Investigating the Unruh Effect at the Event Horizon of a Black Hole

A thesis submitted to Department of Physics and Electronics, CHRIST (Deemed to be University) in partial fulfillment of the requirements for the degree of Masters of Science in Physics

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Declaration

I,Tharakesh S, hereby declare that the project report titled "Investigating the Unruh Effect at the event horizon of a Black Hole", submitted to the Department of Physics & Electronics, CHRIST (Deemed to be University), Bengaluru, is an original work done by me under the guidance of Dr.Kenath Arun, Associate Professor, CHRIST (Deemed to be University), Bengaluru. I confirm that this project has not been the basis for the award of any degree, diploma, associateship, or any other similar title from any university or institution.

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CERTIFICATE

This is to certify that the Mini-project report submitted by Tharakesh S, 2447354, entitled "INVESTIGATING THE UNRUH EFFECT AT THE EVENT HORIZON OF A BLACK HOLE" in partial fulfillment of the requirements for the Degree of Master of Science in Physics is a record of original work carried out by him during the academic year 2025 to 2026 under my supervision. This mini-project report has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles. I hereby confirm the originality of the work and that there is no plagiarism in any part of the report.

Place:Bengaluru
Date: 17.10.2025

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Abstract

Based on the propositions of S. W. Hawking, black holes are known to emit radiation with a thermal spectrum. This phenomenon reveals that black holes can be treated as thermodynamic systems characterized by well-defined temperature and entropy. Interestingly, a similar effect arises for an accelerating observer in flat spacetime: a quantum field that appears as vacuum in a stationary Minkowski frame is perceived as a thermal bath in a uniformly accelerating frame, described by Rindler coordinates.This project aims to present the thermodynamic interpretation of black holes, discuss the laws governing their thermal emission, and derive the Unruh temperature from first principles in non-inertial (Rindler) frames. Although the underlying spacetime remains flat, the accelerating observer experiences an event horizon analogous to that of a black hole. By establishing this parallel, the work highlights the deep connection between Hawking radiation and the Unruh effect.

Contents

1 Introduction

The dynamics of Quantum field theory in curved spacetime, gives rise to particle creation. The QFT aspects of particle creation has been discussed for gravitational field and acceleration.

Hawking radiation is the thermal emission of particles from a Black hole proposed by S W Hawking[1] due to Quantum mechanical effects at the event horizon. Unruh in his work during 1976 [2], proposed that these Quantum mechanical effects can be experienced by an accelerating observer. The Quantum Field which is perceived as vacuum in stationary frame, is observed as thermal bath in the accelerating frame. This effect is called Unruh effect. A static detector at a distance from the black hole, perceives a radiation which is equivalent to a accelerated observer in Rindler spacetime.

In the context of accelerated frames,the validity of equivalence principle in the domain of Unruh effect became questionable. The reading [3] tries to provide the distinction between Unruh effect and Hawking radiation using a free falling observer. The work[4] primarily argues that the equivalence principle is violated because of the non-local entanglement existing between the particles in Quantum field, and indeed it is a surprise how the principle is restores at the horizon limit. It also emphasizes the choice of Vacuum and the metric which defines acceleration plays a crucial factor in defining the effect.

Thus, The aim of this project is to understand the relation between Unruh effect and Hawking radiation using first principles of non-inertial frames by introducing Rindler coordinates and thus derive Unruh temperature for that metric. This project explicitly does not deal with the concept of equivalence principle and how it is restored in the near horizon limit.

Objectives

- To study the Unruh effect at the event horizon of a Black hole
- To investigate the relation between Unruh effect and Hawking radiation

The report is structured in the following manner. In Section 2 the basic requisites of Quantum field theory and metrics would be addressed and details the basic operators and tools required. In Section 3 the concept of Black holes would be introduced and the thermodynamics of the Black hole would be detailed. With having introduced the

Thermodynamical equivalence for a black hole, the thermal emission of particles leading to Hawking radiation is detailed in Section 4. Then to address the equivalence between Unruh effect and Hawking radiation, we begin with deriving the coordinates of the Rindler metric, and proceeding on to use the Bogoliubov transformations to derive the Unruh temperature for that metric in Section 5

2 Quantum Field theory in curved spacetime

2.1 Bogoliubov transformations

This transformation relates the scalar field observed by different observers. This is also an effective diagonalization method used in quantum theory. The free scalar field of a vacuum could be expressed as

$$\phi = \sum_{k=0}^{\infty} \left(a_k u_k + a_k^{\dagger} u_k^* \right) \tag{1}$$

The field observed by any observer must be the same.Then the observer in a surved spacetime will see the field in a different basis.Thus we would have

$$\phi = \sum_{k=0}^{\infty} \left(b_k v_k + b_k^{\dagger} v_k^* \right) \tag{2}$$

The b_j and $b_j^{\dagger}$ modes are expressed as a linear combination of the a_j and $a_j^{\dagger}$. The positive k modes evolve differently from the -k modes, the space and time coordinate is different for the observers.We know that,

$$\langle u_i | u_j \rangle = \delta_{ij} \qquad \langle u_i^* | u_j^* \rangle = \delta_{ij} \qquad \langle u_i | u_j^* \rangle = 0$$

