A python simulator for the pressure wave propagation in the circle of Willis based on conservation of mass and momentum

by

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THESIS

for the degree of

MASTER OF SCIENCE

(Master in Computational physics)

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March 2008

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Chapter 1

Introduction

According to the World Health Organization, the cardiovascular diseases are the number one cause of death globally with the 80% of the deaths taking place in low and middle income countries. By 2005, an stimated of 30% (17.5 millions people) of deaths in the world were caused by CVDs, of which 7.6 millions were due to coronary heart disease and 5.7 millions were due to stroke. It is stimated that, if current tends are allowed to continue, by 2015 allmost 20 millions people will die from CVDs, mainly from hearth diseases and strokes, projecting the CVDs to remain the single leading causes of death[14].

Because of the treatments of such diseases and the study of the cardiovascular system with traditional methods is even limited, new analytical tools need to be developed to get a better insight of the problem[4], [15]. A prominent field emerging with increasing interest is that of computational haemodynamics. It combines mathematical modeling with computer simulation for studying the cardiovascular system and its pathologies, and it has already begun becoming important for diagnosis, prevention and treatment of diseases, and for development of instruments, devices and clinical pratices.

However, the deveploment and implementation of computational tools presents some difficulties and limitations. The prescription of suitable boundary conditions for the tridimensional (3D) Navier Stokes equations is critical [8] and the implementation of such a model of the whole circulatory system would require of a large set of morphological data, quit difficult to obtain. Moreover, the richness of details intrinsic to 3D models may not be neccesary when we only are interested in simulating the global flow features. At the same time, from the point of view of implementation, 3D models for the whole cardiovascular system are prohibitive in terms of computational calculus, storage and time execution.

A valid alternative consists in incorporating multiscaling modeling in the development of solvers. One-dimensional (1D) models of the cardiovascular system are atractive for simulation of the whole systemic tree. Because the imposibility to include all of the ramifications, zero (0D) dimensional models (lumped models) that resemble electrical circuits are created to account for the global behaviour of the terminal parts. The outputs of these models are then used as boundary conditions for 3D equations modeling local phenomena of interest where details are required [3, 13].

A 1D-model for the pulse wave propagation in the cardiovascular system have already been developed and valided experimentally *invivo* in the doctoral thesis by Alastruey(2006)[1]. It shown be able to simulate physiological pressure and flow waveforms in the largest conduit arteries and of capturing the main features of pulse wave propagation along an experimental aorta. The model was also used to investigate the effect of anastomoses and partial or total occlusions of arteries in two arterial networks: the arm and the cerebral circulation.

In a more recent article by Alastruey et. al[2] it is shown how that 1D description of the cardiovascular system is capable to capture the main wave features observed in vivo in the aorta, cerebral arteries and the arteries supplying them. The model was used to study the effects on the flow rates and pulse waveforms of the most frequent anatomical variations in the circle of Willis. The results shown be similar to other numerical simulations and observations in vivo.

Here we attempt to implement a python based solver for the cardiovascular systemic tree based on the ideas by mmmmmm cite mmmmmm. The aim is to provide the model mmm by martin of suitable condtios for simulate the circle of Willis. The interest for simulating the circle of Willis derives from mmmmm studios muestran que tanto por ciento mueron nmmmm. nacimenntosmmmm y mmmm . Este estudio es importante como una herramienta para mmmm mmmm cirujanosmpmmmm. kAquij poroommm smmm con moo boundari contitions for mmmm ..... martin mmm.

Chapter 2

Mathematical model

The derivation of a non-linear one-dimensional (1D) model for the cardiovascular system is well discused in [11]. It describes the pulse wave propagation of volume flux and pressure in the systemic tree, including only the large arteries, which are simulated as a network of flexible tubes. Moreover, because the lengths of the arterial pulse waves are large compared to arterial diameters, it is assumed that the propagation takes place mainly in the axial direction, while the arterial walls deforms only in the radial direction by the action of the internal (averaged) pressure $p(x,t)$, considered as constant over the cross-section of the lumen.

