Practical applications of the compound pendulum

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The compound pendulum is a standard topic in most intermediate physics courses,[1] and is generally included in freshman laboratory programs.[2-4] Traditionally, this experiment has required the acquisition of a large amount of repetitive data, and a complex analysis to determine two parameters, i.e., the position of the center of mass and the moment of inertia. Elementary physics courses tend to concentrate on statics and particle dynamics and thus emphasize the importance of the center of mass, a concept which has many applications within the student's experience and is therefore readily accepted. However, the subtle effects of the mass distribution, which only manifest themselves in angular dynamics, are very much harder to comprehend. Furthermore, this subject is usually introduced at a time when the student's grasp of topics such as torque, moment of inertia, and simple harmonic motion is tenuous at best, so that the experiment does not generally lead to an understanding of how the mass distribution affects the period of oscillation. It is hoped that the following examples of the practical application of compound pendulum theory to the determination of the mass distributions of astronauts,[5] farm tractors[6,7] and sailing boats,[8,9] will help to make this subject more interesting.

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Theory

The period of small amplitude oscillations of a finite body, of mass M, about a horizontal axis through the pivot point "O" (which need not be within the body), see Fig. 1, depends on the moment of inertia I and the gravitational torque, which is

$$\tau = Mga \sin \theta \tag{1}$$

for a displacement θ from the equilibrium position. Here a is the distance from the pivot to the center of gravity. Then from Newton's second law applied to angular motion

$$I \frac{d^2\theta}{dt^2} = Mga \sin \theta \approx Mga\, \theta \qquad \text{for } \theta \ll 1$$

which is the equation of simple harmonic motion with a period of

$$T = 2\pi \sqrt{\frac{I}{Mga}} \tag{2}$$

I is the moment of inertia for rotations about a horizontal axis through the pivot point O and is related to the moment of inertia $I_o = Mk^2$ about a parallel axis through the center of mass by

$$I = Ma^2 + I_o = M(a^2 + k^2) \tag{3}$$

where k is called the radius of gyration, or gyradius, and is a measure of the distribution of the mass about the center of mass. Conceptually the gyradius can be thought of as the half-length of a dumbbell of the same mass and moment of inertia as the body. Combining Eq. (2) and (3) the period is

$$T_1 = 2\pi \sqrt{\frac{a^2 + k^2}{ag}} \tag{4}$$

Both I and τ depend on the mass which therefore cancels, and the period depends only on the two geometrical parameters a and k which are measures of the mass distribution. For $k \ll a$ the period reduces to that of a simple pendulum.

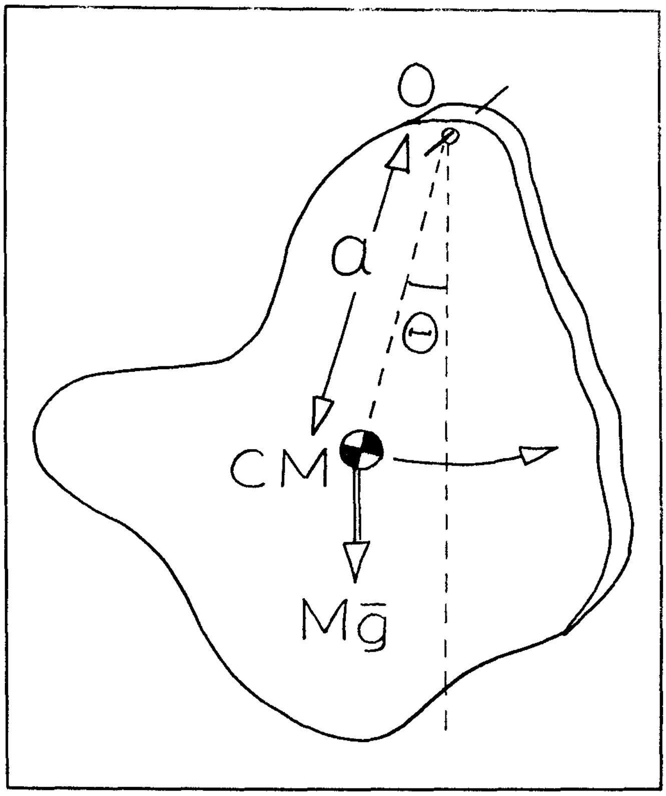


Fig. 1. For an extended object of moment of inertia $I = M(a^2 + k^2)$ about the pivot O, the period of small oscillation depends only on the distance a from the pivot to the center of mass, and the gyradius k.

