1. **Complex Numbers** [7]

(a) Express the term $(1+i)^4$ in the form $re^{i\theta}$, where r and θ are real variables.

(b) Express the complex number $\tan^{-1}(2i)$ in the form $x+iy$ where x, y are real.

(c) Given that $z = z_1 + z_2$, where $z_1 = \mathrm{e}^{i\theta_1}$, $z_2 = \mathrm{e}^{i\theta_2}$ and θ_1, θ_2 are real variables, find an expression for $|z|$ in terms of $\Delta\theta = \theta_1 - \theta_2$.

(a)

$$(1+i)^4 = \left(\sqrt{2}\mathrm{e}^{i\pi/4}\right)^4 = \boxed{4\mathrm{e}^{i\pi}}$$

(b) Let $w = \tan^{-1}(2i)$, then $\tan(w) = 2i$. Using the identity $\tan(w) = \dfrac{\sin(w)}{\cos(w)}$, we have

$$\begin{aligned}
\frac{\mathrm{e}^{iw} - \mathrm{e}^{-iw}}{i(\mathrm{e}^{iw} + \mathrm{e}^{-iw})} &= 2i \\
\mathrm{e}^{iw} - \mathrm{e}^{-iw} &= -2i(\mathrm{e}^{iw} + \mathrm{e}^{-iw}) \\
3\mathrm{e}^{iw} + \mathrm{e}^{-iw} &= 0 \\
\mathrm{e}^{2iw} &= -\frac{1}{3} \\
2iw &= \ln\left(-\frac{1}{3}\right) \\
w &= -\frac{i}{2}\left(\ln\frac{1}{3} + i\pi\right) = \boxed{\frac{\pi}{2} + \frac{i}{2}\ln 3}
\end{aligned}$$

(c)

$$\begin{aligned}
|z|^2 &= z\bar{z} = (z_1 + z_2)(\bar{z_1} + \bar{z_2}) \\
&= |z_1|^2 + |z_2|^2 + z_1\bar{z_2} + \bar{z_1}z_2 \\
&= 2 + \mathrm{e}^{i(\theta_1-\theta_2)} + \mathrm{e}^{-i(\theta_1-\theta_2)} \\
&= 2 + 2\cos(\Delta\theta) \\
\Rightarrow |z| &= \boxed{\sqrt{2\left(1+\cos(\Delta\theta)\right)}}
\end{aligned}$$

2. **Vectors** [8]

(a) Write down the equation of the plane

$$3x + 4y + 5z = 10$$

in the vector form

$$\mathbf{r} = \mathbf{r}_0 + t_1\mathbf{a} + t_2\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors in the plane, t_1 and t_2 are real parameters and $\mathbf{r}_0$ is a constant vector. What is the distance between the plane and the origin?

(b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors. Show that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{b}\cdot\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c}).$$

(a) A particular solution to the plane equation is $\mathbf{r_0} = (0,0,2)^T$. Two independent vectors in the plane are $\mathbf{a} = (0,-2.5,2)^T$ and $\mathbf{b} = (-10/3,0,2)^T$. Thus the vector form of the plane is

$$\mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + t_1\begin{pmatrix}0\\-2.5\\2\end{pmatrix} + t_2\begin{pmatrix}-10/3\\0\\2\end{pmatrix}.$$

The distance from the origin to the plane is given by

$$d = \frac{|\mathbf{r_0}\cdot\mathbf{n}|}{|\mathbf{n}|},$$

where $\mathbf{n} = (3, 4, 5)$ is the normal vector to the plane. Substituting $\mathbf{r_0} = (0, 0, 2)$ and $\mathbf{n} = (3, 4, 5)$,

$$d = \frac{|(0,0,2)\cdot(3,4,5)|}{\sqrt{3^2+4^2+5^2}} = \frac{|10|}{\sqrt{50}} = \frac{10}{\sqrt{50}} = \boxed{\sqrt{2}}.$$