Each artery is approximated by a 1D impermeable tube of length l, cross-section $A(x,t)$ (depending of the longitudinal coordinate x and time t)[1] and thickness $h(x)$ with its wall modeled as thin, homogeneous, isotropic and elastic. The blood is considered as being homogeneous, incompressible, isotropic and Newtonian[2] and its flow as laminar and flowing only in the axial direction with average velocity $U(x,t) = \frac{1}{R^2}\int_0^R 2rudr$, where r describes the radial coordinate, R the radius of the lumen, and $u(r,t)$ the axial velocity of each particle (assumed to be axisymmetric) in the cross-section $A(x,t)$.

2.1 Governing equations

The governing set of equations governing the pulse wave propagation in the vascular network results from considerations on conservation of mass and momentum applied to a system as that described above. Following the ideas presented in [1], [2] and [11] and ignoring body forces they take the form

[1]It is assumed that the curvature of each segment of artery is small enough such that it becomes possible to approximate the axial direction with the coordinate x as the tube was straight. Therefore, the model is not suitable for the aortic arch.

[2]About 90% of the plasma in the blood is water with materials in suspention that are several order of magnitude smaller than the diameter of the systemic arteries. On the other hand, the non-Newtonian behaviour of the blood is insignificant for typical shear rates values in the large systemic arteries as shown by Caro et.al(1978)[**?**]

$$\frac{\partial A}{\partial t} + \frac{\partial (AU)}{\partial x} = 0 \tag{2.1}$$
$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{f}{\rho A} \tag{2.2}$$

where t stands for time, ρ is the density of the blood (assumed constant), $f(x,t)$ the friction force per unit length.

Equations (2.1) and (2.2) are to be solved for the pressure p, velocity U and cross-section $A(x,t)$ on each arterial segment. In order to close the system, an explicit pressure-area relationship[3] given by

$$p(x,t) = p_{ext} + \frac{\beta}{A_0}(\sqrt{A} - \sqrt{A_0}), \qquad \beta = \frac{\sqrt{\pi} h E}{1-\nu^2} \tag{2.3}$$

where $\beta = \beta(x)$ represents the elastic properties of the vessel wall and is a parameter related to the speed of pulse wave propagation according to $c^2 = \frac{\beta}{2\rho A_0}A^{1/2}$. $h = h(x)$ and $A_0 = A_0(x,t)$ denote the vessel thickness and sectional area, respectively, at the equilibrium state $(p_{ext}, U) = (p_0, 0)$. p_{ext} is the pressure outside the arterial wall (assumed to be independent of time and space), $E = E(x)$ is the Young modulus and ν is the Poisson ratio.

2.2 Characteristic variables and the Riemann problem

An analysis of the conservative form of equations (2.1) and (2.3) shows that the non-linear system is strictly hyperbolic[4], which permits to interpret the model in terms of characteristic variables like travelling waves carrying information on pressure and velocity of the system in the forward and backward directions. When the relation (2.3) is used, they take the form

$$W_{f,b} = U - U_0 \pm 4\sqrt{\frac{\beta}{2\rho A_0}}(A^{1/4} - A_0^{1/4}) \tag{2.4}$$

where the subscripts f and b stands for forward and backward, respectively.

The extension of the single vessel model (2.1)-(2.3) to a network of vessels is achieved using the characteristic decomposition above in combination with

[3] In the derivation of the tube law (2.3) static equilibrium in the radial direction of a cylinder is assumed. Moreover, the thickness of the arterial wall is assumed small compared to the radius vessel, so that the external forces applied to the arterial wall are reduced to stresses acting in the circunferential direction. Details are given in [1].

[4] For details on the theorical background on hyperbolicity, characteristic analysis and Riemann problems see for example [6] and [] mmmmmmmmmmmmriemann problems book

conservation of mass (continuity) and momentum (expressed as total pressure). As shown in section (3.1), the choice of discontinuous discrete solution and test functions in the spatial discretization of the governing system of equations decouples the numerical problem on each elemental region. In order to allow the propagation of information to the whole domain Ω, the flux on the boundaries is upwinded.

