

Basics

Notations

- $\binom{V}{k} := \{A : A \subseteq V \wedge |A| = k\}$
- $[n] := \{1, \dots, n\} \subset \mathbb{N}$
- **Power set** $2^X := \{A : A \subseteq X\}$

Graph

- **Definition:** $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$
- **Vertex set:** $V(G)$
- **Edge set:** $E(G)$
- **Isomorphic** (G_1 to another graph G_2): if \exists bijection $f : V_1 \rightarrow V_2$ with $\{u, v\} \in E_1 \Leftrightarrow \{f(u), f(v)\} \in E_2$
- **Order:** $= |V(G)|$, short $|G|$
- **Size:** $= |E(G)|$, short $\|G\|$
- **Complement:** $\bar{G} = (V(G), \binom{V}{2} - E(G))$
- **Degree sequence:** multiset of degrees of vertices in $V(G)$
 - *graphic:* deg. seq. (d_1, \dots, d_n) , iff
 1. $d_1 + \dots + d_n$ even
 2. $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k) \quad (\forall 1 \leq k \leq n)$
- **Degree sum:** $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$
- **Minimum degree:** $\delta(G)$ = degree of $v \in V(G)$ with smallest degree
- **Maximum degree:** $\Delta(G)$ = degree of $v \in V(G)$ with largest degree
- **Adjacency matrix:** $A(G) = \mathbb{R}^{n \times n} \ni A_{i,j} = \begin{cases} 1, & ij \in E \\ 0, & \text{else} \end{cases}$
- **Eulerian:** if it contains an Eulerian tour
- **Connected:** for any two vertices there is a link between them
 - *spanning tree:* if G is connected, then it has a spanning tree
 - *peeling leaves:* vertices can be ordered v_1, \dots, v_n s.t. $G[\{v_1, \dots, v_i\}]$ is connected for $i \in \{1, \dots, n\}$

Digraph

- **Definition:** $G = (V, E)$ with vertex set V and edge set $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$

Multigraph

- **Definition:** $G = (V, E)$ with vertex set V and multiset E of V -pairs

Hypergraph

- **Definition:** $G = (V, E)$ with vertex set V and edge set $E \subseteq 2^V = \{A : A \subseteq V\}$

Vertex

- **Incident** to $e \in E(G)$ if $v \in e$
- **Adjacent** to $\tilde{v} \in V(G)$ if $\{v, \tilde{v}\} \in E(G)$
- **Neighborhood:** $N(v) = \{u : uv \in E(G)\}$
- **Degree:** $\deg(v) = d(v) = |N(v)|$
- **Isolated:** vertex with $\deg(v) = 0$
- **Leaf:** vertex with $\deg(v) = 1$

Subgraph

- **Definition:** H subgraph of G (write $H \subseteq G$) if $V(H) \subseteq V(G) \wedge E(H) \subseteq E(G)$
- **Induced subgraph:** H induced subgraph of G (write $H \subseteq_{\text{ind}} G$), if $H \subseteq G$ and $E(H)$ contains all edges from $E(G)$ between vertices in $V(H)$
- **Edge-induced subgraph:** subgraph induced by $X \subseteq E(G)$, note $G[X]$
- **Subgraph separation:** $X \subset V(G)$ separates $A, B \subset V(G) \Leftrightarrow$ any A - B -path has vertex in X

Spanning graph

- **Definition:** Subgraph with same vertex set as supergraph

Line graph

- **Definition:** $L(G) = (E, \{\{e, e'\} : e \cap e' \neq \emptyset\})$
- **Graphic:** L is line graph of some G , if it doesn't contain one of 9 specific induced subgraphs

Vertex cover

- **Definition:** $V' \subseteq V(G)$ s.t. any $e \in E(G)$ is incident to a vertex in V'

Cycle

- **Definition:** $C_n := (\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\})$
- **Shorthand:** (v_1, \dots, v_n, v_1)
- **Length** (of cycle): $= |V| \equiv |E|$
- **Cyclic subgraph:** If $\delta(G) \geq 2$, then G has cycle with length $\geq \delta + 1$

