# Basics

## Notations

- $\binom{V}{k} := \{A : A \subseteq V \land |A| = k\}$
- $[n] := \{1, \ldots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

## Graphs

- **Definition**: $G = (V, E)$ with $E \subseteq V^2, V \cap E = \emptyset$
- **Vertex**: $v \in V$ for graph $G = (V, E)$
  - $v$ *incident with* $e \Longleftrightarrow v \in e$
  - $v_1, v_2$ *ends of* $e \Longleftrightarrow e = v_1 v_2$
  - $v_1, v_2$ *adjacent/neighbors* $\Longleftrightarrow v_1 v_2 \in E$
- **Edge**: $e = \{x, y\} \in E$ for graph $G = (V, E)$ (short $e = xy$)
  - $e$ *edge at* $v \Longleftrightarrow v$ incident with $e$
  - $e$ *joins* $v_1, v_2 \Longleftrightarrow e = v_1 v_2$
  - $xy$ is *X-Y-edge* $\Longleftrightarrow x \in X \land y \in Y$
  - $e_1, e_2$ *adjacent/neighbors* $\Longleftrightarrow \exists v : v \in e_1 \land v \in e_2$
- **Vertex sets**:
  - $V(G) = V$ for graph $G = (V, E)$
  - $X \subset V(G)$ *independent* $\Longleftrightarrow$ no $x_1, x_2 \in X$ are adjacent
  - *neighborhood of* $v \in V(G)$: $N(v) = \{u \in V(G) : uv \in E(G)\}$
- **Edge sets**:
  - $E(G) = E$ for graph $G = (V, E)$
  - $E(X, Y)$: set of edges between $X \subset V(G)$ and $Y \subset V(G)$
  - $E(x, Y)$: set of edges between vertex $x \in V(G)$ and $Y \subset V(G)$
  - $E(v)$: set of edges at $v \in V(G)$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $\|G\|$
- **Trivial graph**: graph of order 0 or 1
- **Incidence graph** of $G$: $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Isomorphic** ($G_1$ to another graph $G_2$, write $G_1 \cong G_2$ or even $G_1 = G_2$):
  $$\exists \text{ bijection } f : V_1 \to V_2 : \{u, v\} \in E_1 \Longleftrightarrow \{f(u), f(v)\} \in E_2$$
- **Graph union**: $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$
- **Graph intersection**: $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$
- **Graph multiplication**: $G * G'$: join all $v \in G$ with all $v' \in G'$
  (with $V(G) \cap V(G') = \emptyset$)
- **Subgraph** $G'$ of $G$ (write $G' \subseteq G$): if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$
  - $G$ *contains* $G'$
  - $G'$ *proper subgraph* of $G$: if $G' \subseteq G$ and $G' \neq G$
  - $G'$ *induced subgraph* of $G$: $G' \subseteq G$ and $E(G')$ contains all edges of $G$ with both ends in $V(G')$, $V(G')$ *induces* $G'$, write $G' = G[X]$ (with $X = V(G')$)
  - *Edge-induced subgraph*: subgraph induced by $X \subseteq E(G)$, note $G[X]$
  - $G'$ *spanning subgraph* of $G$: $V(G') = V(G)$
- **Supergraph**: $G$ of $G'$ (write $G \supseteq G'$): as above.
- **Vertex cover**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in $V'$
- **Graph subtraction**:
  - $G - U = G[V(G) \setminus U]$ for some vertex set $U$
  - $G - v = G[V(G) \setminus \{v\}]$ for some vertex $v$
  - $G - G' = G[V(G) \setminus V(G')]$ for some graph $G'$
- **Edge addition**: $G + F = (V(G), V(E) \cup F)$ for some $F \subseteq V(G)^2$
- **Complement**: $\overline{G} = (V(G), V^2 \setminus E(G))$
- **Line graph** of $G$: $L(G) = (E(G), \{xy \in E(G)^2 : x, y \text{ adjacent in } G\})$
- **Complete graph**: $(X, X^2)$ with vertex set $X$
  - $K_n$: complete graph on $n$ vertices

## Vertex degrees

- **Degree** of $v \in V$: $d(v) = \deg(v) = |N(v)|$
  - $v \in V(G)$ *isolated*: $d(v) = 0$
  - $v \in V(G)$ *leaf*: $d(v) = 1$
  - number of vertices of odd degree is even
- **Minimum degree** of graph $G$: $\delta(G) = \min\{d(v) : v \in V(G)\}$
- **Maximum degree** of graph $G$: $\Delta(G) = \max\{d(v) : v \in V(G)\}$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Average degree** of graph $G$: $d(G) = \frac{1}{|G|} \sum_{v \in V} d(v)$
  - $\delta(G) \leq d(G) \leq \Delta(G)$
- **k-regular graph**: $\forall v \in V(G) : d(v) = k$
  - *cubic graph*: 3-regular graph
- **Vertex-Edge-ratio** of graph $G$: $\varepsilon(G) = \frac{\|G\|}{|G|}$
  - $\varepsilon(G) = \frac{1}{2} d(G)$
  - every graph with $\|G\| \geq 1$ has $H \subseteq G$ with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$