For boundaries into a single vessel, the upwinded flux is evaluated by determining the upwinded characteristic variables at the elemental interface. Discontinuities in the material properties (represented in β) and A_0 across the interface, results in discontinuities in the characteristic information propagating between elemental regions. If we only take into account the information arriving at both sides of the interface, and denote its constant states by (A_L, U_L) and (A_R, U_R) at a time t, two new states (A_L^u, U_L^u) and (A_R^u, U_R^u) will originate on each side of the interface at a time $t + \Delta t$. Under the assumption of inviscid fluid ($f = 0$),

$$W_f(A_L, U_L) = W_f(A_L^u, U_L^u) \tag{2.5}$$
$$W_b(A_R, U_R) = W_b(A_R^u, U_R^u) \tag{2.6}$$

and enforcing conservation of mass ($Q_L^u = Q_R^u$) and continuity of total pressure at the interface we get

$$A_L^u U_L^u = A_R^u U_R^u \tag{2.7}$$
$$\rho \frac{(U_L^u)^2}{2} + p(A_L^u) = \rho \frac{(U_R^u)^2}{2} + p(A_R^u) \tag{2.8}$$

that are the equations to be solved iteratively by Newton-Raphson method in order to get the upwinded states (A_L^u, U_L^u) and (A_R^u, U_R^u) on each side of the interface with their respective upwinded fluxes calculated as $\mathbf{F_L^u} = \mathbf{F}(A_L^u, U_L^u)$ and $\mathbf{F_R^u} = \mathbf{F}(A_R^u, U_R^u)$.

In the derivation of (2.5) and (2.6) only the characteristic information arriving at both sides of the interface have been taken into account. The one moving away from the interface can be neglected under the assumption that it does not have time enough to interact with adjacent quadratoure points. This is reached by demanding that

$$\Delta t \max(|U \pm c|) \leq \frac{1}{2} \Delta x \tag{2.9}$$

with Δx representing the distance between two successive quadrature points.

When the properties of the interface (β) and A_0 are constant across the interface, we get the same updated state $(A_L^u, U_L^u) = (A_R^u, U_R^u) = (A^u, U^u)$ at both sides of the interface and it can be shown that

$$A^u = \left[\frac{W_f(A_L, U_L) - W_b(A_R, U_R)}{8} \sqrt{\frac{2\rho A_0}{\beta}} + A_0^{1/4} \right]^4 \tag{2.10}$$

$$U^u = \frac{W_f(A_L, U_L) + W_b(A_R, U_R)}{2} \tag{2.11}$$

The fluxes at the inlet Ω_1 and outlet Ω_{Nel} of each arterial domain Ω are calculated solving a Riemann problem in combination with suitable boundary conditions as indicated later.

2.3 Boundary conditions

Depending of the location of the domain, the boundary conditions (BC) can be classified as (exterior) inflow BC, junctions BC for bifurcating and merging flows and terminal BC for the terminal branch coupled to a lumped parameter model.

2.3.1 Inflow/outflow BC

In order to define the flow in the extremes of an artery, we prescribe the inflow area $A_{bc} = A_{bc}(t)$, velocity $U_{bc} = U_{bc}(t)$ or flow rate $Q_{bc} = Q_{bc}(t)$ and determine the other variables by solving a Riemann problem. Since one side of the interface is located outside the arterial domain, we think of the states (A_L, U_L) at the inflow (Ω_1) or (A_R, U_R) at the outflow (Ω_{Nel}) as belonging a virtual region with the same A_0 and β as the adjacent state in Ω_1 or Ω_{Nel}, so that a single upwinded state (A^u, U^u) is obtained by solving (2.10) and (2.11). Then, we need to determine the state in the virtual region producing $A^u = A_{bc}$, $U^u = U_{bc}$ or $Q^u = A^u U^u = Q_{bc}$ when combined with the state at the inlet/outlet of the arterial domain.