Path

- **Definition:** $(\{v_1, \dots, v_n\}, \{\{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}\})$
- **Shorthand:** (v_1, \dots, v_n)
- **Length** (of path): $= |E| \neq |V|$
- $v_0 v_k$ -**path:** path starting at v_0 and ending at v_k
- **Independent:** two ab -paths are independent \Leftrightarrow they only share a and b

Walk

- **Definition:** non-empty alternating sequence of vertices and edges
$$v_0 e_0 \dots e_{k-1} v_k$$
with $e_i = v_i v_{i+1}$, length $k \in \mathbb{N}$
 - *closed:* if $v_0 = v_k$
 - *even:* if k is even
 - *odd:* if k is odd
- **Eulerian tour:**
 - *Definition:* closed walk with
 - no edges of G are repeatedly used
 - all edges of G are used
 - *Even degrees:* G connected has Euler tour $\Leftrightarrow \forall v \in V(G) : \deg(v)$ even
- $v_0 v_k$ -**walk:** walk starting at v_0 and ending at v_k
- **Induces path:** $\exists uv$ -walk $\Rightarrow \exists uv$ -path
- **Odd closed walk, odd cycle:** G has *odd* closed walk $\Rightarrow G$ has odd cycle

Connected component

- **Definition:** *maximal* connected subgraph (connected, but any supergraph isn't)

Block

- **Block:** maximal 2-connected subgraph or bridge
 - share ≤ 1 vertices with one another
- **Block-cut-vertex graph**
 - V = set of blocks \cup set of vertices
 - $E = \{\{v, B\} : v \in V(B), \text{cut-vertex } v, \text{block } B\}$
 - block-cut-vertex graph of connected graph is tree

Acyclic graph, Forest

- **Definition:** Graph with no cycle as subgraph

Tree

- **Definition:** Graph that is connected and acyclic
 - $\Leftrightarrow G$ is connected and $\forall e \in E(G) : G - e$ is disconnected (*minimal-connected*)
 - $\Leftrightarrow G$ is acyclic and $\forall xy \notin E(G) : G \cup xy$ has cycle (*maximal-acyclic*)
 - $\Leftrightarrow G$ is connected and *1-degenerate* ($\forall G' \subseteq G : \delta(G') \leq 1$)
 - $\Leftrightarrow G$ is connected and $\|G\| = |G| - 1$
 - $\Leftrightarrow G$ is acyclic and $\|G\| = |G| - 1$
 - $\Leftrightarrow \forall u, v \in V(G) \exists$ unique uv -path
- **Special trees:** path, star, spider, caterpillar, broom
- **Leaf existence:** Tree $T, |T| \geq 2 \Rightarrow T$ has leaf
- **Edge count:** Tree $T, |T| = n \Rightarrow \|T\| = n - 1$

k-regular graph

- **Definition:** Graph with $\deg(v) = k \in \mathbb{N}_0 \quad (\forall v \in V(G))$

Bipartite graph

- **Definition:** G is bipartite $\Leftrightarrow G$ contains no cycles of odd length
 - *complete bipartite:* $K_{m,n} = (A \cup B, \{a, b\} : a \in A, b \in B)$
- **Matchings:**
 - *saturating:* $G = (A \cup B, E)$ has matching saturating A
$$\Leftrightarrow \forall S \subseteq A : N(S) \geq |S| \quad (N(S) := \{b \in B : ab \in E, a \in S\})$$

- *nearly*: $G = (A \cup B, E), \forall S \subseteq A : |N(S)| \geq |S| - d \quad (d \geq 1).$
 $\Rightarrow \exists$ matching M saturating all but at most d vertices of A
- **Matching vs vertex cover**: size of largest matching = size of smallest vertex cover

