

# Basics

## Notations

- $\binom{V}{k} := \{A : A \subseteq V \wedge |A| = k\}$
- $[n] := \{1, \dots, n\} \subset \mathbb{N}$
- **Power set**  $2^X := \{A : A \subseteq X\}$

## Graphs

- **Definition:**  $G = (V, E)$  with  $E \subseteq V^2, V \cap E = \emptyset$
- **Vertex:**  $v \in V$  for graph  $G = (V, E)$ 
  - $v$  incident with  $e \Leftrightarrow v \in e$
  - $v_1, v_2$  ends of  $e \Leftrightarrow e = v_1 v_2$
  - $v_1, v_2$  adjacent/neighbors  $\Leftrightarrow v_1 v_2 \in E$
- **Edge:**  $e = \{x, y\} \in E$  for graph  $G = (V, E)$  (short  $e = xy$ )
  - $e$  edge at  $v \Leftrightarrow v$  incident with  $e$
  - $e$  joins  $v_1, v_2 \Leftrightarrow e = v_1 v_2$
  - $xy$  is  $X$ - $Y$ -edge  $\Leftrightarrow x \in X \wedge y \in Y$
  - $e_1, e_2$  adjacent/neighbors  $\Leftrightarrow \exists v : v \in e_1 \wedge v \in e_2$
- **Vertex sets:**
  - $V(G) = V$  for graph  $G = (V, E)$
  - $X \subset V(G)$  independent  $\Leftrightarrow$  no  $x_1, x_2 \in X$  are adjacent
  - neighborhood of  $v \in V(G)$ :  $N(v) = \{u \in V(G) : uv \in E(G)\}$
- **Edge sets:**
  - $E(G) = E$  for graph  $G = (V, E)$
  - $E(X, Y)$ : set of edges between  $X \subset V(G)$  and  $Y \subset V(G)$
  - $E(x, Y)$ : set of edges between vertex  $x \in V(G)$  and  $Y \subset V(G)$
  - $E(v)$ : set of edges at  $v \in V(G)$
- **Order:**  $= |V(G)|$ , short  $|G|$
- **Size:**  $= |E(G)|$ , short  $\|G\|$
- **Trivial graph:** graph of order 0 or 1
- **Incidence graph** of  $G$ :  $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Isomorphic** ( $G_1$  to another graph  $G_2$ , write  $G_1 \cong G_2$  or even  $G_1 = G_2$ ):
  - $\exists$  bijection  $f : V_1 \rightarrow V_2 : \{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$
- **Graph union:**  $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$
- **Graph intersection:**  $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$
- **Graph multiplication:**  $G * G'$ : join all  $v \in G$  with all  $v' \in G'$  (with  $V(G) \cap V(G') = \emptyset$ )
- **Subgraph**  $G'$  of  $G$  (write  $G' \subseteq G$ ): if  $V(G') \subseteq V(G)$  and  $E(G') \subseteq E(G)$ 
  - $G$  contains  $G'$
  - $G'$  proper subgraph of  $G$ : if  $G' \subseteq G$  and  $G' \neq G$
  - $G'$  induced subgraph of  $G$ :  $G' \subseteq G$  and  $E(G')$  contains all edges of  $G$  with both ends in  $V(G')$ ,  $V(G')$  induces  $G'$ , write  $G' = G[X]$  (with  $X = V(G')$ )
  - Edge-induced subgraph: subgraph induced by  $X \subseteq E(G)$ , note  $G[X]$
  - $G'$  spanning subgraph of  $G$ :  $V(G') = V(G)$
- **Supergraph:**  $G$  of  $G'$  (write  $G \supseteq G'$ ): as above.
- **Vertex cover:**  $V' \subseteq V(G)$  s.t. any  $e \in E(G)$  is incident to a vertex in  $V'$
- **Graph subtraction:**
  - $G - U = G[V(G) \setminus U]$  for some vertex set  $U$
  - $G - v = G[V(G) \setminus \{v\}]$  for some vertex  $v$
  - $G - G' = G[V(G) \setminus V(G')]$  for some graph  $G'$
- **Edge addition:**  $G + F = (V(G), V(E) \cup F)$  for some  $F \subseteq V(G)^2$
- **Complement:**  $\overline{G} = (V(G), V^2 \setminus E(G))$
- **Line graph** of  $G$ :  $L(G) = (E(G), \{xy \in E(G)^2 : x, y \text{ adjacent in } G\})$
- **Complete graph:**  $(X, X^2)$  with vertex set  $X$ 
  - $K_n$ : complete graph on  $n$  vertices