## Paths

- **Path**: $(\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\})$ (read: $v_0 v_n$-path)
  - *shorthand*: $v_1 \ldots v_n$
  - $v_0, v_n$ *linked* by path
  - $v_0, v_n$ *end-vertices/ends* of path
  - $v_1, \ldots, v_{n-1}$ *inner vertices* of path
- **Length**: $|E(P)| \neq |V(P)|$
- **Shorthands** ($0 \leq i \leq j \leq k$):
  - $P = x_0 \ldots x_k$, $\mathring{P} = x_1 \ldots x_{k-1}$
  - $Px_i = x_0 \ldots x_i$, $P\mathring{x}_i = x_0 \ldots x_{i-1}$
  - $x_i P = x_i \ldots x_k$, $\mathring{x}_i P = x_{i+1} \ldots x_k$
  - $x_i P x_j = x_i \ldots x_j$, $\mathring{x}_i P \mathring{x}_j = x_{i+1} \ldots x_{j-1}$
- **Path concatenation**: $Px \cap xQy \cap yR = PxQyR$
- **A-B-path**: $V(P) \cap A = \{x_0\} \land V(P) \cap B = \{x_n\}$
- **H-path**: graph $H$, $P$ meets $H$ exactly in its ends
- **Independent**: two $ab$-paths are independent $\Longleftrightarrow$ they only share $a$ and $b$
- **Path existence**: Every $G$ with $\delta(G) \geq 2$ contains path of length $\delta(G)$
- **Distance**: $d_G(x, y) = \min(\{k : \exists x\text{-}y\text{-path of length } k\} \cup \{\infty\})$
- **Central**: $v \in V(G)$ where $\text{cen} = \max\{d_G(v, x) : v \neq x \in V(G)\}$ is minimal
- **Radius**: $\text{rad}(G) = \text{minimal cen} = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$
- **Diameter** of $G$: $\text{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$
  - *radius-diameter-relation*: $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$
  - *radius-degree-vertex-restriction*:
    $$\text{rad}(G) \leq k \land \Delta(G) \leq d \geq 3 \Longrightarrow |G| \leq \frac{d}{d-2}(d-1)^k$$
- **Walk**: alternating sequence $v_0 e_0 \ldots e_{k-1} v_k$ s.t. $e_i = v_i v_{i+1}$ ($\forall i < k$)
  - *closed walk*: $v_k = v_0$
  - *walk-path-relation*: all vertices in walk distinct $\rightsquigarrow$ path
  - *walk-path-induction*: $\exists v_0 v_k$-walk $\Longrightarrow \exists v_0 v_k$-path

## Cycles

- **Cycle**: $C = P + x_{k-1}x_0$ with path $P = x_0 \ldots x_{k-1}$ ($k \geq 3$)
  - *shorthand*: $x_0 \ldots x_{k-1}x_0$
- **Length**: $= |C| = \|C\|$
- **k-cycle**: $C_k$ = cycle of length $k$
- **Girth** of graph $G$: $g(G) = \min(\{k : G \text{ contains } C_k\} \cup \{\infty\})$
  - *girth-diameter-relation*: $g(G) \leq 2\text{diam}(G) + 1$
  - *girth-vertex-relation*: $\delta(G) \geq 3 \Longrightarrow g(G) < 2\log|G|$
- **Circumference** of graph $G$: $= \max(\{k : G \text{ contains } C_k\} \cup \{0\})$
- **Chord** of cycle $C \subseteq G$: $= xy \in E(G)$ with $xy \notin E(C)$, but $x, y \in V(C)$
- **Induced cycle**: induced subgraph of $G$ that is a cycle (= cycle in $G$ with no chords)
- **Cycle existence**: Every $G$ with $\delta(G) \geq 2$ contains cycle of length $\geq \delta(G) + 1$
- **Odd closed walk, odd cycle**: $G$ has *odd* closed walk $\Longrightarrow G$ has odd cycle

## Connectivity

- **Connected** graph $G$: $\forall x, y \in V(G) : \exists xy$-path
  - *connected subset* $U \subseteq V(G)$: if $G[U]$ is connected
- **Vertex enumeration**: $G$ connected $\Longrightarrow$ vertices can be enumerated $v_1, \ldots, v_n$ s.t. $G_i := G[v_1, \ldots, v_i]$ is connected ($\forall i \leq n$)
- **Component**: maximal connected subgraph
  - *graph partitioning*: components partition $G$
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \Longleftrightarrow$ any $A$-$B$-path has vertex in $X$
- *separator $X$*
- **Cut-Vertex**: vertex separating two other vertices of the component
- **Bridge**: edge separating its ends (= edges of component not lying on any cycle)
- **k-connected**: if $|G| > k \land G - X$ is connected $\forall X \subseteq V(G)$ with $|X| < k$
  - $\rightsquigarrow$ no two vertices in $G$ are separated by fewer than $k$ other vertices
  - $\rightsquigarrow$ any two vertices can be joined by $k$ independent paths
- **k-linked**: if for any $2k$ vertices $(s_1, \ldots, s_k, t_1, \ldots, t_k) \exists$ pairwise disjoint $s_i t_i$-paths (*note*: $k$-connected $\not\Rightarrow k$-linked)
- **Connectivity**: $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected**: if $|G| > 1 \land G - F$ is connected $\forall F \subseteq E(G)$ with $|F| < l$
- **Edge-connectivity**: $\kappa'(G) = \lambda(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Connectivity and smallest degree**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Connectivity and average degree**: $d(G) \geq 4k \Longrightarrow G$ has $k$-connected subgraph
- **Menger's theorem**: for $A, B \subseteq V(G)$:
  min # of vertices separating $A$ and $B$ = max # of disjoint $A$-$B$-paths
  - *edge corollary*: minimum number of edges separating $a$ and $b$ = maximum number of edge-disjoint $a$-$b$-paths
- **Menger global**:
  - *k-connected* $\Longleftrightarrow \forall a, b \in V(G) \exists k$ pairwise independent $ab$-paths
  - *k-edge-connected* $\Longleftrightarrow \forall a, b \in V(G) \exists k$ pairwise edge-disjoint $ab$-paths

## Trees and forests

- **Forest**: Graph with no cycle as subgraph
- **Tree**: Graph that is connected and acyclic
- $\iff G$ is connected and $\forall e \in E(G) : G - e$ is disconnected
  (*minimal-connected*)
- $\iff G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle
  (*maximal-acyclic*)
- $\iff G$ is connected and *1-degenerate* $(\forall G' \subseteq G : \delta(G') \leq 1)$
- $\iff G$ is connected and $\|G\| = |G| - 1$
- $\iff G$ is acyclic and $\|G\| = |G| - 1$
- $\iff \forall u, v \in V(G) \, \exists$ unique $uv$-path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree $T$, $|T| \geq 2 \Rightarrow T$ has leaf
- **Edge count**: Tree $T$, $|T| = n \Rightarrow \|T\| = n - 1$