We can see that the transformation equations could be given by, the following

$$b_j = \sum_i \alpha_{ji} a_i + \beta_{ji} a_i^{\dagger} \tag{3}$$

and

$$b_j^{\dagger} = \sum_i \alpha_{ji}^* a_i^{\dagger} + \beta_{ji}^* a_i \tag{4}$$

The transformation matrices preserve symmetry and Unitarity. These conditions are a consequence of the commutation relations.

$$[b_i, b_j] = 0 \ [b_i, b_j^\dagger] = \delta_{ij}$$

Thus,the conditions are given by

$$\sum_k \alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk} = \delta_{ij} \tag{5}$$

$$\sum_k \alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk} = 0 \tag{6}$$

Thus, the creation and annihilation operators transform accordingly, from one coordinates to another. This transformation brings about the difference in the number of particles in the state

2.2 Particles in a state

The vacuum is the lowest possible energy state represented as $|0\rangle$. The important property of the vacuum state is the following:

$$a_k|0\rangle = 0$$

which clearly defines the statement in a mathematical form. The stationary observer who perceives a vacuum will have no particles associated with the vacuum. The number of particles in a state is given by the number operator which is given by $a_k^\dagger a_k$[1]. Clearly the expectation of number operator in vacuum is given by

$$\langle 0|a_k^\dagger a_k|0\rangle = 0 \tag{7}$$

This is because of the normal ordering. Now let's switch the frame of reference to the Rindler observer.He will tend to see the same vacuum, but the operators would undergo Bogoliubov transformation. Then from equation 3 and 4we know the transformation equations. Thus the number of particles would be given by

$$\langle 0|b_k^\dagger b_k|0\rangle \tag{8}$$

[1]The Number operator is normal ordered

3 Black holes

Black holes are massive structures,with a strong gravitational field which even light can't escape. These structures are formed due to the gravitational collapse of massive stars or neutron stars. The surface beyond which the light can't escape the gravitational field of a black hole is called the Event Horizon of a Black Hole.

3.1 Entropy of a Black hole

The Entropy of the Black hole is always understood in terms of its Area. If the Black hole is considered to be a thermodynamic system, then area of the Black hole denotes the Entropy. An irreversible process increases the mass of a Black hole whereas a reversible process leaves the mass unchanged. It is seen that the irreducible mass[2] of a Black hole is proportional to the square root of its area.

$$M_{ir} = \sqrt{\frac{A}{16\pi}} \tag{9}$$

Bekenstein[5] argues with the idea of information and Entropy. He considers the matter captured by a Black hole as loss of information. Since, information loss makes things more uncertain, it is considered to be increase in the entropy (i.e)

$$\Delta I = -\Delta S \tag{10}$$

Thus, we can realize how the system's area accounts for the entropy of a system. In fact, Bekenstein shows a result using the Black Hole variables[3] Charge(Q),Mass(M),angular momentum(L) and area(A)which is equivalent to the first law of Thermodynamics.

$$dM = \Theta dA + \vec{\Omega}.d\vec{L} + \Phi dQ \tag{11}$$

where,

$\Theta = \frac{1}{4}(r_+ - r_-)/A \quad : r_\pm = M \pm (M^2 - Q^2 - \frac{L^2}{M^2})^{1/2}$

$\vec{\Omega} = \frac{\vec{L}}{MA}$

$\Phi = Qr_+/A$

[2]The Mass of a black hole after possible extraction of energy

[3]For Kerr Black holes

This equation is equivalent to the first law of thermodynamics,

$$dU = TdS - PdV \tag{12}$$

with $dA \sim dS$ and $-(\vec{\Omega}.d\vec{L} + \Phi dQ)$ as work done on the system. These parameters Q,M and L in turn define the Black hole itself. No hair theorem states that a stable Black hole could be completely defined in terms of the 3 parameters Mass, Charge and Angular momentum of the Black hole.

3.2 Generalized second law

A particle loss into a black hole, causes a decrease in the entropy of the universe. This is a violation of the second law of thermodynamics. The entropy of the Universe should always increase in a thermodynamical process. This was initially resolved by Bekenstein by arguing that the collective entropy of the Black hole and rest of the Universe will increase. If the entropy of the Black hole is taken to be S_{BH} and the rest of the universe as S_c. Then S_U is given by

$$S_U = S_{BH} + S_c \geq 0 \tag{13}$$

Here,S_c decreases, but the black hole entropy increases. As we discussed earlier, now we have reasons to observe an increase in the area of the Black hole, signifying the increase in entropy. As expected, later Hawking proved that the Entropy of a Black hole was indeed proportional to the area of the horizon, thus leading to the Hawking-Bekenstein relation

$$S_{BH} = \frac{kc^3 A}{4G\hbar} \propto A \tag{14}$$

3.3 Thermodynamics Laws of a Black hole

We could see from the above arguments on how a Black hole could be considered as an equivalent of a thermodynamical system. These laws are in resemblance with the laws of thermodynamics.The work of Bardeen, Bekenstein and Hawking in 1973 formulated these laws [6][7]

First Law of Thermodynamics

Analogous to the first law (i.e) Conservation of energy, the following equation described the Black hole thermodynamics.

