Microeconomics 2 (386D #34535)

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Contents

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1 Part 1: Static Games of Complete Information

1.1 Preliminaries

Definition 1.1. **[Static]** A static game is one in which a single (or one-shot) decision is made by each player, and each player has no knowledge of the decision made by the other players before making their own decision. In other words, decisions are made simultaneously (or the order of play is irrelevant).

Definition 1.2. **[Complete information]** A complete information game is one where all players' payoff functions (and all other aspects of the game) are common knowledge.

1.2 Strategic (Normal) Form Game

A strategic form game, G, is defined as a triplet, $< N, \boldsymbol{S}, \boldsymbol{u} >$.

- A (finite) set of players: $N = \{1, 2, \dots, n\}$
- A (finite) set of pure strategies (or actions): For each $i \in N$, let S_i denote the set of player i's pure strategies, and call s_i a typical element of S_i. [1] Then, a pure strategy profile $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$ lists one pure strategy for each player. Thus, we can define the set of all pure strategy profiles $\boldsymbol{S}$ such that $\boldsymbol{S} := \times_{i \in N} S_i$, where $\times$ denotes the Cartesian product. [2]
- Payoffs: Let $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$ denote a list of payoff functions such that $u_i : \boldsymbol{S} \to \mathbb{R}$ for each $i \in N$ where u_i is a Bernoulli utility index function. Or equivalently, you may consider $\boldsymbol{u} : \boldsymbol{S} \to \mathbb{R}^n$ as a vector-valued function consisting of Bernoulli utility index functions. Note that these payoffs essentially reflect the rules of the game.

Example 1.1. **[Rock, paper, scissors; RPS]**

- $N = \{1, 2\}$, $S_i = \{R, P, S\}$ for $i \in N$, and hence $\boldsymbol{S} = \times_{i \in N} S_i = \{(R, R), (R, P), \dots, (S, P), (S, S)\}$ where $\mid \boldsymbol{S} \mid = 9$.
- Let the payoffs for the outcomes "win", "lose", "draw" be $1, -1$, and 0, respectively. We then have $u_1((R, R)) = 0, u_1((R, P)) = -1, \dots, u_2((S, P)) = -1, u_2((S, S)) = 0$. Rather than listing each player's payoff to all possible profiles, it is convenient to represent the two payoff functions, u_1 and u_2, in a **payoff matrix** as follows. Notice that player 1 is also referred to as the "row player" and player 2 as the "column player". By convention, the payoff of the row player is the first entry in each cell of the payoff matrix.

		Player 2		
		R	P	S
Player 1	R	$0, 0$	$-1, 1$	$1, -1$
	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

[1] S_i needs not be the same with S_j when $i \neq j$, although they could be so.

[2] A convenient way to compute the cardinality of $\boldsymbol{S}$ is to notice that $\mid \times_{i \in N} S_i \mid = \Pi_{i \in N} \mid S_i \mid$. That is, the cardinality of the Cartesian product is equal to the product of the cardinalities of the constituting sets.

Remark 1.1. In a strategic form game, a player's strategy is a simple non-contingent choice on action, $A_i = S_i$. The RPS game has the property of **zero-sum game**. That is, for all $\boldsymbol{s} \in \boldsymbol{S}$, $\sum_{i=1}^{2} u_i(\boldsymbol{s}) = 0$. The key feature of a zero-sum game is that the players' payoffs always add up to the same **constant** (Need not be zero, zero is just a normalization). Zero-sum games are usually used to model situation of extreme antagonism; I win if and only you lose.

1.3 Mixed Extension

Let $G =< N, \boldsymbol{S}, \boldsymbol{u} >$ be given. Up to this point, we have assumed that players make their actions with certainty. Now, we want to extend this game, G, by allowing mixed strategies. Let us denote the mixed extension of G by $\Gamma =< N, \boldsymbol{\Sigma}, \boldsymbol{U} >$ in order to distinguish the latter from the former.

Definition 1.3. [Mixed strategy] Let Σ_i denote the set of mixed strategies of player $i \in N$. A typical element of $\boldsymbol{\sigma}_i$, is a probability distribution over player i's pure strategies S_i. That is,

$$\begin{aligned}\Sigma_i &=: \Delta(S_i)\\ &= \left\{\boldsymbol{\sigma}_i : (\sigma_i(s_1), \sigma_i(s_2), \ldots, \sigma_i(s_k)) \in \mathbb{R}^k \middle| \sigma_i(s_m) \geq 0,\ \forall m \in \{1, \ldots, k\} \text{ and } \sum_{j=1}^{k} \sigma_i(s_m) = 1\right\}\end{aligned}$$

where $\mid S_i \mid= k$. The set $\Delta(S_i)$ is a simplex in $\mathbb{R}^{|S_i|}$. Observe that the dimension of this simplex is $\mid S_i \mid -1$, because of the constraint that probabilities must sum up to one. The n-dimensional simplex is often denoted by Δ^n.

Remark 1.2. A pure strategy is a degenerate mixed strategy that puts all probability mass on a specific action.

Now, Γ can be summarized as follows.

- A (finite) set of players: $N = \{1, 2, \ldots, n\}$
- For each $i \in N$, let Σ_i denote the set of mixed strategies of player $i \in N$, and call $\boldsymbol{\sigma}_i$ a typical element of Σ_i. Then, a mixed strategy profile $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_n)$ lists one mixed strategy for each player. Thus, the set of all mixed strategy profiles is $\boldsymbol{\Sigma} = \times_{i\in N}\Sigma_i$.
- Player i's payoff, $U_i(\boldsymbol{\sigma})$, from a mixed strategy profile $\boldsymbol{\sigma}$, is a Von Neumann-Morgenstern expected utility;

$$U_i(\boldsymbol{\sigma}) = \sum_{\boldsymbol{s}\in\boldsymbol{S}} \left(\prod_{i\in N} \sigma_i(s_i)\right) u_i(\boldsymbol{s}) =: \mathbb{E}_{\boldsymbol{\sigma}}[u_i(\boldsymbol{s})]$$

Remark 1.3. Since we assume that each player randomizes on her own, each player's randomization is statistically independent of those her opponents. Thus, the probability under $\boldsymbol{\sigma}$ that the pure strategy $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is played is $\prod_{i\in N} \sigma_i(s_i)$.

Note that we always allow mixed strategies unless otherwise stated. Thus, even if we are given G, we will actually consider the mixed extension version of G, i.e. Γ.

Example 1.2. [RPS]

- $\Sigma_i = \{\boldsymbol{\sigma}_i = (\sigma_i(R), \sigma_i(P), \sigma_i(S)) \in \mathbb{R}^3 \mid \sigma_i(R) \geq 0, \sigma_i(P) \geq 0, \sigma_i(S) \geq 0, \text{ and } \sigma_i(R) + \sigma_i(P) + \sigma_i(S) = 1\}$. Thus, $\Sigma_i = \Delta^2$, the two-dimensional simplex.

- Player 2's expected utility from playing paper against player 1's mixed strategy $\boldsymbol{\sigma}_1$ is

$$
\begin{aligned}
U_2(\boldsymbol{\sigma}) &= \sum_{\boldsymbol{s}\in\boldsymbol{S}}\left(\prod_{i\in N}\sigma_i(s_i)\right)u_2(\boldsymbol{s}) \\
&= \sigma_1(R)\sigma_2(P)u_2((R,P)) + \sigma_1(P)\sigma_2(P)u_2((P,P)) + \sigma_1(S)\sigma_2(P)u_2((S,P)) \ (\because \sigma_2(R)=\sigma_2(S)=0) \\
&= \sigma_1(R)\times 1 + \sigma_1(P)\times 0 + \sigma_1(S)\times(-1) \ (\because \sigma_2(P)=1) \\
&= \sigma_1(R)\times 1 + [1-\sigma_1(R)-\sigma_2(P)]\times(-1) \\
&= 2\sigma_1(R)+\sigma_1(P)-1
\end{aligned}
$$

1.4 Strict Dominance and Iterated Strict Dominance

Notation 1. $(s_i, \boldsymbol{s}_{-i}) = (s_1,\dots,s_i,\dots,s_n)$ and $(s_i', \boldsymbol{s}_{-i}) = (s_1,\dots,s_i',\dots,s_n)$.

Definition 1.4. [Strict dominance] $\boldsymbol{\sigma}_i\in\Sigma_i$ is strictly dominated if there exists $\boldsymbol{\sigma}_i'\in\Sigma_i$ such that

$$\forall \boldsymbol{s}_{-i}\in\boldsymbol{S}_{-i},\ U_i(\boldsymbol{\sigma}_i', \boldsymbol{s}_{-i}) > U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i})$$

Proposition 1.1. *$U_i(\boldsymbol{\sigma}_i', \boldsymbol{s}_{-i}) > U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i}),\ \forall \boldsymbol{s}_{-i}\in\boldsymbol{S}_{-i}$ if and only if $U_i(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_{-i}) > U_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}),\ \forall\boldsymbol{\sigma}_{-i}\in\boldsymbol{\Sigma}_{-i}$.*

Proof. If part is trivial because $\boldsymbol{S}_{-i}\subset\boldsymbol{\Sigma}_{-i}$. Only if part follows because

$$U_i(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_{-i}) - U_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) = \sum_{\boldsymbol{s}_{-i}\in\boldsymbol{S}_{-i}}\left[\prod_{j\neq i}\sigma_j(s_j)\right][U_i(\boldsymbol{\sigma}_i', \boldsymbol{s}_{-i}) - U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i})]$$

□

Proposition 1.2. *If a pure strategy s_i is strictly (weakly) dominated, then any mixed strategy that plays s_i with a positive probability is also strictly (weakly) dominated.*

Proof. Suppose that $\widehat{s}_i$ is strictly dominated. Then, there exists $\boldsymbol{\sigma}_i^*\in\sum_i$ such that

$$\forall \boldsymbol{s}_{-i}\in\boldsymbol{S}_{-i},\ U_i(\boldsymbol{\sigma}_i^*, \boldsymbol{s}_{-i}) > u_i(\widehat{s}_i, \boldsymbol{s}_{-i})$$

Now, consider any mixed strategy $\widehat{\boldsymbol{\sigma}}_i$ such that $\widehat{\sigma}_i(\widehat{s}_i)>0$. Then, for all pure strategies $\boldsymbol{s}_{-i}\in\boldsymbol{S}_{-i}$

$$
\begin{aligned}
U_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{s}_{-i}) &= \sum_{s_i\in S_i}\widehat{\sigma}_i(s_i)u_i(s_i,\boldsymbol{s}_{-i}) \\
&= \widehat{\sigma}_i(\widehat{s}_i)u_i(\widehat{s}_i,\boldsymbol{s}_{-i}) + \sum_{s_i\in S_i\setminus\{\widehat{s}_i\}}\widehat{\sigma}_i(s_i)u_i(s_i,\boldsymbol{s}_{-i})
\end{aligned}
$$

Then, consider a mixed strategies $\boldsymbol{\sigma}_i^{**}$ such that for each $s_i\in S_i$

$$\sigma_i^{**}(s_i) = \widehat{\sigma}_i(\widehat{s}_i)\sigma_i^*(s_i) + [1-\widehat{\sigma}_i(\widehat{s}_i)]\widehat{\sigma}_i(s_i)$$

First, notice that $\boldsymbol{\sigma}_i^{**}$ is an indeed mixed strategy because

$$\sum_{s_i \in S_i} \sigma_i^{**}(s_i) = \widehat{\sigma}_i(\widehat{s}_i) \sum_{s_i \in S_i} \sigma_i^*(s_i) + [1 - \widehat{\sigma}_i(\widehat{s}_i)] \sum_{s_i \in S_i \setminus \{\widehat{s}_i\}} \widehat{\sigma}_i(s_i)$$

$$= 1$$

Second, observe that for all pure strategies $\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$

$$U_i(\boldsymbol{\sigma}_i^{**}, \boldsymbol{s}_{-i}) = \widehat{\sigma}_i(s_i) U_i(\boldsymbol{\sigma}_i^*, \boldsymbol{s}_{-i}) + \sum_{s_i \in S_i \setminus \{\widehat{s}_i\}} \widehat{\sigma}_i(s_i) u_i(s_i, \boldsymbol{s}_{-i})$$

Since $u_i(\boldsymbol{\sigma}_i^*, \boldsymbol{s}_{-i}) > u_i(\widehat{s}_i, \boldsymbol{s}_{-i})$ for all pure strategies $\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$, we then have

$$\forall \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i},\ u_i(\boldsymbol{\sigma}_i^{**}, \boldsymbol{s}_{-i}) > u_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{s}_{-i})$$

That is, $u_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{s}_{-i})$ is strictly dominated.

By the same manner, we can also show that this argument is still valid for weakly dominance case. □

Example 1.3. Consider a following strategic form game.

Player 2

Player 1		L	R
	T	4, 3	0, 0
	M	0, 2	5, 1
	B	1, 1	1, 6

Consider $\boldsymbol{\sigma}_1' = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\boldsymbol{\sigma}_1 = (0, 0, 1)$ where $\boldsymbol{\sigma}_1$ is a pure strategy choosing B. Observe

$$U_1(\boldsymbol{\sigma}_1', L) = 2 > 1 = U_1(\boldsymbol{\sigma}_1, L) \text{ and } U_1(\boldsymbol{\sigma}_1', R) = 2.5 > 1 = U_1(\boldsymbol{\sigma}_1, R)$$

Thus, $\boldsymbol{\sigma}_1'$ strictly dominates $\boldsymbol{\sigma}_1$, i.e. B.

Remark 1.4. In the example above, since B is strictly dominated, can we eliminate it from player 1's action set? We would need to assume something like "player 1 is rational". Would that be sufficient? No! It is true that if player 1 is rational, she will not choose B. But if player 2 does not know that player 1 is rational, he may think that player 1 might choose B. So we need to assume something like "player 2 knows that player 1 is rational". For the next steps of iterated deletion of strictly dominated strategies (at which we would like to delete R, then M), we will also need something like "player 1 know that player 2 knows that player 1 is rational" and so on. Otherwise, player 1 would need to consider the possibility that player 2 (thinking that player 1 is not rational and chooses B) might play R, in which case player 1 might be tempted to play M.

Definition 1.5. **[Iterated deletion of strictly dominated strategies]** Consider $\Gamma =< N, \boldsymbol{\Sigma}, \boldsymbol{U} >$. Let $\boldsymbol{\Sigma}^0 = \boldsymbol{\Sigma}$. For every $i \in N$, for every $k \geq 1$, let

$$\Sigma_i^k = \left\{ \boldsymbol{\sigma}_i \text{ is not strictly dominated in } \Gamma \middle| \Sigma_i^{k-1} \right\}$$

The solution in iterated strict dominance is

$$\mathbf{\Sigma}^{\infty} = \times_{i \in N} \bigcap_{k=1}^{\infty} \Sigma_i^k$$

Remark 1.5. Once we have determined the set of undominated pure strategies for player i using proposition 1.1, we need to consider which mixed strategies are undominated. We can immediately eliminate any mixed strategy that uses a dominated pure strategy by proposition 1.2. Be careful that a mixed strategy that randomizes over undominated pure strategies may be dominated. Therefore, after deleting pure strategies, we need to look at mixed strategies. [3]

Remark 1.6. For the kth iteration, we need $k+1$ levels of common knowledge of rationality. If we assume common knowledge of rationality (i.e. all levels, all the way to infinity), then we can apply the iterated deletion of strictly dominated strategies.

Proposition 1.3. *The order of deletion does not matter for the set of strategies surviving a process of iterated deletion of strictly dominated strategies.*

Proof. Consider the kth round, given Σ_i^{k-1}. Suppose $\boldsymbol{\sigma}_i \in \Sigma_i^{k-1}$ is strictly dominated, then there exists a strategy $\boldsymbol{\sigma}_i^* \in \Sigma_i^{k-1}$ such that

$$\forall \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i},\ U_i(\boldsymbol{\sigma}_i^*, \boldsymbol{s}_{-i}) > U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i})$$

and $\boldsymbol{\sigma}_i^*$ is not a strictly dominated given $\mathbf{\Sigma}_{-i}^{k-1}$. Then, $\boldsymbol{\sigma}_i^* \in \Sigma_i^k$. Let us not delete the dominated strategy $\boldsymbol{\sigma}_i$ at the kth round, so that $\boldsymbol{\sigma}_i \in \Sigma_i^k$. Since $\boldsymbol{S}_{-i}^k \subset \boldsymbol{S}_{-i}^{k-1}$, at the $k+1$th round it must still be the case that $\boldsymbol{\sigma}_i^*$ strictly dominates $\boldsymbol{\sigma}_i$, so we can delete it at the $k+1$th round. $\square$

Remark 1.7. The order and speed at which dominated strategies are eliminated have no effect on the set of strategy profiles that survive. This is not true for the iterated deletion of weakly dominated strategies. See example 1.6.

Example 1.4. [Prisoner's dilemma] Note that payoff is the number of years, out of the next 3, that you don't have to go to prison.

		Player 2: C	Player 2: D
Player 1	C	$2,2$	$0,3$
	D	$3,0$	$1,1$

For each player, D strictly dominates C. Thus, (D,D) is the unique strategy profile surviving the iterated deletion of strictly dominated strategies. That is, $\mathbf{\Sigma}^{\infty} = \{(D,D)\}$. If by iterated elimination of strictly dominated strategies there is only one strategy for each player, the game is called a **dominance-solvable**.

Remark 1.8. When the solution is a unique strategy profile, this solution concept is very strong; In any other outcome, at least one player must have accepted less than what she could have obtained. Still, this could happen; For instance, a player might care about the payoff of her accomplice. Or she might think her accomplice is irrational.

[3] Notice that although neither U nor strategy D would be strictly dominated, the randomized strategy $\frac{1}{2}U + \frac{1}{2}D$ would be strictly dominated by strategy M.

		Player 2: L	Player 2: R
Player 1	U	$10,1$	$0,4$
	M	$6,2$	$6,3$
	D	$0,5$	$10,2$

Example 1.5. **[Matching pennies]** In this game, we cannot eliminate any strategy, and so $\boldsymbol{\Sigma}^\infty = \boldsymbol{\Sigma}$. In this sense, IDSDS is weak as the concept solution because in some games, it really tells nothing.

		Player 2	
		H	T
Player 1	H	$1,-1$	$-1,1$
	T	$-1,1$	$1,-1$

1.5 Weak Dominance and Iterated Weak Dominance

Definition 1.6. **[Weak dominance]** $\boldsymbol{\sigma}_i \in \Sigma_i$ is weakly dominated if there exists $\boldsymbol{\sigma}_i' \in \Sigma_i$ such that

$$\forall \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i},\ U_i(\boldsymbol{\sigma}_i', \boldsymbol{s}_{-i}) \geq U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i}) \text{ and}$$
$$\exists \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i},\ U_i(\boldsymbol{\sigma}_i', \boldsymbol{s}_{-i}) > U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i})$$

Assume [4] that players never use a weakly dominated strategy. Proceed by undertaking the iterated [5] deletion of weakly dominated strategies. The algorithm is the analogue of the one defined for the iterated deletion of strictly dominated strategies.

Example 1.6. **[Order of deletion for weakly dominated strategies]**

		Player 2		
		L	C	R
Player 1	T	$1,2$	$2,3$	$0,3$
	M	$2,2$	$2,1$	$3,2$
	B	$2,1$	$0,0$	$1,0$

Deleting strategies in the following order, we obtain; $T,R,B,C \to (M,L)$; $B,L,C,T \to (M,R)$; $T,C,R \to (M,L)$ or (B,L); $T,B,C \to (M,L)$ or (M,R).

Remark 1.9. Different orders for IDWDS can lead to different results, and even different number of strategy profiles surviving the iterated deletion of weakly dominated strategies.

1.6 Nash Equilibrium

The notion of best-response is weaker than the notion of dominance; a strategy is a best response against a given strategy profile for the other players. For a strategy to be dominant, it must do better another strategy against every possible strategy profile for the other players.

