

# Answers for Collection of Mathematics Methods

JTST

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## Section A

### 1. Complex Numbers [7]

- (a) Express the term  $(1+i)^4$  in the form  $re^{i\theta}$ , where  $r$  and  $\theta$  are real variables.  
(b) Express the complex number  $\tan^{-1}(2i)$  in the form  $x+iy$  where  $x, y$  are real.  
(c) Given that  $z = z_1 + z_2$ , where  $z_1 = e^{i\theta_1}$ ,  $z_2 = e^{i\theta_2}$  and  $\theta_1, \theta_2$  are real variables, find an expression for  $|z|$  in terms of  $\Delta\theta = \theta_1 - \theta_2$ .

(a)

$$(1+i)^4 = \left(\sqrt{2}e^{i\pi/4}\right)^4 = \boxed{4e^{i\pi}}$$

- (b) Let  $w = \tan^{-1}(2i)$ , then  $\tan(w) = 2i$ . Using the identity  $\tan(w) = \frac{\sin(w)}{\cos(w)}$ , we have

$$\begin{aligned}\frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} &= 2i \\ e^{iw} - e^{-iw} &= -2i(e^{iw} + e^{-iw}) \\ 3e^{iw} + e^{-iw} &= 0 \\ e^{2iw} &= -\frac{1}{3} \\ 2iw &= \ln\left(-\frac{1}{3}\right) \\ w &= -\frac{i}{2}\left(\ln\frac{1}{3} + i\pi\right) = \boxed{\frac{\pi}{2} + \frac{i}{2}\ln 3}\end{aligned}$$

(c)

$$\begin{aligned}|z|^2 &= z\bar{z} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 \\ &= 2 + e^{i(\theta_1 - \theta_2)} + e^{-i(\theta_1 - \theta_2)} \\ &= 2 + 2\cos(\Delta\theta) \\ \Rightarrow |z| &= \boxed{\sqrt{2(1 + \cos(\Delta\theta))}}\end{aligned}$$

### 2. Vectors [8]

- (a) Write down the equation of the plane

$$3x + 4y + 5z = 10$$

in the vector form

$$\mathbf{r} = \mathbf{r}_0 + t_1\mathbf{a} + t_2\mathbf{b},$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are constant vectors in the plane,  $t_1$  and  $t_2$  are real parameters and  $\mathbf{r}_0$  is a constant vector. What is the distance between the plane and the origin?

- (b) Let  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  be unit vectors. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}).$$

- (a) A particular solution to the plane equation is  $\mathbf{r}_0 = (0, 0, 2)^T$ . Two independent vectors in the plane are  $\mathbf{a} = (0, -2.5, 2)^T$  and  $\mathbf{b} = (-10/3, 0, 2)^T$ . Thus the vector form of the plane is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -2.5 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -10/3 \\ 0 \\ 2 \end{pmatrix}.$$

The distance from the origin to the plane is given by

$$d = \frac{|\mathbf{r}_0 \cdot \mathbf{n}|}{|\mathbf{n}|},$$

where  $\mathbf{n} = (3, 4, 5)$  is the normal vector to the plane. Substituting  $\mathbf{r}_0 = (0, 0, 2)$  and  $\mathbf{n} = (3, 4, 5)$ ,

$$d = \frac{|(0, 0, 2) \cdot (3, 4, 5)|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{|10|}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \boxed{\sqrt{2}}.$$

- (b) Using the vector triple product identity  $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$ , let  $\mathbf{x} = \mathbf{a}$ ,  $\mathbf{y} = \mathbf{b}$  and  $\mathbf{z} = \mathbf{a} \times \mathbf{c}$ , we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{a} \times \mathbf{c})]$$

Since  $(\mathbf{b} \times (\mathbf{a} \times \mathbf{c}))_i = \epsilon_{ijk} b_j (a \times c)_k = \epsilon_{ijk} b_j \epsilon_{klm} a_k c_m = \epsilon_{ijk} \epsilon_{klm} b_j a_k c_m$ , and using the identity  $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$ , we get  $\epsilon_{ijk} \epsilon_{klm} b_j a_k c_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) b_j a_k c_m = b_j a_i c_m \delta_{il} \delta_{jm} - b_j a_k c_m \delta_{im} \delta_{jl} = a_i (b_j c_j) - c_i (a_j b_j)$ , which is just  $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ . Continuing from above,

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) &= \mathbf{a} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] \\ &= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b}) \\ &= 1 \cdot (\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \end{aligned}$$

Q.E.D.

### 3. Matrix and linear equation [5]

Consider the set of linear equations

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4, \end{aligned}$$

where  $x, y, z$  are real variables and  $a, b$  are real parameters.

