0.1 Properties of the Ginocchio potential

One of many challenges in modern quantum mechanics has been the problem of solving the Schrödinger equation for a system exactly. Whether we are able to solve the Schrödinger equation for an isolated system with n mutually interacting particles depends firstly on the number of interacting particles and secondly on the nature and specific form of the interactions between the particles. It turns out that solving a many body system where the number of particles exceeds $n = 3$ and where a particle only interacts with one other particle at the same time; i.e. a two body force, is an extremely difficult mathematical task.

For two and three body systems the problem has been solved analytically for specific two body interactions. Considering now the binary case, it turns out that the Schrödinger equation can only be solved for a limited class of potentials. It has been a challenge through the history of quantum mechanics to try and classify this class of exactly solvable potentials. The problem of identifying the general class of exactly solvable potentials has in fact been solved only recently by Natanzon [1]. Natanzon proposed that instead of trying to solve the Schrödinger differential equation, one should introduce a variable transformation such that the Schrödinger equation transforms into a hypergeometric differential equation. The advantage of this procedure is that the hypergeometric differrential equation has been studied and investigated for a long time, and its solutions are well known and given in terms of hypergeometric functions. It turned out that the general class of potentials which Natanzon identified included all of the well known solvable potentials that had been discovered in the early days. The Natanzon potential class contains for example the Eckart potential, the Pöschl-Teller potential, the Manning-Rosen potential and the Rosen-Morse potential. These potentials can be said to have been rediscovered as special members of the Natanzon class. It should be emphasized that when we say that the potential is exactly solvable in the above meaning, it does not imply that the Schrödinger equation is solvable for all angular momemtum states. In fact the Natanzon potential class is exactly solvable for angular momemtum zero. If we want to solve the Schrödinger equation for higher angular momentum, some sort of "approximation" of the angular momentum barrier must be done. This can be done by letting a $1/r^2$ singular like function at the origin have angular momentum dependence, and in this respect we say that the potential is solvable for all angular momentum states.

The general Ginocchio potential which will be presented here, is a potential with five parameters (including angular momentum) defining the shape and the nature of its eigen spectrum. Although the potential may look complicated in form, it belongs to the class of potentials which are exactly solvable. And the solutions can be expressed in a rather simple way. For a specific choice of potetential parameters the Ginocchio potential reduces to the Pöschl-Teller potential. So in a sense we can say that the Ginocchio potential is a generalisation of the PT potential, to include a larger amount of shapes. For specific choices of parameters the potential shapes can be said to lie between the Woods-Saxon/Square-well potential with a rather sharp surface, and on the other side potentias with a very diffuse surface, describing loosely bound binary systems.

0.2 Ginocchio potential characteristics

We start by writing the radial Schrödinger equation with correct physical dimensions

$$\left[-\frac{\hbar^2}{2}\frac{d}{dR}\frac{1}{M}\frac{d}{dR}+\frac{\hbar^2}{2}\frac{1}{M}\frac{l(l+1)}{R^2}+W_l(R)\right]\psi_{nl}(R)=E_{nl}\psi_{nl}(R) \tag{1}$$

where the angular momentum dependent potential $W_l(R)$ is written

$$W_l(R)=Vu_l(r) \tag{2}$$

where $u_l(r)$ is the dimensionless Ginocchio potential, and V is a potential strength parameter in units of $[MeV]$. We define a dimension scale in units of $[fm^{-1}]$

$$s=\left[\frac{2MV}{\hbar^2}\right]^{1/2} \tag{3}$$

and introduce the dimensionless coordinate

$$r=Rs \tag{4}$$

The dimensionless effective mass

$$\mu(r)=M(r)/m \tag{5}$$

and the dimensionless energy

$$\varepsilon_{nl}=E_{nl}/V \tag{6}$$

We will study cases for constant effective mass $\mu(r)$, and the corresponding dimensionless equation, for $\psi(R)=s^{1/2}\Phi(r)$, is then given as