[4] This assumption can be justified using trembling hand: Assume that the players' hands "tremble" in a way that is parameterised by some $\epsilon > 0$, so that every pure strategy may be played with strictly positive probability. For instance, in the following example, assume that player 2 has a trembling hand; $\sigma_2^\epsilon = (\epsilon, 1-\epsilon)$. Then, $U_1(T, \boldsymbol{\sigma}_2^\epsilon) = 2-\epsilon$, and $U_1(B, \boldsymbol{\sigma}_2^\epsilon) = 2$. Thus, player 1 facing a player 2 who has trembling hand will not choose T, which is weakly dominated by B, even if we let $\epsilon \to 0$.

		Player 2	
		L	R
Player 1	T	$1,2$	$2,3$
	B	$2,2$	$2,0$

[5] This is hard to justify; As we have already indicated, the argument for deletion of a weakly dominated strategy for player i is that he contemplates the possibility that every strategy combination of his rival occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur. This inconsistency leads the iterative elimination of weakly dominated strategies to have the undesirable feature that it can depend on the order of deletion.

Definition 1.7. [Best-response] $\boldsymbol{\sigma}_i \in \Sigma_i$ is a best-response to $\boldsymbol{\sigma}_{-i} \in \boldsymbol{\Sigma}_{-i}$ if

$$\forall s_i' \in S_i,\ U_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq U_i(s_i', \boldsymbol{\sigma}_{-i})$$

Notice that this definition also implies that $\forall \boldsymbol{\sigma}_i' \in \Sigma_i,\ U_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq U_i(\boldsymbol{\sigma}_i', \boldsymbol{\sigma}_{-i})$ if $\boldsymbol{\sigma}_i$ is a best-response. Because, $U_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) - U_i(\boldsymbol{\sigma'}_i, \boldsymbol{\sigma}_{-i}) = \sum\limits_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} \left[\prod_{j \neq i} \sigma_j(s_j)\right] \sum\limits_{s_i' \in S_i} \sigma_i'(s_i')[U_i(\boldsymbol{\sigma}_i, \boldsymbol{s}_{-i}) - U_i(s'_i, \boldsymbol{s}_{-i})]$. In other words, we only need to check for deviation in pure strategies in order to verify whether a strategy profile constitutes a NE.

Definition 1.8. [Nash equilibrium] $\boldsymbol{\sigma}^* \in \boldsymbol{\Sigma}$ is a Nash equilibrium if $\forall i \in N$,

$$\forall s_i' \in S_i,\ U_i(\boldsymbol{\sigma^*}_i, \boldsymbol{\sigma}^*_{-i}) \geq U_i(s_i', \boldsymbol{\sigma}^*_{-i})$$

Example 1.7. [Chicken game]

		Player 2	
		H	D
Player 1	H	$-1,-1$	$2,1$
	D	$1,2$	$0,0$

Pure strategy NE are (D, H) and (H, D). Using proposition 1.4, let's solve for mixed strategy equilibria of this game. Let $\boldsymbol{\sigma}_1 := (p, 1-p)$ and $\boldsymbol{\sigma}_1 := (q, 1-q)$. Given $\boldsymbol{\sigma}_2$, player 1 must be indifferent between H and D, in order to play $\boldsymbol{\sigma}_1$ with $0 < p < 1$;

$$U_1(H, \boldsymbol{\sigma}_2) = U_1(D, \boldsymbol{\sigma}_2) \ \Leftrightarrow\ q(-1) + (1-q)(2) = q(1) + (1-q)(0)$$

Thus, we have $q = \frac{1}{2}$. Similarly, we obtain that $p = \frac{1}{2}$. Hence, we have the following mixed strategy equilibria which includes the two pure strategy profiles;

$$\{(\frac{1}{2}, \frac{1}{2}), (0,1), (1,0)\}$$

Proposition 1.4. *If $\boldsymbol{\sigma}^*$ is a NE in non-degenerate mixed strategies ($\exists\ s_i, s_i' \in S_i$ such that $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(s_i') > 0$.), it must be that*

$$U_i(s_i, \boldsymbol{\sigma}^*_{-i}) = U_i(s_i', \boldsymbol{\sigma}^*_{-i})$$

That is, all the pure strategies in the support of the mixed strategy $\boldsymbol{\sigma}^*$ must yield the same payoff.

Proof. Suppose not. That is, we have $\sigma_i^*(s_i') > 0$ and $\sigma_i^*(s_i'') > 0$ such that

$$u_i(s_i', \boldsymbol{\sigma}^*_{-i}) > u_i'(s_i'', \boldsymbol{\sigma}^*_{-i})$$

Consider a mixed strategy $\widehat{\boldsymbol{\sigma}}_i \in \sum_i$ such that

$$\widehat{\sigma}_i(s_i) = \begin{cases} \sigma_i^*(s_i') + \sigma_i^*(s_i'') & \text{if } s_i = s_i' \\ 0 & \text{if } s_i = s_i'' \ = 0 \\ \sigma_i^*(s_i) & \text{otherwise} \end{cases}$$

Now observe that

$$\begin{aligned}
u_i(\widehat{\sigma}_i, \boldsymbol{\sigma}^*_{-i}) &= [\sigma^*_i(s'_i) + \sigma^*_i(s''_i)]u_i(s'_i, \boldsymbol{\sigma}^*_{-i}) + 0 \times u'_i(s''_i, \boldsymbol{\sigma}^*_{-i}) + \sum_{s_i \neq s'_i, s''_i} {\sigma^*}_i(s_i)u_i(s_i, \boldsymbol{\sigma}^*_{-i}) \\
&= \sigma^*_i(s''_i)u_i(s'_i, \boldsymbol{\sigma}^*_{-i}) + \sum_{s_i \neq s''_i} {\sigma^*}_i(s_i)u_i(s_i, \boldsymbol{\sigma}^*_{-i})
\end{aligned}$$

Given that $u_i(s'_i, \boldsymbol{\sigma}^*_{-i}) > u'_i(s''_i, \boldsymbol{\sigma}^*_{-i})$, and hence $\sigma^*_i(s''_i)u_i(s'_i, \boldsymbol{\sigma}^*_{-i}) > \sigma^*_i(s''_i)u_i(s''_i, \boldsymbol{\sigma}^*_{-i})$, we must have

$$\begin{aligned}
u_i(\widehat{\sigma}_i, \boldsymbol{\sigma}^*_{-i}) &= \sigma^*_i(s''_i)u_i(s'_i, \boldsymbol{\sigma}^*_{-i}) + \sum_{s_i \neq s''_i} {\sigma^*}_i(s_i)u_i(s_i, \boldsymbol{\sigma}^*_{-i}) \\
&> \sigma^*_i(s''_i)u_i(s''_i, \boldsymbol{\sigma}^*_{-i}) + \sum_{s_i \neq s''_i} {\sigma^*}_i(s_i)u_i(s_i, \boldsymbol{\sigma}^*_{-i}) \\
&= \sum_{s_i \in S_i} {\sigma^*}_i(s_i)u_i(s_i, \boldsymbol{\sigma}^*_{-i}) \\
&= u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}^*_{-i})
\end{aligned}$$

This is a contradiction since $\boldsymbol{\sigma}^*$ is a NE and hence there cannot exist such a profitable deviation $\widehat{\boldsymbol{\sigma}}_i$ from $\boldsymbol{\sigma}^*{}_{-i}$. Thus, the given argument is true. □

Proposition 1.5. *Let $S_i^+ \subseteq S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$. Strategy profile $\boldsymbol{\sigma}$ is a NE if and only if for all $i \in N$,*

1) $u_i(s_i, \boldsymbol{\sigma}_{-i}) = u_i(s'_i, \boldsymbol{\sigma}_{-i})$ for all $s_i, s'_i \in S_i^+$
2) $u_i(s_i, \boldsymbol{\sigma}_{-i}) \geq u_i(s'_i, \boldsymbol{\sigma}_{-i})$ for all $s_i \in S_i^+$ and all $s'_i \notin S_i^+$

Remark 1.10. Proposition 1.5 tells that a necessary and sufficient condition for mixed strategy profile $\boldsymbol{\sigma}$ to be a NE is that each player, given the distribution of strategies played by his opponents, is indifferent among all the pure strategies that he plays with positive probability and that these pure strategies are at least as good as any pure strategy he plays with zero probability.

Remark 1.11. A NE may be a profile of weakly dominated strategies. The following example illustrates this;

		Player 2	
		L	R
Player 1	T	$2,2$	$0,2$
	B	$2,0$	$1,1$

Here, B weakly dominates T and R weakly dominates L. Yet, (T, L) is also a NE.

1.6.1 Nash Equilibrium Existence

Definition 1.9. $\Gamma =< N, \boldsymbol{\Sigma}, \boldsymbol{U} >$ is finite if N and each S_i is finite.

Definition 1.10. [Best response correspondence] For all $i \in N$, let $r_i(\boldsymbol{\sigma}_{-i}) \subseteq \Sigma_i$ be the set of player i's best responses to the strategy profile $\boldsymbol{\sigma}_{-i} \in \boldsymbol{\Sigma}_{-i}$. Let

$$\boldsymbol{r}(\boldsymbol{\sigma}) := (r_1(\boldsymbol{\sigma}_{-1}), \ldots, r_{|N|}(\boldsymbol{\sigma}_{-|N|})) \subseteq \boldsymbol{\Sigma}$$

Then, $\boldsymbol{r} : \boldsymbol{\Sigma} \rightrightarrows \boldsymbol{\Sigma}$ is the best-response correspondence. [6]

Definition 1.11. A fixed point of $\boldsymbol{r}$ is a strategy profile $\boldsymbol{\sigma}^* \in \boldsymbol{\Sigma}$ such that $\boldsymbol{\sigma}^* \in \boldsymbol{r}(\boldsymbol{\sigma}^*)$. It is a Nash equilibrium.

Theorem 1.1. **[Kakutani's fixed point theorem]** *Let X be a compact, convex set in $\mathbb{R}^k$. Suppose that $\forall x \in X$, the set $\boldsymbol{j}(x)$ is a non-empty, convex subset of X, and that the graph $\mathcal{G} : \{(x, y) : y \in \boldsymbol{j}(x)\}$ is a closed subset of $X \times X$. Then, $\boldsymbol{j} : X \rightrightarrows X$ admits a fixed point in X.*

Definition 1.12. **[Upper hemi-continuity]** Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Let $\boldsymbol{f} : X \rightrightarrows Y$. The correspondence $\boldsymbol{f}$ is upper hemi-continuous if, for every $(\boldsymbol{x}, \boldsymbol{y}) \in X \times Y$, if $\boldsymbol{x}^n \to \boldsymbol{x}$, $\boldsymbol{x}^n \in X$, and $\boldsymbol{y}^n \to \boldsymbol{y}$, $\boldsymbol{y}^n \in \boldsymbol{f}(\boldsymbol{x}^n)$, then $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{x})$.

Theorem 1.2. **[Existence of NE]** *Every finite strategic form game, Γ, has a mixed strategy NE.*

Proof. Let us show that we can apply Kakutani's fixed point theorem to $\boldsymbol{r}$.

[Step 1] $X = \boldsymbol{\Sigma}$ is a compact and non-empty. Given that N and S_i are finite, for each $i \in N$, Σ_i is a $\mid S_i \mid -1$ dimensional simplex. Notice that Σ_i is compact and convex in $\mathbb{R}^{|S_i|}$. Thus, $\boldsymbol{\Sigma} = \times_{i \in N} \Sigma_i$ is a compact and convex set in $\mathbb{R}^{\prod_{i \in N} |S_i|}$.

[Step 2] $\boldsymbol{r}$ is non-empty. Given that S_i is finite, at least one of its elements does better than the other against $\boldsymbol{\sigma}_{-i}$.

[Step 3] $\boldsymbol{r}$ is convex valued. Let $\boldsymbol{\sigma}_i \in r_i(\boldsymbol{\sigma}_{-i})$ and $\widehat{\boldsymbol{\sigma}}_i \in r_i(\boldsymbol{\sigma}_{-i})$. Since $\boldsymbol{r}$ is a best response correspondence, for any $\boldsymbol{\sigma}'_i \in \Sigma_i$, we have

$$u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_{-i}) \text{ and } u_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_{-i})$$

Notice that for any $\lambda \in [0, 1]$ and for any $\boldsymbol{\sigma}'_i \in \Sigma_i$,

$$\underbrace{\lambda u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i}) + (1 - \lambda) u_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i})}_{= u_i(\lambda \boldsymbol{\sigma}_i + (1-\lambda)\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i})} \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}_{-i})$$

Thus, $\lambda \boldsymbol{\sigma}_i + (1 - \lambda)\widehat{\boldsymbol{\sigma}}_i \in r_i(\boldsymbol{\sigma}_{-i})$.

[Step 4] $\boldsymbol{r}$ has a closed graph. Given that $\boldsymbol{\Sigma}$ is bounded, we only need to show that $\boldsymbol{r}$ is upper hemi-continuous. Consider a sequence of strategy profile $(\boldsymbol{\sigma}^n, \widetilde{\boldsymbol{\sigma}}^n) \to (\boldsymbol{\sigma}, \widetilde{\boldsymbol{\sigma}})$ such that $\widetilde{\boldsymbol{\sigma}}^n \in \boldsymbol{r}(\boldsymbol{\sigma}^n)$. Then, $\boldsymbol{r}$ is upper hemi-continuous if and only if $\widetilde{\boldsymbol{\sigma}} \in \boldsymbol{r}(\boldsymbol{\sigma})$. By way of contradiction, suppose not, i.e. $\widetilde{\boldsymbol{\sigma}} \notin \boldsymbol{r}(\boldsymbol{\sigma})$. Then, there exists $i \in N$, $\boldsymbol{\sigma}^*_i$, and $\epsilon > 0$ such that

$$u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}_{-i}) > u_i(\widetilde{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i}) + \epsilon$$

Along the sequence, $\widetilde{\boldsymbol{\sigma}}^n_i$ is a best response to $\boldsymbol{\sigma}^n$, so

$$u_i(\widetilde{\boldsymbol{\sigma}}^n_i, \boldsymbol{\sigma}^n_{-i}) \geq u_i(\boldsymbol{\sigma}'_i, \boldsymbol{\sigma}^n_{-i}), \ \forall \boldsymbol{\sigma}'_i \in \Sigma_i$$

In particular,

$$u_i(\widetilde{\boldsymbol{\sigma}}^n_i, \boldsymbol{\sigma}^n_{-i}) \geq u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}^n_{-i})$$

By continuity of u_i, $u_i(\widetilde{\boldsymbol{\sigma}}^n_i, \boldsymbol{\sigma}^n_{-i}) \to u_i(\widetilde{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i})$ and $u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}^n_{-i}) \to u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}_{-i})$, and hence we have

$$u_i(\widetilde{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i}) \geq u_i(\boldsymbol{\sigma}^*_i, \boldsymbol{\sigma}_{-i})$$

which yields a contradiction.

[6] Note that $\rightrightarrows$ represents a correspondence.

[**Step 5**] By step 1 through step 4, $\boldsymbol{r}$ admits a fixed point, $\boldsymbol{\sigma}^*$, in $\boldsymbol{\Sigma}$ by Kakutani's theorem. $\square$

1.7 Rationalisability

Definition 1.13. **[Never a Best response]** $\boldsymbol{\sigma}_i \in \sum_i$ is never a best response if there exists no $\boldsymbol{\sigma}_{-i} \in \sum_{-i}$ for which $\boldsymbol{\sigma}_i$ is a best response.

Remark 1.12. If $\boldsymbol{\sigma}_i$ is strictly dominated, then $\boldsymbol{\sigma}_i$ is never a best response. However, even if $\boldsymbol{\sigma}_i$ is weakly dominated, we cannot say that $\boldsymbol{\sigma}_i$ is never a best response as shown remark 1.11.

Definition. **[Rationalisability]** Rationalisablity is the iterated deletion of strategies that are never a best response.

Remark 1.13. A strategy profile is rationalisable if and only if it is a best response to some strategy profile that is itself rationalisable.

Remark 1.14. The order of deletion does not matter.

Remark 1.15. For $\mid N \mid > 2$, the set of rationalisable strategies $\subseteq$ the set strategies surviving IDSDS. For $\mid N \mid = 2$, the two sets are identical.

Remark 1.16. A NE is a rationalisable profile. The converse is not true. For example, matching pennies game shows that every strategy profile is rationalisable, but derive the unique mixed strategy NE.

1.8 Correlated Equilibrium

Example 1.8. Consider the following 2×2 game.

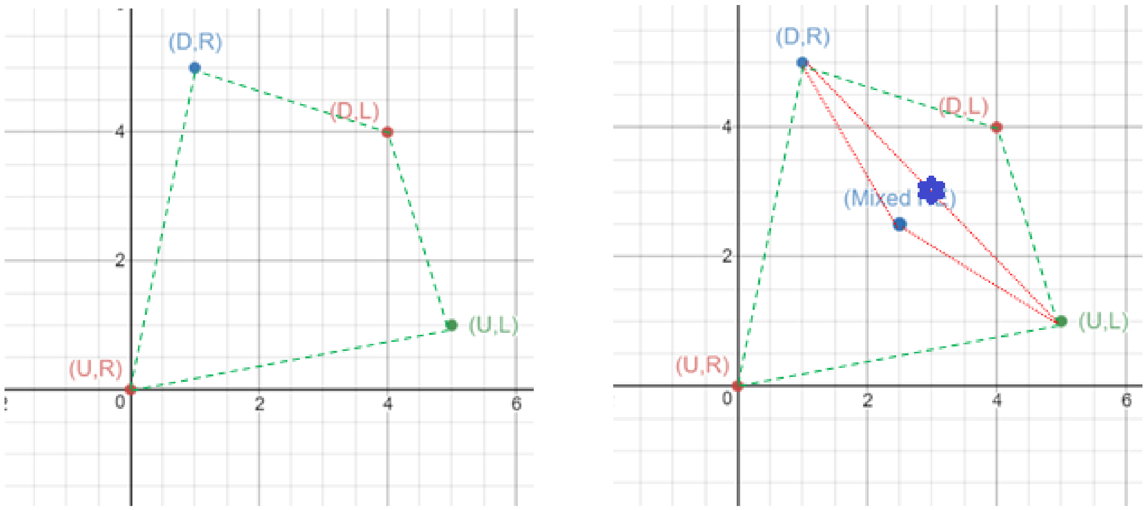
		Player 2	
		L	R
Player 1	U	$5, 1$	$0, 0$
	D	$4, 4$	$1, 5$

It is easy to show that there are three NE, two pure strategy NE and one mixed strategy NE. [7]

$$NE = \left\{ (U, L), (D, R), (\frac{1}{2}, \frac{1}{2}) \right\}$$

With regard to this game, we can illustrate a set of feasible payoffs and convex hull of set of NE payoffs as below.

[7] Rigorously, it should be written as $\boldsymbol{\sigma}_1(\sigma_1(U), \sigma_1(D)) = (\frac{1}{2}, \frac{1}{2})$, and $\boldsymbol{\sigma}_2(\sigma_2(L), \sigma_2(R)) = (\frac{1}{2}, \frac{1}{2})$.



(a) Set of feasible payoffs (in green)

(b) Convex hull of set of NE payoffs (in red)

1.8.1 Public Randomization

We will call publicly observable random variable sun spot.

Proposition 1.6. *If players can jointly observe a sun spot before play, they can achieve the payoffs* $(3, 3)$ *in example 1.8 by jointly randomization between two pure strategy NE.*

Proof. Consider the following strategies that map each realization of the coin flip into a pure strategy.

$$\text{Player 1} = \begin{cases} U & \text{if heads} \\ D & \text{if tails} \end{cases} \quad \text{and} \quad \text{Player 2} = \begin{cases} L & \text{if heads} \\ R & \text{if tails} \end{cases}$$

Then, (U, L) is played with probability $\frac{1}{2}$ and (D, R) is played with probability $\frac{1}{2}$, so that each player's expected payoff is 3. Also notice that two players have no incentive to deviate. This is because, the public randomization device serves to coordinate on a particular NE. For example, suppose H is publicly observed. Given that player 2 follows the suggested direction, i.e. playing L, player 1's best response is to player U, which means that player 1 never deviates from the suggested direction. □

Remark 1.17. By using a public randomization device, players can achieve any payoff in the convex hull of the st of NE payoffs. Conversely, using public randomization does not allow players to obtain payoffs outside the convex hull of the set of NE payoffs.