Find the values of  $a$  and  $b$  for which the set of equations have:

- (i) a unique solution,
- (ii) infinitely many solutions,
- (iii) no solution.

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4. \end{aligned}$$

Let the coefficient matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2b \\ 2a \\ 4 \end{pmatrix}.$$

The determinant of  $A$  is

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2(a - 6).$$

- (i) **Unique solution.** If  $\det A \neq 0$ , i.e.  $a \neq 6$ , the system has a unique solution for all values of  $b$ .

(ii) **Infinitely many solutions.** Let  $a = 6$ . The system becomes

$$\begin{aligned}2x + y + z &= 2b, \\6x + 3y + 2z &= 12, \\2x + y + 3z &= 4.\end{aligned}$$

Subtracting the first equation from the third gives

$$2z = 4 - 2b \quad \Rightarrow \quad z = 2 - b.$$

Dividing the second equation by 3 yields

$$2x + y + \frac{2}{3}z = 4.$$

From the first equation,  $2x + y = 2b - z$ . Substituting,

$$2b - z + \frac{2}{3}z = 4 \quad \Rightarrow \quad 2b - \frac{1}{3}z = 4 \quad \Rightarrow \quad z = 6b - 12.$$

Consistency requires

$$2 - b = 6b - 12 \quad \Rightarrow \quad b = 2.$$

Hence  $z = 0$  and the remaining equation is

$$2x + y = 4.$$

Letting  $x = t$ , the solutions are

$$(x, y, z) = (t, 4 - 2t, 0), \quad t \in \mathbb{R},$$

so there are infinitely many solutions when  $(a, b) = (6, 2)$ .

(iii) **No solution.** If  $a = 6$  and  $b \neq 2$ , the two expressions for  $z$  are inconsistent. Hence the system has no solution.

Unique solution	$a \neq 6$ (any $b$ ),
Infinitely many solutions	$a = 6, b = 2,$
No solution	$a = 6, b \neq 2.$

#### 4. Differential equation [5]

Find the general solution for the differential equation

$$\frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = n(n+1)y,$$

where  $x$  is a real variable and  $n$  is a real constant.

Using the product rule, we have

$$\frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = 2x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2}.$$

Thus the differential equation can be rewritten as

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0.$$

This is an Euler-Cauchy equation. We try a solution of the form  $y = x^m$ , where  $m$  is a constant to be determined. Substituting this into the differential equation, we get

$$x^2 \cdot m(m-1)x^{m-2} + 2x \cdot mx^{m-1} - n(n+1)x^m = 0.$$

Simplifying, we have  $x^m [m(m-1) + 2m - n(n+1)] = 0$ , which gives us the characteristic equation

$$m^2 + m - n(n+1) = 0.$$

Solving for  $m$ , we get  $m = \frac{-1 \pm (2n+1)}{2}$ , so  $m_1 = n$  and  $m_2 = -(n+1)$ .

The general solution is then

$$y(x) = Ax^n + Bx^{-(n+1)},$$

where  $A$  and  $B$  are arbitrary constants.

## 5. Matrix and properties [7]

Let  $A$  and  $B$  be  $n \times n$  Hermitian matrices and  $U$  an  $n \times n$  unitary matrix.

- (a) Show that the modulus of each of the eigenvalues of  $U$  is equal to one ( $|\lambda| = 1$ ).
- (b) Show that the eigenvalues of  $A$  are real.
- (c) Assuming that  $U = A + iB$ , show that
  - (i)  $A^2 + B^2 = I$ , where  $I$  is the identity matrix,
  - (ii)  $AB - BA = 0$ .

- (a) Let  $\lambda$  be an eigenvalue of  $U$  with corresponding eigenvector  $\mathbf{v}$ , so that  $U\mathbf{v} = \lambda\mathbf{v}$ . Taking the conjugate transpose of both sides, we have

$$\mathbf{v}^\dagger U^\dagger = \lambda^* \mathbf{v}^\dagger.$$

$$\mathbf{v}^\dagger U^\dagger U = \mathbf{v}^\dagger = \lambda^* \mathbf{v}^\dagger U.$$

Multiplying both sides by  $U\mathbf{v}$  from the right, we get

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda^* \mathbf{v}^\dagger U U \mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda \mathbf{v}^\dagger \mathbf{v}.$$

Equating the two expressions for  $\mathbf{v}^\dagger U\mathbf{v}$ , we have

$$\lambda \mathbf{v}^\dagger \mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

Since  $\mathbf{v}$  is a non-zero eigenvector,  $\mathbf{v}^\dagger \mathbf{v} \neq 0$ , we can divide both sides by  $\mathbf{v}^\dagger \mathbf{v}$  to get

$$\lambda = \lambda^*.$$

Thus, we have

$$|\lambda|^2 = \lambda \lambda^* = 1.$$

Q.E.D.

