Basics

Notations

- $\binom{V}{k} := \{A : A \subseteq V \land |A| = k\}$
- $[n] := \{1, \dots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

Graphs

- **Definition**: $G = (V, E)$ with $E \subseteq V^2, V \cap E = \varnothing$
- **Vertex**: $v \in V$ for graph $G = (V, E)$
  - *v incident with e* $\iff v \in e$
  - *v_1, v_2 ends of e* $\iff e = v_1 v_2$
  - *v_1, v_2 adjacent/neighbors* $\iff v_1 v_2 \in E$
- **Edge**: $e = \{x, y\} \in E$ for graph $G = (V, E)$ (short $e = xy$)
  - *e edge at v* $\iff v$ incident with e
  - *e joins v_1, v_2* $\iff e = v_1 v_2$
  - *xy is X-Y-edge* $\iff x \in X \land y \in Y$
  - *e_1, e_2 adjacent/neighbors* $\iff \exists\, v : v \in e_1 \land v \in e_2$
- **Vertex sets**:
  - $V(G) = V$ for graph $G = (V, E)$
  - $X \subset V(G)$ *independent* $\iff$ no $x_1, x_2 \in X$ are adjacent
  - *neighborhood* of $v \in V(G)$: $N(v) = \{u \in V(G) : uv \in E(G)\}$
- **Edge sets**:
  - $E(G) = E$ for graph $G = (V, E)$
  - $E(X, Y)$: set of edges between $X \subset V(G)$ and $Y \subset V(G)$
  - $E(x, Y)$: set of edges between vertex $x \in V(G)$ and $Y \subset V(G)$
  - $E(v)$: set of edges at $v \in V(G)$
- **Order**: $= |V(G)|$, short $|G|$
- **Size**: $= |E(G)|$, short $\|G\|$
- **Trivial graph**: graph of order 0 or 1
- **Incidence graph** of G: $IG = (V \cup E, \{\{v, e\} : v \in e, e \in E\})$
- **Isomorphic** (G_1 to another graph G_2, write $G_1 \cong G_2$ or even $G_1 = G_2$):
  $\exists$ bijection $f : V_1 \to V_2 : \{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$
- **Graph union**: $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$
- **Graph intersection**: $G \cap G' = (V(G) \cap V(G'), E(G) \cap E(G'))$
- **Graph multiplication**: $G * G'$: join all $v \in G$ with all $v' \in G'$ (with $V(G) \cap V(G') = \varnothing$)
- **Subgraph** G' of G (write $G' \subseteq G$): if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$
  - *G contains G'*
  - *G' proper subgraph* of G: if $G' \subseteq G$ and $G' \neq G$
  - *G' induced subgraph* of G: $G' \subseteq G$ and $E(G')$ contains all edges of G with both ends in $V(G')$, *$V(G')$ induces G'*, write $G' = G[X]$ (with $X = V(G')$)
  - *Edge-induced subgraph*: subgraph induced by $X \subseteq E(G)$, note $G[X]$
  - *G' spanning subgraph* of G: $V(G') = V(G)$
- **Supergraph**: G of G' (write $G \supseteq G'$): as above.
- **Vertex cover**: $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in V'
- **Graph subtraction**:
  - $G - U = G[V(G) \setminus U]$ for some vertex set U
  - $G - v = G[V(G) \setminus \{v\}]$ for some vertex v
  - $G - G' = G[V(G) \setminus V(G')]$ for some graph G'
- **Edge addition**: $G + F = (V(G), V(E) \cup F)$ for some $F \subseteq V(G)^2$
- **Complement**: $\overline{G} = (V(G), V^2 \setminus E(G))$
- **Line graph** of G: $L(G) = (E(G), \{xy \in E(G)^2 : x, y \text{ adjacent in } G\})$
- **Complete graph**: (X, X^2) with vertex set X
  - K_n: complete graph on n vertices

Vertex degrees

- **Degree** of $v \in V$: $d(v) = \deg(v) = |N(v)|$
  - $v \in V(G)$ *isolated*: $d(v) = 0$
  - $v \in V(G)$ *leaf*: $d(v) = 1$
  - number of vertices of odd degree is even
- **Minimum degree** of graph G: $\delta(G) = \min\{d(v) : v \in V(G)\}$
- **Maximum degree** of graph G: $\Delta(G) = \max\{d(v) : v \in V(G)\}$
- **Degree sum**: $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Average degree** of graph G: $d(G) = \frac{1}{|G|} \sum_{v \in V} d(v)$
  - $\delta(G) \le d(G) \le \Delta(G)$
- **k-regular graph**: $\forall v \in V(G) : d(v) = k$
  - *cubic graph*: 3-regular graph
- **Vertex-Edge-ratio** of graph G: $\varepsilon(G) = \frac{\|G\|}{|G|}$
  - $\varepsilon(G) = \frac{1}{2} d(G)$
  - every graph with $\|G\| \ge 1$ has $H \subseteq G$ with $\delta(H) > \varepsilon(H) \ge \varepsilon(G)$

