Answers for Collection of *Mathematics Methods*

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Section A

1. **Complex Numbers** [7]

   (a) Express the term $(1+i)^4$ in the form $re^{i\theta}$, where r and θ are real variables.
   (b) Express the complex number $\tan^{-1}(2i)$ in the form $x+iy$ where x, y are real.
   (c) Given that $z = z_1 + z_2$, where $z_1 = \mathrm{e}^{i\theta_1}$, $z_2 = \mathrm{e}^{i\theta_2}$ and θ_1, θ_2 are real variables, find an expression for $|z|$ in terms of $\Delta\theta = \theta_1 - \theta_2$.

(a)

$$(1+i)^4 = \left(\sqrt{2}\mathrm{e}^{i\pi/4}\right)^4 = \boxed{4\mathrm{e}^{i\pi}}$$

(b) Let $w = \tan^{-1}(2i)$, then $\tan(w) = 2i$. Using the identity $\tan(w) = \dfrac{\sin(w)}{\cos(w)}$, we have

$$\begin{aligned}
\frac{\mathrm{e}^{iw} - \mathrm{e}^{-iw}}{i(\mathrm{e}^{iw} + \mathrm{e}^{-iw})} &= 2i \\
\mathrm{e}^{iw} - \mathrm{e}^{-iw} &= -2i(\mathrm{e}^{iw} + \mathrm{e}^{-iw}) \\
3\mathrm{e}^{iw} + \mathrm{e}^{-iw} &= 0 \\
\mathrm{e}^{2iw} &= -\frac{1}{3} \\
2iw &= \ln\left(-\frac{1}{3}\right) \\
w &= -\frac{i}{2}\left(\ln\frac{1}{3} + i\pi\right) = \boxed{\frac{\pi}{2} + \frac{i}{2}\ln 3}
\end{aligned}$$

(c)

$$\begin{aligned}
|z|^2 &= z\bar{z} = (z_1 + z_2)(\bar{z_1} + \bar{z_2}) \\
&= |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + \bar{z_1}z_2 \\
&= 2 + \mathrm{e}^{i(\theta_1 - \theta_2)} + \mathrm{e}^{-i(\theta_1 - \theta_2)} \\
&= 2 + 2\cos(\Delta\theta) \\
\Rightarrow |z| &= \boxed{\sqrt{2\,(1 + \cos(\Delta\theta))}}
\end{aligned}$$

2. **Vectors** [8]

   (a) Write down the equation of the plane

   $$3x + 4y + 5z = 10$$

   in the vector form

   $$\mathbf{r} = \mathbf{r}_0 + t_1\mathbf{a} + t_2\mathbf{b},$$

   where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors in the plane, t_1 and t_2 are real parameters and $\mathbf{r}_0$ is a constant vector. What is the distance between the plane and the origin?
   (b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors. Show that

   $$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}).$$

(a) A particular solution to the plane equation is $\mathbf{r_0} = (0, 0, 2)^T$. Two independent vectors in the plane are $\mathbf{a} = (0, -2.5, 2)^T$ and $\mathbf{b} = (-10/3, 0, 2)^T$. Thus the vector form of the plane is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -2.5 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -10/3 \\ 0 \\ 2 \end{pmatrix}.$$

The distance from the origin to the plane is given by

$$d = \frac{|\mathbf{r_0} \cdot \mathbf{n}|}{|\mathbf{n}|},$$

where $\mathbf{n} = (3, 4, 5)$ is the normal vector to the plane. Substituting $\mathbf{r_0} = (0, 0, 2)$ and $\mathbf{n} = (3, 4, 5)$,

$$d = \frac{|(0, 0, 2) \cdot (3, 4, 5)|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{|10|}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \boxed{\sqrt{2}}.$$

(b) Using the vector triple product identity $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$, let $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$ and $\mathbf{z} = \mathbf{a} \times \mathbf{c}$, we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{a} \times \mathbf{c})]$$

Since $(\mathbf{b} \times (\mathbf{a} \times \mathbf{c}))_i = \epsilon_{ijk} b_j (a \times c)_k = \epsilon_{ijk} b_j \epsilon_{klm} a_k c_m = \epsilon_{ijk}\epsilon_{klm} b_j a_k c_m$, and using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, we get $\epsilon_{ijk}\epsilon_{klm} b_j a_k c_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) b_j a_k c_m = b_j a_i c_m \delta_{il}\delta_{jm} - b_j a_k c_m \delta_{im}\delta_{jl} = a_i(b_j c_j) - c_i(a_j b_j)$, which is just $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Continuing from above,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) &= \mathbf{a} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] \\ &= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b}) \\ &= 1 \cdot (\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})\end{aligned}$$

Q.E.D.

3. **Matrix and linear equation** [5]

Consider the set of linear equations

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4, \end{aligned}$$

where x, y, z are real variables and a, b are real parameters.

Find the values of a and b for which the set of equations have:

(i) a unique solution,

(ii) infinitely many solutions,

(iii) no solution.

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4. \end{aligned}$$

Let the coefficient matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2b \\ 2a \\ 4 \end{pmatrix}.$$

The determinant of A is

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2(a - 6).$$

(i) **Unique solution.** If $\det A \neq 0$, i.e. $a \neq 6$, the system has a unique solution for all values of b.

