_1) = 0$ and $L_2(v_2) = 0\}$. Then V is the solution space of a monic homogeneous linear differential equation $L_3(y) = 0$, with coefficients in F. Furthermore, $L_3(y)$ does not depend on M.*

*Proof.*  (a) From [Ko 3], we know that there exists a field $F^+$ containing $F$ and having the same field of constants as $F$ such that $F^+$ contains

a copy of every Picard-Vessiot extension of $F$ (in the proof of (b), we shall also need the fact that if $N_1 \subset F^+$ and $N_2 \subset F^+$ are Picard-Vessiot extensions of $F$ such $N_1$ is isomorphic to $N_2$ over $F$, then $N_1 = N_2$). Therefore we may assume that $M_1$ and $M_2$ are in $F^+$. Let $V_1$ and $V_2$ be the solution spaces of $L_1(y) = 0$ and $L_2(y) = 0$ respectively, and let $V_3 = V_1 + V_2 = \{v_1 + v_2 | v_1 \in V \text{ and } v_2 \in V_2\}$. Let $z_1, \ldots, z_m$ be a basis for $V_3$ and let $M_3 = F\langle V_3 \rangle$. Any $F$-homomorphism of $M_3$ into a fixed universal differential field $U$ will act linearly on $V_1$ and $V_2$ and so will act linearly on $V_3$. Therefore the coefficients of

$$L_3(y) = \frac{\mathrm{Wr}(y, z_1, \ldots, z_n)}{\mathrm{Wr}(z_1, \ldots, z_n)}$$

will be left fixed by such a map (Wr stands for the Wronskian determinant). By the corollary on p. 388 of [Ko 2], we see that $L_3(y)$ has coefficients in $F$. Since $V_3$ is the solution space of $L_3(y) = 0$, $M_3$ is a Picard-Vessiot extension of $F$.

(b) Any automorphism of $M$ over $F$ will act linearly on $V$. If $w_1, \ldots, w_m$ is a basis for $V$, then the monic linear differential polynomial

$$L_3(y) = \frac{\mathrm{Wr}(y, w_1, \ldots, w_m)}{\mathrm{Wr}(w_1, \ldots, w_m)}$$

has coefficients left fixed by all the elements of the galois group of $M$ over $F$ and so the coefficients are in $F$. Let $F^+$ be the field mentioned in (a). We may assume $M \subset K^+$. If $N$ is any other Picard-Vessiot extension of $F$ containing $M_1$ and $M_2$, then we can embed $N$ into $K^+$. This embedding will restrict to automorphisms of $M_1$ and $M_2$ and so leave $V_1$ and $V_2$ invariant. Since the above construction of $L_3(y)$ just depends on a basis of $V_1 \cdot V_2$, we get the same operator for $N$ as for $M$.          ■

We will call the operator $L_3$ constructed in Lemma 3.1, the *symmetric product* of $L_1$ and $L_2$ and designate it by $L_1 \circledS L_2$. One can easily show $(L_1 \circledS L_2) \circledS L_3 = L_1 \circledS (L_2 \circledS L_3)$. We can furthermore define $L^{\circledS n}$ for $n \geq 1$ by $L^{\circledS 1} = L$ and $L^{\circledS n} = L^{\circledS n-1} \circledS L$. $L^{\circledS n}$ is called the $n^{\text{th}}$ symmetric power of $L$.

We note here that one can effectively calculate $L^{\circledS m}(y)$ given $L(y)$. The most efficient way is to let $v = y^m$ and differentiate $v \binom{m+n-1}{n-1}$ times. Each time a derivative $y^{(k)}$ appears with $k \geq n$, use $L(y) = 0$ to replace it with an

expression which only involves the $y^{(k)}$ with $k < n$. Each $v^{(i)}$ with $0 \le i \le \binom{m+n-1}{n-1}$ will then be a homogeneous form of degree $m$ in $y, \ldots, y^{(n-1)}$ with coefficients in $F$. Since the $F$-space of such forms has dimension $\binom{m+n-1}{n-1}$ and we have $\binom{m+n-1}{n-1} + 1$ of these forms, the forms corresponding to the $v^{(i)}$ will be linearly dependent over $F$. Let $N$ be the smallest integer so that the forms corresponding to $v, v', \ldots, v^{(N)}$ are linearly dependent over $F$. We can then find $a_{N-1}, \ldots, a_0$ in $F$ such that $v^{(N)} + a_{N-1}^{(N-1)} v + \cdots + a_0 v = 0$. This differential equation will be satisfied by all $y^m$, where $y$ is a solution of $L(y) = 0$, and no homogeneous linear differential equation of smaller order will have this property. Since $\{y^m \mid L(y) = 0\}$ spans the space of homogeneous forms of degree $m$ in solutions of $L(y) = 0$, this operator must be $L^{\circledS m}$.

*Example.* Let $L_2(y) = y'' - xy$ and let $v = y^2$. Then

$$v = y^2$$

$$v' = 2yy'$$

$$v'' = 2(y')^2 + 2xy^2$$

$$v''' = 8xyy' + 2y^2$$

We therefore have $v''' - 4xv' - 2v = 0$, so

$$L_2^{\circledS 2}(y) = y''' - 4xy' - 2y.$$

**LEMMA 3.2.** *Let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in a differential field $F$, with algebraically closed field of constants $C$.*

(a) *If $L(y)$ has order $n$, then $L^{\circledS m}$ has order at most $\binom{m+n-1}{n-1}$.*

(b) *If $L(y)$ has order 2 and $\{y_1, y_2\}$ is a basis for the solution space of $L(y) = 0$, then $\{y_1^m, y_1^{m-1} y_2, \ldots y_1 y_2^m\}$ is a basis for the solution space of $L^{\circledS m}$. In particular, $L^{\circledS m}$ will have order precisely $m + 1$.*

*Proof.*

(a) Follows from the discussion preceding the lemma.

(b) If there would exist constants $c_0, \ldots c_n$, not all zero, such that $c_0 y_1^n + c_1 y_1^{n-1} y_2 + \cdots + c_n y_2^n = 0$, then $c_0 z^n + \cdots + c_{n-1} z + c_0 = 0$

where $z = y_1/y_2$. This would imply that $z = c \in C$ so $y_1 - cy_2 = 0$, contradicting the fact that $y_1$ and $y_2$ are linearly independent over $C$.  ∎

Note that even for third order equations, $L^{\circledS m}$ may have order strictly less than $\binom{m+n-1}{n-1}$. Consider $L(y) = y''' - 4xy' - 2y$. We have seen that $L(y) = L_2^{\circledS 2}(y)$ where $L_2(y) = y'' - xy$. Hence $L^{\circledS 2}(y) = L_2^{\circledS 4}(y)$ has order equal to 5 which is strictly less than $\binom{2+3-1}{3-1} = 6$.

In later sections we shall need criteria to determine if a homogeneous third order linear differential equation is a second symmetric power of a homogeneous second order equation. These are contained in Lemma 3.4.

First, let us recall some basic facts from the representation theory of $\mathbf{SL}(2, C)$ ([Sp], p. 43). If $V$ is a vector space and $\rho: \mathbf{SL}(2, C) \to \mathbf{GL}(V)$ is a representation then $V = V_1 \oplus \cdots \oplus V_m$, where each $V_i$ is irreducible and invariant under the action of $\mathbf{SL}(2, C)$. If we let $R = C[X, Y]$, then $\mathbf{SL}(2, C)$ acts as a group of linear substitutions on $R$ in the obvious way. If $R_n$ is the $C$-space of homogeneous elements of $R$ of degree $n$, then $R_n$ is an irreducible $\mathbf{SL}(2, C)$ module and furthermore, any irreducible $\mathbf{SL}(2, C)$ module is isomorphic to some $R_n$. For example, when $n = 3$, $R_3$ is a vector space of dimension 3 and relative to the basis $X^2, XY, Y^2$, we have a representation $\rho_3: \mathbf{SL}(2, C) \to \mathbf{SL}(3, C)$ given by:

$$\rho_3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

Note that the kernel of $\rho_3$ is $\{\pm I\}$ so $\rho_3$ induces on isomorphism of $\mathbf{PSL}(2, C)$ into $\mathbf{SL}(3, C)$. To prove Lemma 3.4 and Proposition 4.1, we need the following:

LEMMA 3.3.  (a) *The normalizer of* $\rho_3(\mathbf{SL}(2, C))$ *in* $\mathbf{SL}(3, C)$ *is* $\rho_3(\mathbf{SL}(2, C)) \times H$, *where H is the 3 element subgroup of* $\mathbf{SL}(3, C)$ *consisting of constant matrices.*

(b) *The group G of elements in* $\mathbf{SL}(3, C)$ *that induce automorphisms of the variety* $V = \{(v_1, v_2, v_3) | v_1v_3 - v_2^2 = 0\}$ *is precisely* $\rho_3(\mathbf{SL}(2, C)) \times H$.