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+u_l(r)\right]\Phi_{nl}(r)=\frac{E_{nl}}{V}\Phi_{nl}(r)=\varepsilon_{nl}\Phi_{nl}(r) \tag{7}$$

where u_l is the Ginocchio potential with constant mass. Following [3], the original potential is given in two parts

$$u_l(r)=\frac{v_l(r)}{\mu(r)}+\frac{c_l(r)}{\mu(r)} \tag{8}$$

where the first part is given by

$$\begin{aligned}v_l(r)=&-\lambda^2\nu_l(\nu_l+1)(1-y^2)+\left(\frac{1-\lambda^2}{4}\right)(1-y^2)(2-(7-\lambda^2)y^2+5(1-\lambda^2)y^4)\\&+\frac{a}{\mu^2}(1-y^2)(1-(1-\lambda^2)y^2)[1-a+(a(4-3\lambda^2)-3(2-\lambda^2))y^2\\&+5(1-\lambda^2)(1-a)y^4+2a(1-\lambda^2)y^6]\end{aligned} \tag{9}$$

and the second part by

$$c_l(r) = \left[\left(\alpha_l^2 - \frac{1}{4} \right) \left(\frac{1-y^2}{y^2} \right) (1 + (\lambda^2 - 1)y^2) - \frac{l(l+1)}{r^2} \right] \tag{10}$$

The parameters that determine the form of the potential will be given explicit meaning and definition according to Ginocchio. ν_l is the potential depth parameter, and is the same as the parameter occuring in the PT-potential. It determines the number of bound states. λ is a shape parameter, and determines the nature of the eigenspectrum, it has been shown in [3] that the number of bound states is independent of λ. a is the effective mass parameter, and also affects the nature of the eigenspectrum. α_l is called the centrifugal parameter, and determines the behaviour of the potential at the origin. $\mu(r)$ is the effective mass and depends in general on the radius of the system being studied. From the expression for the potential we see that the angular dependent potential $u_l(r)$ is expressed in the coordinate y. Following [3] this coordinate is defined implicitly as

$$r = \frac{1}{\lambda^2}[arc\tanh(y) + [\lambda^2 - 1]^{1/2} arc\tan([\lambda^2 - 1]^{1/2} y)] \tag{11}$$

where r is the dimensionless radial coordinate and is defined on the positive real axis. When r runs from zero to infinity, the variable y runs through the domain $[0, 1)$. It is easy to see that $r = 0$ is a one to one mapping onto $y = 0$, and when r grows very large y approaches one. The behaviour of y at the boundary as the radial coordinate approaches zero or infinity is given by

$$y \longrightarrow r, \text{ as } r \longrightarrow 0 \tag{12}$$

$$y \longrightarrow 1 - 2e^{-2\lambda^2(r-r_0)}, \text{ as } r \longrightarrow \infty \tag{13}$$

where

$$r_0 = \frac{[\lambda^2 - 1]^{1/2}}{\lambda^2} arc\tan([\lambda^2 - 1]^{1/2}) \tag{14}$$

The effective mass $\mu(r)$ is defined by the radial dependent coordinate y in the way

$$\mu(r) = 1 - a + ay^2 \tag{15}$$

with the mass parameter a taking the values

$$0 \leq a < 1 \tag{16}$$

We will restrict the discussion to a specific choice of parameters. This specific choice of parameters will simplify the potential and the eigensolutions a great deal. If we consider *the Ginocchio potential with constant mass, i.e.* $a = 0$, and assume no angular momentum dependence in the depth parameters ν, we see that the first part of the potential $v_l(r) = v(r)$ simplifies to

$$v(r) = -\lambda^2 \nu(\nu + 1)(1 - y^2) + \left(\frac{1 - \lambda^2}{4} \right) (1 - y^2)(2 - (7 - \lambda^2)y^2 + 5(1 - \lambda^2)y^4) \tag{17}$$

while the second angular momentum dependent part $c_l(r)$ remains in its original form.

