$k_{ab} = k_a^2 + k_b^2$

①

$\langle k_a l_a j_a k_b l_b j_b J T | V | k_c l_c j_c k_d l_d j_d J T \rangle$

$$= \int k^2 dk \int K^2 dK \quad \delta\left(k^2 + \tfrac{1}{4}K^2 - \tfrac{1}{2}k_{ab}\right)$$

$$\theta\left(1 - \frac{(k_a^2 - k^2 - \frac{1}{4}K^2)^2}{k^2K^2}\right) A\left(\frac{k_a^2 - k^2 - \frac{1}{4}K^2}{kK}\right)$$

$$\int k'^2 dk' \int K'^2 dK' \quad \delta\left(k'^2 + \tfrac{1}{4}K'^2 - \tfrac{1}{2}k_{cd}\right)$$

$$\theta\left(1 - \frac{(k_c^2 - k'^2 - \frac{1}{4}K'^2)^2}{k'^2K'^2}\right) A\left(\frac{k_c^2 - k'^2 - \frac{1}{4}K'^2}{k'K'}\right)$$

$$\langle k | V | k' \rangle \quad \delta(K, K')$$

$$= \int k^2 dk \int K^4 dk \quad \delta\left(k^2 + \tfrac{1}{4}K^2 - \tfrac{1}{2}k_{ab}\right)$$

$$\theta\left(1 - \frac{(k_a^2 - k^2 - \frac{1}{4}K^2)^2}{k^2K^2}\right) A\left(\frac{k_a^2 - k^2 - \frac{1}{4}K^2}{kK}\right)$$

$$\int k'^2 dk' \quad \delta\left(k'^2 + \tfrac{1}{4}K^2 - \tfrac{1}{2}k_{cd}\right)$$

$$\theta\left(1 - \frac{(k_c^2 - k'^2 - \frac{1}{4}K^2)^2}{k'^2K^2}\right) A\left(\frac{k_c^2 - k'^2 - \frac{1}{4}K^2}{k'K}\right)$$

$$\langle k | V | k' \rangle$$

$$K = 2\left(-k^2 + \tfrac{1}{2}(k_a^2 + k_b^2)\right)^{1/2}$$

$$k^2 \leq k_F^2 - \left(-k^2 + \tfrac{1}{2}(k_a^2 + k_b^2)\right)$$

②

$$s = \frac{1}{4}K^2 + k^2 - \frac{1}{2}k_{ab}$$

$$t = k'^2 + \frac{1}{4}K^2 - \frac{1}{2}k_{cd}$$

$$\frac{\partial s}{\partial k'} = 0 \quad , \quad \frac{\partial s}{\partial K} = \frac{1}{4}\cdot 2K = \frac{1}{2}K$$

$$\frac{\partial t}{\partial k'} = 2k', \quad \frac{\partial t}{\partial K} = \frac{1}{4}\cdot 2K = \frac{1}{2}K$$

$$\left| \frac{\partial s}{\partial k'}\cdot\frac{\partial t}{\partial K} - \frac{\partial s}{\partial K}\cdot\frac{\partial t}{\partial k'} \right|_{s=t=0} = \left| \frac{1}{2}K\cdot 2k' \right|_{s=t=0} = Kk'\Big|_{s=t=0}$$

$$\frac{1}{4}K^2 = s - k^2 + \frac{1}{2}k_{ab}$$

$$K_0 = 2\left(-k^2 + \frac{1}{2}k_{ab}\right)^{1/2}$$

$$k'^2 = t - \frac{1}{4}K^2 + \frac{1}{2}k_{cd} \overset{s=t=0}{=} t - \left(s - k^2 + \frac{1}{2}k_{ab}\right) + \frac{1}{2}k_{cd}$$