*Proof.*  (a) (For undefined terms and concepts see [Hu 2].) First note that the Lie algebra $\mathbf{g}$ of $\rho_3(\mathbf{SL}(2, C))$ is isomorphic to $\mathbf{sl}(2, C)$. If $g$ is in the normalizer of $\rho_3(\mathbf{SL}(2, C))$ then $\mathrm{Ad}\,g$ acts as a derivation on $\mathbf{g}$. Since

$\mathbf{sl}(2, C)$ is simple, the only derivations of $\mathbf{g}$ are inner. Therefore, there exists an $s$ in $\rho_3(\mathbf{SL}(2, C))$ such that $\mathrm{Ad}(gs^{-1})$ is the identity on $\mathbf{g}$. This means that $gs^{-1}$ commutes with every element of $\rho_3(\mathbf{SL}(2, C))$. Since $\rho_3$ is an irreducible representation, Schur's Lemma implies that $gs^{-1}$ is a constant matrix $h$. Therefore $g = s \cdot h$, so the normalizer of $\rho_3(\mathbf{SL}(2, C))$ is contained in $\rho_3(\mathbf{SL}(2, C)) \times H$. The opposite inclusion is obvious and so (a) is proved.

(b) Let $\pi : C^3 \to CP^2$ be the canonical map identifying lines in $C^3$ with points in projective 2-space $CP^2$. $\pi(V)$ is a nonsingular conic and so isomorphic to $CP^1$, whose automorphism group is $\mathbf{PSL}(2, C)$. We can consider $G$ as acting on $\pi(v)$ and so have a map $\phi$ of $G$ into $\mathbf{PSL}(2, C)$. $G$ contains $\rho_3(\mathbf{SL}(2, C))$ which, being isomorphic to $\mathbf{PSL}(2, C)$ is a simple group. Therefore $\phi$ is injective on $\rho_3(\mathbf{SL}(2, C))$. Comparing dimensions, we see that $\phi$ is surjective. If $g \in G$ is in the kernel of $\phi$, then $g$ must act on $V$ by scalar multiplication. Since $V$ contains a basis of $C^3$, $g$ must be a scalar matrix. Therefore $G = \rho_3(\mathbf{SL}(2, C)) \times H$. .        ■

LEMMA 3.4.    *Let $F$ be a differential field with algebraically closed field of constants $C$ and let $L(y) = y''' - my'' - py' - qy$ with $m$, $p$, and $q$ in $F$.*

(a) *$L(y) = L_2^{\circledS 2}(y)$ for some homogeneous second order linear differential operator $L_2$ with coefficients in some differential extension of $F$ if and only if*

$$(3.4.1) \qquad q = \frac{1}{54}[27p' + 24mm' - 9mm'' - 36mp - 8m^2]$$

*in which case*

$$(3.4.2) \qquad L_2(y) = y'' - \frac{m}{3}y' - \frac{1}{4}\left[p - \frac{m'}{3} + \frac{2}{9}m^2\right]$$

*which has coefficients in $F$.*

(b) *If $m = 0$, $L(y) = L_2^{\circledS}(y)$ if and only if $q = p'/2$.*

(c) *Let $M$ be the Picard-Vessiot extension of $F$ associated with $L(y) = 0$ and assume that the galois group of $M$ over $F$ is contained in $\mathbf{SL}(3, C)$. There exists a basis $y_1, y_2, y_3$ of the solution space of $L_2(y) = 0$ in $M$ such that $y_1 y_3 - y_2^2 = 0$ if and only if $L(y) = L_2^{\circledS 2}(y)$ for some homogeneous second order linear differential equation $L_2(y)$ with coefficients in $F$.*

*Proof.* (a) Let $K$ be a differential extension of $F$ and let $L_2(y) = y'' - ry' - sy$. Then $L_2^{\circledS 2}(y) = y''' - 3ry'' - (4s + r' - 2r^2)y' - (2s' - 4rs)y$. If $L(y) = L_2^{\circledS 2}(y)$, we equate coefficients, solve for $r$ and $s$ in terms of $m$ and $p$ and find that (3.4.1) holds. Conversely, if (3.4.1) holds, then, calculating, we find that $L(y) = L_2^{\circledS 2}(y)$ where $L_2(y)$ is as in (3.4.2).

(b) This is a special case of (a).

(c) Assume $L(y) = L_2^{\circledS 2}(y)$. Letting $\{u, v\}$ be a basis for the solution space of $L_2(y) = 0$, we see from Lemma 3.2(b), that $\{u^2, uv, v^2\}$ is a basis for the solution space of $L(y) = 0$. This basis satisfies $y_1 y_3 - y_2^2 = 0$.

Now assume that there exist linearly independent solutions $y_1, y_2, y_3$ of $L(y) = 0$ such that $y_1 y_3 - y_2^2 = 0$ and let $G$ be the galois group of $M$ over $F$. With respect to this basis, $G$ can be considered a subgroup of $\mathbf{SL}(3, C)$. By Lemma 3.3(b), we have that $G \subset \rho_3(\mathbf{SL}(2, C)) \times H$. Let $G^0$ be the component of the identity in $G$ and let $F_0$ be its fixed field. Let $U$ be a universal differential field containing $F_0$ and let $C$ be its field of constants. Let $u = \sqrt{y_1}$ and $v = \sqrt{y_3}$. Since $y_1 y_3 - y_2^2 = 0$, we can adjust the signs of $u$ and $v$ so that $y_2 = uv$. Let $\sigma$ be an $F_0$-isomorphism of $M(u, v)$ into $U$. $\sigma$ will restrict to an $F_0$-isomorphism of $M$ into $U$ so there exist $a, b, c, d$ in $C$ with $ad - bc = 1$ such that

$$\sigma(u^2) = \sigma(y_1) = a^2 y_1 + 2ab y_2 + b^2 y_3 = (au + bv)^2$$

$$\sigma(uv) = \sigma(y_2) = ac y_1 + (ad + bc)y_2 + bd y_3 = (au + bv)(cu + dv)$$

$$\sigma(v^2) = \sigma(y_3) = c^2 y_1 + 2cd y_2 + d^2 y_3 = (cu + dv)^2$$

Therefore $\sigma(u) = \pm(au + bv)$ and $\sigma(v) = \pm(cu + dv)$. Let

$$L_2(y) = \frac{\mathrm{Wr}(y, u, v)}{\mathrm{Wr}(u, v)}.$$

The coefficients of $L_2(y)$ are left fixed by all $F_0$-isomorphisms of $M(u, v)$ into $U$, so they must be in $F_0$. Therefore $L(y)$ is the second symmetric power of a homogeneous second order linear differential equation with coefficients in $F_0$. By part (a) of this Lemma, we can conclude that $L(y)$ has this property with respect to $F$. ∎

Symmetric products of linear operators have been considered by mathematicians in the 19th century. Lemma 3.4(a) and (b) was known to

Halphen and is the starting point of his investigations into differential invariants [Ha]. Lemma 3.4(c) appears implicitly in Fano's work [Fa]. Fano investigates the situation when an $n^{\text{th}}$ symmetric power of a linear operator of order $n$ has order less than $\binom{m+n-1}{n-1}$.

**4. Third order equations.** In this section we shall use the material developed in the last two sections to completely characterize those third order homogeneous linear differential equations having eulerian solutions. We will then be in a position to prove Theorem 4.6 which gives necessary and sufficient conditions for a homogeneous linear differential equation to have eulerian solutions.

The next proposition gives us the group theoretic information needed to deal with the third order case. Before proving this result we review some nomenclature from the theory of group representations. Let $V$ be a vector space over a field $C$ and $G$ a subgroup of $\mathbf{GL}(V)$. A group homomorphism $\lambda: G \to \mathbf{G}_m$, the multiplicative subgroup of $C$, is called a weight. The subspace $V_\lambda = \{v \in V \mid \sigma v = \lambda(v)v \text{ for all } \sigma \text{ in } G\}$ is called the weight space associated with $\lambda$. Note that if $\lambda_1, \ldots, \lambda_n$ are distinct weights, then $V_{\lambda_1}, \ldots, V_{\lambda_n}$ are linearly independent ([Hu 1], p. 81).