(b) Using the vector triple product identity $\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z}) = \mathbf{y}\cdot(\mathbf{z}\times\mathbf{x}) = \mathbf{z}\cdot(\mathbf{x}\times\mathbf{y})$, let $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$ and $\mathbf{z} = \mathbf{a}\times\mathbf{c}$, we have

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) = \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{a}\times\mathbf{c})]$$

Since $(\mathbf{b}\times(\mathbf{a}\times\mathbf{c}))_i = \epsilon_{ijk}b_j(a\times c)_k = \epsilon_{ijk}b_j\epsilon_{klm}a_kc_m = \epsilon_{ijk}\epsilon_{klm}b_ja_kc_m$, and using the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, we get $\epsilon_{ijk}\epsilon_{klm}b_ja_kc_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})b_ja_kc_m = b_ja_ic_m\delta_{il}\delta_{jm} - b_ja_kc_m\delta_{im}\delta_{jl} = a_i(b_jc_j) - c_i(a_jb_j)$, which is just $\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$. Continuing from above,

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c}) &= \mathbf{a}\cdot[\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})] \\ &= (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b}) \\ &= 1\cdot(\mathbf{b}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c}) \\ &= \mathbf{b}\cdot\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c})\end{aligned}$$

Q.E.D.

3. **Matrix and linear equation** [5]

Consider the set of linear equations

$$\begin{aligned}2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4,\end{aligned}$$

where x, y, z are real variables and a, b are real parameters.

Find the values of a and b for which the set of equations have:

(i) a unique solution,
(ii) infinitely many solutions,
(iii) no solution.

$$\begin{aligned}2x + y + z &= 2b, \\ ax + 3y + 2z &= 2a, \\ 2x + y + 3z &= 4.\end{aligned}$$

Let the coefficient matrix be

$$A = \begin{pmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2b \\ 2a \\ 4 \end{pmatrix}.$$

The determinant of A is

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = -2(a-6).$$

(i) **Unique solution.** If $\det A \neq 0$, i.e. $a \neq 6$, the system has a unique solution for all values of b.

(ii) **Infinitely many solutions.** Let $a = 6$. The system becomes

$$\begin{aligned}2x + y + z &= 2b, \\ 6x + 3y + 2z &= 12, \\ 2x + y + 3z &= 4.\end{aligned}$$

Subtracting the first equation from the third gives

$$2z = 4 - 2b \quad \Rightarrow \quad z = 2 - b.$$

Dividing the second equation by 3 yields

$$2x + y + \frac{2}{3}z = 4.$$

From the first equation, $2x + y = 2b - z$. Substituting,

$$2b - z + \frac{2}{3}z = 4 \quad \Rightarrow \quad 2b - \frac{1}{3}z = 4 \quad \Rightarrow \quad z = 6b - 12.$$

Consistency requires

$$2 - b = 6b - 12 \quad \Rightarrow \quad b = 2.$$

Hence $z = 0$ and the remaining equation is

$$2x + y = 4.$$

Letting $x = t$, the solutions are

$$(x, y, z) = (t,\ 4 - 2t,\ 0), \qquad t \in \mathbb{R},$$

so there are infinitely many solutions when $(a, b) = (6, 2)$.

(iii) **No solution.** If $a = 6$ and $b \neq 2$, the two expressions for z are inconsistent. Hence the system has no solution.

Unique solution	$a \neq 6$ (any b),
Infinitely many solutions	$a = 6,\ b = 2,$
No solution	$a = 6,\ b \neq 2.$

4. **Differential equation** [5]

Find the general solution for the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) = n(n+1)y,$$

where x is a real variable and n is a real constant.

