# Basics

### Notations

- $\binom{V}{k} := \{A : A \subseteq V \wedge |A| = k\}$
- $[n] := \{1, \ldots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

### Graphs

- **Definition**: $G = (V, E)$ with $E \subseteq V^2, V \cap E = \varnothing$
- **Vertex**: $v \in V$ for graph $G = (V, E)$
  - *$v$ incident with $e$* $\iff v \in e$
  - *$v_1, v_2$ ends of $e$* $\iff e = v_1 v_2$
  - *$v_1, v_2$ adjacent/neighbors* $\iff v_1 v_2 \in E$
- **Edge**: $e = \{x, y\} \in E$ for graph $G = (V, E)$ (short $e = xy$)
  - *$e$ edge at $v$* $\iff v$ incident with $e$
  - *$e$ joins $v_1, v_2$* $\iff e = v_1 v_2$
  - $xy$ is *X-Y-edge* $\iff x \in X \wedge y \in Y$
  - *$e_1, e_2$ adjacent/neighbors* $\iff \exists\, v : v \in e_1 \wedge v \in e_2$
- **Vertex sets**:
  - $V(G) = V$ for graph $G = (V, E)$
  - $X \subset V(G)$ *independent* $\iff$ no $x_1, x_2 \in X$ are adjacent
  - *neighborhood* of $v \in V(G)$: $N(v) = \{u \in V(G) : uv \in E(G)\}$
- **Edge sets**:
  - $E(G) = E$ for graph $G = (V, E)$
  - $E(X, Y)$: set of edges between $X \subset V(G)$ and $Y \subset V(G)$
  - $E(x, Y)$: set of edges between vertex $x \in V(G)$ and $Y \subset V(G)$
  - $E(v)$: set of edges at $v \in V(G)$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $||G||$
- **Trivial graph**: graph of order 0 or 1
- **Incidence graph** of $G$: $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Isomorphic** ($G_1$ to another graph $G_2$, write $G_1 \cong G_2$ or even $G_1 = G_2$):
  $\exists$ bijection $f : V_1 \to V_2 : \{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$
- **Graph union**: $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$
- **Graph intersection**: $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$
- **Graph multiplication**: $G * G'$: join all $v \in G$ with all $v' \in G'$ (with $V(G) \cap V(G') = \varnothing$)
- **Subgraph** $G'$ of $G$ (write $G' \subseteq G$): if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$
  - *$G$ contains $G'$*
  - *$G'$ proper subgraph* of $G$: if $G' \subseteq G$ and $G' \neq G$
  - *$G'$ induced subgraph* of $G$: $G' \subseteq G$ and $E(G')$ contains all edges of $G$ with both ends in $V(G')$, *$V(G')$ induces $G'$*, write $G' = G[X]$ (with $X = V(G')$)
  - *Edge-induced subgraph*: subgraph induced by $X \subseteq E(G)$, note $G[X]$
  - *$G'$ spanning subgraph* of $G$: $V(G') = V(G)$
- **Supergraph**: $G$ of $G'$ (write $G \supseteq G'$): as above.
- **Vertex cover**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in $V'$
- **Graph subtraction**:
  - $G - U = G[V(G) \setminus U]$ for some vertex set $U$
  - $G - v = G[V(G) \setminus \{v\}]$ for some vertex $v$
  - $G - G' = G[V(G) \setminus V(G')]$ for some graph $G'$
- **Edge addition**: $G + F = (V(G), V(E) \cup F)$ for some $F \subseteq V(G)^2$
- **Complement**: $\overline{G} = (V(G), V^2 \setminus E(G))$
- **Line graph** of $G$: $L(G) = (E(G), \{xy \in E(G)^2 : x, y \text{ adjacent in } G\})$
- **Complete graph**: $(X, X^2)$ with vertex set $X$
  - $K_n$: complete graph on $n$ vertices

