ch 5

standard optimization problem. $\min f_0(x)$

s.t. $f_i(x) \le 0 \quad i=1,\cdots,m.$

$h_i(x)=0 \quad i=1,\cdots,p.$

Prime Problem 原问题.

Lagrangian: $L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p}\nu_i h_i(x)$. Affine for (λ,ν).

λ, ν: Lagrange multiplier. 拉格朗日函数

Lagrange dual function: $g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu)$ → concave for (λ,ν).

$\forall \lambda \succeq 0, \nu$: $g(\lambda,\nu) \le p^*$

dual feasible (λ,ν): $\lambda \succeq 0$ s.t. $g(\lambda,\nu) > -\infty$. $\Rightarrow$ getting a valid lower bound for p^* if take feasible (λ,ν).

Lagrange dual problem: $\max g(\lambda,\nu)$ s.t. $\lambda \succeq 0$ $\Rightarrow$ (λ^*, ν^*) dual optimal. $\Rightarrow$ optimal value $d^* = g(\lambda^*,\nu^*)$

convex problem.

Weak duality: $d^* \le p^*$ [always holds even $\begin{cases} p^* = -\infty \Rightarrow d^* = -\infty & \text{dual problem infeasible.} \\ d^* = +\infty \Rightarrow p^* = +\infty & \text{primal problem infeasible.} \end{cases}$].

opimal dual gap: $p^* - d^* \ge 0$ d^*给出p^*的1个最优下界.

strong duality: $d^* = p^*$ [best bound obtained through dual function is tight ??]

凸问题的强对偶性: $\min f_0(x)$ s.t. $f_i(x) \le 0 \quad i=1,\cdots,m$; $Ax=b$ } convex 通常但不一定有强对偶

a sufficient condition: (Slater's condition): $\exists\, x \in \operatorname{relint} D$ s.t. $f_i(x) < 0 \quad i=1,\cdots,m.\ Ax=b$

domain of f_i & h_i

① 凸问题也有无强对偶的.

② 非凸问题也有有强对偶的

Saddle Point interpretation for duality 鞍点解释.

$$\begin{cases} p^* = \inf_x \sup_{\lambda\succeq 0} L(x,\lambda) \\ d^* = \sup_{\lambda \succeq 0} \inf_x L(x,\lambda) \end{cases} \Rightarrow \begin{cases} \text{weak} & \sup\inf \le \inf\sup \\ \text{strong} & \sup\inf = \inf\sup \end{cases}$$

desired.

Von Neumann's minimax thm: domain X, Y compact convex

$f: X\times Y \to R$ convex_concave $f(x,y)$

$\Rightarrow \min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y)$

a sufficient condition for strong duality ~~convexity~~ [no specific restriction on convexity].

$(x^*, (\lambda^*, \nu^*))$ is a saddle point for $L(x,\lambda,\nu)$

鞍点: $f(x^*, y) \le f(x^*, y^*) \le f(x, y^*)$.

duality gap [对偶间隙] $f_0(x) - p^* \leq f_0(x) - g(\lambda, \nu)$. 要求 x 可行，$(\lambda,\nu)$ 对偶可行。

if $f_0(x) - g(\lambda,\nu) = \varepsilon \Rightarrow x$ is ε-suboptimal. $\Rightarrow (\lambda,\nu)$ dual ε-suboptimal.

$\Rightarrow p^*, d^* \in [g(\lambda,\nu), f_0(x)]$.

can be used to decide terminate condition.

let $f_0(x^{(k)}) - g(\lambda^{(k)}, \nu^{(k)}) \leq \varepsilon_{abs}$

or relatively: $\dfrac{f_0(x^{(k)}) - g(\lambda^{(k)}, \nu^{(k)})}{g(\lambda^{(k)}, \nu^{(k)})} \leq \varepsilon_{rel}$ when $g(\cdot\cdot) > 0$

$\dfrac{f_0(x^{(k)}) - g(\lambda^{(k)}, \nu^{(k)})}{-f_0(x^{(k)})} \leq \varepsilon_{rel}$ when $f_0(x^{(k)}) < 0$

$\Rightarrow \dfrac{f(x^{(k)}) - p^*}{|p^*|} \leq \varepsilon_{rel}$

KKT condition 形式:

$f_i(x^*) \leq 0 \quad i=1,\cdots,m$

$h_i(x^*) = 0 \quad i=1,\cdots,p$

$\lambda_i^* \geq 0 \quad i=1,\cdots,m$.

$\lambda_i^* f_i(x^*) = 0 \quad i=1,\cdots,m$. complementary slackness. $\begin{cases} \lambda_i^* > 0 \Rightarrow f_i(x^*) = 0 \\ f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0 \end{cases}$

$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0 \Rightarrow (x^*, \lambda^*, \nu^*)$ optimize Lagrangian.

KKT 条件的用处:
- differentiable ⊕ Convex problem ⊕ strong duality : optimal solution $\Longleftrightarrow$ KKT. necessary & sufficient.
- differentiable ⊕ any kind of problem : ~~KKT is nece~~ optimal solution $\Rightarrow$ KKT. necessary but not sufficient.

$\Rightarrow$ sometimes, solving KKT gives the optimal solution.

sensitivity analysis of constraints:

原问题: min $f_0(x)$
s.t. $f_i(x) \leq 0 \quad i=1,\cdots,m$.
$h_i(x) = 0 \quad i=1,\cdots,p$.
$\Downarrow$
p^*

$\Rightarrow$ perturbed problem. min $f_0(x)$
s.t. $f_i(x) \leq u_i \quad i=1,\cdots,m$. $u_i > 0$, relaxed; $u_i < 0$ tightened.
$h_i(x) = v_i \quad i=1,\cdots,p$.
$\Downarrow$
$p^*(u,v)$

$p^* = p^*(0,0)$

① original problem convex $\Rightarrow p^*(u,v)$ convex function.

② strong duality ⊕ (λ^*, ν^*) is optimal for the dual of original problem.

$\Rightarrow p^*(u,v) \geq$ ~~$\lambda^{*T}u - \nu^*$~~ $p^*(0,0) - \lambda^{*T}u - \nu^{*T}v$

we would know $p^*(0,0)$, λ^*, ν^*, so conduct sensitivity analysis based on them. & this is a lower bound for $p^*(u,\nu)$.