Basics

Notations

- $\binom{V}{k} := \{A : A \subseteq V \wedge |A| = k\}$
- $[n] := \{1, \ldots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

Graph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$
- **Vertex set**: $V(G)$
- **Edge set**: $E(G)$
- **Isomorphic** (G_1 to another graph G_2): if $\exists$ bijection $f : V_1 \to V_2$ with $\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $\|G\|$
- **Complement**: $\overline{G} = (V(G), \binom{V}{2} - E(G))$
- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \ldots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Minimum degree**: $\delta(G)$ = degree of $v \in V(G)$ with smallest degree
- **Maximum degree**: $\Delta(G)$ = degree of $v \in V(G)$ with largest degree
- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$
- **Eulerian**: if it contains an Eulerian tour
- **Connected**: for any two vertices there is a link between them
  - *spanning tree*: if G is connected, then it has a spanning tree
  - *peeling leaves*: vertices can be ordered $v_1, \ldots, v_n$ s.t. $G[\{v_1, \ldots, v_i\}]$ is connected for $i \in \{1, \ldots, n\}$

Digraph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$

Multigraph

- **Definition**: $G = (V, E)$ with vertex set V and multiset E of V-pairs

Hypergraph

- **Definition**: $G = (V, E)$ with vertex set V and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

Vertex

- **Incident** to $e \in E(G)$ if $v \in e$
- **Adjacent** to $\tilde{v} \in V(G)$ if $\{v, \tilde{v}\} \in E(G)$
- **Neighborhood**: $N(v) = \{u : uv \in E(G)\}$
- **Degree**: $\deg(v) = \mathrm{d}(v) = |N(v)|$
- **Isolated**: vertex with $\deg(v) = 0$
- **Leaf**: vertex with $\deg(v) = 1$

Subgraph

- **Definition**: H subgraph of G (write $H \subseteq G$) if $V(H) \subseteq V(G) \wedge E(H) \subseteq E(G)$
- **Induced subgraph**: H induced subgraph of G (write $H \underset{\text{ind}}{\subseteq} G$), if $H \subseteq G$ and $E(H)$ contains all edges from $E(G)$ between vertices in $V(H)$
- **Edge-induced subgraph**: subgraph induced by $X \subseteq E(G)$, note $G[X]$
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \iff$ any A-B-path has vertex in X

Spanning graph

- **Definition**: Subgraph with same vertex set as supergraph

Line graph

- **Definition**: $L(G) = (E, \{\{e, e'\} : e \cap e' \neq \emptyset\})$
- **Graphic**: L is line graph of some G, if it doesn't contain one of 9 specific induced subgraphs

Vertex cover

- **Definition**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in V'

Cycle

- **Definition**: $C_n := (\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\})$
- **Shorthand**: $(v_1, \ldots, v_n, v_1)$
- **Length** (of cycle): $= |V| \equiv |E|$
- **Cyclic subgraph**: If $\delta(G) \geq 2$, then G has cycle with length $\geq \delta + 1$

Path

- **Definition**: $(\{v_1, \ldots, v_n\}, \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\})$
- **Shorthand**: $(v_1, \ldots, v_n)$
- **Length** (of path): $= |E| \neq |V|$
- $v_0 v_k$**-path**: path starting at v_0 and ending at v_k
- **Independent**: two ab-paths are independent $\iff$ they only share a and b

Walk

- **Definition**: non-empty alternating sequence of vertices and edges
  $v_0 e_0 \ldots e_{k-1} v_k$
  with $e_i = v_i v_{i+1}$, length $k \in \mathbb{N}$
  - *closed*: if $v_0 = v_k$
  - *even*: if k is even
  - *odd*: if k is odd
- **Eulerian tour**:
  - *Definition*: closed walk with
    - no edges of G are repeatedly used
    - all edges of G are used
  - *Even degrees*: G connected has Euler tour $\iff \forall v \in V(G) : \deg(v)$ even
- $v_0 v_k$**-walk**: walk starting at v_0 and ending at v_k
- **Induces path**: $\exists$ uv-walk $\implies$ $\exists$ uv-path
- **Odd closed walk, odd cycle**: G has *odd* closed walk $\implies G$ has odd cycle

Connected component

- **Definition**: *maximal* connected subgraph (connected, but any supergraph isn't)

Block

- **Block**: maximal 2-connected subgraph or bridge
  - share ≤ 1 vertices with one another
- **Block-cut-vertex graph**
  - V = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex v, block $B\}$
  - block-cut-vertex graph of connected graph is tree