Paths

- **Path**: $(\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$ (read: $v_0 v_n$-path)
  - *shorthand*: $v_1 \dots v_n$
  - v_0, v_n *linked* by path
  - v_0, v_n *end-vertices/ends* of path
  - $v_1, \dots, v_{n-1}$ *inner vertices* of path
- **Length**: $|E(P)| \neq |V(P)|$
- **Shorthands** ($0 \le i \le j \le k$):
  - $P = x_0 \dots x_k$, $\mathring{P} = x_1 \dots x_{k-1}$
  - $Px_i = x_0 \dots x_i$, $P\mathring{x}_i = x_0 \dots x_{i-1}$
  - $x_i P = x_i \dots x_k$, $\mathring{x}_i P = x_{i+1} \dots x_k$
  - $x_i P x_j = x_i \dots x_j$, $\mathring{x}_i P \mathring{x}_j = x_{i+1} \dots x_{j-1}$
- **Path concatenation**: $Px \cap xQy \cap yR = PxQyR$
- **A-B-path**: $V(P) \cap A = \{x_0\} \land V(P) \cap B = \{x_n\}$
- **H-path**: graph H, P meets H exactly in its ends
- **Independent**: two ab-paths are independent $\iff$ they only share a and b
- **Path existence**: Every G with $\delta(G) \ge 2$ contains path of length $\delta(G)$
- **Distance**: $d_G(x, y) = \min(\{k : \exists\ x\text{-}y\text{-path of length } k\} \cup \{\infty\})$
- **Central**: $v \in V(G)$ where cen $= \max\{d_G(v, x) : v \neq x \in V(G)\}$ is minimal
- **Radius**: $\operatorname{rad}(G) =$ minimal cen $= \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$
- **Diameter** of G: $\operatorname{diam}(G) = \max\{d_G(x, y) : x, y \in V(G)\}$
  - *radius-diameter-relation*: $\operatorname{rad}(G) \le \operatorname{diam}(G) \le 2\operatorname{rad}(G)$
  - *radius-degree-vertex-restriction*:
    $$\operatorname{rad}(G) \le k \land \Delta(G) \le d \ge 3 \Rightarrow |G| \le \frac{d}{d-2}(d-1)^k$$
- **Walk**: alternating sequence $v_0 e_0 \dots e_{k-1} v_k$ s.t. $e_i = v_i v_{i+1}$ ($\forall i < k$)
  - *closed walk*: $v_k = v_0$
  - *walk-path-relation*: all vertices in walk distinct $\rightsquigarrow$ path
  - *walk-path-induction*: $\exists\ v_0 v_k$-walk $\Rightarrow\ \exists\ v_0 v_k$-path

Cycles

- **Cycle**: $C = P + x_{k-1} x_0$ with path $P = x_0 \dots x_{k-1}$ ($k \ge 3$)
  - *shorthand*: $x_0 \dots x_{k-1} x_0$
- **Length**: $= |C| = \|C\|$
- **k-cycle**: C_k = cycle of length k
- **Girth** of graph G: $g(G) = \min(\{k : G \text{ contains } C_k\} \cup \{\infty\})$
  - *girth-diameter-relation*: $g(G) \le 2\operatorname{diam}(G) + 1$
  - *girth-vertex-relation*: $\delta(G) \ge 3 \Rightarrow g(G) < 2 \log |G|$
- **Circumference** of graph G: $= \max(\{k : G \text{ contains } C_k\} \cup \{0\})$
- **Chord** of cycle $C \subseteq G$: $= xy \in E(G)$ with $xy \notin E(C)$, but $x, y \in V(C)$
- **Induced cycle**: induced subgraph of G that is a cycle (= cycle in G with no chords)
- **Cycle existence**: Every G with $\delta(G) \ge 2$ contains cycle of length $\ge \delta(G) + 1$
- **Odd closed walk, odd cycle**: G has *odd* closed walk $\Rightarrow G$ has odd cycle

Connectivity

- **Connected** graph G: $\forall x, y \in V(G) : \exists\ xy$-path
  - *connected subset* $U \subseteq V(G)$: if $G[U]$ is connected
- **Vertex enumeration**: G connected $\Rightarrow$ vertices can be enumerated $v_1, \dots, v_n$ s.t. $G_i := G[v_1, \dots, v_i]$ is connected ($\forall i \le n$)
- **Component**: maximal connected subgraph
  - *graph partitioning*: components partition G
- **Subgraph separation**: $X \subset V(G)$ separates $A, B \subset V(G) \iff$ any A-B-path has vertex in X
- *separator* X
- **Cut-Vertex**: vertex separating two other vertices of the component
- **Bridge**: edge separating its ends (= edges of component not lying on any cycle)
- **k-connected**: if $|G| > k \land G - X$ is connected $\forall X \subseteq V(G)$ with $|X| < k$
  - $\rightsquigarrow$ no two vertices in G are separated by fewer than k other vertices
  - $\rightsquigarrow$ any two vertices can be joined by k independent paths
- **k-linked**: if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k)$ $\exists$ pairwise disjoint $s_i t_i$-paths (*note*: k-connected $\not\Rightarrow$ k-linked)
- **Connectivity**: $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected**: if $|G| > 1 \land G - F$ is connected $\forall F \subseteq E(G)$ with $|F| < l$
- **Edge-connectivity**: $\kappa'(G) = \lambda(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Connectivity and smallest degree**: $\kappa(G) \le \kappa'(G) \le \delta(G)$
- **Connectivity and average degree**: $d(G) \ge 4k \Rightarrow G$ has k-connected subgraph
- **Menger's theorem**: for $A, B \subseteq V(G)$:
  min # of vertices separating A and B = max # of disjoint A-B-paths
  - *edge corollary*: minimum number of edges separating a and b = maximum number of edge-disjoint a-b-paths
- **Menger global**:
  - *k-connected* $\iff \forall a, b \in V(G)\ \exists\ k$ pairwise independent ab-paths
  - *k-edge-connected* $\iff \forall a, b \in V(G)\ \exists\ k$ pairwise edge-disjoint ab-paths

Trees and forests

- **Forest**: Graph with no cycle as subgraph
- **Tree**: Graph that is connected and acyclic
- $\Leftrightarrow$ G is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
- $\Leftrightarrow$ G is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
- $\Leftrightarrow$ G is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \le 1$)
- $\Leftrightarrow$ G is connected and $\|G\| = |G| - 1$
- $\Leftrightarrow$ G is acyclic and $\|G\| = |G| - 1$
- $\Leftrightarrow$ $\forall u, v \in V(G)\ \exists$ unique uv-path
- **Special trees**: path, star, spider, caterpillar, broom
- **Leaf existence**: Tree T, $|T| \ge 2 \Rightarrow T$ has leaf
- **Edge count**: Tree T, $|T| = n \Rightarrow \|T\| = n - 1$

