~~STe~~ Stein: perform best when eigenvalues are approximately equal. or forms clusters that are ~~need~~ nearly equal inside. generally outperform original minimax, E.B. & original S.

JS: have several variants. (based on correlation matrix...) too tedious to look into.

improved minimax. [Dipak 1985]

use same old setting.

minimizing Risk obtained with Stein's Loss.

invarance. orthogonal matrix.

(1) Recall: original minimax estimator.

$$S = KK^T \Rightarrow \varphi(S) = K\Delta K^T \quad \Delta_i = \frac{1}{n+p-2i+1} \qquad R(\underbrace{\hat{\Sigma}_{JS}(S)}_{\text{already minimax}}, \Sigma)$$

$$\hat{\Sigma}_{JS}(S)$$

(2) improve: $\hat{\Sigma}^m = B\varphi^m(L)B^T \quad \varphi^m(L) = L\Delta \Rightarrow \varphi_i^m(L) = l_i\Delta_i \qquad R(\hat{\Sigma}^m, \Sigma)$.

Claim $\hat{\Sigma}^m$ to be minimax by saying $R(\hat{\Sigma}^m, \Sigma) \leq R(\hat{\Sigma}_{JS}, \Sigma)$.

why: estimator that's invariant under orthorgonal transformation is a special case of estimator invariant under lower triangular.

$$\sup_{\Sigma} R(\hat{\Sigma}_{JS}, \Sigma) = \inf_{\hat{\Sigma} = K\Delta^* K^T} \sup_{\Sigma} R(\hat{\Sigma}, \Sigma)$$

Δ^*: diagonal

$$\vee\!/$$

$$\sup_{\Sigma} R(\hat{\Sigma}^m, \Sigma) \leq \cancel{\sup_{\Sigma} R(\hat{\Sigma}_{JS}, \Sigma) = \inf_{\hat{\Sigma}=K\Delta^*} \sup}$$

$$\hat{\Sigma}^m \in \{\hat{\Sigma} : \hat{\Sigma} = K\Delta^* K^T\}$$

then must equal

otherwise $\hat{\Sigma}^m = B\varphi^m(L)B^T$ is another class of estimator, can't make conclusion.

~~$\hat{\Sigma}^m =$~~ $S = BLB^T = KK^T$

$$= (BL^{\frac{1}{2}})(BL^{\frac{1}{2}})^T \Rightarrow K = BL^{\frac{1}{2}} \Rightarrow \hat{\Sigma}^m = B\varphi^m(L)B^T$$

$$= K\underbrace{L^{-\frac{1}{2}}\varphi^m(L)L^{-\frac{1}{2}}}_{\Delta^*}K^T$$

same worst case scenario but risk dominates JS. original minimax.

(3) other improvement: very tedious to look into

$$\varphi_i^S(L) = \varphi_i^m(L) - \frac{(L_i \log L_i)\tau(u)}{b_1 + u}$$

$u = \sum \log^2 L_i \quad b_1 > 144(p-2)^2 / 25(n+p-1)^2$

$\tau(u)$ function s.t. $\begin{cases} 0 < \tau(u) < 12(p-2)/5(k+p+1)^2 \\ \tau(u) \text{ non-decreasing in } u \quad E[\tau'(u)] < \infty \end{cases}$

Note: still need isotonizing procedure