## Bipartite graphs

- **r-partite** graph $G$: $V(G)$ allows partitioning in $r$ classes s.t. $\forall e = xy \in E(G)$:
  $x$ and $y$ are in different classes
- **Bipartite** graph: 2-partite graph
- $\iff G$ contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$

## Contraction and minors

- **Subdivision** of graph $G$: any graph obtained from $G$ by subdividing edges
- **Topological minor**: $H$ is topological minor if $TH \subseteq G$ where $TH$ is built from
  $H$ by subdividing edges
  - *branch vertices*: original vertices of $H$
  - *subdividing vertices*: vertices placed on edges joining branch vertices
- **MH**: $G \overset{(*)}{=} MH$ is *minor* of $H$ if
  - $V(G) = V_1 \dot\cup \cdots \dot\cup V_n$ with $n = |H|$
  - $G[V_i]$ connected $(\forall i = 1, \ldots, n)$
  - If $V(H) = \{v_1, \ldots, v_n\}$ and $v_i v_j \in E(H)$, then $\exists$ edge between $V_i$ and $V_j$
  $(*)$: *Notation abuse*: $MH$ is class of graphs
- **Branch sets**: $V_i$'s from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** (H of G, noted $H \preccurlyeq G$): $\iff MH \subseteq G$
  $\rightsquigarrow H \preccurlyeq G \iff H$ can be obtained by edge/vertex deletions + **contractions**.
- **Note**: $TH \subseteq MH$
- **Edge contraction**:
  $$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$$
  $$(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
  with $xy \in E(G)$
- **De-contraction**: if $\exists xy \in E(G) : \kappa(G \circ xy) \geq 3$
  (for $G$ with $\kappa(G) \geq 3$, $|G| \geq 5$)

## Euler tours

- **Definition**: closed walk with
  - no edges of $G$ are repeatedly used
  - all edges of $G$ are used
- **Eulerian graph**: graph containing an Euler tour $\iff \forall v \in V(G) : d(v)$ even

## Algebraic assets

- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$

## Other graph notions

- **Digraph**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$
- **Multigraph**: $G = (V, E)$ with vertex set $V$ and multiset $E$ of $V$-pairs
- **Multigraph**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

# Matching, Covering, Packing

## Matching in bipartite graphs

- **Vertex cover**: $U \subseteq V(G)$ s.t. all edges in $G$ are incident to a vertex $\in U$

- **Matching** $M$: set of independent edges in a graph
  - *matching graph*: $\delta(G) = \Delta(G) = 1$
  - *saturating*: $G = (A \cup B, E)$ has matching saturating $A$
    $\iff \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$
  - *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1)$.
    $\Rightarrow \exists$ matching $M$ saturating all but at most $d$ vertices of $A$
- **k-factor**: $k$-regular spanning subgraph
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover
  (Königs theorem)
- **Matching existence**:
  - *neighbor-based*: $G = (A \cup B, E)$ contains matching of $A \iff |N(S)| \geq |S|$
  - *regular + bipartite*: $G$ is $k$-regular + bipartite $(k \geq 1) \Rightarrow G$ has 1-factor
  - *2k-regular*: graph $2k$-regular $(k \geq 1) \Rightarrow$ has 2-factor
- **Marriages**: make matchings based on preferences
  - *preferences*: family $(\leq_v)_{v \in V}$ of linear orderings $\leq_v$ on $E(v)$
  - *stable matching $M$*:
    $\forall e \in E \setminus M \, \exists f \in M : e$ and $f$ have common vertex $v$ with $e <_v f$
  - *stable matching existence*: For every set of preferences, $G$ has stable matching
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$,
  $f : V(G) \to \{0, 1, \ldots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and $k$ divides $|G|$, then $G$ has $K_k$-factor

## Matchings in general graphs

- **Perfect matching**: spanning + matching subgraph of $G$ (aka *1-factor*)
  - *existence (Tutte)*: $G$ has perfect matching $\iff \forall S \subseteq V(G) : q(G - S) \leq S$
    (Tutte's condition, $q(G)$ = number of components in $G$ with odd order)
  - *existence (Petersen)*: $G$ bridgeless + cubic $\Rightarrow G$ has 1-factor

# Connectivity

## 2-connected graphs and subgraphs

- **2-connected construction**: $G$ is 2-connected $\iff$ it can be constructed by successively adding paths to a cycle (removing those paths: *ear-decomposition*)
- **Block**: maximal 2-connected subgraph or bridge
  - share $\leq 1$ vertices with another
  - *cycles* of $G$ = cycles of its blocks
  - *bounds* of $G$ = minimal cuts of its blocks
- **Block-cut-vertex graph**
  - $V$ = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex $v$, block $B\}$
  - block-cut-vertex graph of connected graph is tree

## Structure of 3-connected graphs

- **3-connected + decontraction**: all 3-connected graphs can be built by iteratively
  de-contracting vertices of $K_4$
- **3-connected + contraction**: 3-connected $\iff \exists$ separate $G_0, \ldots, G_k$ with
  $G_0 = K_4, G_k = G, G_i = G_{i+1} \circ xy$
  with $\deg(x), \deg(y) \geq 3$

## Rest

- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \ldots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \leq k(k - 1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$

# Planar graphs

## Basic definitions

- **Plane graph**: graph drawn without intersecting edges
- **Planar graph**: graph that *can be* dran as plane graph
- **Homeomorphism**: $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t. $f^{-1}$ is also continuous
- **Arc**: homeomorphic image of $[0, 1]$ in $\mathbb{R}^2$ under $f$
  - *endpoints*: $f(0)$ and $f(1) \rightsquigarrow$ arc "joins" endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments

- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and $Y$ is maximal $(X \subseteq \mathbb{R}^2)$
- **Boundary** of $X \subseteq \mathbb{R}^2$:
  $$\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon) \text{ contains points of } X \text{ and not of } X\}$$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\overline{x} : \mathrm{dist}(\overline{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions $R_1, R_2$ and $\delta R_1 = X = \delta R_2$.