PROPOSITION 4.1.   *Let $G$ be an algebraic subgroup of $\mathbf{SL}(3, C)$, $C$ an algebraically closed field of characteristic zero, and let $G^0$ be the component of its identity. One of the following four conclusions holds*:

   (a) *$G$ is finite.*
   (b) *$G^0$ has a one-dimensional weight space, in which case there exists a closed normal subgroup $H$ of $G$ with $|G:H| = 1, 2, 3$ or $6$ leaving invariant this subspace.*
   (c) *$G$ leaves a two dimensional space invariant.*
   (d) *$G^0$ leaves no nontrivial subspace invariant. In this case, $G^0$ is semisimple. If, in addition, $G$ is eulerian, then $G^0$ is conjugate to $\rho_3(\mathbf{SL}(2, C))$ and $|G:G^0| = 1$ or $3$. If $|G:G^0| = 3$, then $G = G^0 \times H$ where $H$ is the finite subgroup of $\mathbf{SL}(3, C)$ consisting of constant matrices.*

   *Proof.*   We may assume that $G$ is not finite. If $G^0$ has a nonzero weight space $V$, then $\dim V \neq 3$, for otherwise $G^0$ (and therefore $G$) would be finite. If $G^0$ has a two-dimensional weight space $V$, then using the remark immediately preceding the statement of the Proposition, we see that $V$ is the only two-dimensional weight space. Since $G^0$ is normal in $G$, $G$ must leave $V$ invariant so we have conclusion (c). If $G^0$ has a one-dimensional weight space, let $X$ be the set of all one-dimensional weight spaces.

By the remark immediately preceding the statement of this Proposition, $X$ has at most 3 elements. Since $G^0$ is normal in $G$, each element of $G$ permutes the elements of $X$. This gives a homomorphism of $G$ into the permutation group of $X$. If $H$ is the kernel of this homomorphism, $H$ leaves the element of $X$ fixed and $|G:H| = 1, 2, 3$ or $6$, thus giving (b) above.

Now we can assume that $G^0$ has no nonzero weight spaces. If $G^0$ leaves a nontrivial subspace invariant, then it must leave a two-dimensional subspace $V$ invariant. There can be no other two-dimensional invariant subspaces, since the intersection with $V$ would be contained in a weight space. Therefore $G$ will leave $V$ invariant and we have conclusion (c) above.

Now assume $G^0$ has no nontrivial invariant subspaces. Since we are in characteristic 0, we can make use of the Lie algebra $\mathbf{g}$ of $G^0$. $\mathbf{g}$ will be a subalgebra of $\mathbf{sl}(3, C)$ and $\mathbf{g}$ acts irreducibly on $C^3$. Therefore $\mathbf{g}$ (and so $G^0$) is semisimple ([Hu 2], Proposition 19.1(6), p. 102).

If $G$ (and therefore $G^0$) is eulerian, Lemma 2.2(f) then tells us that $\mathbf{g}$ is isomorphic to the direct product of copies of $\mathbf{sl}(2, C)$. Let $\mathbf{g} = \mathcal{L}_1 + \cdots + \mathcal{L}_n$ where each $\mathcal{L}_i$ is a Lie algebra isomorphic to $\mathbf{sl}(2, C)$ and let $\psi_i : \mathbf{sl}(2, C) \to \mathbf{g}$ be the isomorphism of $\mathbf{sl}(2, C)$ onto the $i^{\text{th}}$ factor of $\mathbf{g}$. We shall show that $n = 1$. We may consider $C^3$ as an $\mathbf{sl}(2, C)$ module via the map $\psi_1$ and as such $C^3$ can be written as a direct sum of irreducible $\mathbf{sl}(2, C)$ modules. Since $\psi_1$ is not identically zero, not all of these irreducible subspaces are one-dimensional. Therefore either $C^3$ is itself irreducible or can be written as the sum of a one-dimensional and a two dimensional irreducible subspace. Note that if $n > 1$, $\psi_2(\mathbf{sl}(2, C))$ lies in the centralizer of $\psi_1(\mathbf{sl}(2, C))$. If $C^3$ is irreducible as a $\psi_1(\mathbf{sl}(2, C))$ module, then by Schur's Lemma, each element in $\psi_2(\mathbf{sl}(2, C))$ is constant and therefore zero. If $C^3$ is the sum of a one-dimensional and a two dimensional $\psi_1(\mathbf{sl}(2, C))$-irreducible subspace, then $\psi_2(\mathbf{sl}(2, C))$ leaves each of these invariant and so must again be zero. In either case we get a contradiction when we assume $n > 1$. We therefore can conclude that $\mathbf{g}$ is isomorphic to $\mathbf{sl}(2, C)$. Since we assumed $C^3$ is irreducible, $G^0$ is conjugate to $\rho_3(\mathbf{SL}(2, C))$. Lemma 3.3(a) then gives us the conclusion of (d).          ∎

Theorem 4.3 will classify third order homogeneous linear differential equations in terms of their galois groups. We first deal with the following special case:

PROPOSITION 4.2.    *Let $K$ be a differential field with algebraically closed field of constants $C$. Let $M$ be the Picard-Vessiot extension of $K$*

*associated with $L(y) = y''' - py' - qy = 0$ with $p, q \in K, q \neq 0$. Assume that there exists a basis $y_1, y_2, y_3$, of the solution space of $L(y) = 0$ such that with respect to this basis the galois group $G$ of $M$ over $K$ is $\rho_3(\mathbf{SL}(2, C))$. Then*

(a) *The elements $I_0 = y_2^2 - y_3 y_1$, $I_1 = (y_2')^2 - y_3' y_1'$ and*

$$I_2 = (y_2'')^2 - y_3'' y_1'' \text{ lie in } K.$$

(b) *If $b_0, b_1, b_2$ are any nonzero solutions, algebraic over $K$, of*

(4.2.1)
$$0 = I_0 b_0^2 + I_1 b_1^2 + I_2 b_2^2 + I_0' b_0 b_1$$

$$+ \frac{1}{q}(I_2' - pI_1')b_0 b_2 + I_1' b_1 b_2$$

*then*

$$z_1 = b_0 y_1 + b_1 y_1' + b_2 y_1''$$

(4.2.2)
$$z_2 = b_0 y_2 + b_1 y_2' + b_2 y_2''$$

$$z_3 = b_0 y_3 + b_1 y_3' + b_2 y_3''$$

*will be a basis for the solution space of $L_2^{\otimes 2}(y) = 0$, where $L_2(y) = 0$ is a homogeneous second order linear differential equation with coefficients in $K(b_0, b_1, b_2)$.*

*Proof.*

(a) $\rho_3(\mathbf{SL}(2, C))$ leaves $I_0, I_1$ and $I_2$ fixed so they must lie in $K$.

(b) Let $z_i = b_0 y_i + b_1 y_i' + b_2 y_i''$ for $i = 1, 2, 3$.

When we expand the expression for $z_2^2 - z_1 z_3$ we get

$$z_2^2 - z_1 z_3 = I_0 b_0^2 + I_1 b_1^2 + I_2 b_2^2 + I_0' b_1 b_2$$

$$+ \frac{1}{q}(I_2' - pI_1')b_0 b_2 + I_1' b_1 b_2$$

so $z_2^2 - z_1 z_0 = 0$. We now wish to apply Lemma 3.4(c). First we claim that $z_1, z_2, z_3$ are linearly independent over $C$. If not, then there exist $c_1, c_2, c_3$ not all zero in $C$ such that

$$0 = c_1 z_1 + c_2 z_2 + c_3 z_3$$

$$b_2(c_1 y_1 + c_2 y_2 + c_3 y_3)'' + b_1(c_1 y_1 + c_2 y_2 + c_3 y_3)'$$

$$+ b_0(c_1 y_1 + c_2 y_2 + c_3 y_3)$$

Therefore $\bar{L}(y) = b_2 y'' + b_1 y' + b_0 y = 0$ has a nonzero solution in common with $L(y) = 0$. The galois group of $M(b_0, b_1, b_2)$ over $K(b_0, b_1, b_2)$ is isomorphic to the galois group of $M$ over $M \cap K(b_0, b_1, b_2)$. Since $\rho_3(\mathbf{SL}(2, C))$ is simple, this group is just $\rho_3(\mathbf{SL}(2, C))$ again. Let $\bar{V}$ be the subspace of the solution space of $L(y) = 0$ consisting of solutions of $\bar{L}(y) = 0$. This is a proper subspace which is left invariant by $\rho_3(\mathbf{SL}(2, C))$. Since $\rho_3(\mathbf{SL}(2, C))$ acts irreducibly, we have $\bar{V} = \{0\}$, a contradiction. Therefore $z_1$, $z_2$ and $z_3$ are linearly independent.

