CPTS_591 Elements of Network Science
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# Midterm

## Problem 1

a) True

b) True

c) True

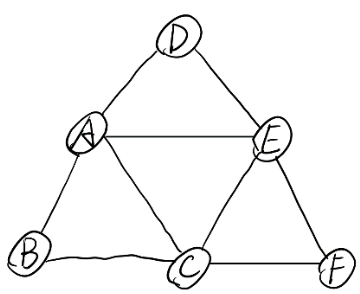
d) False

The global clustering coefficient is based on closed triplets of nodes. The local clustering coefficient of a node in a graph quantifies how close its neighbors are to being a clique. The average clustering coefficient and the global clustering coefficient are not equivalent.

e) True

## Problem 2

a) The derived network can be given by:




b) The triangle on the nodes A, C, and E can reach all other nodes on the graph,  and all the other nodes have to pass one of the nodes A, C, and E to reach other nodes.

**Problem 3**

Construct the adjacency matrix $A$ of Figure 2:

$$A = \begin{bmatrix} 0&0&1&0&0 \\ 0&0&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&1&0&0&0 \\ 0&1&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}$$

We write $h$ for the vector of hub scores and $a$ for the vector of authority scores.

When $k=2$, we get: $a^{\langle 2\rangle} = A^T h^{\langle 1\rangle} = A^T \cdot A \cdot A^T \cdot h^{\langle 0\rangle}$

$$h^{\langle 2\rangle} = A a^{\langle 2\rangle} = A \cdot A^T \cdot A \cdot A^T \cdot h^{\langle 0\rangle} = (A \cdot A^T)^2 \cdot h^{\langle 0\rangle}$$

We assume that the initial hub vector $h^{\langle 0\rangle}$ as 1.

$$\therefore \quad a^{\langle 2\rangle} = A^T \cdot A \cdot A^T \cdot \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 0\\0\\6\\4\\4 \end{bmatrix}$$

$$h^{\langle 2\rangle} = (A \cdot A^T)^2 \cdot \begin{bmatrix} 1\\1\\1\\1\\1 \end{bmatrix} = \begin{bmatrix} 6\\14\\0\\0\\0 \end{bmatrix}$$

Calculate new authority: $0^2 + 0^2 + 6^2 + 4^2 + 4^2 = 78$

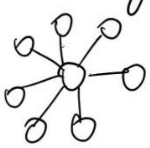
Normalization: $\begin{bmatrix} A\\B\\C\\D\\E \end{bmatrix} \approx \begin{bmatrix} 0\\0\\0.679\\0.453\\0.453 \end{bmatrix}$

Calculate new hub score: $6^2 + 14^2 + 0^2 + 0^2 + 0^2 = 232$

Normalization: $\begin{bmatrix} A\\B\\C\\D\\E \end{bmatrix} \approx \begin{bmatrix} 0.394\\0.919\\0\\0\\0 \end{bmatrix}$

**Problem 4**

star graph



Degree matrix $D = \begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$ Adjacency matrix $A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$

Laplacian matrix $L = D - A$

$$= \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -1 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

$\sum_{i,j=1}^{n} (a_{ij}) = 0 \Rightarrow 0$ is an eigenvalue.

Let $x$ be an eigenvector of $0$

$$\Rightarrow L \cdot x = 0$$

$$\Rightarrow x^T L \cdot x = 0$$

$$\Rightarrow \sum_{(u,v)\in E} [x(u) - x(v)]^2 = 0.$$

Thus, for every adjacent verticles, $x(u) = x(v)$.
As every pair is connected by a path, we get

$$x(u) = x(v) \quad \forall u, v \in V(G)$$

$\Rightarrow x$ is a constant vector $\Rightarrow L$ has eigenvalue $0$ with multiplicity $1$.

$x_i = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}$, $\{x_i\}_{2 \le i \le n}$ are $n-2$ independent vectors of eigenvalue $1$.

To get the last eigenvalue, we calculate the trace of the Laplacian matrix.

$$\text{trace}(L) = 2(n-1)$$

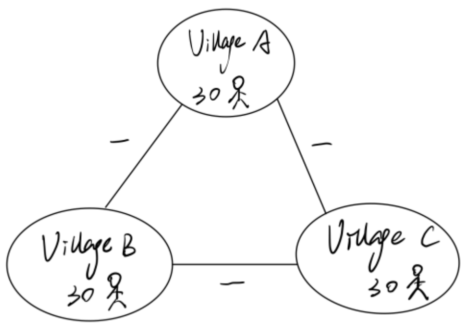
$$\text{eigenvalue} = 2(n-1) - 1 - (n-1) + 1$$

$$= 2n - 2 - n + 2$$

$$= n$$

$\therefore L$ has eigenvalue $0$ with multiplicity $1$, eigenvalue $1$ with multiplicity $n-2$, and eigenvalue $n$ with multiplicity $1$.

**Problem 5**




This network on 90 people is unbalanced since three villages are mutual enemies.

According to the Balance Theorem (Harary, 1953), if a labeled complete graph is balanced, then either (1) all edges are positive, or else (2) the nodes can be divided into 2 groups, X and Y, such that every edge in X is positive, every edge in Y is positive , and every edge running between X and Y is negative. The network described in the problem dose not satisfy any of the above two conditions, so it is unbalanced.