$$dM = \frac{\kappa}{8\pi G} dA + \vec{\Omega}.d\vec{L} + \Phi dQ\dagger \tag{15}$$

where κ is the surface gravity. In a generalized form, we have

$$\delta M = \frac{\kappa}{8\pi}\delta A + \vec{\Omega}_H.\delta\vec{L}_H + \int \tilde{\mu}\delta dN + \int \tilde{\theta}\delta dS + \int \Omega\delta dL \tag{16}$$

Comparing this with the first law of thermodynamics

$$dE = TdS + \sum_i X_i dx_i \tag{17}$$

where the latter term in the above equation corresponds to the generalized work contribution. Upon comparison we can, the surface gravity κ is the equivalent of temperature, and A the equivalent of entropy, and M being the internal energy of the system which perfectly makes sense

Zeroth Law of Thermodynamics

The Zeroth Law of thermodynamics state objects that are in thermal contact attain thermal equilibrium at a temperature T upon transfer of energy. Here, the surface gravity κ would be constant throughout the event horizon of a stationary black hole, making it analogous to the constancy of temperature of a system

Second Law of Thermodynamics

The second law of thermodynamics states that, in an irreversible process the change in entropy should always be greater than zero. Similarly, we find the change in area should always be greater than zero, in an irreversible process

$$\Delta A \geq 0 \tag{18}$$

$\Delta A = 0$ for a reversible process.

Third Law of thermodynamics

In the context of Black holes, surface gravity is the equivalent of temperature. It is impossible to attain zero surface gravity by a finite sequence of operations.

3.4 Thermal emission from a Black hole

Though we consider Black hole to be a thermodynamical system, it is not complete without discussing thermal emission. Though Bekenstein proposed that,Black hole is a thermodynamical system, his explanation ran into inconsistencies without the idea of emission.

Consider the gedanken experiment, Lets consider a black hole at temperature T_H and the surrounding matter to be at temperature T_M. If $T_H < T_M$, the Black hole gains mass and the GSL holds good. But if $T_H > T_M$, the law is violated as there won't be any matter flow from Black hole. But this inconsistency could be addressed in two ways. (i) The black hole is always at $0K$ temperature or (ii) There is thermal emission of particles from the interior of a Black hole.The former case makes S_{BH} infinite and thus meaningless. Therefore, it should be the latter idea that removes the inconsistency in the formalism.

Indeed, Hawking discussed this idea in his paper [8] *"Black Holes and thermodynamics"*.Incorporating the quantum field fluctuations in the Schwarzschild metric helped him reveal the understanding of the emission of particles from a black hole[9]. This radiation is known as the Hawking radiation. He went on to prove that the Black hole emits like a thermodynamic body with temperature

$$T_H = \frac{\hbar c^3}{8\pi G M k_b} \tag{19}$$

In terms of surface gravity,

$$T_H = \frac{\hbar \kappa}{2\pi k_b c} \quad : \kappa = \frac{c^4}{4GM} \tag{20}$$

This result holds good for uncharged, non-rotating Black holes. κ is the surface gravity of the Black hole

4 Hawking radiation

When it was realized that Black holes should thermally emit radiation, Hawking made an account of the detailed calculations for a Non-rotating, Uncharged Black hole whose metric is defined by Schwarzschild metric in his work "Particle creation by Black holes".[1]

4.1 Metric and coordinates

Although the word the metric and coordinates have been used before, this section formally addresses the concept of metric. The metric defines the curvature of a space time.
In the context of inertial frames, the spacetime is not curved (i.e) flat.The metric of the spacetime dictates the norm of a vector in the spacetime .It is like the measuring ruler of spacetime.

The flat spacetime is called the Minkowski spacetime. The metric of that spacetime is

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

The vector in a spacetime is expressed by the metric and the coordinates. In cartesian system, the vector of Minkowski is given by

$$x'^{\mu} = \eta_{\mu\nu} x^{\nu} \qquad : x^{\nu} = (ct, x, y, z) \tag{21}$$

By equivalence principle stated in General theory of relativity, an accelerating system is equivalent to a gravitational field, which in turn is effectively expressed in terms of a curved spacetime. In the context of non-rotating,uncharged Black holes, the solution is given by the Schwarzschild metric

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{22}$$

where $r_s = 2M$ is called the Schwarzschild radius which defines the event horizon of a black hole. It could also be seen that as $r \to 0$ becomes singular. For Charged and rotating black hole, Kerr metric is used.

4.2 Killing vector

The Killing vector is the vector in the spacetime along which the metric doesn't change. Mathematically, Killing vector satisfies the Lie derivative

$$\mathcal{L}_{\xi} g_{\mu\nu} = 0 \tag{23}$$

In terms of coordinates,

$$\nabla_{\nu}\xi^{\mu} + \nabla_{\mu}\xi^{\nu} = 0 \tag{24}$$

Killing vector field generates an Isometry. Noether's theorem states that every continuous symmetry gives rise to a conserved quantity. Timelike Killing vector implies Energy is conserved over time and spacelike Killing vector conserves momentum.

It is indeed possible to find some isometries without computing the partial derivatives. If the metric is not explicitly dependent on time and azimuth, then the Energy and Momentum could be conserved based on the existence of timelike and spacelike Killing vectors.