### Vertex degrees

- **Degree** of $v \in V$: $d(v) = \deg(v) = |N(v)|$
  - $v \in V(G)$ *isolated*: $d(v) = 0$
  - $v \in V(G)$ *leaf*: $d(v) = 1$
  - number of vertices of odd degree is even
- **Minimum degree** of graph $G$: $\delta(G) = \min\{d(v) : v \in V(G)\}$
- **Maximum degree** of graph $G$: $\Delta(G) = \max\{d(v) : v \in V(G)\}$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Average degree** of graph $G$: $d(G) = \frac{1}{|G|} \sum_{v \in V} d(v)$
  - $\delta(G) \leq d(G) \leq \Delta(G)$
- **k-regular graph**: $\forall v \in V(G) : d(v) = k$
  - *cubic graph*: 3-regular graph
- **Vertex-Edge-ratio** of graph $G$: $\varepsilon(G) = \frac{||G||}{|G|}$
  - $\varepsilon(G) = \frac{1}{2} d(G)$
  - every graph with $||G|| \geq 1$ has $H \subseteq G$ with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$

### Paths

- **Path**: $(\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\})$ (read: $v_0 v_n$-path)
  - *shorthand*: $v_1 \ldots v_n$
  - $v_0, v_n$ *linked* by path
  - $v_0, v_n$ *end-vertices/ends* of path
  - $v_1, \ldots, v_{n-1}$ *inner vertices* of path
- **Length**: $|E(P)| \neq |V(P)|$
- **Shorthands** ($0 \leq i \leq j \leq k$):
  - $P = x_0 \ldots x_k$, $\mathring{P} = x_1 \ldots x_{k-1}$
  - $Px_i = x_0 \ldots x_i$, $P\mathring{x}_i = x_0 \ldots x_{i-1}$
  - $x_i P = x_i \ldots x_k$, $\mathring{x}_i P = x_{i+1} \ldots x_k$
  - $x_i P x_j = x_i \ldots x_j$, $\mathring{x}_i P \mathring{x}_j = x_{i+1} \ldots x_{j-1}$
- **Path concatenation**: $Px \cap xQy \cap yR = PxQyR$
- **A-B-path**: $V(P) \cap A = \{x_0\} \wedge V(P) \cap B = \{x_n\}$
- **H-path**: graph $H$, $P$ meets $H$ exactly in its ends
- **Independent**: two $ab$-paths are independent $\iff$ they only share $a$ and $b$
- **Path existence**: Every $G$ with $\delta(G) \geq 2$ contains path of length $\delta(G)$
- **Distance**: $d_G(x, y) = \min(\{k : \exists\ x\text{-}y\text{-path of length } k\} \cup \{\infty\})$
- **Central**: $v \in V(G)$ where cen $= \max\{d_G(v, x) : v \neq x \in V(G)\}$ is minimal
- **Radius**: $\mathrm{rad}(G)$ = minimal cen $= \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$
- **Diameter** of $G$: $\mathrm{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$
  - *radius-diameter-relation*: $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$
  - *radius-degree-vertex-restriction*:
    $$\mathrm{rad}(G) \leq k \wedge \Delta(G) \leq d \geq 3 \Rightarrow |G| \leq \frac{d}{d-2}(d-1)^k$$
- **Walk**: alternating sequence $v_0 e_0 \ldots e_{k-1} v_k$ s.t. $e_i = v_i v_{i+1}$ ($\forall i < k$)
  - *closed walk*: $v_k = v_0$
  - *walk-path-relation*: all vertices in walk distinct $\rightsquigarrow$ path
  - *walk-path-induction*: $\exists\ v_0 v_k$-walk $\Rightarrow\ \exists\ v_0 v_k$-path

### Cycles

- **Cycle**: $C = P + x_{k-1} x_0$ with path $P = x_0 \ldots x_{k-1}$ ($k \geq 3$)
  - *shorthand*: $x_0 \ldots x_{k-1} x_0$
- **Length**: $= |C| = ||C||$
- **k-cycle**: $C_k$ = cycle of length $k$
- **Girth** of graph $G$: $g(G) = \min(\{k : G \text{ contains } C_k\} \cup \{\infty\})$
  - *girth-diameter-relation*: $g(G) \leq 2\mathrm{diam}(G) + 1$
  - *girth-vertex-relation*: $\delta(G) \geq 3 \Rightarrow g(G) < 2 \log |G|$
- **Circumference** of graph $G$: $= \max(\{k : G \text{ contains } C_k\} \cup \{0\})$
- **Chord** of cycle $C \subseteq G$: $= xy \in E(G)$ with $xy \notin E(C)$, but $x, y \in V(C)$
- **Induced cycle**: induced subgraph of $G$ that is a cycle (= cycle in $G$ with no chords)
- **Cycle existence**: Every $G$ with $\delta(G) \geq 2$ contains cycle of length $\geq \delta(G) + 1$
- **Odd closed walk, odd cycle**: $G$ has *odd* closed walk $\Rightarrow G$ has odd cycle