## Plane graphs

- **Definition**: graph such that $E(G)$ is set of arcs in $\mathbb{R}^2$ and endpoints of arcs in $E(G)$ are vertices and:
  - $\forall e, e' \in E, e \neq e' : e$ and $e'$ have distinct sets of edge sets
  - $\forall e \in E, \mathring{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus \left(\bigcup_{e \in E} e \cup V\right)$
  - *face set* of graph $G$: $F(G)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle $\iff$ graph is maximally plane
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no $\triangle$: $\|G\| \leq 2|G| - 4$
- **Properties**: Let $G$ be plane graph and $H \subseteq G$.
  - *face inheritance*: $\forall f \in F(G) \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*: $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*: $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \lor \delta f \cap \mathring{e} = \varnothing$
  - *edges in circles*:
    $$e \in E(G) \text{ is edge of a cycle} \Rightarrow e \text{ is on boundary of exactly 2 faces}$$
    $$\text{not edge of a cycle} \Rightarrow e \text{ is on boundary of exactly 1 face}$$
  - *faces in cycles*: $f_1, f_2 \in F(G)$. $f_1 \neq f_2 \land \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
  - *cyclic boundaries*: $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
  - *2-connected*: 2-connected plane graph $\Rightarrow \forall f \in F(G)$ bounded by cycle
  - *3-connected*: boundaries of 3-connected plane graph = its non-separating induced cycles
  - *order $\geq 3$*: plane graph of order $\geq 3$ maximally plane $\iff$ is plane triangulation
- **Euler's formula**: If $G$ is connected plane graph with $f$ faces, then
  $$|G| - \|G\| + f = 2$$
- **Dual multigraph**: Given plane $G$:
  1. Insert vertex in each face
  2. Put edge $\tilde{e}$ between vertices if respective faces share $e$ (s.t. $\tilde{e}$ and $e$ cross once)
  3. *Result*: Dual graph $G'$ of $G$ (plane multigraph)
  $\rightsquigarrow$ faces of $G$ properly $k$-colored $\iff \exists$ proper $k$-coloring of vertices of $G'$

## Planar graphs

- **Definition**: graph s.t. $\exists$ plane graph $G'$ and bijection $f : V(G) \to V(G')$ s.t. $\forall u, v \in V(G), uv \in E(G) : f(u), f(v)$ are endpoints of arc in $G'$
- **Planar embedding** of $G$: $f$ from the definition
- **Planar because of minors**: The following statements are equivalent:
  - $G$ is planar
  - $G \not\supseteq MK_5 \land G \not\supseteq MK_{3,3}$
  - $G \not\supseteq TK_5 \land G \not\supseteq TK_{3,3}$
- **$\delta(G)$ limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: $K_5$ and $K_{3,3}$ are not planar
- **Kuratowski's lemmas**:
  1. $(TK_5 \subseteq G \lor TK_{3,3} \subseteq G) \iff MK_5 \subseteq G \lor MK_{3,3} \subseteq G$
  2. $\kappa(G) \geq 3 \land MK_5 \not\subseteq G \land MK_{3,3} \not\subseteq G \Rightarrow G$ is planar
  3. $\kappa(G) \geq 3$, $G$ edge-maximal wrt not containing $TX$. If $S$ is vertex-cut of $G$, $|S| \leq 2 \land G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$, then $G_i$ is edge-maximal with no $TX$ and $S$ induces an edge
  4. $|G| \geq 3$, $G$ edge-maximal wrt not containing $TK_5$ and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of $G$ on surface $S$): any closed simple curve in any region of $S - G$ is continuously contractible into a point
- **Euler characteristic**: $G$ embedded on surface $S \Rightarrow n - e + f = $ *Euler characteristic* is invariant
- **Euler genus**: $n - e + f = 2 - 2\gamma \rightsquigarrow$ *Euler genus $2\gamma$ of $S$*
- **Heawood's formula**: $\chi(G) \leq \left\lfloor \dfrac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor$
  $$\underbrace{\qquad\qquad}_{f(\gamma), \text{ Heawoods number}}$$
  (for $G$ embedded on $S$ with Euler char $2 - 2\gamma$)
- **Klein bottle**: $K_{f(\gamma)}$ is embeddable on $S$, unless $S$ is *klein bottle*

## Algebraic planarity criteria — Posets

- **Definition**: antisymmetric, reflexive, transitive relation on $X$ (write $x \leq y$ instead of $(x, y)$)
- **Incidence poset** of $G$: poset whose cover diagram is represented by $IG$ with vertices all below the edges
- **Poset dimension**: $\dim(R) = $ smallest $k \in \mathbb{N} : R$ is intersection of $k$ total orders

- **Poset dimension in planar graphs**: $G$ planar $\iff \dim(\text{incidence poset}) \leq 3$

# Coloring

## Base definitions

- **Vertex coloring**: map $c : V(G) \to S$ with $c(v) \neq c(w)$ for adjacent $v, w$
  - *color set* $S$
  - *$k$-coloring*: coloring $c : V(G) \to S$ with $|S| = k$
- **(Vertex) chromatic number**: $= \chi(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-coloring}\}$
  - $\chi(G) \geq \omega(G)$
  - $\chi(G) \geq \dfrac{|G|}{\alpha(G)}$
  - $\chi(G) \leq \Delta(G) + 1$ (*greedy coloring*)
  - $G$ connected, not complete, no odd cycles $\Rightarrow \chi(G) \leq \Delta(G)$
  - *$k$-chromatic* graph: $\chi(G) = k$
  - *$k$-colorable* graph: $\chi(G) \leq k$
- **Color classes**: partitions of $V(G)$ with same color
- **Equitable coloring**: proper coloring + color classes have almost ($\pm 1$) equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors
- **$ij$-flip**: $c' : V(G) \to [k]$ is $ij$-flip at $v \in V(G)$
  $\iff c'$ obtained by flipping colors $i$ and $j$ in max. conn. component containing $v$
- **Edge coloring**: map $c : E(G) \to S$ with $c(e) \neq c(f)$ for adjacent $e, f$
  - edge coloring of $G \iff$ vertex coloring of $L(G)$
  - *$k$-edge-coloring*: edge-coloring $c : E(G) \to S$ with $|S| = k$
- **Edge chromatic number**: $= \chi'(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-edge-coloring}\}$

