

Basics

Notations

- $\binom{V}{k} := \{A : A \subseteq V \wedge |A| = k\}$
- $[n] := \{1, \dots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

Graphs

- **Definition:** $G = (V, E)$ with $E \subseteq V^2, V \cap E = \emptyset$
- **Vertex:** $v \in V$ for graph $G = (V, E)$
 - v incident with $e \Leftrightarrow v \in e$
 - v_1, v_2 ends of $e \Leftrightarrow e = v_1 v_2$
 - v_1, v_2 adjacent/neighbors $\Leftrightarrow v_1 v_2 \in E$
- **Edge:** $e = \{x, y\} \in E$ for graph $G = (V, E)$ (short $e = xy$)
 - e edge at $v \Leftrightarrow v$ incident with e
 - e joins $v_1, v_2 \Leftrightarrow e = v_1 v_2$
 - xy is X - Y -edge $\Leftrightarrow x \in X \wedge y \in Y$
 - e_1, e_2 adjacent/neighbors $\Leftrightarrow \exists v : v \in e_1 \wedge v \in e_2$
- **Vertex sets:**
 - $V(G) = V$ for graph $G = (V, E)$
 - $X \subset V(G)$ independent \Leftrightarrow no $x_1, x_2 \in X$ are adjacent
 - neighborhood of $v \in V(G)$: $N(v) = \{u \in V(G) : uv \in E(G)\}$
- **Edge sets:**
 - $E(G) = E$ for graph $G = (V, E)$
 - $E(X, Y)$: set of edges between $X \subset V(G)$ and $Y \subset V(G)$
 - $E(x, Y)$: set of edges between vertex $x \in V(G)$ and $Y \subset V(G)$
 - $E(v)$: set of edges at $v \in V(G)$
- **Order:** $= |V(G)|$, short $|G|$
- **Size:** $= |E(G)|$, short $\|G\|$
- **Trivial graph:** graph of order 0 or 1
- **Incidence graph** of G : $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Isomorphic** (G_1 to another graph G_2 , write $G_1 \cong G_2$ or even $G_1 = G_2$):
 - \exists bijection $f : V_1 \rightarrow V_2 : \{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$
- **Graph union:** $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$
- **Graph intersection:** $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$
- **Graph multiplication:** $G * G'$: join all $v \in G$ with all $v' \in G'$ (with $V(G) \cap V(G') = \emptyset$)
- **Subgraph** G' of G (write $G' \subseteq G$): if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$
 - G contains G'
 - G' proper subgraph of G : if $G' \subseteq G$ and $G' \neq G$
 - G' induced subgraph of G : $G' \subseteq G$ and $E(G')$ contains all edges of G with both ends in $V(G')$, $V(G')$ induces G' , write $G' = G[X]$ (with $X = V(G')$)
 - Edge-induced subgraph: subgraph induced by $X \subseteq E(G)$, note $G[X]$
 - G' spanning subgraph of G : $V(G') = V(G)$
- **Supergraph:** G of G' (write $G \supseteq G'$): as above.
- **Vertex cover:** $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in V'
- **Graph subtraction:**
 - $G - U = G[V(G) \setminus U]$ for some vertex set U
 - $G - v = G[V(G) \setminus \{v\}]$ for some vertex v
 - $G - G' = G[V(G) \setminus V(G')]$ for some graph G'
- **Edge addition:** $G + F = (V(G), V(E) \cup F)$ for some $F \subseteq V(G)^2$
- **Complement:** $\overline{G} = (V(G), V^2 \setminus E(G))$
- **Line graph** of $L(G) = (E(G), \{xy \in E(G)^2 : x, y \text{ adjacent in } G\})$
- **Complete graph:** (X, X^2) with vertex set X
 - K_n : complete graph on n vertices

