

Answers for Collection of Mathematics Methods

JTST

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Section A

1. Complex Numbers [7]

- (a) Express the term $(1+i)^4$ in the form $r e^{i\theta}$, where r and θ are real variables.
- (b) Express the complex number $\tan^{-1}(2i)$ in the form $x+iy$ where x,y are real.
- (c) Given that $z = z_1 + z_2$, where $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are real variables, find an expression for $|z|$ in terms of $\Delta\theta = \theta_1 - \theta_2$.

(a)

$$(1+i)^4 = \left(\sqrt{2}e^{i\pi/4}\right)^4 = \boxed{4e^{i\pi}}$$

(b) Let $w = \tan^{-1}(2i)$, then $\tan(w) = 2i$. Using the identity $\tan(w) = \frac{\sin(w)}{\cos(w)}$, we have

$$\begin{aligned} \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} &= 2i \\ e^{iw} - e^{-iw} &= -2i(e^{iw} + e^{-iw}) \\ 3e^{iw} + e^{-iw} &= 0 \\ e^{2iw} &= -\frac{1}{3} \\ 2iw &= \ln\left(-\frac{1}{3}\right) \\ w &= -\frac{i}{2} \left(\ln \frac{1}{3} + i\pi \right) = \boxed{\frac{\pi}{2} + \frac{i}{2} \ln 3} \end{aligned}$$

(c)

$$\begin{aligned} |z|^2 &= z\bar{z} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1z_2 \\ &= 2 + e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)} \\ &= 2 + 2\cos(\Delta\theta) \\ \Rightarrow |z| &= \boxed{\sqrt{2(1+\cos(\Delta\theta))}} \end{aligned}$$

2. Vectors [8]

- (a) Write down the equation of the plane

$$3x + 4y + 5z = 10$$

in the vector form

$$\mathbf{r} = \mathbf{r}_0 + t_1 \mathbf{a} + t_2 \mathbf{b},$$

where \mathbf{a} and \mathbf{b} are constant vectors in the plane, t_1 and t_2 are real parameters and \mathbf{r}_0 is a constant vector. What is the distance between the plane and the origin?

- (b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}).$$

- (a) A particular solution to the plane equation is $\mathbf{r}_0 = (0, 0, 2)^T$. Two independent vectors in the plane are $\mathbf{a} = (0, -2.5, 2)^T$ and $\mathbf{b} = (-10/3, 0, 2)^T$. Thus the vector form of the plane is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ -2.5 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} -10/3 \\ 0 \\ 2 \end{pmatrix}.$$

The distance from the origin to the plane is given by

$$d = \frac{|\mathbf{r}_0 \cdot \mathbf{n}|}{|\mathbf{n}|},$$

where $\mathbf{n} = (3, 4, 5)$ is the normal vector to the plane. Substituting $\mathbf{r}_0 = (0, 0, 2)$ and $\mathbf{n} = (3, 4, 5)$,

$$d = \frac{|(0, 0, 2) \cdot (3, 4, 5)|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{|10|}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \boxed{\sqrt{2}}.$$

- (b) Using the vector triple product identity $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$, let $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$ and $\mathbf{z} = \mathbf{a} \times \mathbf{c}$, we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{a} \times \mathbf{c})]$$

Since $(\mathbf{b} \times (\mathbf{a} \times \mathbf{c}))_i = \epsilon_{ijk} b_j (a \times c)_k = \epsilon_{ijk} b_j \epsilon_{klm} a_k c_m = \epsilon_{ijk} \epsilon_{klm} b_j a_k c_m$, and using the identity $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$, we get $\epsilon_{ijk} \epsilon_{klm} b_j a_k c_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) b_j a_k c_m = b_j a_i c_m \delta_{il} \delta_{jm} - b_j a_k c_m \delta_{im} \delta_{jl} = a_i (b_j c_j) - c_i (a_j b_j)$, which is just $\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$. Continuing from above,

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) &= \mathbf{a} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})] \\ &= (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b}) \\ &= 1 \cdot (\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \\ &= \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \end{aligned}$$

Q.E.D.

3. Matrix and linear equation [5]

Consider the set of linear equations

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4, \end{aligned}$$

where x, y, z are real variables and a, b are real parameters.

Find the values of a and b for which the set of equations have:

- (i) a unique solution,
- (ii) infinitely many solutions,
- (iii) no solution.

$$\begin{aligned} 2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4. \end{aligned}$$

Let the coefficient matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2b \\ 2a \\ 4 \end{pmatrix}.$$

The determinant of A is

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2(a - 6).$$

- (i) **Unique solution.** If $\det A \neq 0$, i.e. $a \neq 6$, the system has a unique solution for all values of b .

