HW5. part II

1. $\Sigma = a^2 I$ so $\Sigma^{-1} = \frac{1}{a^2} I$.

$$var(\beta | a^2, x, y) = [x^T \Sigma^{-1} x]^{-1} + \Sigma_\beta^{-1}$$
$$= [x^T \frac{1}{a^2} I x]^{-1} + \Sigma_\beta^{-1}$$
$$= [\frac{1}{a^2} x^T x]^{-1} + \Sigma_\beta^{-1}$$

$$E(\beta | a^2, x, y) = var(\beta | a^2, x, y)^{-1} (x^T \Sigma^{-1} y + \Sigma_\beta^{-1} \beta_0)$$
$$= [[\frac{1}{a^2} x^T x]^{-1} + \Sigma_\beta^{-1}][x^T \frac{1}{a^2} I y + \Sigma_\beta^{-1} \beta_0]$$
$$= [\frac{1}{a^2} x^T x + \Sigma_\beta]^{-1} [\frac{1}{a^2} x^T y + \Sigma_\beta^{-1} \beta_0]$$

$$P(\beta | a^2, x, y) = MVN\Big(E(\beta | a^2, x, y) = [\frac{1}{a^2} x^T x + \Sigma_\beta]^{-1} [\frac{1}{a^2} x^T y + \Sigma_\beta^{-1} \beta_0],$$
$$var(\beta | a^2, x, y) = [\frac{1}{a^2} x^T x]^{-1} + \Sigma_\beta^{-1}$$

2. we get the $E(\beta | a^2, x, y)$ in Q_1 which is

$$E(\beta | a^2, x, y) = [\frac{1}{a^2} x^T x + \Sigma_\beta]^{-1} [\frac{1}{a^2} x^T y + \Sigma_\beta^{-1} \beta_0]$$

3. if hyperparameter is illegal which means $\Sigma_\beta = 0$.

we can get

$$E[\beta | a^2, x, y] = [\frac{1}{a^2} x^T x + \Sigma_\beta]^{-1} [\frac{1}{a^2} x^T y + \Sigma_\beta^{-1} \beta_0]$$
$$= [\frac{1}{a^2} x^T x + 0]^{-1} [\frac{1}{a^2} x^T y + 0 \beta_0]$$
$$= \cancel{a^2} (x^T x)^{-1} \frac{1}{\cancel{a^2}} x^T y$$
$$= (x^T x)^{-1} x^T y$$

4. $E(\hat{y} = x\beta | a^2, x, y) = x E(\hat{y} | a^2, x, y)$

if hyperparameter is illegal which means $\Sigma_\beta = 0$

so we get $E[\beta | a^2, X, y] = (X^T X)^{-1} X^T y$ from Q3.

$$X E(\hat{y} | a^2, X, y) = X (X^T X)^{-1} X^T y$$

6. We get the $var(\beta | a^2, X, y)$ in Q_1 which is

$$\begin{aligned} var(\beta | a^2, X, y) &= [X^T \Sigma^{-1} X]^{-1} + \Sigma_\beta^{-1} \\ &= [X^T \tfrac{1}{a^2} I X]^{-1} + \Sigma_\beta^{-1} \\ &= [\tfrac{1}{a^2} X^T X]^{-1} + \Sigma_\beta^{-1} \end{aligned}$$