Vertex degrees

- **Degree** of $v \in V$: $d(v) = \deg(v) = |N(v)|$
 - $v \in V(G)$ isolated: $d(v) = 0$
 - $v \in V(G)$ leaf: $d(v) = 1$
 - number of vertices of odd degree is even
- **Minimum degree** of graph G : $\delta(G) = \min\{d(v) : v \in V(G)\}$
- **Maximum degree** of graph G : $\Delta(G) = \max\{d(v) : v \in V(G)\}$
- **Degree sum:** $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Average degree** of graph G : $d(G) = \frac{1}{|G|} \sum_{v \in V} d(v)$
 - $\delta(G) \leq d(G) \leq \Delta(G)$
- **k-regular graph:** $\forall v \in V(G) : d(v) = k$
 - cubic graph: 3-regular graph
- **Vertex-Edge-ratio** of graph G : $\varepsilon(G) = \frac{\|G\|}{|G|}$
 - $\varepsilon(G) = \frac{1}{2} d(G)$
 - every graph with $\|G\| \geq 1$ has $H \subseteq G$ with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$

Paths

- **Path:** $(\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$ (read: $v_0 v_n$ -path)
 - shorthand: $v_1 \dots v_n$
 - v_0, v_n linked by path
 - v_0, v_n end-vertices/ends of path
 - v_1, \dots, v_{n-1} inner vertices of path
- **Length:** $|E(P)| \neq |V(P)|$
- **Shorthands** ($0 \leq i \leq j \leq k$):
 - $P = x_0 \dots x_k, \hat{P} = x_1 \dots x_{k-1}$
 - $Px_i = x_0 \dots x_i, Px_i = x_0 \dots x_{i-1}$
 - $x_i P = x_i \dots x_k, \hat{x}_i P = x_{i+1} \dots x_k$
 - $x_i P x_j = x_i \dots x_j, \hat{x}_i P \hat{x}_j = x_{i+1} \dots x_{j-1}$
- **Path concatenation:** $Px \cap xQy \cap yR = PxQyR$
- **A-B-path:** $V(P) \cap A = \{x_0\} \wedge V(P) \cap B = \{x_n\}$
- **H-path:** graph H, P meets H exactly in its ends
- **Independent:** two ab -paths are independent \Leftrightarrow they only share a and b
- **Path existence:** Every G with $\delta(G) \geq 2$ contains path of length $\delta(G)$
- **Distance:** $d_G(x, y) = \min(\{k : \exists x\text{-}y\text{-path of length } k\} \cup \{\infty\})$
- **Central:** $v \in V(G)$ where $\text{cen} = \max\{d_G(v, x) : v \neq x \in V(G)\}$ is minimal
- **Radius:** $\text{rad}(G) = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$
- **Diameter** of G : $\text{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$
 - radius-diameter-relation: $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$
 - radius-degree-vertex-restriction:

$$\text{rad}(G) \leq k \wedge \Delta(G) \leq d \geq 3 \Rightarrow |G| \leq \frac{d}{d-2} (d-1)^k$$
- **Walk:** alternating sequence $v_0 e_0 \dots e_{k-1} v_k$ s.t. $e_i = v_i v_{i+1}$ ($\forall i < k$)
 - closed walk: $v_k = v_0$
 - walk-path-relation: all vertices in walk distinct \leadsto path
 - walk-path-induction: $\exists v_0 v_k\text{-walk} \Rightarrow \exists v_0 v_k\text{-path}$

Cycles

- **Cycle:** $C = P + x_{k-1} x_0$ with path $P = x_0 \dots x_{k-1}$ ($k \geq 3$)
 - shorthand: $x_0 \dots x_{k-1} x_0$
- **Length:** $= |C| = \|C\|$
- **k-cycle:** $C_k =$ cycle of length k
- **Girth** of graph G : $g(G) = \min(\{k : G \text{ contains } C_k\} \cup \{\infty\})$
 - girth-diameter-relation: $g(G) \leq 2\text{diam}(G) + 1$
 - girth-vertex-relation: $\delta(G) \geq 3 \Rightarrow g(G) < 2 \log |G|$
- **Circumference** of graph G : $= \max(\{k : G \text{ contains } C_k\} \cup \{0\})$
- **Chord** of cycle $C \subseteq G$: $xy \in E(G)$ with $xy \notin E(C)$, but $x, y \in V(C)$
- **Induced cycle:** induced subgraph of G that is a cycle (= cycle in G with no chords)
- **Cycle existence:** Every G with $\delta(G) \geq 2$ contains cycle of length $\geq \delta(G) + 1$
- **Odd closed walk, odd cycle:** G has odd closed walk $\Rightarrow G$ has odd cycle