1.8.2 Correlated Private Signals

With a public randomization device, the "signal" that the players observe are perfectly correlated. Now, we want to allow for other types of correlation.

Definition 1.14. **[Correlation device]** A correlation device is a triple $(\Omega, \pi, \{\mathcal{P}_i\}_{i \in N})$, where Ω is a finite state space corresponding to the outcomes of the device, and π is a probability measure on the state space Ω. For each $i \in N$, $\mathcal{P}_i$ is a partition of Ω representing player i's information.

Example 1.9. $\Omega = \{A, B, C\}$ and $\pi(\{A\}) = \pi(\{B\}) = \pi(\{C\}) = 1/3$. $\mathcal{P}_1 = \{\{A\}, \{B, C\}\}$ and $\mathcal{P}_2 = \{\{A, B\}, \{C\}\}$. If state A occurs, player 1 learns that state A occurs, and player 2 learns $\{A, B\}$, i.e. that the event has occurred that either state A or state B has occurred. Player 2 does not know whether it is state A or state B that has occurred, she just knows it is one of the two. If state B occurs, player 2 learns that either state A or state B has occurred, and player 1 learns that state B or state C has occurred. Then, what is the beliefs about the state and the other player's private signal?

- If player 1 observes the signal $\{A\}$, player 1 knows that the state is $\{A\}$, and therefore she also knows that player 2 observes $\{A, B\}$.
- If player 1 observes the signal $\{B, C\}$, player 1 does know whether the state is $\{B\}$or $\{C\}$. Given that she is Bayesian, she holds the following beliefs about the realized state
  - $\pi(\{B\} \mid \{B, C\}) = \frac{\pi(\{B\})}{\pi(\{B,C\})} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$ and $\pi(\{C\} \mid \{B, C\}) = \frac{\pi(\{B\})}{\pi(\{B,C\})} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$
  - She therefore believes that player 2 is equally likely to have observed the private signal $\{A, B\}$ or $\{C\}$.

Proposition 1.7. *The following strategy is an equilibrium* [8] *in example 1.8.*

$$\text{Player 1} = \begin{cases} U & \text{if } \{A\} \text{ received} \\ D & \text{if } \{B, C\} \text{ received} \end{cases} \quad \text{and} \quad \text{Player 2} = \begin{cases} L & \text{if } \{A, B\} \text{ received} \\ R & \text{if } \{C\} \text{ received} \end{cases}$$

Proof. Let's check that no player has a profitable deviation incentive.

[Step 1] If player 1 receives $\{A\}$, she knows that player 2 observes $\{A, B\}$ and therefore will play L. Thus, player 1's best response is to play U, and hence there does not exist a profitable deviation.

[Step 2] If player 1 receives $\{B, C\}$, she believes that player 2 is equally likely to have observed the private signal $\{A, B\}$ or $\{C\}$. Thus, player 1 thinks that player 2 is equally likely to play L and R. Thus, no profitable deviation for player 1 means $u_1(D, (\frac{1}{2}, \frac{1}{2})) \geq u_1(U, (\frac{1}{2}, \frac{1}{2}))$, and indeed it is satisfied since $2.5 = u_1(D, (\frac{1}{2}, \frac{1}{2})) \geq u_1(U, (\frac{1}{2}, \frac{1}{2})) = 2.5$.

[Step 3] By the same argument, player 2 also does not have any profitable deviations. $\square$

Remark 1.18. The strategy in proposition 1.7 yields the distribution where (U, L), (D, L) and (D, R) are played with equal probability $\frac{1}{3}$. Thus, the expected payoff to each player is $\frac{1}{3}(4 + 5 + 1) = \frac{10}{3}$. Notice that we have constructed an equilibrium in which the player's choices are correlated and their expected payoffs are outside the convex hull of NE payoffs.

1.8.3 Correlated Equilibrium

Definition 1.15. [Correlated strategy] A correlated strategy p is a probability distribution over $\times_{i \in N} S_i$, i.e. $p \in \Delta(\times_{i \in N} S_i)$. Then, for each $\boldsymbol{s} \in \times_{i \in N} S_i$, $p(\boldsymbol{s})$ is a probability that the strategy profile $\boldsymbol{s}$ is selected by the correlated strategy p. We also denote the marginal probability that an action $s_i \in S_i$ is selected for player i by $p(s_i) = \sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} p(s_i, \boldsymbol{s}_{-i})$.

Example 1.10. For example, the correlated strategy in proposition 1.7 can be written as

$$p = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \in \Delta(S_1 \times S_2)$$

In this case, $p(s_1 = U) = \frac{1}{3} + 0 = 0$ and $p(s_2 = L) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

[8]No deviation incentives exist for the time being.

Definition 1.16. [Conditional distribution and payoff] When p prescribes that player i play s_i, the probability that player i assigns to the event that other players play the $\boldsymbol{s}_{-i}$ under p is

$$p(\boldsymbol{s}_{-i} \mid s_i) = \frac{p(s_i, \boldsymbol{s}_{-i})}{p(s_i)}$$

and hence player i's conditional expected payoff from playing s_i when prescribed by p is

$$\sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} p(\boldsymbol{s}_{-i} \mid s_i) u_i(s_i, \boldsymbol{s}_{-i}) = \sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} \frac{p(s_i, \boldsymbol{s}_{-i})}{p(s_i)} u_i(s_i, \boldsymbol{s}_{-i})$$

Notice that player i prefers obeying the correlated strategy p to deviating to $s_i' \in S_i$ if and only if

$$\sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} \frac{p(s_i, \boldsymbol{s}_{-i})}{p(s_i)} u_i(s_i, \boldsymbol{s}_{-i}) \geq \sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} \frac{p(s_i, \boldsymbol{s}_{-i})}{p(s_i)} u_i(s_i', \boldsymbol{s}_{-i})$$

Definition 1.17. [Correlated equilibrium] A correlated strategy $p \in \Delta(\times_{i \in N} S_i)$ is a correlated equilibrium if

$$\sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} p(s_i, \boldsymbol{s}_{-i}) u_i(s_i, \boldsymbol{s}_{-i}) \geq \sum_{\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}} p(s_i, \boldsymbol{s}_{-i}) u_i(s_i', \boldsymbol{s}_{-i}), \ \forall i \in N, \ \forall s_i, s_i' \in S_i$$

Remark 1.19. A pure strategy NE is a correlated equilibrium with a degenerate p. For example, two NE in example 1.8 can be written as correlated equilibria $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Remark 1.20. A mixed strategy NE is also a correlated equilibrium such that p is the joint distribution implied by mixed strategies. For example, a mixed NE in example 1.8 can be written as a correlated equilibrium $p = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

Remark 1.21. The set of correlated equilibria payoffs is convex, and includes the convex hull of set of NE payoffs. In other words, the payoffs outside the convex hull of set of NE payoffs may be obtained by using private and imperfectly correlated signals.

Using an example 1.8, let us illustrate the set of correlated payoffs equilibria. Consider any correlated strategy $p \in \Delta(S_1 \times S_2)$ such that

$$p = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For p to be correlated equilibrium, it must satisfy the followings.

- Player 1 prefers playing U when told to do so. That is,

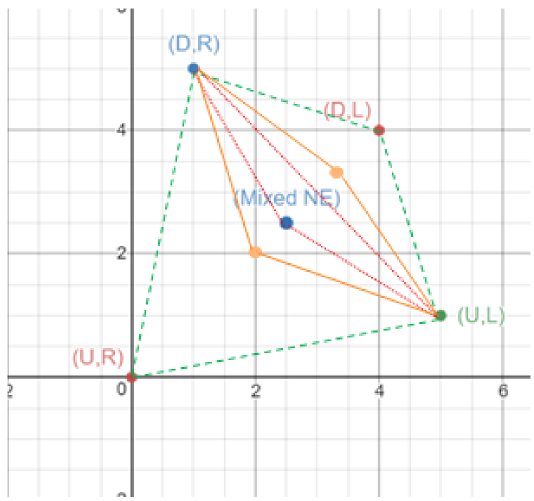
$$\frac{a}{a+b} u_1(U, L) + \frac{b}{a+b} u_1(U, R) \geq \frac{a}{a+b} u_1(D, L) + \frac{b}{a+b} u_1(D, R) \ \Leftrightarrow \ a \geq b$$

- Player 1 prefers playing D when told to do so, and hence $d \geq c$.
- Player 2 prefers playing L when told to do so, and hence $a \geq c$.
- Player 2 prefers playing R when told to do so, and hence $d \geq b$.
- Since $p \in \Delta^3$, $a + b + c + d = 1$.

Summarizing these four conditions yields

$$c \leq \min\{a, d\} \quad \text{and} \quad b \leq \min\{a, d\}$$

To compute the worst equilibrium, we need to minimize c and maximize b, and hence $a = b = d = \frac{1}{3}$ and $c = 0$. Thus, $p = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$. Conversely, if we want to compute the best equilibrium, we need to maximize c and minimize b, and hence $a = c = d = \frac{1}{3}$ and $b = 0$. The illustrated set of correlated equilibrium payoffs is as below.



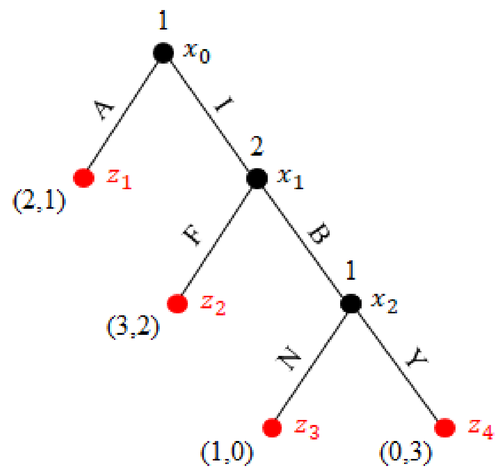
(c) Set correlated equilibria payoffs (in orange)

2 Part 2: Extensive Games of Complete Information

2.1 Preliminaries

It is often convenient to represent an extensive form game as a **game tree** as below.

Example 2.1. **[Movie watching game]**



In order to fully describe extensive form games, we need to introduce more elements.

Definition 2.1. **[Nodes]** In an extensive form game, there are two types of nodes: decision nodes and terminal nodes.

- **Decision nodes**: A decision node is a node at which a new decision is to be made. Let us denote the set of all decision nodes by X and a typical element by x. It should be also noted that each decision node corresponds to a sequence of actions, or a **history** that have been made so far. Let X_i denote the set of player i's decision nodes.

- **Terminal nodes**: A terminal node is a node at which the game has ended and payoffs for each player are specified. Let us denote the set of all terminal nodes by Z and a typical element by z. Any terminal node also corresponds to a specific history.

- In our example 2.1, $X = \{x_0, x_1, x_2\}$ and $Z = \{z_1, z_2, z_3, z_4\}$. Also notice that x_0 corresponds to the empty history ϕ, or z_2 corresponds to the history (I, F), and so on.

2.2 Associated Strategic (Normal) Form Game

For each extensive form game, we can construct the **associated strategic (normal) form game**, $< N, \boldsymbol{S}, \widetilde{\boldsymbol{u}} >$. For example, an extensive form game in example 2.1 can be represented as the one in left below

		Player 2	
		F	B
Player 1	AN	2, 1	2, 1
	AY	2, 1	2, 1
	IN	3, 2	1, 0
	IY	3, 2	0, 3

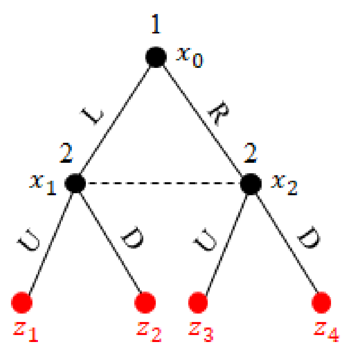
$\overset{\Longrightarrow}{\text{reduced}}$

		Player 2	
		F	B
Player 1	A	2, 1	2, 1
	IN	3, 2	1, 0
	IY	3, 2	0, 3

Notice that player 1's strategies (AN) and (AY) generate the same payoff for each player 2's strategy. We say that these two strategies are **equivalent** (in player 1's perspective). This is because, once A is played at x_0, then x_2 is never reached. Letting every set of equivalent strategies be represented by a single representative, we would obtain the reduced strategic (normal) form of the game as the one in right above.
In the above game, there are three pure strategy NE, $\{AN, B\}$, $\{AY, B\}$ and $\{IN, F\}$. Note that these are NE in terms of best responses to each other but they never consider the sequence in the original extensive form game.

2.3 Imperfect Information

Example 2.2. **[Moving game]** Consider an extensive form game below. The dotted line indicates that for player 2, the decision nodes (or histories) x_1 and x_2 belong to the same **information set**. This signifies that when player 2 chooses U or D, she does not know whether player 1 has previously chosen L or R.



Definition 2.2. **[Information set]** Consider a partition H of X. Let $h \in H$ be an element of partition H. H is said to partition X into information sets if $x, x' \in h$, then $\iota(x) = \iota(x')$ and $A(x) = A(x') =: A(h)$. [9]

Definition 2.3. **[Imperfect game]** A game is said to be **imperfect** if there exists at least one information set containing two decision nodes. In other words, some players do not know the actions taken before their turn. We can also distinguish perfect games and imperfect games by comparing cardinality of X and H. The game is perfect if and only if $\mid X \mid = \mid H \mid$, i.e. all information sets are singleton sets.

Notice that in example 2.2, neither player knows the action taken by her opponent when choosing her own action. Therefore, it is just an extensive form representation of the following simultaneous game.

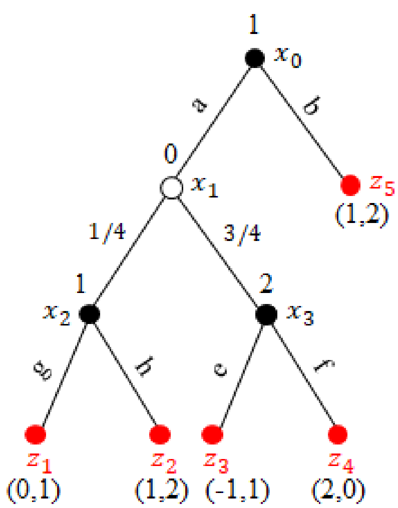
[9]If $x' \in h(x)$, then player $\iota(x)$ cannot distinguish between history x' and x.

		Player 2	
		U	D
Player 1	L	$u_1(L,U), u_2(L,U)$	$u_1(L,D), u_2(L,D)$
	R	$u_1(R,U), u_2(R,U)$	$u_1(R,D), u_2(R,D)$

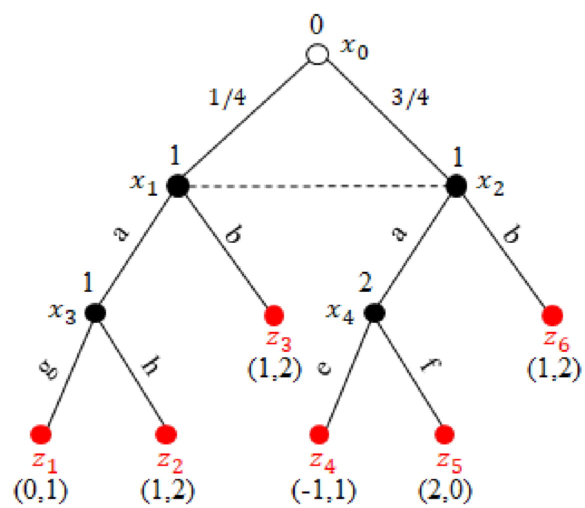
2.4 Moves of Nature

In many texts, extensive form games incorporate an $n+1$th player, "**nature**", called player 0, and whose actions are the possible outcomes of a lottery. The convention is to label nodes at which nature moves as $\circ$, and the actions of nature as the probability with which nature selects the corresponding history. Without loss of generality, we may summarize all moves of nature in a game as occurring first. That is, $\iota(x_0) = 0$ and $\pi \in \Delta^{|A(x_0)|-1}$ is the compound lottery of all possible moves in this game.

Example 2.3. [Extensive form game with a move of nature]



(a) Perfect Info. Extensive form game with nature



(b) Imperfect Info. Extensive form game with nature

Notice that both are equivalent in that they describe exactly the same strategic interaction. The two extensive form games admit the same reduced strategic game using VNM expected utility functions as below.

		Player 2	
		e	f
Player 1	ag	$-\frac{3}{4}, 1$	$\frac{3}{2}, \frac{1}{4}$
	ah	$-\frac{1}{2}, \frac{5}{4}$	$\frac{7}{4}, \frac{1}{2}$
	bg	$1, 2$	$1, 2$
	bh	$1, 2$	$1, 2$

$\underset{\text{reduced}}{\Longrightarrow}$

		Player 2	
		e	f
Player 1	ag	$-\frac{3}{4}, 1$	$\frac{3}{2}, \frac{1}{4}$
	ah	$-\frac{1}{2}, \frac{5}{4}$	$\frac{7}{4}, \frac{1}{2}$
	b	$1, 2$	$1, 2$

2.5 Extensive Form Game

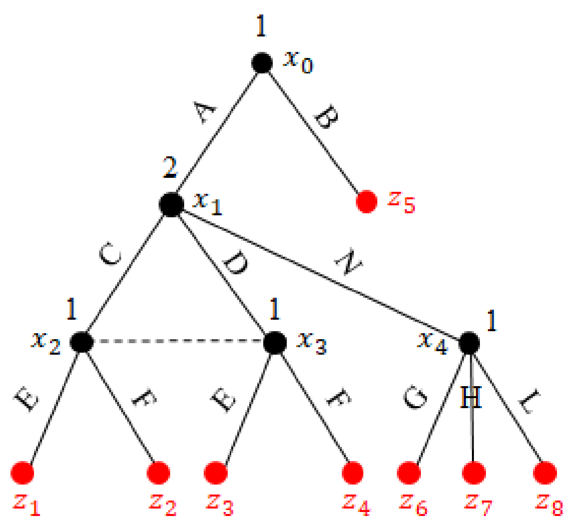
An extensive form game is a collection of $\Gamma =< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$

- A (finite) set of players: $N = \{1, 2, \ldots, n\}$
- A function maps each decision node to a player $\iota : X \to N$. The player $\iota(x)$ is to move at x. In our example 2.1, $\iota(x_0) = \iota(x_2) = 1$ and $\iota(x_1) = 2$.
- A feasible action set $A(x)$ for each $x \in X$. Let $A = \cup_{x \in X} A(x)$ and $A_i = \cup_{x \in X_i} A(x)$. In our example 2.1, $A(x_0) = \{A, I\}$, $A(x_1) = \{F, B\}$, $A(x_2) = \{N, Y\}$ and hence $A_1 = \{AF, AB, IF, IB\}$ and $A_2 = \{N, Y\}$.
- Payoffs: Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ denote a list of payoff functions such that $u_i : Z \to \mathbb{R}$ for each $i \in N$ where u_i is a Bernoulli utility index function. In our example 2.1, $\boldsymbol{u}(z_1) = (2, 1)$.

Remark 2.1. If A and X are finite, we say that the above describes a finite extensive form game.

Definition 2.4. [Strategy] Let H be the collection of information sets. Let $H_i = \{h \in H : \; x \in h \text{ and } \iota(x) = i\}$ denote the collection of player i's information sets, and $A_i = \cup_{h \in H_i} A(h)$ denote the set of all cations for player i. A pure strategy is a mapping $s_i : H_i \to A_i$ such that $s_i(h) \in A(h)$ for all $h \in H_i$. Then, the set of player i's pure strategy is $S_i = \times_{h \in H_i} A(h)$. In words, a strategy is a **complete contingent plan** that says what a player will do at each of her information sets if she is called on to play there. Also notice that $\mid S_i \mid = \prod_{h \in H_i} \mid A(h) \mid$. [10] [11]

Example 2.4.