Now consider the equation

$$L_3(y) = \frac{\mathrm{Wr}(y, z_1, z_2, z_3)}{\mathrm{Wr}(z_1, z_2, z_3)}$$

Since $G$ acts linearly on the span of $\{z_1, z_2, z_3\}$, this equation has coefficients in $K(b_0, b_1, b_2)$. We now apply Lemma 3.4(c) and conclude that $L_3$ is a second symmetric power. ∎

THEOREM 4.3. *Let $F$ be a differential field with algebraically closed field of constants. Let $L(y) = y''' - py' - qy$ with $p, q \in F$ and let $M$ be the associated Picard-Vessiot extension of $F$. If the galois group of $M$ over $L$ is eulerian then one of the following four cases apply:*

(a) *All solutions of $L(y) = 0$ in $M$ are algebraic over $F$.*

(b) *There exists as subfield $K$ of $M$ with $[K:F] = 1, 2, 3$ or $6$ and $a, b, c$ in $K$ such that*

$$L(y) = L_2(L_1(y))$$

*where*

$$L_2(y) = y'' + ay' + by$$

$$L_1(y) = y' + cy$$

(c) *There exist $a, b, c$ in $F$ such that*

$$L(y) = L_1(L_2(y))$$

*where*

$$L_2(y) = y'' + ay' + by$$

*and*

$$L_1(y) = y' + cy$$

(d) *The component $G^0$ of the identity in the galois group is conjugate to $\rho_3(\mathbf{SL}(2, C))$. In this case, there exist a subfield $K$ of $M$ with $[K:F] = 1$ or $3$, an algebraic extension $N$ of $K$ with $[N:K] = 1$ or $2$ and elements $a_0$, $a_1, a_2, b, c$ in $N$ such that if $\{u, v\}$ is a basis for the solution space of $y'' + by' + cy = 0$, then*

$$y_1 = a_0 u^2 + a_1(u^2)' + a_2(u^2)''$$

$$y_2 = a_0 uv + a_1(uv)' + a_2(uv)''$$

$$y_3 = a_0 v^2 + a_1(v^2)' + a_2(v^2)'' \quad \cdot$$

*is a basis for the solution space of $L(y) = 0$. If $F$ is an algebraic extension of $C(x)$, $x' = 1$, we may take $N = K$.*

*Conversely if any of the above hold, all solutions of $L(y) = 0$ are eulerian and so the galois group of $M$ over $K$ is eulerian.*

*Proof.* Since there is no $y''$ term in $L(y)$, we may assume that the galois group $G$ of $M$ over $U$ lies in $\mathbf{SL}(3, C)$, ([Ka], Corollary 6.2, p. 41). We now apply Proposition 4.1. If $G$ is finite, then $M$ is an algebraic extension of $F$ so (a) above follows.

If Proposition 4.1(b) holds, then the fixed field $K$ of $G^0$ is algebraic over $F$ of degree equal to $|G:G^0| = 1, 2, 3$ or $6$. Furthermore if $u$ spans a one-dimensional $G^0$ invariant subspace then for all $\sigma$ in $G^0$, there exists a constant $c_\sigma$ such that $\sigma(u) = c_\sigma u$. Therefore $\sigma(u'/u) = u'/u$, so $u'/u$ is in $K$. Since $L(y) = 0$ and $L_1(y) = y' - (u'/u)y = 0$ share a common non-zero solution, we may write $L(y) = L_2(L_1(y))$ where $L_2(y) = y'' + ay' + by$ with $a, b$ in $K$ ([Po], p. 34). This yields (b) above.

If Proposition 4.1(c) holds, then there exist linearly independent solutions $y_1, y_2$ of $L(y) = 0$ such that $G$ leaves the span of $y_1$ and $y_2$ invariant. Therefore

$$L_2(y) = \frac{\mathrm{Wr}(y, y_1, y_2)}{\mathrm{Wr}(y_1, y_2)} = y'' + ay' + by$$

has coefficients $a$ and $b$ in $F$. Since every solution of $L_2(y) = 0$ is a solution of $L(y) = 0$, we have $L(y) = L_1(L_2(y))$ where $L_1(y) = y' + cy$ for some $c$ in $F$. This yields (c).

If Proposition 4.1(d) holds, then let $K$ be the fixed field of $G^0$. Let $I_0$, $I_1$, $I_2$ be as in Proposition 4.2. We can clearly find a solution $b_0, b_1, b_2$ of (4.2.1) in (at worst) a quadratic extension $N$ of $K$. We then have the conclusion of Proposition 4.2(b). We wish to find $a_0, a_1, a_2$ in $N$ such that $y_i = a_0 z_i + a_1 z_i' + a_2 z_i''$ for $i = 1, 2, 3$. If we use Cramer's Rule, we get

$$a_0 = \frac{\begin{vmatrix} y_1 & z_1' & z_1'' \\ y_2 & z_2' & z_2'' \\ y_3 & z_3' & z_3'' \end{vmatrix}}{\mathrm{Wr}(z_1, z_2, z_3)}$$

and similar expressions for $a_1$ and $a_2$. Using (4.2.2) we see that $G^0$ acts on $(z_1, z_2, z_3)$ in the same way as it acts on $(y_1, y_2, y_3)$ so $G^0$ leaves $a_0, a_1, a_2$ invariant. Therefore $a_0, a_1, a_2$ are in $N$.

If $F$ is an algebraic extension of $C(x)$, then Theorem 3.6 on p. 22 of [Gr] says that (4.2.1) will have a nonzero solution in $K$, so we may let $N = K$. We make a remark here that will be used in Section 6. If $F$ is an algebraic extension of $C(x)$, we can choose $b_0, b_1, b_2$ to be in $F(I_0, I_1, I_2)$. Since $|G:G^0| = 1$ or 3, $[F(I_0, I_1, I_2):F] = 1$ or 3. If $[F(I_0, I_1, I_2):F] = 1$, then $b_0 b_1, b_2$ will lie in $F$ so *all* of $G$ leaves $a_0$ (and similarly $a_1$ and $a_2$) fixed. Therefore, $a_0, a_1$ and $a_2$ will lie in $F$. If $[F(I_0, I_1, I_2):F] = 3$, then $F(I_0, I_1, I_2) = K$. In either case $a_0, a_1, a_2$ will lie in $F(I_0, I_1, I_2)$. This is the remark we need in Section 6. ∎

COROLLARY 4.4.   *Let $F$, $M$ and $L(y)$ be as in Theorem 4.3. If $L(y) = 0$ has a nonzero solution in some eulerian extension of $F$, then the conclusions of Theorem 4.3 hold.*

*Proof.*   If $L(y) = 0$ has a solution $u$ in an eulerian extension $E$ of $F$, then there exist $a$, $b$ in $E$ such that $L(y) = L_2(L_1(y))$ where $L_2(y) = y'' + ay' + by$ and $L_1(y) = y' - (u'/u)y$. Let $\{y_1, y_2\}$ be two linearly independent solutions of $L_2(y) = 0$ and let $z_i$, $i = 1, 2$ be solutions $L_1(y) = y_i$. $\{u, z_1, z_2\}$ forms a basis for the solution space of $L(y) = 0$ and $\{u, z_1, z_2\}$ lies in an eulerian extension of $F$. By Proposition 2.3, the galois group of $M$ over $F$ is eulerian, so the hypotheses of Theorem 4.3 are satisfied. ∎

COROLLARY 4.5. *Let F be a differential field with algebraically closed field of constants, $L(y) = y''' + ry'' + sy' + ty$ with $r, s, t$ in $F$, and M the Picard-Vessiot extension of F corresponding to $L(y) = 0$. If the galois group of M over F is eulerian, then M lies in an eulerian extension of F.*

*Proof.* Let $K = F < \exp(\int r) >$ and let $y_1, y_2, y_3$ be a basis for the solution space of $L(y) = 0$. We have that $MK = K \langle u_1, u_2, u_3 \rangle$, where $u_i = y_i(\exp \int r)$. The $u_i$ satisfy a third order homogeneous linear differential equation, with coefficients in $F$, having no $y''$ term. The galois group of $MK$ over $K$ is isomorphic to a subgroup of the galois group of $M$ over $K$ and so is eulerian. Theorem 4.3 implies that $MK$ lies in an eulerian extension of $K$, so $M$ lies in an eulerian extension of $F$. ■

We are now ready to prove the converse of Proposition 2.3. We state this proposition and its converse in the following theorem.

THEOREM 4.6. *Let F be a differential field with algebraically closed field of constants C and let M be a Picard-Vessiot extension of F. A necessary and sufficient condition that M lie in an eulerian extension of F is that the galois group of M of F is eulerian.*

*Proof.* If $M$ lies in an eulerian extension of $F$, then by Proposition 2.3, the galois group of $M$ over $F$ is eulerian.

If the galois group of $M$ over $F$ is eulerian, then by Proposition 2.5, there are subfields $E_0, E_1, \ldots, E_m, E$ of $M$ such that $E_0$ is algebraic over $F$, $M$ is a liouvillian extension of $E$ and $E_0 = E_1 \cdot \cdots \cdot E_m$ where each $E_i$ is a Picard-Vessiot extension of $E_0$ with galois group isomorphic to $\mathbf{SL}(2, C)$ or $\mathbf{PSL}(2, C)$. To show that $M$ lies in an eulerian extension of $F$, it will be enough to show that each $E_i$ lies in an eulerian extension of $E_0$. To do this we shall rely heavily on differential galois cohomology ([Ko 2], p. 421).