(ii) **Infinitely many solutions.** Let $a = 6$. The system becomes

$$\begin{aligned} 2x + y + z &= 2b, \\ 6x + 3y + 2z &= 12, \\ 2x + y + 3z &= 4. \end{aligned}$$

Subtracting the first equation from the third gives

$$2z = 4 - 2b \quad \Rightarrow \quad z = 2 - b.$$

Dividing the second equation by 3 yields

$$2x + y + \frac{2}{3}z = 4.$$

From the first equation, $2x + y = 2b - z$. Substituting,

$$2b - z + \frac{2}{3}z = 4 \quad \Rightarrow \quad 2b - \frac{1}{3}z = 4 \quad \Rightarrow \quad z = 6b - 12.$$

Consistency requires

$$2 - b = 6b - 12 \quad \Rightarrow \quad b = 2.$$

Hence $z = 0$ and the remaining equation is

$$2x + y = 4.$$

Letting $x = t$, the solutions are

$$(x, y, z) = (t,\, 4 - 2t,\, 0), \qquad t \in \mathbb{R},$$

so there are infinitely many solutions when $(a, b) = (6, 2)$.

(iii) **No solution.** If $a = 6$ and $b \neq 2$, the two expressions for z are inconsistent. Hence the system has no solution.

Unique solution	$a \neq 6$ (any b),
Infinitely many solutions	$a = 6,\ b = 2$,
No solution	$a = 6,\ b \neq 2$.

4. **Differential equation** [5]

Find the general solution for the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) = n(n+1)y,$$

where x is a real variable and n is a real constant.

Using the product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

Thus the differential equation can be rewritten as

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - n(n+1)y = 0.$$

This is an Euler-Cauchy equation. We try a solution of the form $y = x^m$, where m is a constant to be determined. Substituting this into the differential equation, we get

$$x^2 \cdot m(m-1)x^{m-2} + 2x \cdot mx^{m-1} - n(n+1)x^m = 0.$$

Simplifying, we have $x^m\left[m(m-1) + 2m - n(n+1)\right] = 0$, which gives us the characteristic equation

$$m^2 + m - n(n+1) = 0.$$

Solving for m, we get $m = \frac{-1 \pm (2n+1)}{2}$, so $m_1 = n$ and $m_2 = -(n+1)$.
The general solution is then

$$y(x) = Ax^n + Bx^{-(n+1)},$$

where A and B are arbitrary constants.

5. **Matrix and properties** [7]

Let A and B be $n \times n$ Hermitian matrices and U an $n \times n$ unitary matrix.

(a) Show that the modulus of each of the eigenvalues of U is equal to one ($|\lambda| = 1$).

(b) Show that the eigenvalues of A are real.

(c) Assuming that $U = A + iB$, show that

(i) $A^2 + B^2 = I$, where I is the identity matrix,

(ii) $AB - BA = 0$.

(a) Let λ be an eigenvalue of U with corresponding eigenvector $\mathbf{v}$, so that $U\mathbf{v} = \lambda\mathbf{v}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{v}^\dagger U^\dagger = \lambda^* \mathbf{v}^\dagger.$$

$$\mathbf{v}^\dagger U^\dagger U = \mathbf{v}^\dagger = \lambda^* \mathbf{v}^\dagger U.$$

Multiplying both sides by $U\mathbf{v}$ from the right, we get

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda^* \mathbf{v}^\dagger UU\mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda \mathbf{v}^\dagger \mathbf{v}.$$

Equating the two expressions for $\mathbf{v}^\dagger U\mathbf{v}$, we have

$$\lambda \mathbf{v}^\dagger \mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

Since $\mathbf{v}$ is a non-zero eigenvector, $\mathbf{v}^\dagger \mathbf{v} \neq 0$, we can divide both sides by $\mathbf{v}^\dagger \mathbf{v}$ to get

$$\lambda = \lambda^*.$$

Thus, we have

$$|\lambda|^2 = \lambda\lambda^* = 1.$$

Q.E.D.

(b) Let μ be an eigenvalue of A with corresponding eigenvector $\mathbf{w}$, so that $A\mathbf{w} = \mu\mathbf{w}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{w}^\dagger A^\dagger = \mu^* \mathbf{w}^\dagger.$$

Since A is Hermitian, $A^\dagger = A$. Thus,

$$\mathbf{w}^\dagger A = \mu^* \mathbf{w}^\dagger.$$

Multiplying both sides by $\mathbf{w}$ from the right, we get

$$\mathbf{w}^\dagger A\mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{w}^\dagger A\mathbf{w} = \mu \mathbf{w}^\dagger \mathbf{w}.$$

Equating the two expressions for $\mathbf{w}^\dagger A\mathbf{w}$, we have

$$\mu \mathbf{w}^\dagger \mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

Since $\mathbf{w}$ is a non-zero eigenvector, $\mathbf{w}^\dagger \mathbf{w} \neq 0$, we can divide both sides by $\mathbf{w}^\dagger \mathbf{w}$ to get

$$\mu = \mu^*.$$

Q.E.D.