## Coloring maps and planar graphs

- **4-color-theorem**: every planar graph is 4-colorable
- **3-color-theorem**: every triangle-free planar graph is 3-colorable

## Coloring vertices

- **Chromatic number upper bound**: $\chi(G) \leq \frac{1}{2} + \sqrt{2\|G\| + \frac{1}{4}}$
- **Greedy coloring**: sort vertices $v_1, \ldots, v_n$, color them with the smallest possible color starting at $v_1$
  $\rightsquigarrow$ never uses more than $\Delta(G) + 1$ colors
- **coloring number** of graph $G$: $\mathrm{col}(G) := $ smallest $k$ s.t. $G$ has vertex enumeration where each vertex is preceded by $< k$ neighbors

## Coloring edges

- **Vizing's theorem**: for every graph $G$, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$
- **Bipartite graphs**: $\chi'(G) = \Delta(G)$

## List coloring

- **L-list-colorable**: if $\exists c : V \to \mathbb{N} \ \forall v \in V : c(v) \in L(v)$
  (for *list of colors $L(v) \subseteq \mathbb{N}$ for each vertex*, adjacent vertices receive different colors)
- **$k$-list-colorable/-choosable**: if $G$ is $L$-list-colorable for each list $L$
- **List chromatic number**: $\chi_l(G) = \mathrm{ch}(G)$
  $= \min\left\{k : G \text{ is } L\text{-colorable } \forall L : V \to 2^\mathbb{N} : |L(v)| = k \forall v \in V(G)\right\}$
  - $\chi_l(G) \geq \chi(G)$ because we can choose $L(v) = \{1, \ldots, k\}$ ($\forall v \in V(G)$)
  - often $\chi_l(G) \gg \chi(G)$ (see $K_{m,n}$: $\chi = 2, \chi_l \approx \log n$)
- **Planar graphs**: $\chi_l(G) \leq 5$
- **Locally planar graphs**: $\chi_l(G) \leq 5$

## Perfect graphs

- **Clique number** of graph $G$: $\omega(G) := \max\{k \in \mathbb{N} : K_k \subseteq G\}$
- **Co-clique number** of graph $G$: $\alpha(G) := $ largest order of independent set in $G$
- **Independence number** of graph $G$: size of largest independent vertex set
- **Perfect graph**: $\forall H \underset{\mathrm{ind}}{\subseteq} G : \chi(H) = \omega(H)$
- **Perfect complement**: $G$ is perfect $\iff \overline{G}$ is perfect
- **Perfect graph conjecture**: $G$ is perfect $\iff$
  $$C_{2k+1} \not\subseteq G \text{ for } k \geq 2 \land \overline{C_{2k+1}} \not\subseteq G$$

# Extremal Graph Theory

## Base definitions

- **Sparse** graph: $\|G\| \sim |G|$
- **Dense** graph: $\|G\| \sim |G|^2$
- **Density**: $\|X,Y\| := \#$ edges between $X$ and $Y$, $d(X,Y) := \frac{\|X,Y\|}{|X||Y|}$
- **Edge density** of graph $G$: $\|G\| / \binom{|G|}{2}$
- **$\varepsilon$-regular pair** $(X,Y)$: if $|d(X,Y) - d(A,B)| \le \varepsilon$ for $\varepsilon > 0$ and all $A \subseteq X$, $B \subseteq Y$ with $|A| \ge \varepsilon|X|, |B| \ge \varepsilon|Y|$
- **$\varepsilon$-regular partition**: $= V_0 \dot\cup \cdots \dot\cup V_k = V$ with
  1. $|V_0| \le \varepsilon|V|$
  2. $|V_1| = \cdots = |V_k|$
  3. all but at most $\varepsilon k^2$ of $(V_i, V_j)$-pairs $(1 \le i < j \le k)$ are $\varepsilon$-regular