Connectivity

- **Connected** graph G : $\forall x, y \in V(G) : \exists xy\text{-path}$
 - connected subset $U \subseteq V(G)$: if $G[U]$ is connected
- **Vertex enumeration:** G connected \Rightarrow vertices can be enumerated v_1, \dots, v_n s.t. $G_i := G[v_1, \dots, v_i]$ is connected ($\forall i \leq n$)
- **Component:** maximal connected subgraph
 - graph partitioning: components partition G
- **Subgraph separation:** $X \subset V(G)$ separates $A, B \subset V(G) \Leftrightarrow$ any A - B -path has vertex in X
- separator X
- **Cut-Vertex:** vertex separating two other vertices of the component
- **Bridge:** edge separating its ends (= edges of component not lying on any cycle)
- **k-connected:** if $|G| > k \wedge G - X$ is connected $\forall X \subseteq V(G)$ with $|X| < k$
 - \leadsto no two vertices in G are separated by fewer than k other vertices
 - \leadsto any two vertices can be joined by k independent paths
- **k-linked:** if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k) \exists$ pairwise disjoint $s_i t_i$ -paths (note: k -connected $\not\Rightarrow$ k -linked)
- **Connectivity:** $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected:** if $|G| > 1 \wedge G - F$ is connected $\forall F \subseteq E(G)$ with $|F| < l$
- **Edge-connectivity:** $\kappa'(G) = \lambda(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Connectivity and smallest degree:** $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Connectivity and average degree:** $d(G) \geq 4k \Rightarrow G$ has k -connected subgraph
- **Menger's theorem:** for $A, B \subseteq V(G)$:
 - $\min \#$ of vertices separating A and $B = \max \#$ of disjoint A - B -paths
 - edge corollary: minimum number of edges separating a and $b = \max$ number of edge-disjoint a - b -paths
- **Menger global:**
 - k -connected $\Leftrightarrow \forall a, b \in V(G) \exists k$ pairwise independent ab -paths
 - k -edge-connected $\Leftrightarrow \forall a, b \in V(G) \exists k$ pairwise edge-disjoint ab -paths

Trees and forests

- **Forest:** Graph with no cycle as subgraph
- **Tree:** Graph that is connected and acyclic
 - ⇔ G is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
 - ⇔ G is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
 - ⇔ G is connected and 1-degenerate ($\forall G' \subseteq G : \delta(G') \leq 1$)
 - ⇔ G is connected and $\|G\| = |G| - 1$
 - ⇔ G is acyclic and $\|G\| = |G| - 1$
 - ⇔ $\forall u, v \in V(G) \exists$ unique uv -path
- **Special trees:** path, star, spider, caterpillar, broom
- **Leaf existence:** Tree $T, |T| \geq 2 \Rightarrow T$ has leaf
- **Edge count:** Tree $T, |T| = n \Rightarrow \|T\| = n - 1$

Bipartite graphs

- **r-partite** graph $G: V(G)$ allows partitioning in r classes s.t. $\forall e = xy \in E(G) : x$ and y are in different classes
- **Bipartite** graph: 2-partite graph
 - ⇔ G contains no cycles of odd length
 - *complete bipartite:* $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$