(c) Given that $U = A + iB$ is unitary, we have

$$U^\dagger U = I.$$

Calculating $U^\dagger U$, we get

$$(A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2 + i(AB - BA).$$

Setting this equal to the identity matrix I, we have

$$A^2 + B^2 + i(AB - BA) = I.$$

Q.E.D.

6. **Matrix and geometry** [8]

The rotation matrix A in $\mathbb{R}^3$ is given by

$$A = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}.$$

(a) Show that the matrix A is orthogonal.

(b) Calculate $\cos\theta$, where θ is the angle of rotation, and find a unit vector in the direction of the axis of rotation.

(a) To show that A is orthogonal, we need to verify that $A^TA = I$. Calculating A^T,

$$A^T = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \\ -1 & -\sqrt{2} & 1 \end{pmatrix}.$$

Now, calculating A^TA,

$$A^TA = \frac{1}{4}\begin{pmatrix} 2+0+2 & -\sqrt{2}+0-\sqrt{2} & -\sqrt{2}+0-\sqrt{2} \\ -\sqrt{2}+0-\sqrt{2} & 1+2+1 & 1-2+1 \\ -\sqrt{2}+0-\sqrt{2} & 1-2+1 & 1+2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus A is orthogonal.

(b) For an orthogonal matrix representing a rotation in $\mathbb{R}^3$, in 2D dimensions, the matrix can be represented by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus Trace can be used to find the angle of rotation. For a rotation matrix in $\mathbb{R}^3$,

$$\operatorname{tr}(A) = 1 + 2\cos\theta.$$

Here

$$\operatorname{tr}(A) = \frac{1}{2}(\sqrt{2}+\sqrt{2}+1) = \sqrt{2}+\frac{1}{2},$$

so

$$\cos\theta = \frac{\operatorname{tr}(A)-1}{2} = \frac{\left(\sqrt{2}+\frac{1}{2}\right)-1}{2} = \frac{\sqrt{2}}{2} - \frac{1}{4}.$$

This gives

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} - \frac{1}{4}\right) \approx \boxed{62.8^\circ}.$$

The rotation axis is the eigenspace for eigenvalue 1, i.e. solutions of

$$(A - I)\mathbf{v} = 0.$$

A nonzero solution is

$$\mathbf{v} = \begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix},$$

so a unit vector along the axis is

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{(1+\sqrt{2})^2 + (1+\sqrt{2})^2 + 1}}\begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix} = \boxed{\frac{1}{\sqrt{7+4\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ -(1+\sqrt{2}) \\ 1 \end{pmatrix}}.$$

Section B

7.
(a) State de Moivre's theorem and show that

(i)

$$\sum_{n=0}^{N-1} \cos n\theta = \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cos \frac{(N-1)\theta}{2},$$

(ii)

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,$$

where n, N are integers and θ is a complex variable. [8]
(b) Find all the roots for the equation

$$\left(\frac{z-1}{z+1}\right)^n = -1. \qquad (\dagger)$$

Verify your solution for the special case $n = 3$, by finding the roots of the resulting third order equation.

Use the general solution to $(\dagger)$ to calculate the product

$$\prod_{r=1}^{n} \cot\left(\frac{(2r+1)\pi}{2n}\right),$$

where r is an integer, for both odd and even values of n. [8]
(c) Show that if the complex numbers z and u satisfy the relation

$$\left|\frac{z+u}{z+u^*}\right| = 1,$$

then either u or z must be real.

[The $(*)$ stands for the complex conjugate.] [4]

(a) **de Moivre's theorem** states that for any real number θ and integer n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

(i) Using the formula for the sum of a geometric series, we have

$$\begin{aligned}
\sum_{n=0}^{N-1} \cos n\theta &= \sum_{n=0}^{N-1} \Re(\mathrm{e}^{in\theta}) \\
&= \Re\left(\sum_{n=0}^{N-1} \mathrm{e}^{in\theta}\right) \\
&= \Re\left(\frac{1-\mathrm{e}^{iN\theta}}{1-\mathrm{e}^{i\theta}}\right) \\
&= \Re\left(\frac{\mathrm{e}^{i(N-1)\theta/2}}{\mathrm{e}^{i\theta/2}} \cdot \frac{\mathrm{e}^{iN\theta/2}-\mathrm{e}^{-iN\theta/2}}{\mathrm{e}^{i\theta/2}-\mathrm{e}^{-i\theta/2}}\right) \\
&= \Re\left(\frac{\mathrm{e}^{i(N-1)\theta/2}}{\mathrm{e}^{i\theta/2}} \cdot \frac{\sin(N\theta/2)}{\sin(\theta/2)}\right) \\
&= \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cdot \Re(\mathrm{e}^{i(N-1)\theta/2}) \\
&= \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cos((N-1)\theta/2).
\end{aligned}$$

Q.E.D.

(ii) Using de Moivre's theorem, we have

$$\begin{aligned}
\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta.
\end{aligned}$$

Equating the real parts, we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Using the identity $\sin^2\theta = 1 - \cos^2\theta$, we can rewrite the equation as

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

Q.E.D.