Bipartite graphs

- **r-partite** graph G: $V(G)$ allows partitioning in r classes s.t. $\forall e = xy \in E(G)$: x and y are in different classes
- **Bipartite** graph: 2-partite graph
- $\Leftrightarrow$ G contains no cycles of odd length
  - *complete bipartite*: $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$

Contraction and minors

- **Subdivision** of graph G: any graph obtained from G by subdividing edges
- **Topological minor**: H is topological minor if $TH \subseteq G$ where TH is built from H by subdividing edges
  - *branch vertices*: original vertices of H
  - *subdividing vertices*: vertices placed on edges joining branch vertices
- **MH**: $G \overset{(*)}{=} MH$ is *minor* of H if
  - $V(G) = V_1 \dot\cup \cdots \dot\cup V_n$ with $n = |H|$
  - $G[V_i]$ connected ($\forall i = 1, \ldots, n$)
  - If $V(H) = \{v_1, \ldots, v_n\}$ and $v_i v_j \in E(H)$, then $\exists$ edge between V_i and V_j

  $(*)$: *Notation abuse*: MH is class of graphs
- **Branch sets**: V_i's from above
- **Extended branch graph**: Branch set together with incident edges
- **Minor** (H of G, noted $H \preccurlyeq G$): $\Leftrightarrow MH \subseteq G$
  $\rightsquigarrow H \preccurlyeq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + **contractions**.
- **Note**: $TH \subseteq MH$
- **Edge contraction**:
  $$G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$$
  $$(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$$
  with $xy \in E(G)$
- **De-contraction**: if $\exists\, xy \in E(G) : \kappa(G \circ xy) \ge 3$ (for G with $\kappa(G) \ge 3$, $|G| \ge 5$)

Euler tours

- **Definition**: closed walk with
  - no edges of G are repeatedly used
  - all edges of G are used
- **Eulerian graph**: graph containing an Euler tour $\Leftrightarrow \forall v \in V(G) : d(v)$ even

Algebraic assets

- **Adjacency matrix**: $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$

Other graph notions

- **Digraph**: $G = (V, E)$ with vertex set V and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$
- **Multigraph**: $G = (V, E)$ with vertex set V and multiset E of V-pairs
- **Multigraph**: $G = (V, E)$ with vertex set V and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

Matching, Covering, Packing

Matching in bipartite graphs

- **Vertex cover**: $U \subseteq V(G)$ s.t. all edges in G are incident to a vertex $\in U$
- **Matching** M: set of independent edges in a graph
  - *matching graph*: $\delta(G) = \Delta(G) = 1$
  - *saturating*: $G = (A \cup B, E)$ has matching saturating A
    $\Leftrightarrow \forall S \subseteq A : N(S) \ge |S| \quad (N(S) \coloneqq \{b \in B : ab \in E, a \in S\})$
  - *nearly*: $G = (A \cup B, E)$, $\forall S \subseteq A : |N(S)| \ge |S| - d \quad (d \ge 1)$.
    $\Rightarrow\ \exists$ matching M saturating all but at most d vertices of A
- **k-factor**: k-regular spanning subgraph
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover (Königs theorem)
- **Matching existence**:
  - *neighbor-based*: $G = (A \cup B, E)$ contains matching of $A \Leftrightarrow |N(S)| \ge |S|$
  - *regular + bipartite*: G is k-regular + bipartite ($k \ge 1$) $\Rightarrow$ G has 1-factor
  - *2k-regular*: graph $2k$-regular ($k \ge 1$) $\Rightarrow$ has 2-factor
- **Marriages**: make matchings based on preferences
  - *preferences*: family $(\le_v)_{v \in V}$ of linear orderings $\le_v$ on $E(v)$
  - *stable matching* M:
    $\forall e \in E \setminus M\ \exists\, f \in M : e$ and f have common vertex v with $e <_v f$
  - *stable matching existence*: For every set of preferences, G has stable matching
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v)$, $f : V(G) \to \{0, 1, \ldots\}$ with $f(v) \le \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
  - *existence*: if $\delta(G) \ge \left(1 - \frac{1}{k}|V(G)|\right)$ and k divides $|G|$, then G has K_k-factor

Matchings in general graphs

- **Perfect matching**: spanning + matching subgraph of G (aka *1-factor*)
  - *existence (Tutte)*: G has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \le S$ (Tutte's condition, $q(G)$ = number of components in G with odd order)
  - *existence (Petersen)*: G bridgeless + cubic $\Rightarrow G$ has 1-factor

Connectivity

2-connected graphs and subgraphs

- **2-connected construction**: G is 2-connected $\Leftrightarrow$ it can be constructed by successively adding paths to a cycle (removing those paths: *ear-decomposition*)
- **Block**: maximal 2-connected subgraph or bridge
  - share ≤ 1 vertices with another
  - *cycles* of G = cycles of its blocks
  - *bounds* of G = minimal cuts of its blocks
- **Block-cut-vertex graph**
  - V = set of blocks $\cup$ set of vertices
  - $E = \{\{v, B\} : v \in V(B)$, cut-vertex v, block $B\}$
  - block-cut-vertex graph of connected graph is tree

Structure of 3-connected graphs

- **3-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of K_4
- **3-connected + contraction**: 3-connected $\Leftrightarrow\ \exists$ separate $G_0, \ldots, G_k$ with
  $$G_0 = K_4,\ G_k = G,\ G_i = G_{i+1} \circ xy$$
  with $\deg(x), \deg(y) \ge 3$