### Connectivity

- **Connected** graph $G$: $\forall x, y \in V(G) : \exists\ xy$-path
  - *connected subset* $U \subseteq V(G)$: if $G[U]$ is connected
- **Vertex enumeration**: $G$ connected $\Rightarrow$ vertices can be enumerated $v_1, \ldots, v_n$ s.t. $G_i := G[v_1, \ldots, v_i]$ is connected ($\forall i \leq n$)
- **Component**: maximal connected subgraph
  - *graph partitioning*: components partition $G$
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \iff$ any $A$-$B$-path has vertex in $X$
- *separator* $X$
- **Cut-Vertex**: vertex separating two other vertices of the component
- **Bridge**: edge separating its ends (= edges of component not lying on any cycle)
- **k-connected**: if $|G| > k \wedge G - X$ is connected $\forall X \subseteq V(G)$ with $|X| < k$
- $\rightsquigarrow$ no two vertices in $G$ are separated by fewer than $k$ other vertices
- **Connectivity**: $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected**: if $|G| > 1 \wedge G - F$ is connected $\forall F \subseteq E(G)$ with $|F| < l$
- **Edge-connectivity**: $\kappa'(G) = \lambda(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Connectivity and smallest degree**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Connectivity and average degree**: $d(G) \geq 4k \Rightarrow G$ has $k$-connected subgraph

### Trees and forests

- **Forest**: Graph with no cycle as subgraph
- **Tree**: Graph that is connected and acyclic
- $\iff$ $G$ is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
- $\iff$ $G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
- $\iff$ $G$ is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \leq 1$)
- $\iff$ $G$ is connected and $||G|| = |G| - 1$
- $\iff$ $G$ is acyclic and $||G|| = |G| - 1$
- $\iff$ $\forall u, v \in V(G)\ \exists$ unique $uv$-path

- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree $T$, $|T| \geq 2 \Rightarrow T$ has leaf
- **Edge count**: Tree $T$, $|T| = n \Rightarrow \|T\| = n - 1$

### Bipartite graphs

- **r-partite** graph $G$: $V(G)$ allows partitioning in $r$ classes s.t. $\forall e = xy \in E(G)$ : $x$ and $y$ are in different classes
- **Bipartite** graph: 2-partite graph

$\Leftrightarrow$ $G$ contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$

### Contraction and minors

- **Subdivision** of graph $G$: any graph obtained from $G$ by subdividing edges
- **Topological minor**: $H$ is topological minor if $TH \subseteq G$ where $TH$ is built from $H$ by subdividing edges
  - *branch vertices*: original vertices of $H$
  - *subdividing vertices*: vertices placed on edges joining branch vertices
- **MH**: $G \overset{(\star)}{=} MH$ is *minor* of $H$ if
  - $V(G) = V_1 \dot\cup \cdots \dot\cup V_n$ with $n = |H|$
  - $G[V_i]$ connected ($\forall i = 1, \ldots, n$)
  - If $V(H) = \{v_1, \ldots, v_n\}$ and $v_i v_j \in E(H)$, then $\exists$ edge between $V_i$ and $V_j$

  ($\star$): *Notation abuse*: $MH$ is class of graphs
- **Branch sets**: $V_i$'s from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** ($H$ of $G$, noted $H \preccurlyeq G$): $\Leftrightarrow MH \subseteq G$

  $\rightsquigarrow H \preccurlyeq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + **contractions**.
- **Note**: $TH \subseteq MH$

### Euler tours

- **Definition**: closed walk with
  - no edges of $G$ are repeatedly used
  - all edges of $G$ are used
- **Eulerian graph**: graph containing an Euler tour $\Leftrightarrow \forall v \in V(G) : d(v)$ even

### Algebraic assets

- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$

### Other graph notions

- **Digraph**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$
- **Multigraph**: $G = (V, E)$ with vertex set $V$ and multiset $E$ of $V$-pairs
- **Multigraph**: $G = (V, E)$ with vertex set $V$ and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

# Matching, Covering, Packing

### Matching in bipartite graphs

- **Vertex cover**: $U \subseteq V(G)$ s.t. all edges in $G$ are incident to a vertex $\in U$
- **Matching** $M$: set of independent edges in a graph
  - *matching graph*: $\delta(G) = \Delta(G) = 1$
  - *saturating*: $G = (A \cup B, E)$ has matching saturating $A$

    $\Leftrightarrow \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$
  - *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1)$.