## Vertex degrees

- **Degree** of  $v \in V$ :  $d(v) = \deg(v) = |N(v)|$ 
  - $v \in V(G)$  isolated:  $d(v) = 0$
  - $v \in V(G)$  leaf:  $d(v) = 1$
  - number of vertices of odd degree is even
- **Minimum degree** of graph  $G$ :  $\delta(G) = \min\{d(v) : v \in V(G)\}$
- **Maximum degree** of graph  $G$ :  $\Delta(G) = \max\{d(v) : v \in V(G)\}$
- **Degree sum:**  $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Average degree** of graph  $G$ :  $d(G) = \frac{1}{|G|} \sum_{v \in V} d(v)$ 
  - $\delta(G) \leq d(G) \leq \Delta(G)$
- **k-regular graph:**  $\forall v \in V(G) : d(v) = k$ 
  - cubic graph: 3-regular graph
- **Vertex-Edge-ratio** of graph  $G$ :  $\varepsilon(G) = \frac{\|G\|}{|G|}$ 
  - $\varepsilon(G) = \frac{1}{2}d(G)$
  - every graph with  $\|G\| \geq 1$  has  $H \subseteq G$  with  $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$

## Paths

- **Path:**  $(\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$  (read:  $v_0 v_n$ -path)
  - shorthand:  $v_1 \dots v_n$
  - $v_0, v_n$  linked by path
  - $v_0, v_n$  end-vertices/ends of path
  - $v_1, \dots, v_{n-1}$  inner vertices of path
- **Length:**  $|E(P)| \neq |V(P)|$
- **Shorthands** ( $0 \leq i \leq j \leq k$ ):
  - $P = x_0 \dots x_k, \hat{P} = x_1 \dots x_{k-1}$
  - $Px_i = x_0 \dots x_i, P\hat{x}_i = x_0 \dots x_{i-1}$
  - $x_i P = x_i \dots x_k, \hat{x}_i P = x_{i+1} \dots x_k$
  - $x_i P x_j = x_i \dots x_j, \hat{x}_i P \hat{x}_j = x_{i+1} \dots x_{j-1}$
- **Path concatenation:**  $Px \cap xQy \cap yR = PxQyR$
- **A-B-path:**  $V(P) \cap A = \{x_0\} \wedge V(P) \cap B = \{x_n\}$
- **H-path:** graph  $H, P$  meets  $H$  exactly in its ends
- **Independent:** two  $ab$ -paths are independent  $\Leftrightarrow$  they only share  $a$  and  $b$
- **Path existence:** Every  $G$  with  $\delta(G) \geq 2$  contains path of length  $\delta(G)$
- **Distance:**  $d_G(x, y) = \min(\{k : \exists x\text{-}y\text{-path of length } k\} \cup \{\infty\})$
- **Central:**  $v \in V(G)$  where  $\text{cen} = \max\{d_G(v, x) : v \neq x \in V(G)\}$  is minimal
- **Radius:**  $\text{rad}(G) = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$
- **Diameter** of  $G$ :  $\text{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$ 
  - radius-diameter-relation:  $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$
  - radius-degree-vertex-restriction:
 
$$\text{rad}(G) \leq k \wedge \Delta(G) \leq d \geq 3 \Rightarrow |G| \leq \frac{d}{d-2}(d-1)^k$$
- **Walk:** alternating sequence  $v_0 e_0 \dots e_{k-1} v_k$  s.t.  $e_i = v_i v_{i+1}$  ( $\forall i < k$ )
  - closed walk:  $v_k = v_0$
  - walk-path-relation: all vertices in walk distinct  $\leadsto$  path
  - walk-path-induction:  $\exists v_0 v_k\text{-walk} \Rightarrow \exists v_0 v_k\text{-path}$