If the galois group $G$ of $E_i$ over $E_0$ is isomorphic to $\mathbf{SL}(2, C)$, then the map $f$ identifying $G$ with $\mathbf{SL}(2, C)$ is an element of $H^1(E_i/E_0, \mathbf{SL}(2, C))$. Since $H^1(E_i/E_0, \mathbf{SL}(2, C))$ is trivial ([Ko 2], Corollary 2, p. 425]) there exists an element $y = (y_{ij})$ in $\mathbf{SL}(2, E_i)$ such that for any $\sigma$ in $G$,

$$\sigma(y) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} f(\sigma)$$

for some $a, b, cd$ in $C$ with $ad - bc = 1$. $G$ acts nontrivially on the span of $\{y_{11}, y_{12}\}$ so this span must have dimension 2. Therefore $\text{Wr}(y_{11}, y_{12}) \neq 0$. The coefficients of

$$L(y) = \frac{\mathrm{Wr}(y, y_{11}, y_{12})}{\mathrm{Wr}(y_{11}, y_{12})}$$

are left fixed by all automorphisms in $G$ and so lie in $E_0$. We claim that $E_i = E_0 \langle y_{11}, y_{12} \rangle$. Let $\sigma$ be an element of $G$ leaving $y_{11}$ and $y_{12}$ fixed. $f(\sigma)$ must be the identity matrix, so $\sigma$ must be the identity. Therefore, the subgroup of $G$ leaving $E_0 \langle y_{11}, y_{12} \rangle$ fixed is trivial and so $E_i = E_0 \langle y_{11}, y_{12} \rangle$. This means $E_i$ is an eulerian extension of $E_0$.

Now assume that the galois group $G$ of $E_i$ over $E_0$ is isomorphic to **PSL**(2, $C$). This implies that there exists a map $f$ of $G$ into **SL**(3, $C$) such that the image of $G$ is $\rho_3(\mathbf{SL}(2, C))$. This map is an element of $H^1(E_i/E_0,$ **SL**(3, $C$)). Since $H^1(E_i/E_0,$ **SL**(3, $C$)) is trivial, there exists an element $y = (y_{ij})$ in **SL**(3, $E_i$) such that for all $\sigma$ in $G$, $\sigma(y) = yf(\sigma)$. $G$ acts nontrivially on the span of $\{y_{11}, y_{12}, y_{13}\}$. If this span had dimension less than 3, we would have a representation of **PSL**(2, $C$) of dimension less than 3. Since any such representation is trivial, we get a contradiction. Therefore $\{y_{11}, y_{12}, y_{13}\}$ forms a linearly independent set. Furthermore, any $\sigma$ in $G$ that leaves $y_{11}$, $y_{12}$, and $y_{13}$ fixed must be the identity. Therefore $E_i = E_0 \langle y_{11}, y_{12}, y_{13} \rangle$. The third order linear differential polynomial

$$L(y) = \frac{\mathrm{Wr}(y, y_{11}, y_{12}, y_{13})}{\mathrm{Wr}(y_{11}, y_{12}, y_{13})}$$

has coefficients which are left fixed by all members of G, so these coefficients are in $E_0$. We can now apply Corollary 4.5 to $E_i$ over $E_0$ and conclude that $E_i$ lies in an eulerian extension of $E_0$.    ∎

5. **An example.**    In this section we will show that $y''' - xy = 0$ has no eulerian solutions. We shall give a sufficient condition for a third order homogeneous linear differential equation to have no eulerian solutions and verify that this condition holds for $y''' - xy = 0$. We shall need the concept of the formal adjoint of a linear operator which we now review.

Let $F$ be a differential field and let

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y,$$

$a_i \in F, i = 0, \ldots, n$. The formal adjoint $L^*(y)$ of $L(y)$ is defined to be ([Po] p. 38)

$$L^*(y) = (-1)^n (a_n y)^{(n)} + (-1)^{n-1} (a_{n-1} y)^{(n-1)} + \cdots + a_0 y.$$

For example, if $L(y) = y''' - xy$, then $L*(y) = -y''' - xy$. One can show ([Po] p. 38) that $L**(y)$ (the adjoint of the adjoint of $L(y)$) is just $L(y)$ and if $L(y) = L_k(L_{n-k}(y))$ where $L_k(y)$ and $L_{n-k}(y)$ are of order $k$ and $n - k$ respectively, then $L*(y) = L^*_{n-k}(L^*_k(y))$.

We say that an $n^{\text{th}}$ order linear operator $L(y)$ *has a right first order factor over $F$* if we can write $L(y) = L_{n-1}(L_1(y))$ where $L_{n-1}$ and $L_1$ have coefficients in $F$ and are of order $n - 1$ and $1$ respectively. Note that $L(y)$ has a right first order factor over $F$ if and only if $L(y) = 0$ has a solution $u \neq 0$ in some extension of $F$ such that $u'/u \in F$.

PROPOSITION 5.1.    *Let $F$ be a differential field with algebraically closed field of constants and let $L(y) = y''' - py' - qy$ with $p$, $q$ in $F$. Assume*:

(1)  $L(y) = 0$ *has no nonzero solutions algebraic over $F$.*

(2)  $p'/2 - q \neq 0$

(3)  $L(y), L*(y), L^{\circledS 2}(y)$ *and $L^{\circledS 3}(y)$ all have no right first order factors of $F$.*

*Then $L(y) = 0$ has no nonzero eulerian solutions over $F$.*

*Proof.*    Let $M$ be the Picard-Vessiot extension of $F$ associated with $L(y) = 0$, $G$ be the galois group of $M$ over $F$ and $V$ the solution space of $L(y) = 0$. If $L(y) = 0$ had a nonzero eulerian solution, then Corollary 4.4 implies that $M$ lies in an eulerian extension of $F$ and so $G$ is an eulerian group. Using this assumption and Proposition 4.1 we shall derive a contradiction.

If conclusion (a) of Proposition 4.1 holds, then $M$ would be a finite extension of $F$ and so $L(y) = 0$ would have nonzero algebraic solutions, contradicting (1) above.

If $G^0$ has a one-dimensional weight space, let $X$ be the set of one dimensional weight spaces for $G^0$. Say that $X$ contains $n$ elements, $1 \leq n \leq 3$ and let $y_1, \ldots, y_n$ be bases for the distinct members of $X$. Let $u = \Pi^n_{i=1} y_i$ For any $\sigma$ in $G$, there is a $c_\sigma$ in $C$ such that $\sigma(u) = c_\sigma u$ so $u'/u$ is left fixed by all elements of $G$. Therefore $u$ is solution of either $L(y) = 0$ ($n = 1$), $L^{\circledS 2}(y) = 0$ ($n = 2$) or $L^{\circledS 3}(y) = 0$ ($n = 3$) with $u'/u$ in $F$ contradicting the fact that none of these equations has a right first order factor over $F$.

If $G$ leaves a two-dimensional space invariant, let $y_1, y_2$ be a basis for this space. The differential polynomial

$$L_2(y) = \frac{\text{Wr}(y, y_1, y_2)}{\text{Wr}(y_1, y_2)}$$

therefore has coefficients in $F$ and shares two linearly independent solutions with $L(y) = 0$. Therefore, $L(y) = L_1(L_2(y))$ where $L_1$ is a first order operator with coefficients in $F$. This means that $L^*(y) = L_2^*(L_1^*(y))$, contradicting the fact that $L^*(y)$ has no right first order factors over $F$.

The only possibility left is that conclusion (d) of Proposition 4.1 holds. In this case, there exists a basis $\{y_1, y_2, y_3\}$ for the solution space of $L(y) = 0$ such that the component of the identity $G^0$ of the galois group of $L(y)$ is $\rho_3(\mathbf{SL}(2, C))$ with respect to this basis. In particular $y_1 y_3 - y_2^2$ will be left invariant by all elements in $G^0$. If $y_1 y_3 - y_2^2 = 0$, then Lemma 3.4(c) would imply that $L(y) = L_2^{\circledS 2}(y)$ for some second order $L_2(y)$ with coefficients in $F$. Since $p'/2 - q \neq 0$, Lemma 3.4(b) says that this is impossible. Therefore we can assume that $u = y_1 y_3 - y_2^2 \neq 0$ and that this element lies in the fixed field of $G^0$. If $|G:G^0| = 1$, then $u$ is in $F$. If $|G:G^0| = 3$, then $G = G^0 \times H$ where $H$ is the 3 element subgroup of $\mathbf{SL}(3, C)$ consisting of constant matrices. For any $\sigma \in H$ we have $\sigma(y_i) = c_\sigma y_i$ for $i = 1, 2, 3$ and some $c_\sigma$ in $C$, so $\sigma(y_1 y_3 - y_2^2) = c_\sigma^2(y_1 y_3 - y_2^2)$. Therefore, for any $\sigma$ in $G$ we have $\sigma(u'/u) = u'/u$ and so $u'/u$ is in $F$. In either case, $|G:G^0| = 1$ or $3$, $u'/u$ will be in $F$ and $u$ will satisfy $L^{\circledS 2}(y) = 0$ contradicting the fact that $L^{\circledS 2}(y)$ has no right first order factors.                                                                        ■

We are now ready to show that $L(y) = y''' - xy = 0$ has no eulerian solutions. We shall verify conditions (1), (2) and (3) of Proposition 5.1. Since the coefficients of $L(y)$ are analytic everywhere on the complex plane, any solution of $L(y) = 0$ will also be everywhere analytic. Any algebraic everywhere analytic function is a polynomial and it is easy to see $L(y) = 0$ has no nonzero polynomial solutions. Therefore condition (1) is satisfied. Since $p'/2 - q = -x \neq 0$, condition (2) is also satisfied.

