IEOR 4106 : Foundations of Financial Engineering.

Optional project 1 : CCR model and American options.

1. American options

The goal of the this project is to study two problems linked to American options. Unlike European options, they allow their buyer to exercise his rights at any time prior to the maturity of the option. In order to both price and hedge such financial products, it is necessary for the seller to be able to understand and compute which instants are optimal for the buyer to potentially exercise the aforementioned rights.

We first study the so-called "Perpetual" American options, which have the desirable feature that their pricing and hedging admit explicit solutions, before going back to the more standard case in finite horizon which we will treat numerically.

1.1. **Perpetual American Options.** We consider Black-Scholes model for which the underlying asset on which the American options will be written has the following dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{1.1}$$

We denote by S_t^x the price of the asset at t, which started from the value x at 0. The perpetual American option with payoff h allows its owner to exercise his rights at any moment in $[0, +\infty)$. It is possible to prove that the price of such an option is given by

$$v(x) := \sup_{\tau \geq 0} \mathbb{E}\left[e^{-r\tau} h(S_\tau^x)\right]. \tag{1.2}$$

We will always assume that $r \geq \mu \geq 0$. We will also admit that the function v defined above satisfies the following so-called Hamilton-Jacobi-Bellman equation

$$\min\left\{rv(x) - \mu x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x),\ v(x) - h(x)\right\} = 0. \tag{1.3}$$

We now focus on the case of the American Put option and therefore enforce $h(x) = (K-x)^+$.

1. Justify heuristically that we should expect an exercise region (that is to say the values of x such that if the asset price reaches them, it is then optimal to exercise the option) of the form $\mathcal{S} = (0, x^\star]$ for some $x^\star \in (0, K]$.

2. Under the hypothesis of the previous question, show that v must satisfy

$$rv(x) - \mu x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = 0,\ x > x^*.$$

Give the general solution of this ODE under the form

$$v(x) = Ax^m + Bx^n,$$

where $m \leq 0$ and $n \geq 1$ should be given explicitly.

3. Justify that v is bounded and deduce that $B = 0$.

4. Assuming that v is C^1 at $x^\star$, determine both A and $x^\star$. When is it optimal to exercise ? What happens when r goes to 0 ?

1.2. **Pricing and exercise frontier for the American Put in the CRR model.** Recall that the CRR model is a discrete-time model on a fixed grid $t_i := iT/n$, $i = 0..n$, stipulating at each node the following potential evolution of the value of the risky asset

$$S_{t_{i+1}} = S_{t_i} U^i,$$

where U^i equals u (state "up") or d (state "down") with

$$u := e^{\frac{\sigma}{\sqrt{n}}} \text{ and } d := e^{-\frac{\sigma}{\sqrt{n}}}$$

1. Create the tree of the possible values of the risky asset thanks to a triangular matrix SS of size $(n+1) \times (n+1)$ such that $SS(i,j)$ for $i = 0..n$ et $j = 0..i$ represents the value of S at t_i when it has increased j times.

In the CRR model, the unique risk-neutral measure $\mathbb{Q}$ is given by

$$q := \mathbb{Q}[U^i = u] = \frac{e^{r\frac{T}{n}} - d}{u - d},$$

q representing the probability of an increase of S between two successive times. In addition, the price P^i of a European option with payoff h is given at any time by the backward induction

$$P^n = h(S_{t_n}) \text{ and } P^i = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\frac{T}{n}} P^{i+1} \middle| \mathcal{F}_{t_i}\right].$$

2. Use the above formula to write a program computing the price of a European Call and a European Put option at every node of the tree.

3. For American options, give a similar recursion formula, and use it to write a similar code giving the price of an American Call and an American Put. Examine in particular the case $r = 0$.

4. In order to obtain the exercise frontier of the American Put, define a matrix EPA(i,j) whose components are equal to 1 at the points (i,j) of the tree situated below the the exercise frontier, and 0 at the others.

Notice that the Matlab command plot2d$(i, SS(i,j), -2\text{EPA}(i,j) - 2)$ allows to draw the points $(i, S(i,j))$ using a different symbol depending on whether EPA(i,j) is equal to 1 or 0 (see the online help if you're using another language than Matlab).

Draw the exercise frontier and study its dependence on r and n.