Matching

- **Definition**: graph with $\delta(G) = \Delta(G) = 1$
- **Perfect matching**: spanning + matching subgraph of G (aka *1-factor*)
 - *existence*: G has perfect matching $\Leftrightarrow \forall S \subseteq V(G) : q(G - S) \leq S$
 $(q(G) = \text{number of components in } G \text{ with odd order})$

Coloring

- **Proper coloring**: $= c : V(G) \rightarrow [k]$ with $c(u) \neq c(v) \quad (\forall uv \in E(G))$
- **Equitable coloring**: proper coloring + color classes have almost (± 1) equal size
 - *existence*: any graph has equitable coloring in $(\Delta(G) + 1)$ colors

Chromatic number

- **Definition**: $\chi(G) = \min\{k : G \text{ has proper coloring with } k \text{ colors}\}$
- **Examples**: $\chi(C_{2n}) = 2, \chi(C_{2n+1}) = 3$

Factors

- **k-factor**: spanning k -regular subgraph (easy to find)
- **f-factor**: spanning subgraph $H \subseteq G$ with $\deg_H(v) = f(v),$
 $f : V(G) \rightarrow \{0, 1, \dots\}$ with $f(v) \leq \deg(v) \quad (\forall v \in V)$
- **H-factor** (aka *perfect H-packing*): spanning subgraph s.t. each component is $\cong H$
 - *existence*: if $\delta(G) \geq (1 - \frac{1}{k} |V(G)|)$ and k divides $|G|$, then G has K_k -factor

Connectivity

- **k-connected**: if $|G| > k$ and deleting $< k$ vertices does not disconnect G
- **k-linked**: if for any $2k$ vertices $(s_1, \dots, s_k, t_1, \dots, t_k) \exists$ pairwise disjoint $s_i t_i$ -paths (*note*: k -connected $\nRightarrow k$ -linked)
- **Vertex-connectivity**: $\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}$
- **l-edge-connected**: if deleting $< l$ edges does not disconnect G
- **Edge-connectivity**: $\kappa'(G) = \max\{l : G \text{ is } l\text{-edge-connected}\}$
- **Vertex- vs Edge-connectivity**: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
- **Three-connected + contraction**: $3\text{-connected} \Leftrightarrow \exists$ separate G_0, \dots, G_k with
 $G_0 = K_4, G_k = G, G_i = G_{i+1} \circ xy$
 with $\deg(x), \deg(y) \geq 3$
- **Three-connected + decontraction**: all 3-connected graphs can be built by iteratively de-contracting vertices of K_4
- **Average degree ≥ 4** : has k -connected subgraph ($k \geq 2$)

Cuts

- **Cut-Set**: $X \subseteq V(G) \cup E(G)$ s.t. #components in $(G - X)$ greater than in G
- **Cut-Vertex**: Cut-Set consisting of single vertex
- **Cut-Edge** (or *bridge*): Cut-Set consisting of single edge
- **Menger's theorem**: for $A, B \subseteq V(G)$:
 $\min \# \text{ of vertices separating } A \text{ and } B = \max \# \text{ of disjoint } A\text{-}B\text{-paths}$
- **Menger global**:
 1. $k\text{-connected} \Leftrightarrow \forall a, b \in V(G) \exists k$ pairwise independent ab -paths
 2. $k\text{-edge-connected} \Leftrightarrow \forall a, b \in V(G) \exists k$ pairwise edge-disjoint ab -paths

Ear-decomposition

- **Definition**: G has *ear-decomposition* $\Leftrightarrow \exists$ sequence of graphs G_0, \dots, G_k with
 $G_k = G, G_0 = \text{cycle}, G_{i+1}$ obtained from G_i by attaching "ear" (path that shares only endpoints with G_i)
- **2-connected** $\Leftrightarrow \forall$ cycles C in G there is ear-decomposition starting at C

