# Basics

### Notations

- $\binom{V}{k} \coloneqq \{A : A \subseteq V \wedge |A| = k\}$
- $[n] \coloneqq \{1, \dots, n\} \subset \mathbb{N}$
- **Power set** $2^X \coloneqq \{A : A \subseteq X\}$

### Graph

- **Definition**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$
- **Vertex set**: $V(G)$
- **Edge set**: $E(G)$
- **Isomorphic** ($G_1$ to another graph $G_2$): if $\exists$ bijection $f : V_1 \to V_2$ with $\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $\|G\|$
- **Complement**: $\overline{G} = (V(G), \binom{V}{2} - E(G))$
- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \dots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Minimum degree**: $\delta(G)$ = degree of $v \in V(G)$ with smallest degree
- **Maximum degree**: $\Delta(G)$ = degree of $v \in V(G)$ with largest degree
- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$
- **Eulerian**: if it contains an Eulerian tour
- **Connected**: for any two vertices there is a link between them
  - *spanning tree*: if $G$ is connected, then it has a spanning tree
  - *peeling leaves*: vertices can be ordered $v_1, \dots, v_n$ s.t. $G[\{v_1, \dots, v_i\}]$ is connected for $i \in \{1, \dots, n\}$

### Digraph

- **Definition**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$

### Multigraph

- **Definition**: $G = (V, E)$ with vertex set $V$ and multiset $E$ of $V$-pairs

### Hypergraph

- **Definition**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

### Vertex

- **Incident** to $e \in E(G)$ if $v \in e$
- **Adjacent** to $\tilde{v} \in V(G)$ if $\{v, \tilde{v}\} \in E(G)$
- **Neighborhood**: $N(v) = \{u : uv \in E(G)\}$
- **Degree**: $\deg(v) = \mathrm{d}(v) = |N(v)|$
- **Isolated**: vertex with $\deg(v) = 0$
- **Leaf**: vertex with $\deg(v) = 1$

### Subgraph

- **Definition**: $H$ subgraph of $G$ (write $H \subseteq G$) if $V(H) \subseteq V(G) \wedge E(H) \subseteq E(G)$
- **Induced subgraph**: $H$ induced subgraph of $G$ (write $H \underset{\text{ind}}{\subseteq} G$), if $H \subseteq G$ and $E(H)$ contains all edges from $E(G)$ between vertices in $V(H)$
- **Edge-induced subgraph**: subgraph induced by $X \subseteq E(G)$, note $G[X]$
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \iff$ any $A$-$B$-path has vertex in $X$

### Spanning graph

- **Definition**: Subgraph with same vertex set as supergraph

### Line graph

- **Definition**: $L(G) = (E, \{\{e, e'\} : e \cap e' \neq \emptyset\})$
- **Graphic**: $L$ is line graph of some $G$, if it doesn't contain one of 9 specific induced subgraphs

### Vertex cover

- **Definition**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in $V'$

### Cycle

- **Definition**: $C_n \coloneqq (\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\})$
- **Shorthand**: $(v_1, \dots, v_n, v_1)$
- **Length** (of cycle): $= |V| \equiv |E|$
- **Cyclic subgraph**: If $\delta(G) \geq 2$, then $G$ has cycle with length $\geq \delta + 1$

### Path

- **Definition**: $(\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$
- **Shorthand**: $(v_1, \dots, v_n)$
- **Length** (of path): $= |E| \neq |V|$
- $v_0 v_k$**-path**: path starting at $v_0$ and ending at $v_k$
- **Independent**: two $ab$-paths are independent $\iff$ they only share $a$ and $b$

### Walk

- **Definition**: non-empty alternating sequence of vertices and edges
  $v_0 e_0 \dots e_{k-1} v_k$
  with $e_i = v_i v_{i+1}$, length $k \in \mathbb{N}$
  - *closed*: if $v_0 = v_k$
  - *even*: if $k$ is even
  - *odd*: if $k$ is odd
- **Eulerian tour**:
  - *Definition*: closed walk with
    - no edges of $G$ are repeatedly used
    - all edges of $G$ are used
  - *Even degrees*: $G$ connected has Euler tour $\iff \forall v \in V(G) : \deg(v)$ even
- $v_0 v_k$**-walk**: walk starting at $v_0$ and ending at $v_k$
- **Induces path**: $\exists$ $uv$-walk $\implies$ $\exists$ $uv$-path
- **Odd closed walk, odd cycle**: $G$ has *odd* closed walk $\implies G$ has odd cycle

### Connected component

- **Definition**: *maximal* connected subgraph (connected, but any supergraph isn't)

### Block

- **Block**: maximal 2-connected subgraph or bridge
  - share $\leq 1$ vertices with one another
- **Block-cut-vertex graph**
  - $V$ = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex $v$, block $B\}$
  - block-cut-vertex graph of connected graph is tree