## Subgraphs

- **Extremal number**: $\mathrm{ex}(n,H) := \max\{\|G\| : |G| = n \wedge H \nsubseteq G\}$
- **Extremal set**: $\mathrm{EX}(n,H) := \{G : |G| = n \wedge \|G\| = \mathrm{ex}(n,H) \wedge H \nsubseteq G\}$
- **Turán graph**: $T(n,r) =$ unique complete $r$-partite graph with $|T(n,r)| = n$, partite sets differing at most by $1$ $(1 \le r \le n)$
  - $K_{r+1} \nsubseteq T(n,r)$
  - size: $\|T(n,r)\| =: t(n,r)$
  - *special Turán graph*: $K_r^s := T(n,r)$ if $n = r * s$
  - *Turán-graphs edge-maximal*: among all $r$-partite graphs of order $n$, $T(n,r)$ has largest size
  - $t(n,r) = t(n-r,r) + (n-r)(r-1) + \binom{r}{2}$
  - *size difference to complete graph*:
    $$\lim_{n \to \infty} \frac{t(n-r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right)$$
- **Turán's theorem**: $\forall r > 1, n \ge 1$, any graph $G$ with $|G| = n$, $\|G\| = \mathrm{ex}(n,K_r)$ and $K_r \nsubseteq G$ is a $T(n,r-1)$
  $\Leftrightarrow \mathrm{EX}(n,K_r) = \{T(n,r-1)\}$
- **Szemerédi's regularity lemma**: $\forall \varepsilon > 0 \forall 1 \le m \in \mathbb{N} \exists M \in \mathbb{N}$: every graph $G$ with $|G| \ge m$ has $\varepsilon$-regular partition $V_0 \dot\cup \cdots \dot\cup V_k$ with $m \le k \le M$.
- **Erdős-Stone theorem**: $\forall$ integers $r > s \ge 1$ and any $\varepsilon > 0 \exists n_0 \in \mathbb{N}$: every graph with $|G| = n \ge n_0$ and $\|G\| \ge t(n,r-1) + \varepsilon n^2$ has $K_r^s \subseteq G$.
  - *corollary*: the theorem together with the size difference to complete graph yields
    $$\lim_{n \to \infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$
- **Chvátal-Szemerédi theorem**: $\forall \varepsilon > 0$ and any integer $r \ge 3$, any graph with $|G| = n$ and $\|G\| \ge \left(1 - \frac{1}{r-1} + \varepsilon\right)\binom{n}{2}$ has $K_r^t \subseteq G$ with $t = \frac{\log n}{500 \log(\frac{1}{\varepsilon})}$
  - *existance*: $\exists G$ with $|G| = n$ and $\|G\| = \left(1 - \frac{1+\varepsilon}{r-1}\right)\binom{n}{2}$ with $G \nsubseteq K_r^t$ for $t = \frac{5 \log n}{\log(\frac{1}{\varepsilon})}$
- **Zarankiewicz function**: $z(m,n;s,t) =$ maximum $\#$ of edges that bipartite graph with parts of size $m$ and $n$ can have without containing $K_{s,t}$
- **Kővári-Sós-Turán**: $z(m,n;s,t) \le (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$
  - $m = n, t = s$: $z(n,n;t,t) \le c_1 n n^{1-\frac{1}{t}} + c_2 n = \mathcal{O}(n^{2-\frac{1}{t}})$
- **Bound for $\mathrm{ex}(n,K_{t,s})$**: $\le \frac{1}{2}z(n,n;s,t) \le cn^{2-\frac{1}{s}}$ $(t \ge s \ge 1)$
  - $t = s = 2$: $\mathrm{ex}(n,C_4) \le \frac{n}{4}(1 + \sqrt{4n-3})$
- **Bound for $\mathrm{ex}(n,K_{r,r})$**: $\ge cn^{2-\frac{2}{r+1}}$ $(\forall n, r \in \mathbb{N})$
- **Bound for $\mathrm{ex}(n,P_{k+1})$**: $\le \frac{n(k-1)}{2}$

## Minors

- **Hadwiger conjecture**: $\chi(G) \ge r \Rightarrow MK_r \subseteq G$
  - $r \in \{1,2,3,4\}$: easy to see
  - $r \in \{5,6\}$: proven using 4-color theorem
  - $r \ge 7$: still open
- **Bollobás-Thomason theorem**: $d(G) \ge cr^2 \Rightarrow TK_r \subseteq G$
- **Minimum degree + girth = minor**: $\delta(G) \ge d, g(G) \ge 8k+3$ $(d,k \in \mathbb{N}, d \le 3)$. Then $MH \subseteq G$ with $\delta(H) \ge d(d-1)^k$.
- **Thomassen's theorem**: $\forall r \in \mathbb{N} \exists f : \mathbb{N} \to \mathbb{N}$ s.t. every $G$ with $\delta(G) \ge 3$ and $g(G) \ge f(r)$ has $K_r$ minor
- **Kühn-Osthus theorem**: $\forall r \in \mathbb{N} \exists g \in \mathbb{N} : TK_r \subseteq G$ for all $G$ with $\delta(G) \ge r - 1$ and $g(G) \ge g$

# Ramsey theory

## Base definitions

- **Monochromatic** edge coloring: all edges have same color
- **Rainbow** edge coloring: no two edges have same color
- **Lexical** edge coloring: two edges have same color $\Longleftrightarrow$ have same lower endpoint in some vertex ordering
- **Ramsey number** $R(k) \in \mathbb{N}$: smallest $n$ s.t. every 2-edge-coloring of $K_n$ contains monochromatic $K_k$ $(n \in \mathbb{N})$
- **Asymmetric Ramsey number** $R(k,l)$: smallest $n \in \mathbb{N}$ s.t. every 2-edge-coloring of $K_n$ contains red $K_k$ or blue $K_l$ $(k,l \in \mathbb{N})$
- **Graph Ramsey number** $R(G,H)$: smallest $n \in \mathbb{N}$ s.t. every 2-edge-coloring of $K_n$ contains red $G$ or blue $H$
- **Hypergraph Ramsey number** $R_r(l_1,\ldots,l_k)$: smallest $n \in \mathbb{N}$ s.t. for every $k$-coloring of $\binom{[n]}{r} \exists i \in \{1,\ldots,k\}$ and a $V \subseteq [n]$ with $|V| = l$ s.t. all sets in $\binom{V}{r}$ have color $i$
- **Induced Ramsey number** $R_{\mathrm{ind}}(G,H)$: smallest $n \in \mathbb{N}$ s.t. $\exists$ graph $F$ with $|F| = n$ with every 2-coloring of it containing red $G$ or blue $H$
- **Anti Ramsey number** $AR(n,H)$: maximum number of colors that edge-coloring on $K_n$ can have without containing rainbow copy of $H$
- **r-regular matrix**: if there is a monochromatic solution of $Ax = 0$ for any $r$-coloring $c : \mathbb{N} \to [r]$ of $\mathbb{N}$
- **column condition**: matrix fulfills it if there is partition $C_1 \dot\cup \cdots \dot\cup C_l$ of $A$-columns s.t. the following holds:
  Let $s_i := \sum_{c \in C_i} c$ for $i \in [l]$. Then $s_1 = 0$ and every $s_i$ is linear combination of columns in $C_1 \dot\cup \cdots \dot\cup C_{i-1}$ $(2x_1 + x_2 + x_3 - 4x_4$ fulfills: $2 + 1 + 1 - 4 = 0)$