## Cycles

- **Cycle:**  $C = P + x_{k-1} x_0$  with path  $P = x_0 \dots x_{k-1}$  ( $k \geq 3$ )
  - shorthand:  $x_0 \dots x_{k-1} x_0$
- **Length:**  $= |C| = \|C\|$
- **k-cycle:**  $C_k =$  cycle of length  $k$
- **Girth** of graph  $G$ :  $g(G) = \min(\{k : G \text{ contains } C_k\} \cup \{\infty\})$ 
  - girth-diameter-relation:  $g(G) \leq 2\text{diam}(G) + 1$
  - girth-vertex-relation:  $\delta(G) \geq 3 \Rightarrow g(G) < 2 \log |G|$
- **Circumference** of graph  $G$ :  $= \max(\{k : G \text{ contains } C_k\} \cup \{0\})$
- **Chord** of cycle  $C \subseteq G$ :  $= xy \in E(G)$  with  $xy \notin E(C)$ , but  $x, y \in V(C)$
- **Induced cycle:** induced subgraph of  $G$  that is a cycle (= cycle in  $G$  with no chords)
- **Cycle existence:** Every  $G$  with  $\delta(G) \geq 2$  contains cycle of length  $\geq \delta(G) + 1$
- **Odd closed walk, odd cycle:**  $G$  has odd closed walk  $\Rightarrow G$  has odd cycle

## Connectivity

- **Connected** graph  $G$ :  $\forall x, y \in V(G) : \exists xy\text{-path}$ 
  - connected subset  $U \subseteq V(G)$ : if  $G[U]$  is connected
- **Vertex enumeration:**  $G$  connected  $\Rightarrow$  vertices can be enumerated  $v_1, \dots, v_n$  s.t.  $G_i := G[v_1, \dots, v_i]$  is connected ( $\forall i \leq n$ )
- **Component:** maximal connected subgraph
  - graph partitioning: components partition  $G$
- **Subgraph separation:**  $X \subset V(G)$  separates  $A, B \subset V(G) \Leftrightarrow$  any  $A$ - $B$ -path has vertex in  $X$
- separator  $X$
- **Cut-Vertex:** vertex separating two other vertices of the component
- **Bridge:** edge separating its ends (= edges of component not lying on any cycle)
- **k-connected:** if  $|G| > k \wedge G - X$  is connected  $\forall X \subseteq V(G)$  with  $|X| < k$ 
  - $\leadsto$  no two vertices in  $G$  are separated by fewer than  $k$  other vertices
  - $\leadsto$  any two vertices can be joined by  $k$  independent paths
- **k-linked:** if for any  $2k$  vertices  $(s_1, \dots, s_k, t_1, \dots, t_k) \exists$  pairwise disjoint  $s_i t_i$ -paths (note:  $k$ -connected  $\not\Rightarrow$   $k$ -linked)
- **Connectivity:**  $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected:** if  $|G| > 1 \wedge G - F$  is connected  $\forall F \subseteq E(G)$  with  $|F| < l$
- **Edge-connectivity:**  $\kappa'(G) = \lambda(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Connectivity and smallest degree:**  $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Connectivity and average degree:**  $d(G) \geq 4k \Rightarrow G$  has  $k$ -connected subgraph
- **Menger's theorem:** for  $A, B \subseteq V(G)$ :
  - $\min \#$  of vertices separating  $A$  and  $B = \max \#$  of disjoint  $A$ - $B$ -paths
  - edge corollary: minimum number of edges separating  $a$  and  $b = \max$  number of edge-disjoint  $a$ - $b$ -paths
- **Menger global:**
  - $k$ -connected  $\Leftrightarrow \forall a, b \in V(G) \exists k$  pairwise independent  $ab$ -paths
  - $k$ -edge-connected  $\Leftrightarrow \forall a, b \in V(G) \exists k$  pairwise edge-disjoint  $ab$ -paths

Trees and forests

- **Forest**: Graph with no cycle as subgraph
- **Tree**: Graph that is connected and acyclic
  - $\Leftrightarrow G$  is connected and  $\forall e \in E(G) : G - e$  is disconnected (*minimal-connected*)
  - $\Leftrightarrow G$  is acyclic and  $\forall xy \notin E(G) : G \cup xy$  has cycle (*maximal-acyclic*)
  - $\Leftrightarrow G$  is connected and 1-degenerate ( $\forall G' \subseteq G : \delta(G') \leq 1$ )
  - $\Leftrightarrow G$  is connected and  $\|G\| = |G| - 1$
  - $\Leftrightarrow G$  is acyclic and  $\|G\| = |G| - 1$
  - $\Leftrightarrow \forall u, v \in V(G) \exists$  unique  $uv$ -path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree  $T, |T| \geq 2 \Rightarrow T$  has leaf
- **Edge count**: Tree  $T, |T| = n \Rightarrow \|T\| = n - 1$