Prescribing the inlet area $A_{bc}(t) = A_{bc}$ and assuming $U_L = U_R$ in (2.10), the value of A_L yielding the desired $A^u = A_{bc}$ is given by

$$A_L = [2(A_{bc})^{1/4} - (A_L)^{1/4}]^4 \tag{2.12}$$

In a similar way, prescribing $U_{bc}(t) = U_{bc}$ and assuming $A_L = A_R$ in (2.11), the value of U_L yielding the desired $U^u = U_{bc}$ is determined by

$$U_L = 2U_{bc} - U_R \tag{2.13}$$

To prescribe the inflow volume flux Q_{bc} we substitute $U_{bc} = \frac{Q_b c}{A_R}$ in the equation above such that

$$U_L = 2\frac{Q_b c}{A_R} - U_R \tag{2.14}$$

The same is to be done for the outlet of the arterial domain (right hand side), but with L substituted by R in the equations above.

2.3.2 Connection between arteries

The calculation of the upwinded states of the elemental regions connecting two arteries follows the same procedure as that describe in section (2.2) to connect elements in a single artery.

2.3.3 Junction BC

For arterial geometries splitting the flow we let (A_i, p_i, U_i) and (U_i^u, p_i^u, U_i^u) for $i = 1, \ldots, 3$ be, respectively, the initial and upwinded states at the points of each elemental region adjacent to a splitting flow junction. Taking into account figure (mmmmm) left, the information from the parent vessel 1 arrives at the junction as a forward characteristic, while the one from the daughter vessels 2 and 3 do it as backward characteristics. Assuming inviscid flow within the points adjacent to the junction and because of the conservation of the characteristics variables moving towards the junction we get

$$
\begin{aligned}
W_f(A_1^u, U_1^u) &= W_f(A_1, U_1) && (2.15)\\
W_b(A_2^u, U_2^u) &= W_b(A_2, U_2) && (2.16)\\
W_b(A_3^u, U_3^u) &= W_b(A_3, U_3) && (2.17)
\end{aligned}
$$

Enforcing conservation of mass and continuity of the momentum flux at the bifurcation applied to the upwinded variables yields

$$
\begin{aligned}
A_1^u U_1^u &= A_2^u U_2^u + A_3^u U_3^u && (2.18)\\
p(A_1^u) + \frac{1}{2}\rho (U_1^u)^2 &= p(A_2^u) + \frac{1}{2}\rho (U_2^u)^2 && (2.19)\\
p(A_1^u) + \frac{1}{2}\rho (U_1^u)^2 &= p(A_3^u) + \frac{1}{2}\rho (U_3^u)^2 && (2.20)
\end{aligned}
$$

Adopting the notation of figure () mmmmmm right, for merging flow junctions the upwinded states are related to their respective initial states by

$$
\begin{aligned}
W_b(A_1^u, U_1^u) &= W_b(A_1, U_1) && (2.21)\\
W_f(A_2^u, U_2^u) &= W_f(A_2, U_2) && (2.22)\\
W_f(A_3^u, U_3^u) &= W_f(A_3, U_3) && (2.23)
\end{aligned}
$$

The nonlinear system of algebraic equations (2.16)-(2.20) and (2.18)-(2.23) are to be solved for the upwinded states (A_i^u, U_i^u) for $i = 1, \ldots, 3$ using the Newton-Raphson method.

In the application of the Bernoulli's law we have ignored the mmmmmmm-mmm PERDIDAS occuring at junctions. However, in a recent work by mmmm, it has been shown found that hhh estas perdidas no son tan imoprtantes.

2.3.4 Terminal BC

In order to account for the effect of flow resistance R, compliance C and inertia L of the small vessels on the pulse wave propagation, the terminal branches of the 1D model are coupled to a linear lumped parameter or zero-dimensional (0D) model, more precisely to a three-element (RCR) windkessel model, in which the pressure and flow rate are only time-dependent. The terminal branch is conected directly to a resistance R_{μ_1}, followed by a CR_{μ_2} model. The first is done by imposing that the upwinded state (A^u, U^u) satisfies

$$\mathcal{F}(A^u) = R_{\mu_1}\tilde{W}_f A^u - 4R_\mu\sqrt{\frac{\beta_L}{2\rho A_{0L}}}(A^u)^{\frac{5}{4}} - p_0 - \frac{\beta_L}{A_{0L}}(\sqrt{A^u} - \sqrt{A_{0L}}) + p_C = 0 \tag{2.24}$$

