CptS 591: Elements of Network Science

Similarity

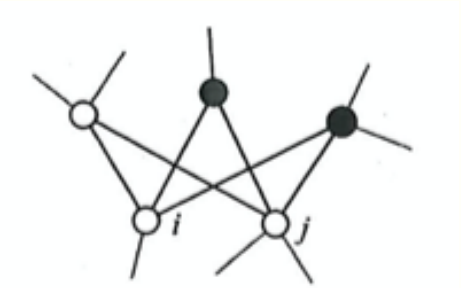
Similarity

- In what ways can vertices in a network be similar?
- How can we quantify that similarity?
- Which vertices in a given network are most similar to one another?
- Which vertex v is most similar to a given vertex u?

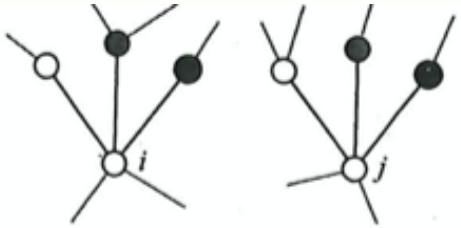
Answers to such questions can help tease apart the types and relationships of vertices in social and information networks.

Two types of similarity

- Structural equivalence
  - Vertices share many of the same neighbors



- Regular equivalence
  - Vertices have neighbors who are themselves similar



We will look at some mathematical measures that quantify these ideas of similarity

Cosine similarity

- Simplest measure: count number of common neighbors
- In an undirected network, the number of common neighbors between i and j is

$$n_{ij} = \sum_k A_{ik} A_{kj}$$

which is the ij-th element of A^2, where A is the adj. matrix

- This, however, does not tell us much on its own.

  We need some sort of normalization.
- Cosine similarity (suggested by Salton (89)) is one such normalized quantity.

Cosine similarity

- Idea: the inner product between two vectors x and y is

$$x \cdot y = |x||y| \cos\theta$$

where θ is the angle between the two vectors.

Rearranging,

$$\cos\theta = \frac{x \cdot y}{|x| \cdot |y|}$$

- Salton proposed to regard the i-th and j-th row (or column) of the adjacency matrix as two vectors and use the cosine of the angle between them as the similarity measure.
- Noting that the dot product is $\sum_k A_{ik} A_{kj}$

The similarity measure is

$$\sigma_{ij} = \cos\theta = \frac{\sum_k A_{ik} A_{kj}}{\sqrt{\sum_k A^2{}_{ik}}\sqrt{\sum_k A^2{}_{jk}}}$$

- Assuming unweighted graph, entries of A are either zero or one, thus

$$\sigma_{ij} = \frac{\sum_k A_{ik} A_{kj}}{\sqrt{d_i}\sqrt{d_j}} = \frac{n_{ij}}{\sqrt{d_i d_j}}$$

Pearson coefficient

- Normalization factor: the expected value the count would take on a network in which vertices choose their neighbors at random.
- Suppose vertices i and j have degrees d_i and d_j
- Suppose further that vertex i chooses the d_i neighbors uniformly at random from the n possibilities, and vertex j similarly chooses d_j neighbors at random.
- For the first neighbor that j chooses, there is a probability of d_i/n that it will choose one of the ones i chose, and similarly for each succeeding choice.
- Then in total the expected number of common neighbors between the two vertices is $d_i d_j/n$

Pearson coefficient

- A reasonable measure of similarity between two vertices is the actual number of common neighbors they have *minus* the expected number that they would have if they chose their neighbors at random:

$$\sum_k A_{ik}A_{jk} - \frac{d_i d_j}{n} = \sum_k A_{ik}A_{jk} - 1/n \sum_k A_{ik} \sum_l A_{jl}$$
$$= \sum_k A_{ik}A_{jk} - n\bar{A}_i\bar{A}_j$$
$$= \sum_k [A_{ik}A_{jk} - \bar{A}_i\bar{A}_j]$$
$$= \sum_k (A_{ik} - \bar{A}_i)(A_{jk} - \bar{A}_j)$$

- This equation is simply n times the *covariance* $\mathrm{cov}(A_i, A_j)$ of the two rows of the matrix.