$$= k^2 - \frac{1}{2}(k_{ab} - k_{cd})$$

$$k'_0 = \sqrt{k^2 + \frac{1}{2}(k_{cd} - k_{ab})}$$

$$\langle -|V|- \rangle = \int dk\, k^2 K^3 k' \; \theta\left(1 - \frac{(k_a^2 - k^2 - \frac{1}{4}K^2)^2}{k^2K^2}\right)$$

$$\times A\left(\frac{k_a^2 - k^2 - \frac{1}{4}K^2}{kK}\right) \theta\left(1 - \frac{(k_c^2 - k'^2 - \frac{1}{4}K^2)^2}{k'^2K^2}\right)$$

$$\times A\left(\frac{k_c^2 - k'^2 - \frac{1}{4}K^2}{k'K}\right) \langle k|V|k' \rangle$$

③ $f(k) =$

$$1 - \frac{(k_a^2 - k^2 - \frac{1}{4}K^2)^2}{k^2 K^2}$$

$$= 1 - \frac{(k_a^2 - k^2 - (-k^2 + \frac{1}{2}k_{ab}))^2}{k^2 K^2}$$

$$= 1 - \frac{(k_a^2 - \frac{1}{2}(k_a^2 + k_b^2))^2}{k^2 4(-k^2 + \frac{1}{2}k_{ab})}$$

$$= 1 - \frac{1}{16}\frac{(k_a^2 - k_b^2)^2}{k^2(-k^2 + \frac{1}{2}k_{ab})}$$

$f'(k) \geq 0$, if $0 \leq k \leq \frac{1}{2}k_{ab}^{1/2}$

$f'(k) < 0$, if $k \geq \frac{1}{2}k_{ab}^{1/2}$

1. Assume $k_{min} = 0$:

$\lim_{k \to 0+} f(k) = -\infty < 0 \Rightarrow k_{min} = k_0,$

$f(k_0) = 0$

$$(k_a^2 - k_b^2)^2 = 16k^2(-k^2 + \tfrac{1}{2}k_{ab})$$

$\vdots$

$k_0 = \frac{1}{2}|k_a \pm k_b|$ We assume $k \geq 0$

Assume $k_{max} = \infty$:

$f(\infty) = 1 > 0 \Rightarrow k_{max} = \infty$

Not physically justifiable!

Assume $k_{max} = k_{max}$, and $f(k_{max}) > 0$

$\Rightarrow k_{max} = k_{max}$

④

We get

$$\langle - | V | - \rangle = \int_{\frac{1}{2}|k_a - k_b|}^{\frac{1}{2}k_{ab}^{1/2}} + \int_{\frac{1}{2}k_{ab}^{1/2}}^{k_{max}} dk\, k^2 K^3 k'$$

$$\times A\left(\frac{k_c^2 - k^2 - \frac{1}{4}K^2}{kK}\right) \theta\left(1 - \frac{(k_c^2 - k'^2 - \frac{1}{4}K^2)^2}{k'^2 K^2}\right)$$

$$A\left(\frac{k_c^2 - k'^2 - \frac{1}{4}K^2}{k'K}\right) \langle k | V | k' \rangle$$

$$g(k) = 1 - \frac{(k_c^2 - k'^2 - \frac{1}{4}K^2)^2}{k'^2 K^2}$$

$$= 1 - \frac{(k_c^2 - (k^2 - \frac{1}{2}(k_{ab} - k_{cd})) - (-k^2 + \frac{1}{2}k_{ab}))^2}{(k^2 - \frac{1}{2}(k_{ab} - k_{cd}))\, 4(-k^2 + \frac{1}{2}k_{ab})}$$

$$= 1 - \frac{(k_c^2 - \cancel{k^2} + \cancel{\frac{1}{2}k_{ab}} - \frac{1}{2}(k_c^2 + k_d^2) + \cancel{k^2} - \cancel{\frac{1}{2}k_{ab}})^2}{4(k^2 - \frac{1}{2}k_{ab} + \frac{1}{2}k_{cd})(-k^2 + \frac{1}{2}k_{ab})}$$