For such metrics ∂_t and ∂_ϕ are Killing vector field.

In Flat Spacetime, all translational and rotational symmetry denotes linear and angular momentum conservation. When we think of it, intuitively the gravitational field remains unchanged along Killing Vector.

For the metric defined in subsection 4.1, the Killing vectors could be classified based on their norm $||\xi||^2$

- if $||\xi||^2 > 0$ the Killing vector is said to be spacelike Killing vector
- if $||\xi||^2 = 0$ the Killing vector is said to be null Killing vector
- if $||\xi||^2 < 0$ the Killing vector is said to be timelike Killing vector

4.3 Thermal Radiation and mode mixing in Black holes

The Schwarzschild metric is used to express the spacetime curvature. The Schwarzschild metric does not have a global timelike Killing vector.In QFT, we could express the field in terms of the wave modes. This is possible when there exists a global timelike Killing vector. In curved spacetime like Schwarzschild metric, it is not possible to write the field globally using a linear combination of creation and annihilation operators. Rather we require the operators to be related through Bogoliubov transformation
Outside the Event Horizon (i.e) $r > 2M$ we have a timelike Killing vector.The energy is conserved outside the Horizon and the wave modes do not mix. But in the region $r < 2M$, we have a mode mixing leading to Vacuum fluctuations[4].

A free falling observer often does not observe a any mixing in the modes. The emission of particles from the Black Hole, results in a decrease of the area and increase in the temperature of the Black Hole. Classically the area of the event Horizon cannot decrease. This violation could be due to the negative energy flux into the Black Hole. Hawking poses some heuristic arguments to interpret the inflow of negative energy. he argues that just outside the event horizon, there will be virtual pairs of particles with positive and negative energy. The particle with negative energy crosses the horizon and the other one goes to infinity. The negative particle is in a region where it is classically forbidden and it tunnels through the horizon where the Killing vector is spacelike.They could be regarded as positive energy particles traveling through past directed worldlines in the region with a

[4]refer section 2 for further reading on Bogoliubov transformation

timelike Killing vector. Now it could be understood that decrease in the area is due to the violation of the weak energy condition.

5 Relativity in Non-inertial Frames

From the work of Einstein[10], it is emphasized that the event should be the same in all inertial frames of references. But how do we perceive the event in non-inertial or accelerating frame? The inertial coordinates are related by the Lorentz transformation. This transformation could be thought of as a rotational transformation in spacetime. In rotation, the norm of the vector in the Euclidean space is unchanged.

$$(dx)^2 + (dy)^2 + (dz)^2 = constant \tag{25}$$

But in Lorentz transformation, the norm is no longer defined by Euclidean space but by Minkowski space, which is given by

$$-c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 = constant^5 \tag{26}$$

The above-mentioned is an invariant quantity under the Lorentz transformation. Minkowski is the flat spacetime continuum is used to define an event occurring in the inertial frames. Let us consider a frame S, which is at rest and another frame S', which is moving at a constant velocity v with respect to S. An accelerating particle is observed in either of these frame. The S' momentarily coincides with the particle's rest frame,such that u'= 0. We consider this to be confined along 1 space direction, thus making it effectively 1+1 dimensional space. The inertial frames are connected via Lorentz transformations

$$x' = \gamma(x - vt) \qquad : \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{27}$$

$$t' = \gamma(t - \frac{v}{c^2}x) \tag{28}$$

Thus the coordinates are related. Differentiating equation 27 with respect to t'

$$u' = \gamma(u - v)\frac{dt}{dt'} \tag{29}$$

[5] The Minkowski metric considered here is of the sign convention $(-, +, +, +)$

And upon differentiating equation 28 with respect to t, we get

$$\frac{dt}{dt'} = \frac{1}{\gamma(1-\frac{uv}{c^2})} \tag{30}$$

Using this relation, we obtain,

$$u' = (u-v)/(1-\frac{uv}{c^2}) \tag{31}$$

Upon Differentiating equation 31 with respect to t', we obtain

$a' = \left(1/(1-\frac{uv}{c^2}) + \frac{v(u-v)}{c^2}(1/(1-\frac{uv}{c^2})^2)\right)\frac{du}{dt'}$

And when the particle meets with the instantaneous rest frame, u=v and u'=0 Thus, we have

$a' = (1/(1-\frac{u^2}{c^2}))\frac{du}{dt}\frac{dt}{dt'}$

Since $\frac{du}{dt} = a$ and $\frac{dt}{dt'}$ is known from equation 30, we can rewrite things as

$a' = (1/(1-\frac{u^2}{c^2}))^2\frac{a}{\gamma}$

Rearranging the above equation we arrive at,

$$a' = \gamma'^3 a : \gamma' = \frac{1}{\sqrt{1-u^2/c^2}} \tag{32}$$

The acceleration perceived by S' is a' and is just a parameter in the transformation.Rewriting equation 32 we see