For complex objects such as boats, cars, and human beings the gyradius is normally measured empirically. It should be pointed out that this must involve an angular acceleration, and that the gyradius cannot be determined from static measurements alone. Equation (4) contains the two unknowns a and k, therefore two periods of oscillation must be measured. The period T_2 about a second horizontal axis which is a measured distance b below the first axis, see Fig. 2, is

$$T_2 = 2\pi \sqrt{\frac{(a-b)^2 + k^2}{(a-b)g}} \tag{5}$$

Equations (4) and (5) can be solved for the center of mass position

$$a = b\left[\frac{4\pi^2 b + g\,T_2{}^2}{8\pi^2 b + g(T_2{}^2 - T_1{}^2)}\right] \tag{6}$$

and for the gyradius

$$k = a\sqrt{\frac{gT_1{}^2}{4\pi^2 a} - 1} \tag{7}$$

Thus by measuring the two periods T_1 and T_2 and the displacement of the axis both the vertical position of the CM and the gyradius can be calculated. Note that a depends on the difference between the squares of the two periods, which must therefore be measured quite precisely. Sufficient precision can easily be achieved with modern electronic timers, or by timing many oscillations. Alternatively, if the distance a has been determined from static measurements, this value can be used and only a single period need be measured.

Human moment of inertia

Detailed knowledge of the moment of inertia of the human body, and its separate parts, is required in a wide

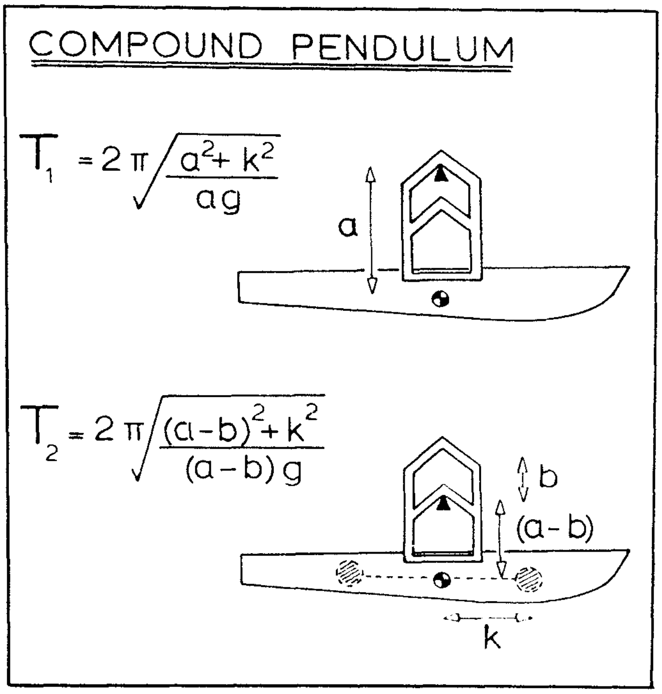
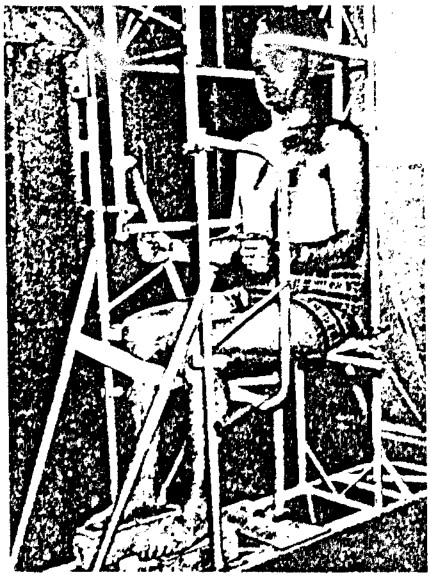


Fig. 2. The gyradius k of the boat is determined from measurements of the periods of oscillation about two axes, and the distance b between these axes.

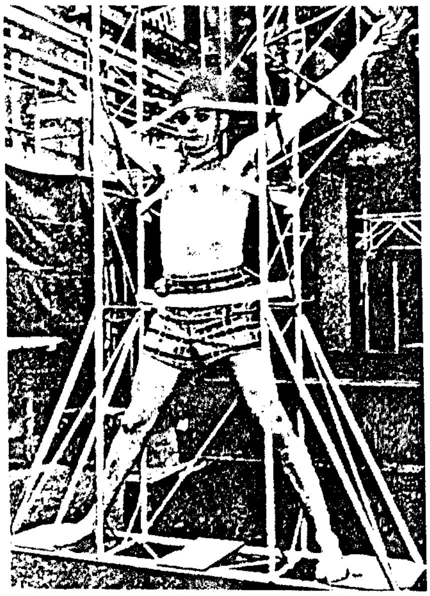
variety of studies. In the investigation of automobile safety anthropomorphic dummies are filmed to study the effects of restraints and improvements in interior design. For the dummies to model human motions not only must the masses of the body segments be correct but also the moments of inertia, otherwise the angular motions are not reproduced. Computer models are extensively used in the design of aircraft cockpits and other control systems where the human response must be optimized. The parameters required for such models of human motion include the moments of inertia.