If we consider constant effective mass $\mu(r)(a = 0)$ and a shape parameter λ equal to unity, it is easy to show that we get the general PT-potential. By rewriting the potential (9) with the parameters chosen above we get the first part of the potential

$$v(r) = -\nu(\nu+1)(1-y^2) \tag{18}$$

which is the angular momentum independent part of the potential, while the second part (10) is given by

$$c_l = l(l+1)\left[\left(\frac{1-y^2}{y^2}\right) - \frac{1}{r^2}\right] \tag{19}$$

which contains the angular momentum dependence. The y coordinate is related to the radial coordinate for this choice of potential parameters by (see 11)

$$r = arc\tanh(y) \tag{20}$$

By transforming from the y coordinate to the r coordinate we get the potential expressed in the coordinate r. We do this by using the relation

$$1 - y^2 = 1 - \tanh^2(r) = \frac{1}{\cosh^2(r)} \tag{21}$$

By substituting this into (18) and (19), we get for the first potential

$$v(r) = -\frac{\nu(\nu+1)}{\cosh^2(r)} \tag{22}$$

which we immediately recognize as the modified PT-potential, and for the second potential (19) we get

$$c_l(r) = l(l+1)\left[\frac{1}{\sinh^2(r)} - \frac{1}{r^2}\right] \tag{23}$$

Fig(1) shows the Ginocchio potential with constant mass and $\lambda = 1, 3/2, 2$ and with depth parameter $\nu = 2$. The centrifugal potential is not included.

0.3 Analytic bound state solutions

The general bound state solutions to the Schrödinger equation (7) with constant mass are given by [3] [1]

$$\Phi_{nl}(r) = N(n,l)\frac{1}{2^{1/4}}[1+\lambda^2-(1-\lambda^2)x]^{1/4} \times \\ \left(\frac{1+x}{2}\right)^{\beta_{nl}/2}\left(\frac{1-x}{2}\right)^{(\alpha_l+1/2)/2} P_n^{(\alpha_l,\beta_{nl})}(x) \tag{24}$$

[1] Notice printing errrors in Ginocchio's formula for the wave functions.

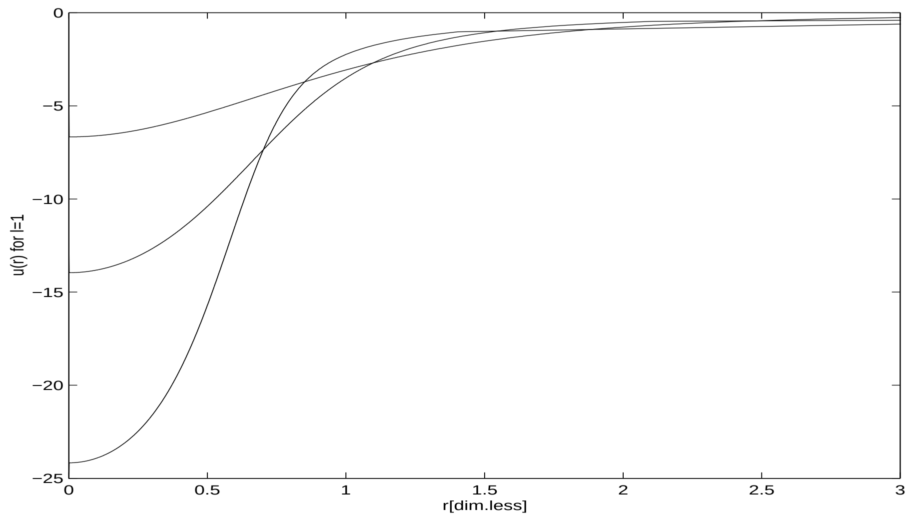

Figure 1: Plot of the Ginocchio potential $u_l(r)$ for angular momentum l=1 and with the choice of parameters $\nu = 2$, and $a = 0$. The deepest potential corresponds to $\lambda = 2$, the next potential $\lambda = 3/2$ and the shallow potential corresponds to $\lambda = 1$

where

$$x = \frac{1-(1+\lambda^2)y^2}{1-(1-\lambda^2)y^2} \tag{25}$$

which ranges through

$$1 \geq x > -1 \tag{26}$$

as

$$0 \leq y < 1 \tag{27}$$

The corresponding dimensionless eigenenergies are given by

$$\varepsilon_{nl} = -\lambda^4 \beta_{nl}^2 \tag{28}$$

where

$$\beta_{nl} = \frac{[(2n+\alpha_l+1)^2(1-\lambda^2)+\lambda^2(\nu+1/2)^2]^{1/2}-(2n+\alpha_l+1)}{\lambda^2} \tag{29}$$