$a'dt = du/(1-u^2/c^2)^{\frac{3}{2}}$

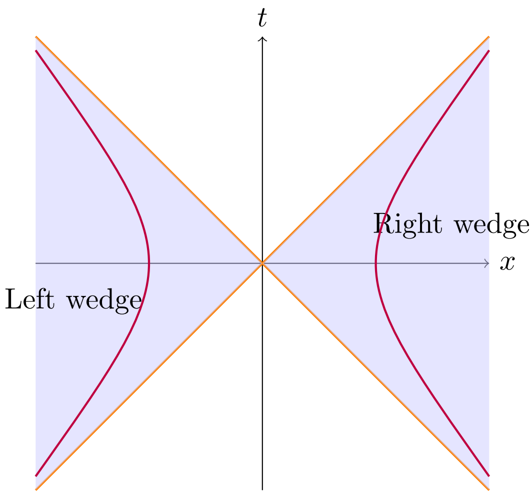
putting $u = c\sin(\theta)$ in the above equation and integrating we get, with the initial condition the particle was at rest initially. Then we have,

$$u(t) = a'ct/\sqrt{c^2+(a't)^2} \tag{33}$$

We integrate equation 33 to obtain the position of the particle, with the initial position as $x(0) = c^2/a$. Substituting $t = \frac{a'}{c}\tan\theta$ simplifies the algebra. Thus the position of the particle with the equation

$$x^2 - (ct)^2 = (\frac{c^2}{a'})^2 \tag{34}$$

Graphically, the accessible region of the coordinates was described in[11] as



5.1 Proper time

Proper time is the time experienced from a frame, in which the particular event is stationary. The proper time is expressed in special theory of relativity as

$$\tau = \int \frac{dt}{\gamma} \tag{35}$$

The value of γ is given by $1/\sqrt{1 - u^2/c^2}$. The value of u(t) is given from equation 32. Substituting and integrating it, we get

$$\tau(t) = \frac{c}{a'} \ln \left| \frac{a'}{c} t + \sqrt{1 + (\frac{a'}{c} t)^2} \right| \tag{36}$$

Expressing in τ in terms of t

$$ct(\tau) = \frac{c^2}{a'} \sinh(\frac{a'}{c}\tau) \tag{37}$$

Using equation 33,we can obtain $x(\tau)$ as

$$x(\tau) = \frac{c^2}{a'} \cosh(\frac{a'}{c}\tau) \tag{38}$$

5.2 Particle creation due to acceleration

As discussed before in the previous section, the operators in different metrics are related by Bogoliubov transformations. The state of the vacuum still remains same.Let the creation and annihilation operators be $a_j^\dagger$ and a_j for the stationary observer and the Rindler observer perceives operators $b_j^\dagger$ and b_j for the j^{th} wave mode. Thus the transformation equation is given by

$$b_j = \sum_i \alpha_{ji} a_i + \beta_{ji} a_i^\dagger \tag{39}$$

and

$$b_j^\dagger = \sum_i \alpha_{ji}^* a_i^\dagger + \beta_{ji}^* a_i \tag{40}$$

Using these transformation equations, we can calculate the expectation of the Number operator. The Number operator is defined as $a_j^\dagger a_j$. The no of particles in the stationary observer's frame is

$$\langle 0| a_j^\dagger a_j |0\rangle = 0 \tag{41}$$

This is very well known in the case of vacuum. Now let's try computing it for the accelerating observer. Then the equation goes by

$$\langle 0| b_j^\dagger b_j |0\rangle \tag{42}$$

Now using the Bogoliubov transformation, we get

$$N = \langle 0| \sum_i \sum_k (\alpha_{ji}^* a_i^\dagger + \beta_{ji}^* a_i)(\alpha_{jk} a_k + \beta_{jk} a_k^\dagger) |0\rangle \tag{43}$$

Since $a_k|0\rangle = 0$ and $\langle 0|a_k^\dagger = 0$ we have only the $a_k a_k^\dagger$ contribution. Thus,

$$N = \sum_i \sum_k \beta_{ji}^* \beta_{jk} \langle 0| a_i a_k^\dagger |0\rangle \tag{44}$$

We can use the commutator $[a_i, a_k^\dagger] = \delta_{ik}$ to solve the above expectation

$$N = \sum_i \sum_k \beta_{ji}^* \beta_{jk} \langle 0| \delta_{ik} |0\rangle \tag{45}$$

Using the properties of Kronecker delta , we obtain

$$N = \sum_i \beta_{ji}^* \beta_{ji} \tag{46}$$

Rewriting we have,

$$N = \sum_{i} |\beta_{ij}|^2 \tag{47}$$

Thus, we can see the number of particles is non-zero in the Rindler metric coordinates. For the same vacuum state used, we obtain a zero expectation in the stationary coordinates. This proves that for the same vacuum we can see a thermal bath in an accelerated frame

5.3 Creation of particles and temperature in the accelerated coordinates

The general expression of a non-interacting scalar field for 3-dimensions is given by

$$\phi = \int_{-\infty}^{\infty} d^3k \frac{1}{(2\pi)^{3/2}\sqrt{2E_k}} (e^{i(kx-E_k t)} a_k + e^{-i(kx-E_k t)} a_k^\dagger) \tag{48}$$

where $E_k = \sqrt{k^2 + m^2}$ Let us consider, that the observers are moving in a non-interacting massless scalar field in 1 dimension. Then, the field is written as

$$\phi = \int_{-\infty}^{+\infty} dk \frac{1}{\sqrt{4\pi|k|}} (e^{i(kx-|k|t)} a_k + e^{-i(kx-|k|t)} a_k^\dagger) \tag{49}$$