## Observations

- $R(3) = 6$
- $R(2,k) = R(k,2) = k$
- **Ramsey theorem**: $\forall k \in \mathbb{N} : \sqrt{2}^k \le R(k) \le 4^k$
  $\to$ (Asymmetric) Ramsey numbers and graph Ramsey numbers are finite
- **Induction theorem**: $\forall k,l \in \mathbb{N} : R(k,l) \le R(k-1,l) + R(k,l-1)$
  $\to R(k,l) \le \binom{k+l-2}{k-1}$
- **Hypergraph recursion**:
  $\forall r,p,q \in \mathbb{N} : R_r(p,q) \le R_{r-1}(R_r(q-1,q), R_r(p,q-1)) + 1$
- **2-Hypergraph boundary**: $c_1 2^k \le R_2(\underbrace{3 * \cdots * 3}_{k \text{ times}}) \le c_2 k!$ for some $c_1, c_2 > 0$

## Ramsey theory applications

- **Erdős-Szekeres subsequences**: Any sequence of $(r-1)(s-1) + 1$ distinct real numbers contains increasing subsequence of length $r$ or a decreasing subsequence of length $s$
- **Erdős-Szekeres m-gons**: $\forall m \in \mathbb{N} \exists N \in \mathbb{N}$: every set of $\ge N$ points in general position in $\mathbb{R}^2$ contains the vertex set of a convex $m$-gon
- **Schur**: Let $c : \mathbb{N} \to [r]$ be coloring of the natural numbers with $r \in \mathbb{N}$ colors. Then there are $x,y,z \in \mathbb{N}$ of same color with $x + y = z$
- **Rado theorem**: $A$ fulfills column condition $\Rightarrow A$ is $r$-regular $\forall r \in \mathbb{N}$
  $(A \in \mathbb{Z}^{n \times k})$
- $\forall s,t \in \mathbb{N}$ with $s \ge t \ge 1$: $R(sK_2, tK_2) = 2s + t - 1$
- $\forall s,t \in \mathbb{N}$ with $s \ge t \ge 1$: $R(sK_3, tK_3) = 3s + 2t$
- **Chvátal-Harary**: $R(G,H) \ge (\chi(G) - 1)(c(H) - 1) + 1$
  $(c(H)$ order of largest component of $H)$
- $R_{\mathrm{ind}}(G,H)$ is finite for all graphs $G, H$
- **Canonical Ramsey theorem**: $\forall k \in \mathbb{N} \exists n \in \mathbb{N}$: any edge coloring of $K_n$ with arbitrarily many colors contains monochromatic, rainbow or lexical $K_k$
- $\forall \Delta \in \mathbb{N} \exists c \in \mathbb{N}$: for every graph $H$ with $\Delta(H) = \Delta$ we have $R(H,H) \le c|V(H)|$
- For any $n$-vertex graph $H$ with $\Delta(H) = 3$ we have $R(H,H) \le cn$ for some $c > 0$, which grows way slower than $R(K_n, K_n) \ge \sqrt{2}^n$
- **Anti-Ramsey theorem**:
  $$\forall n,r \in \mathbb{N} : AR(n,K_r) = \binom{n}{2}\left(1 - \frac{1}{r-2}\right)(1 - o(1))$$

# Flows

## Circulations

- **Circulation**:
  - $H :=$ abelian semigroup, $G$ multigraph, $\widetilde{E} := \{(x,y) : xy \in E(G)\}$
  - $f : \widetilde{E} \to H, X, Y \subseteq V \rightsquigarrow f(X,Y) := \sum_{(x,y) \in (X \times Y) \cap \widetilde{E}} f(x,y)$

- $f : \widetilde{E} \to H$ is *circulation* on $G \Leftrightarrow$
  1. $f(x, y) = -f(y, x) \ (\forall xy \in E(G))$,
  2. $f(v, V) = 0 \ (\forall v \in v)$.
- **H-flow**: circulation $f : \widetilde{E} \to H$ with abelian *group* $H$
- **nowhere-zero-flow**: $\forall xy \in E : f(x, y) \neq 0$
- **k-flow**: $\mathbb{Z}$-flow $f$ with $\forall xy \in E : 0 < |f(x, y)| < k$
- **flow number** $\varphi(G)$: $\min \{k \in \mathbb{N} : G \text{ has } k\text{-flow}\}$

## Networks

- **Network**:
  - $s, t \in V, s \neq t, c : \widetilde{E} \to \mathbb{N}_0$
  - *network* $(G, s, t, c)$ with
    - *source $s$*
    - *sink $t$*
    - *capacity function $c$*
- **Network flow**: $f : \widetilde{E} \to \mathbb{R}$ with $\forall x, y \in V$:
  1. $f(x, y) = -f(y, x)$
  2. $x \notin \{s, t\} \Rightarrow f(x, V) = 0$
  3. $f(x, y) \leq c(x, y)$
- **Cut**: $(S, V \setminus S)$ with $s \in S, t \notin S$ (for any $S \subseteq V$)
  - *capacpity $c(S, V \setminus S)$*
- **Value** of $f :$ $= |f| := f(s, V)$
- **Basic network properties**:
  - $\forall$ circulation $f, X \subseteq V$: $\underline{f(X, X) = f(X, V) = f(X, V \setminus X) = 0}$
  - $\forall$ network flow $f$, cut $S, \overline{S}$: $f(S, \overline{S}) = |f|$
- **Ford-Fulkerson**: $\forall$ networks:
  - max value of a flow = min capacity of a cut
  - $\exists$ integral flow $f : \widetilde{E} \to \mathbb{N}_0$ with max flow value
- **Tutte**: $\forall$ multigraph $G \ \exists$ polynomial $P \in \mathbb{Z}[X] : \forall$ finite Abelian group $H$: number of nowhere-zero $H$-flows on $G$ is $P(|H| - 1)$
- **Abelian group can be exchanged**: $H$-flow on $G$ exists (Abelian group $H$) $\Rightarrow$ $\exists \widetilde{H}$-flow on $G$ ($\forall$ finite Abelian groups $\widetilde{H}$ with $|\widetilde{H}| = |H|$)
  - $\rightsquigarrow \mathbb{Z}_4$-flow exists $\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$-flow exists
- **Flow can be $\mathbb{Z}_k$-substituted**: multigraph admits $k$-flow $\Leftrightarrow$ admits $\mathbb{Z}_k$-flow
- **Flows in planar graphs**: planar graph $G$, dual $G^*$: $\chi(G) = \varphi(G^*)$
- **2-flow only on even degrees**: graph has 2-flow $\Leftrightarrow$ all degrees are even
- **3-flow on bipartite graphs**: 3-regular graph has 3-flow $\Leftrightarrow$ bipartite
- **Tutte on 5-flows**: every bridgeless multigraph has flow number $\leq 5$
- **Seymour on 6-flows**: every brigeless graph has flow number $\leq 6$