The normalization constant is given by [3]

$$N(n,l) = \left[\frac{2\lambda^2 n!\Gamma(\alpha_l+\beta_{nl}+n+1)\beta_{n,l}(\alpha_l+\beta_{nl}+2n+1)}{\Gamma(\alpha_l+n+1)\Gamma(\beta_{nl}+n+1)(\beta_{nl}\lambda^2+\alpha_l+2n+1)}\right]^{1/2} \tag{30}$$

The number of bound states for given angular momentum l and potential depth parameter ν is given by

$$n = 0,1,2,...,\left\{\frac{\nu-l-1}{2}\right\} \tag{31}$$

whrere t means the largest integer smaller than t. The eigenfunctions are orthonormal for different nodal number n

$$\int_0^\infty dr\ \Phi^*_{nl}(r)\Phi_{n'l}(r) = \delta_{n,n'} \tag{32}$$

0.4 Scattering matrix and poles

Studying the positive energy regime, exact scattering solutions with outgoing boundary conditions can be derived, by analytic continuing the parameter β_{nl} to complex values. The scattering solutions for the Ginocchio potential with outgoing and incoming spherical waves asymptotically, and with constant mass are given

$$\Phi^{\pm}_{nl}(r) = [1+\lambda^2-(1-\lambda^2)x]^{1/4}\times \tag{33}$$

$$\left(\frac{1+x}{2}\right)^{\tilde\beta/2}\left(\frac{1-x}{2}\right)^{(\alpha_l+1/2)/2}\times \tag{34}$$

$${}_2F_1\left(\frac{\alpha_l+1\pm\tilde\beta+\tilde{nu}}{2},\frac{\alpha_l+1\pm\tilde\beta-\tilde{nu}}{2};1\pm\tilde\beta;\frac{1+x}{2}\right) \tag{35}$$

where

$$\tilde\beta = -ik/\lambda^2 \tag{36}$$

From the partial wave analysis we know that the l'th partial wave will in the asymptotic region have the form

$$\Phi_{nl}(r) \longrightarrow C(k,l)[e^{-i(kr-l\pi/2)} - S_l(k)e^{i(kr-l\pi/2)}],\ \text{ as } r\rightarrow\infty \tag{37}$$

Since the exact continuum solutions for the Ginocchio potential are given, the scattering matrix can be derived by investigating how the exact scattering solutions behave asymptotically. The scattering matrix is given by

$$S_l(k) = (-)^{l+1}e^{-2ikr_1}\frac{\Gamma(-\tilde\beta)\Gamma(\frac{\alpha_l+1+\tilde\beta+\tilde\nu}{2})\Gamma(\frac{\alpha_l+1+\tilde\beta-\tilde\nu}{2})}{\Gamma(\tilde\beta)\Gamma(\frac{\alpha_l+1-\tilde\beta+\tilde\nu}{2})\Gamma(\frac{\alpha_l+1-\tilde\beta-\tilde\nu}{2})} \tag{38}$$

where

$$\tilde\nu = \left[(\nu+1/2)^2+\tilde\beta^2(1-\lambda^2)\right]^{1/2} \tag{39}$$

and

$$r_1 = r_0 - (1/\lambda^2)\ln(\lambda/2) \tag{40}$$

r_0 is given by (14). The poles of the scattering matrix are easily found by noting that the Gamma function $\Gamma(z)$ has simple poles for $z=-n, n=0,1,2,....$ The poles for bound, virtual and resonant states are then given by

$$k_{n,l} = i\{\pm\left[(2n+\alpha_l+1)^2+\lambda^2((v+1/2)^2-(2n+\alpha_l+1)^2)\right]^{1/2}-(2n+\alpha_l+1)\} \tag{41}$$

In order for the potential to support resonant states we must have

$$\lambda^2 > 2 \tag{42}$$

If this condition is fulfilled, resonances will occur for integers n such that

$$n > \frac{1}{2}\left\{\lambda\left[\frac{1}{\lambda^2-2}\right]^{1/2}(\nu+1/2)-\alpha_l-1\right\}$$