The occupant of an aircraft ejection seat contributes significantly to the moment of inertia and must therefore be included in the design. Space-walking astronauts maneuver using small rocket engines. If the thrust is not through the center of mass it will exert a torque as well as a force, and produce an unwanted tumbling motion. To predict such motions the astronaut's moment of inertia must be known for a variety of postures.

Athletic coaches film winning performances and use computer analysis of the films to understand the mechanics involved and so improve coaching methods and athletic performance.[10,11] Examples are gymnasts, long jumpers, high jumpers (the Fosbury flop), and runners. In the latter the swinging of the arms helps to compensate for the horizontal angular momentum of the legs. A spectacular example is the motion of Olympic divers. Once they leave the diving board their angular momentum is conserved, however, they can change the magnitude and direction of their angular velocity by appropriate changes of their moment of inertia.[12] For quantitative studies the human moment of inertia must be known for many different positions and for people of different heights and weights, thus a reliable and simple measurement technique is required. Human moments of inertia can be determined from measurements on live subjects or by calculation from the moments of inertia of the separate body segments of



3A



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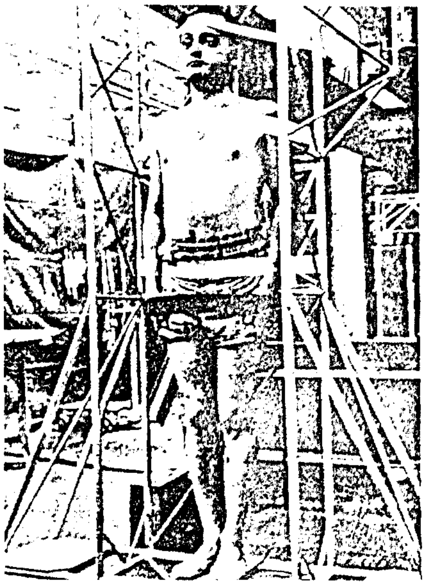

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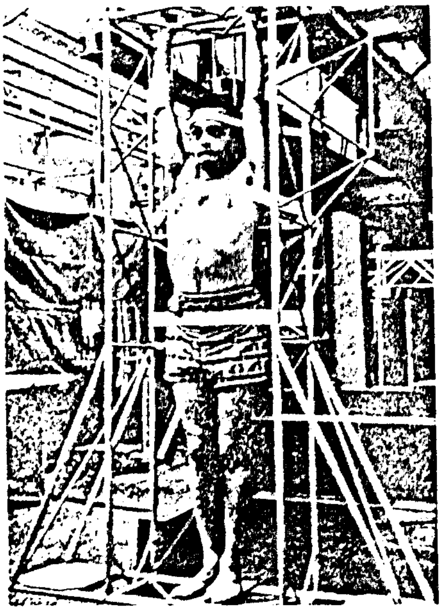

3D

Fig. 3. An airforce "volunteer" attached to a compound pendulum for measurements of his moment of inertia. Courtesy Wright-Patterson A.F.B.

frozen cadavers. The compound pendulum method has been used for both types of measurement[5] and Fig. 3 shows an Air Force "volunteer" taped to a compound pendulum. One advantage of this technique is that the forces applied to the volunteer are small enough not to change the mass distribution being measured.

Farm tractors

Farm tractors are designed for slow speed use over uneven ground and therefore, unlike cars and trucks, do not have a spring suspension. The rough ride can lead to back injuries and other health problems, and so a computer model, Fig. 4, has been used to optimize the weight distribution and seat design. The computer model[6] treated the tractor as a rigid body supported by two damped springs which simulate the front and rear tires. The spring constants and damping were determined from measurements on the tires. The operator and seat were modeled by a mass m and a third damped spring, and his motion was studied as the tractor progressed over steps and over sinusoidally varying terrain at different speeds. The front and rear wheels do not remain at the same level so angular motion

