Basics

Notations

- $\binom{V}{k} \coloneqq \{A : A \subseteq V \wedge |A| = k\}$
- $[n] \coloneqq \{1, \ldots, n\} \subset \mathbb{N}$
- **Power set** $2^X \coloneqq \{A : A \subseteq X\}$

Graph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$
- **Vertex set**: $V(G)$
- **Edge set**: $E(G)$
- **Isomorphic** (G_1 to another graph G_2): if $\exists$ bijection $f : V_1 \to V_2$ with $\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $\|G\|$
- **Complement**: $\overline{G} = (V(G), \binom{V}{2} - E(G))$
- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \ldots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Minimum degree**: $\delta(G)$ = degree of $v \in V(G)$ with smallest degree
- **Maximum degree**: $\Delta(G)$ = degree of $v \in V(G)$ with largest degree
- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$
- **Incidence graph** of G: $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Eulerian**: if it contains an Eulerian tour
- **Connected**: for any two vertices there is a link between them
  - *spanning tree*: if G is connected, then it has a spanning tree
  - *peeling leaves*: vertices can be ordered $v_1, \ldots, v_n$ s.t. $G[\{v_1, \ldots, v_i\}]$ is connected for $i \in \{1, \ldots, n\}$

Digraph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$

Multigraph

- **Definition**: $G = (V, E)$ with vertex set V and multiset E of V-pairs

Hypergraph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

Vertex

- **Incident** to $e \in E(G)$ if $v \in e$
- **Adjacent** to $\tilde{v} \in V(G)$ if $\{v, \tilde{v}\} \in E(G)$
- **Neighborhood**: $N(v) = \{u : uv \in E(G)\}$
- **Degree**: $\deg(v) = \mathrm{d}(v) = |N(v)|$
- **Isolated**: vertex with $\deg(v) = 0$
- **Leaf**: vertex with $\deg(v) = 1$

Subgraph

- **Definition**: H subgraph of G (write $H \subseteq G$) if $V(H) \subseteq V(G) \wedge E(H) \subseteq E(G)$
- **Induced subgraph**: H induced subgraph of G (write $H \underset{\text{ind}}{\subseteq} G$), if $H \subseteq G$ and $E(H)$ contains all edges from $E(G)$ between vertices in $V(H)$
- **Edge-induced subgraph**: subgraph induced by $X \subseteq E(G)$, note $G[X]$
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \Leftrightarrow$ any A-B-path has vertex in X

Spanning graph

- **Definition**: Subgraph with same vertex set as supergraph

Line graph

- **Definition**: $L(G) = (E, \{\{e, e'\} : e \cap e' \neq \varnothing\})$
- **Graphic**: L is line graph of some G, if it doesn't contain one of 9 specific induced subgraphs

Vertex cover

- **Definition**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in V'

Cycle

- **Definition**: $C_n \coloneqq (\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\})$
- **Shorthand**: $(v_1, \ldots, v_n, v_1)$
- **Length** (of cycle): $= |V| \equiv |E|$
- **Cyclic subgraph**: If $\delta(G) \geq 2$, then G has cycle with length $\geq \delta + 1$

Path

- **Definition**: $(\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\})$
- **Shorthand**: $(v_1, \ldots, v_n)$
- **Length** (of path): $= |E| \neq |V|$
- $v_0 v_k$**-path**: path starting at v_0 and ending at v_k
- **Independent**: two ab-paths are independent $\Leftrightarrow$ they only share a and b

Walk

- **Definition**: non-empty alternating sequence of vertices and edges
  $v_0 e_0 \ldots e_{k-1} v_k$
  with $e_i = v_i v_{i+1}$, length $k \in \mathbb{N}$
  - *closed*: if $v_0 = v_k$
  - *even*: if k is even
  - *odd*: if k is odd
- **Eulerian tour**:
  - *Definition*: closed walk with
    - no edges of G are repeatedly used
    - all edges of G are used
  - *Even degrees*: G connected has Euler tour $\Leftrightarrow \forall v \in V(G) : \deg(v)$ even
- $v_0 v_k$**-walk**: walk starting at v_0 and ending at v_k
- **Induces path**: $\exists\ uv$-walk $\Rightarrow\ \exists\ uv$-path
- **Odd closed walk, odd cycle**: G has *odd* closed walk $\Rightarrow G$ has odd cycle