We can split the integral into positive k and negative k modes,thus making it

$$\phi = \int_{-\infty}^{0} dk \frac{1}{\sqrt{-4\pi k}} (e^{i(kx+kt)} a_k + e^{-i(kx+kt)} a_k^\dagger) + \int_{0}^{+\infty} dk \frac{1}{\sqrt{4\pi k}} (e^{i(kx-kt)} a_k + e^{-i(kx-kt)} a_k^\dagger) \tag{50}$$

Therefore the field experienced by both the stationary and Rindler observers can be written in their respective coordinates.Let (x,t) be the coordinates of the stationary observer and (x',t') be the coordinates of the accelerating observer. Then we can define a coordinate,

$$\begin{aligned} u &= t - x \qquad & u' &= t' - x' \\ v &= t + x \qquad & v' &= t' + x' \end{aligned}$$

Using the above set of equations and substituting it in equation 35, we would obtain the field experienced by the Stationary and Rindler Observer. The equation are as follows

$$\phi = \int_{-\infty}^{0} dk \frac{1}{\sqrt{-4\pi k}} (e^{ikv} a_k + e^{-ikv} a_k^\dagger) + \int_{0}^{+\infty} dk \frac{1}{\sqrt{4\pi k}} (e^{-iku} a_k + e^{iku} a_k^\dagger) \tag{51}$$

and

$$\phi = \int_{-\infty}^{0} dk' \frac{1}{\sqrt{-4\pi k'}} (e^{ik'v'} b_{k'} + e^{-ik'v'} b_{k'}^{\dagger}) + \int_{0}^{+\infty} dk' \frac{1}{\sqrt{4\pi k'}} (e^{-ik'u'} b_{k'} + e^{ik'u'} b_{k'}^{\dagger}) \quad (52)$$

Substituting $k \to -k$ and $k' \to -k'$, we can write the integral in terms of positive and negative k modes. Thus for the stationary and Rindler observer's metric are given as follows:

$$\phi = \int_{0}^{+\infty} dk \frac{1}{\sqrt{4\pi k}} (e^{-ikv} a_{-k} + e^{+ikv} a_{-k}^{\dagger} + e^{-iku} a_{k} + e^{iku} a_{k}^{\dagger}) \quad (53)$$

and

$$\phi = \int_{0}^{+\infty} dk' \frac{1}{\sqrt{4\pi k'}} (e^{-ik'v'} b_{-k'} + e^{+ik'v'} b_{-k'}^{\dagger} + e^{-ik'u'} b_{k}' + e^{ik'u'} b_{k'}^{\dagger}) \quad (54)$$

Now the field is written in terms of decoupled modes, with creation and annihilation for positive k and negative k modes. Independent of the basis chosen to represent the field, it should be same. Therefore the positive modes of the field could be equated for both the observers.

$$\int_{0}^{+\infty} dk \frac{1}{\sqrt{4\pi k}} (e^{-iku} a_{k} + e^{iku} a_{k}^{\dagger}) = \int_{0}^{+\infty} dk' \frac{1}{\sqrt{4\pi k}} (e^{-ik'u'} b_{k'} + e^{ik'u'} b_{k'}^{\dagger}) \quad (55)$$

Now taking the Inverse fourier transform for k', we obtain the RHS to be

$$= \int_{0}^{\infty} \int_{-\infty}^{\infty} du' dk' \frac{1}{\sqrt{4\pi k}} (e^{i(\omega - k')u'} b_{k'} + e^{i(\omega + k')u'} b_{k'}^{\dagger}) \quad (56)$$

The dirac delta function could be represented as

$$\delta(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} dx \quad (57)$$

Now making use of this, we can express the RHS as

$$\int_{0}^{\infty} dk' \frac{2\pi}{\sqrt{4\pi k}} \delta(\omega - k') b_{k'} + \delta(\omega + k') b_{k'}^{\dagger}) \quad (58)$$

Evaluating this we get,

$$= \sqrt{\frac{\pi}{\omega}} b_{\omega} \quad (59)$$

This is the result for $k > 0$ and for $k < 0$, we would have the creation operator coming out as the result

Now, lets compute the LHS

$$\int_0^{+\infty}\int_{-\infty}^{\infty} du'dk\frac{1}{\sqrt{4\pi k}}(e^{i(\omega u'-ku)}a_k + e^{i(\omega u'+ku)}a_k^\dagger) \tag{60}$$

let's express the integral

$$\int_{-\infty}^{\infty} du' e^{i(\omega u'-ku)} \tag{61}$$

as $f(\omega,-k)$ and the other one as $f(\omega,k)$. Then we can write the LHS as

$$= \int_0^{\infty} dk\frac{1}{\sqrt{4\pi k}}(f(\omega,-k)a_k + f(\omega,k)a_k^\dagger) \tag{62}$$

Thus, we can use the computed LHS and RHS to express b operator in terms of a and $a^\dagger$.Then we have the expression

$$b_\omega = \int_0^{\infty}\frac{1}{2\pi}\sqrt{\frac{\omega}{k}}f(\omega,-k)a_k dk + \int_0^{\infty}\frac{1}{2\pi}\sqrt{\frac{\omega}{k}}f(\omega,k)a_k^\dagger dk \tag{63}$$