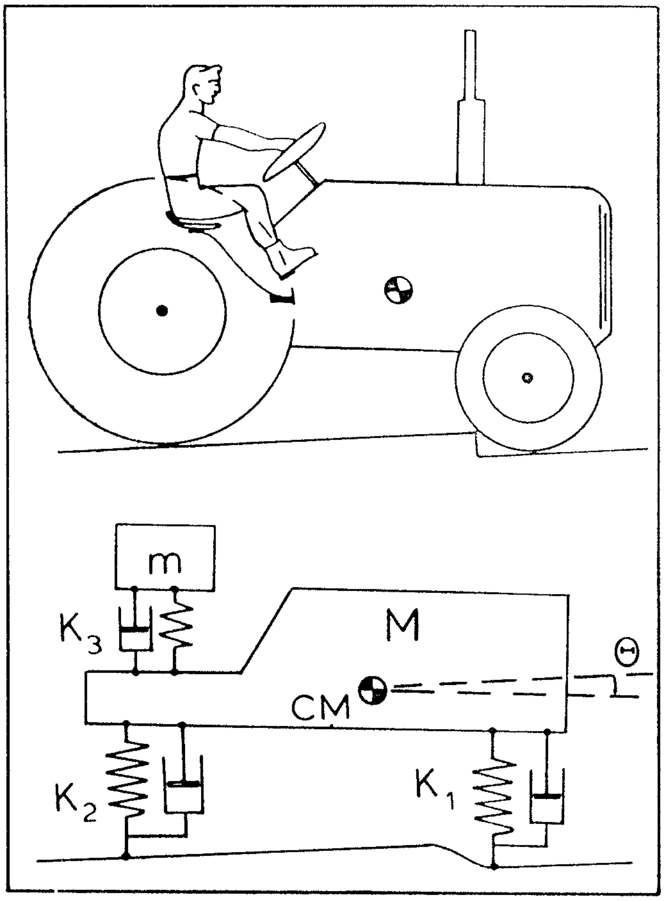

Fig. 4. Dynamic model of a tractor. The tractor of mass M and moment of inertia I is supported by the tires, represented by damped springs k_1 and k_2. The motion of the operator m supported by the seat, damped spring k_3, is studied as the tractor goes over uneven ground.

is involved, and the moment of inertia of the tractor must be known.

The tractor was mounted on a horizontally-pivoted

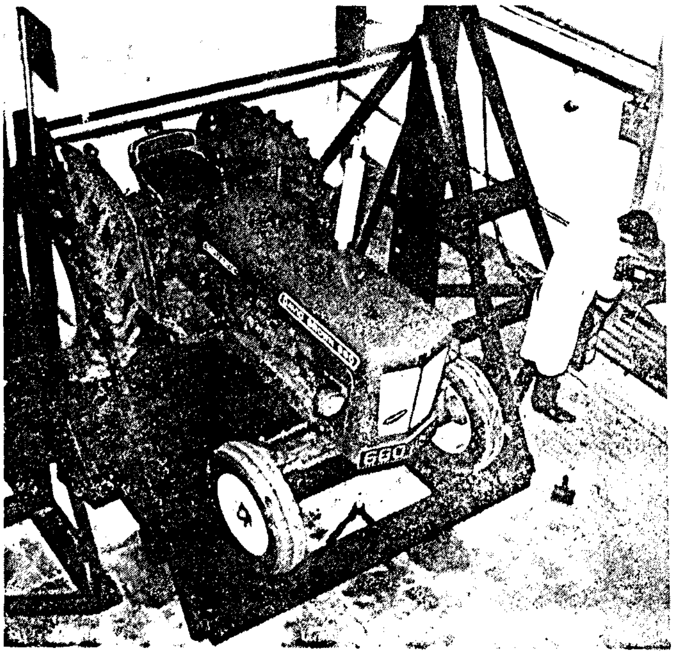

Fig. 5. A farm tractor mounted on a platform which is free to oscillate about a horizontal axis. From a static determination of the CM, and the period of oscillation, the moment of inertia can be calculated.