Acyclic graph, Forest

- **Definition**: Graph with no cycle as subgraph

Tree

- **Definition**: Graph that is connected and acyclic
  - $\iff G$ is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
  - $\iff G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
  - $\iff G$ is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \leq 1$)
  - $\iff G$ is connected and $\|G\| = |G| - 1$
  - $\iff G$ is acyclic and $\|G\| = |G| - 1$
  - $\iff \forall u, v \in V(G)\ \exists$ unique uv-path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree T, $|T| \geq 2 \implies T$ has leaf
- **Edge count**: Tree T, $|T| = n \implies \|T\| = n - 1$

k-regular graph

- **Definition**: Graph with $\deg(v) = k \in \mathbb{N}_0 \quad (\forall v \in V(G))$

Bipartite graph

- **Definition**: G is bipartite $\iff G$ contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$
- **Matchings**:
  - *saturating*: $G = (A \cup B, E)$ has matching saturating A
    $\iff \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$

- *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1)$.
  $\Rightarrow\ \exists$ matching M saturating all but at most d vertices of A
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover

Matching

- **Definition**: graph with $\delta(G) = \Delta(G) = 1$
- **Perfect matching**: spanning + matching subgraph of G (aka *1-factor*)
  - *existence*: G has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq S$
    ($q(G)$ = number of components in G with odd order)

Coloring

- **Proper coloring**: = $c : V(G) \to [k]$ with $c(u) \neq c(v) \quad (\forall uv \in E(G))$
- **Equitable coloring**: proper coloring + color classes have almost (± 1) equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors

Chromatic number

- **Definition**: $\chi(G) = \min\{k : G$ has proper coloring with k colors$\}$
- **Examples**: $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$

Factors

- **k-factor**: spanning k-regular subgraph (easy to find)
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$,
  $f : V(G) \to \{0, 1, \dots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \geq \left(1 - \frac{1}{k}|V(G)|\right)$ and k divides $|G|$, then G has K_k-factor

Connectivity

- k-**connected**: if $|G| > k$ and deleting $< k$ vertices does not disconnect G
- k-**linked**: if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k)$ $\exists$ pairwise disjoint $s_i t_i$-paths (*note*: k-connected $\not\Rightarrow$ k-linked)
- **Vertex-connectivity**: $\kappa(G) = \max\{k : G$ is k-connected$\}$
- l-**edge-connected**: if deleting $< l$ edges does not disconnect G
- **Edge-connectivity**: $\kappa'(G) = \max\{l : G$ is l-edge-connected$\}$
- **Vertex- vs Edge-connectivity**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Three-connected + contraction**: 3-connected $\Leftrightarrow\ \exists$ separate $G_0, \dots, G_k$ with
  $G_0 = K_4,\ G_k = G,\ G_i = G_{i+1} \circ xy$
  with $\deg(x), \deg(y) \geq 3$
- **Three-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of K_4
- **Average degree** ≥ 4: has k-connected subgraph ($k \geq 2$)

Cuts

- **Cut-Set**: $X \subseteq V(G) \cup E(G)$ s.t. #components in $(G - X)$ greater than in G
- **Cut-Vertex**: Cut-Set consisting of single vertex
- **Cut-Edge** (or *bridge*): Cut-Set consisting of single edge
- **Menger's theorem**: for $A, B \subseteq V(G)$:
  min # of vertices separating A and B = max # of disjoint A-B-paths
- **Menger global**:
  1. *k-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise independent ab-paths
  2. *k-edge-connected* $\Leftrightarrow \forall a, b \in V(G)\ \exists\ k$ pairwise edge-disjoint ab-paths

Ear-decomposition

- **Definition**: G has *ear-decomposition* $\Leftrightarrow\ \exists$ sequence of graphs $G_0, \dots, G_k$ with $G_k = G, G_0$ = cycle, G_{i+1} obtained from G_i by attaching "ear" (path that shares only endpoints with G_i)
- **2-connected** $\Leftrightarrow\ \forall$ cycles C in G there is ear-decomposition starting at C

Edge contraction

- **Contraction**:
  $$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$$
  $$(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
  with $xy \in E(G)$
- **De-contraction**: if $\exists\ xy \in E(G) : \kappa(G \circ xy) \geq 3$
  (for G with $\kappa(G) \geq 3$, $|G| \geq 5$)