We shall now show that $L(y)$ has no right first order factors over $\mathbf{C}(x)$, where $\mathbf{C}$ is the complex numbers. Assume not and let $u$ be a solution of $y''' - xy = 0$ such that $u'/u$ is in $\mathbf{C}(x)$. If we write $u = \exp(\int v)$ where $v$ is in $\mathbf{C}(x)$, we see that $v$ satisfies

$$(5.1) \qquad\qquad v'' + 3vv' + v^3 - x = 0.$$

Let $v = f/g$ where $f$ and $g$ are in $\mathbf{C}[x]$. Using this expression in (5.1) and clearing denominators, we get:

$$(5.2)\ (f''g - gf'')g - 2gg'f + 2(g')^2 f - 3f(f'g - g'f) + f^3 = g^3 x$$

If deg $f$ > deg $g$, then the term of highest degree on the left hand side of equation (5.2) comes from $f^3$ and is of degree $3(\deg f)$. The term of highest degree on the right hand side of (5.2) is of degree $3(\deg g) + 1 < 3 \deg f$ so these terms cannot cancel. If deg $g \geq$ deg $f$, all terms on the left hand side of (5.2) have degree $\leq 3 \deg g$ while on the right hand side of (5.2) we get a term of degree $3(\deg g) + 1$. This also yields a contradiction so $L(y)$ has no right first order factors.

To show that $L*(y) = -y''' - xy$ has no right first order factors, it is sufficient to show that $y''' + xy$ has none. Since the proof proceeds in the same fashion as the proof for $L(y)$, this proof is omitted.

All that remains is to show that $L^{\circledS 2}(y)$ and $L^{\circledS 3}(y)$ have no right first order factors. To do this we shall use the following Lemma.

LEMMA 5.2.   *Let $y = \exp(\int v)$ with $v$ in $\mathbf{C}(x)$ and let $v = f/g$ with $f$ and $g$ in $\mathbf{C}[x]$.*

(a) *For each $i \geq 1$*

(5.3) $$y^{(i)} = v^i y + y R_i(v, \ldots, v^{(i-1)})$$

*where $R_i$ is a polynomial with rational coefficients and of total degree $\leq i - 1$.*

(b) *If deg $f$ > deg $g$, then for each $i \geq 1$ there exist $c_j$ in $\mathbf{C}$ such that*

$$\frac{y^{(i)}}{y} = c_{im} x^{im} + \sum_{j=-im-1}^{\infty} c_j x^{-j}$$

*where $m = \deg f - \deg g$ and $c_{im} \neq 0$.*

*Proof.*   (a) We proceed by induction on $i$. For $i = 1$, we have $y' = vy$. Assume that (5.3) holds for $i$ and differentiate:

$$y^{(i+1)} = v^{i+1}y + y[iv'v^{i-1} + vR_i(v, \ldots, v^{(i-1)})$$

$$+ (R_i(v, \ldots, v^{(i-1)}))']$$

Since the expression in the brackets is a polynomial with rational coefficients in $v, \ldots, v^{(i)}$ of total degree $\leq i$, we have verified (a) for $i + 1$.

(b) If deg $f$ − deg $g$ = $m$ > 0, we can write $v = a_0 x^m + a$ series involving smaller powers of $x$. Using (a), we see that

$$\frac{y^{(i)}}{y} = v^i + R_i(v, \ldots, v^{(i-1)})$$

The term with the largest power of $x$ appearing on the right hand side of this equation must appear in $v^i$. It is $cx^{im}$ for some $c \neq 0$ and does not cancel with any other term.                                        □

Let us now consider $L^{\circledS 2}(y)$. We can calculate this symmetric power using the recipe given after Lemma 3.1. Doing this we find that

$$(5.4) \qquad L^{\circledS 2}(y) = y^{(vi)} - \frac{1}{x}y^{(v)} + 7xy''' - 7y'' + \frac{7}{x}y' - 8x^2y$$

Note that since $L(y) = 0$ has a basis for its solution space consisting of functions analytic in the complex plane, $L^{\circledS 2}(y) = 0$ will have a basis with the same property. Now assume $L^{\circledS 2}(y) = 0$ has a solution $y = \exp(\int u)$ with $u$ in $\mathbf{C}(x)$ and let $u = f/g$ with $f$ and $g$ in $\mathbf{C}[x]$. If $\deg f \leq \deg g$ then, since $y$ is analytic in the complex plane, we have

$$\frac{f}{g} = c + \Sigma \frac{\alpha_i}{(x - \beta_i)}$$

where $c, \beta_i$ are in $\mathbf{C}$ and the $\alpha_i$'s are positive integers. Therefore, $y$ would be a polynomial in $x$. Since it is easy to see that $L^{\circledS 2}(y) = 0$ has no nonzero polynomial solutions, we get a contradiction. If $\deg f > \deg g$, then we can use Lemma 5.2(b). Dividing (5.4) by $y$, we see that the term $y^{(vi)}/y$ contributes a power of $x$ of degree $6m$ where $m = \deg f - \deg g$. This term is not cancelled by any other term in (5.4) so we get a contradiction. Therefore $L^{\circledS 2}(y)$ has no right first order factor.

Now consider $L^{\circledS 3}(y)$. Again using the recipe following Lemma 3.1 (and the MACSYMA computer system), we found that

$$(5.5) \quad L^{\circledS 3}(y) = (4077x^5 + 2800x)^{-1}[(4077x^5 + 2800x)y^{(x)}$$

$$- (17010x^4 + 2800)y^{(ix)} + 43110x^3y^{(viii)}$$

$$- (110079x^6 + 133080x^2)y^{(vii)} - (85050x^5 + 282240x)y^{(vi)}$$

$$+ (628236x^4 + 282240)y^{(v)} + (110079x^7 - 1735020x^3)y^{(iv)}$$

$$+ (56133x^6 + 2825760x^2)y''' + (288117x^5 - 73920x)y''$$

$$- (2972133x^8 + 2911302x^4 - 73920)y'$$

$$- (4503033x^7 + 8113050x^3)y]$$

Again, $L^{\circledS 3}(y) = 0$ has a basis for its solution space consisting of functions analytic in the complex plane. Let $u = \exp(\int v)$ be a solution of $L^{\circledS 3}(y) = 0$ where $f = f/g, f, g \in \mathbf{C}[x]$. If deg $f \le$ deg $g$, we can see as before that $v$ must be a polynomial and one easily shows that $L^{\circledS 3}(y)$ has no nonzero polynomial solutions. If deg $f >$ deg $g$, we divide by $y$ and see from Lemma 5.2(b) that $y^{(x)}/y$ contributes a power of $x$ that is not cancelled by any other term. Therefore $L^{\circledS 3}(y)$ has no right first order factors.

**6. A decision procedure.** In this section we shall give a procedure to decide if a third order homogeneous linear differential equation $L(y) = 0$ with coefficients in $k_0(x)$, $k_0$ a finitely generated extension of $\mathbf{Q}$, the rationals, has eulerian solutions and if so, find a basis for the vector space of such solutions. To do this we shall show that one can effectively test to see if each of the conclusions of Theorem 4.3 holds. We shall rely heavily on [Si], which describes how one can effectively find liouvillian solutions of homogeneous linear differential equations.