Using the product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 2x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Thus the differential equation can be rewritten as

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - n(n+1)y = 0.$$

This is an Euler-Cauchy equation. We try a solution of the form $y = x^m$, where m is a constant to be determined. Substituting this into the differential equation, we get

$$x^2 \cdot m(m-1)x^{m-2} + 2x \cdot mx^{m-1} - n(n+1)x^m = 0.$$

Simplifying, we have $x^m\left[m(m-1) + 2m - n(n+1)\right] = 0$, which gives us the characteristic equation

$$m^2 + m - n(n+1) = 0.$$

Solving for m, we get $m = \frac{-1 \pm (2n+1)}{2}$, so $m_1 = n$ and $m_2 = -(n+1)$.
The general solution is then

$$y(x) = Ax^n + Bx^{-(n+1)},$$

where A and B are arbitrary constants.

5. **Matrix and properties** [7]

Let A and B be $n \times n$ Hermitian matrices and U an $n \times n$ unitary matrix.

(a) Show that the modulus of each of the eigenvalues of U is equal to one ($|\lambda| = 1$).

(b) Show that the eigenvalues of A are real.

(c) Assuming that $U = A + iB$, show that

(i) $A^2 + B^2 = I$, where I is the identity matrix,

(ii) $AB - BA = 0$.

(a) Let λ be an eigenvalue of U with corresponding eigenvector $\mathbf{v}$, so that $U\mathbf{v} = \lambda\mathbf{v}$. Taking the norm on both sides, we have

$$\|U\mathbf{v}\| = \|\lambda\mathbf{v}\|.$$

Since U is unitary, it preserves the norm, so $\|U\mathbf{v}\| = \|\mathbf{v}\|$. Also, $\|\lambda\mathbf{v}\| = |\lambda|\|\mathbf{v}\|$. Therefore,

$$\|\mathbf{v}\| = |\lambda|\|\mathbf{v}\|.$$

Since $\mathbf{v}$ is a non-zero eigenvector, $\|\mathbf{v}\| \neq 0$, we can divide both sides by $\|\mathbf{v}\|$ to get

$$1 = |\lambda|.$$

Thus, the modulus of each eigenvalue of U is equal to one.

(b) Let μ be an eigenvalue of A with corresponding eigenvector $\mathbf{w}$, so that $A\mathbf{w} = \mu\mathbf{w}$. Taking the conjugate transpose of both sides, we have

$$\mathbf{w}^\dagger A^\dagger = \mu^* \mathbf{w}^\dagger.$$

Since A is Hermitian, $A^\dagger = A$. Thus,

$$\mathbf{w}^\dagger A = \mu^* \mathbf{w}^\dagger.$$

Multiplying both sides by $\mathbf{w}$ from the right, we get

$$\mathbf{w}^\dagger A\mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

On the other hand, from the original eigenvalue equation,

$$\mathbf{w}^\dagger A\mathbf{w} = \mu \mathbf{w}^\dagger \mathbf{w}.$$

Equating the two expressions for $\mathbf{w}^\dagger A\mathbf{w}$, we have

$$\mu \mathbf{w}^\dagger \mathbf{w} = \mu^* \mathbf{w}^\dagger \mathbf{w}.$$

Since $\mathbf{w}$ is a non-zero eigenvector, $\mathbf{w}^\dagger \mathbf{w} \neq 0$, we can divide both sides by $\mathbf{w}^\dagger \mathbf{w}$ to get

$$\mu = \mu^*.$$

Thus, the eigenvalues of A are real.

(c) Given that $U = A + iB$ is unitary, we have

$$U^\dagger U = I.$$

Calculating $U^\dagger U$, we get

$$(A - iB)(A + iB) = A^2 + iAB - iBA + B^2 = A^2 + B^2 + i(AB - BA).$$

Setting this equal to the identity matrix I, we have

$$A^2 + B^2 + i(AB - BA) = I.$$

Equating the real and imaginary parts, we obtain the two equations:

(i) Real part:

$$A^2 + B^2 = I.$$

(ii) Imaginary part:

$$AB - BA = 0.$$

6.