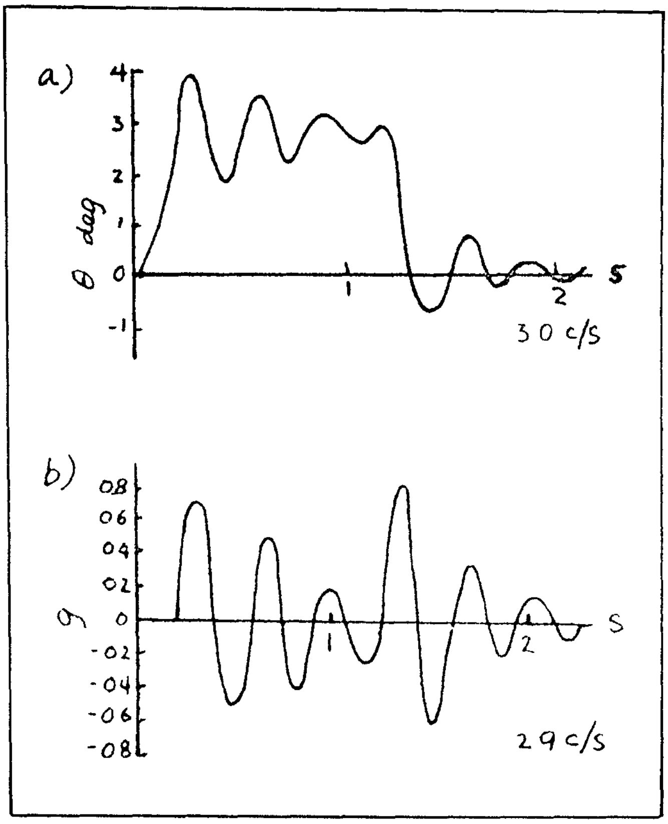

Fig. 6. The angular motion (a), and the vertical acceleration of the operator (b), as a tractor is driven at 3.6 mph over a 2-in. step.

platform and the period of oscillation was measured (Fig. 5). The CM position was determined statically, so only one period was measured, however the moment of inertia of the platform also had to be determined. A 50-horsepower tractor weighing 5950 lbs was found to have a gyradius of 2.74 ft, and thus a moment of inertia of 4.47×10^4 lb ft^2. The angular motion of the tractor, and the resulting vertical acceleration of the operator as the tractor is driven over a 2-in. step at 3.6 mph is shown in Fig. 6. The computer simulations agreed well with measurements on a real tractor, and were then used to study the effects of varying the weight distribution, wheelbase, and incorporating a suspension.

Sailboats

A third example of the compound pendulum method of measuring weight distribution is its application to sailboats.[8] The total weight of a sailboat has a major effect on its speed. A heavy boat floats lower and, therefore, causes a greater disturbance when moving through the water, also the larger immersed area increases the frictional drag. The dramatic and exhilarating increase in speed when a modern racing dinghy planes or surfs is crucial to success in racing, and even small increases in the weight can delay the onset of planing. Most Olympic sailors go to extreme lengths to ensure that their boats are at exactly the minimum allowed weight.

It is not, however, quite so obvious that the distribution of the weight can also affect the speed.[13-15] Wind generates waves and the boat responds to the waves by oscillating up and down (heave) and also by pitching, i.e., there is an angular oscillation about a horizontal athwartships axis (Fig. 7). The pitching is an oscillation with a natural period

$$T = 2\pi\sqrt{\frac{I}{K}}$$

where I is the pitching moment of inertia and the buoyant restoring torque is

$$\tau \approx -K\theta$$

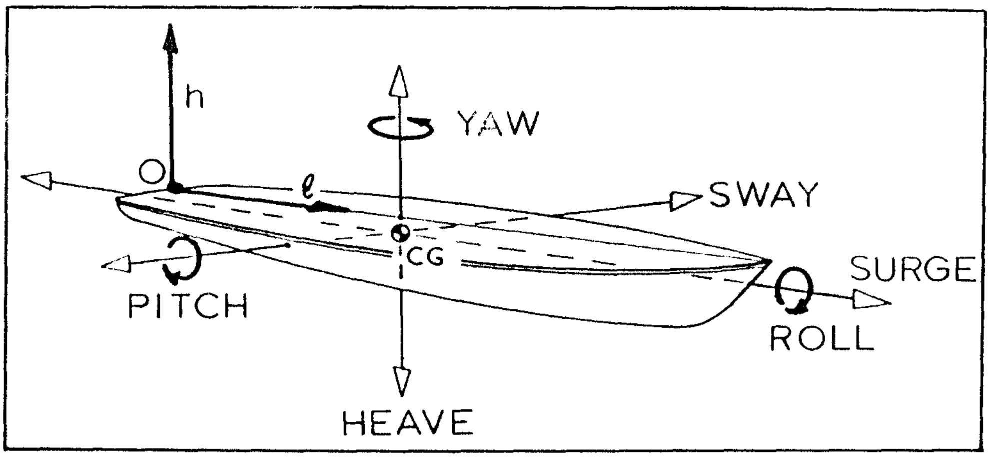

Fig. 7. The motions of a boat. The dominant effect of waves is to generate oscillations in heave and pitch. The data in Table I uses the top of the transom, O, as the origin of coordinates.