Rest

- **Degree sequence**: multiset of degrees of vertices in $V(G)$
  - *graphic*: deg. seq. $(d_1, \ldots, d_n)$, iff
    1. $d_1 + \cdots + d_n$ even
    2. $\sum_{i=1}^{k} d_i \le k(k-1) + \sum i = k + 1^n \min(d_i, k) \quad (\forall 1 \le k \le n)$

Planar graphs

Basic definitions

- **Plane graph**: graph drawn without intersecting edges
- **Planar graph**: graph that *can be* dran as plane graph
- **Homeomorphism**: $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous s.t. f^{-1} is also continuous
- **Arc**: homeomorphic image of $[0, 1]$ in $\mathbb{R}^2$ under f
  - *endpoints*: $f(0)$ and $f(1) \rightsquigarrow$ arc "joins" endpoints
  - *polynomial arc*: arc that is union of finitely many straight line segments

- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and Y is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
  $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon)$ contains points of X and not of $X\}$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\overline{x} : \text{dist}(\overline{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions R_1, R_2 and $\delta R_1 = X = \delta R_2$.

Plane graphs

- **Definition**: graph such that $E(G)$ is set of arcs in $\mathbb{R}^2$ and endpoints of arcs in $E(G)$ are vertices and:
  - $\forall e, e' \in E, e \neq e'$: e and e' have distinct sets of edge sets
  - $\forall e \in E, \mathring{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus \left(\bigcup_{e \in E} e \cup V\right)$
  - *face set* of graph G: $F(G)$
- **Maximally plane**: no edges can be added without breaking planarity
  - *plane triangulation*: every face is bounded by triangle $\Longleftrightarrow$ graph is maximally plane
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no $\triangle$: $\|G\| \leq 2|G| - 4$
- **Properties**: Let G be plane graph and $H \subseteq G$.
  - *face inheritance*: $\forall f \in F(G) \, \exists f' \in F(H) : f' \supseteq f$
  - *border inheritance*: $\delta f \subseteq H \Rightarrow f' = f$
  - *edge-border relations*: $e \in E(G), f \in F(G) \Rightarrow e \subseteq \delta f \vee \delta f \cap \mathring{e} = \emptyset$
  - *edges in circles*:
    $e \in E(G)$ is edge of a cycle $\Rightarrow e$ is on boundary of exactly 2 faces
    not edge of a cycle $\Rightarrow e$ is on boundary of exactly 1 face
  - *faces in cycles*: $f_1, f_2 \in F(G)$. $f_1 \neq f_2 \wedge \delta f_1 = \delta f_2 \Rightarrow G$ is cycle
  - *cyclic boundaries*: $\kappa(G) \geq 2 \Rightarrow$ each face is bounded by cycle
  - *plane forests*: plane forests have exactly 1 face
  - *2-connected*: 2-connected plane graph $\Rightarrow \forall f \in F(G)$ bounded by cycle
  - *3-connected*: boundaries of 3-connected plane graph = its non-separating induced cycles
  - *order* ≥ 3: plane graph of order ≥ 3 maximally plane $\Longleftrightarrow$ is plane triangulation
- **Euler's formula**: If G is connected plane graph with f faces, then
  $|G| - \|G\| + f = 2$
- **Dual multigraph**: Given plane G:
  1. Insert vertex in each face
  2. Put edge $\tilde{e}$ between vertices if respective faces share e (s.t. $\tilde{e}$ and e cross once)
  3. *Result*: Dual graph G' of G (plane multigraph)

  $\rightsquigarrow$ faces of G properly k-colored $\Longleftrightarrow$ $\exists$ proper k-coloring of vertices of G'

Planar graphs

- **Definition**: graph s.t. $\exists$ plane graph G' and bijection $f : V(G) \rightarrow V(G')$ s.t. $\forall u, v \in V(G),\ uv \in E(G) : f(u),\ f(v)$ are endpoints of arc in G'
- **Planar embedding** of G: f from the definition
- **Planar because of minors**: The following statements are equivalent:
  - G is planar
  - $G \not\supseteq MK_5 \wedge G \not\supseteq MK_{3,3}$
  - $G \not\supseteq TK_5 \wedge G \not\supseteq TK_{3,3}$
- **$\delta(G)$ limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: K_5 and $K_{3,3}$ are not planar
- **Kuratowski's lemmas**:
  1. $(TK_5 \subseteq G \vee TK_{3,3} \subseteq G) \Longleftrightarrow MK_5 \subseteq G \vee MK_{3,3} \subseteq G$
  2. $\kappa(G) \geq 3 \wedge MK_5 \nsubseteq G \wedge MK_{3,3} \nsubseteq G \Rightarrow G$ is planar
  3. $\kappa(G) \geq 3$, G edge-maximal wrt not containing TX. If S is vertex-cut of G, $|S| \leq 2 \wedge G = G_1 \cup G_2, S = V(G_1) \cap V(G_2)$, then G_i is edge-maximal with no TX and S induces an edge
  4. $|G| \geq 3$, G edge-maximal wrt not containing TK_5 and $TK_{3,3} \Rightarrow \kappa(G) \geq 3$
- **2-cell** (embedding of G on surface S): any closed simple curve in any region of $S - G$ is continuously contractible into a point
- **Euler characteristic**: G embedded on surface $S \Rightarrow n - e + f$ = *Euler characteristic* is invariant
- **Euler genus**: $n - e + f = 2 - 2\gamma \rightsquigarrow$ *Euler genus* 2γ of S
- **Heawood's formula**: $\chi(G) \leq \underbrace{\left\lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \right\rfloor}_{f(\gamma),\text{ Heawoods number}}$
  (for G embedded on S with Euler char $2 - 2\gamma$)
- **Klein bottle**: $K_{f(\gamma)}$ is embeddable on S, unless S is *klein bottle*

