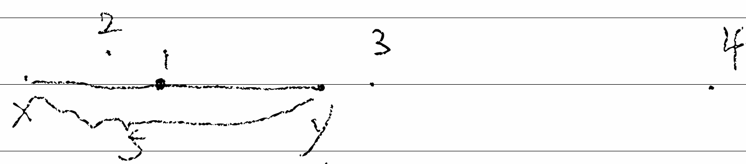
PaLD U_{xy} size:

U_{xy} is a local conflict focus that contains points that are closer to x or y than the distance between x and y. For example, $d(x,y) = 5$, then any point z that has $d(x,z) \leq 5$ or $d(y,z) \leq 5$ should be count into U_{xy}.



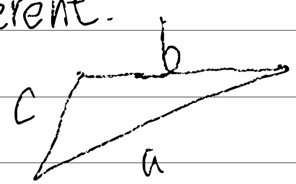
* 1, 2, 3 are in U_{xy} but 4 is not.

Assuming we have a data set contains n points, then the total number of unique pairs of points are $C_n^2 = \frac{n(n-1)}{2}$. Since the size of U_{xy} is betwen 2 and n (x and y are also include in U_{xy}), we know that the sum of size of all U_{xy} is between $\frac{n(n-1)}{2} \cdot 2 = n(n-1) \to \Theta(n^2)$ and $\frac{n(n-1)}{2} \cdot n = \frac{1}{2}(n^3 - n^2) \to \Theta(n^3)$.

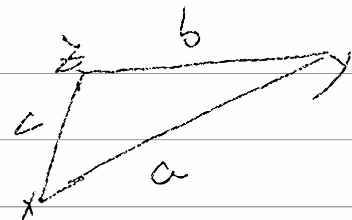
To simplif the process, we take 3 points at a time to analyze the size of U_{xy}. Another advantage of taking 3 points is that no matter which dimension space those points live in they all keep the nice properties of a triangle if they were not locate in the same direction. Therefore we have following situations,

① three points x, y, z can form a triangle.

a. $d(x,y)$, $d(x,z)$, $d(z,y)$ they are all different.



Here we ignore the exact location of each point. Instead, we focus on the distances between two points. let a denote the largest distance, and b denote the second largest distance, and c denote the smallest distance. Then, we have $a > b > c$



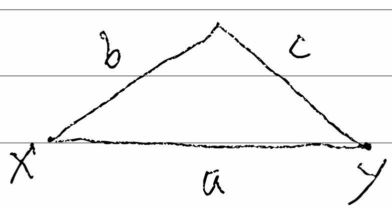
To make explanation clear, let's assume x and y locate on the side of a, and x, z locate on the side of c, and z, y locate on the side of b. Then for different local conflict focus, we have

		size	
U_{xy} :	x, y, z	3	有重复算 x, y
U_{xz} :	x, z	2	x, z
U_{yz} :	x, y, z	3	z, y

b. only two of $d(x,y)$, $d(x,z)$ and $d(y,z)$ are different

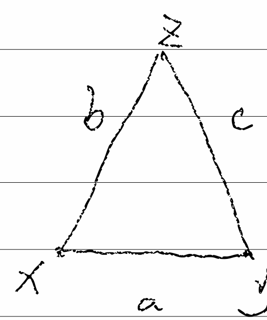
Using similar assumptions in a, we propose the following graph

I.



$a > b = c$

II.



$a < b = c$

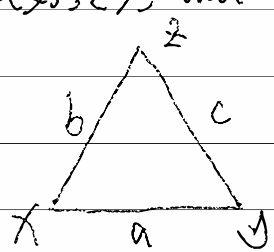
I. $a > b = c$

		size
U_{xy} :	x, y, z	3
$U_{x,z}$:	x, z, y	3
$U_{y,z}$:	x, z, y	3

II. $a < b = c$

		size
U_{xy} :	x, y	2
U_{xz} :	x, y, z	3
U_{yz} :	x, y, z	3

C. $d(x, y)$, $d(x, z)$, and $d(y, z)$ are the same



$a = b = c$

		size
U_{xy}:	x, y, z	3
U_{xz}:	x, y, z	3
$U_{z,y}$:	x, y, z	3

② Three points x, y, z are in the same direction that form a line

Assuming x and y have the largest distance
$d(x, y) = a$

x $0.5a$ $0.5a$ y

a. z is at the middle of x, y

x z y

		size
U_{xy} :	x, y, z	3
U_{xz} :	x, z, y	3
U_{zy} :	x, z, y	3

b. z is not at the middle of x, y

z z^*

x $\frac{x+y}{2}$ y

U_{xy} :	x, y, z	3	U_{xy} :	x, y, z	3
U_{xz} :	x, z	2	U_{xz^*} :	x, y, z^*	3
U_{yz} :	x, y, z	3	U_{z^*y} :	z^*, y	2

So far, we list all the possible location situations of three points. One thing need to be careful is that x, y (x,z) (y,z) will be count many times for U_{xy}, U_{xz}, U_{yz}.

The total number of unique three points out of n points is

$$C_n^3 = \frac{n(n-1)(n-2)}{3\times 2\times 1} = \frac{n(n-1)(n-2)}{6}$$

Assuming each situation has a equal chance to happen. (7 situations) Then each situation has $\frac{n(n-1)(n-2)}{6\times 7} = \frac{n(n-1)(n-2)}{42}$ unique ways to form three points.

pairs \ size	① a	bI	bII	c	② a	bI	bII
U_{xy}	3	3	2	3	3	3	3
U_{xz}	2	3	3	3	3	2	3
U_{yz}	3	3	3	3	3	3	2
Total	8	9	8	9	9	8	8

Then, the total sum of size of local conflict focus (without remove redundant) are

$$\frac{n(n-1)(n-2)}{42} \times 8 \times 4 + \frac{n(n-1)(n-2)}{42} \times 9 \times 3$$
$$= \frac{59}{42} n(n-1)(n-2)$$

Now lets remove the redundant in local conflict focus. We only need to count each unique pair once in it's local conflict. For example, for x, y the size of U_{xy} only count x, y for once. But in three points set up we count x, y n-2 times since there are n-2 to form three points with x, y include. The same redundant rule also applies to x, z and y, z. Therefore, the total size of sum of local conflict focus without redundancy is

$$\frac{\frac{59}{42} n(n-1)(n-2)}{(n-2)} = \frac{59}{42} n(n-1)$$
$$\sim O(n^2)$$