Pearson coefficient

- It is common to normalize this quantity so that its maximum value is 1
- The maximum value of the covariance of any two sets occurs when the sets are exactly the same, in which case their covariance is equal to the variance of either set, $\sigma_i \sigma_j$
- Normalizing by this quantity gives us the standard Pearson correlation coefficient:

$$r_{ij} = \frac{cov(A_i, A_j)}{\sigma_i \sigma_j}$$

- This quantity lies between -1 and 1

Euclidean distance

- Measures the number of vertices that are neighbors of i but not of j

(more of a dissimilarity measure)

- In terms of the adjacency matrix, Euclidean distance can be written as

$$d_{ij} = \sum_k (A_{ik} - A_{jk})^2$$

- Convenient to normalize by dividing by maximum possible value
- The maximum value of d_{ij} occurs when two vertices have no neighbor in common, in which case $d_{ij} = d_i + d_j$
- Dividing by this quantity, the normalized distance becomes

$$1 - 2\frac{n_{ij}}{d_i + d_j}$$

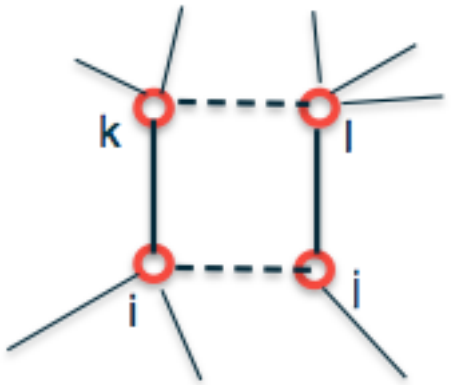
Regular equivalence

- Similarity score σ_{ij} such that i and j have high similarity if they have neighbors k and l that themselves have high similarity. For an undirected network,

$$\sigma_{ij} = \alpha \sum_{kl} A_{ik} A_{jl} \sigma_{kl}$$

Or in matrix terms

$$\sigma = \alpha A \sigma A$$



A type of eigenvector equation.

- This formula has two problems:
  - It does not necessarily give a high value for self similarity
  - It does not necessarily give a high similarity score to vertex pairs that have a lot of common neighbors.

Regular equivalence

- Fix to the two problems: introduce an extra diagonal term in the similarity. Thus

Or in matrix notation

$$\sigma_{ij} = \alpha \sum_{kl} A_{ik} A_{jl} \sigma_{kl} + \delta_{ij}$$

$$\sigma = \alpha A \sigma A + I$$

- When we solve this via repeated iterations, only paths of even length get counted
- Can be further generalized (for all paths) as

$$\sigma = \sum_{m=0}^{\infty} (\alpha A)^m = (I - \alpha A)^{-1}$$

(reminiscent of katz centrality)

Homophily

- People form friendships with those that are similar to them
- This is called *homophily* or *assortative mixing*
- More rarely one also sees *disassortative mixing*