Connected component

- **Definition**: *maximal* connected subgraph (connected, but any supergraph isn't)

Block

- **Block**: maximal 2-connected subgraph or bridge
  - share ≤ 1 vertices with one another
- **Block-cut-vertex graph**
  - V = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex v, block $B\}$
  - block-cut-vertex graph of connected graph is tree

Acyclic graph, Forest

- **Definition**: Graph with no cycle as subgraph

Tree

- **Definition**: Graph that is connected and acyclic
  - $\Leftrightarrow G$ is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
  - $\Leftrightarrow G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
  - $\Leftrightarrow G$ is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \leq 1$)
  - $\Leftrightarrow G$ is connected and $\|G\| = |G| - 1$
  - $\Leftrightarrow G$ is acyclic and $\|G\| = |G| - 1$
  - $\Leftrightarrow \forall u, v \in V(G)\ \exists$ unique uv-path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree T, $|T| \geq 2 \Rightarrow T$ has leaf
- **Edge count**: Tree T, $|T| = n \Rightarrow \|T\| = n - 1$

k-regular graph

- **Definition**: Graph with $\deg(v) = k \in \mathbb{N}_0 \quad (\forall v \in V(G))$

Bipartite graph

- **Definition**: G is bipartite $\Leftrightarrow$ G contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a,b\} : a \in A, b \in B)$
- **Matchings**:
  - *saturating*: $G = (A \cup B, E)$ has matching saturating A
    $\Leftrightarrow \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$
  - *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1)$.
    $\Rightarrow \exists$ matching M saturating all but at most d vertices of A
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover

Matching

- **Definition**: graph with $\delta(G) = \Delta(G) = 1$
- **Perfect matching**: spanning + matching subgraph of G (aka *1-factor*)
  - *existence*: G has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq S$
    ($q(G)$ = number of components in G with odd order)

Factors

- **k-factor**: spanning k-regular subgraph (easy to find)
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$,
  $f : V(G) \to \{0, 1, \dots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and k divides $|G|$, then G has K_k-factor

Connectivity

- **k-connected**: if $|G| > k$ and deleting $< k$ vertices does not disconnect G
- **k-linked**: if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k)$ $\exists$ pairwise disjoint $s_i t_i$-paths (*note*: k-connected $\not\Rightarrow$ k-linked)
- **Vertex-connectivity**: $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected**: if deleting $< l$ edges does not disconnect G
- **Edge-connectivity**: $\kappa'(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Vertex- vs Edge-connectivity**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Three-connected + contraction**: 3-connected $\Leftrightarrow$ $\exists$ separate $G_0, \dots, G_k$ with
  $G_0 = K_4,\ G_k = G,\ G_i = G_{i+1} \circ xy$
  with $\deg(x), \deg(y) \geq 3$
- **Three-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of K_4
- **Average degree** ≥ 4: has k-connected subgraph ($k \geq 2$)

Cuts

- **Cut-Set**: $X \subseteq V(G) \cup E(G)$ s.t. #components in $(G - X)$ greater than in G
- **Cut-Vertex**: Cut-Set consisting of single vertex
- **Cut-Edge** (or *bridge*): Cut-Set consisting of single edge
- **Menger's theorem**: for $A, B \subseteq V(G)$:
  min # of vertices separating A and B = max # of disjoint A-B-paths
- **Menger global**:
  1. *k-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise independent ab-paths
  2. *k-edge-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise edge-disjoint ab-paths