First of all we may assume that $L(y) = 0$ is of the form $y''' - py' - qy = 0$ since, as we have seen, it can always be transformed into such an equation without effecting the property of having or not having eulerian solutions. Using Theorem 4.1 of [Si] we can explicitly find a basis for the space of liouvillian solutions of $L(y) = 0$. In particular, Corollary 4.3 of [Si] allows us to decide if all solutions are algebraic. If $L(y) = L_2(L_1(y))$ where $L_2$ and $L_1$ are of orders 2 and 1 respectively and have coefficients in some algebraic extension of $k_0(x)$, then $L(y) = 0$ will have a liouvillian solution (corresponding to the solution of $L_1(y) = 0$). Therefore Theorem 4.1 of [Si] allows us to decide if conclusion (2) of Theorem 4.3 above holds. If $L(y) = L_1(L_2(y))$ where $L_1$ and $L_2$ are of first and second order with coefficients in $F$, then $L^*(y) = L_2^*(L_1^*(y))$ so Theorem 4.1 of [Si] again will let us decide if this holds. Note that we have actually shown how to decide if $L(y)$ can be written as the composite of lower order operators over any algebraic extension of $k_0(x)$.

We now assume that we have decided that conclusions (1), (2) and (3) of Theorem 4.3 do not hold and wish to decide if conclusion (4) holds. To

do this, we need the following four lemmas. The first two are just effective versions of special cases of results appearing in ([Gr] pp. 18–23). We say a field $k$ is a computable field with splitting algorithm (c.f. [Ra]) if we can effectively perform the field operations and if we can effectively factor polynomials in one variable over $k$. $Q$ is computable with a splitting algorithm. If $k$ is computable with a splitting algorithm, so is any finitely generated extension as is the algebraic closure of $k$. If $k$ is a computable field with splitting algorithm and $k$ is algebraically closed than we can apply elimination theory and decide if any system of polynomial equations with coefficients in $k$ has a solution in $k$ and find such a solution when it exists.

LEMMA 6.1. *Let $k$ be a computable algebraically closed field with splitting algorithm and let $f_1, \ldots, f_e$ be homogeneous polynomials of degree $d$ in $n$ variables with coefficients in $k[x]$, $x$ transcendental over $k$. Assume that $n > ed$. One can then effectively find a nontrivial solution $(g_1, \ldots, g_n)$ in $k[x]$ of $f_1 = 0, \ldots, f_e = 0$.*

*Proof.* Let $T_1, \ldots, T_n$ be the variables in the $f_i$ and let $r$ be the largest power of $x$ appearing in any $f_i$. Define new variables $\{\bar{T}_{ij}\}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, s$ where $s = er$ and let

$$T_i = \bar{T}_{i,0} + \bar{T}_{i,1}x + \cdots + \bar{T}_{i,s}x^s.$$

We then have, for each $j$, $1 \le j \le e$,

$$f_j(T_1, \ldots, T_n) = f_{j,0}(\bar{T}) + f_{j,1}(\bar{T})x + \cdots + f_{j,ds+r}(\bar{T})x^{ds+r}$$

where each $f_{j,k}$ is homogeneous in the $n(s + 1)$ variables $\{\bar{T}_{i,j}\}$. The equations $\{f_{j,k} = 0\}$ form a system of $e(ds + r + 1)$ polynomial equations in $n(s + 1)$ unknowns. If we can show that $n(s + 1) > e(ds + r + 1)$, we could find in $k$ ([Wa], p. 11) a nontrivial zero $\{t_{ij}\}$ of these equations. In this case $t_i = t_{i0} + t_{i1}x + \cdots + t_{is}x^s$, $1 \le i \le n$ will be a nontrivial zero of $f_1 = 0, \ldots, f_e = 0$. In order that $n(s + 1) > e(ds + r + 1)$ we must have $s(n - ed) > er - n$. Since $n - ed > 0$ and $s = er$, we are done. ∎

LEMMA 6.2. *Let $k$ be a computable algebraically closed field with division algorithm and let $K$ be a finite algebraic extension of $k(x)$, $x$ transcendental over $k$. Let $f$ be a homogeneous polynomial of degree $d$ in $n$ variables with coefficients in $K$ and assume that $n > d$. One can then effectively find a nontrivial solution $(g_1, \ldots, g_n)$ in $K$ of $f = 0$.*

*Proof.* Let $w_1, \ldots, w_e$ be a basis for $K$ over $k(x)$ and let $f = f(T_1, \ldots, T_n)$. By clearing denominators, we can assume that $f$ has coefficients in $k[x, w_1, \ldots, w_e]$. Let $T_i = \bar{T}_{i1}w_1 + \cdots + \bar{T}_{ie}w_e$ for $i = 1, \ldots, n$. We can then write $f(T_1, \ldots, T_n) = f_1(\bar{T})w_1 + \cdots + f_e(\bar{T})w_e$ where each $f_i$ is a homogeneous polynomial of degree $d$ in the $en$ variables $\bar{T}_{ij}$. Since $n > d$, we have $en > ed$, so we can apply the previous lemma. ∎

Let $k$ be an algebraically closed field that is computable with a splitting algorithm and let $F$ be a finitely generated algebraic extension of $k(x)$, $x$ transcendental over $k$. $F$ can be made into a differential field with derivation trivial on $k$ and $x' = 1$.

**LEMMA 6.3.** *Let $F$ be as above, let $L(y) = 0$ be a homogeneous linear differential equation with coefficients in $F$ and let $m$ be in $\mathbf{Z}_l$. One can effectively find an integer $N(m)$ such that if $v$ is in $k(x)$ and $y = v^m$ is a solution of $L(y) = 0$, then the degrees of the numerator and denominator of $v$ are bounded by $N(m)$.*

*Proof.* Let $v = c \prod_{i=1}^{\ell}(x - \alpha_i)^{n_i}$ with $c$, $\alpha_i$ in $k$ and $n_i$ in $Z$. We then have

$$\frac{y'}{y} = \frac{1}{m}\frac{v'}{v} = \frac{1}{m}\sum_{i=1}^{\ell}\frac{n_i}{x - \alpha_i}$$

By Proposition 3.6 of [Si] (which also holds when $\mathbf{Q}$ is replaced by $k$), we can find $M$ and $N$ such that the degrees of the numerator and denominator of $y'/y$ are bounded by $M$ and the residue of $y'/y$ at any point has absolute value $\leq N$. Letting $N(m) = mNM$, we get the conclusion of this lemma. ∎

We say a differential field is computable if we can effectively perform the field operations and differentiation. Let $L$ be a linear differential operator with coefficients in a differential field $F$. $L$ is said to be irreducible over $F$ if we cannot write $L$ as the composite of two operators of smaller order with coefficients in $F$. If $L(y) = 0$ shares a nonzero solution with a homogeneous linear equation of smaller order, it is not irreducible ([Po], p. 34).

**LEMMA 6.4.** *Let $F$ be a computable differential field and let $L$ be an $n^{th}$ order linear differential operator with coefficients in $F$. Furthermore, assume that $L$ is irreducible over $F$. Given $a_0, \ldots, a_{n-1}$, not all zero, in $F$,*

*one can effectively find $B_0, \ldots, B_{n-1}$ in $F$ such that if $\{y_1, \ldots, y_n\}$ is a basis for the solution space of $L(y) = 0$, then*

$$
\begin{aligned}
z_1 &= a_0 y_1 + a_1 y_1' + \cdots + a_{n-1} y_1^{(n-1)} \\
&\vdots \\
z_n &= a_0 y_n + a_1 y_n' + \cdots + a_{n-1} y_n^{(n-1)}
\end{aligned}
$$

*is a basis for the solution space of*

$$
z^{(n)} + B_{n-1} z^{(n-1)} + \cdots + B_0 z = 0
$$

*Proof.* Let $y$ be a formal solution of $L(y) = 0$ and let $z = a_{0,0} y + \cdots + a_{0,n-1} y^{(n-1)}$ where $a_{0,i} = a_i$. If we differentiate this $n$ times and use the fact that $L(y) = 0$ we get $n + 1$ equations

$$
\begin{aligned}
z &= a_{0,0} y + \cdots + a_{0,n-1} y^{(n-1)} \\
&\vdots \\
z^{(n)} &= a_{n,0} y + \cdots + a_{n,n-1} y^{(n-1)}
\end{aligned}
$$

where the $a_{ij}$ are in $F$. This gives $n + 1$ linear equations in the $n$ quantities $y, \ldots, y^{(n-1)}$, so we can find $B_0, \ldots, B_{n-1}$ in $F$ such that $z^{(n)} + B_{n-1} z^{(n-1)} + \cdots + B_0 z = 0$. If $y_1, \ldots, y_n$ are linearly independent solutions of $L(y) = 0$ then $z_i = a_0 y_i + a_1 y_i' + \cdots + a_{n-1} y_i^{(n-1)}$ is a solution of $z^{(n)} + B_{n-1} z^{(n-1)} + \cdots B_0 z = 0$ for $i = 1, \ldots, n$. If there exist constants $c_1, \ldots, c_n$, not all zero in $F$ such that $c_1 z_1 + c_2 z_2 + \cdots + c_n z_n = 0$, then

$$
a_{n-1} \left( \sum_{i=1}^{n} c_i y_i \right)^{(n-1)} + a_{n-2} \left( \sum_{i=1}^{n} c_i y_i \right)^{(n-2)} + \cdots + a_0 \left( \sum_{i=1}^{n} c_i y_i \right) = 0
$$