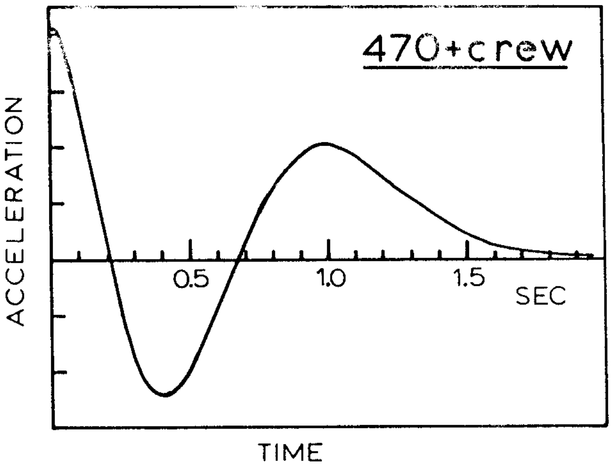

Fig. 8. The vertical acceleration at the bow of a 470 (two man Olympic sailing dinghy) after release from a bow-down position. Taken from Ref. 15.

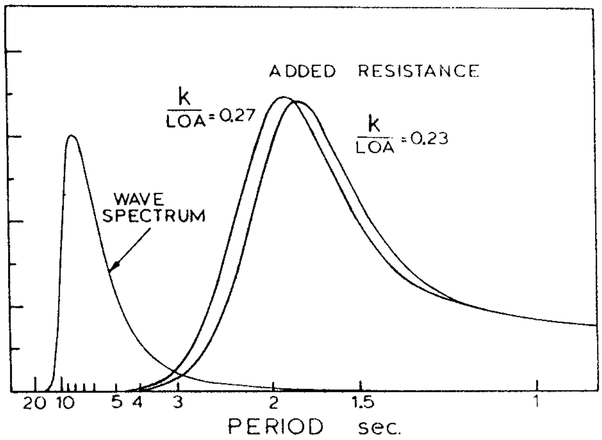

Fig. 9. The calculated added resistance due to waves for a yacht. Curves for two values of the ratio of the gyradius to the length overall are shown, together with the wave spectral density. For the larger gyradius the resonant frequency is closer to the peak in the wave spectrum and the pitching motion will be greater. After J. Gerritsma and G. Moeyes Ref. 14.

If the bow of the boat is pushed down, the center of buoyancy moves forward and there is a buoyant restoring torque which is a function of the angle θ with the horizontal [$\tau \approx -K\theta$ is the first term in the expansion]. This torque depends only on the shape of the boat, which is rigidly controlled for Olympic competition. Due to the energy expended in wavemaking and the viscous drag the pitching oscillation is heavily damped (Fig. 8).

The frequency at which the boat encounters waves depends on their wavelength, their velocity, and that of the boat. The boat's velocity Doppler-shifts the frequency. In waves, the boat can be thought of as a forced oscillator with a resonant frequency equal to the natural pitching frequency $1/T$. The amplitude of pitching can be reduced by decreasing the moment of inertia, thus increasing the resonant frequency and so moving it farther from the encounter frequency (Fig. 9). Thus weight should be kept out of the ends of a boat to reduce pitching. Pitching not only causes a large increase in resistance to forward motion but also reduces the drive generated by the sails. The oscillatory motion at the top of the mast can reverse the instantaneous airflow and destroy the lift.

This resonance theory of pitching has been applied to supertankers[14] at sea and to the design of America's Cup yachts, for which speed differences of a few seconds per mile due to pitching response have been calculated. However, for light sailing dinghies which are extremely asymmetrical and have very large damping, the resonance will be so broad that the changes in pitching response due to changes in gyradius are likely to be very small. Notwithstanding these and other theoretical objections many Olympic sailors believe that they can feel the difference in the motions of dinghies with light and heavy ends, and that the former are faster. It must be remembered that sailboat races are won and lost by extremely small margins, so that

Table I

Component	Mass kg	Mass %	Gyradius m	CM position Horiz. m	CM position Vert. m	Moment of inertia kg m^2	Moment of inertia %
Hull	125.1	74.2	1.526	2.897	−0.300	308.9	50.3
Mast	14.1	8.4	2.068	3.667	2.996	189.3	30.8
Boom	3.7	2.2	0.821	2.219	0.765	5.9	1.0
Mainsail	3.2	1.9	1.650	2.692	2.933	35.0	5.7
Genoa	2.6	1.5	1.424	3.715	1.750	14.4	2.3
Rudder and Tiller	5.1	3.0	0.545	−0.050	−0.265	46.0	7.5
Centerboard	6.6	3.9	0.387	3.000	−0.880	7.0	1.2
Anchor, paddles, etc.	8.1	4.8	0.904	3.200	−0.220	8.1	1.3
Complete boat	168.5	100.0	1.910	2.885	0.074	614.6	100.0
Clothed helmsman	90.0	53.4	0.130	1.800	0.150	46.4	7.5
Clothed crew	97.0	57.6	0.130	2.500	0.150	1.8	0.3
Total	355.5	211.0	1.390	2.505	0.114	687.3	111.8

The contributions to the moment of inertia of a "Flying Dutchman" sailboat. The CM positions are relative to the top of the transom (Fig. 7). The moments of inertia of the components are relative to the CM of the "complete boat," while those of the helmsman and crew are relative to the CM of the total.