Ear-decomposition

- **Definition**: G has *ear-decomposition* $\Leftrightarrow$ $\exists$ sequence of graphs $G_0, \dots, G_k$ with $G_k = G, G_0$ = cycle, G_{i+1} obtained from G_i by attaching "ear" (path that shares only endpoints with G_i)
- **2-connected** $\Leftrightarrow \forall$ cycles C in G there is ear-decomposition starting at C

Edge contraction

- **Contraction**:
  $$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$$
  $$(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
  with $xy \in E(G)$
- **De-contraction**: if $\exists\, xy \in E(G) : \kappa(G \circ xy) \geq 3$
  (for G with $\kappa(G) \geq 3, |G| \geq 5$)

Planar graph tools

- **Homeomorphism**: $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t. f^{-1} is also continuous
- **Arc**: homeomorphic image of $[0, 1]$ in $\mathbb{R}^2$ under f
  - *endpoints*: $f(0)$ and $f(1) \rightsquigarrow$ arc "joins" endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments
- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and Y is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
  $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon)$ contains points of X and not of $X\}$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\overline{x} : \text{dist}(\overline{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions R_1, R_2 and $\delta R_1 = X = \delta R_2$.

Plane graph

- **Definition**: graph such that $E(G)$ is set of arcs in $\mathbb{R}^2$ and endpoints of arcs in $E(G)$ are vertices and:
  - $\forall e, e' \in E, e \neq e'$: e and e' have distinct sets of edge sets
  - $\forall e \in E, \mathring{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus \left(\bigcup_{e \in E} e \cup V\right)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle $\Leftrightarrow$ graph is maximally plane
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow ||G|| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no $\triangle$: $||G|| \leq 2|G| - 4$
- **Properties**: Let G be plane graph and $H \subseteq G$.
  - *face inheritance*: $\forall f \in F(G)\ \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*: $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*: $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \mathring{e} = \varnothing$
  - *edges in circles*:
    $e \in E(G)$ is edge of a cycle $\Rightarrow e$ is on boundary of exactly 2 faces
    not edge of a cycle $\Rightarrow e$ is on boundary of exactly 1 face
  - *faces in cycles*: $f_1, f_2 \in F(G)$. $f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
  - *cyclic boundaries*: $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
- **Dual multigraph**: Given plane G:
  1. Insert vertex in each face
  2. Put edge $\tilde{e}$ between vertices if respective faces share e (s.t. $\tilde{e}$ and e cross once)
  3. *Result*: Dual graph G' of G (plane multigraph)

  $\rightsquigarrow$ faces of G properly k-colored $\Leftrightarrow$ $\exists$ proper k-coloring of vertices of G'

Planar graph

- **Definition**: graph s.t. $\exists$ plane graph G' and bijection $f : V(G) \to V(G')$ s.t. $\forall u, v \in V(G),\ uv \in E(G) : f(u),\ f(v)$ are endpoints of arc in G'
- **Planar embedding** of G: f from the definition
- **Planar because of minors**: The following statements are equivalent:
  - G is planar
  - $G \not\supseteq MK_5 \wedge G \not\supseteq MK_{3,3}$
  - $G \not\supseteq TK_5 \wedge G \not\supseteq TK_{3,3}$
- **Euler's formula**: If G is connected plane graph with f faces, then
  $|G| - ||G|| + f = 2$
- **$\delta(G)$ limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: K_5 and $K_{3,3}$ are not planar
- **Kuratowski's lemmas**:
  1. $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Leftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
  2. $\kappa(G) \geq 3 \wedge MK_5 \nsubseteq G \wedge MK_{3,3} \nsubseteq G \Rightarrow G$ is planar
  3. $\kappa(G) \geq 3$, G edge-maximal wrt not containing TX. If S is vertex-cut of G, $|S| \leq 2 \wedge G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$, then G_i is edge-maximal with no TX and S induces an edge
  4. $|G| \geq 3$, G edge-maximal wrt not containing TK_5 and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of G on surface S): any closed simple curve in any region of $S - G$ is continuously contractible into a point
- **Euler characteristic**: G embedded on surface $S \Rightarrow n - e + f$ = *Euler characteristic* is invariant
- **Euler genus**: $n - e + f = 2 - 2\gamma \rightsquigarrow$ *Euler genus* 2γ of S
- **Heawood's formula**: $\chi(G) \leq \underbrace{\left\lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor}_{f(\gamma),\ \text{Heawoods number}}$
  (for G embedded on S with Euler char $2 - 2\gamma$)
- **Klein bottle**: $K_{f(\gamma)}$ is embeddable on S, unless S is *klein bottle*