- $N = \{1, 2\}$, $A = \{A, B, C, D, E, F, G, H, L, N\}$, $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$.
- $\iota(x_0) = \iota(x_2) = \iota(x_3) = \iota(x_4) = 1$ and $\iota(x_1) = 2$.
- $H_1 = \{\{x_0\}, \{x_2, x_3\}, \{x_4\}\}$, $H_2 = \{x_1\}$ and $H = H_1 \cup H_2$.
- $S_1 = \{AEG, AEH, AEL, AFG, AFH, AFL, BEG, BEH, BEL, BFG, BFH, BFL\}$, $S_2 = \{C, D, N\}$ and $S = S_1 \cup S_2$.

[10] This definition is very general in the sense that it can be applied to static games as well as perfect/imperfect extensive form games.
[11] Observe that in a normal form game, strategies and actions were the same. This is because, each player has a single information set.

2.6 Backwards Induction

There are two ways to show that there exists a NE in a "**finite**" extensive form game of "**perfect**" information.

- Represent an extensive from game in an associated strategic form. The existence of a NE follows from Nash theorem 1.2.
- Use the backwards induction algorithm.

Definition 2.5. [Backwards Induction] For given an extensive form game, let $Y_0 = Z$. For every $j \geq 1$, let

$$Y_j = \{x \in X \setminus \cup_{k=0}^{j-1} Y_k :\ \nexists\, x' \in X \setminus \cup_{k=0}^{j-1} Y_k,\ x \prec x'\}$$

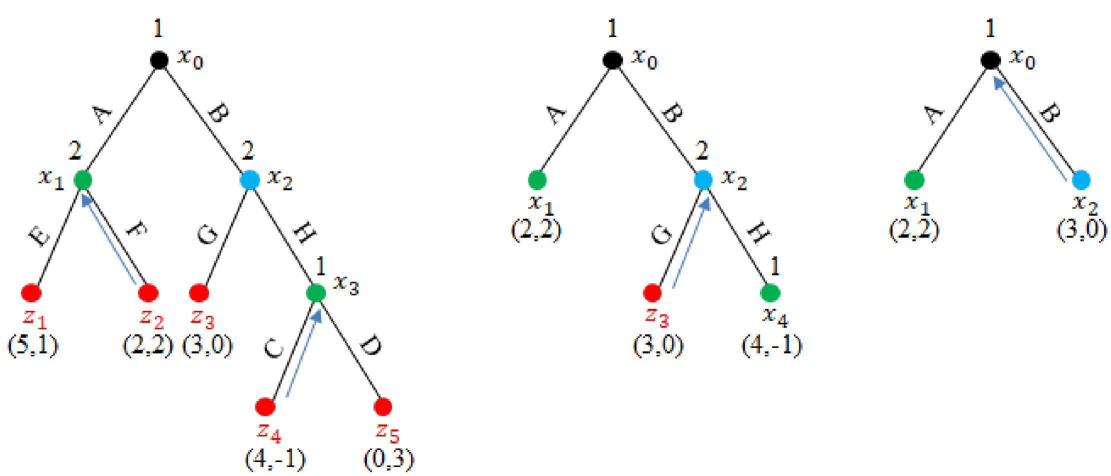
where $x \prec x'$ signifies x precedes x'. The backwards induction solution $\boldsymbol{a}^*$ is defined as follows: Let $\boldsymbol{u}^*(z) = \boldsymbol{u}(z)$ for every $z \in Z$. For every $j \geq 1$, and every $x \in Y_j$, let

$$a_x^* = \operatorname*{argmax}_{a \in A(x)} u_{\iota(x)}^*(x, a) \quad \text{and} \quad \boldsymbol{u}^*(x) = \boldsymbol{u}^*(x, a_x^*)$$

The BI algorithm starts with the terminal nodes. At each step, treat those decision nodes that are not followed by other decision nodes in the pruned tree, and prune the branches already treated.

- At each step, the BI algorithm attributes to the node x being treated they payoff vector that maximizes player $\iota(x)$'s payoff, when maximizing over the actions $a \in A(x)$ available to play $\iota(x)$ at node x. The terminal nodes of the new pruned tree are Y_j. Also observe that they payoffs that are being maximized are those of the truncated trees.
- The vector $\boldsymbol{a}^*$ record all the maximizers.
- The BI algorithm chooses a player's optimal action at each node and does this recursively through the entire game tree.

Example 2.5.



- $Y_0 = \{z_1, z_2, z_3, z_4, z_5\}$, $Y_1 = \{x_1, x_3\}$, $Y_2 = \{x_2\}$ and $Y_3 = \{x_0\}$.

- When $j = 1$, $a^*_{x_1} = \underset{a \in A(x)}{\text{argmax}}\ u^*_{\iota(x)}(x, a) = \underset{\{E,F\}}{\text{argmax}}\ u^*_2(x_1, a) = F$, and $\boldsymbol{u}^*(x_1) = (2, 2)$. Similarly, $a^*_{x_3} = \underset{a \in A(x)}{\text{argmax}}\ u^*_{\iota(x)}(x, a) = \underset{\{C,D\}}{\text{argmax}}\ u^*_1(x_3, a) = C$, and $\boldsymbol{u}^*(x_3) = (4, -1)$.
- From the truncated tree in the middle above, when $j = 2$, $a^*_{x_2} = \underset{a \in A(x)}{\text{argmax}}\ u^*_{\iota(x)}(x, a) = \underset{\{G,H\}}{\text{argmax}}\ u^*_2(x_2, a) = G$, and $\boldsymbol{u}^*(x_2) = (3, 0)$.
- From the truncated tree in the right above, when $j = 3$, $a^*_{x_0} = \underset{a \in A(x)}{\text{argmax}}\ u^*_{\iota(x)}(x, a) = \underset{\{A,B\}}{\text{argmax}}\ u^*_1(x_0, a) = B$, and $\boldsymbol{u}^*(x_0) = (3, 0)$.
- Thus, the BI strategy is $\boldsymbol{a}^* = \{BC, FG\}$.

Notice that an associated strategic form game of example 2.5 can be represented by

		Player 2			
		EG	EH	FG	FH
	AC	$5,1$	$5,1$	$2,2$	$2,2$
Player 1	AD	$5,1$	$5,1$	$2,2$	$2,2$
	BC	$3,0$	$4,-1$	$3,0$	$4,-1$
	BD	$3,0$	$0,3$	$3,0$	$0,3$

- Notice that $\{BC, FG\}$ is indeed a NE.
- Iterated deletion of weakly dominated strategies also yields $\{BC, FG\}$. Why? $BC \succsim BD$, $FG \succsim EG$, $FH \succsim EH$, $FG \succsim FH$, and $BC \succsim AC, AD$.
- This illustrates how iterated deletion of weakly dominated strategies can be equivalent to the BI.
- Recall that for the IDWDS, the order of deletion matters. Thus, for the equivalence between the IDWDS and the BI to hold, the IDWDS should follow the order of the BI.

2.7 Backwards Induction and Games of Imperfect Information

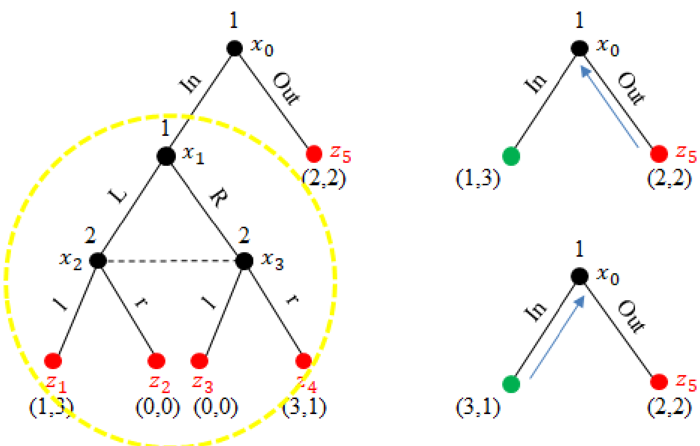
Applying backwards induction may not work in games of imperfect information. First, games of imperfect information are not guaranteed to have a last information set, in which case it is not even clear how to get the BI algorithm started. But even if there exists a last information set, the BI algorithm still may not work in games of imperfect information. Let us illustrate this with the following example.

- The game depicted below does have a last information set. However, it is impossible to say what is a best response player 2 without assuming that player 2 knows whether player 1 previously played L or R.
- Move up one more step and consider a game in dotted circle in yellow as a simultaneous move game as we discussed before. This game has two pure strategy NE, (L, l) and (R, r). [12]

		Player 2	
		l	r
Player 1	L	$1,3$	$0,0$
	R	$0,0$	$3,1$

[12] Ignore a mixed strategy NE at this moment.

- Resume the BI. If (L, r) is played, the truncated tree is the one on the upper right, and hence we have $\{(Out, L), l\}$. Similarly, if (R, r) is played, then we have $\{(In, R), r\}$.
- Both strategy profiles are NE of the original game. This approach works because we identify a proper subgame. Also note that this approach may not find all the NE. In fact, the original game has three pure strategy NE.

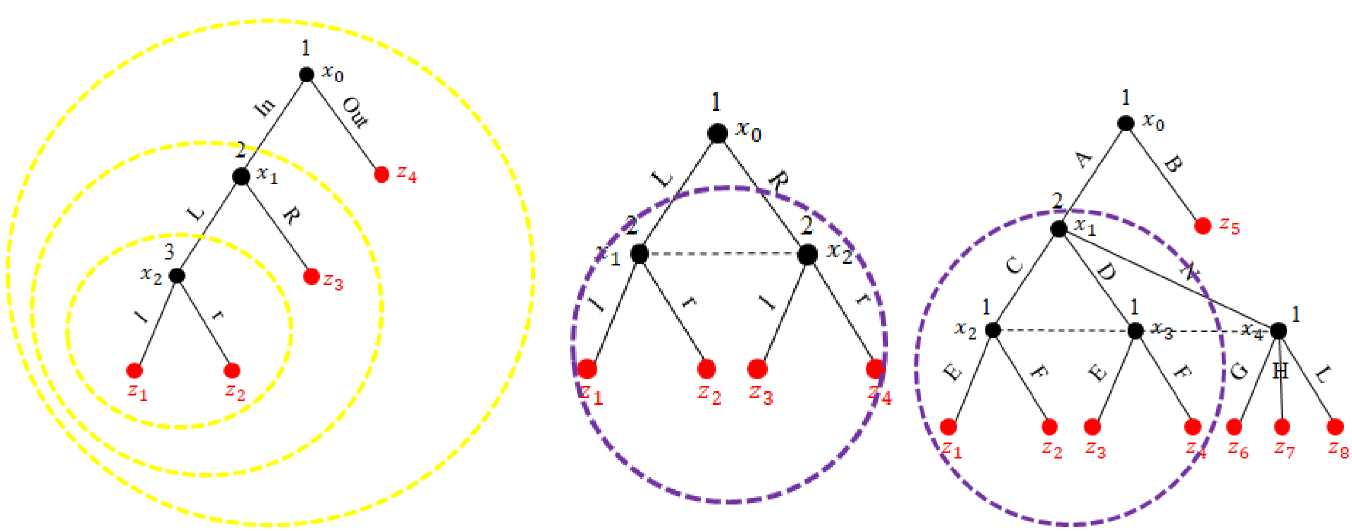


2.8 Proper Subgames

Definition 2.6. [Proper subgames] A decision node $x \in X$ defines a proper subgame of an extensive form game if

1) $h(x) = \{x\}$
2) For every node $y \in X$ following x, if $y' \in h(y)$, then y' also follows x

We say that $y \in X$ follows $x \in X$ if y is defined by the history $(x, a_1, a_2, \dots, a_k)$, $k \geq 1$, with $a_1 \in A(x)$, $a_2 \in A(x, a_1)$, etc; or if $x = y$. In words, a proper subgame includes a node and all its successors without severing an information set. It should be also noted that the entire game is itself a proper subgame.

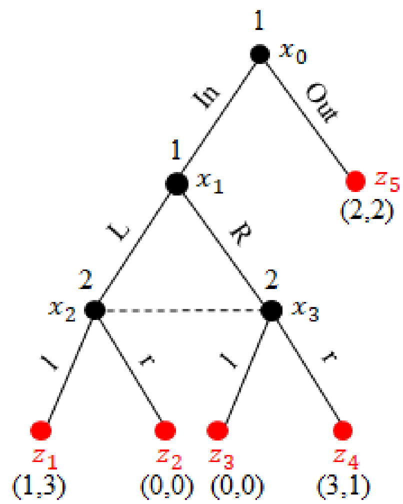


(c) Correctly defined proper subgames (d) Violating Def. 2.6-1) (e) Violating Def. 2.6-2)

2.9 Pure Strategy Subgame Perfect Equilibrium

Definition 2.7. Let $< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$ be given. A pure strategy $\boldsymbol{s} \in \boldsymbol{S}$ constitutes a pure strategy subgame perfect equilibrium if $\boldsymbol{s}$ induces a Nash equilibrium in every proper subgame of a given extensive form game.

Recall an extensive form game we consider in subsection 2.7.



- Using an associated strategic form, it is easy to check that it has three pure strategy NE, namely, $\{(In, R), r\}$, $\{(Out, L), l\}$ and $\{(Out, R), l\}$.
- However, the last NE, $\{(Out, R), l\}$ is not a SPE because it does not induce a NE in the subgame. Thus, we have two pure strategy SPE in this game.
- In general, NE is more general than SPE, and hence the set of NE $\supseteq$ the set of SPE. In other words, SPE is a refinement of NE in the sense that it eliminates those strategy profiles that rely on **incredible threats**. This is because SPE requires players to play in all subgames, even those that are not reached under the strategy profile. As such, SPE satisfies a requirement of sequential rationality. That is, it is not enough that strategies are best responses to one another, the sequentiality of moves in an extensive form game also needs to be taken into account.
- Conceptually, SPE is more general than the BI in the sense that it can be applied to an extensive form game of imperfect information. Note that under perfect information, those two approaches are equivalent.

2.10 Mixed Strategy in an Extensive Form Game

Consider the following extensive form game in which there does not exist a pure strategy NE in the last proper subgame. In this case, there are two ways in which a player might randomize her actions. One is a **mixed strategy**, and the other one is a **behavior strategy**.

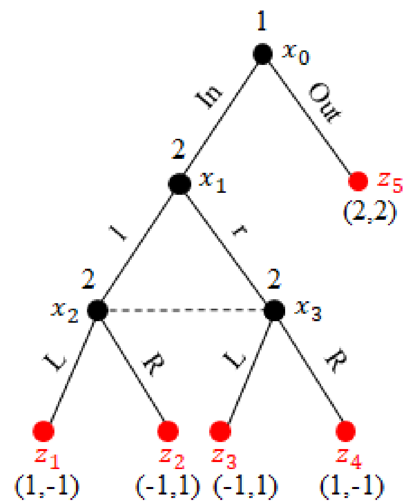
Definition 2.8. **[Mixed strategy]** Let $< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$ be given. A mixed strategy for player $i \in N$,

$$\boldsymbol{\sigma}_i \in \Sigma_i = \Delta(S_i) = \Delta(\times_{h \in H_i} A(h))$$

Definition 2.9. [Behavioral strategy] Let $< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$ be given. A behavioral strategy for player $i \in N$ is defined by

$$\boldsymbol{b}_i \in \times_{h \in H_i} \Delta(A(h))$$

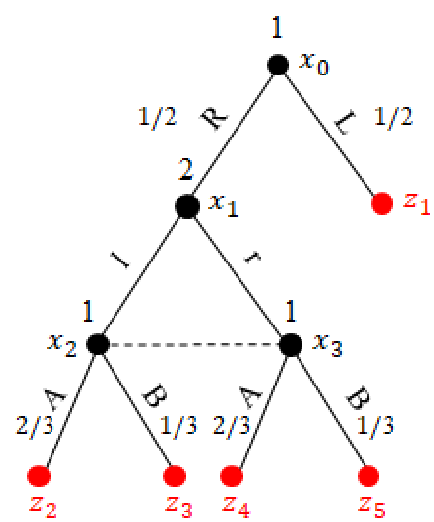
that is, at each information set $h \in H_i$, player i uses an independent randomization on $\Delta(A(h))$. The main difference between a behavioral strategy and a mixed strategy is that the former requires independent randomization at each information set.



- In this game, $\Sigma_1 = \Delta^3$ and $\Sigma_2 = \Delta^1$ but $\boldsymbol{b}_1 \in \Delta^1 \times \Delta^1$ and $\boldsymbol{b}_2 \in \Delta^1$.

2.11 Perfect Recall and Kuhn's Theorem

In this subsubsection, we will show that it does not matter whether we consider mixed or behavior strategies under perfect recall. Consider the following extensive form game, a behavioral strategy $b_1(L) = \frac{1}{2}$ and $b_1(A) = \frac{2}{3}$, and a mixed strategy $\boldsymbol{\sigma}_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})$ over $S_1 = \{LA, LB, RA, RB\}$ obtained by taking the product of a given behavioral strategy.



- For any possible $\sigma_2(l) \in [0,1]$, it is easy to check that $\boldsymbol{b}_1$ and $\boldsymbol{\sigma}_1$ yield the same probability distribution over Z, namely, $z_1 = \frac{1}{2}$, $z_2 = \frac{\sigma_2(l)}{3}$, $z_3 = \frac{\sigma_2(l)}{6}$, $z_4 = \frac{1-\sigma_2(l)}{3}$ and $z_5 = \frac{1-\sigma_2(l)}{6}$.

- It is also easy to verify that $\boldsymbol{\sigma}_1' = (0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ also generates the same distribution as $\boldsymbol{b}_1$ and $\boldsymbol{\sigma}_1$.

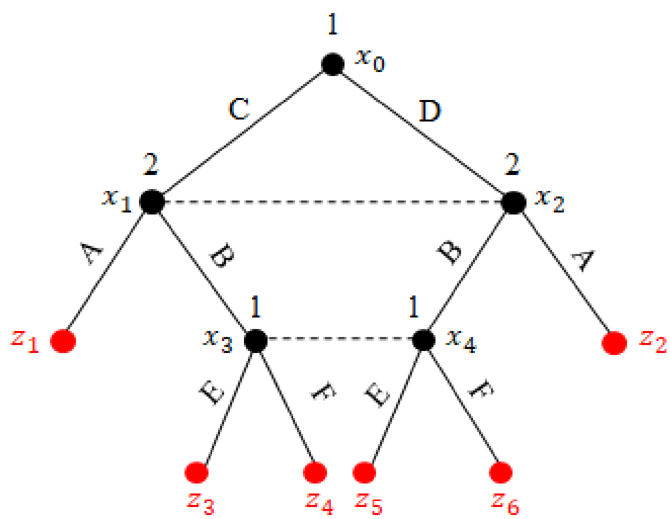
Definition 2.10. [Equivalence of strategies] For player $i \in N$, two strategies are said to be equivalent if they lead to the same distribution over outcomes for all strategies of all opponents.

Definition 2.11. [Perfect recall] Let $x, x' \in h (\in H_i)$, $(x_0, a_0 \ldots, x_n, a_n, x)$ and $(x_0', a_0' \ldots, x_m', a_m', x')$ be the record of decision nodes and actions on the path from the initial node x_0 to x and x', respectively. Player i has perfect recall if for any $h \in H_i$,

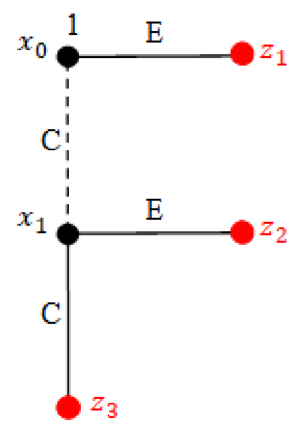
1) $n = m$
2) For all $0 \leq j \leq n$, x_j and x_j' are in the same information set for player i.
3) For all $0 \leq j \leq n$, if $\iota(x_j) = i$, then $a_j(x_j) = a_j'(x_j')$

We say that Γ is a game of perfect recall if every player has perfect recall in it.