(b) Rearranging the equation, we have

$$\frac{z-1}{z+1} = \mathrm{e}^{i(2k+1)\pi/n}, \quad k = 0, 1, 2, \ldots, n-1.$$

Solving for z, we get

$$\boxed{z = \frac{1+\mathrm{e}^{i(2k+1)\pi/n}}{1-\mathrm{e}^{i(2k+1)\pi/n}}}.$$

For the special case $n = 3$, we have

$$z_k = \frac{1+\mathrm{e}^{i(2k+1)\pi/3}}{1-\mathrm{e}^{i(2k+1)\pi/3}}, \quad k = 0, 1, 2.$$

The three roots are

$$\begin{aligned}z_0 &= \frac{1+\mathrm{e}^{i\pi/3}}{1-\mathrm{e}^{i\pi/3}},\\ z_1 &= \frac{1+\mathrm{e}^{i\pi}}{1-\mathrm{e}^{i\pi}},\\ z_2 &= \frac{1+\mathrm{e}^{i5\pi/3}}{1-\mathrm{e}^{i5\pi/3}}.\end{aligned}$$

Simplifying these expressions, we get

$$\begin{aligned}z_0 &= \sqrt{3}i,\\ z_1 &= 0,\\ z_2 &= -\sqrt{3}i.\end{aligned}$$

These are the three roots of the equation for $n = 3$.
For the product, we have the equation

$$\left(\frac{z-1}{z+1}\right)^n = -1. \tag{†}$$

Let

$$w = \frac{z-1}{z+1}.$$

Then $w^n = -1 = e^{i(2k+1)\pi}$, so

$$w_k = e^{i(2k+1)\pi/n}, \qquad k = 0, 1, \ldots, n-1.$$

Solving $z - 1 = w(z+1)$ gives

$$z = \frac{1+w}{1-w}.$$

With $w = e^{i\phi}$ we have the standard identity

$$\frac{1+e^{i\phi}}{1-e^{i\phi}} = -i\cot\left(\frac{\phi}{2}\right),$$

hence

$$z_k = -i\cot\left(\frac{(2k+1)\pi}{2n}\right).$$

Since the set of angles

$$\left\{\frac{(2r+1)\pi}{2n} : r = 1, \dots, n\right\}$$

is the same as

$$\left\{\frac{(2k+1)\pi}{2n} : k = 0, \dots, n-1\right\}$$

(up to a permutation, using $\cot(\pi + \alpha) = \cot\alpha$), the required product is

$$P := \prod_{r=1}^{n} \cot\left(\frac{(2r+1)\pi}{2n}\right) = \prod_{k=0}^{n-1} \cot\left(\frac{(2k+1)\pi}{2n}\right).$$

From $z_k = -i\cot(\cdot)$ we get $\cot(\cdot) = iz_k$, so

$$P = i^n \prod_{k=0}^{n-1} z_k.$$

Now (†) is equivalent to

$$(z-1)^n = -(z+1)^n \iff (z-1)^n + (z+1)^n = 0,$$

Expanding the left-hand side using the binomial theorem, we see that only the even powers of z survive, with coefficients

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1},$$

so the polynomial is of degree n with leading coefficient 2^{n-1}. The z_k are the n roots of this polynomial,

$$F(z) := (z-1)^n + (z+1)^n = 0.$$

We know that

$$2(z - z_0)(z - z_1)\cdots(z - z_{n-1}) = F(z)/2^{n-1}.$$

Setting $z = 0$ gives

$$\prod_{k=0}^{n-1} z_k = (-1)^n \frac{F(0)}{2^n} = (-1)^n \frac{(-1)^n + 1}{2}.$$

Therefore

$$P = i^n (-1)^n \frac{(-1)^n + 1}{2}.$$

If n is odd, then $(-1)^n + 1 = 0$ and so $P = 0$. If n is even, then $(-1)^n + 1 = 2$ and $\prod z_k = 1$, so

$$P = i^n = i^{2(n/2)} = (-1)^{n/2}.$$

$$\boxed{\prod_{r=1}^{n} \cot\left(\frac{(2r+1)\pi}{2n}\right) = \begin{cases} 0, & n \text{ odd}, \\ (-1)^{n/2}, & n \text{ even}. \end{cases}}$$

(c) Given that

$$\left|\frac{z+u}{z+u^*}\right| = 1,$$

we have

$$\left|\frac{z+u}{z+u^*}\right|^2 = \frac{(z+u)(z^*+u^*)}{(z+u^*)(z^*+u)} = 1.$$

Expanding both sides, we get

$$\begin{aligned}
(z+u)(z^*+u^*) &= (z+u^*)(z^*+u) \\
zz^* + zu^* + uz^* + uu^* &= zz^* + zu + u^*z^* + u^*u \\
zu^* + uz^* &= zu + u^*z^* \\
zu^* - zu &= u^*z^* - uz^* \\
z(u^* - u) &= z^*(u^* - u) \\
(z - z^*)(u^* - u) &= 0.
\end{aligned}$$

Thus, either $z - z^* = 0$ or $u^* - u = 0$, which means either $z \in \mathbb{R}$ or $u \in \mathbb{R}$. Q.E.D.

