Answers for Collection of Mathematics Methods

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3. Februar 2026

Section A

1. **Complex Numbers** [7]

   (a) Express the term $(1+i)^4$ in the form $re^{i\theta}$, where r and θ are real variables.
   (b) Express the complex number $\tan^{-1}(2i)$ in the form $x+iy$ where x, y are real.
   (c) Given that $z = z_1 + z_2$, where $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are real variables, find an expression for $|z|$ in terms of $\Delta\theta = \theta_1 - \theta_2$.

   (a)

$$(1+i)^4 = \left(\sqrt{2}e^{i\pi/4}\right)^4 = \boxed{4e^{i\pi}}$$

   (b) Let $w = \tan^{-1}(2i)$, then $\tan(w) = 2i$. Using the identity $\tan(w) = \dfrac{\sin(w)}{\cos(w)}$, we have

$$\begin{aligned}
\frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} &= 2i \\
e^{iw} - e^{-iw} &= -2i(e^{iw} + e^{-iw}) \\
3e^{iw} + e^{-iw} &= 0 \\
e^{2iw} &= -\frac{1}{3} \\
2iw &= \ln\left(-\frac{1}{3}\right) \\
w &= -\frac{i}{2}\left(\ln\frac{1}{3} + i\pi\right) = \boxed{\frac{\pi}{2} + \frac{i}{2}\ln 3}
\end{aligned}$$

   (c)

$$\begin{aligned}
|z|^2 &= z\bar{z} = (z_1 + z_2)(\bar{z_1} + \bar{z_2}) \\
&= |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + \bar{z_1}z_2 \\
&= 2 + e^{i(\theta_1 - \theta_2)} + e^{-i(\theta_1 - \theta_2)} \\
&= 2 + 2\cos(\Delta\theta) \\
\Rightarrow |z| &= \boxed{\sqrt{2\left(1 + \cos(\Delta\theta)\right)}}
\end{aligned}$$

2. **Vectors** [8]

   (a) Write down the equation of the plane

$$3x + 4y + 5z = 10$$

   in the vector form

$$\mathbf{r} = \mathbf{r}_0 + t_1\mathbf{a} + t_2\mathbf{b},$$

   where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors in the plane, t_1 and t_2 are real parameters and $\mathbf{r}_0$ is a constant vector. What is the distance between the plane and the origin?
   (b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}).$$

(a) A particular solution to the plane equation is $\mathbf{r_0} = (0, 0, 2)^T$. Two independent vectors in the plane are $\mathbf{a} = (0, -2.5, 2)^T$ and $\mathbf{b} = (-10/3, 0, 2)^T$. Thus the vector form of the plane is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -2.5 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -10/3 \\ 0 \\ 2 \end{pmatrix}.$$

The distance from the origin to the plane is given by

$$d = \frac{|\mathbf{r_0} \cdot \mathbf{n}|}{|\mathbf{n}|},$$

where $\mathbf{n} = (3, 4, 5)$ is the normal vector to the plane. Substituting $\mathbf{r_0} = (0, 0, 2)$ and $\mathbf{n} = (3, 4, 5)$,

$$d = \frac{|(0,0,2) \cdot (3,4,5)|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{|10|}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \boxed{\sqrt{2}}.$$

(b) Using the vector triple product identity $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$, let $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$ and $\mathbf{z} = \mathbf{a} \times \mathbf{c}$, we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{a} \times \mathbf{c})]$$

Since $(\mathbf{b} \times (\mathbf{a} \times \mathbf{c}))_i = \epsilon_{ijk} b_j (a \times c)_k = \epsilon_{ijk} b_j \epsilon_{klm} a_k c_m = \epsilon_{ijk} \epsilon_{klm} b_j a_k c_m$, and using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, we get $\epsilon_{ijk}\epsilon_{klm} b_j a_k c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) b_j a_k c_m = b_j a_i c_m \delta_{il}\delta_{jm} - b_j a_k c_m \delta_{im}\delta_{jl} = a_i(b_j c_j) - c_i(a_j b_j)$, which is just $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Continuing from above,

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) &= \mathbf{a} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] \\
&= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b}) \\
&= 1 \cdot (\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \\
&= \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})
\end{aligned}$$

Q.E.D.