Algebraic planarity criteria — Posets

- **Definition**: antisymmetric, reflexive, transitive relation on X
  (write $x \leq y$ instead of (x, y))
- **Incidence poset** of G: poset whose cover diagram is represented by IG with vertices all below the edges
- **Poset dimension**: $\dim(R)$ = smallest $k \in \mathbb{N} : R$ is intersection of k total orders
- **Poset dimension in planar graphs**: G planar $\Longleftrightarrow \dim(\text{incidence poset}) \leq 3$

Coloring

Base definitions

- **Vertex coloring**: map $c : V(G) \rightarrow S$ with $c(v) \neq c(w)$ for adjacent v, w
  - *color set* S
  - *k-coloring*: coloring $c : V(G) \rightarrow S$ with $|S| = k$
- **(Vertex) chromatic number**: $= \chi(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-coloring}\}$
  - $\chi(G) \geq \omega(G)$
  - $\chi(G) \geq \frac{|G|}{\alpha(G)}$
  - $\chi(G) \leq \Delta(G) + 1$ (*greedy coloring*)
  - G connected, not complete, no odd cycles $\Rightarrow \chi(G) \leq \Delta(G)$
  - *k-chromatic* graph: $\chi(G) = k$
  - *k-colorable* graph: $\chi(G) \leq k$
- **Color classes**: partitions of $V(G)$ with same color
- **Equitable coloring**: proper coloring + color classes have almost (± 1) equal size
  - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors
- **ij-flip**: $c' : V(G) \rightarrow [k]$ is ij-flip at $v \in V(G)$
  $\Longleftrightarrow c'$ obtained by flipping colors i and j in max. conn. component containing v
- **Edge coloring**: map $c : E(G) \rightarrow S$ with $c(e) \neq c(f)$ for adjacent e, f
  - edge coloring of $G \Longleftrightarrow$ vertex coloring of $L(G)$
  - *k-edge-coloring*: edge-coloring $c : E(G) \rightarrow S$ with $|S| = k$
- **Edge chromatic number**: $= \chi'(G) := \min\{k \in \mathbb{N} : G \text{ has } k\text{-edge-coloring}\}$

Coloring maps and planar graphs

- **4-color-theorem**: every planar graph is 4-colorable
- **3-color-theorem**: every triangle-free planar graph is 3-colorable

Coloring vertices

- **Chromatic number upper bound**: $\chi(G) \leq \frac{1}{2} + \sqrt{2\|G\| + \frac{1}{4}}$
- **Greedy coloring**: sort vertices $v_1, \ldots, v_n$, color them with the smallest possible color starting at v_1
  $\rightsquigarrow$ never uses more than $\Delta(G) + 1$ colors
- **coloring number** of graph G: $\text{col}(G) :=$ smallest k s.t. G has vertex enumeration where each vertex is preceded by $< k$ neighbors

Coloring edges

- **Vizing's theorem**: for every graph G, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$
- **Bipartite graphs**: $\chi'(G) = \Delta(G)$

List coloring

- **L-list-colorable**: if $\exists\, c : V \rightarrow \mathbb{N}\ \forall v \in V : c(v) \in L(v)$
  (for *list of colors* $L(v) \subseteq \mathbb{N}$ for each vertex, adjacent vertices receive different colors)
- **k-list-colorable/-choosable**: if G is L-list-colorable for each list L
- **List chromatic number**: $\chi_l(G) = \text{ch}(G)$
  $= \min\left\{k : G \text{ is } L\text{-colorable } \forall L : V \rightarrow 2^{\mathbb{N}} : |L(v)| = k \forall v \in V(G)\right\}$
  - $\chi_l(G) \geq \chi(G)$ because we can choose $L(v) = \{1, \ldots, k\}$ $(\forall v \in V(G))$
  - often $\chi_l(G) \gg \chi(G)$ (see $K_{m,n}$: $\chi = 2, \chi_l \approx \log n$)
- **Planar graphs**: $\chi_l(G) \leq 5$
- **Locally planar graphs**: $\chi_l(G) \leq 5$

Perfect graphs

- **Clique number** of graph G: $\omega(G) := \max\{k \in \mathbb{N} : K_k \subseteq G\}$
- **Independence number** of graph G: size of largest independent vertex set
- **Perfect graph**: $\forall H \underset{\text{ind}}{\subseteq} G : \chi(H) = \omega(H)$
- **Perfect complement**: G is perfect $\Longleftrightarrow \overline{G}$ is perfect
- **Perfect graph conjecture**: G is perfect $\Longleftrightarrow$
  $C_{2k+1} \nsubseteq G$ for $k \geq 2 \wedge \overline{C_{2k+1}} \nsubseteq G$

Extremal Graph Theory

Base definitions

- **Sparse** graph: $||G|| \sim |G|$
- **Dense** graph: $||G|| \sim |G|^2$
- **Density**: $||X,Y|| := \#$ edges between X and Y, $d(X,Y) := \frac{||X,Y||}{|X||Y|}$
- **Edge density** of graph G: $||G|| / \binom{|G|}{2}$
- **ε-regular pair** (X,Y): if $|d(X,Y) - d(A,B)| \leq \varepsilon$ for $\varepsilon > 0$ and all $A \subseteq X$, $B \subseteq Y$ with $|A| \geq \varepsilon|X|$, $|B| \geq \varepsilon|Y|$
- **ε-regular partition**: $= V_0 \dot\cup \cdots \dot\cup V_k = V$ with
  1. $|V_0| \leq \varepsilon|V|$
  2. $|V_1| = \cdots = |V_k|$
  3. all but at most εk^2 of (V_i, V_j)-pairs ($1 \leq i < j \leq k$) are ε-regular