Bipartite graphs

- **r-partite** graph  $G: V(G)$  allows partitioning in  $r$  classes s.t.  $\forall e = xy \in E(G) : x$  and  $y$  are in different classes
- **Bipartite** graph: 2-partite graph
  - $\Leftrightarrow G$  contains no cycles of odd length
    - *complete bipartite*:  $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$

Contraction and minors

- **Subdivision** of graph  $G$ : any graph obtained from  $G$  by subdividing edges
- **Topological minor**:  $H$  is topological minor if  $TH \subseteq G$  where  $TH$  is built from  $H$  by subdividing edges
  - *branch vertices*: original vertices of  $H$
  - *subdividing vertices*: vertices placed on edges joining branch vertices
- **MH**:  $G \stackrel{(\star)}{=} MH$  is *minor* of  $H$  if
  - $V(G) = V_1 \dot{\cup} \dots \dot{\cup} V_n$  with  $n = |H|$
  - $G[V_i]$  connected ( $\forall i = 1, \dots, n$ )
  - If  $V(H) = \{v_1, \dots, v_n\}$  and  $v_i v_j \in E(H)$ , then  $\exists$  edge between  $V_i$  and  $V_j$  ( $\star$ ): *Notation abuse*:  $MH$  is class of graphs
- **Branch sets**:  $V_i$ 's from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** ( $H$  of  $G$ , noted  $H \preceq G$ ):  $\Leftrightarrow MH \subseteq G$ 
  - $\sim H \preceq G \Leftrightarrow H$  can be obtained by edge/vertex deletions + **contractions**.
- **Note**:  $TH \subseteq MH$
- **Edge contraction**:
$$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy}, (E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy}z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
with  $xy \in E(G)$
- **De-contraction**: if  $\exists xy \in E(G) : \kappa(G \circ xy) \geq 3$  (for  $G$  with  $\kappa(G) \geq 3, |G| \geq 5$ )

Euler tours

- **Definition**: closed walk with
  - no edges of  $G$  are repeatedly used
  - all edges of  $G$  are used
- **Eulerian graph**: graph containing an Euler tour  $\Leftrightarrow \forall v \in V(G) : d(v)$  even

Algebraic assets

- **Adjacency matrix**:  $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$

Other graph notions

- **Digraph**:  $G = (V, E)$  with vertex set  $V$  and edge set  $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$
- **Multigraph**:  $G = (V, E)$  with vertex set  $V$  and multiset  $E$  of  $V$ -pairs
- **Multigraph**:  $G = (V, E)$  with vertex set  $V$  and edge set  $E \subseteq 2^V = \{A : A \subseteq V\}$

Matching, Covering, Packing

Matching in bipartite graphs

- **Vertex cover**:  $U \subseteq V(G)$  s.t. all edges in  $G$  are incident to a vertex  $\in U$

- **Matching**  $M$ : set of independent edges in a graph
  - *matching graph*:  $\delta(G) = \Delta(G) = 1$
  - *saturating*:  $G = (A \cup B, E)$  has matching saturating  $A \Leftrightarrow \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$
  - *nearly*:  $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1) \Rightarrow \exists$  matching  $M$  saturating all but at most  $d$  vertices of  $A$
- **k-factor**:  $k$ -regular spanning subgraph
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover (Königs theorem)
- **Matching existence**:
  - *neighbor-based*:  $G = (A \cup B, E)$  contains matching of  $A \Leftrightarrow |N(S)| \geq |S|$
  - *regular + bipartite*:  $G$  is  $k$ -regular + bipartite ( $k \geq 1$ )  $\Rightarrow G$  has 1-factor
  - *2k-regular*: graph  $2k$ -regular ( $k \geq 1$ )  $\Rightarrow$  has 2-factor
- **Marriages**: make matchings based on preferences
  - *preferences*: family  $(\leq_v)_{v \in V}$  of linear orderings  $\leq_v$  on  $E(v)$
  - *stable matching*  $M$ :
$$\forall e \in E \setminus M \exists f \in M : e \text{ and } f \text{ have common vertex } v \text{ with } e <_v f$$
  - *stable matching existence*: For every set of preferences,  $G$  has stable matching
- **f-factor**: spanning subgraph  $H \subseteq G$  with  $\deg_H(v) = f(v)$ ,  $f : V(G) \rightarrow \{0, 1, \dots\}$  with  $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is  $\cong H$ 
  - *existence*: if  $\delta(G) \geq (1 - \frac{1}{k}|V(G)|)$  and  $k$  divides  $|G|$ , then  $G$  has  $K_k$ -factor