Example 2.6. [Perfect recall violation]



(f) Perfect recall violation 1



(g) Perfect recall violation 2(Piccione & Rubistein 1997)

- In violation 1, player 1 knows that player 2 played B, but she cannot remember whether she herself played C or D at her first information set. Condition 3) is violated.

- In violation 2, player 1 forgets that she has previously chose C at x_1. But at x_0, player 1 is more forgetful in the sense that she even does not know whether she has already played or not. Condition 1) is violated.

Theorem 2.1. [Kuhn's theorem] *Under perfect recall, for each player $i \in N$*

1) Every mixed strategy $\boldsymbol{\sigma}_i$ can be associated with a unique equivalent behavioral strategy $\boldsymbol{b}_i$

2) Every behavioral strategy $\boldsymbol{b}_i$ can be associated with at least one (and possibly many) equivalent strategy $\boldsymbol{\sigma}_i$

2.11.1 The Existence of Subgame Perfect Equilibrium

Definition 2.12. [Mixed strategy SPE] A behavior strategy profile $\boldsymbol{b} = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ is a SPE of the finite extensive form game if it induces a NE in every proper subgame.

Theorem 2.2. **[Selten's theorem]** *Every finite extensive form game with perfect recall has a SPE.*

2.12 One Shot Deviation Principle

2.12.1 Finite Horizon games of Perfect Information

Consider Γ, $< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$, a finite extensive form game of perfect information.

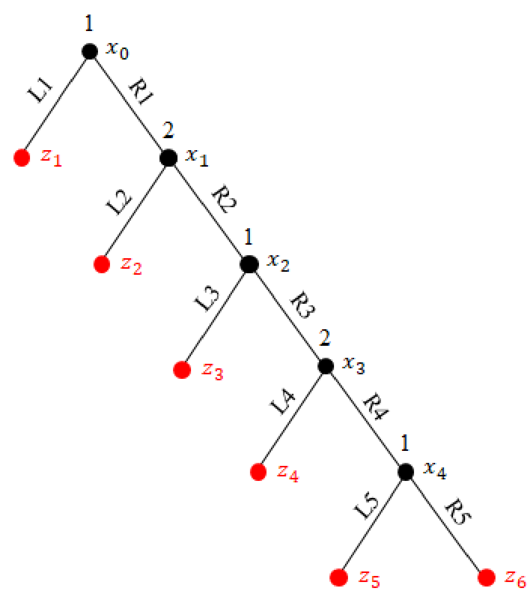
Definition 2.13. **[Continuation strategy]** Consider $s_i \in S_i$ and $h \in H$. [13] Let $s_i \mid_h$ be the strategy that s_i induces over the continuation game $\Gamma(h)$ [14] such that

$$s_i \mid_h (h') \equiv s_i(h, h'),\ \forall h' \in H_i \mid_h$$

where $H_i \mid_h$ is the set of all continuation information set h' following h, i.e. such that $(h, h') \in H_i$.

Definition 2.14. **[Outcome function]** For every $\boldsymbol{s} \in \boldsymbol{S}$, define $O(\boldsymbol{s})$ to be the terminal node that is reached when $\boldsymbol{s}$ is played. Let O_h denote the outcome function in the continuation game $\Gamma(h)$.

Example 2.7.



- $H = \{\phi, (R1), (R1, R2), (R1, R2, R3), (R1, R2, R3, R4)\}$. [15]
- Let $s_1 = (L_1, L_3, L_5)$ and $s_2 = (R_2, R_4)$. Consider $\Gamma(h = (R1))$, then $s_1 \mid_h = (L3, L5)$, $s_2 \mid_h = (R_2, R_4)$, $O(s_1, s_2) = z_1$ and $O_h((s_1, s_2) \mid_h) = z_3$.

Theorem 2.3. **[One shot deviation principle, OSD]** *Let $< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$ be a finite extensive form game of perfect information. Then, $\boldsymbol{s}^*$ is a SPE if and only if there exists no profitable one shot deviation. That is, for every $i \in N$, and for every $h \in H_i$, we have*

$$u_i(O_h(s_i^* \mid_h, \boldsymbol{s}_{-i}^* \mid_h)) \geq u_i(O_h(s_i \mid_h, \boldsymbol{s}_{-i}^* \mid_h))$$

[13] Since we assume that Γ is perfect information, each information set h corresponds to a specific decision node.
[14] In this lecture notes, $\Gamma(h)$ always defines a proper game.
[15] Here, we denote each information set h using the history path, e.g. $h = \{x_2\} = (R1, R2)$.

for every strategy $s_i \in S_i$ restricted to the subgame $\Gamma(h)$ that differs from $s_i^* \mid_h$ ONLY in the action it prescribes at the initial information set of $\Gamma(h)$, i.e. $s_i^*(h) \neq s_i(h)$ and $s_i^*(h,h') = s_i(h,h')$ for every $h' \in H_i \mid_h$.

Proof.
[**Necessity**] Let $\boldsymbol{s}^*$ be a SPE but allow a profitable OSD. Then, it admits a profitable deviation for some subgames, and hence $\boldsymbol{s}^*$ cannot be a SPE. Thus, if $\boldsymbol{s}^*$ is a SPE, then there is no profitable one shot deviation./

[**Sufficiency**] [16] We want to show that if $\boldsymbol{s}^*$ admits no profitable one shot deviation, then $\boldsymbol{s}^*$ is a SPE. By way of contradiction, suppose that $\boldsymbol{s}^*$ is not a SPE. Then, there must exist a subgame $\Gamma(h')$ and a profitable deviation s_i such that $s_i \mid_{h'} (h) \neq s_i^* \mid_{h'} (h)$ for at most all information sets $h \in H \mid_{h'}$ which is no longer than the length of $\Gamma(h')$.

In order to find the shortest (the smallest number of) profitable deviation, choose s_i that does better for player i than s_i^* but differs from it as seldom as possible in $\Gamma(h')$. If it differs from s_i^* once, then we are done. Otherwise proceed to the next step.

Let $h^* \in H$ be the longest history at which s_i and s_i^* differ. Consider a subgame $\Gamma(h^*)$. In this continuation game, s_i^* and s_i differ only at the first information set of $\Gamma(h^*)$ and coincide thereafter. Moreover, $s_i \mid_{h^*}$ does strictly better than $s_i^* \mid_{h^*}$ in $\Gamma(h^*)$, otherwise we could have found a shorter profitable deviation $s_i' \in \Gamma(h')$ that differs from s_i^* for fewer steps than s_i, contradicting that s_i is the shortest profitable deviation. Hence, we have found a profitable OSD, establishing the contradiction. [17] □

2.12.2 Infinite Horizon games of Perfect Information

Definition 2.15. [**Continuity at infinity**] Let $h, h' \in H$ be two infinite horizon histories. A game is continuous at infinity if for every $i \in N$,

$$\lim_{t\to\infty} \sup_{(h,h')|h_t=h'_t} \mid u_i(h) - u_i(h') \mid = 0$$

where u_i defined over realized histories, and h_t denote the restriction of h to the first t periods.

- Loosely speaking, continuity at infinity means payoff relevant events that occur very far in the future do not matter very much from the point of view of today.

- This condition is satisfied in particular if payoffs are the discounted sum of per period utilities, provided those per period utilities are bounded. That is,

$$U_i(h) = (1-\delta)\sum_{t\geq 0} \delta^t u_i(\boldsymbol{a}^t) \quad \text{where} \quad \delta \in (0,1) \quad \text{and} \quad \max_{\boldsymbol{a}^t \in A} \mid u_i(\boldsymbol{a}^t) \mid < B < \infty$$

[16] It turns out that whenever a strategy profile admits a profitable deviation, it also admits a profitable one shot deviation, and hence it suffices to check whether there exists any one shot deviation.

[17] Using 2.7, we may review the second part of this proof. Let $s_1^* = (L_1, L_3, L_5)$ and $s_2^* = (R_2, R_2)$ be a SPE. For player 1, there are three one shot deviations from $s_1^* = (L_1, L_2, L_3)$, namely, (R_1, L_3, L_5), (L_1, R_3, L_5) and (L_1, L_3, R_5). Notice that (s_1^*, s_2^*) is not a SPE if and only if there exists a profitable deviation. Let s_1 be the shortest profitable deviation available in this game. If it differs from s_1^* at only one information set, it is a OSD and we are done. Assume that it differs from s_1^* at more than one information set. For instance, suppose that it differs from s_1^* at two information sets, namely ϕ and (R_1, R_2), i.e. $s_1 = (R_1, R_3, L_5)$, so that $s_1(h) \neq s_1^*(h)$ for $h \in \{\phi, (R_1, R_2)\}$. The longest of those two information sets is (R_1, R_2). Consider a subgame $\Gamma(h^*)$ with $h^* = (R_1, R_2)$. Then, $s_1 \mid_{h^*}$ is a strictly profitable OSD in $\Gamma(h^*)$. If s_1 did not do strictly better than s_1^* at h^*, then there would have been a shorter profitable deviation at the previous step.

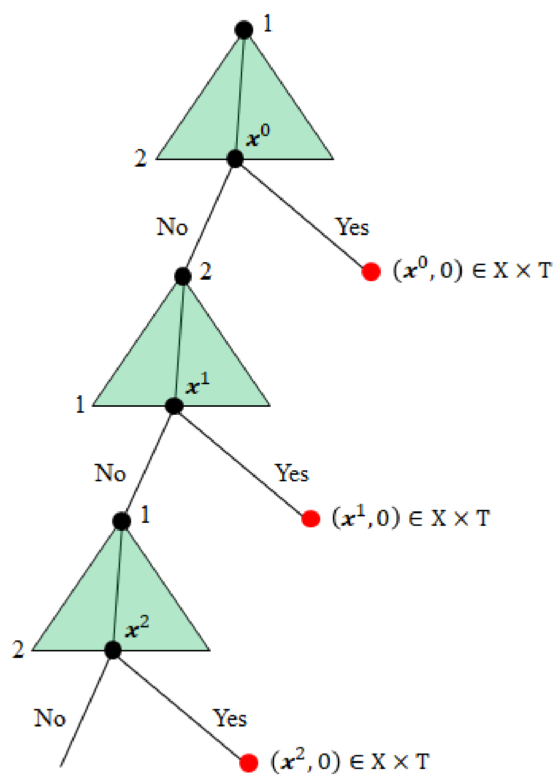
2.13 Rubinstein Bargaining Model

2.13.1 Model Description

- Two parties alternate in making offers. Given a party's offer, the other party may either accept and the game end or reject and the game proceeds to the next round.
- There is no bound on the number of possible rounds $T = \{0, 1, 2, \ldots\}$.
- **Action**: Let X denote the set of possible offers at every stage, $X = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \mid \ x_i \geq 0 \text{ and } x_1 + x_2 = 1\}$ where x_i denotes player i's share.
- **Payoffs**
  - Perpetual disagreement yields 0 to both players.
  - If agreement made at date t, we denote the $(\boldsymbol{x}, t) \in X \times T$ as the outcome and each player i has $u_i(\boldsymbol{x}, t) = \delta_i^t x_i$ where $i \in \{1, 2\}$ and $\delta_i \in (0, 1)$ is player i's discount factor.

2.13.2 Extensive Form Representation

Representing the first two periods is as follows.



2.13.3 Nash Equilibrium

Proposition 2.1. *Let $\overline{\boldsymbol{x}} \in X$. Then, there exists a NE of the following form; for each $i \in \{1, 2\}$*

$$\textit{player } i \textit{ offers } \overline{\boldsymbol{x}} \textit{ for all } h_t \in B_i \textit{ and player } i \textit{ accpets } \boldsymbol{x} \textit{ iff } x_i \geq \overline{x}_i \textit{ for all } h_t \in B_j$$

where B_i is the set of all histories at which player i makes a proposal and player j responds, i.e. $h_t \in B_1$ if and only if t is even, and $h_t \in B_2$ if and only if t is odd. (Notice that in this NE, agreement is reached immediately, i.e. $t = 0$.)

Proof. Fix player j's strategy, and suppose that player i uses a different strategy. Since player j never makes an offer different from $\overline{\boldsymbol{x}}$, and never accepts an offer $\boldsymbol{x}$ with $x_j < \overline{x}_j$, player i will never get more than $\overline{x}_i$ at any date t. Perpetual disagreement is the worst outcome. Hence the best thing for player i is to get $\overline{x}_i$ as soon as possible, i.e. the best outcome she can achieve is $(\overline{\boldsymbol{x}}, 0)$. □

- However, a NE, $\overline{\boldsymbol{x}}$, described above may involves an incredible threat, and hence may not be a SPE.
- Assume that player 1 were to deviate and offer $\boldsymbol{x}^0$ such that $\delta_2 \overline{x}_2 < x_2^0 < \overline{x}_2$. Player 2's strategy prescribes that she would reject $\boldsymbol{x}^0$, and at the next period respond with the offer $\overline{\boldsymbol{x}}$, which player 1 would then accept. Player 2's payoff from adhering to her strategy would be $\delta_2 \overline{x}_2$, but her payoff from accepting $\boldsymbol{x}^0$ would be x_2^0 which is greater than $\delta_2 \overline{x}_2$ by construction.

2.13.4 Subgame Perfect Equilibrium

Consider a stationary strategy profile as a candidate for our SPE;

$$\text{player 1 offers } \boldsymbol{x}^* = (x_1^*, x_2^*) \text{ for all } h_t \in B_1 \text{ and accpets } \boldsymbol{y}^* \text{ iff } y_1 \geq y_1^* \text{ for all } h_t \in B_2$$
$$\text{player 2 offers } \boldsymbol{y}^* = (y_1^*, y_2^*) \text{ for all } h_t \in B_2 \text{ and accpets } \boldsymbol{x}^* \text{ iff } x_2 \geq x_2^* \text{ for all } h_t \in B_1$$

To exclude incredible thereat, we also need

$$u_1(\boldsymbol{y}^*, t) = u_1(\boldsymbol{x}^*, t+1) \text{ and } u_2(\boldsymbol{x}^*, t) = u_2(\boldsymbol{y}^*, t+1)$$

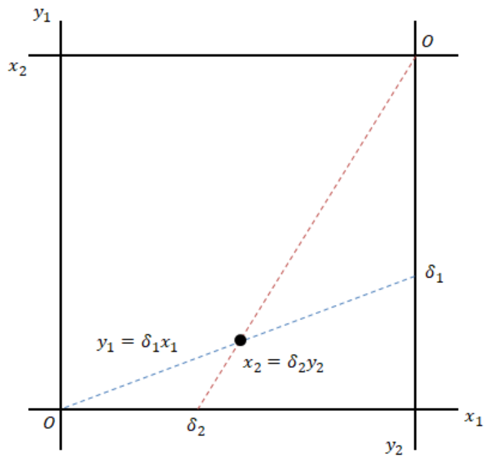
Using this condition, we can find that

$$u_1(\boldsymbol{y}^*, t) = u_1(\boldsymbol{x}^*, t+1) \;\Leftrightarrow\; \delta_1^t y_1^* = \delta_1^{t+1} x_1^* \;\Leftrightarrow\; y_1^* = \delta_1 x_1^*$$
$$u_2(\boldsymbol{x}^*, t) = u_2(\boldsymbol{y}^*, t+1) \;\Leftrightarrow\; \delta_2^t x_2^* = \delta_2^{t+1} y_2^* \;\Leftrightarrow\; x_2^* = \delta_2 y_2^*$$

Since we also know that $x_1^* + x_2^* = 1$ and $y_1^* + y_2^* = 1$, solving this system of equations yields

$$\boldsymbol{x}^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right) \text{ and } \boldsymbol{y}^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2} \right)$$

Representing these indifference conditions in the unit square is



Now, we want to check that our candidate profile is an indeed SPE using the OSD principle. Consider each history $h_t \in B_1$.

- **On path**: At t, player 1 offers $\boldsymbol{x}^*$ and player 2 accepts, and hence we have $(\boldsymbol{x}^*, t)$. Then, continuation utilities are
$$\left(\frac{1}{\delta_1^t} u_1(\boldsymbol{x}^*, t), \frac{1}{\delta_2^t} u_2(\boldsymbol{x}^*, t)\right) = (x_1^*, x_2^*)$$
- No profitable one shot deviation exists.
  - Player 1's perspective
    * Suppose that player 1 offers $\boldsymbol{x}'$ such that $x_1' > x_1^*$. Then, Player 2 will reject and offers $\boldsymbol{y}^*$ and player 1 accepts it at $t+1$, and hence we have $(\boldsymbol{y}^*, t+1)$. This OSD is not profitable for player 1 because continuation utility is
    $$\frac{1}{\delta_1^{t+1}} u_1(\boldsymbol{y}^*, t+1) = \frac{1}{\delta_1^t} \delta_1^{t+1} \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} = \delta_1^2 x_1^* < x_1^*$$
    * Suppose that player 1 offers $\boldsymbol{x}'$ such that $x_1' < x_1^*$. Then, Player 2 accepts, and hence we have $(\boldsymbol{x}', t)$. This OSD is also not profitable for player 1 because continuation utility is
    $$\frac{1}{\delta_1^t} u_1(\boldsymbol{x}', t) = x_1' < x_1^*$$
  - Player 2's perspective
    * Suppose that player 2 reject $\boldsymbol{x}^*$ and offer $\boldsymbol{y}^*$ at $t+1$. Then, player 1 will accept it, and hence we have $(\boldsymbol{y}^*, t+1)$. This OSD is not profitable for player 2 because
    $$\frac{1}{\delta_2^t} u_2(\boldsymbol{y}^*, t+1) = \frac{1}{\delta_2^t} \delta_2^{t+1} \frac{1-\delta_1}{1-\delta_1\delta_2} = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} = x_2^*$$

For each history $h_t \in B_2$, we can also show that there does not exist any profitable deviation. Thus, our candidate strategy profile is a SPE.

2.13.5 Uniqueness of Subgame Perfect Equilibrium

Let $G_i := \Gamma(h)$ for all $h \in B_i$. Notice that all these subgames are isomorphic. Consider for each $i \in \{1,2\}$ the set of SPE strategies

$$\Sigma_i^* = \{\boldsymbol{\sigma}_i \mid \boldsymbol{\sigma} \text{ is a SPE of subgame } G_i\} \neq \phi$$

Using continuation utility of player $i \in \{1,2\}$ over all subgames G_i, define m_i and M_i such that

$$m_i = \inf_{\boldsymbol{\sigma} \in \Sigma_i^*} \{u_i(\boldsymbol{x}, \tau) \mid (\boldsymbol{x}, \tau) \text{ is the outcome of } \boldsymbol{\sigma} \text{ in } G_i\}$$

$$M_i = \sup_{\boldsymbol{\sigma} \in \Sigma_i^*} \{u_i(\boldsymbol{x}, \tau) \mid (\boldsymbol{x}, \tau) \text{ is the outcome of } \boldsymbol{\sigma} \text{ in } G_i\}$$

- In any SPE, player j is guaranteed the payoff m_j in a subgame G_j, thus in a subgame G_i, she rejects all offers that give her less than $\delta_j m_j$. Therefore, in a subgame G_i, player i can get at most $M_i \leq 1 - \delta_j m_j$.
- In any SPE, player j's best possible payoff is M_j in a subgame G_j, thus in a subgame G_i, she accepts all offers that giver her more than $\delta_j M_j$. Therefore, in a subgame G_i, player i can get at least $m_i \geq 1 - \delta_j M_j$.
- Replacing the subscript in the second inequality, we have $m_j \geq 1 - \delta_i M_i$. Augmenting this with the first inequality, $M_i \leq 1 - \delta_j m_j$ yields $m_j \geq \frac{1-\delta_i}{1-\delta_i\delta_j}$.
- Similarly, replacing the subscript in the first inequality, we have $M_j \leq 1 - \delta_i m_i$. Augmenting this with the second inequality, $m_i \geq 1 - \delta_j M_j$. yields $M_j \leq \frac{1-\delta_i}{1-\delta_i\delta_j}$.
- By construction, we must have $M_j \geq m_j$, and therefore $m_j = M_j = \frac{1-\delta_i}{1-\delta_i\delta_j}$. Thus, in every continuation game G_i where player i makes the offer, the SPE payoff to player i is unique, and equal to $\frac{1-\delta_j}{1-\delta_i\delta_j}$.

