## 0.1 Implementation of Rontani expression

This function computes almost the exchange part in the anti-symmetrized Coulomb matrix element $\langle 12|V|34\rangle_{as}$.

Then while the complete anti-symmetrized Coulomb matrix element reads

$$\langle 12|V|34\rangle_{as} = \langle 12|V|34\rangle - \langle 12|V|43\rangle, \tag{1}$$

with

$$\langle 12|V|43\rangle = \delta_{m_{s1},m_{s4}}\ \delta_{m_{s2},m_{s3}}\ V_{n_1,m_{s1},n_2,m_{s2};n_3,m_{s3},n_4,m_{s4}} \tag{2}$$

then the function $Anisimovas(n_1, m_1, n_2, m_2, n_3, m_3, n_4, m_4)$ only computes $V_{n_1,m_1,n_2,m_2;n_3,m_3,n_4,m_4}$.

$$\begin{aligned} V_{n_1,m_1,n_2,m_2;n_3,m_3,n_4,m_4} &= \delta_{m_1+m_2,m_3+m_4} \sqrt{\left[\prod_{i=1}^{4} \frac{n_i!}{(n_i+|m_i|!)}\right]} \\ &\times \sum_{j_1=0,\ldots,j_4=0}^{n_1,\ldots,n_4} \left[ \frac{(-1)^{j_1+j_2+j_3+j_4}}{j_1!j_2!j_3!j_4!} \left[\prod_{k=1}^{4} \binom{n_k+|m_k|}{n_k-j_k}\right] \frac{1}{2^{\frac{G+1}{2}}} \right. \\ &\times \left. \sum_{l_1=0,\ldots,l_4=0}^{\gamma_1=0,\ldots,\gamma_4=0} \left( \delta_{l_1,l_2}\ \delta_{l_3,l_4}\ (-1)^{\gamma_2+\gamma_3-l_2-l_3} \left[\prod_{t=1}^{4} \binom{\gamma_t}{l_t}\right] \Gamma\left(1+\frac{\Lambda}{2}\right) \Gamma\left(\frac{G-\Lambda+1}{2}\right)\right)\right] \end{aligned} \tag{3}$$

where

$$\gamma_1 = j_1 + j_4 + \frac{|m_1|+m_1}{2} + \frac{|m_4|-m_4}{2}, \tag{4}$$

$$\gamma_2 = j_2 + j_3 + \frac{|m_2|+m_2}{2} + \frac{|m_3|-m_3}{2}, \tag{5}$$

$$\gamma_3 = j_3 + j_2 + \frac{|m_3|+m_3}{2} + \frac{|m_2|-m_2}{2}, \tag{6}$$

$$\gamma_4 = j_4 + j_1 + \frac{|m_4|+m_4}{2} + \frac{|m_1|-m_1}{2} \tag{7}$$

and

$$G = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \tag{8}$$

$$\Lambda = l_1 + l_2 + l_3 + l_4. \tag{9}$$

In the code *Anisimovas(...)* includes the following subfunctions:

- *minusPower(int k)* which computes $(-1)^k$
- *LogFac(int n)* which computes $log(n!)$
- *LogRatio1(int* $j_1$*,int* $j_2$*,int* $j_3$*,int* $j_4$*)* which computes the $log$ of $\frac{1}{j_1!j_2!j_3!j_4!}$
- *LogRatio2(int G)* which computes the $log$ of $\frac{1}{2^{\frac{G+1}{2}}}$
- *Product1 (int* $n_1$*,int* $m_1$*,int* $n_2$*,int* $m_2$*, int* $n_3$*,int* $m_3$*,int* $n_4$*,int* $m_4$*)* which computes the explicit (not the $log$) product $\sqrt{\left[\prod_{i=1}^{4} \frac{n_i!}{(n_i+|m_i|!)}\right]}$

- *LogProduct2(int $n_1$,int $m_1$,int $n_2$,int $m_2$, int $n_3$,int $m_3$,int $n_4$,int $m_4$, int $j_1$,int $j_2$,int $j_3$,int $j_4$ )* which computes the *log* of $\prod_{k=1}^{4} \binom{n_k + |m_k|}{n_k - j_k}$
- *LogProduct3(int $l_1$,int $l_2$,int $l_3$,int $l_4$, int $\gamma_1$,int $\gamma_2$,int $\gamma_3$,int $\gamma_4$)* which computes the *log* of $\prod_{t=1}^{4} \binom{\gamma_t}{l_t}$
- *lgamma(double x)* which computes the *log* $[\Gamma(x)]$