Matchings in general graphs

- **Perfect matching**: spanning + matching subgraph of  $G$  (aka *1-factor*)
  - *existence (Tutte)*:  $G$  has perfect matching  $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq |S|$  (Tutte's condition,  $q(G)$  = number of components in  $G$  with odd order)
  - *existence (Petersen)*:  $G$  bridgeless + cubic  $\Rightarrow G$  has 1-factor

Connectivity

2-connected graphs and subgraphs

- **2-connected construction**:  $G$  is 2-connected  $\Leftrightarrow$  it can be constructed by successively adding paths to a cycle (removing those paths: *ear-decomposition*)
- **Block**: maximal 2-connected subgraph or bridge
  - share  $\leq 1$  vertices with another
  - *cycles* of  $G$  = cycles of its blocks
  - *bounds* of  $G$  = minimal cuts of its blocks
- **Block-cut-vertex graph**
  - $V$  = set of blocks  $\cup$  set of vertices
  - $E = \{\{v, B\} : v \in V(B), \text{cut-vertex } v, \text{block } B\}$
  - block-cut-vertex graph of connected graph is tree

Structure of 3-connected graphs

- **3-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of  $K_4$
- **3-connected + contraction**: 3-connected  $\Leftrightarrow \exists$  separate  $G_0, \dots, G_k$  with
$$G_0 = K_4, G_k = G, G_i = G_{i+1} \circ xy$$
with  $\deg(x), \deg(y) \geq 3$

Rest

- **Degree sequence**: multiset of degrees of vertices in  $V(G)$ 
  - *graphic*: deg. seq.  $(d_1, \dots, d_n)$ , iff
    1.  $d_1 + \dots + d_n$  even
    2.  $\sum_{i=1}^k d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$

Planar graphs

Basic definitions

- **Plane graph**: graph drawn without intersecting edges
- **Planar graph**: graph that *can be* dran as plane graph
- **Homeomorphism**:  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  continuous s.t.  $f^{-1}$  is also continuous
- **Arc**: homeomorphic image of  $[0, 1]$  in  $\mathbb{R}^2$  under  $f$ 
  - *endpoints*:  $f(0)$  and  $f(1) \rightsquigarrow$  arc “joins” endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments

- **Region**  $Y \subseteq \mathbb{R}^2 \setminus X$ : any two points  $\in Y$  could be joined by arc and  $Y$  is maximal ( $X \subseteq \mathbb{R}^2$ )
- **Boundary** of  $X \subseteq \mathbb{R}^2$ :  
 $\partial X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon) \text{ contains points of } X \text{ and not of } X\}$
- **Jordan curve theorem**: If  $X \subseteq \mathbb{R}^2$  and homeomorphic to  $\{\bar{x} : \text{dist}(\bar{x}, 0) = 1\}$  (*unit circle*), then  $\mathbb{R}^2 \setminus X$  has two regions  $R_1, R_2$  and  $\partial R_1 = X = \partial R_2$ .

