Ch 9 Unconstrained Minimization.

closed function: $\forall \alpha \in R$, sublevel set $\{x \in \text{dom} f \mid f(x) \le \alpha\}$ is closed.
$\iff$ epigraph of f is closed.

proper function: $\text{dom} f \neq \emptyset$ ⊕ $f \not\equiv +\infty$ ⊕ f never takes value $-\infty$.

a proper convex function is closed $\iff$ lower semi-continuous.

---

pick initial point $x^{(0)}$ $\longrightarrow$ all the remaining steps, x is taken from set $S = \{x \in \text{dom} f : f(x) \le f(x^{(0)})\}$ closed.

强凸函数 (strong convexity)

define: $\nabla^2 f(x) \succeq mI$. $\iff \exists\, m>0$ s.t. $g(x) = f(x) - \frac{m}{2}\|x\|^2$ is still convex.

$\iff f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y) - \frac{m}{2}\theta(1-\theta)\|x-y\|^2$

$\iff$ 梯度单调性: $(\nabla f(x) - \nabla f(y))^T (x-y) \ge m\|x-y\|^2 \quad \forall x, y \in \text{dom} f$.

(same for only convex function: convex $\iff (\nabla f(x) - \nabla f(y))^T (x-y) \ge 0$. $\forall x, y \in \text{dom} f$.

properties: ① f strong convex ⊕ minimum exists $\Longrightarrow$ f has unique minimum.

② $f(y) \ge f(x) + \nabla f(x)^T (y-x) + \frac{m}{2}\|y-x\|^2 \quad \forall x, y \in \text{dom} f$.

③ $p^* \ge f(x) - \frac{1}{2m}\|\nabla f(x)\|_2^2$

④ f strong convex, all α-sublevel sets of f are bdd.

proof: let set be $\{y : f(y) \le \alpha\}$

$$\alpha \ge f(y) \ge f(x^*) + \nabla f(x^*)(y - x^*) + \frac{m}{2}(y-x^*)^T(y-x^*)$$
$$= \frac{m}{2}\left[\|y - x^* + \frac{1}{m}\nabla f(x^*)\|^2\right] + f(x^*) - \frac{1}{2m}\|\nabla f(x^*)\|^2$$
$$\Longrightarrow \|y - x^* + \frac{1}{m}\nabla f(x^*)\|^2 \le \frac{2}{m}\left(\alpha - f(x^*) + \frac{1}{2m}\|\nabla f(x^*)\|^2\right)$$

$\Longrightarrow$ y bdd. in a finite circle.

⑤ find a starting point $x^{(0)}$. $\Longrightarrow S = \{x \in \text{dom} f \mid f(x) \le f(x^{(0)})\}$ bdd.

$\Longrightarrow \max\limits_{\substack{\|v\| \le 1 \\ \cancel{\nabla f x}\ x}} v^T \nabla^2 f(x) v \le M$. $\longrightarrow$ continuous function of both x & v.

⑥ $mI \preceq \nabla^2 f(x) \preceq MI$. $\forall x \in S$.

⑦ 上界的对称结论: $f(y) \le f(x) + \nabla f(x)^T (y-x) + \frac{M}{2}\|y-x\|^2 \quad \forall x, y \in \text{dom} f$.

$$\frac{1}{2M}\|\nabla f(x)\|^2 \le f(x) - f(x^*)$$

$$f(x) - \frac{1}{2m}\|\nabla f(x)\|^2 \le f(x^*) \le f(x) - \frac{1}{2M}\|\nabla f(x)\|^2.$$

General descent method. [without constraint].

min $f(x)$. ← 二次可微凸函数，$\exists\, x^*$ 且唯一.

then: $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$.

$t^{(k)}$ ← step; $\Delta x^{(k)}$ ← search direction.

[1] Find search direction: $\nabla f(x^{(k)})^T \Delta x^{(k)} < 0$ 要求.

[2] Find step size. (line search, on line $x^{(k)} + t\Delta x^{(k)}$, what's a plausible t)

[3] stopping criterion: $\|\nabla f(x)\|_2 \le \eta$. [from sub optimality condition. $p^* \ge f(x) - \frac{m}{2}\|\nabla f(x)\|^2$].

⇓

in every step: $f(x^{(k+1)}) < f(x^{(k)})$

more on [2]. for finding optimal t.

① exact line search: $\underset{s}{\operatorname{argmin}} f(x + s\Delta x)$.

② backtracking line search: (回溯直线搜索)

given $x, \Delta x$., set fixed $\alpha \in (0, \frac{1}{2})$, $\beta \in (0,1)$., start from $t=1$.

if $f(x + t\Delta x) > f(x) + \alpha t \nabla f(x)^T \Delta x$, shrink t by $t := \beta t$.