8. The differential equation for the displacement $y(t)$ of a particle executing forced and damped harmonic oscillations with damping factor γ and natural frequency ω_0 may be written as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = F\cos\omega t,$$

where F and ω are the amplitude and frequency of the driving force respectively.
(a) Assuming that $F = 0$, find the displacements $y(t)$ of the particle for the cases $\omega_0 < \gamma$ and $\omega_0 = \gamma$. Sketch and compare the two displacements. [6]
(b) Assume now that $F = F_0 \neq 0$ and $\gamma < \omega_0$. Explain what is meant by a steady state solution of the differential equation and find an expression for the steady state amplitude and phase of the displacement.

For a given value of the natural frequency ω_0, which value of the driving force frequency ω maximises the displacement? For what value of ω is the velocity a maximum? [6]
(c) Explain what is meant by the width of the oscillator resonance. Calculate the width of the resonance for the case $\gamma \ll \omega_0$. [4]

(a) For $F = 0$, the differential equation becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = 0.$$

The characteristic equation is

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

The roots are

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

- For $\omega_0 < \gamma$, the roots are real and distinct. The general solution is

$$y(t) = \mathrm{e}^{-\gamma t}(C_1 \mathrm{e}^{\sqrt{\gamma^2-\omega_0^2}t} + C_2 \mathrm{e}^{-\sqrt{\gamma^2-\omega_0^2}t}),$$

  where C_1 and C_2 are constants determined by initial conditions.
- For $\omega_0 = \gamma$, the roots are real and equal. The general solution is

$$y(t) = (C_1 + C_2 t)\mathrm{e}^{-\gamma t},$$

  where C_1 and C_2 are constants determined by initial conditions.

The two displacements can be sketched as follows:

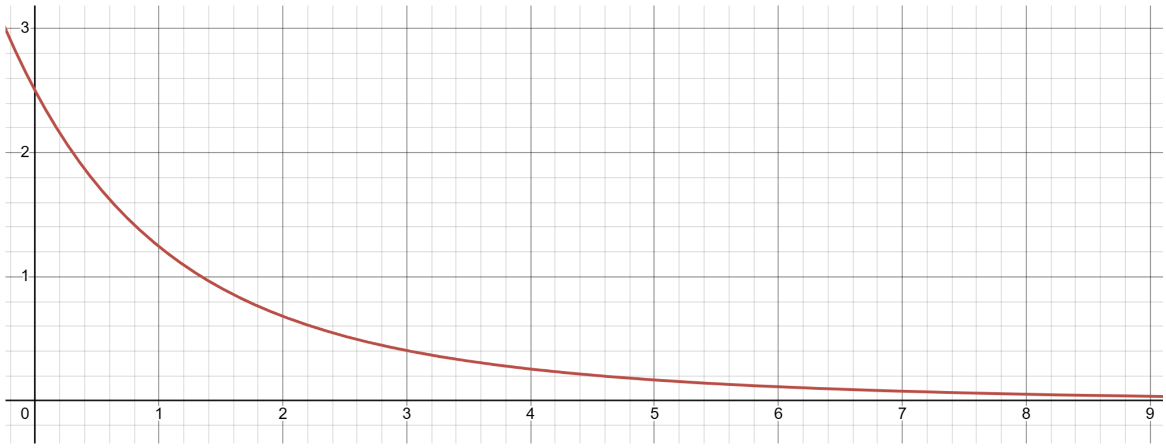

Figure 1: Displacements for $\omega_0 < \gamma$

(b) A steady state solution is a particular solution of the differential equation that represents the long-term behavior of the system after transient effects have died out. It is typically periodic and has the same frequency as the driving force.

To find the steady state solution, we assume a solution of the form

$$y(t) = A\cos(\omega t - \delta),$$

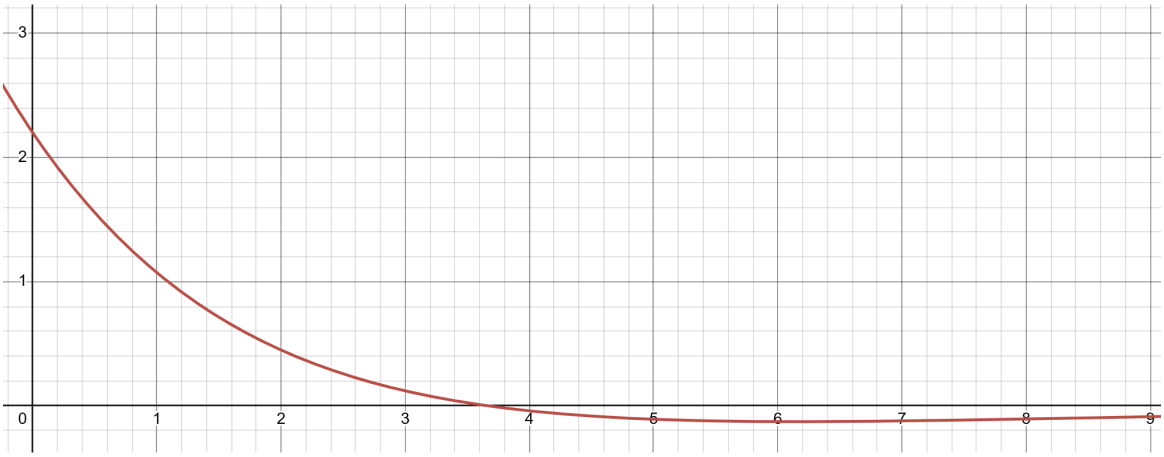

Figure 2: Displacements for $\omega_0 = \gamma$

where A is the amplitude and δ is the phase shift. Substituting this into the differential equation, we get