## Plane graphs

- **Definition**: graph such that  $E(G)$  is set of arcs in  $\mathbb{R}^2$  and endpoints of arcs in  $E(G)$  are vertices and:
  - $\forall e, e' \in E, e \neq e' : e$  and  $e'$  have distinct sets of edge sets
  - $\forall e \in E, \tilde{e} = e \setminus \{\text{endpoints}\}$  doesn't contain any vertices and points from other arcs
- **Faces**: regions of  $\mathbb{R}^2 \setminus (\bigcup_{e \in E} e \cup V)$ 
  - *face set* of graph  $G$ :  $F(G)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle  $\Leftrightarrow$  graph is maximally plane
- **Edge limitation 1**: Plane graph:  $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no  $\Delta$ :  $\|G\| \leq 2|G| - 4$
- **Properties**: Let  $G$  be plane graph and  $H \subseteq G$ .
  - *face inheritance*:  $\forall f \in F(G) \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*:  $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*:  $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \tilde{e} = \emptyset$
  - *edges in circles*:  
 $e \in E(G)$  is edge of a cycle  $\Rightarrow e$  is on boundary of exactly 2 faces  
 not edge of a cycle  $\Rightarrow e$  is on boundary of exactly 1 face
  - *faces in cycles*:  $f_1, f_2 \in F(G). f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$  is cycle
  - *cyclic boundaries*:  $\kappa(G) \geq 2 \Rightarrow$  each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
  - *2-connected*: 2-connected plane graph  $\Rightarrow \forall f \in F(G)$  bounded by cycle
  - *3-connected*: boundaries of 3-connected plane graph = its non-separating induced cycles
  - *order*  $\geq 3$ : plane graph of order  $\geq 3$  maximally plane  $\Leftrightarrow$  is plane triangulation
- **Euler's formula**: If  $G$  is connected plane graph with  $f$  faces, then  
 $|G| - \|G\| + f = 2$
- **Dual multigraph**: Given plane  $G$ :
  1. Insert vertex in each face
  2. Put edge  $\tilde{e}$  between vertices if respective faces share  $e$  (s.t.  $\tilde{e}$  and  $e$  cross once)
  3. *Result*: Dual graph  $G^I$  of  $G$  (plane multigraph)  
 $\leadsto$  faces of  $G$  properly  $k$ -colored  $\Leftrightarrow \exists$  proper  $k$ -coloring of vertices of  $G^I$

## Planar graphs

- **Definition**: graph s.t.  $\exists$  plane graph  $G^I$  and bijection  $f : V(G) \rightarrow V(G^I)$  s.t.  
 $\forall u, v \in V(G), uv \in E(G) : f(u), f(v)$  are endpoints of arc in  $G^I$
- **Planar embedding** of  $G$ :  $f$  from the definition
- **Planar because of minors**: The following statements are equivalent:
  - $G$  is planar
  - $G \not\subseteq MK_5 \wedge G \not\subseteq MK_{3,3}$
  - $G \not\subseteq TK_5 \wedge G \not\subseteq TK_{3,3}$
- $\delta(G)$  **limitation**: Planar graph  $\delta(G) \leq 5$
- **Non-planar graphs**:  $K_5$  and  $K_{3,3}$  are not planar
- **Kuratowski's lemmas**:
  1.  $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Leftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
  2.  $\kappa(G) \geq 3 \wedge MK_5 \not\subseteq G \wedge MK_{3,3} \not\subseteq G \Rightarrow G$  is planar
  3.  $\kappa(G) \geq 3, G$  edge-maximal wrt not containing  $TX$ . If  $S$  is vertex-cut of  $G$ ,  $|S| \leq 2 \wedge G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$ , then  $G_i$  is edge-maximal with no  $TX$  and  $S$  induces an edge
  4.  $|G| \geq 3, G$  edge-maximal wrt not containing  $TK_5$  and  $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of  $G$  on surface  $S$ ): any closed simple curve in any region of  $S - G$  is continuously contractible into a point
- **Euler characteristic**:  $G$  embedded on surface  $S \Rightarrow n - e + f = \text{Euler characteristic}$  is invariant
- **Euler genus**:  $n - e + f = 2 - 2\gamma \leadsto \text{Euler genus } 2\gamma$  of  $S$
- **Heawood's formula**:  $\chi(G) \leq \underbrace{\frac{7 + \sqrt{1 + 48\gamma}}{2}}_{f(\gamma), \text{Heawoods number}}$   
 (for  $G$  embedded on  $S$  with Euler char  $2 - 2\gamma$ )
- **Klein bottle**:  $K_{f(\gamma)}$  is embeddable on  $S$ , unless  $S$  is *klein bottle*

