# 1 Single particle equations

The Hamiltonian for a single particle in a Harmonic oscillator and a magnetic field is given by:

$$H = \frac{1}{2m*}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{1}{2}m*\omega_0^2\left(x^2+y^2\right) + e\phi + \boldsymbol{\rho}\cdot\boldsymbol{B} \tag{1}$$

For minimum gauge the magnetic field is given by:

$$\begin{aligned}\boldsymbol{A} &= \frac{B}{2}\left(-y, x, 0\right)\\ \boldsymbol{B} &= B\boldsymbol{e}_z\end{aligned}$$

- $>$ A commutes with p and we have:

$$\begin{aligned}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2 =& p^2 + \frac{e}{c}\left(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}\right) + \frac{e^2}{c^2}A^2\\ =& p^2 + \frac{Be}{c}\left(xp_y - yp_x\right) + \frac{B^2e^2}{4c^2}\left(x^2+y^2\right)\end{aligned}$$

- $>$ Equation (1) becomes:

$$\begin{aligned}H =& \frac{1}{2m*}\left(p^2 + \frac{Be}{c}(xp_y - yp_x) + \frac{B^2e^2}{4c^2}(x^2+y^2)\right)\\ &+ \frac{1}{2}m*\omega_0^2\left(x^2+y^2\right) + e\phi + \boldsymbol{\rho}\cdot\boldsymbol{B}\\ =& \frac{-\hbar^2}{2m*}\left(\nabla^2 + i\frac{Be}{\hbar c}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\\ &+ \frac{1}{2}m*(\omega_0^2 + \frac{B^2e^2}{4c^2m*^2})(x^2+y^2) + e\phi + \boldsymbol{\rho}\cdot\boldsymbol{B}\end{aligned} \tag{2}$$

Using polar coordinetes $r$, $\theta$: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$, $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial\theta}$ and $\omega^2 = \omega_0^2 + \frac{B^2e^2}{4c^2m*^2}$, $H_S = e\phi + \boldsymbol{\rho}\cdot\boldsymbol{B}$ this equation becomes:

$$H = -\frac{\hbar^2}{2m*}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + i\frac{Be}{\hbar c}\frac{\partial}{\partial\theta}\right) + \frac{1}{2}m*\omega^2r^2 + H_S \tag{3}$$

Separating variables: $\Psi(r,\theta) = e^{im\theta}\psi(r)\chi(s_1)$, $m = 0, \pm1, \pm2....$ The Schrodinger equation $H\Psi = E\Psi \rightarrow$

$$\begin{aligned}&-\frac{\hbar^2}{2m*}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{Bem}{\hbar c}\right)\psi(r) + \frac{1}{2}m*\omega^2r^2\psi(r) = E\psi(r)\\ &\rightarrow\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{m*^2\,\omega^2}{\hbar^2}r^2\right)\psi(r)\\ &\qquad = -\left(E\frac{2m*}{\hbar^2} - \frac{Bem}{\hbar c}\right)\psi(r)\end{aligned} \tag{4}$$

$$\left(e\phi + \boldsymbol{\rho}\cdot\boldsymbol{B}\right)\chi(s_1) = E_S\chi(s_1) \tag{5}$$

Equation (4) can be written as:

$$\rightarrow A\psi(r) = \epsilon_m \psi(r) \tag{6}$$

Where A and $\epsilon_m$ is given by:

$$A = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{m*^2\,\omega^2}{\hbar^2} r^2 \right) \tag{7}$$

$$\epsilon_m = \left( -E\frac{2m*}{\hbar^2} + \frac{Bem}{\hbar c} \right) \tag{8}$$

## 2 Two-particle equations

$$\begin{aligned} H = & \sum_{i=1}^{2} \left[ \frac{1}{2m*} \left( \boldsymbol{p}_i + \frac{e}{c}\boldsymbol{A}_i \right)^2 + \frac{1}{2} m * \omega_0^2 r_i^2 + e\phi_i + \boldsymbol{\rho}_i \cdot \boldsymbol{B}_i \right] \\ & + \frac{e^2}{4\pi\epsilon\epsilon_0} \frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \end{aligned} \tag{9}$$

Using relative and center of mass coordinates defined as:

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}_1 - \boldsymbol{r}_2 & \boldsymbol{R} &= \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2) \\ \boldsymbol{p} &= \frac{1}{2}(\boldsymbol{p}_1 - \boldsymbol{p}_2) & \boldsymbol{P} &= \boldsymbol{p}_1 + \boldsymbol{p}_2 \\ \boldsymbol{A}_r &= \frac{1}{2}(\boldsymbol{A}_1 - \boldsymbol{A}_2) & \boldsymbol{A}_R &= \boldsymbol{A}_1 + \boldsymbol{A}_2 \end{aligned}$$

Giving some relations:

$$\begin{aligned} & r_1^2 + r_2^2 = \frac{1}{2}\left(4R^2 + r^2\right) \qquad p_1^2 + p_2^2 = \frac{1}{2}\left(4p^2 + P^2\right) \\ & \boldsymbol{p}_1 \cdot \boldsymbol{A}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{A}_2 = \frac{1}{2}(4\boldsymbol{p} \cdot \boldsymbol{A}_r + \boldsymbol{P} \cdot \boldsymbol{A}_R) \\ & A_1^2 + A_2^2 = 2A_r^2 + \frac{1}{2}A_R^2 \end{aligned}$$