Subgraphs

- **Extremal number**: $\text{ex}(n,H) := \max\{||G|| : |G| = n \wedge H \nsubseteq G\}$
- **Extremal set**: $\text{EX}(n,H) := \{G : |G| = n \wedge ||G|| = \text{ex}(n,H) \wedge H \nsubseteq G\}$
- **Turán graph**: $T(n,r)$ = unique complete r-partite graph with $|T(n,r)| = n$, partite sets differing at most by 1 ($1 \leq r \leq n$)
  - $K_{r+1} \nsubseteq T(n,r)$
  - *size*: $||T(n,r)|| =: t(n,r)$
  - *special Turán graph*: $K_r^s := T(n,r)$ if $n = r * s$
  - *Turán-graphs edge-maximal*: among all r-partite graphs of order n, $T(n,r)$ has largest size
  - $t(n,r) = t(n-r,r) + (n-r)(r-1) + \binom{r}{2}$
  - *size difference to complete graph*:
    $$\lim_{n\to\infty} \frac{t(n-r)}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right)$$
- **Turán's theorem**: $\forall r > 1, n \geq 1$, any graph G with $|G| = n$, $||G|| = \text{ex}(n,K_r)$ and $K_r \nsubseteq G$ is a $T(n,r-1)$
$\iff \text{EX}(n,K_r) = \{T(n,r-1)\}$
- **Szemerédi's regularity lemma**: $\forall \varepsilon > 0 \forall 1 \leq m \in \mathbb{N}\ \exists\ M \in \mathbb{N}$: every graph G with $|G| \geq m$ has ε-regular partition $V_0 \dot\cup \cdots \dot\cup V_k$ with $m \leq k \leq M$.
- **Erdős-Stone theorem**: $\forall$ integers $r > s \geq 1$ and any $\varepsilon > 0\ \exists\ n_0 \in \mathbb{N}$: every graph with $|G| = n \geq n_0$ and $||G|| \geq t(n,r-1) + \varepsilon n^2$ has $K_r^s \subseteq G$.
  - *corollary*: the theorem together with the size difference to complete graph yields
    $$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$
- **Chvátal-Szemerédi theorem**: $\forall \varepsilon > 0$ and any integer $r \geq 3$, any graph with $|G| = n$ and $||G|| \geq (1 - \frac{1}{r-1} + \varepsilon)\binom{n}{2}$ has $K_r^t \subseteq G$ with $t = \frac{\log n}{500 \log(\frac{1}{\varepsilon})}$
  - *existance*: $\exists\ G$ with $|G| = n$ and $||G|| = (1 - \frac{1+\varepsilon}{r-1})\binom{n}{2}$ with $G \nsubseteq K_r^t$ for $t = \frac{5 \log n}{\log(\frac{1}{\varepsilon})}$
- **Zarankiewicz function**: $z(m,n;s,t)$ = maximum # of edges that bipartite graph with parts of size m and n can have without containing $K_{s,t}$
- **Kővári-Sós-Turán**: $z(m,n;s,t) \leq (s-1)^{\frac{1}{t}}(n-t+1)m^{1-\frac{1}{t}} + (t-1)m$
  - $m = n, t = s$: $\quad z(n,n;t,t) \leq c_1 n n^{1-\frac{1}{t}} + c_2 n = \mathcal{O}(n^{2-\frac{1}{t}})$
- **Bound for ex**$(n, K_{t,s})$: $\leq \frac{1}{2} z(n,n;s,t) \leq cn^{2-\frac{1}{s}}$ ($t \geq s \geq 1$)
  - $t = s = 2$: $\quad \text{ex}(n,C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3})$
- **Bound for ex**$(n, K_{r,r})$: $\geq cn^{2-\frac{2}{r+1}}$ ($\forall n, r \in \mathbb{N}$)
- **Bound for ex**(n, P_{k+1}): $\leq \frac{n(k-1)}{2}$

Minors

- **Hadwiger conjecture**: $\chi(G) \geq r \Rightarrow MK_r \subseteq G$
  - $r \in \{1,2,3,4\}$: easy to see
  - $r \in \{5,6\}$: proven using 4-color theorem
  - $r \geq 7$: still open
- **Bollobás-Thomason theorem**: $d(G) \geq cr^2 \Rightarrow TK_r \subseteq G$
- **Minimum degree + girth = minor**: $\delta(G) \geq d, g(G) \geq 8k+3$ ($d, k \in \mathbb{N}, d \leq 3$). Then $MH \subseteq G$ with $\delta(H) \geq d(d-1)^k$.
- **Thomassen's theorem**: $\forall r \in \mathbb{N}\ \exists\ f : \mathbb{N} \to \mathbb{N}$ s.t. every G with $\delta(G) \geq 3$ and $g(G) \geq f(r)$ has K_r minor
- **Kühn-Osthus theorem**: $\forall r \in \mathbb{N}\ \exists\ g \in \mathbb{N} : TK_r \subseteq G$ for all G with $\delta(G) \geq r-1$ and $g(G) \geq g$

Ramsey theory

Base definitions

- **Monochromatic** edge coloring: all edges have same color
- **Rainbow** edge coloring: no two edges have same color
- **Lexical** edge coloring: two edges have same color $\iff$ have same lower endpoint in some vertex ordering
- **Ramsey number** $R(k) \in \mathbb{N}$: smallest n s.t. every 2-edge-coloring of K_n contains monochromatic K_k ($n \in \mathbb{N}$)
- **Asymmetric Ramsey number** $R(k,l)$: smallest $n \in \mathbb{N}$ s.t. every 2-edge-coloring of K_n contains red K_k or blue K_l ($k, l \in \mathbb{N}$)
- **Graph Ramsey number** $R(G,H)$: smallest $n \in \mathbb{N}$ s.t. every 2-edge-coloring of K_n contains red G or blue H
- **Hypergraph Ramsey number** $R_r(l_1,\ldots,l_k)$: smallest $n \in \mathbb{N}$ s.t. for every k-coloring of $\binom{[n]}{r}\ \exists\ i \in \{1,\ldots,k\}$ and a $V \subseteq [n]$ with $|V| = l$ s.t. all sets in $\binom{V}{r}$ have color i
- **Induced Ramsey number** $R_{\text{ind}}(G,H)$: smallest $n \in \mathbb{N}$ s.t. $\exists$ graph F with $|F| = n$ with every 2-coloring of it containing red G or blue H
- **Anti Ramsey number** $AR(n,H)$: maximum number of colors that edge-coloring on K_n can have without containing rainbow copy of H
- **r-regular matrix**: if there is a monochromatic solution of $Ax = 0$ for any r-coloring $c : \mathbb{N} \to [r]$ of $\mathbb{N}$
- **column condition**: matrix fulfills it if there is partition $C_1 \dot\cup \cdots \dot\cup C_l$ of A-columns s.t. the following holds:
  Let $s_i := \sum_{c \in C_i} c$ for $i \in [l]$. Then $s_1 = 0$ and every s_i is linear combination of columns in $C_1 \dot\cup \cdots \dot\cup C_{i-1}$ ($2x_1 + x_2 + x_3 - 4x_4$ fulfills: $2+1+1-4=0$)