2.13.6 Properties of the Unique SPE

In this model, we have shown that there exits a unique SPE. We also want to list some properties of this unique SPE as follows.

- **Efficiency**: An agreement is reached at $t = 0$ and the pie is not allowed to shrink.
- **Stationary strategy**: This was not assumed. Rather, it emerges as a result.
- **First mover advantage**: Suppose that $\delta_1 = \delta_2 = \delta$. Then, $x_1^* > x_2^*$.
- **Impatience**: A player's equilibrium payoff is an increasing function of her discount factor. That is, $\frac{\partial(\frac{1-\delta_j}{1-\delta_i\delta_j})}{\partial\delta_i} > 0$, which means that more patient players get larger payoffs.

2.14 Repeated Games

2.14.1 Basic Structure

Consider the mixed extension of the stage game $G =< N, A := \times_{i\in N} A_i, \boldsymbol{u} >$ [18] where A_i is finite for all $i \in N$ and $u_i : A \to \mathbb{R}$.

- We will assume that players play this game G at each date $t = 0, 1, \ldots$.

[18] Notice that $A = \boldsymbol{S}$.

- Let $G(\infty)$ denote the infinitely repeated game.
- **Perfect monitoring**: [19] At the end of each period $t \geq 0$, all players observe the realized action profile

$$\boldsymbol{a}^t = (a_1^t, \ldots, a_n^t) \in A$$

- **History** (up to t but not included): $h^t = (\boldsymbol{a}^0, \ldots, \boldsymbol{a}^{t-1}) \in A^t \equiv H^t$ where H^t denotes the set of possible histories up to date t.
- If the game were finitely repeated (for T periods i.e. $0, 1, \ldots, T-1$.), then $H^T = A^T$ would correspond to the set of terminal nodes.
- **The set of all possible histories** in $G(\infty)$ is denoted by $\boldsymbol{H} = \cup_{t=0}^{\infty} H^t$ where $H^0 = \{\phi\}$.
- For player i, a **pure strategy** in $G(\infty)$ is defined by $s_i : \boldsymbol{H} \to A_i$, and let S_i denote the set of pure strategies for player i.
- For player i, a **behavioral strategy** is defined by $\boldsymbol{\sigma}_i : \boldsymbol{H} \to \Delta(A_i)$, and let Σ_i denote the set of behavioral strategies for player i. (Notice that we change the notation for behavioral strategies from $\boldsymbol{b}_i$ to $\boldsymbol{\sigma}_i$.)
- Consider a pure strategy profile $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n)$. At $h^0 = \phi$, $\boldsymbol{s}$ induces $(s_1(h^0), \ldots, s_n(h^0)) = \boldsymbol{a}^0(\boldsymbol{s})$. At $h^1 = \boldsymbol{a}^0(\boldsymbol{s})$, $\boldsymbol{s}$ induces $(s_1(h^1), \ldots, s_n(h^1)) = \boldsymbol{a}^1(\boldsymbol{s})$, and so on. Thus, $\boldsymbol{s}$ induces a unique outcome path $(\boldsymbol{a}^0(\boldsymbol{s}), \boldsymbol{a}^1(\boldsymbol{s}), \ldots)$.
- At any history h^t, the continuation game associated with this h^t constitutes a proper subgame of the original game. More importantly, this continuation game is **homomorphic** and strategically equivalent to the original game. Thus, infinitely repeated games have a **recursive form**.
- Consider $\boldsymbol{s}_i \in S_i$ and $h^t \in H^t$. Then, a **continuation strategy** $s_i \mid_{h^t}$ is defined by for all $\widehat{h}^\tau$ following h^t

$$s_i \mid_{h^t} (\widehat{h}^\tau) = s_i(h^t, \widehat{h}^\tau)$$

  where $h^t = (\boldsymbol{a}^0, \ldots, \boldsymbol{a}^{t-1}) \in H^t$ and $\widehat{h}^\tau = (\widehat{\boldsymbol{a}}^0, \ldots, \widehat{\boldsymbol{a}}^{\tau-1}) \in H^\tau$, so that $(h^t, \widehat{h}^\tau) \in (\boldsymbol{a}^0, \ldots, \boldsymbol{a}^{t-1}, \widehat{\boldsymbol{a}}^0, \ldots, \widehat{\boldsymbol{a}}^{\tau-1}) \in H^{t+\tau}$.
- In the context of infinitely repeated games, $s_i \mid_{h^t}: \boldsymbol{H} \to A_i$ and we have $s_i \mid_{h^t} \in S_i$ for every h^t due to the recursive property.
- For given $\boldsymbol{s} \in \boldsymbol{S}$, payoffs in the infinitely repeated games are given by the **average discounted payoff**

$$U_i(\boldsymbol{s}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i(\boldsymbol{a}^t(\boldsymbol{s}))$$

  where $\delta \in (0, 1)$ is a common discount factor for players.
  - The expected payoff from a mixed or behavioral strategy is defined in the usual way.

[19] In this course, we always assume perfect monitoring and restrict our attention to games of perfect information, and thus each player's information sets are all singleton sets.

– Consider $\boldsymbol{s} \in \boldsymbol{S}$ such that $\boldsymbol{s}$ prescribes the same action profile at all possible histories. Then, we have $\boldsymbol{a} \in A$ such that $\boldsymbol{a}^t(\boldsymbol{s}) = \boldsymbol{a}$ for all $t \geq 0$. Thus,

$$U_i(\boldsymbol{s}) = (1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(\boldsymbol{a}^t(\boldsymbol{s})) = u_i(\boldsymbol{a})$$

– More generally, when using the average discounted payoff we know that for any given $\boldsymbol{s}$, $\{U_i(\boldsymbol{s})\}_{i\in N} \subset \mathcal{F} \equiv \text{conv}\{\boldsymbol{v} \in \mathbb{R}^n \mid \exists\, \boldsymbol{a} \in A \text{ and } \boldsymbol{v} = \boldsymbol{u}(\boldsymbol{a})\}$.

– Notice that

$$\begin{aligned} U_i(\boldsymbol{s}) = (1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(\boldsymbol{a}^t(\boldsymbol{s})) &= (1-\delta)u_i(\boldsymbol{a}^0(\boldsymbol{s})) + (1-\delta)\sum_{t=1}^{\infty}\delta^t u_i(\boldsymbol{a}^t(\boldsymbol{s})) \\ &= (1-\delta)u_i(\boldsymbol{a}^0(\boldsymbol{s})) + \underbrace{\delta(1-\delta)\sum_{t=0}^{\infty}\delta^t u_i(\boldsymbol{a}^{t+1}(\boldsymbol{s}))}_{\text{continuation payoff}} \\ &= (1-\delta)u_i(\boldsymbol{a}^0(\boldsymbol{s})) + \delta U_i(\boldsymbol{s}\mid_{h^1=\boldsymbol{a}^0(\boldsymbol{s})}) \end{aligned}$$

More generally,

$$U_i(\boldsymbol{s}\mid_{h^t}) = (1-\delta)u_i(\boldsymbol{a}^t(\boldsymbol{s})) + \delta U_i(\boldsymbol{s}\mid_{h^{t+1}=(h^t,\boldsymbol{a}^t(\boldsymbol{s}))})$$

2.14.2 Nash Equilibrium and Subgame Perfect Equilibrium

Definition 2.16. [Nash equilibrium in repeated game] The strategy profile $\boldsymbol{\sigma}^* \in \times_{i\in N}\Sigma_i$ is a Nash equilibrium of $G(\infty)$ if $\forall i \in N$ and $\widehat{\boldsymbol{\sigma}}_i : \boldsymbol{H} \to \Delta(A_i)$

$$U_i(\boldsymbol{\sigma}_i^*, \boldsymbol{\sigma}_{-i}^*) \geq U_i(\widehat{\boldsymbol{\sigma}}_i, \boldsymbol{\sigma}_{-i}^*)$$

Notice that the existence of NE in repeated games is guaranteed from playing the stage-NE at each $t \geq 0$. (on path)

Definition 2.17. [Subgame perfect equilibrium in repeated game] The strategy profile $\boldsymbol{\sigma}^* \in \times_{i\in N}\Sigma_i$ is a SPE of $G(\infty)$ if for every $t \geq 0$ and for every history $h^t \in H^t$, $\boldsymbol{\sigma}^*\mid_{h^t} = (\boldsymbol{\sigma}_1^*\mid_{h^t}, \ldots, \boldsymbol{\sigma}_n^*\mid_{h^t})$ is a NE of $G(\infty)$. By the same argument above, the existence of SPE is also guaranteed. (all path)

2.14.3 Illustration: Grim Trigger in Prisoner's Dilemma

Consider the following prisoner's dilemma game where $g, l > 0$.

		Player 2	
		C	D
Player 1	C	$1, 1$	$-l, 1+g$
	D	$1+g, -l$	$0, 0$

Notice that the unique NE is (D, D) for this stage game. The grim trigger strategy is, for all $h^t \in H^t$,

$$s_i^{grim}(h^t) = \begin{cases} C & \text{if } \ h^t = \phi \\ C & \text{if } \ \boldsymbol{a}^\tau = (C, C), \ \forall \tau < t \\ D & \text{otherwise} \end{cases}$$

The points to notice about grim trigger are

- Cooperation is rewarded.
- Any deviation **including a player's own** is punished by all players henceforth playing D forever. Once the punishment phase has been initiated, it never stops.

Now, we want to show that $\boldsymbol{s}^{grim} = (s_1^{grim}, s_2^{grim})$ constitutes a symmetric SPE of the infinitely repeated PD game by OSD principle. First, we partition the possible histories into two sets

$$\boldsymbol{H}_C = H^0 \cup \bigcup_{t=1}^{\infty} \{h^t \in H^t \mid \boldsymbol{a}^\tau = (C, C), \ \forall \tau < t\} \ \text{ and } \boldsymbol{H}_D = \boldsymbol{H} \setminus \boldsymbol{H}_C$$

- At histories $\forall h^t \in \boldsymbol{H}_D$
  - **On path**: If both players adhere to grim trigger, (D, D) is played today and forever thereafter. Thus, player i's payoff is

    $$V_i(\boldsymbol{H}_D) \equiv U_i(\boldsymbol{s}^{grim} \mid_{h^t}) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(D, D) = u_i(D, D) = 0$$

  - No profitable one shot deviation exists. ((D, D) is a stage-NE, thus no reason to deviate)
    * If player i deviates from grim trigger at $h^t \in \boldsymbol{H}_D$, then player i plays C today, and (D, D) is played forever thereafter. Thus, player i's payoff is

      $$(1 - \delta) u_i(C, D) + (1 - \delta) \sum_{t=1}^{\infty} \delta^t u_i(D, D) = (1 - \delta) u_i(C, D) + \delta \underbrace{V_i(\boldsymbol{H}_D)}_{=0}$$

      Since $u_i(C, D) = -l < 0$ and $V_i(\boldsymbol{H}_D) = 0$, the deviation is not profitable for all $\delta \in (0, 1)$.

- At histories $\forall h^t \in \boldsymbol{H}_C$
  - **On path**: If both players adhere to grim trigger, (C, C) is played today and forever thereafter. Thus, player i's payoff is

    $$V_i(\boldsymbol{H}_C) \equiv U_i(\boldsymbol{s}^{grim} \mid_{h^t}) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u_i(C, C) = u_i(C, C) = 1$$

  - No profitable one shot deviation exists only if $\delta \in [\frac{g}{1+g}, 1)$.
    * If player i deviates from grim trigger at $h^t \in \boldsymbol{H}_C$, then player i plays D today, and (C, D) is played

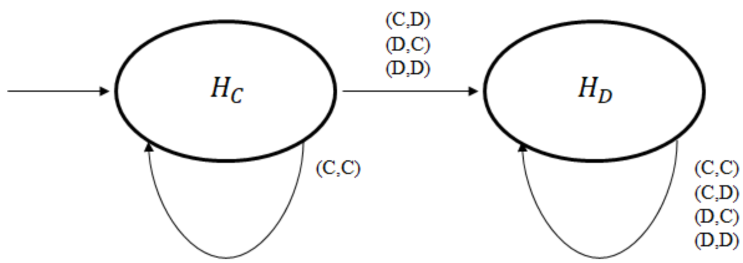
forever thereafter. Thus, player i's payoff is

$$(1-\delta)u_i(D,C) + (1-\delta)\sum_{t=1}^{\infty}\delta^t u_i(D,D) = (1-\delta)u_i(D,C) + \delta V_i(\boldsymbol{H}_D)$$

Then, the deviation is not profitable only if $1 \geq (1-\delta)u_i(D,C) \;\Leftrightarrow \delta \geq \frac{g}{1+g}$.

2.14.4 Strategies as Automata

An automaton consists of a set of states $\mathcal{W}$, an initial state $\omega_0 \in \mathcal{W}$, an output (or decision) function $f : \mathcal{W} \to \Delta(A_i)$ associating a state with a mixed action profile for each player, and a transition function $\tau : \mathcal{W}\times A \to \mathcal{W}$. For example, the automaton for the grim trigger strategy is depicted as below.



- $\mathcal{W} = \{\boldsymbol{H}_C, \boldsymbol{H}_D\}$, $\omega_0 = \boldsymbol{H}_C$, $f(\boldsymbol{H}_C) = C$, $f(\boldsymbol{H}_D) = D$, $\tau(\omega, \boldsymbol{a}) = \boldsymbol{H}_C$ if $\omega = \boldsymbol{H}_C$ and $\boldsymbol{a} = (C,C)$, and $\tau(\omega, \boldsymbol{a}) = \boldsymbol{H}_D$ otherwise.

- Let $V_i(\omega)$ be player i's average discounted payoff from play that begins in state ω. At any state $\omega \in \mathcal{W}$, the payoff $V_i(\omega)$ induced by the automaton is the weighted average of current payoff $u_i(f_i(\omega), f_j(\omega))$ and continuation payoffs $V_i(\tau_i(\omega; f_i(\omega), f_j(\omega)))$, and hence

$$V_i(\omega) = u_i(f_i(\omega), f_j(\omega)) + \delta_i V_i(\tau_i(\omega; f_i(\omega), f_j(\omega)))$$

2.14.5 Folk Theorem

Let $\mathcal{F}$ denote the set of feasible payoffs, i.e.

$$\mathcal{F} = \text{conv}\{\boldsymbol{v} \in \mathbb{R}^n \mid \exists\, \boldsymbol{a} \in A \text{ and } \boldsymbol{v} = \boldsymbol{u}(\boldsymbol{a})\}$$

Definition 2.18. **[Mixed strategy minmax]** Player i's mixed strategy minmax (in stage game) is defined by

$$\underline{v}_i \equiv \min_{\boldsymbol{\sigma}_{-i}\in\times_{j\neq i}\Delta(A_j)} \max_{\boldsymbol{\sigma}_i\in\Delta(A_i)} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i})$$

Theorem 2.4. *Suppose that $\boldsymbol{\sigma}^*$ is a (possibly mixed) NE of $G(\infty)$, then $U_i(\boldsymbol{\sigma}^*) \geq \underline{v}_i$ for all $i \in N$.*

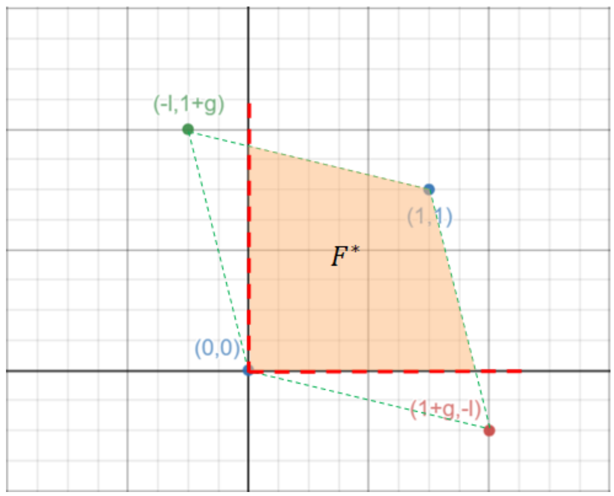
Definition 2.19. **[Individual rational and feasible payoff]** $\mathcal{F}^*$ is defined by

$$\mathcal{F}^* = \mathcal{F} \cap \{\boldsymbol{v} \in \mathbb{R}^n \mid v_i \geq \underline{v}_i,\ \forall i \in N\}$$

Example 2.8. **[Prisoner's dilemma]** Recall the prisoner's dilemma game we consider in previous subsubsection. In this case, for player $i \in \{1,2\}$ we have

$$\underline{v}_i = \underbrace{\min_{\boldsymbol{\sigma}_{-i} \in \times_{j \neq i} \Delta(A_j)} \underbrace{\max_{\boldsymbol{\sigma}_i \in \Delta(A_i)} u_i(\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_{-i})}_{u_i(D, \boldsymbol{\sigma}_{-i})}}_{u_i(D,D)=0}$$

Therefore, $\mathcal{F}^*$ can be depicted as below.

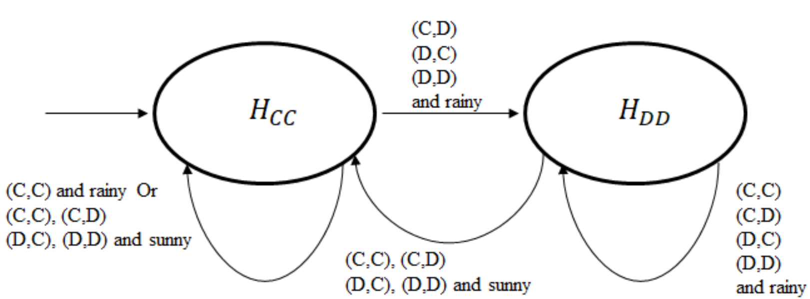


(h) Individual rational and feasible payoff in PD

Theorem 2.5. **[Folk theorem]** *Suppose that $\mathcal{F}^*$ has a non-empty interior in $\mathbb{R}^n$. For all $\boldsymbol{v} \in \mathcal{F}^*$, there exists a $\delta' \in (0,1)$ such that for all $\delta \geq \delta'$, $\boldsymbol{v}$ can be supported as the payoffs of a SPE of the infinitely repeated game. (Notice that this is a Folk theorem which only considers a game in which both players have the same discount factor δ.)*

2.14.6 Strategy Profiles with Public Randomization

Let $\{X(t)\}_{t \geq 0}$ be a random variable such that each $X(t)$ is independently drawn from $U[0,1]$. Assume that at each date, the realization of this random variable is publicly observable for all players. Let $x \in [0,1]$ and define the event $\{X(t) \leq x\}$ as "**sunny**" and $\{X(t) > x\}$ as "**rainy**".
Now, consider a strategy profile represented in the following automaton.



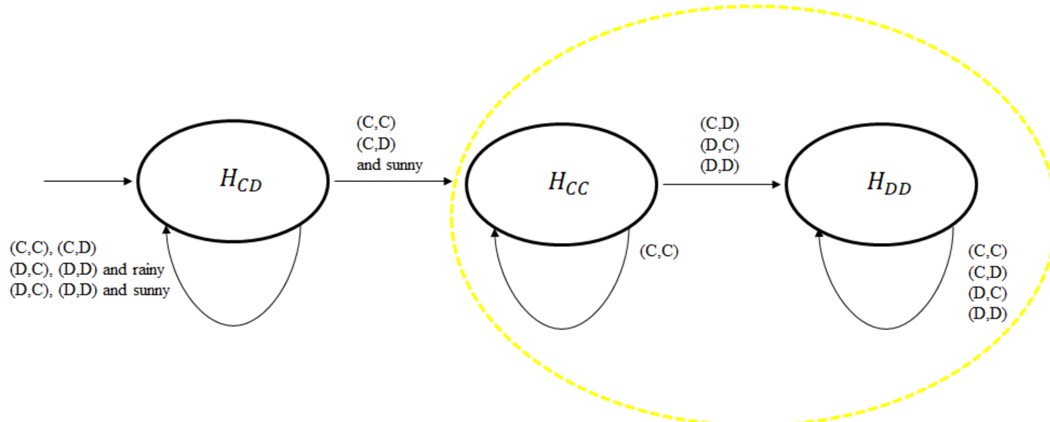
Now, we want to show that this strategy profile constitutes a symmetric SPE of the infinitely repeated PD game. Let $V_i(\boldsymbol{H}_{CC})$ and $V_i(\boldsymbol{H}_{DD})$ denote player i's payoff in states $\boldsymbol{H}_{CC}$ and $\boldsymbol{H}_{DD}$ respectively when adhering to the strategy described by the automaton above.

