Let $X \in \mathcal{R}^D$, and g(X) = $\sqrt[D]{x_1 \dots x_D}$. Then the centered log-ratio transform can be write as

clr(X) = $\begin{pmatrix} \ln \frac{x_1}{g(X)} & \ln \frac{x_2}{g(X)} & \dots & \ln \frac{x_{D-1}}{g(X)} & \ln \frac{x_D}{g(X)} \end{pmatrix}$. We can also present the transform in matrix vector multiplication form. Take $\ln \frac{x_1}{g(X)}$ as an example, we know that

$$\begin{aligned} \ln \frac{x_1}{g(X)} &= \ln x_1 - \ln g(X) = \ln x_1 - \ln{(x_1 \dots x_D)}^{\frac{1}{D}} \\ &= \ln x_1 - \frac{1}{D}(\ln x_1 + \ln x_2 + \dots + \ln x_{D-1} + \ln x_D) \\ &= \frac{1}{D}(D \ln x_1 - \ln x_1 - \ln x_2 - \dots - \ln x_{D-1} - \ln x_D) \\ &= \frac{1}{D} \begin{pmatrix} \ln x_1 & \ln x_2 & \dots & \ln x_{D-1} & \ln x_D \end{pmatrix} \begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \end{pmatrix}^T \end{aligned} \tag{0.1}$$

Therefore, we can write the clr transform in the following format,

$$clr(X) = \frac{1}{D} \begin{pmatrix} \ln x_1 & \ln x_2 & \dots & \ln x_{D-1} & \ln x_D \end{pmatrix} \begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \\ -1 & D-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & D-1 & -1 \\ -1 & -1 & \dots & -1 & D-1 \end{pmatrix}$$

The matrix in the above equation lives in $\mathcal{R}^{D \times D}$. Take one point $b = (x_1, x_2, \dots, x_{D-1}, x_D) \in X$, then for another point c that in the same direction of b we know that $c = \alpha(x_1, x_2, \dots, x_{D-1}, x_D)$, where α is a constant real number. Therefore, the centered log-ratio transform for point b and c will be

$$clr(b) = \frac{1}{D} \begin{pmatrix} \ln x_1 & \ln x_2 & \dots & \ln x_{D-1} & \ln x_D \end{pmatrix} \begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \\ -1 & D-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & D-1 & -1 \\ -1 & -1 & \dots & -1 & D-1 \end{pmatrix}$$

$$clr(c) = \frac{1}{D} \begin{pmatrix} \ln \alpha x_1 & \ln \alpha x_2 & \dots & \ln \alpha x_{D-1} & \ln \alpha x_D \end{pmatrix} \begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \\ -1 & D-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & D-1 & -1 \\ -1 & -1 & \dots & -1 & D-1 \end{pmatrix}$$

Since $\ln \alpha x_1 = \ln \alpha + \ln x_1$, we can rewrite $clr(c)$ as

$$
\begin{aligned}
clr(c) &= \frac{1}{D}\begin{pmatrix}\ln\alpha+\ln x_1 & \ln\alpha+\ln x_2 & \dots & \ln\alpha+\ln x_{D-1} & \ln\alpha+\ln x_D\end{pmatrix}\begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \\ -1 & D-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & D-1 & -1 \\ -1 & -1 & \dots & -1 & D-1 \end{pmatrix} \\
&= \frac{1}{D}\begin{pmatrix}\ln x_1 & \ln x_2 & \dots & \ln x_{D-1} & \ln x_D\end{pmatrix}\begin{pmatrix} D-1 & -1 & \dots & -1 & -1 \\ -1 & D-1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \dots & D-1 & -1 \\ -1 & -1 & \dots & -1 & D-1 \end{pmatrix} = clr(b)
\end{aligned}
\tag{0.2}
$$

Since point b and c are arbitrary, the centered log-ratio transform is not one to one, but it preserves the property of compositional data.