$$-A\omega^2 \cos(\omega t - \delta) - 2\gamma A\omega \sin(\omega t - \delta) + \omega_0^2 A \cos(\omega t - \delta) = F_0 \cos(\omega t).$$

Equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$, we have

$$A(\omega_0^2 - \omega^2) = F_0 \cos \delta,$$

$$2\gamma A\omega = F_0 \sin \delta.$$

Solving for A and δ, we get

$$A = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}},$$

$$\tan \delta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

To maximize the displacement, we differentiate A with respect to ω and set it to zero. The maximum occurs when $(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2$ is minimized. This happens when $\boxed{\omega = \sqrt{\omega_0^2 - 2\gamma^2}}$, provided $\omega_0 > \gamma$. The velocity is given by

$$v = \omega A = \frac{\omega F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}.$$

To maximize the velocity, we differentiate v with respect to ω and set it to zero. The maximum occurs when minimizing

$$\left(\left(\frac{\omega_0}{\omega}\right)^2 - 1\right)^2 + 4\gamma^2.$$

This happens when $\boxed{\omega = \omega_0}$.

(c) The width of the oscillator resonance, also known as the bandwidth, is defined as the range of frequencies over which the amplitude of the oscillation is greater than or equal to $\frac{A_{\max}}{2}$, where $A_{\max}$ is the maximum amplitude at resonance. For $\gamma \ll \omega_0$, the resonance occurs at $\omega \approx \omega_0$. The amplitude at resonance is

$$A_{\max} = \frac{F_0}{2\gamma\omega_0}.$$

To find the frequencies at which the amplitude is half of $A_{\max}$, we set

$$\frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} = \frac{A_{\max}}{2} = \frac{F_0}{4\gamma\omega_0}.$$

Squaring both sides and simplifying, we get

$$(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2 = (4\gamma\omega_0)^2.$$

Solving this quadratic equation for ω, we find the two frequencies ω_1 and ω_2 at which the amplitude is half of $A_{\max}$. The width of the resonance is then given by

$$\Delta\omega = \omega_2 - \omega_1 \approx \boxed{2\gamma}.$$

9. The matrix

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ \alpha - 3 & 0 & 1 \\ 2 & -1 & \alpha + 1 \end{pmatrix}$$

defines the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x) = Ax$, where $x \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

Let A_4 be equal to the matrix A for $\alpha = 4$.

(a) Find a basis for $\ker(f)$ and show that the geometry of the kernel is a straight line in $\mathbb{R}^3$. Write the equation of the line in vector form. [4]

(b) By choosing $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a basis of $\mathbb{R}^3$ with $\mathbf{e}_1 \in \ker(f)$, show that the geometry of the image of f is a plane and find the direction of the normal to this plane. [5]

Now let A_0 be equal to A for $\alpha = 0$.

(c) Assuming that the matrix A_0 was calculated with respect to the basis

$$\mathbf{u}_1 = (1, 0, 0)^T, \quad \mathbf{u}_2 = (0, 1, 0)^T, \quad \mathbf{u}_3 = (0, 0, 1)^T,$$

express the map $f(x)$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the coordinates of the vector $x = (x_1, x_2, x_3)^T$. [3]

(d) Assume now that the matrix A_0' of the map f is calculated with respect to the basis

$$\mathbf{w}_1 = (1, 1, 0)^T, \quad \mathbf{w}_2 = (1, 0, 1)^T, \quad \mathbf{w}_3 = (0, 1, 1)^T.$$

Calculate the matrix A_0' from the relation

$$A_0' = C A_0 C^{-1}$$

by a suitable choice of the matrix C. [8]

(a) For $\alpha = 4$, the matrix A_4 is

$$A_4 = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & 1 \\ 2 & -1 & 5 \end{pmatrix}.$$

To find $\ker(f)$, we solve the equation $A_4 x = 0$. The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 4 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 2 & -1 & 5 & | & 0 \end{pmatrix}.$$

Row reducing, we get

$$\begin{pmatrix} 1 & -1 & 4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

From the row-reduced form, we have the equations:

$$x_1 - x_2 + 4x_3 = 0,$$

$$x_2 - 3x_3 = 0.$$

Solving these equations, we find that

$$x_2 = 3x_3,$$

$$x_1 = x_2 - 4x_3 = -x_3.$$

Letting $x_3 = t$, we have

$$\mathbf{r}(t) = t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

(b) We choose a basis for $\mathbb{R}^3$ starting with $\mathbf{e}_1 \in \ker(f)$:

$$\mathbf{e}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The image of f is spanned by the transformed basis vectors. Since $f(\mathbf{e}_1) = \mathbf{0}$:

$$\mathrm{Im}(f) = \mathrm{span}\{f(\mathbf{e}_2), f(\mathbf{e}_3)\}$$

Calculating the images using A_4:

$$f(\mathbf{e}_2) = A_4 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}, \quad f(\mathbf{e}_3) = A_4 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$$

These two vectors are linearly independent, so the geometry of the image is a **plane**.