$$= 1 - \frac{1}{16}\frac{(k_c^2 - k_d^2)^2}{(k^2 - \frac{1}{2}k_{ab} + \frac{1}{2}k_{cd})(-k^2 + \frac{1}{2}k_{ab})}$$

$$g'(k) = \frac{1}{16}\frac{(k_c^2 - k_d^2)^2\, (-4k^3 + k(2k_{ab} - k_{cd}))}{((k^2 - \frac{1}{2}k_{ab} - \frac{1}{2}k_{cd})(-k^2 + \frac{1}{2}k_{ab}))^2}$$

$g'(k) \geq 0$, if $k \geq 0$

$$-4k^2 + 2k_{ab} - k_{cd} \geq 0$$

$$k^2 \leq \frac{1}{4}(2k_{ab} - k_{cd})$$

$$k \leq \frac{1}{2}(2k_{ab} - k_{cd})^{1/2}$$

(5)

$$\left(\tfrac{1}{2}|k_a-k_b|\right)^2 = \tfrac{1}{4}(k_{ab}-2k_ak_b)$$

$$g\left(\tfrac{1}{2}|k_a-k_b|\right) = 1$$

$$-\frac{1}{16}\frac{(k_c^2-k_d^2)^2}{\left(\frac{1}{4}(k_{ab}-2k_ak_b)-\frac{1}{2}k_{ab}+\frac{1}{2}k_{cd}\right)\left(-\frac{1}{4}(k_{ab}-2k_ak_b)+\frac{1}{4}k_{ab}\right)}$$

$$= 1$$

$$-\frac{1}{16}\frac{(k_c^2-k_d^2)^2}{\left(-\frac{1}{2}(k_a+k_b)^2+\frac{1}{2}k_{cd}\right)k_ak_b}$$

$g\left(\frac{1}{2}|k_a-k_b|\right) \geq 0$, if

$$\frac{1}{16}\frac{(k_c^2-k_d^2)^2}{\left(-\frac{1}{2}(k_a+k_b)^2+\frac{1}{2}k_{cd}\right)k_ak_b} \leq 1$$

?

$$(k_c^2-k_d^2)^2 \leq 8\left(-(k_a+k_b)^2+k_{cd}\right)k_ak_b$$

$$k_c^4+k_d^4-2k_c^2k_d^2 \leq 8(-k_a^2-k_b^2-2k_ak_b+k_c^2+k_d^2)k_ak_b$$

It is at least not fulfilled, if $(k_a+k_b)^2 > k_{cd}$,

i.e., if $k_a^2+k_b^2+2k_ak_b > k_c^2+k_d^2$

$$g\left(\tfrac{1}{2}k_{ab}^{1/2}\right) = 1$$

$$-\frac{1}{16}\frac{(k_c^2-k_d^2)^2}{\left(\frac{1}{4}k_{ab}-\frac{1}{2}k_{ab}+\frac{1}{2}k_{cd}\right)\left(-\frac{1}{4}k_{ab}+\frac{1}{2}k_{ab}\right)}$$

$$= 1-\frac{1}{16}\frac{(k_c^2-k_d^2)^2}{\left(-\frac{1}{4}k_{ab}+\frac{1}{2}k_{cd}\right)\frac{1}{4}k_{ab}}$$

$$= 1-\frac{1}{4}\frac{(k_c^2-k_d^2)^2}{(-k_{ab}+2k_{cd})k_{ab}}$$

$g\left(\frac{1}{2}k_{ab}^{1/2}\right) \geq 0$, if

$$\frac{1}{4}\frac{(k_c^2-k_d^2)^2}{(-k_{ab}+2k_{cd})k_{ab}} \leq 1$$

?

$$(k_c^2-k_d^2)^2 \leq 4(-k_{ab}+2k_{cd})k_{ab}$$

It is at least not fulfilled, if $k_{ab} > 2k_{cd}$.