0.5 Transition and reaction matrices for the Ginocchio potential

One of the advantages of having analytical solutions of a problem, is that we then have an opportunity to test and improve numerical procedures and methods. In this section we will solve for the Reaction and Transition matrices by different numercial procedures, and compare with the analytical results we have available. The T-matrix is defined as

$$T(\mathbf{k},\mathbf{k}') \equiv \langle\mathbf{k}|T|\mathbf{k}'\rangle = \langle\phi_{\mathbf{k}}|V|\psi^+_{\mathbf{k}'}\rangle \equiv \int d^3r \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}} V(\mathbf{r})\psi^+_{\mathbf{k}'}(\mathbf{r}) \tag{43}$$

By decomposing the T-matrix into partial waves, we obtain the 1-dimensional Lippmann-Scwinger equation

$$T_l(k,k') = V_l(k,k') + \frac{2}{\pi}\int_0^\infty \frac{dq q^2 V_l(k,q)T_l(q,k')}{E-E(q)+i\epsilon} \tag{44}$$

This is equation is a Fredholm integral equation of the 2'd type, with a singular kernel $K(q,k)$

$$K(q,k) = \frac{q^2 V_l(k,q)}{E-E(q)+i\epsilon} \tag{45}$$

There exists various types of numerical methods for solutions of these types of integral equations. Solving singular integrals can either be done by Cauchy's Residue theorem, where we integrate over a closed contour enclosing the poles, or by the Cauchy *principal-value prescription* where we integrate up to - but not through - them. If we employ the *principal-value prescription* we get for (44)

$$T_l(k,k') = V_l(k,k') + \frac{2}{\pi}P.V.\int_0^\infty \frac{dq q^2 V_l(k,q)T_l(q,k')}{E-E(q)} - 2ik_0^2 V_l(k,k_0)T_l(k_0,k') \tag{46}$$

The reaction matrix is just the principal value part of the transition matrix

$$R_l(k,k') = V_l(k,k') + \frac{2}{\pi}P.V.\int_0^\infty \frac{dq q^2 V_l(k,q)R_l(q,k')}{E-E(q)} \tag{47}$$

where

$$E = \hbar^2 k_0^2/2\mu \tag{48}$$

$$E(q) = \hbar^2 q^2/2\mu \tag{49}$$

We choose to work in dimensionless coordinates defined in the previous sections for simplicity. The dimensionless bound state energies for the Ginocchio potential are given

$$\varepsilon_{nl} = -\lambda^4 \beta_{nl}^2 = k_{n,l}^2 \tag{50}$$

where $k_{n,l}$ is the dimensionless momentum (or wavenumber). For the positive continuum, $\beta_{n,l}$ is analytic continued to complex values

$$\varepsilon = -\lambda^4 \tilde{\beta}^2 = k^2 \tag{51}$$

where $k \geq 0$ and $\tilde{\beta} = -ik/\lambda^2$. The dimensionless Lippmann-Schwinger equation for the R-matrix is then

$$R_l(k,k') = u_l(k,k') + \frac{2}{\pi} P.V. \int_0^\infty \frac{dq q^2 u_l(k,q) R_l(q,k')}{k_0^2 - q^2} \tag{52}$$

where $u_l(k,k')$ is the dimensionless Ginocchio potential (8) in momentum representation. Before we discuss numerical methods for solving (52), we will need to obtain the potential in momentum representation. For a local potential, the potential in k space is given by a Fourier - Bessel transform

$$u_l(k,k') = \int_0^\infty dr\, r^2 j_l(kr) u_l(r) j_l(k'r) \tag{53}$$

where $j_l(kr)$ is the spherical Besselfunction, defined in terms of ordinary Bessel functions of the first kind by

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z) \tag{54}$$

The Bessel functions of order equal to an integer plus one-half are given by the expression ([4]

$$J_{n+1/2}(z) = (-1)^n z^{1/2} \sqrt{\frac{2}{\pi}} \frac{d^n}{dz^n} \left(\frac{\sin(z)}{z} \right) \tag{55}$$