Contraction and minors

- **Subdivision** of graph G : any graph obtained from G by subdividing edges
- **Topological minor:** H is topological minor if $TH \subseteq G$ where TH is built from H by subdividing edges
 - *branch vertices:* original vertices of H
 - *subdividing vertices:* vertices placed on edges joining branch vertices
- **MH:** $G \stackrel{(\star)}{=} MH$ is *minor* of H if
 - $V(G) = V_1 \dot{\cup} \dots \dot{\cup} V_n$ with $n = |H|$
 - $G[V_i]$ connected ($\forall i = 1, \dots, n$)
 - If $V(H) = \{v_1, \dots, v_n\}$ and $v_i v_j \in E(H)$, then \exists edge between V_i and V_j
- **(\star):** *Notation abuse:* MH is class of graphs
- **Branch sets:** V_i 's from above
- **Extended branch graph:** Branch set together with incident edges
- **Minor** (H of G , noted $H \preceq G$): $\Leftrightarrow MH \subseteq G$
 - $\sim H \preceq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + **contractions**.
- **Note:** $TH \subseteq MH$
- **Edge contraction:**
$$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy}, \\ (E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy}z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
with $xy \in E(G)$
- **De-contraction:** if $\exists xy \in E(G) : \kappa(G \circ xy) \geq 3$ (for G with $\kappa(G) \geq 3, |G| \geq 5$)

Euler tours

- **Definition:** closed walk with
 - no edges of G are repeatedly used
 - all edges of G are used
- **Eulerian graph:** graph containing an Euler tour $\Leftrightarrow \forall v \in V(G) : d(v)$ even

Algebraic assets

- **Adjacency matrix:** $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$

Other graph notions

- **Digraph:** $G = (V, E)$ with vertex set V and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$
- **Multigraph:** $G = (V, E)$ with vertex set V and multiset E of V -pairs
- **Multigraph:** $G = (V, E)$ with vertex set V and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

Matching, Covering, Packing

Matching in bipartite graphs

- **Vertex cover:** $U \subseteq V(G)$ s.t. all edges in G are incident to a vertex $\in U$
- **Matching** M : set of independent edges in a graph
 - *matching graph:* $\delta(G) = \Delta(G) = 1$

- *saturating:* $G = (A \cup B, E)$ has matching saturating A
 - $\Leftrightarrow \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$
- *nearly:* $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1).$
 - $\Rightarrow \exists$ matching M saturating all but at most d vertices of A
- **k-factor:** k -regular spanning subgraph
- **Matching vs vertex cover:** size of largest matching = size of smallest vertex cover (Königs theorem)
- **Matching existence:**
 - *neighbor-based:* $G = (A \cup B, E)$ contains matching of $A \Leftrightarrow |N(S)| \geq |S|$
 - *regular + bipartite:* G is k -regular + bipartite ($k \geq 1$) $\Rightarrow G$ has 1-factor
 - *2k-regular:* graph $2k$ -regular ($k \geq 1$) \Rightarrow has 2-factor
- **Marriages:** make matchings based on preferences
 - *preferences:* family $(\leq_v)_{v \in V}$ of linear orderings \leq_v on $E(v)$
 - *stable matching* M :
 - $\forall e \in E \setminus M \exists f \in M : e$ and f have common vertex v with $e <_v f$
 - *stable matching existence:* For every set of preferences, G has stable matching
- **f-factor:** spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v), f : V(G) \rightarrow \{0, 1, \dots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
 - *existence:* if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and k divides $|G|$, then G has K_k -factor

Matchings in general graphs

- **Perfect matching:** spanning + matching subgraph of G (aka *1-factor*)
 - *existence (Tutte):* G has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq |S|$ (Tutte's condition, $q(G)$ = number of components in G with odd order)
 - *existence (Petersen):* G bridgeless + cubic $\Rightarrow G$ has 1-factor

Connectivity

2-connected graphs and subgraphs

- **2-connected construction:** G is 2-connected \Leftrightarrow it can be constructed by successively adding paths to a cycle (removing those paths: *ear-decomposition*)
- **Block:** maximal 2-connected subgraph or bridge
 - share ≤ 1 vertices with another
 - *cycles* of G = cycles of its blocks
 - *bounds* of G = minimal cuts of its blocks
- **Block-cut-vertex graph**
 - V = set of blocks \cup set of vertices
 - $E = \{\{v, B\} : v \in V(B), \text{cut-vertex } v, \text{block } B\}$
 - block-cut-vertex graph of connected graph is tree