Thus by the definition, we can see the nature of Bogoliubov transformations underlying in these equations. Then we have

$$\alpha_{\omega k} = \frac{1}{2\pi}\sqrt{\frac{\omega}{k}}f(\omega,-k) \tag{64}$$

$$\beta_{\omega k} = \frac{1}{2\pi}\sqrt{\frac{\omega}{k}}f(\omega,k) \tag{65}$$

The number of particles in the accelerated coordinate is given by

$$N = \int_0^{\infty} dk|\beta_{\omega k}|^2 \tag{66}$$

In order to evaluate this we require $f(\omega,k)$

$$f(\omega,k) = \int_{-\infty}^{\infty} du' e^{i(\omega u'+ku)} \tag{67}$$

From the above expression we see that both u' and u are involved, which is from the Rindler and the stationary observer frames respectively. From the equations 13 and 14, we note that

$$x = \tfrac{c}{a}\cosh(\tfrac{a}{c}\tau) \qquad t = \tfrac{c}{a}\sinh(\tfrac{a}{c}\tau)$$

Then $u = t - x$ would be given by

$$u = \frac{c}{a}(\sinh(\frac{a}{c}\tau) - \cosh(\frac{a}{c}\tau))$$

$$u = \frac{c}{a}(e^{\frac{a}{c}\tau} - e^{-\frac{a}{c}\tau} - e^{\frac{a}{c}\tau} - e^{-\frac{a}{c}\tau})/2$$

Thus,

$$u = \frac{-c}{a} e^{\frac{-a}{c}\tau} \tag{68}$$

which effectively can be written as

$$u = \frac{-c}{a} e^{\frac{-a}{c}u'} \tag{69}$$

Then the integral could be written as

$$f(\omega, -k) = \int_{-\infty}^{\infty} du' e^{i(\omega u' - k\frac{c}{a}e^{\frac{-a}{c}u'})} \tag{70}$$

Using $e^{\frac{-au'}{c}=p}$ and consider $-i\frac{\omega c}{a} = s$ and $-i\frac{kc}{a} = w$ variable substitution for integration we get the integral to be

$$f(\omega, -k) = \int_0^{\infty} \frac{c}{a} p^{s-1} e^{-wp} dp \tag{71}$$

The integration gives

$$f(\omega, -k) = \frac{c}{a} w^{-s} \Gamma(s) \tag{72}$$

since w and s are complex nos, we can use logarithm form to express it (i.e) $w^{-s} = e^{-s\ln(w)}$. Now we can express $\ln(w)$ as

$$\ln(w) = \ln\Big| -i\frac{kc}{a}\Big| = \ln|\frac{kc}{a}| - i\frac{\pi}{2} sgn(\frac{kc}{a}) \tag{73}$$

Thus, the function $f(\omega, k)$ is

$$f(\omega, -k) = \frac{c}{a} \exp\left[i\frac{\omega c}{a}\ln\Big|\frac{kc}{a}\Big| + \frac{\pi\omega c}{2a} sgn(\frac{kc}{a})\right]\Gamma(-i\frac{kc}{a}) \tag{74}$$

Now we have to compute $f(w, k)$. This can be done using the following relations

$$f(\omega, -k) = \exp(\frac{\omega\pi c}{a}) f(\omega, k)$$

$$f^*(\omega, -k) = f(-\omega, k)$$

The Number of particles is given by equation 66, where $\beta_{\omega k}$ is given by equation 65

$$N = \int_{-\infty}^{\infty} dk \frac{\omega}{4\pi^2 k} f^*(\omega, k) f(\omega, k) \tag{75}$$

This integral can be computed using the properties of Bogoliubov transformations

$$\alpha\alpha^\dagger - \beta\beta^\dagger = I$$

Thus,

$$\int_{-\infty}^{\infty} dk \frac{\sqrt{\omega\omega'}}{4\pi^2 k}(f(\omega,-k)f^*(\omega',-k) - f(\omega,k)f^*(\omega',k)) = \delta(\omega-\omega') \tag{76}$$

From the above defined properties of $f(\omega,-k)$ we have

$$[\exp(\tfrac{2\pi\omega c}{a}) - 1] \textstyle\int_{-\infty}^{\infty} dk \frac{\omega}{4\pi^2 k}(f(\omega,k)f^*(\omega',k)) = \delta(\omega-\omega')$$

Thus,

$$\int_{-\infty}^{\infty} dk \frac{\omega}{4\pi^2 k}(f(\omega,k)f^*(\omega',k)) = \delta(0)/(e^{\frac{2\pi\omega c}{a}} - 1) \tag{77}$$