Minors

- **MH**: $G \overset{(\star)}{=} MH$ is *minor* of H if
  - $V(G) = V_1 \,\dot\cup \cdots \dot\cup\, V_n$ with $n = |H|$
  - $G[V_i]$ connected ($\forall i = 1, \dots, n$)
  - If $V(H) = \{v_1, \dots, v_n\}$ and $v_i v_j \in E(H)$, then $\exists$ edge between V_i and V_j

  ($\star$): *Notation abuse*: MH is class of graphs
- **Branch sets**: V_i's from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** (H of G, noted $H \preccurlyeq G$): $\Leftrightarrow MH \subseteq G$
  $\rightsquigarrow H \preccurlyeq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + contractions.
- **Topological minor**: H is topological minor if $TH \subseteq G$ where TH is built from H by subdividing edges

- **Note**: $TH \subseteq MH$

Coloring

- $\alpha(G)$ = size of largest independent set
- $\omega(G)$ = clique number
- **Proper coloring**: = $c : V(G) \to [k]$ with $c(u) \neq c(v) \quad (\forall uv \in E(G))$
- **Equitable coloring**: proper coloring + color classes have almost (±1) equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors
- **4-color-theorem**: G planar $\Rightarrow \chi(G) \leq 4$
- **ij-flip**: $c' : V(G) \to [k]$ is ij-flip at $v \in V(G)$
  $\Leftrightarrow c'$ obtained by flipping colors i and j in max. conn. component containing v
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Chromatic number

- **Definition**: $\chi(G) = \min\{k : G$ has proper coloring with k colors$\}$
- **Examples**: $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$
- **Properties**:
  - $\chi(G) \geq \omega(G)$
  - $\chi(G) \geq \frac{|G|}{\alpha(G)}$
  - $\chi(G) \leq \Delta(G) + 1$ (*greedy coloring*)
  - G connected, not complete, no odd cycles $\Rightarrow \chi(G) \leq \Delta(G)$

Posets

- **Definition**: antisymmetric, reflexive, transitive relation on X
  (write $x \leq y$ instead of (x, y))
- **Incidence poset** of G: poset whose cover diagram is represented by IG with vertices all below the edges
- **Poset dimension**: $\dim(R)$ = smallest $k \in \mathbb{N} : R$ is intersection of k total orders
- **Poset dimension in planar graphs**: G planar $\Leftrightarrow \dim(\text{incidence poset}) \leq 3$

List-colorings

- **L-list-colorable**: if $\exists\, c : V \to \mathbb{N}\ \forall v \in V : c(v) \in L(v)$
  (for *list of colors* $L(v) \subseteq \mathbb{N}$ for each vertex, adjacent vertices receive different colors)
- **k-list-colorable/-choosable**: if G is L-list-colorable for each list L
- **List chromatic number**: $\chi_l(G) = \text{ch}(G)$
  $= \min\left\{k : G \text{ is } L\text{-colorable } \forall L : V \to 2^{\mathbb{N}} : |L(v)| = k \forall v \in V(G)\right\}$
  - $\chi_l(G) \geq \chi(G)$ because we can choose $L(v) = \{1, \ldots, k\}$ $(\forall v \in V(G))$
  - often $\chi_l(G) \gg \chi(G)$ (see $K_{m,n}$: $\chi = 2$, $\chi_l \approx \log n$)
- **Planar graphs**: $\chi_l(G) \leq 5$
- **Locally planar graphs**: $\chi_l(G) \leq 5$