Structure of 3-connected graphs

- **3-connected + decontraction:** all 3-connected graphs can be built by iteratively de-contracting vertices of K_4
- **3-connected + contraction:** 3-connected $\Leftrightarrow \exists$ separate G_0, \dots, G_k with $G_0 = K_4, G_k = G, G_i = G_{i+1} \circ xy$ with $\deg(x), \deg(y) \geq 3$

Rest

- **Degree sequence:** multiset of degrees of vertices in $V(G)$
 - *graphic:* deg. seq. (d_1, \dots, d_n) , iff
 1. $d_1 + \dots + d_n$ even
 2. $\sum_{i=1}^k d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$

Planar graphs

Basic definitions

- **Plane graph:** graph drawn without intersecting edges
- **Planar graph:** graph that *can be* dran as plane graph
- **Homeomorphism:** $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous s.t. f^{-1} is also continuous
- **Arc:** homeomorphic image of $[0, 1]$ in \mathbb{R}^2 under f
 - *endpoints:* $f(0)$ and $f(1) \rightsquigarrow$ arc “joins” endpoints
 - *polynomial arc:* arc that is union of finitely many straight line segments
- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and Y is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
 - $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon) \text{ contains points of } X \text{ and not of } X\}$
- **Jordan curve theorem:** If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\bar{x} : \text{dist}(\bar{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions R_1, R_2 and $\delta R_1 = X = \delta R_2$.

Plane graphs

- **Definition:** graph such that $E(G)$ is set of arcs in \mathbb{R}^2 and endpoints of arcs in $E(G)$ are vertices and:
 - $\forall e, e' \in E, e \neq e' : e$ and e' have distinct sets of edge sets
 - $\forall e \in E, \hat{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces:** regions of $\mathbb{R}^2 \setminus (\bigcup_{e \in E} e \cup V)$
 - *face set* of graph G : $F(G)$
- **Maximally plane:** no edges can be added without breaking planarity
 - *plane triangulation:* every face is bounded by triangle \Leftrightarrow graph is maximally plane
- **Edge limitation 1:** Plane graph: $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2:** Plane graph with no Δ : $\|G\| \leq 2|G| - 4$
- **Properties:** Let G be plane graph and $H \subseteq G$.
 - *face inheritance:* $\forall f \in F(G) \exists f' \in F(H) : f' \supseteq f$
 - *border inheritance:* $\delta f \subseteq H \Rightarrow f' = f$
 - *edge-border relations:* $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \hat{e} = \emptyset$
 - *edges in circles:*
 - $e \in E(G)$ is edge of a cycle $\Rightarrow e$ is on boundary of exactly 2 faces
 - not edge of a cycle $\Rightarrow e$ is on boundary of exactly 1 face
 - *faces in cycles:* $f_1, f_2 \in F(G). f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
 - *cyclic boundaries:* $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
 - *plane forests:* plane forests have exactly 1 face
 - *2-connected:* 2-connected plane graph $\Rightarrow \forall f \in F(G)$ bounded by cycle
 - *3-connected:* boundaries of 3-connected plane graph = its non-separating induced cycles
 - *order ≥ 3 :* plane graph of order ≥ 3 maximally plane \Leftrightarrow is plane triangulation
- **Euler's formula:** If G is connected plane graph with f faces, then $|G| - \|G\| + f = 2$
- **Dual multigraph:** Given plane G :
 1. Insert vertex in each face
 2. Put edge \tilde{e} between vertices if respective faces share e (s.t. \tilde{e} and e cross once)
 3. *Result:* Dual graph G' of G (plane multigraph) \leadsto faces of G properly k -colored $\Leftrightarrow \exists$ proper k -coloring of vertices of G'