3. **Matrix and linear equation** [5]

Consider the set of linear equations

$$\begin{aligned}
2x + y + z &= 2b, \\
ax + 3y + 2z &= 2a, \\
2x + y + 3z &= 4,
\end{aligned}$$

where x, y, z are real variables and a, b are real parameters.

Find the values of a and b for which the set of equations have:

(i) a unique solution,

(ii) infinitely many solutions,

(iii) no solution.

$$\begin{aligned}
2x + y + z &= 2b, \\
ax + 3y + 2z &= 2a, \\
2x + y + 3z &= 4.
\end{aligned}$$

Let the coefficient matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2b \\ 2a \\ 4 \end{pmatrix}.$$

The determinant of A is

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2(a - 6).$$

(i) **Unique solution.** If $\det A \neq 0$, i.e. $a \neq 6$, the system has a unique solution for all values of b.

(ii) **Infinitely many solutions.** Let $a = 6$. The system becomes

$$\begin{aligned} 2x + y + z &= 2b, \\ 6x + 3y + 2z &= 12, \\ 2x + y + 3z &= 4. \end{aligned}$$

Subtracting the first equation from the third gives

$$2z = 4 - 2b \quad \Rightarrow \quad z = 2 - b.$$

Dividing the second equation by 3 yields

$$2x + y + \frac{2}{3}z = 4.$$

From the first equation, $2x + y = 2b - z$. Substituting,

$$2b - z + \frac{2}{3}z = 4 \quad \Rightarrow \quad 2b - \frac{1}{3}z = 4 \quad \Rightarrow \quad z = 6b - 12.$$

Consistency requires

$$2 - b = 6b - 12 \quad \Rightarrow \quad b = 2.$$

Hence $z = 0$ and the remaining equation is

$$2x + y = 4.$$

Letting $x = t$, the solutions are

$$(x, y, z) = (t,\, 4 - 2t,\, 0), \qquad t \in \mathbb{R},$$

so there are infinitely many solutions when $(a, b) = (6, 2)$.

(iii) **No solution.** If $a = 6$ and $b \neq 2$, the two expressions for z are inconsistent. Hence the system has no solution.

Unique solution	$a \neq 6$ (any b),
Infinitely many solutions	$a = 6,\ b = 2$,
No solution	$a = 6,\ b \neq 2$.

4. **Differential equation** [5]

Find the general solution for the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{dy}}{\mathrm{d}x}\right) = n(n+1)y,$$

where x is a real variable and n is a real constant.

Using the product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{dy}}{\mathrm{d}x}\right) = 2x\frac{\mathrm{dy}}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

Thus the differential equation can be rewritten as

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{dy}}{\mathrm{d}x} - n(n+1)y = 0.$$

This is an Euler-Cauchy equation. We try a solution of the form $y = x^m$, where m is a constant to be determined. Substituting this into the differential equation, we get

$$x^2 \cdot m(m-1)x^{m-2} + 2x \cdot mx^{m-1} - n(n+1)x^m = 0.$$

Simplifying, we have $x^m\left[m(m-1) + 2m - n(n+1)\right] = 0$, which gives us the characteristic equation

$$m^2 + m - n(n+1) = 0.$$

Solving for m, we get $m = \frac{-1 \pm (2n+1)}{2}$, so $m_1 = n$ and $m_2 = -(n+1)$.
The general solution is then

$$y(x) = Ax^n + Bx^{-(n+1)},$$

where A and B are arbitrary constants.

5. **Matrix and properties** [7]

Let A and B be $n \times n$ Hermitian matrices and U an $n \times n$ unitary matrix.

(a) Show that the modulus of each of the eigenvalues of U is equal to one ($|\lambda| = 1$).

(b) Show that the eigenvalues of A are real.

(c) Assuming that $U = A + iB$, show that

(i) $A^2 + B^2 = I$, where I is the identity matrix,

(ii) $AB - BA = 0$.

(a) Let λ be an eigenvalue of U with corresponding eigenvector $\mathbf{v}$, so that $U\mathbf{v} = \lambda\mathbf{v}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{v}^\dagger U^\dagger = \lambda^* \mathbf{v}^\dagger.$$

$$\mathbf{v}^\dagger U^\dagger U = \mathbf{v}^\dagger = \lambda^* \mathbf{v}^\dagger U.$$

Multiplying both sides by $U\mathbf{v}$ from the right, we get

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda^* \mathbf{v}^\dagger UU\mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda \mathbf{v}^\dagger \mathbf{v}.$$

Equating the two expressions for $\mathbf{v}^\dagger U\mathbf{v}$, we have

$$\lambda \mathbf{v}^\dagger \mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

Since $\mathbf{v}$ is a non-zero eigenvector, $\mathbf{v}^\dagger \mathbf{v} \neq 0$, we can divide both sides by $\mathbf{v}^\dagger \mathbf{v}$ to get

$$\lambda = \lambda^*.$$

Thus, we have

$$|\lambda|^2 = \lambda\lambda^* = 1.$$

Q.E.D.