### Acyclic graph, Forest

- **Definition**: Graph with no cycle as subgraph

### Tree

- **Definition**: Graph that is connected and acyclic
  - $\iff G$ is connected and $\forall e \in E(G) : G - e$ is disconnected
    (*minimal-connected*)
  - $\iff G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle
    (*maximal-acyclic*)
  - $\iff G$ is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \leq 1$)
  - $\iff G$ is connected and $\|G\| = |G| - 1$
  - $\iff G$ is acyclic and $\|G\| = |G| - 1$
  - $\iff \forall u, v \in V(G) \,\exists$ unique $uv$-path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree $T$, $|T| \geq 2 \implies T$ has leaf
- **Edge count**: Tree $T$, $|T| = n \implies \|T\| = n - 1$

### k-regular graph

- **Definition**: Graph with $\deg(v) = k \in \mathbb{N}_0 \quad (\forall v \in V(G))$

### Bipartite graph

- **Definition**: $G$ is bipartite $\iff G$ contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$
- **Matchings**:
  - *saturating*: $G = (A \cup B, E)$ has matching saturating $A$
    $\iff \forall S \subseteq A : N(S) \geq |S| \quad (N(S) \coloneqq \{b \in B : ab \in E, a \in S\})$

- *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1)$.
  $\Rightarrow\ \exists$ matching $M$ saturating all but at most $d$ vertices of $A$
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover

## Matching

- **Definition**: graph with $\delta(G) = \Delta(G) = 1$
- **Perfect matching**: spanning + matching subgraph of $G$ (aka *1-factor*)
  - *existence*: $G$ has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq S$
    ($q(G)$ = number of components in $G$ with odd order)

## Coloring

- **Proper coloring**: $= c : V(G) \to [k]$ with $c(u) \neq c(v) \quad (\forall uv \in E(G))$
- **Equitable coloring**: proper coloring + color classes have almost ($\pm 1$) equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors

## Chromatic number

- **Definition**: $\chi(G) = \min\{k : G$ has proper coloring with $k$ colors$\}$
- **Examples**: $\chi(C_{2n}) = 2, \chi(C_{2n+1}) = 3$

## Factors

- **k-factor**: spanning $k$-regular subgraph (easy to find)
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$,
  $f : V(G) \to \{0, 1, \dots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect $H$-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and $k$ divides $|G|$, then $G$ has $K_k$-factor

## Connectivity

- $k$-**connected**: if $|G| > k$ and deleting $< k$ vertices does not disconnect $G$
- $k$-**linked**: if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k)$ $\exists$ pairwise disjoint $s_i t_i$-paths (*note*: $k$-connected $\not\Rightarrow$ $k$-linked)
- **Vertex-connectivity**: $\kappa(G) = \max\{k : G$ is $k$-connected$\}$
- $l$-**edge-connected**: if deleting $< l$ edges does not disconnect $G$
- **Edge-connectivity**: $\kappa'(G) = \max\{l : G$ is $l$-edge-connected$\}$
- **Vertex- vs Edge-connectivity**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Three-connected + contraction**: 3-connected $\Leftrightarrow\ \exists$ separate $G_0, \dots, G_k$ with
  $G_0 = K_4,\ G_k = G,\ G_i = G_{i+1} \circ xy$
  with $\deg(x), \deg(y) \geq 3$
- **Three-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of $K_4$
- **Average degree** $\geq 4$: has $k$-connected subgraph ($k \geq 2$)

## Cuts

- **Cut-Set**: $X \subseteq V(G) \cup E(G)$ s.t. #components in $(G - X)$ greater than in $G$
- **Cut-Vertex**: Cut-Set consisting of single vertex
- **Cut-Edge** (or *bridge*): Cut-Set consisting of single edge
- **Menger's theorem**: for $A, B \subseteq V(G)$:
  min # of vertices separating $A$ and $B$ = max # of disjoint $A$-$B$-paths
- **Menger global**:
  1. *k-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise independent $ab$-paths
  2. *k-edge-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise edge-disjoint $ab$-paths

## Ear-decomposition

- **Definition**: $G$ has *ear-decomposition* $\Leftrightarrow\ \exists$ sequence of graphs $G_0, \dots, G_k$ with $G_k = G, G_0$ = cycle, $G_{i+1}$ obtained from $G_i$ by attaching "ear" (path that shares only endpoints with $G_i$)
- **2-connected** $\Leftrightarrow \forall$ cycles $C$ in $G$ there is ear-decomposition starting at $C$

## Edge contraction

- **Contraction**:
  $$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$$
  $$(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
  with $xy \in E(G)$
- **De-contraction**: if $\exists\ xy \in E(G) : \kappa(G \circ xy) \geq 3$
  (for $G$ with $\kappa(G) \geq 3, |G| \geq 5$)

## Planar graph tools

- **Homeomorphism**: $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t. $f^{-1}$ is also continuous
- **Arc**: homeomorphic image of $[0, 1]$ in $\mathbb{R}^2$ under $f$
  - *endpoints*: $f(0)$ and $f(1) \rightsquigarrow$ arc "joins" endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments
- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and $Y$ is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
  $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon)$ contains points of $X$ and not of $X\}$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\overline{x} : \mathrm{dist}(\overline{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions $R_1, R_2$ and $\delta R_1 = X = \delta R_2$.