(ii) **Infinitely many solutions.** Let $a = 6$. The system becomes

$$\begin{aligned} 2x + y + z &= 2b, \\ 6x + 3y + 2z &= 12, \\ 2x + y + 3z &= 4. \end{aligned}$$

Subtracting the first equation from the third gives

$$2z = 4 - 2b \Rightarrow z = 2 - b.$$

Dividing the second equation by 3 yields

$$2x + y + \frac{2}{3}z = 4.$$

From the first equation, $2x + y = 2b - z$. Substituting,

$$2b - z + \frac{2}{3}z = 4 \Rightarrow 2b - \frac{1}{3}z = 4 \Rightarrow z = 6b - 12.$$

Consistency requires

$$2 - b = 6b - 12 \Rightarrow b = 2.$$

Hence $z = 0$ and the remaining equation is

$$2x + y = 4.$$

Letting $x = t$, the solutions are

$$(x, y, z) = (t, 4 - 2t, 0), \quad t \in \mathbb{R},$$

so there are infinitely many solutions when $(a, b) = (6, 2)$.

(iii) **No solution.** If $a = 6$ and $b \neq 2$, the two expressions for z are inconsistent. Hence the system has no solution.

| | |
|---------------------------|------------------------|
| Unique solution | $a \neq 6$ (any b), |
| Infinitely many solutions | $a = 6, b = 2$, |
| No solution | $a = 6, b \neq 2$. |

4. Differential equation [5]

Find the general solution for the differential equation

$$\frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) = n(n+1)y,$$

where x is a real variable and n is a real constant.

Using the product rule, we have

$$\frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) = 2x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2}.$$

Thus the differential equation can be rewritten as

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0.$$

This is an Euler-Cauchy equation. We try a solution of the form $y = x^m$, where m is a constant to be determined. Substituting this into the differential equation, we get

$$x^2 \cdot m(m-1)x^{m-2} + 2x \cdot mx^{m-1} - n(n+1)x^m = 0.$$

Simplifying, we have $x^m [m(m-1) + 2m - n(n+1)] = 0$, which gives us the characteristic equation

$$m^2 + m - n(n+1) = 0.$$

Solving for m , we get $m = \frac{-1 \pm (2n+1)}{2}$, so $m_1 = n$ and $m_2 = -(n+1)$.

The general solution is then

$$y(x) = Ax^n + Bx^{-(n+1)},$$

where A and B are arbitrary constants.

5. Matrix and properties [7]

Let A and B be $n \times n$ Hermitian matrices and U an $n \times n$ unitary matrix.

- (a) Show that the modulus of each of the eigenvalues of U is equal to one ($|\lambda| = 1$).
- (b) Show that the eigenvalues of A are real.
- (c) Assuming that $U = A + iB$, show that
 - (i) $A^2 + B^2 = I$, where I is the identity matrix,
 - (ii) $AB - BA = 0$.

- (a) Let λ be an eigenvalue of U with corresponding eigenvector \mathbf{v} , so that $U\mathbf{v} = \lambda\mathbf{v}$. Taking the conjugate transpose of both sides, we have

$$\begin{aligned}\mathbf{v}^\dagger U^\dagger &= \lambda^* \mathbf{v}^\dagger. \\ \mathbf{v}^\dagger U^\dagger U &= \mathbf{v}^\dagger = \lambda^* \mathbf{v}^\dagger U.\end{aligned}$$

Multiplying both sides by $U\mathbf{v}$ from the right, we get

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda^* \mathbf{v}^\dagger UU\mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{v}^\dagger U\mathbf{v} = \lambda \mathbf{v}^\dagger \mathbf{v}.$$

Equating the two expressions for $\mathbf{v}^\dagger U\mathbf{v}$, we have

$$\lambda \mathbf{v}^\dagger \mathbf{v} = \lambda^* \mathbf{v}^\dagger \mathbf{v}.$$

Since \mathbf{v} is a non-zero eigenvector, $\mathbf{v}^\dagger \mathbf{v} \neq 0$, we can divide both sides by $\mathbf{v}^\dagger \mathbf{v}$ to get

$$\lambda = \lambda^*.$$

Thus, we have

$$|\lambda|^2 = \lambda \lambda^* = 1.$$

Q.E.D.

- (b) Let μ be an eigenvalue of A with corresponding eigenvector \mathbf{w} , so that $A\mathbf{w} = \mu\mathbf{w}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{w}^\dagger A^\dagger = \mu^* \mathbf{w}^\dagger.$$

Since A is Hermitian, $A^\dagger = A$. Thus,

$$\mathbf{w}^\dagger A = \mu^* \mathbf{w}^\dagger.$$

Multiplying both sides by \mathbf{w} from the right, we get

$$\mathbf{w}^\dagger A\mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{w}^\dagger A\mathbf{w} = \mu \mathbf{w}^\dagger \mathbf{w}.$$

Equating the two expressions for $\mathbf{w}^\dagger A\mathbf{w}$, we have

$$\mu \mathbf{w}^\dagger \mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

Since \mathbf{w} is a non-zero eigenvector, $\mathbf{w}^\dagger \mathbf{w} \neq 0$, we can divide both sides by $\mathbf{w}^\dagger \mathbf{w}$ to get

$$\mu = \mu^*.$$

Q.E.D.

- (c) Given that $U = A + iB$ is unitary, we have

$$U^\dagger U = I.$$

Calculating $U^\dagger U$, we get

$$(A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2 + i(AB - BA).$$

Setting this equal to the identity matrix I , we have

$$A^2 + B^2 + i(AB - BA) = I.$$

Q.E.D.