with $\tilde{W}_f = U_L + 4\sqrt{\frac{\beta_L}{2\rho A_{0L}}}(A_L)^{\frac{1}{4}}$ and A^u the unknown to be determined by a Newton's Raphson method setting $A^u = A_L$ as the initial guess. U^u is determined from

$$U^u = \frac{p(A^u) - p_C}{R_{\mu_1}A^u} \tag{2.25}$$

The terminal boundary conditions is set through $U_R = 2U_{bc} - U_L$ with $U_{bc} = U^u$ and the assumption that $A_R = A_L$. The term p_C is determined for every time step n from a first-order time discretized scheme of the continuous governing equation for the CR_{μ_2} part, which is given by

$$p_C^n = p_C^{n-1} + \frac{\Delta t}{C}(Q_{in}^{n-1} - Q_{out}^{n-1}) \tag{2.26}$$

with $Q_{in}^{n-1} = (A^u)^{n-1}(U^u)^{n-1}$, $Q_{out}^{n-1} = \frac{p_C^{n-1} - p_{out}}{R_{\mu_2}}$ and the pressure $p_C^{n-1} = 0$ for $n = 1$.

Chapter 3

Numerical formulation and algorithm

The solutions of the non-linear hyperbolic system (2.1)-(2.2) are pressure and flow waves that are reflected in the cardiovascular system (for example at branch points, the aortic valve and arterioles) due to changes in the geometry and elastic properties of the arterial wall and because the viscous attenuation and effect of tapering, affecting the waveform of the pulse and its amplitude and shape. Hence, the selection of a suitable numerical method has to pay attention to its capability of propagating waves of different frequencies without suffering from excesive dispersion and diffusion errors.

Under physiological conditions, i.e for typical values of velocity, vessel area and elastic parameter β encountered in arteries, the flow is subcritical, i.e, the flow velocity is much less than the wave velocity ($u \ll c$)[1] and does not produce shocks under physiological conditions, making the high order methods atractive due to the fast convergence of the phase and diffusion properties with the polynomial order of the scheme when the solution remains smooth as the one here[12].

Following the ideas exposed in [1], [11] and [12] we atend to discretize the 1D non-linear model (2.1)-(2.2) by a discontinuous Galerkin scheme, with a high-order spectral/hp element method for the spatial discretization and a second-order Adams-Bashforth time integration. The interfaces within each artery, at the boundaries and at the junctions involves the solution of a Riemann problem.

System (2.1)-(2.2) can be expressed in conservative form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \tag{3.1}$$

with

[1] The pulse wave travles mmmm while the heart pumps mmm so that the wave has time enought to propagate in both directions in a mmmmm

$$\mathbf{U} = \begin{bmatrix} A \\ U \end{bmatrix}, \qquad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} AU \\ \frac{U^2}{2} + \frac{p}{\rho} \end{bmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} 0 \\ \frac{1}{\rho}\left(\frac{f}{A} - \frac{\partial p}{\partial \beta}\frac{d\beta}{dx} - \frac{\partial p}{\partial A_0}\frac{dA_0}{dx}\right) \end{bmatrix}$$

and where $\mathbf{F}(\mathbf{U})$ is the flux vector containing the volum flux and the energy per unit of mass, and $\mathbf{S}(\mathbf{U})$ is the source term. The last equals zero when the assumption on inviscid fluid with constant β and A_0 parameters is introduced.

3.1 Spatial discretization

In order to get the weak formulation of the system, we start decomposing the domain of each artery $\Omega = (a, b)$ into a mesh of N_{el} elemental non-overlapping regions $\Omega_e = (x_e^L, x_e^R)$ such that $x_e^R = x_{e+1}^L$ for $e = 1, \ldots, N_{el} - 1$ and $\cup_{e=1}^{N_{el}} \Omega_e = \Omega$, with the superscripts L and R refering to the left and right boundary of the elemental regions, respectively. Then, we multiply (3.1) (strong formulation) by a vector $\mathbf{\Phi}$ of test functions and integrate over the whole domain Ω. After using the standard definition of $\mathcal{L}^2$ inner product $(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{uv} d\Omega$ we get

$$\left(\frac{\partial \mathbf{U}}{\partial t}, \mathbf{\Phi}\right)_\Omega + \left(\frac{\partial \mathbf{F}}{\partial x}, \mathbf{\Phi}\right)_\Omega = (\mathbf{S}, \mathbf{\Phi})_\Omega$$