(b) Let μ be an eigenvalue of A with corresponding eigenvector $\mathbf{w}$, so that $A\mathbf{w} = \mu\mathbf{w}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{w}^\dagger A^\dagger = \mu^* \mathbf{w}^\dagger.$$

Since A is Hermitian, $A^\dagger = A$. Thus,

$$\mathbf{w}^\dagger A = \mu^* \mathbf{w}^\dagger.$$

Multiplying both sides by $\mathbf{w}$ from the right, we get

$$\mathbf{w}^\dagger A\mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{w}^\dagger A\mathbf{w} = \mu \mathbf{w}^\dagger \mathbf{w}.$$

Equating the two expressions for $\mathbf{w}^\dagger A\mathbf{w}$, we have

$$\mu \mathbf{w}^\dagger \mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

Since $\mathbf{w}$ is a non-zero eigenvector, $\mathbf{w}^\dagger \mathbf{w} \neq 0$, we can divide both sides by $\mathbf{w}^\dagger \mathbf{w}$ to get

$$\mu = \mu^*.$$

Q.E.D.

(c) Given that $U = A + iB$ is unitary, we have

$$U^\dagger U = I.$$

Calculating $U^\dagger U$, we get

$$(A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2 + i(AB - BA).$$

Setting this equal to the identity matrix I, we have

$$A^2 + B^2 + i(AB - BA) = I.$$

Q.E.D.

6. **Matrix and geometry** [8]

The rotation matrix A in $\mathbb{R}^3$ is given by

$$A = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}.$$

(a) Show that the matrix A is orthogonal.

(b) Calculate $\cos\theta$, where θ is the angle of rotation, and find a unit vector in the direction of the axis of rotation.

(a) To show that A is orthogonal, we need to verify that $A^T A = I$. Calculating A^T,

$$A^T = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \\ -1 & -\sqrt{2} & 1 \end{pmatrix}.$$

Now, calculating $A^T A$,

$$A^T A = \frac{1}{4}\begin{pmatrix} 2+0+2 & -\sqrt{2}+0-\sqrt{2} & -\sqrt{2}+0-\sqrt{2} \\ -\sqrt{2}+0-\sqrt{2} & 1+2+1 & 1-2+1 \\ -\sqrt{2}+0-\sqrt{2} & 1-2+1 & 1+2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus A is orthogonal.

(b) For an orthogonal matrix representing a rotation in $\mathbb{R}^3$, in 2D dimensions, the matrix can be represented by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus Trace can be used to find the angle of rotation. For a rotation matrix in $\mathbb{R}^3$,

$$\mathrm{tr}(A) = 1 + 2\cos\theta.$$

Here

$$\mathrm{tr}(A) = \frac{1}{2}(\sqrt{2}+\sqrt{2}+1) = \sqrt{2} + \frac{1}{2},$$

so

$$\cos\theta = \frac{\mathrm{tr}(A)-1}{2} = \frac{\left(\sqrt{2}+\frac{1}{2}\right)-1}{2} = \frac{\sqrt{2}}{2} - \frac{1}{4}.$$

This gives

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} - \frac{1}{4}\right) \approx \boxed{62.8^\circ}.$$

The rotation axis is the eigenspace for eigenvalue 1, i.e. solutions of

$$(A - I)\mathbf{v} = 0.$$

A nonzero solution is

$$\mathbf{v} = \begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix},$$

so a unit vector along the axis is

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{(1+\sqrt{2})^2 + (1+\sqrt{2})^2 + 1}}\begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix} = \boxed{\frac{1}{\sqrt{7+4\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix}}.$$

Section B

Section B

7.
(a) State de Moivre's theorem and show that

(i)
$$\sum_{n=0}^{N-1} \cos n\theta = \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cos \frac{(N-1)\theta}{2},$$

(ii)
$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,$$

where n, N are integers and θ is a complex variable. [8]

(b) Find all the roots for the equation

$$\left(\frac{z-1}{z+1}\right)^n = -1. \qquad (\dagger)$$

Verify your solution for the special case $n = 3$, by finding the roots of the resulting third order equation.

