CptS 591: Elements of Network Science, Spring 2021
Spectral Analysis, Part I
Lecture Notes

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Two of the topics that we will cover in future lectures in this course are (a) the notion of centrality in a network and the different measures used to quantify it and (b) methods for link analysis. In both of these contexts, eigenvectors and eigenvalues of matrices show up — and analysis that relies on them is called *spectral analysis*. The purpose of these notes is to review very basic linear algebra concepts we need for spectral analysis. In a follow up lecture, we will take up spectral analysis as a topic on its own and see different types of matrices that can be associated with a graph and how spectra of the matrices can be used to analyze the graph.

Review of Basic Linear Algebra

We begin by reviewing basic concepts we need.

Let $M \in \mathbb{C}^{n\times n}$ be a square matrix whose entries are complex numbers. A nonzero vector $x \in \mathbb{C}^n$ is an *eigenvector* of M, and $\lambda \in \mathbb{C}$ is its corresponding *eigenvalue*, if

$$Mx = \lambda x. \tag{1}$$

The set of all the eigenvalues of a matrix M is the *spectrum* of M, a subset of $\mathbb{C}$ denoted by $\Lambda(M)$.

Characteristic polynomial

The *characteristic polynomial* of a matrix $M \in \mathbb{C}^{n\times n}$, denoted by p_M, is the degree n polynomial defined by

$$p_M(z) = det(zI - M). \tag{2}$$

Theorem 1 *λ is an eigenvalue of M if and only if $p_M(\lambda) = 0$.*

Theorem 1 implies that even if a matrix is real, some of its eigenvalues may be complex.

Algebraic multiplicity

Using the fundamental theorem of algebra, we can write the characteristic polynomial p_M in the form

$$p_M(z) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$$

for some numbers $\lambda_j \in \mathbb{C}$. By Theorem 1, each λ_j is an eigenvalue of M, and all eigenvalues of M appear somewhere in the list. In general, an eigenvalue could appear more than once. The *algebraic multiplicity* of an eigenvalue λ of M is its multiplicity as a root of p_M.

The characteristic polynomial gives an easy way to count the number of eigenvalues of a matrix:

Theorem 2 *If $M \in \mathbb{C}^{n\times n}$, then M has n eigenvalues, counted with algebraic multiplicity.*

Eigenvalue Decomposition

An *eigenvalue decomposition* of a square matrix M, when it exists, is a factorization

$$M = X\Lambda X^{-1}. \tag{3}$$

This can equivalently be written as

$$MX = X\Lambda,$$

which makes it clear that if x_j is the jth column of X and λ_j is the jth diagonal entry of Λ, then $Mx_j = \lambda_j x_j$. Thus the jth column of X is an eigenvector of M and the jth entry of Λ is the corresponding eigenvalue.

Geometric multiplicity

The *maximum number of linearly independent eigenvectors* that can be found, each with the same eigenvalue λ, is called the *geometric multiplicity* of λ.

The geometric multiplicity and the algebraic multiplicity of an eigenvalue are related as follows: The algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity. An eigenvalue whose algebraic multiplicity exceeds its geometric multiplicity is a *defective eigenvalue.* A matrix that has one or more defective eigenvalues is a *defective matrix.*

Diagonalizability

Any diagonal matrix is nondefective. For a diagonal matrix, both the algebraic and the geometric multiplicities of an eigenvalue λ are equal to the number of its occurrences along the diagonal. The class of nondefective matrices is precisely the class of matrices that have an eigenvalue decomposition (another term for nondefective is *diagonizable*):

Theorem 3 *An $n \times n$ matrix M is nondefective if and only if it has an eigenvalue decomposition $M = X\Lambda X^{-1}$.*

Similarity

Two matrices M and M' are said to be *similar* if there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that $M' = X^{-1}MX$. Similar matrices share many properties.

Theorem 4 *Two similar matrices have the same characteristic polynomial, eigenvalues, and algebraic and geometric multiplicities.*

Determinant and Trace

The *trace* of $M \in \mathbb{C}^{n\times n}$ is the sum of its diagonal elements: $tr(M) = \sum_{j=1}^{n} m_{jj}$. Both the trace and the determinant are related simply to the eigenvalues.

Theorem 5 *The determinant $det(M)$ and trace $tr(M)$ are equal to the product and the sum of the eigenvalues of M, respectively, counted with algebraic multiplicity:*

$$det(M) = \prod_{j=1}^{n} \lambda_j, \qquad tr(M) = \sum_{j=1}^{n} \lambda_j. \tag{4}$$

Symmetric matrices

In the lectures in this course, we will mostly be concerned with symmetric matrices. We can restate the implication of the above results for a real, symmetric matrix $M \in \mathbb{R}^{n\times n}$ in the following manner (this is also known as the Spectral Theorem):

Theorem 6 *If M is an $n \times n$ symmetrix matrix, then M has real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and n orthonormal eigenvectors forming a basis of $\mathbb{R}^n$.*

Eigenvalues and eigenvectors of symmetric matrices have many characterizations. Some are connected to optimizing the Rayleigh quotient.

Definition 1 *The Rayleigh quotient of a vector y with respect to a matrix M is the ratio*

$$\frac{y^T M y}{y^T y}.$$

Observe that if x is an eigenvector of M of eigenvalue λ, then

$$\frac{x^T M x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \frac{\lambda x^T x}{x^T x} = \lambda.$$

The following is an important result in spectral theory.

Theorem 7 *Let M be a symmetric matrix and let y be the non-zero vector that maximizes the Rayleigh quotient with respect to M. Then, y is an eigenvector of M with eigenvalue equal to the Rayleigh quotient. Moreover, this eigenvalue is the largest eigenvalue of M.*

References

Numerical Linear Algebra, Lloyd N. Trefethen and David Bau, III, SIAM, 1997.

Dan Speilman's class on Spectral Graph Theory, Fall 2015, Lecture 1:
`https://www.cs.yale.edu/homes/spielman/561/`.