Integrating the second term by parts and expressing the integrals as the sum of integrals over elemental regions we arrive to

$$\sum_{e=1}^{N_{el}} \left[\left(\frac{\partial \mathbf{U}}{\partial t}, \mathbf{\Phi}\right)_{\Omega_e} - \left(\mathbf{F}, \frac{\partial \mathbf{\Phi}}{\partial x}\right)_{\Omega_e} + [\mathbf{F} \bullet \mathbf{\Phi}]_{x_e^L}^{x_e^R}\right] = \sum_{e=1}^{N_{el}} (\mathbf{S}, \mathbf{\Phi})_{\Omega_e} \tag{3.2}$$

We approximate the solution by a discretized expansion $U(x,t) \approx U^\delta(x,t)$ in the finite space of $\mathcal{L}^2$ functions. Furthermore, we let the test functions $\Phi(x)$ to be in the same discrete space as the numerical solution. U_e^δ and Φ_e^δ must be continuous within each element, but may be discontinuous across the interface of every elemental region, where they are denoted U^δ and Φ^δ. Because we have to attain the global solution in the domain Ω, we allow the information to propagate between elements Ω_e by upwinding the boundary flux in (3.2) such that

$$\sum_{e=1}^{N_{el}} \left[\left(\frac{\partial \mathbf{U}_\mathbf{e}^\delta}{\partial t}, \mathbf{\Phi}_\mathbf{e}^\delta\right)_{\Omega_e} - \left(\mathbf{F}(\mathbf{U}_\mathbf{e}^\delta), \frac{\partial \mathbf{\Phi}_\mathbf{e}^\delta}{\partial x}\right)_{\Omega_e} + [\mathbf{F}^\mathbf{u} \bullet \mathbf{\Phi}_\mathbf{e}^\delta]_{x_e^L}^{x_e^R}\right] = \sum_{e=1}^{N_{el}} \left(\mathbf{S}(\mathbf{U}_\mathbf{e}^\delta, \mathbf{\Phi}_\mathbf{e}^\delta\right)_{\Omega_e}$$

From an implementation point of view it is better to have a derivative on $\mathbf{U}_\mathbf{e}^\delta$ rather than on $\mathbf{\Phi}_\mathbf{e}^\delta$ in the second term above. Therefore, we integrate it by parts to get

$$\sum_{e=1}^{N_{el}} \left[\left(\frac{\partial \mathbf{U}_\mathbf{e}^\delta}{\partial t}, \mathbf{\Phi}_\mathbf{e}^\delta \right)_{\Omega_e} + \left(\frac{\partial \mathbf{F}(\mathbf{U}_\mathbf{e}^\delta)}{\partial x}, \mathbf{\Phi}_\mathbf{e}^\delta \right)_{\Omega_e} \right.$$
$$\left. + \left[(\mathbf{F^u} - \mathbf{F}(\mathbf{U}_\mathbf{e}^\delta)) \bullet \mathbf{\Phi}_\mathbf{e}^\delta \right]_{x_e^l}^{x_e^u} \right] = \sum_{e=1}^{N_{el}} \left(\mathbf{S}(\mathbf{U}_\mathbf{e}^\delta, \mathbf{\Phi}_\mathbf{e}^\delta \right)_{\Omega_e} \quad (3.3)$$

The third term in the equation above propagates the information between elements throught out the difference between the upwinded and the local fluxes. The value of $\mathbf{F}(\mathbf{U}_\mathbf{e}^\delta)$ at the interface is to be understood as the restriction of the flux from the element Ω_e [10].