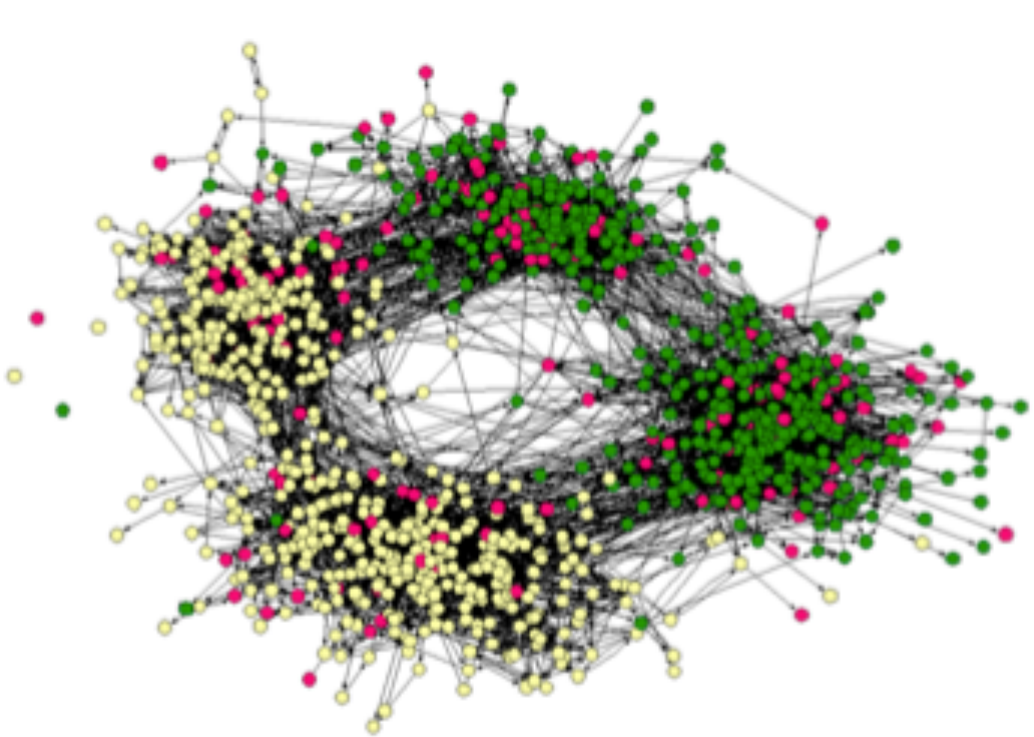
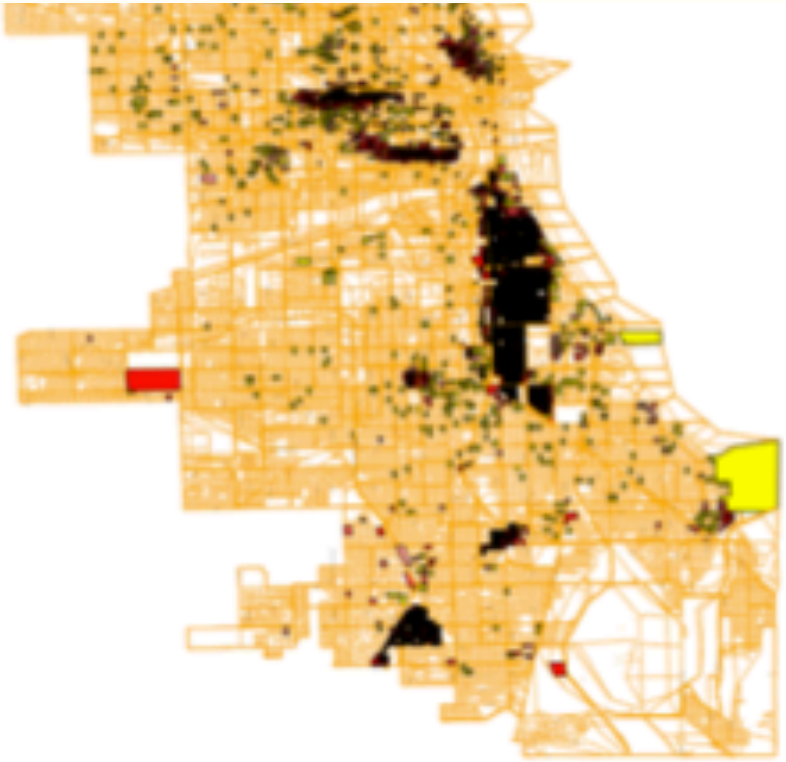


Figure 4.1: Homophily can produce a division of a social network into densely-connected, homogeneous parts that are weakly connected to each other. In this social network from a town's middle school and high school, two such divisions in the network are apparent: one based on race (with students of different races drawn as differently colored circles), and the other based on friendships in the middle and high schools respectively [304].

Homophily



(a) *Chicago, 1940*

(b) *Chicago, 1960*

Further readings

- Section 7.12 and 7.13 of the book Networks, An Introduction by Newman
- Chapter 4 of the book Networks, Crowds and Markets by Easley and Kleinberg.