    $\Rightarrow \exists$ matching $M$ saturating all but at most $d$ vertices of $A$
- **k-factor**: $k$-regular spanning subgraph
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover (Königs theorem)
- **Matching existence**:
  - *neighbor-based*: $G = (A \cup B, E)$ contains matching of $A \Leftrightarrow |N(S)| \geq |S|$
  - *regular + bipartite*: $G$ is $k$-regular + bipartite ($k \geq 1$) $\Rightarrow$ $G$ has 1-factor
  - *2k-regular*: graph $2k$-regular ($k \geq 1$) $\Rightarrow$ has 2-factor
- **Marriages**: make matchings based on preferences
  - *preferences*: family $(\leq_v)_{v \in V}$ of linear orderings $\leq_v$ on $E(v)$
  - *stable matching* $M$:

    $\forall e \in E \setminus M \; \exists f \in M : e$ and $f$ have common vertex $v$ with $e <_v f$
  - *stable matching existence*: For every set of preferences, $G$ has stable matching

### Matchings in general graphs

- **Perfect matching**: spanning + matching subgraph of $G$ (aka *1-factor*)
  - *existence (Tutte)*: $G$ has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq S$ (Tutte's condition, $q(G)$ = number of components in $G$ with odd order)
  - *existence (Petersen)*: $G$ bridgeless + cubic $\Rightarrow G$ has 1-factor

# Connectivity

### Rest

- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \ldots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$

### Block

- **Block**: maximal 2-connected subgraph or bridge
  - share $\leq 1$ vertices with one another
- **Block-cut-vertex graph**
  - $V$ = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex $v$, block $B\}$
  - block-cut-vertex graph of connected graph is tree

### Factors

- **k-factor**: spanning $k$-regular subgraph (easy to find)
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$,

  $f : V(G) \to \{0, 1, \ldots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect $H$-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and $k$ divides $|G|$, then $G$ has $K_k$-factor

### Connectivity

- $k$-**connected**: if $|G| > k$ and deleting $< k$ vertices does not disconnect $G$
- $k$-**linked**: if for any $2k$ vertices $(s_1, \ldots, s_k, t_1, \ldots, t_k)$ $\exists$ pairwise disjoint $s_i t_i$-paths (*note*: $k$-connected $\not\Rightarrow$ $k$-linked)
- **Vertex-connectivity**: $\kappa(G) = \max\{k : G$ is $k$-connected$\}$
- $l$-**edge-connected**: if deleting $< l$ edges does not disconnect $G$
- **Edge-connectivity**: $\kappa'(G) = \max\{l : G$ is $l$-edge-connected$\}$
- **Vertex- vs Edge-connectivity**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Three-connected + contraction**: 3-connected $\Leftrightarrow$ $\exists$ separate $G_0, \ldots, G_k$ with

  $G_0 = K_4,\; G_k = G,\; G_i = G_{i+1} \circ xy$

  with $\deg(x), \deg(y) \geq 3$
- **Three-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of $K_4$

### Cuts

- **Cut-Set**: $X \subseteq V(G) \cup E(G)$ s.t. #components in $(G - X)$ greater than in $G$
- **Cut-Vertex**: Cut-Set consisting of single vertex
- **Cut-Edge** (or *bridge*): Cut-Set consisting of single edge
- **Menger's theorem**: for $A, B \subseteq V(G)$:

  min # of vertices separating $A$ and $B$ = max # of disjoint $A$-$B$-paths
- **Menger global**:
  1. *k-connected* $\Leftrightarrow \forall a, b \in V(G) \; \exists \; k$ pairwise independent $ab$-paths
  2. *k-edge-connected* $\Leftrightarrow \forall a, b \in V(G) \; \exists \; k$ pairwise edge-disjoint $ab$-paths