It turns out that for the case $\lambda = 1$ and $l = 0$, (53) can be evaluated exactly. In this case the Ginocchio potential reduces to the modified PT potential (22). The Fourier-Bessel transform of (22)for angular momentum zero is given by

$$u(k,q) = -\frac{\nu(\nu+1)}{kq} \int_0^\infty dr \frac{\sin(kr)\sin(qr)}{\cosh^2(r)} \tag{56}$$

$$= \frac{\pi\, v\, (1+v) \left(q \cosh(\frac{\pi q}{2}) \sinh(\frac{k\pi}{2}) - k \cosh(\frac{k\pi}{2}) \sinh(\frac{\pi q}{2}) \right)}{kq(\cosh(k\,\pi) - \cosh(\pi\, q))} \tag{57}$$

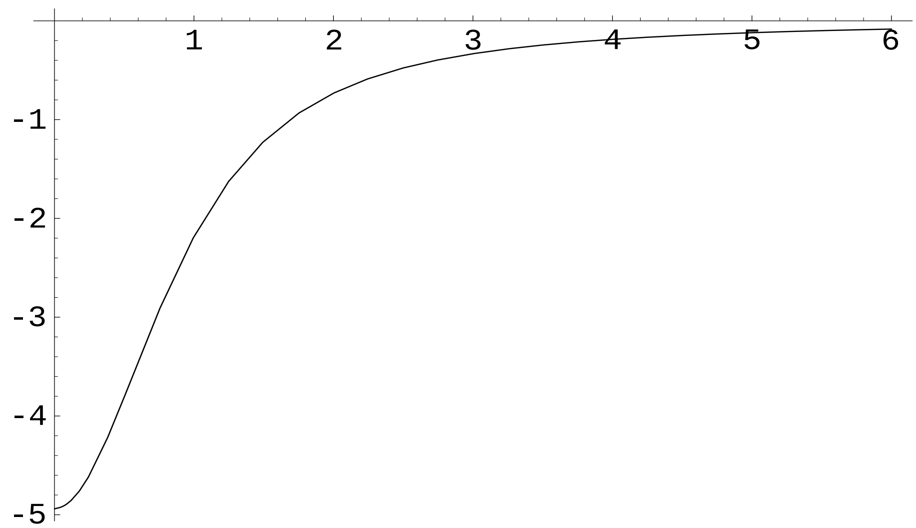

Figure 2: Plot of the diagonal Ginocchio potential in momentum representation fo$\lambda = 1, l = 0$ and potential depth $\nu = 2$

See figure (2) for a plot of the diagonal part of the PT potential in momentum representation. In the other cases the Fourier - Bessel transform has to be evaluated by some numerical integration routine. Since the potential (8) is defined in terms of y on the interval $[0, 1)$, and only implicitly in terms of the radial coordinate r, see (11), at first sight it seems as good idea to either try and solve explictly for y in terms of r or solvefor the roots of the equation for given r numerically. Both of these procedures fails. To solve for y in terms of r seems like an impossible task. To solve for the roots for given radii r does not work either due to limited numerical precision. For example $r = 5$ could correspond to $y = 0.99999999999999231...$ See equation (12) for the behaviour of y as r approaches zero or infinity. It seems like transforming the integral (53) into an integral over the variable y is the only reasonable choice. The differential dr can be shown to be given by

$$dr = \frac{dy}{(1-y^2)(1+y^2(\lambda^2-1))} \tag{58}$$

notice that the term $1/(1-y^2)$ get cancelled by the potential (8). The integral (53), with $\tilde{u}(r) = u(r)/(1-y^2)$, then reads

$$u_l(k,k') = \int_0^1 \frac{dy}{(1+y^2(\lambda^2-1))}(r(y))^2 j_l(kr(y))\tilde{u}_l(y) j_l(k'r(y)) \tag{59}$$

which is symmetric in the exchange of variables $u_l(k,k') = u_l(k',k)$. It can also be shown that the integrand in (59) is even in y,which allows us to extend the integration limits from $[0,1)$ to $(-1,1)$ and multiply by $1/2$. The integrand is then singular in the limit $y \to -1$ and $y \to 1$. If we are able to separate a factor $(1-y^2)^{-1/2}$ from the integrand,