## Random Graphs

### Erdős-Rényi Model

- **Erdős-Renyi Model**: $\mathcal{G}(n, p)$ probability space on $n$-vertex graphs resulted by independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0, 1]$
- **Property** $\mathcal{P}$: set of graphs (e.g. $\mathcal{P} = \{G : G \text{ is } k\text{-connected}\}$)
- **Almost always**: let $(p_n) \in [0, 1]^{\mathbb{N}}$. $G \in \mathcal{G}(n, p_n)$ has property $\mathcal{P}$ *almost always* $\Leftrightarrow \lim_{n \to \infty} \text{Prob}(G \in \mathcal{G}(n, p_n) \cap \mathcal{P}) = 1$
- **Almost all**: like almost always, but $(p_n) = $ constant $p$
- **Threshold function**: $f(n) : \mathbb{N} \to [0, 1]$ *threshold function* for property $\mathcal{P}$ if
  1. $\forall (p_n) \in [0, 1]^{\mathbb{N}}, p_n / f(n) \xrightarrow{n \to \infty} 0 : G \in \mathcal{G}(n, p_n)$ almost always does **not** have property $\mathcal{P}$
  2. $\forall (p_n) \in [0, 1]^{\mathbb{N}}, p_n / f(n) \xrightarrow{n \to \infty} \infty : G \in \mathcal{G}(n, p_n)$ almost always **has** propertpy $\mathcal{P}$
  ! not all properties $\mathcal{P}$ have a threshold function

### Basic probability properties

- $G$ with $|G| = n, \|G\| = m$: $\text{Prob}(G = \mathcal{G}(n, p)) = p^m (1 - p)^{\binom{n}{2} - m}$
- $\text{Prob}(G \in \mathcal{G}(n, p), \alpha(G) \geq k) \leq \binom{n}{k}(1 - p)^{\binom{k}{2}} \quad (n \geq k \geq 2)$
- $\text{Prob}(G \in \mathcal{G}(n, p), \omega(G) \geq k) \leq \binom{n}{k} p^{\binom{n}{k}}$

### More complex results

- $\text{Exp}(\#k\text{-cycles in } G \in \mathcal{G}(n, p)) = \frac{n_k}{2k} p^k \ (n_k = n(n-1)\cdots(n-k+1))$
- **Erdős**: $\forall k \in \mathbb{N} \ \exists$ graph $H : g(H) \geq k \wedge \chi(H) \geq k$
- $\forall p \in (0, 1)$, graph $H$, almost all $G \in \mathcal{G}(n, p)$ contain $H$ as induced subgraph
- $\forall p \in (0, 1), \varepsilon > 0$, almost all $G \in \mathcal{G}(n, p)$ fulfill
$$\chi(G) > \frac{\log(1/(1-p))}{2 + \varepsilon} \frac{n}{\log n}$$

- **Asymptotic behaviour of $\mathcal{G}(n, p)$ for some properties**:
  - $p_n = \sqrt{n}/n^2 \Rightarrow G$ almost always has component with $> 2$ vertices
  - $p_n = 1/n \Rightarrow G$ almost always has a cycle
  - $p_n = \log n/n \Rightarrow G$ is almost always connected
  - $p_n = (1 + \varepsilon) \log n/n \Rightarrow G$ almost always has Hamiltonian cycle
  - $p_n = n^{-2/(k-1)}$ is the threshold function for containing $K_k$
- **Lovász Local**: $A_1, \ldots, A_n$ events in some probabilistic space. If $\text{Prob}(A_i) \leq p \in (0, 1)$, each $A_i$ is mutually independent from all but at most $d \in \mathbb{N} \ A_i$s and $\text{ep}(d + 1) \leq 1$, then
$$\text{Prob}\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0$$
- **Van der Waerden's number**: $W(k) = $ smallest $n$ s.t. any 2-coloring of $[n]$ contains monochromatic arithmetic progression of length $k$
  - $\rightarrow$ using Lovász Local Lemma: $W(k) \geq 2k - 1/(ek^2)$

# Hamiltonian Cycles

## Definitions

- **Hamiltonian cycle**: Closed walk on $G$ containing all $v \in V(G)$ exactly once
- **Hamiltonian graph**: graph containing Hamiltonian cycle
- **Hamiltonian path**: Path on $G$ containing all $v \in V(G)$ exactly once
- **Square** of graph $G$: $G^2 := (V, E')$ with $E' := \{uv : u, v \in V, d_G(u, v) \leq 2\}$

## Results

- **Necessary condition**: $G$ has Hamiltonian cycle $\Rightarrow \forall \varnothing \neq S \subseteq V$: graph $G - S$ cannot have more than $|S|$ components
- **Dirac**: Every graph with $|G| \geq 3$ and $\delta(G) \geq \frac{n}{2}$ has Hamiltonian cycle
- **Clique**: Every graph with $|G| \geq 3$ with $\alpha(G) \leq \kappa(G)$ is Hamiltonian
- **Fleischner**: $G$ is 2-connected $\Rightarrow G^2$ is Hamiltonian
- **Chvátal**: graph with degree sequence $a_1, \ldots, a_n$ is Hamiltonian $\Leftrightarrow a_i \leq i$ implies $a_{n-i} \geq n - 1 \ (\forall i < n/2, 0 \leq a_1 \leq \cdots \leq a_n < n)$