### Ear-decomposition

- **Definition**: $G$ has *ear-decomposition* $\Leftrightarrow$ $\exists$ sequence of graphs $G_0, \ldots, G_k$ with $G_k = G, G_0$ = cycle, $G_{i+1}$ obtained from $G_i$ by attaching "ear" (path that shares only endpoints with $G_i$)
- **2-connected** $\Leftrightarrow \forall$ cycles $C$ in $G$ there is ear-decomposition starting at $C$

### Edge contraction

- **Contraction**:

  $G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$

  $(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$

  with $xy \in E(G)$
- **De-contraction**: if $\exists \, xy \in E(G) : \kappa(G \circ xy) \geq 3$

  (for $G$ with $\kappa(G) \geq 3$, $|G| \geq 5$)

## Planar graph tools

- **Homeomorphism**: $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t. $f^{-1}$ is also continuous
- **Arc**: homeomorphic image of $[0,1]$ in $\mathbb{R}^2$ under $f$
  - *endpoints*: $f(0)$ and $f(1) \rightsquigarrow$ arc "joins" endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments
- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and $Y$ is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
  $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon)$ contains points of $X$ and not of $X\}$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\overline{x} : \text{dist}(\overline{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions $R_1$, $R_2$ and $\delta R_1 = X = \delta R_2$.

## Plane graph

- **Definition**: graph such that $E(G)$ is set of arcs in $\mathbb{R}^2$ and endpoints of arcs in $E(G)$ are vertices and:
  - $\forall e, e' \in E, e \neq e'$: $e$ and $e'$ have distinct sets of edge sets
  - $\forall e \in E$, $\mathring{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus \left(\bigcup_{e \in E} e \cup V\right)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle $\Leftrightarrow$ graph is maximally plane
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no $\triangle$: $\|G\| \leq 2|G| - 4$
- **Properties**: Let $G$ be plane graph and $H \subseteq G$.
  - *face inheritance*: $\forall f \in F(G)\ \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*: $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*: $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \mathring{e} = \emptyset$
  - *edges in circles*:
    $e \in E(G)$ is edge of a cycle $\Rightarrow e$ is on boundary of exactly 2 faces
    not edge of a cycle $\Rightarrow e$ is on boundary of exactly 1 face
  - *faces in cycles*: $f_1, f_2 \in F(G)$. $f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
  - *cyclic boundaries*: $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
- **Dual multigraph**: Given plane $G$:
  1. Insert vertex in each face
  2. Put edge $\tilde{e}$ between vertices if respective faces share $e$ (s.t. $\tilde{e}$ and $e$ cross once)
  3. *Result*: Dual graph $G'$ of $G$ (plane multigraph)
  
  $\rightsquigarrow$ faces of $G$ properly $k$-colored $\Leftrightarrow$ $\exists$ proper $k$-coloring of vertices of $G'$

## Planar graph

- **Definition**: graph s.t. $\exists$ plane graph $G'$ and bijection $f : V(G) \to V(G')$ s.t. $\forall u, v \in V(G),\ uv \in E(G) : f(u),\ f(v)$ are endpoints of arc in $G'$
- **Planar embedding** of $G$: $f$ from the definition
- **Planar because of minors**: The following statements are equivalent:
  - $G$ is planar
  - $G \not\supseteq MK_5 \wedge G \not\supseteq MK_{3,3}$
  - $G \not\supseteq TK_5 \wedge G \not\supseteq TK_{3,3}$
- **Euler's formula**: If $G$ is connected plane graph with $f$ faces, then
  $|G| - \|G\| + f = 2$
- **$\delta(G)$ limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: $K_5$ and $K_{3,3}$ are not planar
- **Kuratowski's lemmas**:
  1. $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Leftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
  2. $\kappa(G) \geq 3 \wedge MK_5 \nsubseteq G \wedge MK_{3,3} \nsubseteq G \Rightarrow G$ is planar
  3. $\kappa(G) \geq 3$, $G$ edge-maximal wrt not containing $TX$. If $S$ is vertex-cut of $G$, $|S| \leq 2 \wedge G = G_1 \cup G_2$, $S = V(G_1) \cap V(G_2)$, then $G_i$ is edge-maximal with no $TX$ and $S$ induces an edge
  4. $|G| \geq 3$, $G$ edge-maximal wrt not containing $TK_5$ and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of $G$ on surface $S$): any closed simple curve in any region of $S - G$ is continuously contractible into a point
- **Euler characteristic**: $G$ embedded on surface $S \Rightarrow n - e + f$ = *Euler characteristic* is invariant
- **Euler genus**: $n - e + f = 2 - 2\gamma \rightsquigarrow$ *Euler genus* $2\gamma$ of $S$
- **Heawood's formula**: $\chi(G) \leq \underbrace{\left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor}_{f(\gamma),\ \text{Heawoods number}}$
  (for $G$ embedded on $S$ with Euler char $2 - 2\gamma$)
- **Klein bottle**: $K_{f(\gamma)}$ is embeddable on $S$, unless $S$ is *klein bottle*