Therefore, the expectation of number operator gives

$$N = \frac{\delta(0)}{(e^{\frac{2\pi\omega c}{a}} - 1)} \tag{78}$$

It is evident that the particles follow Bose-Einstein statistics. From the exponential term we can see,

$$\tfrac{\hbar\omega}{k_B T} = \tfrac{2\pi\omega c}{a}$$

Then we can write the temperature of the system to be

$$T = \frac{\hbar a}{2\pi k_B c} \tag{79}$$

6 Conclusions

It is to be noted that the principle of Hawking radiation and Unruh effect has been detailed throughout the text. The absence of a Global timelike Killing vector leads to a non local stress energy tensor, which in turn leads to particle creation in the accelerated systems. However, the temperature derived from Hawking radiation is given by equation 20 results in

$$T_H = \tfrac{\hbar\kappa}{2\pi k_b c}$$

This resembles the Unruh temperature derived in equation 79, with a being replaced by the surface gravity. For a Black Hole of order $\sim M_\odot$ we get the surface gravity to be around $\approx 1.518 \times 10^{13} ms^{-2}$. This would result in a Hawking temperature of $10^{-8}K$. Whereas for a mass of 0.5% of Earth's mass, we get the acceleration to be of 10^{21} orders of magnitude. This would result in a temperature of $4K$.

To detect a $4K$ temperature rise in the vacuum, purely due to acceleration, we require acceleration of order 10^{21} magnitudes. The particles emitted could be Fermions or Bosons, where equation 79 changes respectively as $e^{\frac{2\pi\omega c}{a}} \pm 1$.

This is purely due to mixing of modes in the metric as there does not exist a global timelike killing vector. Though we can address the correspondence between the Unruh effect and Hawking radiation, this work has not yet addressed the equivalence principle upon which Einstein's theory of relativity is built. Though current works suggest that the Unruh effect becomes Hawking radiation in the horizon limit, it is not yet clear on how things switch between coordinates.

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Appendix

Conditions satisfied under Bogoliubov transformation-Proof of Symmetry condition

The commutation has to be preserved under the transformation of coordinates. Thus, similar to a_j and $a_j^\dagger$, the operators b_j and $b_j^\dagger$ also satisfy

$$[b_i, b_j] = 0 \qquad [b_i, b_j^\dagger] = \delta_{ij}$$

Consider the first commutation.Now applying the transformation rules, we get

$$[b_i, b_j] = [\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger, \sum_l \alpha_{jl} a_l + \beta_{jl} a_l^\dagger] \tag{80}$$

$$= \left(\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger\right)\left(\sum_l \alpha_{jl} a_l + \beta_{jl} a_l^\dagger\right) - \left(\sum_l \alpha_{jl} a_l + \beta_{jl} a_l^\dagger\right)\left(\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger\right)$$

Since α commutes with itself, the contributions of creation and annihilation alone become zero, leaving behind the operation due to creation on annihilation and vice versa. Thus the above equation becomes,

$$\sum_k \sum_l \alpha_{ik} \beta_{jl} a_k a_l^\dagger - \beta_{jl} \alpha_{ik} a_l^\dagger a_k + \sum_k \sum_l \beta_{ik} \alpha_{jl} a_k^\dagger a_l - \alpha_{jl} \beta_{ik} a_l a_k^\dagger$$

This further leads to

$$\sum_k \sum_l \alpha_{ik} \beta_{jl} [a_k, a_l^\dagger] + \sum_k \sum_l \beta_{ik} \alpha_{jk} [a_k^\dagger, a_l] \tag{81}$$

But, $[a_k, a_l^\dagger] = \delta_{ij}$. Then we obtain

$$[b_i, b_j] = \sum_k \alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} \tag{82}$$

But the commutator has to be zero, thus proving the first condition

$$\sum_k \alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} = 0 \tag{83}$$

Thus the Symmetry condition has been proved.

Proof of Unitarity condition

Now, to prove the second condition we use the other commutator relation

$$[b_i, b_j^\dagger] = [\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger, \sum_l \alpha_{jl}^* a_l^\dagger + \beta_{jl}^* a_l] \tag{84}$$

Expanding it we obtain

$$\left(\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger\right)\left(\sum_l \alpha_{jl}^* a_l^\dagger + \beta_{jl}^* a_l\right) - \left(\sum_l \alpha_{jl}^* a_l^\dagger + \beta_{jl}^* a_l\right)\left(\sum_k \alpha_{ik} a_k + \beta_{ik} a_k^\dagger\right)$$

As argued before only the non-commuting terms remain, leading to

$$\sum_k \sum_l \alpha_{ik} \alpha_{jl}^* [a_k, a_l^\dagger] - \beta_{ik} \beta_{jl}^* [a_k^\dagger, a_l] \tag{85}$$

Now, based on our previous knowledge, we can reduce this down to

$$[b_i, b_j^\dagger] = \sum_k \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jl}^* \tag{86}$$

Since, we know that the RHS of the equation 86 should satisfy the commutation relation, we have

$$\sum_k \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jl}^* = \delta_{ij} \tag{87}$$

Thus, the unitarity condition has been proved using the commutation relations.An important note to understand is that, in case of fermions the anti-commutator will be preserved.

Representing the condition in matrix form

Symmetry condition

$$\sum_k \alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk} = 0$$

In matrix notation the symmetry condition is given by

$$\alpha \beta^T - \beta \alpha^T = 0 \tag{88}$$

Unitarity condition

$$\sum_k \alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jl}^* = \delta_{ij}$$

Writing the unitarity condition in matrix form

$$\alpha\alpha^{\dagger} - \beta\beta^{\dagger} = I \tag{89}$$