1.3. **Finite differences for American options.** In this section, we look for a numerical approximation $p(t,x)$ of the price of an American option with payoff g, for $(t,x) \in [0,T] \times$

$[0, S_{\max}]$, which we admit to be solution of the following PDE

$$\begin{cases} \min\left\{\partial_t p - \frac{\sigma^2}{2}\partial_{xx}p - rx\partial_x p + rp, p - g\right\} = 0,\ (t,x) \in [0,T] \times [0, S_{\max}], \\ p(t, S_{\max}) = 0,\ 0 < t < T, \\ p(0,x) = h(x). \end{cases}$$

Notice here that we reversed time so as to have forward algorithms (the value of the option is now known at time 0 instead of time T. The price of the American option when the underlying asset starts from the value x is thus here $p(T,x)$. For numerical experiments, we will take

$$K = 100,\ S_{\max} = 200,\ T = 1,\ \sigma = 0.2,\ r = 0.1.$$

1.3.1. *Explicit Euler scheme.* We use the following time-space discretisation

$$x_j := jh,\ j = 0, \dots, M+1 \text{ with } h := \frac{S_{\max}}{M+1} \text{ and } t_n := n\Delta t,\ n = 0, \dots, N \text{ with } \Delta t := \frac{T}{N}.$$

We define $P_j^n := P(t_n, x_j)$ and $P^n = (P^n)_{0 \leq j \leq M+1}$. Notice that $P_j^0 = g(x_j)$, $j = 0, \dots, M+1$.

We first consider the following explicit scheme, for $n \geq 0$ and $0 \leq j \leq M$

$$\begin{cases} \min\left\{\frac{P_j^{n+1} - P_j^n}{\Delta t} + \frac{\sigma^2 x_j^2}{2}\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2} - rx_j\frac{P_{j+1}^n - P_j^n}{h} + rP_j^n, P_j^{n+1} - g(x_j)\right\} = 0, \\ P_{M+1}^{n+1} = 0. \end{cases}$$

1. Denote by A the $(M+1) \times (M+1)$ matrix such that

$$(AP)_j = \frac{\sigma^2 x_j^2}{2}\frac{-P_{j-1}^n + 2P_j^n - P_{j+1}^n}{h^2} - rx_j\frac{P_{j+1}^n - P_j^n}{h} + rP_j^n,\ 0 \leq j \leq M,$$

with the convention $P_{-1} = 0$.

Compute and generate in the language of your choice the matrix A.

2. Define also $\phi := (g(x_j))_{0 \leq j \leq M+1}$. Prove that the discretisation scheme studied here takes the form

$$P^{n+1} = \max\left\{P^n - \Delta t A P^n, \phi\right\}. \tag{1.4}$$

3. Use the above scheme for $M = 20$ and $N = 20$. What happens if you now take $M = 50$ and $N = 20$? This is the so-called instability of explicit schemes.

1.3.2. *Implicit Euler scheme.* To solve the above stability issues, we now consider the following scheme

$$\begin{cases} \min\left\{\frac{P^{n+1} - P^n}{\Delta t} + AP^{n+1}, P^{n+1} - \phi\right\} = 0,\ n = 0, \dots, N-1, \\ P^0 = \phi. \end{cases}$$

Define then $B = I_{M+1} + \Delta t A$ and $b = P^n$. For every n, we therefore now need to find a solution x to the following non-linear system

$$F(x) := \min\{Bx - b, x - \phi\} = 0.$$

We then take $P^{n+1} = x$.

In order to solve this equation, we propose a Newton-Raphson algorithm. We recall that the idea is to start from an arbitrary initial value x^0 and to iterate

$$x^{k+1} = x^k - F'(x_k)^{-1} F(x^k),$$

until a maximum number of iterations (fixed a priori) has been reached, or the difference $x^{k+1} - x^k$ becomes smaller than a fixed error.

4. Prove that the derivative of F can be written

$$F'(x)_{i,j} = \begin{cases} B_{i,j}, \text{ if } (Bx - b)_i \leq (x - \phi)_i \\ \delta_{i,j}, \text{ otherwise}, \end{cases}$$

where $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise.

5. Write codes for the Newton algorithm and then the implicit Euler scheme. Check that it is indeed stable.