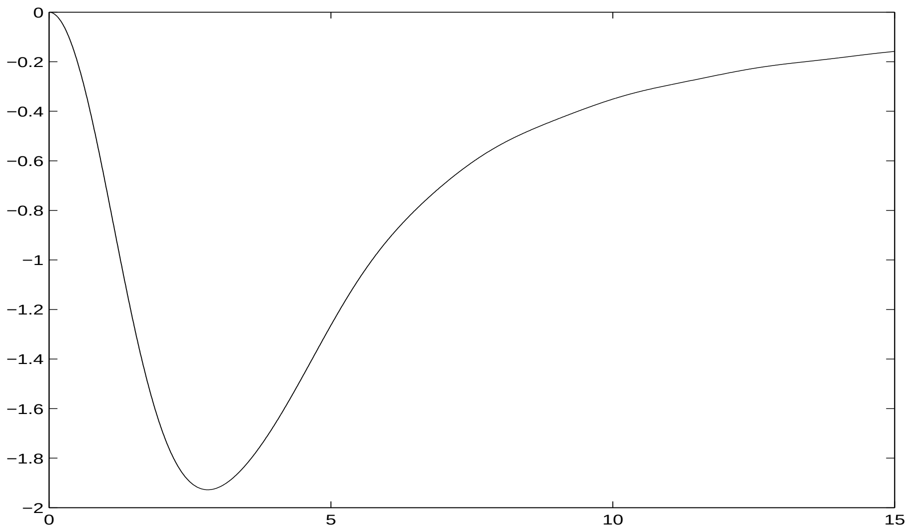

Figure 3: Plot of the diagonal Ginocchio potential in momentum representation fo$\lambda = 2, l = 1$ and potential depth $\nu = 5$. The calculation has been performed with Gauss-Legendre quadrature with 30 meshpoints and weights over the interval (0,1)

this would suggest a Gaussian quadrature rule of the Chebyshev type, when evaluating the integral. But in fact the integrand gets even more singular when separating out this factor. And when comparing Gauss-Legendre and Gauss-Chebyshev quadrature, the Gauss-Legendre quadrature converges more rapidly. See figure(3) for a plot of the diagonal part of the Ginocchio potential in momentum representation, calculated with Gauss-Legendre quadrature.

Turning now to the numerical evaluation of the R-matrix for the Ginocchio potential. We first have to deal with the singular kernel, this is dealt with by using a subraction trick

$$P.V. \int_0^\infty dk \, \frac{f(k)}{k^2 - k_0^2} = \int_0^\infty dk \, \frac{(f(k) - f(k_0))}{k^2 - k_0^2} \tag{60}$$

Using this, we can rewrite the R-matrix as

$$R_l(k, k') = u_l(k, k') + \frac{2}{\pi} \int_0^\infty \frac{dq q^2 u_l(k, q) R_l(q, k') - k_0^2 u_l(k, k_0) R(k_0, k')}{k_0^2 - q^2} \tag{61}$$

This integral can be converted into a set of linear equations by approximating the integral as a sum over N Gaussian quadrature points $(k_j; j = 1, ...N)$, each weighted by w_i.

$$R_l(k, k') = u_l(k, k') + \frac{2}{\pi} \sum_{j=1}^{N} \frac{w_j k_j^2 u_l(k, k_j) R_l(k_j, k')}{(k_0^2 - k_j^2)} - \frac{2}{\pi} k_0^2 u_l(k, k0) R(k_0, k') \sum_{n=1}^{N} \frac{w_n}{(k_0^2 - k_n^2)} \tag{62}$$

Solving this for the on-shjell point k_0, requires solving it for all k_j. Thus we have a $(N+1) \times (N+1)$ dimensional problem. We define the matrix A by

$$A_{i,j} = \delta_{i,j} - u_l(k_i, k_j) u_j \tag{63}$$

where

$$u_j = \frac{2}{\pi} \frac{w_j k_j^2}{(k_0^2 - k_j^2)} \quad , j = 1, ..N \tag{64}$$

and

$$u_{N+1} = -\frac{2}{\pi} \sum_{j=1}^{N} \frac{w_j k_0^2}{(k_0^2 - k_j^2)} \tag{65}$$