Fig. 10. A Flying Dutchman hull suspended from two brackets ready for a "Lamboley Test," at the 1976 Olympic regatta.

very small improvements count, even if the advantage is only psychological!

In order to prevent the trend towards light ends being taken to extremes, and thus produce weak and unseaworthy boats, a nondestructive method of control had to be introduced. The method should control only those factors which could affect the speed, and leave the widest latitude for developments in new materials and construction techniques. In 1970 Gibert Lamboley[8] introduced the compound pendulum test to control weight distribution in Finn dinghy hulls (the Olympic single-handed sailboat) and this test is known by his name in sailing circles. A nomogram, or programmable pocket calculator, can be used to evaluate Eq. (6) and (7) to determine rapidly the height of the center of mass and the gyradius of the hull. Reliable results can be obtained under regatta conditions by volunteers with little scientific background.

In 1975 it was suggested that the "Lamboley Test" should be applied to all the Olympic dinghy classes. In the Flying Dutchman (Olympic two-man centerboard class) the helmsman and crewman, who constitute more than half the total weight, can move independently. They can therefore adjust both the trim, which depends on the position of the CG, and the moment of inertia, separately. It was not clear *a priori* that these changes in the moment of inertia would not far outweigh the differences due to hull construction. Furthermore, a wide variety of masts and equipment are allowed, and as it is not feasable to measure the gyradius of a complete boat (even small air currents perturb the oscillations of the mast), quantitative data on their effect was required in order to evaluate the effectiveness of a rule controlling only the gyradius of the hull.

The compound pendulum technique was used to measure the gyradii and center of mass positions of a variety of masts, rudders, etc., as well as the hulls of all the Flying Dutchmen competing in the 1976 Olympics[9] (Fig. 10). The moments of inertia of the sails were calculated, and the data for the helmsman and crewman were obtained from the previously mentioned Air Force study.[5] The results for an average Flying Dutchman together with the CM positions, relative to the top of the transom, are given in Table 1. The application of the parallel axis theorem to the evaluation of the total moment of inertia, and its dependence on each parameter, turned out to be a very successful student computer-programming project.

The results lead to a number of interesting conclusions. Despite their large contribution to the total mass, the helmsman and crewman cannot significantly change the moment of inertia while sailing the boat effectively to windward. Figure 11 shows the effect of the crewman's

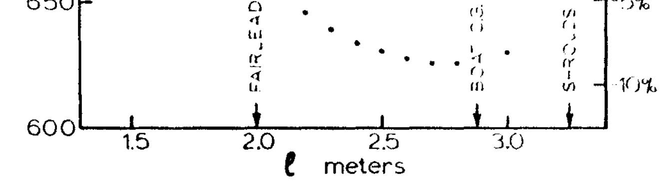

Fig. 11. The effect of the helmsman's and crew's position on the total longitudinal moment of inertia of a Flying Dutchman sailboat. Note that within the practical range, solid curve, the crew's position has only a small effect, despite the fact that he constitutes more than 25% of the total mass.

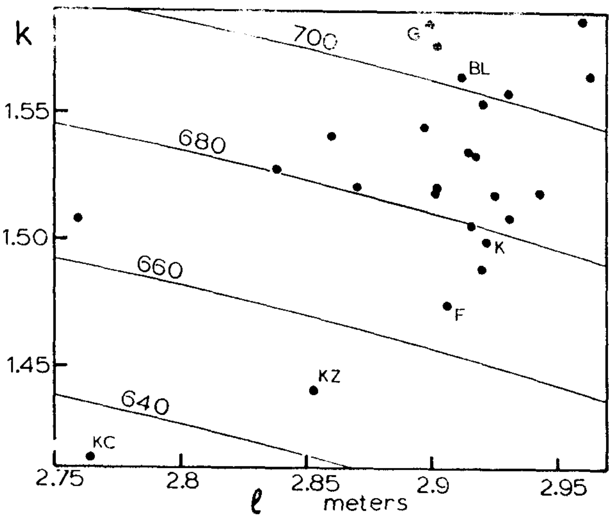

Fig. 12. The hull gyradius *k* is plotted versus the distance *l* from the transom to the CM, for the Flying Dutchmen present at the 1976 Olympics. The lines are contours of constant total moment of inertia. The finishing order G, K, Bl, KC shows no correlation with these parameters.

position with the helmsman in his optimum position, and vice versa. The crewman is close to the CM of the "boat plus helmsman," i.e., at the minimum in the total moment of inertia as a function of the crewman's position. Thus, even quite large changes in the crewman's position have only a small effect on the total moment of inertia. The effect of changes in the helmsman's position are much greater, because for practical boathandling reasons, he cannot sit far enough forward to be at the CM of the "boat plus crew."