- At state $\boldsymbol{H}_{DD}$
  - **On path**: For player 1,
  $$V_1(\boldsymbol{H}_{DD}) = (1-\delta)u_1(D,D) + \delta\Big[(1-x)V_1(\boldsymbol{H}_{DD}) + xV_1(\boldsymbol{H}_{CC})\Big]$$
  Then, we have
  $$V_1(\boldsymbol{H}_{DD}) = \frac{\delta x}{1-\delta(1-x)}V_1(\boldsymbol{H}_{CC})$$
  - No profitable one shot deviation exists.
    * If player 1 deviates, then
    $$(1-\delta)u_1(C,D) + \delta\Big[(1-x)V_1(\boldsymbol{H}_{DD}) + xV_1(\boldsymbol{H}_{CC})\Big]$$
    Since that $u_1(C,D) < u_1(D,D)$, this deviation is not profitable for player 1 for all $\delta \in (0,1)$.
- At state $\boldsymbol{H}_{CC}$
  - **On path**: For player 1,
  $$V_1(\boldsymbol{H}_{CC}) = (1-\delta)u_1(C,C) + \delta V_1(\boldsymbol{H}_{CC})$$
  Then, we have
  $$V_1(\boldsymbol{H}_{CC}) = 1$$
  - No profitable one shot deviation exists if $x \leq \frac{1}{1+g}$.
    * If player 1 deviates, then
    $$(1-\delta)u_1(D,C) + \delta\Big[(1-x)V_1(\boldsymbol{H}_{DD}) + xV_1(\boldsymbol{H}_{CC})\Big]$$
    For this OSD not to be profitable, we require
    $$1 \geq (1-\delta)\underbrace{u_1(D,C)}_{=1+g} + \delta\Big[(1-x)\underbrace{V_1(\boldsymbol{H}_{DD})}_{=\frac{\delta x}{1-\delta(1-x)}} + x\underbrace{V_1(\boldsymbol{H}_{CC})}_{=1}\Big]$$
    which is equivalent to
    $$\gamma\delta^2 - (\gamma+g)\delta + g \leq 0$$
    where $\gamma = (1-x)(1+g)$. Solving this inequality equation yields $\delta_1 = \frac{g}{\gamma}$ and $\delta_2 = 1$. To make SPE supporting δ exist, we need $\delta_1 = \frac{g}{\gamma} \leq 1$, and hence $x \leq \frac{1}{1+g}$. [20]
- Under this condition on x, the strategy profile is a SPE of the infinitely repeated game for all $\delta \in [\frac{g}{\gamma}, 1) = [\frac{g}{(1-x)(1+g)}, 1)$.

[20] In general, we also need to require $x \leq 1$.

2.14.7 Generating Individually Rational and Feasible Payoffs using Public Randomization

Suppose that we want to generate the payoff profile in which player 1 receives her minmax (i.e. 0) and player 2 receives the highest possible individually rational and feasible payoff. Consider a strategy profile represented in the following automaton.



Notice that the part in yellow dotted line is a grim trigger strategy, and we already know that this strategy profile can be supported as a SPE if $\delta \geq \frac{g}{1+g}$. So, we just need to consider the state $\boldsymbol{H}_{CD}$.

- At state $\boldsymbol{H}_{CD}$
  - **On path**: For player 2,

$$V_2(\boldsymbol{H}_{CD}) = (1-\delta)u_2(C,D) + \delta\Big[(1-x)V_2(\boldsymbol{H}_{CD}) + xV_2(\boldsymbol{H}_{CC})\Big]$$

  - No profitable one shot deviation exists.
    * If player 2 deviates, then

$$(1-\delta)u_2(C,C) + \delta\Big[(1-x)V_2(\boldsymbol{H}_{CD}) + xV_2(\boldsymbol{H}_{CC})\Big]$$

      This deviation cannot be profitable since $u_2(C,D) > u_2(C,C)$.
  - **On path**: For player 1,

$$V_1(\boldsymbol{H}_{CD}) = (1-\delta)u_1(C,D) + \delta\Big[(1-x)V_1(\boldsymbol{H}_{CD}) + xV_1(\boldsymbol{H}_{CC})\Big]$$

    Then, we have

$$V_1(\boldsymbol{H}_{CD}) = \frac{-l(1-\delta)+\delta x}{1-\delta(1-x)}$$

  - No profitable one shot deviation exists if $x \geq \frac{l(1-\delta)}{\delta}$.
    * If player 1 deviates, then

$$(1-\delta)u_1(D,D) + \delta V_1(\boldsymbol{H}_{CD})$$

For this OSD not to be profitable, we require

$$V_1(\boldsymbol{H}_{CD}) \geq (1-\delta)u_1(D,D) + \delta V_1(\boldsymbol{H}_{CD}) \iff x \geq \frac{l(1-\delta)}{\delta}$$

– Player 1 receives her minmax payoff if and only if $V_1(\boldsymbol{H}_{CD}) = 0 \iff x = \frac{l(1-\delta)}{\delta}$.

– Lastly, we need $x \leq 1 \iff \delta \geq \frac{l}{l+1}$ and recall that we also need $\delta \geq \frac{g}{g+1}$ for the grim trigger strategy part. Thus, if δ is greater than these two thresholds, the strategy profile is a SPE in which player 1 receives her minmax payoff 0.

- It should be notice that in public randomization case, we also need to find a proper range or value of x. If x is well-characterized by given parameters(e.g. g, l), (subsubsection 2.14.6 case), then δ would be characterized by x. Conversely, if δ is well-characterized by given parameters (this subsubsection case), then x would be characterized by δ.

3 Part 3: Static Games of Incomplete Information

3.1 Illustration

In games of incomplete information, we relax the assumption that every player is perfectly informed about the payoffs of all other players. For example, suppose that player 1 does know player 2's type, type I or type II. Then, player 1 could be playing either of the following games.

Player 1 vs. Player 2(I):

		B	F
Player 1	B	$2, 1$	$0, 0$
	F	$0, 0$	$1, 2$

Player 1 vs. Player 2(II):

		B	F
Player 1	B	$2, 0$	$0, 2$
	F	$0, 1$	$1, 0$

Player 2 knows what her preferences are and which game she is playing. Further assume that player 1 thinks that either type of player 2 is equally likely.

We model each type of player 2 as a separate player, so that $N = \{1, I, II\}$, $S_i = \{B, F\}$ for all $i \in N$, and a strategy profile $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_I, \boldsymbol{\sigma}_{II}) \in \times_{i\in N}\Delta(S_i)$. All players evaluate payoffs from a strategy profile by taking expectation over the resulting payoffs with respect to their updated priors. (However, in this example, this is not informative at all.)

Consider a pure strategy profile $\boldsymbol{s} = (B, B, F)$. Both type I and II are best responding. In terms of player 1, her expected payoff is

$$\frac{1}{2}u_1(B, B) + \frac{1}{2}u_1(B, F) = 1$$

and that from deviating to F is

$$\frac{1}{2}u_1(F, B) + \frac{1}{2}u_1(F, F) = \frac{1}{2}$$

Thus, $\boldsymbol{s}$ is a Bayes-Nash equilibrium.

3.2 Preliminaries

A game of incomplete information (or Bayesian game) $G =< N, \boldsymbol{S}, \boldsymbol{u}, \boldsymbol{\Theta}, p >$ consists of

- A (finite) set of players: $N = \{1, 2, \dots, n\}$
- A set of pure action profiles: $A = \times_{i\in N} A_i$ where A_i is the set of player i's pure actions.
- A set of possible types: $\boldsymbol{\Theta} = \times_{i\in N}\Theta_i$ where Θ_i is the set of player i's possible types.
- A **pure strategy** for player i is a mapping $s_i : \Theta_i \to A_i$ prescribing an action for each possible type of player i.
- A **common prior**: A joint probability distribution p over $\boldsymbol{\Theta}$ such that $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$.
  - Ex-ante: Before $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is realized, each player's belief about $\boldsymbol{\theta}$ must be identical and equal to p.
  - Interim: When player i knows her own type $\theta_i \in \Theta_i$, her belief about the types of the others is

$$p(\boldsymbol{\theta}_{-i} \mid \theta_i) = \frac{p(\theta_i, \boldsymbol{\theta}_{-i})}{\sum_{\boldsymbol{\theta}_{-i}\in\boldsymbol{\Theta}_{-i}} p(\theta_i, \boldsymbol{\theta}_{-i})}, \ \forall \boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}$$

- Ex-post: $\boldsymbol{\theta}$ is known to every player.

- A payoff function for each player: $u_i : A \times \boldsymbol{\Theta} \to \mathbb{R}$. [21]
  - When player i knows her own type, her (expected) payoff from $\boldsymbol{s} = (s_i, \boldsymbol{s}_{-i})$ is

$$U_i(s_i, \boldsymbol{s}_{-i}; \theta_i) = \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} p(\boldsymbol{\theta}_{-i} \mid \theta_i) u_i(s_i(\theta_i), \boldsymbol{s}_{-i}(\boldsymbol{\theta}_{-i}); \theta_i, \boldsymbol{\theta}_{-i})$$

- If $\mid A_i \mid$ and $\mid \Theta_i \mid$ are finite for each $i \in N$, and $\mid N \mid$ is finite, we say that G is a finite game of incomplete information.

3.3 Bayes-Nash Equilibrium

Definition 3.1. [Bayes Nash equilibrium] A strategy profile $\boldsymbol{s}$ is a pure strategy Bayes Nash equilibrium if $\forall i \in N$ and $\forall \theta_i \in \Theta_i$ we have that

$$s_i(\theta_i) \in \underset{\widehat{s}_i \in S_i}{\operatorname{argmax}} \sum_{\boldsymbol{\theta}_{-i} \in \boldsymbol{\Theta}_{-i}} p(\boldsymbol{\theta}_{-i} \mid \theta_i) u_i(\widehat{s}_i, \boldsymbol{s}_{-i}(\boldsymbol{\theta}_{-i}); \theta_i, \boldsymbol{\theta}_{-i}) = \underset{\widehat{s}_i \in S_i}{\operatorname{argmax}}\, U_i(\widehat{s}_i, \boldsymbol{s}_{-i}, \theta_i)$$

The definition extends to mixed strategy profiles in the usual way.

Theorem 3.1. *Let $G =< N, \boldsymbol{S}, \boldsymbol{u}, \boldsymbol{\Theta}, p >$ be finite. Then, a mixed strategy BNE exists.*

Theorem 3.2. *Consider a Bayesian game with continuous action spaces and continuous types. If action sets and type sets are compact, and payoff functions are continuous and concave in own actions, then there exists a pure strategy BNE.*

3.4 State of the World

Consider a following voting game where players have aligned preference in the sense that they wish to elect the correct candidate.

- The electorate consists of two people $N = \{1, 2\}$, and there are two candidates $n \in \{\mathcal{L}, \mathcal{R}\}$.
- The state of the world $\omega \in \{L, R\}$ is drawn by nature with $p_0 = P(\omega = L)$, and ω is not observable for each player.
- Player i privately observes a noisy signal $k_i \in \{l, r\}$ with the precision $q_i = P(k_i = l \mid \omega = L) = P(k_i = r \mid \omega = R)$ where $q_i \in (0, 1)$.
- If candidate n is elected in state ω, player i's payoff is $u(\mathcal{R}, R) = u(\mathcal{L}, L) = 1$ and $u(\mathcal{R}, L) = u(\mathcal{L}, R) = -1$.
- Casting a vote yields a cost $c \in (0, 1)$.
- Player i's type is realization of the signal she observes. Hence, a strategy is mapping from $\{l, r\} \to \{l, r, 0\}$ where 0 denotes abstention.
- The resulting prior distribution over the type space $\boldsymbol{\Theta} = \{l, r\} \times \{l, r\}$ is

[21] u_i depends on the realized pure action profile, her own realized type, and the realized types of all other players.

		Player 2(k_2)	
		l	r
Player 1(k_1)	l	$p_0q_1q_2 + (1-p_0)(1-q_1)(1-q_2)$	$p_0q_1(1-q_2) + (1-p_0)(1-q_1)q_2$
	r	$p_0(1-q_1)q_2 + (1-p_0)q_1(1-q_2)$	$p_0(1-q_1)(1-q_2) + (1-p_0)q_1q_2$

- Player i's interim posterior about the state of the world, for each $k \in \{l, r\}$ is

$$P(\omega = L \mid k_i = k) = \frac{p_0 P(k_i = k \mid \omega = L)}{p_0 P(k_i = k \mid \omega = L) + (1-p_0)P(k_i = k \mid \omega = R)}$$

- Player i's interim posterior about player j's type, for each $(k, k') \in \{l, r\}^2$ is

$$P(k_j = k' \mid k_i = k) = P(\omega = L \mid k_i = k)P(k_j = k' \mid \omega = L) + P(\omega = R \mid k_i = k)P(k_j = k' \mid \omega = R)$$

3.5 Auction Theory

In this lecture, we consider two auction formats.

- **First price auction** (FPA): The highest bid wins the good, and pays her own bid.
- **Second price auction** (SPA): The highest bid wins the good, and pays the second highest bid.

In both cases, we commonly assume that

- A single object for sale, and N bidders.
- A bidder i assigns value v_i to the object, which is private information. ($\Rightarrow$ **private value auction**)
- Each v_i is independently and identically distributed on $[0, \overline{v}]$ with CDF F and pdf f having full support on $[0, \overline{v}]$. ($\Rightarrow$ **independent private value auction**)
- The probability distribution F is common knowledge.
- A strategy for bidder i is a mapping $b_i : [0, \overline{v}] \to \mathbb{R}$.

3.5.1 Second Price Auction

Let $b_i \in \mathbb{R}$ denote the bid submitted by player i, and $(b_i, \boldsymbol{b}_{-i}) \in \mathbb{R}^N$ denote a bidding profile. Bidder i's payoff is

$$U_i(b_i, \boldsymbol{b}_{-i}; v_i) = \begin{cases} v_i - \max\limits_{j \neq i} b_j & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \frac{v_i - \max\limits_{j \neq i} b_j}{M} & \text{if } b_i = \max\limits_{j \neq i} b_j \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

where $M = \mid \{k \mid b_k = \max\limits_{i \in \{1,2,\ldots,N\}} b_i\} \mid$ denotes the number of bidders who have placed the highest bid.

Lemma 3.1. *In a SPA, bidding her own valuation $b_i(v_i) = v_i$ is a weakly dominant strategy.*

Proof. Let $B_i = \max\limits_{j \neq i} b_j$ and $b_i'(v_i) \neq b_i(v_i) = v_i$. Observe that

	$B_i < v_i$	$B_i = v_i$	$v_i < B_i < b'_i$	$B_i = b'_i$	$b'_i < B_i$
$b'_i > v_i$	$v_i - B_i > 0$	$v_i - B_i = 0$	$v_i - B_i < 0$	$\frac{v_i - B_i}{M} < 0$	0
$b_i = v_i$	$v_i - B_i > 0$	$\frac{v_i - B_i}{M} = 0$	0	0	0

	$B_i < b''_i$	$B_i = b''_i$	$b''_i < B_i < v_i$	$B_i = v_i$	$v_i < B_i$
$b''_i < v_i$	$v_i - B_i > 0$	$\frac{v_i - B_i}{M} > 0$	0	0	0
$b_i = v_i$	$v_i - B_i > 0$	$v_i - B_i > 0$	$v_i - B_i$	$\frac{v_i - B_i}{M} = 0$	0

Thus, we conclude that given $b_i = v_i$ for all $\widetilde{b}_i \neq v_i$, $U_i(b_i, \boldsymbol{b}_{-i}; v_i) \geq U_i(\widetilde{b}_i, \boldsymbol{b}_{-i}; v_i)$ for every $\boldsymbol{b}_{-i}$ and there exists $\boldsymbol{b}_{-i}$ such that the inequality is strict. $\square$

Theorem 3.3. *In a SPA, if N bidders have independent private values, there exists a unique BNE in weakly dominant strategies which involves each player using the bidding strategy $b_i(v_i) = v_i$.* [22]

3.5.2 First Price Sealed Bid Auction

To analyze the first price auction, we further assume that

- A bidding function $b_i(v_i) : [0,1] \to \mathbb{R}$ is strictly increasing and differentiable. [23]
- Players have a symmetric bidding function for all $i \in \{1, 2, \ldots, N\}$, and thus $b_i(v_i) = b(v_i)$.

Now, let $\widehat{b}(\cdot)$ denote the symmetric BNE bidding strategy.

Lemma 3.2. *Bidder i is (weakly) better off bidding $\widehat{b}(v_i)$ than $\widehat{b}(r)$ for all $r \neq v_i$ where $r \in [0,1]$.*

Proof. Suppose not. Then, for some v_i, there exists $r (\neq v_i)$ such that $\widetilde{b} = \widehat{b}(r)$ which yields a higher payoff than $\widehat{b}(v_i)$. Then, $\widehat{b}(\cdot)$ is not a BNE strategy. $\square$

Now, we want to find a closed form of $\widehat{b}(\cdot)$. Player i's payoff from report $r \in [0,1]$ in the BNE is

$$u_i(\widehat{b}(r), \widehat{b}((v_j))_{j \neq i}; v_i, (v_j)_{j \neq i}) = \begin{cases} v_i - \widehat{b}(r) & \text{if } \widehat{b}(r) \geq \max\limits_{j \neq i} \widehat{b}(v_j) \\ 0 & \text{otherwise} \end{cases}$$

In this case, the equilibrium probability of player i winning the auction with report r is

$$\begin{aligned} P\Big(\widehat{b}(r) \geq \max_{j \neq i} \widehat{b}(v_j)\Big) &= P\Big(\widehat{b}(r) \geq \widehat{b}\big(\max_{j \neq i} v_j\big)\Big) \ (\because \ \widehat{b}'(\cdot) > 0) \\ &= P\Big(r \geq \max_{j \neq i} v_j\Big) \ (\because \ \widehat{b}'(\cdot) > 0) \\ &= \big[P(v_j \leq r)\big]^{N-1} \ (\because \ \text{i.i.d}) \\ &= F(r)^{N-1} \equiv G(r) \end{aligned}$$

Thus, player i's expected payoff under the symmetric BNE profile $\widehat{b}(\cdot)$ from making report r given that her type is v_i is

$$U_i(r, v_i) = [v_i - \widehat{b}(r)] G(r)$$

[22] It should be carefully noted that that $\{b_i(v_i) = v_i\}_{i=1}^N$ is not a unique BNE. For example, $\{b_1(v_1) = \overline{v}, b_j(v_j) = 0\}_{j=2}^N$ is also a BNE. In this sense, keep in mind that $\{b_i(v_i) = v_i\}_{i=1}^N$ is unique only in terms of using weakly dominant strategies. Yo

[23] Note that we normalize $\overline{v}$ as 1.