Edge contraction

- **Contraction**:
 $G \circ xy = ((V \setminus \{x, y\}) \cup v_{xy},$
 $(E \setminus \{e : x \in E \vee y \in e\}) \cup \{v_{xy} z : z \in (N_G(x) \cup N_G(y)) \setminus \{x, y\}\})$
 with $xy \in E(G)$
- **De-contraction**: if $\exists xy \in E(G) : \kappa(G \circ xy) \geq 3$
 (for G with $\kappa(G) \geq 3, |G| \geq 5$)

Planar graph tools

- **Homeomorphism**: $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous s.t. f^{-1} is also continuous
- **Arc**: homeomorphic image of $[0, 1]$ in \mathbb{R}^2 under f

- *endpoints*: $f(0)$ and $f(1) \leadsto$ arc "joins" endpoints
- *polynomial arc*: arc that is union of finitely many straight line segments
- **Region** $Y \subseteq \mathbb{R}^2 \setminus X$: any two points $\in Y$ could be joined by arc and Y is maximal ($X \subseteq \mathbb{R}^2$)
- **Boundary** of $X \subseteq \mathbb{R}^2$:
 $\delta X = \{y \mid \forall \varepsilon > 0 : B(y, \varepsilon) \text{ contains points of } X \text{ and not of } X\}$
- **Jordan curve theorem**: If $X \subseteq \mathbb{R}^2$ and homeomorphic to $\{\bar{x} : \text{dist}(\bar{x}, 0) = 1\}$ (*unit circle*), then $\mathbb{R}^2 \setminus X$ has two regions R_1, R_2 and $\delta R_1 = X = \delta R_2$.

Plane graph

- **Definition**: graph such that $E(G)$ is set of arcs in \mathbb{R}^2 and endpoints of arcs in $E(G)$ are vertices and:
 - $\forall e, e' \in E, e \neq e' : e$ and e' have distinct sets of edge sets
 - $\forall e \in E, \hat{e} = e \setminus \{\text{endpoints}\}$ doesn't contain any vertices and points from other arcs
- **Faces**: regions of $\mathbb{R}^2 \setminus (\bigcup_{e \in E} e \cup V)$
- **Edge limitation 1**: Plane graph: $|G| \geq 3 \Rightarrow \|G\| \leq 3n - 6$
- **Edge limitation 2**: Plane graph with no Δ : $\|G\| \leq 2|G| - 4$

Planar graph

- **Definition**: graph s.t. \exists plane graph G' and bijection $f : V(G) \rightarrow V(G')$ s.t.
 $\forall u, v \in V(G), uv \in E(G) : f(u), f(v)$ are endpoints of arc in G'
- **Planar embedding** of G : f from the definition
- **Planar because of minors**: The following statements are equivalent:
 - G is planar
 - $G \not\preceq MK_5 \wedge G \not\preceq MK_{3,3}$
 - $G \not\preceq TK_5 \wedge G \not\preceq TK_{3,3}$
- **Euler's formula**: If G is connected plane graph with f faces, then
 $|G| - \|G\| + f = 2$
- $\delta(G)$ **limitation**: Planar graph $\delta(G) \leq 5$
- **Non-planar graphs**: K_5 and $K_{3,3}$ are not planar

Minors

- **MH**: $G \stackrel{(\star)}{=} MH$ is *minor* of H if
 - $V(G) = V_1 \dot{\cup} \dots \dot{\cup} V_n$ with $n = |H|$
 - $G[V_i]$ connected ($\forall i = 1, \dots, n$)
 - If $V(H) = \{v_1, \dots, v_n\}$ and $v_i v_j \in E(H)$, then \exists edge between V_i and V_j
- (\star): *Notation abuse*: MH is class of graphs
- **Branch sets**: V_i 's from above
- **Minor** (H of G , noted $H \preceq G$): $\Leftrightarrow MH \subseteq G$
 $\leadsto H \preceq G \Leftrightarrow H$ can be obtained by edge/vertex deletions + contractions.
- **Topological minor**: H is topological minor if $TH \subseteq G$ where TH is built from H by subdividing edges
- **Note**: $TH \subseteq MH$