The gyradius, and fore and aft position of the CM, for each hull are plotted in Fig. 12, which also includes contours of constant total moment of inertia. The CM of the "helmsman plus crew" is behind that of the hull, thus moving the CM of the hull aft reduces the total moment of inertia, even if the change involves some increase in the gyradius of the hull. This corroborates the empirically known fact that weight in the bow is more harmful than weight in the stern.

The finishing order in the 1976 Olympic regatta was West Germany (G), Britain (K), Brazil (Bl) and Canada (KC) and showed no correlation with the gyradius (Fig. 12). This was confirmed by a statistical analysis of the six races,[9] which hardly constitute a good sample! The hull with by far the lowest moment of inertia was a strong seaworthy boat, and so the Flying Dutchman class has not adopted a gyradius rule. The compound pendulum test is, however, performed at all major regattas for the Finn dinghy.

Conclusion

Examples of the application of compound-pendulum theory to the practical measurement of the moments of inertia of human beings, farm tractors, and sailboats serve to increase interest in this subject. It is suggested that laboratory experiments employing this method to measure the moments of inertia of model cars or aircraft,[16] of sports equipment such as racquets,[17] baseball bats, hockey sticks, golf clubs, footballs, frisbies, boomerangs, or just about anything that interests the students, would help to generate enthusiasm for and understanding of this subject. The advent of inexpensive battery-powered timers, with a pendulum mode,[18] make the technique applicable to projects outside the laboratory. Anyone who would like more information or a bibliography is encouraged to contact the author.

Acknowledgments

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References

1. D. G. Ivey and J. N. Patterson Hume, *Physics*, p. 434 (Ronald Press, New York, 1974).
2. C. J. Overbeck, R. R. Palmer, R. J. Stephenson and M.W. White, "Cenco Selective Experiments in Physics," Nos. 154 and 157 (Central Scientific Co., Chicago, 1941).
3. G. O. Kolodiy, Phys. Teach. 17, 52 (1979).
4. M. Iona, Phys. Teach. 17, 224 (1979), Am. J. Phys. 14, 252 (1946).
5. W. R. Santschi, J. Dubois and C. Omoto. Report No. AMRL-TDR-63-36. Aerospace Medical Division, Wright-Patterson A.F.B., Ohio, 1963.
6. J. Matthews and J. C. C. Talamo, J. Agric. Eng. Res. 10, No. 2, 93 (1965).
7. G. C. Sneed, "School Technology Programme," (Inst. Ed. Tech., University of Surrey, Guildford, Surrey, U.K. 1971).
8. G. Lamboley, *Yachts and Yachting*, March 26 (1971) 790, April 9 (1971) 921, International Yacht Racing Union Finn Class Rulebook (IYRU, 60 Knightsbrige, Westminster, London, 1971).
9. P. F. Hinrichsen, *Sail* 9, 28 (1978), *Yachts and Yachting* 63, Feb. (1978) p. 347.
10. G. Dyson, *The Mechanics of Athletics* (University of London Press, London 1962).
11. J. G. Hay, *The Biomechanics of Sports Techniques*, 2nd Ed. (Prentice Hall, Englewood Cliffs, 1978).
12. C. Frohlich, Am. J. Phys. 47, 583 (1979). Sci. Am. 242, 154, March (1980).
13. W. G. Van Dorn, *Oceanography and Seamanship* (Dodd Mead and Co., New York, 1974), Ch. 23.
14. J. Gerritsma and G. Moeyes, "Third HISWA Symposium," Amsterdam 1973. *Sail* 4 April (1973) p. 104.
15. R. Compton, B. Johnson, and C. Van Duyne, SNAME, Second Chesapeake Sailing Yacht Symposium, Annapolis, MD 1975.
16. L. F. Minkler, Phys. Teach. 13, 46 (1975).
17. H. Brody, Am. J. Phys. 47, 482 (1979) and *AAPT Announcer* 10, 4 (1980) C7, p. 60.
18. Pasco scientific, Model 9201 Photogale Timer, 1933 Republic Ave., San Leandro, CA 94577.