Then H transforms:

$$\begin{aligned} H = & \frac{1}{2m*} \left( p_1^2 + p_2^2 + \frac{2e}{c}(\boldsymbol{p}_1 \cdot \boldsymbol{A}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{A}_2) + \frac{e^2}{c^2}(A_1^2 + A_2^2) \right) \\ & + \frac{1}{2} m * \omega_0^2 (r_1^2 + r_2^2) + \frac{e^2}{4\pi\epsilon\epsilon_0} \frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} + \sum_{i=1}^{2} [e\phi_i + \boldsymbol{\rho}_i \cdot \boldsymbol{B}_i] \end{aligned} \tag{10}$$

$$
\begin{aligned}
H =& \frac{1}{2m*}\left(2p^2 + \frac{4e}{c}\boldsymbol{p}\cdot\boldsymbol{A}_r + \frac{2e^2}{c^2}A_r^2 + \frac{1}{2}P^2 + \frac{e}{c}\boldsymbol{P}\cdot\boldsymbol{A}_R + \frac{e^2}{2c^2}A_R^2\right) \\
&+ \frac{1}{2}m*\omega_0^2\frac{1}{2}(4R^2 + r^2) + \frac{e^2}{4\pi\epsilon\epsilon_0}\frac{1}{r} + \sum_{i=1}^{2}\left[e\phi_i + \boldsymbol{\rho}_i\cdot\boldsymbol{B}_i\right] \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
H =& \frac{1}{m*}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}_r\right) + \frac{1}{4}m*\omega_0^2 r^2 + const.\frac{1}{r} \\
&+ \frac{1}{4m*}\left(\boldsymbol{P} + \frac{e}{c}\boldsymbol{A}_R\right) + m*\omega_0^2 R^2 \\
&+ \sum_{i=1}^{2}\left[e\phi_i + \boldsymbol{\rho}_i\cdot\boldsymbol{B}_i\right] \qquad (12) \\
=& H_r + H_R + H_S \qquad (13)
\end{aligned}
$$

Using separation of variables: $\Psi(\boldsymbol{r}, \boldsymbol{R}, s_1, s_2) = \phi(\boldsymbol{r})\gamma(\boldsymbol{R})\chi(s_1, s_2)$

$$
H\Psi = E\Psi \rightarrow \frac{H_r\phi(\boldsymbol{r})}{\phi(\boldsymbol{r})} + \frac{H_R\gamma(\boldsymbol{R})}{\gamma(\boldsymbol{R})} + \frac{H_S\chi(s_1, s_2)}{\chi(s_1, s_2)} = E \qquad (14)
$$

-> can be divided in separate equation which must each be a constant:

$$
\begin{aligned}
&H_r\phi(\boldsymbol{r}) = E_r\phi(\boldsymbol{r}) && (15) \\
&H_R\gamma(\boldsymbol{R}) = E_R\gamma(\boldsymbol{R}) && (16) \\
&H_S\chi(s_1, s_2) = E_S\chi(s_1, s_2) && (17) \\
&E = E_r + E_R + E_S && (18)
\end{aligned}
$$

$$
\begin{aligned}
H_r &= \frac{1}{m*}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}_r\right) + \frac{1}{4}m*\omega_0^2 r^2 + \frac{e^2}{4\pi\epsilon\epsilon_0}\frac{1}{r} && (19) \\
H_R &= \frac{1}{4m*}\left(\boldsymbol{P} + \frac{e}{c}\boldsymbol{A}_R\right) + m*\omega_0^2 r^2 && (20) \\
H_S &= (\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)\cdot\boldsymbol{B} + e(\phi + \phi_2) && (21) \\
& && (22)
\end{aligned}
$$

- > have two single particle equations that should be solved as in section 1. It is possible to set up a general equation to be solved.

## 3 General One-particle radial equation

The general equation become:

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - K\frac{1}{r} - \Omega^2 r^2\right)\psi(r) = \epsilon_m\psi(r) \qquad (23)
$$

With constants given for each problem.
Single particle: $\Omega^2 = \frac{m*^2\omega^2}{\hbar^2}$ , $K = 0$, $\epsilon_m = \frac{Bem}{\hbar c} - \frac{2m*E}{\hbar^2}$. This gives equation(4).
Relative: $\Omega^2 = \frac{m*^2\omega^2}{4\hbar^2}$ , $K = \frac{e^2}{4\pi\epsilon\epsilon_0}$, $\epsilon_m = \frac{Bem}{\hbar c} - \frac{m*E}{\hbar^2}$.
COM: $\Omega^2 = \frac{4m*^2\omega^2}{\hbar^2}$ , $K = 0$, $\epsilon_m = \frac{Bem}{\hbar c} - \frac{4m*E}{\hbar^2}$.

## 4 Finite difference equations

The general equation to be solved numerically is:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - K\frac{1}{r} - \Omega^2 r^2\right)\psi(r) = \epsilon_m \psi(r) \tag{24}$$

$$r_i = ih \qquad u_i = \psi(r_i) \qquad h = \frac{1}{N-1} \qquad i = 0, 1, ...N-1$$

$N$ is the number of points.

For the derivatives we write:

$$\frac{\partial^2}{\partial r^2}u_i \simeq \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$
$$\frac{\partial}{\partial r}u_i \simeq \frac{u_{i+1} - u_{i-1}}{2h}$$

Using this on (24) we get a non-symmetric tridiagonal matrix eigenvalueproblem:

$$u_{i-1}\left(1 - \frac{1}{2i}\right) + u_i\left(-2 - \frac{m^2}{i^2} - \Omega^2 i^2 h^4 - \frac{Kh}{i}\right) + u_{i+1}\left(1 + \frac{1}{2i}\right) = \epsilon_m h^2 u_i \tag{25}$$