To find the normal $\mathbf{n}$, we take the cross product:

$$\mathbf{n} = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}-1\\0\\-1\end{pmatrix} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 2\\ -1 & 0 & -1\end{vmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

(c) Let $\alpha = 0$. The matrix is:

$$A_0 = \begin{pmatrix}1 & -1 & 0\\ -3 & 0 & 1\\ 2 & -1 & 1\end{pmatrix}$$

For a vector $x = (\alpha_1, \alpha_2, \alpha_3)^T$:

$$f(x) = A_0 x = \begin{pmatrix}\alpha_1 - \alpha_2\\ -3\alpha_1 + \alpha_3\\ 2\alpha_1 - \alpha_2 + \alpha_3\end{pmatrix}$$

Expressed in terms of the basis $\mathbf{u}_i$:

$$f(x) = (\alpha_1 - \alpha_2)\mathbf{u}_1 + (-3\alpha_1 + \alpha_3)\mathbf{u}_2 + (2\alpha_1 - \alpha_2 + \alpha_3)\mathbf{u}_3$$

(d) We are given the new basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Let P be the transition matrix whose columns are these basis vectors:

$$P = \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$$

The relationship $A_0' = CA_0C^{-1}$ implies that A_0' is the matrix in the new basis if $C = P^{-1}$. Thus we compute $A_0' = P^{-1}A_0P$.

1. Find P^{-1} (which is C):

$$\det(P) = 1(0-1) - 1(1-0) + 0 = -2$$

$$P^{-1} = \frac{1}{-2}\begin{pmatrix}-1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{pmatrix}$$

Calculate A_0P:

$$A_0P = \begin{pmatrix}1 & -1 & 0\\ -3 & 0 & 1\\ 2 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\\ -3 & -2 & 1\\ 1 & 3 & 0\end{pmatrix}$$

Calculate $A_0' = P^{-1}(A_0P)$:

$$A_0' = \frac{1}{2}\begin{pmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & -1\\ -3 & -2 & 1\\ 1 & 3 & 0\end{pmatrix}$$

Performing the multiplication:

$$\begin{aligned}
\text{Row } 1 &= \tfrac{1}{2}[(0-3-1), (1-2-3), (-1+1+0)] = (-2, -2, 0)\\
\text{Row } 2 &= \tfrac{1}{2}[(0+3+1), (1+2+3), (-1-1+0)] = (2, 3, -1)\\
\text{Row } 3 &= \tfrac{1}{2}[(0-3+1), (-1+2+3), (1+1+0)] = (-1, 0, 1)
\end{aligned}$$

Thus

$$A_0' = \begin{pmatrix}-2 & -2 & 0\\ 2 & 3 & -1\\ -1 & 0 & 1\end{pmatrix}$$

10.
(a) The components of a vector $\mathbf{a}$ in the (x, y) plane of a Cartesian coordinate system (x, y, z) are given by (a_x, a_y). Assume now that the (x, y) axes are rotated about the z axis by an angle θ, anticlockwise, so that the components of $\mathbf{a}$ with respect to the rotated coordinate system (x', y', z) are given by (a'_x, a'_y).

Calculate the elements of the matrix R that relates the vector $(a'_x, a'_y)^T$ to the vector $(a_x, a_y)^T$ and show that R is a rotation matrix. Find the eigenvectors of the matrix R. [7]
(b) The equation of a conical section in the (x, y) coordinate system in $\mathbb{R}^2$ is given by

$$f(x, y) = x^2 + 6xy + y^2 = 4. \qquad (*)$$

Write down the above equation in the matrix form

$$\mathbf{x}^T M \mathbf{x} = 4$$

where M is a symmetric matrix and $\mathbf{x} = (x, y)^T$ is a coordinate vector in $\mathbb{R}^2$. Use matrix diagonalisation to show that the curve in $(*)$ represents a hyperbola. Find the elements of the unitary matrix that diagonalises the matrix M.

Sketch this curve showing the asymptotes and the points of intersection with the (x, y) axes. [8]

(a) The rotation matrix R that relates the components of the vector $\mathbf{a}$ in the rotated coordinate system to the original coordinate system is given by

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

To show that R is a rotation matrix, we need to verify that it preserves lengths and angles. We can check that $R^T R = I$:

$$R^T R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Thus, R is an orthogonal matrix, confirming that it is a rotation matrix.