To facilitate the implementation we map each elemental region onto the standard element $\Omega_{st} = [-1, 1]$ of the dimensionless ξ-axis according to

$$\chi_e(\xi) = x_e^L \frac{(1-\xi)}{2} + x_e^R \frac{(1+\xi)}{2}, \quad \xi \in \Omega_{st}$$

Selecting the expansion basis to be Legendre polynomials $L_p(\xi)$ of degree P on each elemental region Ω_e we may write

$$\mathbf{U} \approx \mathbf{U}_\mathbf{e}^\delta(\chi_e(\xi), t) = \sum_{p=0}^{P} L_p(\xi) \hat{\mathbf{U}}_\mathbf{e}^\mathbf{p}(t) \quad (3.4)$$

where $\hat{\mathbf{U}}_\mathbf{e}^\mathbf{p}(t)$ are the (time-dependent) coefficients of the expansion.

The problem is now decoupled on each elemental region Ω_e because of the choice of discontinuous discrete solution and test functions, with the only link coming through the upwinded boundary fluxes $\mathbf{F^u}$ either in single vessels or at junctions.

Taking into account the traditional Galerkin approach $\mathbf{\Phi}_\mathbf{e}^\delta = \mathbf{U}_\mathbf{e}^\delta$ and after substituting of (3.4) in (3.3) we arrive to

$$J_e \frac{d\hat{U}_{i,e}^p(t)}{dt} + J_e \left(\frac{\partial F_i}{\partial x}, L_p \right)_{\Omega_e} + \left[[F_i^u + F_i(U_e^\delta)] L_p \right]_{x_e^l}^{x_e^u} = J_e \left(S_i(\mathbf{U}_\mathbf{e}^\delta), L_p \right)_{\Omega_e} \quad (3.5)$$

where $J_e = \frac{1}{2}(x_e^R - x_e^L)$ is the Jacobian of the elemental mapping and $i = 1, 2$ denotes the two components belonging the vectors above, indicating that (3.5) yields a system of $2P$ equations (P being the degree of the Lagrange polynomial) to be solved for each elemental domain Ω_e, $e = 1, \ldots, N_{el}$.

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3.2 Temporal discretization

We start rearrenging (3.5) such that

$$\begin{aligned}\frac{d\hat{U}^p_{i,e}(t)}{dt} &= -\left(\frac{\partial F_i}{\partial x}, L_p\right)_{\Omega_e} - \frac{1}{J_e}\left[[F_i^u + F_i(U_e^\delta)]L_p\right]_{x_e^l}^{x_e^u} + \left(S_i(\mathbf{U}_\mathbf{e}^\delta), L_p\right)_{\Omega_e} \\ &= \mathcal{F}(\mathbf{U}_\mathbf{e}^\delta)\end{aligned}$$

and applying a second-order Adams-Bashforth scheme in time we come to

$$(\hat{U}^p_{i,e})^{n+1} = (\hat{U}^p_{i,e})^n + \frac{3\Delta t}{2}\mathcal{F}\left((\mathbf{U}_\mathbf{e}^\delta)^n\right) - \frac{\Delta t}{2}\mathcal{F}\left((\mathbf{U}_\mathbf{e}^\delta)^{n-1}\right) \tag{3.6}$$

where Δt is the time step and n the number of time steps. Each of the integrals above is to be solved with a Gauss quadrature rule.

3.3 Algorithm and implementation

3.3.1 Initialization of the system

Each simulation starts defining a initial state $(A, p, U) = (A_0, p_0, U_0)$ in each arterial domain Ω

Chapter 4

Numerical experiments

4.1 Pulse wave propagation along the aorta

$$l = 0.4\,m$$

Se utilizan datos espeficisos para un paciente?

que hay del trabajo de IIndia??

The density of the blood is taken to be $1050\,kgm^{-3}$

In the work by Alaustrey2007 [2], the friction force per unit length is modelled as $f = -22\pi\mu U$ based in a previous work by Smith et. al [**?**] which was obtained by fitting experimental data measured in different points inthe cardiac cycle to an axisymmetric and constant velocity profile that satisfies the non-slip boundary condition [**?**].

The viscosity of the blood is taken to be $4.5\,mPas$

Since the biological tissue is practically incompressible, the Poisson ratio is taken to be $\nu = 0.5$

4.2 Physiological parameters

4.3 Inputs

4.4 Results and discusion

Como son las predicciones segun otros modelos de una dimension, como son las predicciones segun el modelo tridimensional de la alemana y la noruega con los datos especificos para un paciente??????

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