## Coloring

- **Co-clique number**: $\alpha(G)$ = size of largest independent set
- **Clique number**: $\omega(G)$ = size of largest clique
- **Proper coloring**: $= c : V(G) \to [k]$ with $c(u) \neq c(v) \quad (\forall uv \in E(G))$
- **Equitable coloring**: proper coloring + color classes have almost $(\pm 1)$ equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors
- **4-color-theorem**: $G$ planar $\Rightarrow \chi(G) \leq 4$
- **ij-flip**: $c' : V(G) \to [k]$ is $ij$-flip at $v \in V(G)$
  $\Leftrightarrow c'$ obtained by flipping colors $i$ and $j$ in max. conn. component containing $v$
- 

## Chromatic number

- **Definition**: $\chi(G) = \min\{k : G$ has proper coloring with $k$ colors$\}$
- **Examples**: $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$
- **Properties**:
  - $\chi(G) \geq \omega(G)$
  - $\chi(G) \geq \frac{|G|}{\alpha(G)}$
  - $\chi(G) \leq \Delta(G) + 1$ (*greedy coloring*)
  - $G$ connected, not complete, no odd cycles $\Rightarrow \chi(G) \leq \Delta(G)$

## Perfect graph

- **Definition**: $\forall H \underset{\text{ind}}{\subseteq} G : \chi(H) = \omega(H)$
- **Perfect complement**: $G$ is perfect $\Leftrightarrow \overline{G}$ is perfect
- **Perfect graph conjecture**: $G$ is perfect $\Leftrightarrow$
  $C_{2k+1} \nsubseteq G$ for $k \geq 2 \wedge \overline{C_{2k+1}} \nsubseteq G$

## Posets

- **Definition**: antisymmetric, reflexive, transitive relation on $X$
  (write $x \leq y$ instead of $(x, y)$)
- **Incidence poset** of $G$: poset whose cover diagram is represented by $IG$ with vertices all below the edges
- **Poset dimension**: $\dim(R)$ = smallest $k \in \mathbb{N} : R$ is intersection of $k$ total orders
- **Poset dimension in planar graphs**: $G$ planar $\Leftrightarrow \dim(\text{incidence poset}) \leq 3$

## List-colorings

- **L-list-colorable**: if $\exists\, c : V \to \mathbb{N}\ \forall v \in V : c(v) \in L(v)$
  (for *list of colors* $L(v) \subseteq \mathbb{N}$ for each vertex, adjacent vertices receive different colors)
- **k-list-colorable/-choosable**: if $G$ is $L$-list-colorable for each list $L$
- **List chromatic number**: $\chi_l(G) = \text{ch}(G)$
  $= \min\left\{k : G \text{ is } L\text{-colorable } \forall L : V \to 2^{\mathbb{N}} : |L(v)| = k \forall v \in V(G)\right\}$
  - $\chi_l(G) \geq \chi(G)$ because we can choose $L(v) = \{1, \ldots, k\}$ $(\forall v \in V(G))$
  - often $\chi_l(G) \gg \chi(G)$ (see $K_{m,n}$: $\chi = 2$, $\chi_l \approx \log n$)
- **Planar graphs**: $\chi_l(G) \leq 5$
- **Locally planar graphs**: $\chi_l(G) \leq 5$