Therefore $\Sigma c_i y_i$ would be a solution of $L(y) = 0$ and a homogeneous linear differential equation of lower order, contradicting the fact that $L$ is irreducible. Therefore $z_1, \ldots, z_n$ are linearly independent. ∎

We are now ready to continue with the decision procedure. We can assume that we have shown that conclusions (a), (b) and (c) of Theorem 4.3 do not hold. In particular, we can assume $L(y)$ is irreducible over any algebraic extension of $k_0(x)$. Furthermore if $L(y) = 0$ is to have nonzero eulerian solutions, then by Theorem 4.3(d), the associated galois group is

either $\rho_3(\mathbf{SL}(2, C))$ or $\rho_3(\mathbf{SL}(2, C)) \times H$ where $H$ is the three element subgroup of $\mathbf{SL}(3, C)$ consisting of constant matrices. We will now use Proposition 4.2 to test if this is the case. Given $L(y)$ it is easy to construct $L_a$ and $L_b$, both linear operators with coefficients in $k_0(x)$, such that if $L(y) = 0$, then $L_a(y') = 0$ and $L_b(y'') = 0$ ([Si]), Lemma 3.8(c)). Therefore $I_0$, $I_1$ and $I_2$ of Proposition 4.2 will be solutions of $L^{\circledS 2}(y) = 0$, $L_a^{\circledS 2}(y) = 0$ and $L_b^{\circledS 2}(y) = 0$ respectively. Since $I_0^3$, $I_1^3$, and $I_2^3$ are in $k_0(x)$, Lemma 6.3 allows us to find an integer $N$ such that the degrees of the numerators and denominators of $I_0^3$, $I_1^3$ and $I_2^3$ are bounded by $N$. Let $f_0(\alpha_1, \ldots, \alpha_{2N+2}, x)$, $f_1(\beta_1, \ldots, \beta_{2N+2}, x)$ and $f_2(\gamma_1, \ldots, \gamma_{2N+2}, x)$ be generic rational functions of $x$, where the numerators and denominators have degree $N$ and where the coefficients of $x$ are indeterminants $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{2N+1})$, $\bar{\beta} = (\beta_1, \ldots, \beta_{2N+1})$ and $\bar{\gamma} = (\gamma_1, \ldots, \gamma_{2N+1})$ respectively. Let $K$ be the algebraic closure of $k_0(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and let $K = k(x, \bar{I}_0, \bar{I}_1, \bar{I}_2)$ where $\bar{I}_i = f_i^{1/3}$ for $i = 0, 1, 2$. Note that $F$ is a computable differential field with splitting algorithm. Using Lemma 6.2, we can find in $K$, a nonzero solution $b_0, b_1, b_2$ of

(6.5)

$$0 = \bar{I}_0 b_0^2 + \bar{I}_1 b_1^2 + \bar{I}_2 b_2^2 + \bar{I}_0' b_1 b_2 + \frac{1}{q}(\bar{I}_2' - p\bar{I}_1')b_0 b_2 + \bar{I}_1' b_1 b_2$$

Note that this solution depends algebraically on the $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$. Using Lemma 6.4, we can construct a differential polynomial $z''' + B_2 z'' + B_1 z' + B_0 z$ with $B_0$, $B_1$ and $B_2$ in $F$, satisfied by $z_i = b_0 y_i + b_1 y_i' + B_2 y_i''$, $i = 1, 2, 3$, where $y_1, y_2, y_3$ are any linearly independent solutions of $L(y) = 0$. The expression

(6.6)    $$B_2 - \frac{1}{54}[27B_1' + 24B_2 B_2' - 9B_2'' - 36B_2 B_1 - 8B_1^2]$$

depends algebraically on the $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$. We can decide if there exist constants in $\bar{k}_0$, the algebraic closure of $k_0$, such that when we substitute them for the $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, (6.6) becomes zero. If such constants exist, then $z''' + B_2 z'' + B_1 z' + B_0 z$ will be the second symmetric power of a second order operator with coefficients in an algebraic extension of $k_0(x)$, by Lemma 3.4(a). If the galois group of $L(y) = 0$ is $\rho_3(\mathbf{SL}(2, C))$ or $\rho_3(\mathbf{SL}(2, C)) \times H$, such constants will exist and furthermore, as we have seen in the proof of Theorem 4.3, there will exist $a_0, a_1, a_2$ in $\bar{k}_0(x, \bar{I}_0, \bar{I}_1, \bar{I}_2)$ such that $y_i =$

$a_0 z_i + a_1 z_i' + a_2 z_i''$ for $i = 1, 2, 3$, where $z_1, z_2, z_3$ are linearly independent solutions of $z''' + B_2 z'' + B_1 z' + B_0 z = 0$ with the $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ specialized to our choice of constants. We must now try to find $a_0, a_1, a_2$. Let $\bar{a}_0$, $\bar{a}_1, \bar{a}_2, z$ be indeterminants, let $y = \bar{a}_0 z + \bar{a}_1 z' + \bar{a}_2 z''$, and let $b_0, b_1$, $b_2$ be a nonzero solution in $K$ of (6.5). If we let $z = b_0 y + b_1 y' + b_2 y''$, we get $y = \bar{a}_0(b_0 y + b_1 y' + b_2 y'') + \bar{a}_1(b_0 y + b_1 y' + b_2 y'') + \bar{a}_2(b_0 y + b_1 y' + b_2 y'')''$. Letting $y$ be a formal solution of $L(y) = 0$, we can replace $y^{(i)}$ with $i > 2$ in this expression by terms involving $y^{(i)}$ with $i \leq 2$. In this way we get an expression of the form $c_2 y'' + c_1 y' + c_0 y = 0$, where the $c_i$ are linear in $\bar{a}_0, \bar{a}_1, \bar{a}_2$ with coefficients in $F$. Since we are assuming that $L(y)$ is irreducible over $F$, we must have $c_0 = c_1 = c_2 = 0$. Again, if the galois group of $L(y) = 0$ is $\rho_3(\mathbf{SL}(2, C))$ or $\rho_3(\mathbf{SL}(2, C)) \times H$, then for some specialization of $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$, this system of linear equations will have a solution $a_0, a_1, a_2$ in $\bar{k}_0(x, \bar{I}_0, \bar{I}_1, \bar{I}_2)$. Conversely, if for some specialization of $\bar{\alpha}, \bar{\beta}$, and $\bar{\gamma}$, (6.6) vanishes and $c_0 = c_1 = c_2 = 0$ has a solution $a_0, a_1, a_2$, then $L(y) = 0$ has linearly independent solutions $y_1, y_2, y_3$ where $y_i = a_0 z_i + a_1 z_i' + \dot{a}_2 z_i''$ with $z_1, z_2, z_3$ linearly independent solutions of a second symmetric power $L_2^{\circledS 2}(z) = 0$. The question of the existence of a specialization of $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ in $\bar{k}_0$ such that (6.6) vanishes and $c_0 = c_1 = c_2 = 0$ has a solution, can be reduced to a question in elimination theory and therefore decided.

## 7. Final comments.

(a) In the 19$^{\text{th}}$ Century and early 20$^{\text{th}}$ Century, T. Clausen, E. Goursat, W. N. Bailey and others investigated the problem of expressing generalized hypergeometric functions $_pF_q$ in terms of products of hypergeometric series $F(a, b, c; x)$, ([Er], p. 185). Theorem 4.3 and its consequences shed some light on this problem and offer a way of discovering other relations among these functions.

(b) Proposition 2.5 gives some insight into the problem of what functions are needed to solve a general homogeneous linear differential equation with coefficients say in $C(x)$. This proposition says that to solve such an equation we can start with algebraic functions, add functions defined by linear differential equations with algebraic function coefficients whose galois groups are simple and then integrate and use the exponential function. Furthermore, using differential galois cohomology as in Theorem 4.6, one can show that the linear differential equations $L_i(y) = 0$ with simple galois groups used here can be chosen so that the action of the ga-

lois group on the solution space of $L_i(y) = 0$ corresponds to the smallest faithful irreducible representation of the corresponding simple group.

In [S-T1], using the uniformization theory of Riemann surfaces and Tit's Theorem concerning free subgroups of linear algebraic groups, M. Tretkoff and the author give another approach to the question of what functions are needed to solve a general homogeneous linear differential equation. In [S-T2], we show that there are homogeneous linear differential equations with coefficients in $\mathbf{C}(x)$, having only algebraic solutions, such that these solutions cannot be expressed in terms of functions built up from $\mathbf{C}(x)$ by successively adjoining solutions of second order linear differential equations having previously defined coefficients (but not allowing arbitrary algebraic extensions).

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