To find the eigenvectors of R, we solve the characteristic equation $\det(R - \lambda I) = 0$:

$$\det \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

This simplifies to

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0.$$

$$\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

The corresponding eigenvectors can be found by solving $(R - \lambda I)\mathbf{v} = 0$ for each eigenvalue. Thus,

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(b) The equation of the conic section can be written in matrix form as

$$\mathbf{x}^T M \mathbf{x} = 4,$$

where

$$M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To diagonalize M, we first find its eigenvalues by solving the characteristic equation:

$$\det(M - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 - 9 = 0.$$

This simplifies to

$$(1 - \lambda)^2 = 9 \Rightarrow 1 - \lambda = \pm 3 \Rightarrow \lambda = 1 \pm 3.$$

So the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$.

For $\lambda_1 = 4$:

$$M - 4I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \Rightarrow \mathbf{v}_1 = (1, 1)^T.$$

For $\lambda_2 = -2$:

$$M + 2I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \Rightarrow \mathbf{v}_2 = (1, -1)^T.$$

The unitary matrix P that diagonalizes M is formed by normalizing the eigenvectors:

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The diagonal matrix D is

$$D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then the equation becomes

$$-2x'^2 + 4y'^2 = 4,$$

where $\mathbf{x}' = P^{-1}\mathbf{x}$. This is a hyperbola in the rotated coordinate system. The asymptotes of the hyperbola can be found by setting the equation to zero:

$$x^2 + 6xy + y^2 = 0.$$

Factoring, we get

$$(x + 3y)^2 - 8y^2 = 0 \Rightarrow (x + 3y - 2\sqrt{2}y)(x + 3y + 2\sqrt{2}y) = 0.$$

Thus, the asymptotes are given by the lines

$$x + (3 - 2\sqrt{2})y = 0 \quad \text{and} \quad x + (3 + 2\sqrt{2})y = 0.$$

The points of intersection with the axes can be found by setting $y = 0$ and $x = 0$:

$$x = \pm 2 \quad \text{and} \quad y = \pm 2.$$

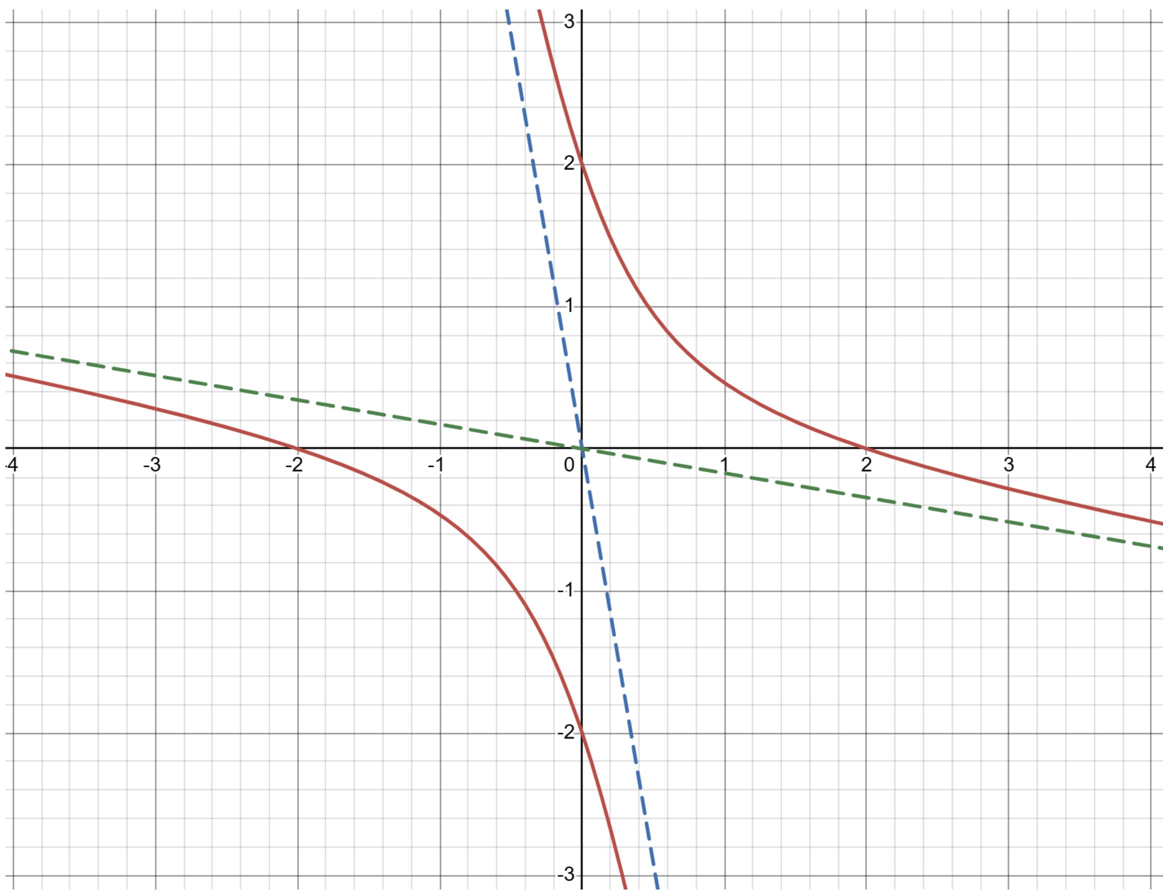

Figure 3: Hyperbola with asymptotes and intersection points