(b)

$$g(k_{max}) = 1 - \frac{1}{16} \frac{(k_c^2 - k_d^2)^2}{(k_{max}^2 - \frac{1}{2} k_{ab} + \frac{1}{2} k_{cd})(-k_{max}^2 + \frac{1}{2} k_{ab})}$$

$$\boxed{\begin{aligned} \langle - | V | - \rangle = & \int_{?}^{?} dk \; k^2 K^3 k' \; A\left(\frac{k_a^2 - k^2 - \frac{1}{4}K^2}{kK} \right) \\ & \times A\left(\frac{k_c^2 - k'^2 - \frac{1}{4}K^2}{k'K} \right) \langle k | V | k' \rangle \end{aligned}}$$

⑦

$$g(k_0) = 0$$

$$(k_c^2 - k_d^2)^2 = 16(k^2 - \tfrac{1}{2}k_{ab} + \tfrac{1}{2}k_{cd})(-k^2 + \tfrac{1}{2}k_{ab})$$

$$16k^4 - 8k^2 k_{ab} - 8(k_{ab} - k_{cd})k^2 + 4k_{ab}(k_{ab} - k_{cd})$$
$$+ (k_c^2 - k_d^2)^2 = 0$$

$$k^2 = \frac{8(k_{ab} + k_{ab} - k_{cd})}{2 \cdot 16}$$
$$\pm \frac{1}{32}\left(64(2k_{ab} - k_{cd})^2 - 4 \cdot 16\left(4k_{ab}(k_{ab} - k_{cd}) + (k_c^2 + k_d^2)^2\right)\right)^{1/2}$$

$$= \tfrac{1}{4}(2k_{ab} - k_{cd}) \pm \tfrac{1}{4}\left(\cancel{4k_{ab}^2} + k_{cd}^2 - \cancel{4k_{ab}k_{cd}} - 4k_{ab}(\cancel{k_{ab}} - \cancel{k_{cd}})\right.$$
$$\left. - (k_c^4 + k_d^4 + 2k_c^2 k_d^2)\right)^{1/2}$$

$$= \tfrac{1}{4}(2k_{ab} - k_{cd}) \pm \tfrac{1}{4}\left(\cancel{k_{cd}^2 - k_{cd}^2}\right)^{1/2}$$

$$= \tfrac{1}{4}(2k_{ab} - k_{cd})$$

$$k_0 = \tfrac{1}{2}(2k_{ab} - k_{cd})^{1/2}$$

$\Rightarrow$ The integration limits are chosen among the following depending on if $g(k_i) \geq 0$ or not:

$$\begin{cases} \tfrac{1}{2}|k_a - k_b| \\ \tfrac{1}{2}(k_a^2 + k_b^2)^{1/2} \\ k_{max} \quad (\text{han vore } \tfrac{1}{2}(k_a + k_b)) \\ k_0 = \tfrac{1}{2}(2(k_a^2 + k_b^2) - k_c^2 - k_d^2)^{1/2} \end{cases}$$

In the program:

If $(p_a + p_b \geq p_c + p_d)$ $\quad g_{min} = \tfrac{1}{2}|p_c - p_d|$

If $(p_c + p_d > p_a + p_b)$ $\quad g_{min} = \left(\tfrac{1}{2}(p_c^2 + p_d^2) - \tfrac{1}{4}(p_a + p_b)^2\right)^{1/2}$

If $(|p_c - p_d| \geq |p_a - p_b|)$ $\quad g_{max} = \tfrac{1}{2}(p_c + p_d)$

If $(|p_a - p_b| > |p_c - p_d|)$ $\quad g_{max} = \left(\left|\tfrac{1}{2}(p_c^2 + p_d^2) - \tfrac{1}{4}(p_a - p_b)^2\right|\right)^{1/2}$