Use the general solution to ($\dagger$) to calculate the product

$$\prod_{r=1}^{n} \cot\left(\frac{(2r+1)\pi}{2n}\right),$$

where r is an integer, for both odd and even values of n. [8]

(c) Show that if the complex numbers z and u satisfy the relation

$$\left|\frac{z+u}{z+u^*}\right| = 1,$$

then either u or z must be real.

[The ($*$) stands for the complex conjugate.] [4]

8. The differential equation for the displacement $y(t)$ of a particle executing forced and damped harmonic oscillations with damping factor γ and natural frequency ω_0 may be written as

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = F\cos\omega t,$$

where F and ω are the amplitude and frequency of the driving force respectively.

(a) Assuming that $F = 0$, find the displacements $y(t)$ of the particle for the cases $\omega_0 < \gamma$ and $\omega_0 = \gamma$. Sketch and compare the two displacements. [6]

(b) Assume now that $F = F_0 \neq 0$ and $\gamma < \omega_0$. Explain what is meant by a steady state solution of the differential equation and find an expression for the steady state amplitude and phase of the displacement.

For a given value of the natural frequency ω_0, which value of the driving force frequency ω maximises the displacement? For what value of ω is the velocity a maximum? [6]

(c) Explain what is meant by the width of the oscillator resonance. Calculate the width of the resonance for the case $\gamma \ll \omega_0$. [4]

9. The matrix

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ \alpha - 3 & 0 & 1 \\ 2 & -1 & \alpha + 1 \end{pmatrix}$$

defines the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x) = Ax$, where $x \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

Let A_4 be equal to the matrix A for $\alpha = 4$.

(a) Find a basis for $\ker(f)$ and show that the geometry of the kernel is a straight line in $\mathbb{R}^3$. Write the equation of the line in vector form. [4]

(b) By choosing $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a basis of $\mathbb{R}^3$ with $\mathbf{e}_1 \in \ker(f)$, show that the geometry of the image of f is a plane and find the direction of the normal to this plane. [5]

Now let A_0 be equal to A for $\alpha = 0$.

(c) Assuming that the matrix A_0 was calculated with respect to the basis

$$\mathbf{u}_1 = (1, 0, 0)^T, \quad \mathbf{u}_2 = (0, 1, 0)^T, \quad \mathbf{u}_3 = (0, 0, 1)^T,$$

express the map $f(x)$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the coordinates of the vector $x = (x_1, x_2, x_3)^T$. [3]

(d) Assume now that the matrix A_0' of the map f is calculated with respect to the basis

$$\mathbf{w}_1 = (1, 1, 0)^T, \quad \mathbf{w}_2 = (1, 0, 1)^T, \quad \mathbf{w}_3 = (0, 1, 1)^T.$$

Calculate the matrix A_0' from the relation

$$A_0' = C A_0 C^{-1}$$

by a suitable choice of the matrix C. [8]

10.

(a) The components of a vector $\mathbf{a}$ in the (x, y) plane of a Cartesian coordinate system (x, y, z) are given by (a_x, a_y). Assume now that the (x, y) axes are rotated about the z axis by an angle θ, anticlockwise, so that the components of $\mathbf{a}$ with respect to the rotated coordinate system (x', y', z) are given by (a_x', a_y').

Calculate the elements of the matrix R that relates the vector $(a_x', a_y')^T$ to the vector $(a_x, a_y)^T$ and show that R is a rotation matrix. Find the eigenvectors of the matrix R. [7]

(b) The equation of a conical section in the (x, y) coordinate system in $\mathbb{R}^2$ is given by

$$f(x, y) = x^2 + 6xy + y^2 = 4. \qquad (*)$$

Write down the above equation in the matrix form

$$\mathbf{x}^T M \mathbf{x} = 4$$

where M is a symmetric matrix and $\mathbf{x} = (x, y)^T$ is a coordinate vector in $\mathbb{R}^2$. Use matrix diagonalisation to show that the curve in $(*)$ represents a hyperbola. Find the elements of the unitary matrix that diagonalises the matrix M.

Sketch this curve showing the asymptotes and the points of intersection with the (x, y) axes. [8]