With the matrix A we obtain the matrix equation

$$A_{i,l} R_{l,j} = u_{i,j} \tag{66}$$

by inverting the matrix A we have solved the reaction matrix R, and the on-shjell r-matrix - elements are given by $R(k_0, k_0) = R(k_{N+1}, k_{N+1})$. When scattering from the Ginocchio potential the phaseshift in the outgoing wave is given by the relation

$$R_l(k_0, k_0) = -\frac{\tan(\delta_l)}{k_0} \tag{67}$$

We can now compare the numerical evaluation of the phaseshift with the analytical result for the phaseshift. The analytical phaseshifts are obtained by the the definition of the scattering matrix (38)

$$S_l(k) = e^{2i\delta_l} \tag{68}$$

and the phase shifts are then given by

$$\delta_l(k) = -\frac{i}{2} \ln e^{2i\delta_l} \tag{69}$$

Figure (4) shows a plot of the exact phaseshift for $\lambda = 1, \nu = 2$ and angular momentum zero. figure (5) shows a plot of the phaseshift evaluated by the numerical method described above for the Ginocchio potential with $\lambda = 1, \nu = 2$ and $l = 0$

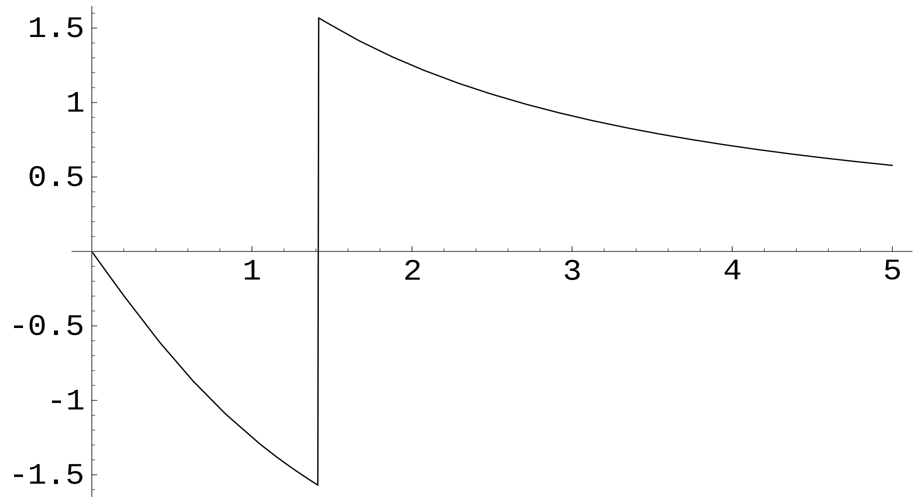

Figure 4: Plot of the exact phaseshift for $\lambda = 1, \nu = 2$ and $l = 0$

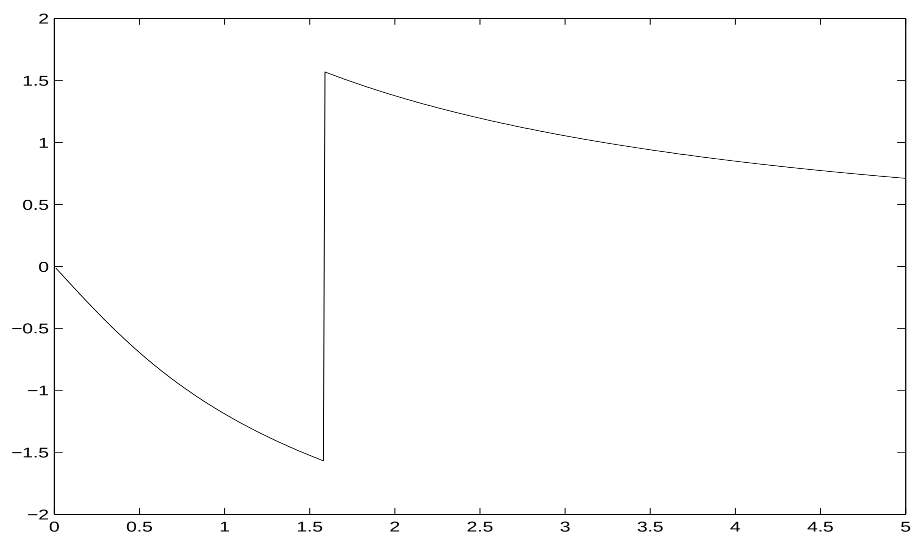

Figure 5: Plot of the numercial evaluated phaseshift for $\lambda = 1, \nu = 2$ and $l = 0$

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