Planar graphs

- **Definition:** graph s.t. \exists plane graph G' and bijection $f : V(G) \rightarrow V(G')$ s.t. $\forall u, v \in V(G), uv \in E(G) : f(u), f(v)$ are endpoints of arc in G'
- **Planar embedding of G :** f from the definition
- **Planar because of minors:** The following statements are equivalent:
 - G is planar
 - $G \not\subseteq MK_5 \wedge G \not\subseteq MK_{3,3}$
 - $G \not\subseteq TK_5 \wedge G \not\subseteq TK_{3,3}$
- $\delta(G)$ **limitation:** Planar graph $\delta(G) \leq 5$
- **Non-planar graphs:** K_5 and $K_{3,3}$ are not planar
- **Kuratowski's lemmas:**
 1. $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Leftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
 2. $\kappa(G) \geq 3 \wedge MK_5 \not\subseteq G \wedge MK_{3,3} \not\subseteq G \Rightarrow G$ is planar
 3. $\kappa(G) \geq 3, G$ edge-maximal wrt not containing TX . If S is vertex-cut of G , $|S| \leq 2 \wedge G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$, then G_i is edge-maximal with no TX and S induces an edge
 4. $|G| \geq 3, G$ edge-maximal wrt not containing TK_5 and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of G on surface S): any closed simple curve in any region of $S - G$ is continuously contractible into a point
- **Euler characteristic:** G embedded on surface $S \Rightarrow n - e + f = \text{Euler characteristic}$ is invariant
- **Euler genus:** $n - e + f = 2 - 2\gamma \leadsto \text{Euler genus } 2\gamma$ of S
- **Heawood's formula:** $\chi(G) \leq \underbrace{\left\lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor}_{f(\gamma), \text{Heawoods number}}$
(for G embedded on S with Euler char $2 - 2\gamma$)
- **Klein bottle:** $K_{f(\gamma)}$ is embeddable on S , unless S is *klein bottle*

Algebraic planarity criteria — Posets

- **Definition:** antisymmetric, reflexive, transitive relation on X
(write $x \leq y$ instead of (x, y))
- **Incidence poset** of G : poset whose cover diagram is represented by IG with vertices all below the edges
- **Poset dimension:** $\dim(R) =$ smallest $k \in \mathbb{N} : R$ is intersection of k total orders
- **Poset dimension in planar graphs:** G planar $\Leftrightarrow \dim(\text{incidence poset}) \leq 3$

Coloring

Base definitions

- **Vertex coloring:** map $c : V(G) \rightarrow S$ with $c(v) \neq c(w)$ for adjacent v, w
 - *color set* S
 - *k-coloring:* coloring $c : V(G) \rightarrow S$ with $|S| = k$
- **(Vertex) chromatic number:** $\chi(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-coloring}\}$
 - $\chi(G) \geq \omega(G)$
 - $\chi(G) \geq \frac{|G|}{\alpha(G)}$
 - $\chi(G) \leq \Delta(G) + 1$ (*greedy coloring*)
 - G connected, not complete, no odd cycles $\Rightarrow \chi(G) \leq \Delta(G)$
 - *k-chromatic* graph: $\chi(G) = k$
 - *k-colorable* graph: $\chi(G) \leq k$
- **Color classes:** partitions of $V(G)$ with same color
- **Equitable coloring:** proper coloring + color classes have almost (± 1) equal size
 - *existence:* any graph has equitable coloring in $(\Delta(G) + 1)$ colors
- **ij-flip:** $c' : V(G) \rightarrow [k]$ is *ij-flip* at $v \in V(G)$
 $\Leftrightarrow c'$ obtained by flipping colors i and j in max. conn. component containing v
- **Edge coloring:** map $c : E(G) \rightarrow S$ with $c(e) \neq c(f)$ for adjacent e, f
 - edge coloring of $G \Leftrightarrow$ vertex coloring of $L(G)$
 - *k-edge-coloring:* edge-coloring $c : E(G) \rightarrow S$ with $|S| = k$
- **Edge chromatic number:** $\chi'(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-edge-coloring}\}$

Coloring maps and planar graphs

- **4-color-theorem:** every planar graph is 4-colorable
- **3-color-theorem:** every triangle-free planar graph is 3-colorable

Coloring vertices

- **Chromatic number upper bound:** $\chi(G) \leq \frac{1}{2} + \sqrt{2\|G\| + \frac{1}{4}}$
- **Greedy coloring:** sort vertices v_1, \dots, v_n , color them with the smallest possible color starting at v_1
 \leadsto never uses more than $\Delta(G) + 1$ colors
- **coloring number** of graph G : $\text{col}(G) :=$ smallest k s.t. G has vertex enumeration where each vertex is preceded by $< k$ neighbors