# Coloring

## Base definitions

- **Vertex coloring**: map  $c : V(G) \rightarrow S$  with  $c(v) \neq c(w)$  for adjacent  $v, w$ 
  - *color set*  $S$
  - $k$ -coloring: coloring  $c : V(G) \rightarrow S$  with  $|S| = k$
- **(Vertex) chromatic number**:  $\chi(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-coloring}\}$ 
  - $\chi(G) \geq \omega(G)$
  - $\chi(G) \geq \frac{|G|}{\alpha(G)}$
  - $\chi(G) \leq \Delta(G) + 1$  (*greedy coloring*)
  - $G$  connected, not complete, no odd cycles  $\Rightarrow \chi(G) \leq \Delta(G)$
  - $k$ -chromatic graph:  $\chi(G) = k$
  - $k$ -colorable graph:  $\chi(G) \leq k$
- **Color classes**: partitions of  $V(G)$  with same color
- **Equitable coloring**: proper coloring + color classes have almost ( $\pm 1$ ) equal size
  - *existence*: any graph has equitable coloring in  $(\Delta(G) + 1)$  colors
- **ij-flip**:  $c^I : V(G) \rightarrow [k]$  is  $ij$ -flip at  $v \in V(G)$   
 $\Leftrightarrow c^I$  obtained by flipping colors  $i$  and  $j$  in max. conn. component containing  $v$
- **Edge coloring**: map  $c : E(G) \rightarrow S$  with  $c(e) \neq c(f)$  for adjacent  $e, f$ 
  - edge coloring of  $G \Leftrightarrow$  vertex coloring of  $L(G)$
  - $k$ -edge-coloring: edge-coloring  $c : E(G) \rightarrow S$  with  $|S| = k$
- **Edge chromatic number**:  $\chi'(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-edge-coloring}\}$

## Coloring maps and planar graphs

- **4-color-theorem**: every planar graph is 4-colorable
- **3-color-theorem**: every triangle-free planar graph is 3-colorable

## Coloring vertices

- **Chromatic number upper bound**:  $\chi(G) \leq \frac{1}{2} + \sqrt{2\|G\| + \frac{1}{4}}$
- **Greedy coloring**: sort vertices  $v_1, \dots, v_n$ , color them with the smallest possible color starting at  $v_1$   
 $\leadsto$  never uses more than  $\Delta(G) + 1$  colors
- **coloring number** of graph  $G$ :  $\text{col}(G) :=$  smallest  $k$  s.t.  $G$  has vertex enumeration where each vertex is preceded by  $< k$  neighbors

## Coloring edges

- **Vizing's theorem**: for every graph  $G, \chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$
- **Bipartite graphs**:  $\chi'(G) = \Delta(G)$

## Perfect graph

- **Definition**:  $\forall H \subseteq_{\text{ind}} G : \chi(H) = \omega(H)$
- **Perfect complement**:  $G$  is perfect  $\Leftrightarrow \overline{G}$  is perfect
- **Perfect graph conjecture**:  $G$  is perfect  $\Leftrightarrow C_{2k+1} \not\subseteq G$  for  $k \geq 2 \wedge \overline{C}_{2k+1} \not\subseteq G$

## Posets

- **Definition**: antisymmetric, reflexive, transitive relation on  $X$   
 (write  $x \leq y$  instead of  $(x, y)$ )
- **Incidence poset** of  $G$ : poset whose cover diagram is represented by  $IG$  with vertices all below the edges
- **Poset dimension**:  $\dim(R) =$  smallest  $k \in \mathbb{N} : R$  is intersection of  $k$  total orders
- **Poset dimension in planar graphs**:  $G$  planar  $\Leftrightarrow \dim(\text{incidence poset}) \leq 3$

## List-colorings

- **L-list-colorable**: if  $\exists c : V \rightarrow \mathbb{N} \forall v \in V : c(v) \in L(v)$   
 (for list of colors  $L(v) \subseteq \mathbb{N}$  for each vertex, adjacent vertices receive different colors)
- **k-list-colorable/-choosable**: if  $G$  is  $L$ -list-colorable for each list  $L$
- **List chromatic number**:  $\chi_l(G) = \text{ch}(G)$   
 $= \min\{k : G \text{ is } L\text{-colorable } \forall L : V \rightarrow 2^{\mathbb{N}} : |L(v)| = k \forall v \in V(G)\}$ 
  - $\chi_l(G) \geq \chi(G)$  because we can choose  $L(v) = \{1, \dots, k\} (\forall v \in V(G))$
  - often  $\chi_l(G) \gg \chi(G)$  (see  $K_{m,n} : \chi = 2, \chi_l \approx \log n$ )
- **Planar graphs**:  $\chi_l(G) \leq 5$
- **Locally planar graphs**:  $\chi_l(G) \leq 5$