- (b) Let  $\mu$  be an eigenvalue of  $A$  with corresponding eigenvector  $\mathbf{w}$ , so that  $A\mathbf{w} = \mu\mathbf{w}$ . Taking the conjugate transpose of both sides, we have

$$\mathbf{w}^\dagger A^\dagger = \mu^* \mathbf{w}^\dagger.$$

Since  $A$  is Hermitian,  $A^\dagger = A$ . Thus,

$$\mathbf{w}^\dagger A = \mu^* \mathbf{w}^\dagger.$$

Multiplying both sides by  $\mathbf{w}$  from the right, we get

$$\mathbf{w}^\dagger A\mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{w}^\dagger A\mathbf{w} = \mu \mathbf{w}^\dagger \mathbf{w}.$$

Equating the two expressions for  $\mathbf{w}^\dagger A\mathbf{w}$ , we have

$$\mu \mathbf{w}^\dagger \mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

Since  $\mathbf{w}$  is a non-zero eigenvector,  $\mathbf{w}^\dagger \mathbf{w} \neq 0$ , we can divide both sides by  $\mathbf{w}^\dagger \mathbf{w}$  to get

$$\mu = \mu^*.$$

Q.E.D.

- (c) Given that  $U = A + iB$  is unitary, we have

$$U^\dagger U = I.$$

Calculating  $U^\dagger U$ , we get

$$(A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2 + i(AB - BA).$$

Setting this equal to the identity matrix  $I$ , we have

$$A^2 + B^2 + i(AB - BA) = I.$$

Q.E.D.

## 6. Matrix and geometry [8]

The rotation matrix  $A$  in  $\mathbb{R}^3$  is given by

$$A = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}.$$

- (a) Show that the matrix  $A$  is orthogonal.  
 (b) Calculate  $\cos \theta$ , where  $\theta$  is the angle of rotation, and find a unit vector in the direction of the axis of rotation.

- (a) To show that  $A$  is orthogonal, we need to verify that  $A^T A = I$ . Calculating  $A^T$ ,

$$A^T = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \\ -1 & -\sqrt{2} & 1 \end{pmatrix}.$$

Now, calculating  $A^T A$ ,

$$A^T A = \frac{1}{4} \begin{pmatrix} 2+0+2 & -\sqrt{2}+0-\sqrt{2} & -\sqrt{2}+0-\sqrt{2} \\ -\sqrt{2}+0-\sqrt{2} & 1+2+1 & 1-2+1 \\ -\sqrt{2}+0-\sqrt{2} & 1-2+1 & 1+2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus,  $A$  is orthogonal.

- (b) For an orthogonal matrix representing a rotation in  $\mathbb{R}^3$ , in 2D dimensions, the matrix can be represented by

$$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus Trace can be used to find the angle of rotation. For a rotation matrix in  $\mathbb{R}^3$ ,

$$\text{tr}(A) = 1 + 2 \cos \theta.$$

Here

$$\text{tr}(A) = \frac{1}{2}(\sqrt{2} + \sqrt{2} + 1) = \sqrt{2} + \frac{1}{2},$$

so

$$\cos \theta = \frac{\text{tr}(A) - 1}{2} = \frac{(\sqrt{2} + \frac{1}{2}) - 1}{2} = \frac{\sqrt{2}}{2} - \frac{1}{4}.$$

This gives

$$\theta = \cos^{-1} \left( \frac{\sqrt{2}}{2} - \frac{1}{4} \right) \approx \boxed{62.8^\circ}.$$

The rotation axis is the eigenspace for eigenvalue 1, i.e. solutions of

$$(A - I)\mathbf{v} = 0.$$

A nonzero solution is

$$\mathbf{v} = \begin{pmatrix} 1 + \sqrt{2} \\ -(1 + \sqrt{2}) \\ 1 \end{pmatrix},$$

so a unit vector along the axis is

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{(1 + \sqrt{2})^2 + (1 + \sqrt{2})^2 + 1}} \begin{pmatrix} 1 + \sqrt{2} \\ -(1 + \sqrt{2}) \\ 1 \end{pmatrix} = \boxed{\frac{1}{\sqrt{7 + 4\sqrt{2}}} \begin{pmatrix} 1 + \sqrt{2} \\ -(1 + \sqrt{2}) \\ 1 \end{pmatrix}}.$$

## Section B