Coloring edges

- **Vizing's theorem:** for every graph $G, \chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$
- **Bipartite graphs:** $\chi'(G) = \Delta(G)$

List coloring

- **L-list-colorable:** if $\exists c : V \rightarrow \mathbb{N} \forall v \in V : c(v) \in L(v)$
(for list of colors $L(v) \subseteq \mathbb{N}$ for each vertex, adjacent vertices receive different colors)
- **k-list-colorable/-choosable:** if G is L -list-colorable for each list L
- **List chromatic number:** $\chi_l(G) = \text{ch}(G)$
 $= \min\{k : G \text{ is } L\text{-colorable } \forall L : V \rightarrow 2^{\mathbb{N}} : |L(v)| = k \forall v \in V(G)\}$
 - $\chi_l(G) \geq \chi(G)$ because we can choose $L(v) = \{1, \dots, k\} (\forall v \in V(G))$
 - often $\chi_l(G) \gg \chi(G)$ (see $K_{m,n} : \chi = 2, \chi_l \approx \log n$)
- **Planar graphs:** $\chi_l(G) \leq 5$
- **Locally planar graphs:** $\chi_l(G) \leq 5$

Perfect graphs

- **Clique number** of graph G : $\omega(G) := \max\{k \in \mathbb{N} : K_k \subseteq G\}$
- **Independence number** of graph G : size of largest independent vertex set
- **Perfect graph:** $\forall H \subseteq G : \chi(H) = \omega(H)$
- **Perfect complement:** G is perfect $\Leftrightarrow \overline{G}$ is perfect
- **Perfect graph conjecture:** G is perfect $\Leftrightarrow C_{2k+1} \not\subseteq G$ for $k \geq 2 \wedge \overline{C_{2k+1}} \not\subseteq G$

Extremal Graph Theory

Base definitions

- **Sparse** graph: $\|G\| \sim |G|$
- **Dense** graph: $\|G\| \sim |G|^2$
- **Density:** $\|X, Y\| := \# \text{ edges between } X \text{ and } Y, d(X, Y) := \frac{\|X, Y\|}{|X||Y|}$
- **Edge density** of graph G : $\|G\| / \binom{|G|}{2}$
- ϵ -**regular** pair (X, Y) : if $|d(X, Y) - d(A, B)| \leq \epsilon$ for $\epsilon > 0$ and all $A \subseteq X, B \subseteq Y$ with $|A| \geq \epsilon|X|, |B| \geq \epsilon|Y|$
- ϵ -**regular** partition: $V_0 \dot{\cup} \dots \dot{\cup} V_k = V$ with

1. $|V_0| \leq \varepsilon |V|$
2. $|V_1| = \dots = |V_k|$
3. all but at most εk^2 of (V_i, V_j) -pairs ($1 \leq i < j \leq k$) are ε -regular

Subgraphs

- **Extremal number:** $\text{ex}(n, H) := \max\{|G| : |G| = n \wedge H \not\subseteq G\}$
- **Extremal set:** $\text{EX}(n, H) := \{G : |G| = n \wedge \|G\| = \text{ex}(n, H) \wedge H \not\subseteq G\}$
- **Turán graph:** $T(n, r)$ = unique complete r -partite graph with $|T(n, r)| = n$, partite sets differing at most by 1 ($1 \leq r \leq n$)
 - $K_{r+1} \not\subseteq T(n, r)$
 - size: $\|T(n, r)\| =: t(n, r)$
 - special Turán graph: $K_r^s := T(n, r)$ if $n = r * s$
 - Turán-graphs edge-maximal: among all r -partite graphs of order n , $T(n, r)$ has largest size
 - $t(n, r) = t(n - r, r) + (n - r)(r - 1) + \binom{r}{2}$
 - size difference to complete graph:

$$\lim_{n \rightarrow \infty} \frac{t(n - r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right)$$