Observations

- $R(3) = 6$
- $R(2,k) = R(k,2) = k$
- **Ramsey theorem**: $\forall k \in \mathbb{N} : \sqrt{2}^k \leq R(k) \leq 4^k$
$\rightarrow$ (Asymmetric) Ramsey numbers and graph Ramsey numbers are finite
- **Induction theorem**: $\forall k, l \in \mathbb{N} : R(k,l) \leq R(k-1,l) + R(k,l-1)$
$\rightarrow R(k,l) \leq \binom{k+l-2}{k-1}$
- **Hypergraph recursion**:
  $$\forall r,p,q \in \mathbb{N} : R_r(p,q) \leq R_{r-1}(R_r(q-1,q), R_r(p,q-1)) + 1$$
- 2-**Hypergraph boundary**: $c_1 2^k \leq R_2(\underbrace{3 * \cdots * 3}_{k \text{ times}}) \leq c_2 k!$ for some $c_1, c_2 > 0$

Ramsey theory applications

- **Erdős-Szekeres subsequences**: Any sequence of $(r-1)(s-1)+1$ distinct real numbers contains increasing subsequence of length r or a decreasing subsequence of length s
- **Erdős-Szekeres m-gons**: $\forall m \in \mathbb{N}\ \exists\ N \in \mathbb{N}$: every set of $\geq N$ points in general position in $\mathbb{R}^2$ contains the vertex set of a convex m-gon
- **Schur**: Let $c : \mathbb{N} \to [r]$ be coloring of the natural numbers with $r \in \mathbb{N}$ colors. Then there are $x, y, z \in \mathbb{N}$ of same color with $x + y = z$
- **Rado theorem**: A fulfills column condition $\Rightarrow A$ is r-regular $\forall r \in \mathbb{N}$ ($A \in \mathbb{Z}^{n \times k}$)
- $\forall s, t \in \mathbb{N}$ with $s \geq t \geq 1$: $R(sK_2, tK_2) = 2s + t - 1$
- $\forall s, t \in \mathbb{N}$ with $s \geq t \geq 1$: $R(sK_3, tK_3) = 3s + 2t$
- **Chvátal-Harary**: $R(G,H) \geq (\chi(G)-1)(c(H)-1)+1$
($c(H)$ order of largest component of H)
- $R_{\text{ind}}(G,H)$ is finite for all graphs G, H
- **Canonical Ramsey theorem**: $\forall k \in \mathbb{N}\ \exists\ n \in \mathbb{N}$: any edge coloring of K_n with arbitrarily many colors contains monochromatic, rainbow or lexical K_k
- $\forall \Delta \in \mathbb{N}\ \exists\ c \in \mathbb{N}$: for every graph H with $\Delta(H) = \Delta$ we have $R(H,H) \leq c|V(H)|$
- For any n-vertex graph H with $\Delta(H) = 3$ we have $R(H,H) \leq cn$ for some $c > 0$, which grows way slower than $R(K_n,K_n) \geq \sqrt{2}^n$
- **Anti-Ramsey theorem**:
  $$\forall n, r \in \mathbb{N} : AR(n,K_r) = \binom{n}{2}\left(1 - \frac{1}{r-2}\right)(1 - o(1))$$

Flows

Circulations

- **Circulation**:
  - $H :=$ abelian semigroup, G multigraph, $\widetilde{E} := \{(x,y) : xy \in E(G)\}$
  - $f : \widetilde{E} \to H, X, Y \subseteq V \rightsquigarrow f(X,Y) := \sum_{(x,y) \in (X \times Y) \cap \widetilde{E}} f(x,y)$

- $f : \widetilde{E} \to H$ is *circulation* on $G \iff$
  1. $f(x,y) = -f(y,x)$ ($\forall xy \in E(G)$),
  2. $f(v,V) = 0$ ($\forall v \in v$).
- **H-flow**: circulation $f : \widetilde{E} \to H$ with abelian *group* H
- **nowhere-zero-flow**: $\forall xy \in E : f(x,y) \neq 0$
- **k-flow**: $\mathbb{Z}$-flow f with $\forall xy \in E : 0 < |f(x,y)| < k$
- **flow number** $\varphi(G)$: $\min\{k \in \mathbb{N} : G$ has k-flow$\}$