## Plane graph

- **Definition**: graph such that $E(G)$ is set of arcs in $\mathbb{R}^2$ and endpoints of arcs in $E(G)$ are vertices and:
  - $\forall e, e' \in E, e \neq e'$: $e$ and $e'$ have distinct sets of edge sets
  - $\forall e \in E, \mathring{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus \left(\bigcup_{e \in E} e \cup V\right)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle $\Leftrightarrow$ graph is maximally plane
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow ||G|| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no $\triangle$: $||G|| \leq 2|G| - 4$
- **Properties**: Let $G$ be plane graph and $H \subseteq G$.
  - *face inheritance*: $\forall f \in F(G)\ \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*: $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*: $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \mathring{e} = \emptyset$
  - *edges in circles*:
    $e \in E(G)$ is edge of a cycle $\Rightarrow e$ is on boundary of exactly 2 faces
    not edge of a cycle $\Rightarrow e$ is on boundary of exactly 1 face
  - *faces in cycles*: $f_1, f_2 \in F(G)$. $f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
  - *cyclic boundaries*: $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
- **Dual multigraph**: Given plane $G$:
  1. Insert vertex in each face
  2. Put edge $\tilde{e}$ between vertices if respective faces share $e$ (s.t. $\tilde{e}$ and $e$ cross once)
  3. *Result*: Dual graph $G'$ of $G$ (plane multigraph)

  $\rightsquigarrow$ faces of $G$ properly $k$-colored $\Leftrightarrow\ \exists$ proper $k$-coloring of vertices of $G'$

## Planar graph

- **Definition**: graph s.t. $\exists$ plane graph $G'$ and bijection $f : V(G) \to V(G')$ s.t. $\forall u, v \in V(G),\ uv \in E(G) : f(u),\ f(v)$ are endpoints of arc in $G'$
- **Planar embedding** of $G$: $f$ from the definition
- **Planar because of minors**: The following statements are equivalent:
  - $G$ is planar
  - $G \not\supseteq MK_5 \wedge G \not\supseteq MK_{3,3}$
  - $G \not\supseteq TK_5 \wedge G \not\supseteq TK_{3,3}$
- **Euler's formula**: If $G$ is connected plane graph with $f$ faces, then
  $|G| - ||G|| + f = 2$
- $\delta(G)$ **limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: $K_5$ and $K_{3,3}$ are not planar
- **Kuratowski's lemmas**:
  1. $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Leftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
  2. $\kappa(G) \geq 3 \wedge MK_5 \nsubseteq G \wedge MK_{3,3} \nsubseteq G \Rightarrow G$ is planar
  3. $\kappa(G) \geq 3$, $G$ edge-maximal wrt not containing $TX$. If $S$ is vertex-cut of $G$, $|S| \leq 2 \wedge G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$, then $G_i$ is edge-maximal with no $TX$ and $S$ induces an edge
  4. $|G| \geq 3$, $G$ edge-maximal wrt not containing $TK_5$ and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$

## Minors

- **MH**: $G \overset{(\star)}{=} MH$ is *minor* of $H$ if
  - $V(G) = V_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} V_n$ with $n = |H|$
  - $G[V_i]$ connected ($\forall i = 1, \dots, n$)
  - If $V(H) = \{v_1, \dots, v_n\}$ and $v_i v_j \in E(H)$, then $\exists$ edge between $V_i$ and $V_j$

  ($\star$): *Notation abuse*: $MH$ is class of graphs
- **Branch sets**: $V_i$'s from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** ($H$ of $G$, noted $H \preccurlyeq G$): $\Leftrightarrow MH \subseteq G$
  $\rightsquigarrow H \preccurlyeq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + contractions.
- **Topological minor**: $H$ is topological minor if $TH \subseteq G$ where $TH$ is built from $H$ by subdividing edges
- **Note**: $TH \subseteq MH$

## Colorings

- **4-color-theorem**: $G$ planar $\Rightarrow \chi(G) \leq 4$

- **ij-flip**: $c' : V(G) \to [k]$ is $ij$-flip at $v \in V(G)$
  $\Leftrightarrow c'$ obtained by flipping colors $i$ and $j$ in max. conn. component containing $v$