- **Turán's theorem:** $\forall r > 1, n \geq 1$, any graph G with $|G| = n, \|G\| = \text{ex}(n, K_r)$ and $K_r \not\subseteq G$ is a $T(n, r - 1)$

$$\Leftrightarrow \text{EX}(n, K_r) = \{T(n, r - 1)\}$$

- **Szemerédi's regularity lemma:** $\forall \varepsilon > 0 \forall 1 \leq m \in \mathbb{N} \exists M \in \mathbb{N}$: every graph G with $|G| \geq m$ has ε -regular partition $V_0 \dot{\cup} \dots \dot{\cup} V_k$ with $m \leq k \leq M$.
- **Erdős-Stone theorem:** \forall integers $r > s \geq 1$ and any $\varepsilon > 0 \exists n_0 \in \mathbb{N}$: every graph with $|G| = n \geq n_0$ and $\|G\| \geq t(n, r - 1) + \varepsilon n^2$ has $K_r^s \subseteq G$.
 - corollary: the theorem together with the size difference to complete graph yields

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

- **Chvátal-Szemerédi theorem:** $\forall \varepsilon > 0$ and any integer $r \geq 3$, any graph with $|G| = n$ and $\|G\| \geq (1 - \frac{1}{r-1} + \varepsilon) \binom{n}{2}$ has $K_r^t \subseteq G$ with $t = \frac{\log n}{500 \log(\frac{1}{\varepsilon})}$

- existence: $\exists G$ with $|G| = n$ and $\|G\| = (1 - \frac{1+\varepsilon}{r-1}) \binom{n}{2}$ with $G \not\subseteq K_r^t$ for $t = \frac{5 \log n}{\log(\frac{1}{\varepsilon})}$

- **Zarankiewicz function:** $z(m, n; s, t)$ = maximum # of edges that bipartite graph with parts of size m and n can have without containing $K_{s,t}$

- **Kővári-Sós-Turán:** $z(m, n; s, t) \leq (s - 1)^{\frac{1}{t}} (n - t + 1) m^{1 - \frac{1}{t}} + (t - 1)m$
 - $m = n, t = s$: $z(n, n; t, t) \leq c_1 n n^{1 - \frac{1}{t}} + c_2 n = \mathcal{O}(n^{2 - \frac{1}{t}})$

- **Bound for $\text{ex}(n, K_{t,s})$:** $\leq \frac{1}{2} z(n, n; s, t) \leq c n^{2 - \frac{1}{s}} (t \geq s \geq 1)$
 - $t = s = 2$: $\text{ex}(n, C_4) \leq \frac{n}{4} (1 + \sqrt{4n - 3})$

- **Bound for $\text{ex}(n, K_{r,r})$:** $\geq c n^{2 - \frac{2}{r+1}} (\forall n, r \in \mathbb{N})$

- **Bound for $\text{ex}(n, P_{k+1})$:** $\leq \frac{n(k-1)}{2}$

Minors

- **Hadwiger conjecture:** $\chi(G) \geq r \Rightarrow MK_r \subseteq G$
 - $r \in \{1, 2, 3, 4\}$: easy to see
 - $r \in \{5, 6\}$: proven using 4-color theorem
 - $r \geq 7$: still open
- **Bollobás-Thomason theorem:** $d(G) \geq c r^2 \Rightarrow TK_r \subseteq G$
- **Minimum degree + girth = minor:** $\delta(G) \geq d, g(G) \geq 8k + 3 (d, k \in \mathbb{N}, d \leq 3)$.
Then $MH \subseteq G$ with $\delta(H) \geq d(d - 1)^k$.
- **Thomassen's theorem:** $\forall r \in \mathbb{N} \exists f : \mathbb{N} \rightarrow \mathbb{N}$ s.t. every G with $\delta(G) \geq 3$ and $g(G) \geq f(r)$ has K_r minor
- **Kühn-Osthus theorem:** $\forall r \in \mathbb{N} \exists g \in \mathbb{N} : TK_r \subseteq G$ for all G with $\delta(G) \geq r - 1$ and $g(G) \geq g$