Networks

- **Network**:
  - $s,t \in V, s \neq t, c : \widetilde{E} \to \mathbb{N}_0$
  - *network* (G,s,t,c) with
    - *source* s
    - *sink* t
    - *capacity function* c
- **Network flow**: $f : \widetilde{E} \to \mathbb{R}$ with $\forall x,y \in V$:
  1. $f(x,y) = -f(y,x)$
  2. $x \notin \{s,t\} \Rightarrow f(x,V) = 0$
  3. $f(x,y) \leq c(x,y)$
- **Cut**: $(S, V \setminus S)$ with $s \in S, t \notin S$ (for any $S \subseteq V$)
  - *capacpity* $c(S, V \setminus S)$
- **Value** of f: $= |f| := f(s,V)$
- **Basic network properties**:
  - $\forall$ circulation $f, X \subseteq V$: $f(X,X) = f(X,V) = f(X, V \setminus X) = 0$
  - $\forall$ network flow f, cut $S, \overline{S}$: $f(S,\overline{S}) = |f|$
- **Ford-Fulkerson**: $\forall$ networks:
  - max value of a flow = min capacity of a cut
  - $\exists$ integral flow $f : \widetilde{E} \to \mathbb{N}_0$ with max flow value
- **Tutte**: $\forall$ multigraph G $\exists$ polynomial $P \in \mathbb{Z}[X]$: $\forall$ finite Abelian group H: number of nowhere-zero H-flows on G is $P(|H| - 1)$
- **Abelian group can be exchanged**: H-flow on G exists (Abelian group H) $\Rightarrow$ $\exists$ $\widetilde{H}$-flow on G ($\forall$ finite Abelian groups $\widetilde{H}$ with $|\widetilde{H}| = |H|$)
  $\rightsquigarrow$ $\mathbb{Z}_4$-flow exists $\Rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$-flow exists
- **Flow can be $\mathbb{Z}_k$-substituted**: multigraph admits k-flow $\iff$ admits $\mathbb{Z}_k$-flow
- **Flows in planar graphs**: planar graph G, dual G^*: $\chi(G) = \varphi(G^*)$
- **2-flow only on even degrees**: graph has 2-flow $\iff$ all degrees are even
- **3-flow on bipartite graphs**: 3-regular graph has 3-flow $\iff$ bipartite
- **Tutte on 5-flows**: every bridgeless multigraph has flow number ≤ 5
- **Seymour on 6-flows**: every brigeless graph has flow number ≤ 6

Random Graphs

Erdős-Rényi Model

- **Erdős-Renyi Model**: $\mathcal{G}(n,p)$ probability space on n-vertex graphs resulted by independently deciding whether to include each of the $\binom{n}{2}$ possible edges with fixed probability $p \in [0,1]$
- **Property $\mathcal{P}$**: set of graphs (e.g. $\mathcal{P} = \{G : G$ is k-connected$\}$)
- **Almost always**: let $(p_n) \in [0,1]^{\mathbb{N}}$. $G \in \mathcal{G}(n,p_n)$ has property $\mathcal{P}$ *almost always* $\iff \lim_{n\to\infty} \text{Prob}(G \in \mathcal{G}(n,p_n) \cap \mathcal{P}) = 1$
- **Almost all**: like almost always, but (p_n) = constant p
- **Threshold function**: $f(n) : \mathbb{N} \to [0,1]$ *threshold function* for property $\mathcal{P}$ if
  1. $\forall (p_n) \in [0,1]^{\mathbb{N}}, p_n/f(n) \xrightarrow{n\to\infty} 0 : G \in \mathcal{G}(n,p_n)$ almost always does **not** have property $\mathcal{P}$
  2. $\forall (p_n) \in [0,1]^{\mathbb{N}}, p_n/f(n) \xrightarrow{n\to\infty} \infty : G \in \mathcal{G}(n,p_n)$ almost always **has** propertpy $\mathcal{P}$

  ! not all properties $\mathcal{P}$ have a threshold function

Basic probability properties

- G with $|G| = n$, $||G|| = m$: $\text{Prob}(G = \mathcal{G}(n,p)) = p^m(1-p)^{\binom{n}{2}-m}$
- $\text{Prob}(G \in \mathcal{G}(n,p), \alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$ $(n \geq k \geq 2)$
- $\text{Prob}(G \in \mathcal{G}(n,p), \omega(G) \geq k) \leq \binom{n}{k}p^{\binom{n}{k}}$

More complex results

- $\text{Exp}(\#k\text{-cycles in } G \in \mathcal{G}(n,p)) = \frac{n_k}{2k}p^k$ $(n_k = n(n-1)\cdots(n-k+1))$
- **Erdős**: $\forall k \in \mathbb{N}$ $\exists$ graph $H : g(H) \geq k \wedge \chi(H) \geq k$
- $\forall p \in (0,1)$, graph H, almost all $G \in \mathcal{G}(n,p)$ contain H as induced subgraph
- $\forall p \in (0,1), \varepsilon > 0$, almost all $G \in \mathcal{G}(n,p)$ fulfill
  $$\chi(G) > \frac{\log(1/(1-p))}{2+\varepsilon}\frac{n}{\log n}$$
- **Asymptotic behaviour of $\mathcal{G}(n,p)$ for some properties**:
  - $p_n = \sqrt{n}/n^2 \Rightarrow G$ almost always has component with > 2 vertices
  - $p_n = 1/n \Rightarrow G$ almost always has a cycle
  - $p_n = \log n/n \Rightarrow G$ is almost always connected
  - $p_n = (1+\varepsilon)\log n/n \Rightarrow G$ almost always has Hamiltonian cycle
  - $p_n = n^{-2/(k-1)}$ is the threshold function for containing K_k
- **Lovász Local**: $A_1, \ldots, A_n$ events in some probabilistic space. If $\text{Prob}(A_i) \leq p \in (0,1)$, each A_i is mutually independent from all but at most $d \in \mathbb{N}$ A_is and $\text{e}p(d+1) \leq 1$, then
  $$\text{Prob}\left(\bigwedge_{i=1}^{n} \overline{A_i}\right) > 0$$
- **Van der Waerden's number**: $W(k)$ = smallest n s.t. any 2-coloring of $[n]$ contains monochromatic arithmetic progression of length k
  → using Lovász Local Lemma: $W(k) \geq 2k - 1/(ek^2)$