As claimed in previous lemma, reporting her type truthfully is optimal for player i with valuation $v_i \in [0,1]$, and hence the first order condition requires

$$
\begin{aligned}
\frac{\partial U_i(r,v_i)}{\partial r}\Big|_{r=v_i} = 0 \;\Leftrightarrow\; & \underbrace{\frac{\partial v_i G(r)}{\partial r}\Big|_{r=v_i}}_{\text{marginal benefit}} = \underbrace{\frac{\partial \widehat{b}(r)G(r)}{\partial r}\Big|_{r=v_i}}_{\text{margial cost}} \\
\Leftrightarrow\; & v_i g(v_i) = \frac{\partial \widehat{b}(v_i)G(v_i)}{\partial v_i} \\
\Leftrightarrow\; & \int_0^{v_i} x g(x) dx = \int_0^{v_i} \frac{\partial \widehat{b}(x)G(x)}{\partial x} dx \\
\Leftrightarrow\; & \int_0^{v_i} x g(x) dx = \widehat{b}(v_i)G(v_i) - \underbrace{\widehat{b}(0)}_{=0}G(0) \\
\Leftrightarrow\; & \widehat{b}(v_i) = \int_0^{v_i} x \frac{g(x)}{G(v_i)} dx = \mathbb{E}\Big[\max_{j\neq i} v_j \mid \max_{j\neq i} v_j \le v_i\Big]
\end{aligned}
$$

Lastly, we want to check whether the second order condition is also satisfied. Observe that

$$U_i(r,v_i) = v_i G(r) - \widehat{b}(r)G(r)$$

Since

$$
\begin{aligned}
\frac{\partial U_i(r,v_i)}{\partial r} &= v_i g(r) - \frac{d}{dr}\underbrace{\widehat{b}(r)G(r)}_{=rg(r)} \\
&= (v_i - r) g(r)
\end{aligned}
$$

Thus,

$$\frac{\partial U_i(r,v_i)}{\partial r} > 0 \;\text{ if } v_i > r \;\text{ and }\; \frac{\partial U_i(r,v_i)}{\partial r} < 0 \;\text{ if } v_i < r$$

Example 3.1. Let $v_i \sim U[0,1]$, so that $F(v_i) = v_i$. Then, we have

$$
\begin{aligned}
\widehat{b}(v_i) &= \int_0^{v_i} x \frac{g(x)}{G(v_i)} dx \\
&= \int_0^{v_i} x \frac{(N-1)x^{N-2}}{{v_i}^{N-1}} dx \\
&= \frac{N-1}{N} v_i
\end{aligned}
$$

3.5.3 Expected Revenue Equivalence

Proposition 3.1. *With independent private values, FPA and SPA are revenue equivalent.*

Proof. Consider the event that highest bidder is i and that her value is v. What is her expected payment, conditional on this event? Under both formats, it is the expected value of the second highest bidder. In a FPA, this is i's bid, and in a SPA, i pays the second highest bid and each player bids her value. Finally the probability of the even that i is the highest bidder, and that her valuation is v is the same under both formats. □

4 Part 4: Extensive Form Equilibrium Refinements

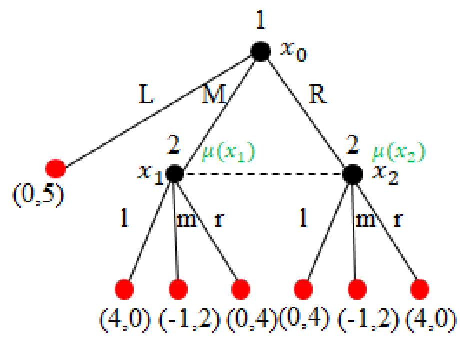
4.1 Preliminaries

Consider a finite extensive form game with perfect recall $\Gamma =< N, A, X, Z, \iota, H, \pi, \boldsymbol{u} >$ and a behavioral strategy $\boldsymbol{b} = (\boldsymbol{b}_i)_{i \in N}$ where $\boldsymbol{b}_i \in \times_{h \in H_i} \Delta(A(h))$.

Definition 4.1. [Belief] Let $\mu_h(x)$ denote player $\iota(h)$'s belief that node $x \in h$ is reached conditional on information set $h \in H$ being reached. A system of belief $\boldsymbol{\mu} = (\boldsymbol{\mu}_h)_{h \in H}$ is a collection of beliefs such that $\forall h \in H$ and $\boldsymbol{\mu}_h \in \Delta(h)$.

Definition 4.2. [Assessment] A pair of $(\boldsymbol{b}, \boldsymbol{\mu})$ is called an assessment.

Example 4.1.



Player 2's belief should satisfy $\mu(x_1) + \mu(x_2) = 1$. Then, player 2's expected continuation utility at h from her pure strategy l is $\mathbb{E}[u_2(l; \boldsymbol{\mu})] = \mu(x_1) \times 0 + \mu(x_2) \times 4 = 4\mu(x_2)$.

4.2 Perfect Bayesian Equilibrium

Definition 4.3. Let $p(x \mid \boldsymbol{b})$ denote the probability that node x is reached under the behavioral strategy profile $\boldsymbol{b}$. Then, $p(x \mid \boldsymbol{b}) = \prod_{a \in \boldsymbol{a}^x} b(a)$ where $\boldsymbol{a}^x$ is the path of actions leading from the initial node x_0 to the node x.

Definition 4.4. Let $p(h \mid \boldsymbol{b})$ denote the probability that information set h is reached under the behavioral strategy profile $\boldsymbol{b}$. Then, $p(h \mid \boldsymbol{b}) = \sum_{x \in h} p(x \mid \boldsymbol{b})$.

Definition 4.5. [Perfect Bayesian Equilibrium] An assessment $(\boldsymbol{b}, \boldsymbol{\mu})$ constitutes a perfect bayesian equilibrium of the game Γ if $\forall h \in H$,

1) **Sequential Rationality**: For given $\boldsymbol{\mu}_h$, for every $\boldsymbol{b}'_{\iota(h)}$

$$\mathbb{E}[u_{\iota(h)}((\boldsymbol{b} \mid h); \boldsymbol{\mu}_h)] \geq \mathbb{E}[u_{\iota(h)}((\boldsymbol{b}'_{\iota(h)}, \boldsymbol{b}_{-\iota(h)}) \mid h; \boldsymbol{\mu}_h)]$$

2) **Bayes Rule**: Whenever possible, beliefs are must be defined by Bayes's rule. That is, if $p(h \mid \boldsymbol{b}) > 0$, then for any $x \in h$,

$$\mu_h(x) = \frac{p(x \mid \boldsymbol{b})}{p(h \mid \boldsymbol{b})}$$

Remark 4.1. If $p(h \mid \boldsymbol{b}) = 0$, PBE imposes no restriction on beliefs.

Remark 4.2. The sequential rationality in PBE does not necessarily imply sequential rationality in the sense of no incredible threat.

Proposition 4.1. *If an assessment $(\boldsymbol{b}, \boldsymbol{\mu})$ is a PBE, then a strategy profile $\boldsymbol{b}$ is a NE.*

Example 4.2. In previous example 4.1, consider the following two NE, [24] $\boldsymbol{b}^* = (b_M^* = b_R^* = \frac{1}{2},\ b_l^* = b_r^* = \frac{1}{2})$ and $\boldsymbol{b}^{**} = (b_L^{**} = 1, b_m^{**} = 1)$. Notice that both of them are SPE since the entire game is the only proper subgame.

- Under $\boldsymbol{b}^*$, player $2's$ information set is reached, and her beliefs must be defined by Bayes rule, thus

$$\mu^*(x_1) = \frac{b_M^*}{b_M^* + b_R^*} = \frac{1}{2} \text{ and } \mu^*(x_2) = \frac{b_R^*}{b_M^* + b_R^*} = \frac{1}{2}$$

  The sequential rationality for player 2 requires

$$\underbrace{\mathbb{E}[u_2(l; \boldsymbol{\mu}^*)]}_{=2} = \underbrace{\mathbb{E}[u_2(r; \boldsymbol{\mu}^*)]}_{=2} \geq \underbrace{\mathbb{E}[u_2(m; \boldsymbol{\mu}^*)]}_{=2}$$

  and this is clearly satisfied.

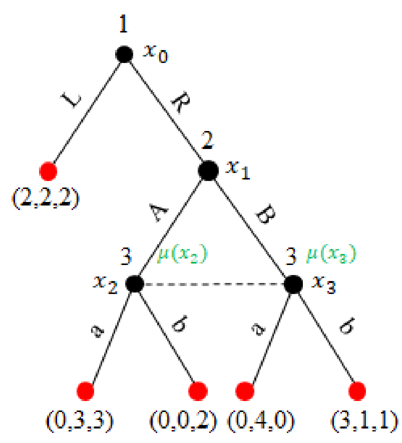
- It is also easy to check that the sequential rationality for player 1 is satisfied, and hence $(\boldsymbol{b}^*, \boldsymbol{\mu}^*)$ is a PBE.

- Under $\boldsymbol{b}^{**}$, player 2's information set is not reached, so there is no restriction on her beliefs. To satisfy sequential rationality for player 2, we need

$$\mathbb{E}[u_2(m; \boldsymbol{\mu}^{**})] = \begin{cases} \geq \mathbb{E}[u_2(l; \boldsymbol{\mu}^{**})] & \Leftrightarrow \ \frac{1}{2} \geq \mu^{**}(x_1) \\ \geq \mathbb{E}[u_2(r; \boldsymbol{\mu}^{**})] & \Leftrightarrow \ \frac{1}{2} \geq \mu^{**}(x_2) \end{cases}$$

- Again, it is also easy to check that the sequential rationality for player 1 is satisfied, and hence $(\boldsymbol{b}^{**}, \boldsymbol{\mu}^{**})$ is also a PBE.

Example 4.3. [Weakness of PBE]



- The game above admits four 4 pure strategy NE, $(L, A, a), (L, A, b), (L, B, a)$ and (R, B, b). But, it is (R, B, b) is the only (pure strategy) SPE.

- A strategy A for player 2 is weakly dominated in the entire game and strictly dominated in the proper subgame.

[24]Notice that there are many other equilibria.

- Will the PBE eliminate all NE that are not SPE? No.
  - Note that (L, A, a) and (L, A, b) cannot be supported as PBE because sequential rationality for player 2 is violated.
  - However, (L, B, a) can be supported as a PBE. To see this
    * For player 1 and 2, the sequential rationality holds.
    * For player 3, her information set is not reached. So, we only require her sequential rationality, and for this
    $$\mathbb{E}[u_3(a; \boldsymbol{\mu}^*)] \geq \mathbb{E}[u_3(b; \boldsymbol{\mu}^*)] \;\Leftrightarrow\; \mu^*(x_2) \geq \mu^*(x_3) \;\Leftrightarrow\; \mu^*(x_2) \geq \frac{1}{2}$$
    * Hence, $((L, B, a), \mu^*(x_2) \geq \frac{1}{2})$ is a PBE.
    * Notice that this PBE is somewhat strange because A is strictly dominated for player 2 in the proper subgame, but player 3 believes that player 2 is more likely to play A.
- This example illustrates that PBE is truly a refinement of NE and not of SPE.

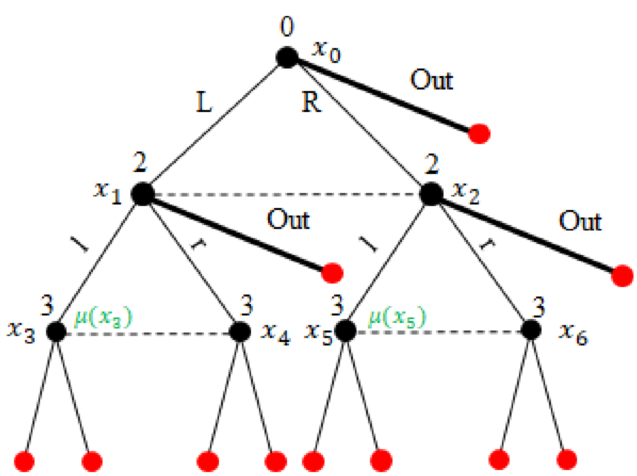
4.3 Sequential Equilibrium

Given that PBE may fail to impose proper beliefs off path, we want to restrict the equilibrium belief off path (informally) as follows.

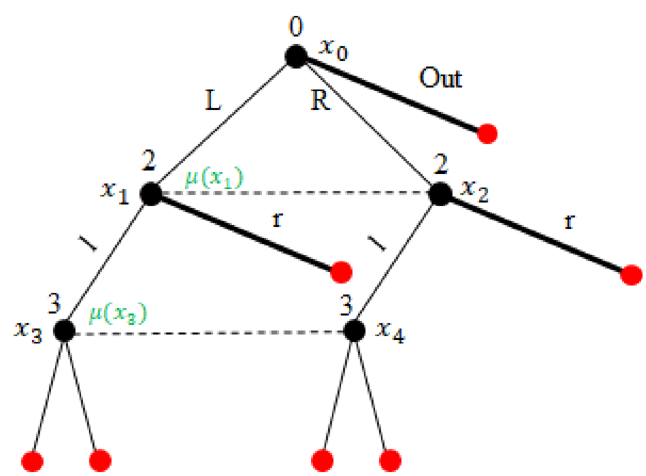
- **Independence**: Beliefs must reflect that players choose their strategies independently.
- **Common beliefs**: Players with identical information must have the identical beliefs.

Notice that whenever the information sets are reached, Bayes rule make independence and common beliefs satisfied.

Example 4.4. Consider games and illustrated (bold) strategy profiles below.



(a) Independence



(b) Common beliefs

- **Independence**: Player 2 does not know whether player 1 has chosen L or R. Player 3 knows whether player 1 has chosen L or R, but does not know whether player 2 has chosen l or r. Given that two information sets are not reached, they are not determined by Bayes Rule. Since $\boldsymbol{b}_2$ is independent of whether player 1 has chosen L or R, knowing whether player 1 has chosen L or R does not provide player 3 with any additional information concerning $\boldsymbol{b}_2$, thus $\mu(x_3) = \mu(x_5)$.

- **Common beliefs**: Neither player 2 nor player 3 know whether player 1 has chosen L or R. Player 3 knows that player 2 has chosen l. Again, given that two information sets are not reached, they are not determined by Bayes Rule. Suppose that player 2 assigns $\mu(x_1)$ to player 1 having chosen L. Since player 3 also has the same information, she must agree with player 2. Moreover, since knowing that player 2 chose l does not provide player 3 with any information on $\boldsymbol{b}_1$ by independence, player 3 must assign $\mu(x_3) = \mu(x_1)$.

Definition 4.6. [**Consistent assessment**] An assessment $(\boldsymbol{b}, \boldsymbol{\mu})$ is consistent if there exists a sequence of **completely mixed** behavioral strategies, $\{\boldsymbol{b}^n\}_{n=1}^{\infty}$ converging to $\boldsymbol{b}$ such that its associated sequence of Bayes rule induced systems of beliefs $\{\boldsymbol{\mu}^n\}_{n=1}^{\infty}$ also converges to $\boldsymbol{\mu}$.

Example 4.5. Recall example 4.4. Let $(\boldsymbol{b}, \boldsymbol{\mu})$ be given and consistent. Now, consider a following a sequence of completely mixed behavioral strategies, $(\boldsymbol{b}_1^n, \boldsymbol{b}_2^n) = (b_1^n(L), b_1^n(R), b_1^n(Out), b_2^n(l), b_2^n(r), b_2^n(Out))$ such that $\{\boldsymbol{b}^n\}_{n=1}^{\infty} \to \boldsymbol{b}$ and its associated $\{\boldsymbol{\mu}^n\}_{n=1}^{\infty} \to \boldsymbol{\mu}$. Under consistency,

$$\mu^n(x_3) = \frac{b_1^n(L)b_2^n(l)}{b_1^n(L)(b_2^n(l)+b_2^n(r))} = \frac{b_2^n(l)}{b_2^n(l)+b_2^n(r)} \quad \text{and} \quad \mu^n(x_5) = \frac{b_1^n(R)b_2^n(l)}{b_1^n(R)(b_2^n(l)+b_2^n(r))} = \frac{b_2^n(l)}{b_2^n(l)+b_2^n(r)}$$

Since $\mu^n(x_3) = \mu^n(x_5)$ for all $n \in N$, we must have $\mu(x_3) = \mu(x_5)$. That is, independence holds. Similarly, now consider a sequence of completely mixed behavioral strategies, $(\boldsymbol{b}_1^n, \boldsymbol{b}_2^n) = (b_1^n(L), b_1^n(R), b_1^n(Out), b_2^n(l), b_2^n(r))$. Under consistency,

$$\mu^n(x_1) = \frac{b_1^n(L)}{b_1^n(L)+b_1^n(R)} \quad \text{and} \quad \mu^n(x_3) = \frac{b_2^n(l)b_1^n(L)}{b_2^n(l)(b_1^n(L)+b_1^n(R))} = \frac{b_1^n(L)}{b_1^n(L)+b_1^n(R)}$$

Since $\mu^n(x_1) = \mu^n(x_3)$ for all $n \in N$, we must have $\mu(x_1) = \mu(x_3)$. That is, common beliefs and independence hold.

Definition 4.7. [**Sequential equilibrium**] [25] An assessment $(\boldsymbol{b}, \boldsymbol{\mu})$ constitutes a sequential equilibrium of the game Γ if $\forall h \in H$, it is **sequentially rational** and **consistent.**

Proposition 4.2. *If $(\boldsymbol{b}, \boldsymbol{\mu})$ is a sequential equilibrium, then $(\boldsymbol{b}, \boldsymbol{\mu})$ is a PBE. So, derive the set of PBE, and test those that have unreached information sets to find the set of SE.*

Remark 4.3. If all information sets are reached, the set of SE and that of PBE coincide.

Example 4.6. Recall example 4.3. We want to show that a NE (L, B, a) which is not a SPE cannot be supported as a SE. We already have shown that sequential rationality for player 1 and player 2 is satisfied. Consider any sequence of completely mixed behavioral strategy profile $(b_1^n(L), b_2^n(A), b_3^n(a)) \to (1, 0, 1)$. Then, for each $n \in \mathbb{N}$, we have

$$\mu(x_2)^n = \frac{(1-b_1^n(L))b_2^n(A)}{(1-b_1^n(L))(b_2^n(A)+1-b_2^n(A))} = b_2^n(A)$$

Thus, if consistency is required, we must have $\mu(x_2)^n \to 0 < \frac{1}{2}$, which make the sequential rationality for player 3 never be satisfied.

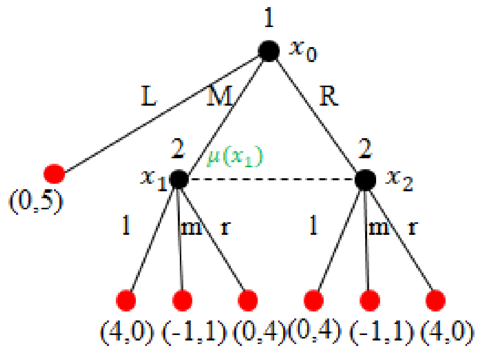
Proposition 4.3. *If $(\boldsymbol{b}, \boldsymbol{\mu})$ is a sequential equilibrium, then $\boldsymbol{b}$ is a SPE. In this sense, SE are refinement of SPE that guarantee sequential rationality in games of imperfect and incomplete information.*

[25] Kreps and Wilson (1982, Econometrica)

Remark 4.4. If a game is perfect information, SPE and SE are equivalent.

Remark 4.5. If a game is the only proper subgame itself, NE must be SPE. However, this SPE may not be SE.

Example 4.7.



- Consider a NE, $\boldsymbol{b}^* = (b_L^* = 1, b_m^* = 1)$. Since the entire game is the only proper subgame, $\boldsymbol{b}$ is also a SPE.
  - For player 2 to be sequentially rational, we need

$$\mathbb{E}[u_2(m, \boldsymbol{\mu})] = 1 \geq \begin{cases} \mathbb{E}[u_2(l, \boldsymbol{\mu})] = 4(1 - \mu(x_1)) \\ \mathbb{E}[u_2(m, \boldsymbol{\mu})] = 4\mu(x_1) \end{cases}$$

    That is

$$\mu(x_1) \geq \frac{3}{4} \quad \text{and} \quad \mu(x_1) \leq \frac{1}{4}$$

    Thus, there does not exists $\boldsymbol{\mu}$ which can support $\boldsymbol{b}^*$ as a PBE, and hence it cannot be a SE as well.

- Consider a NE, $\boldsymbol{b}^{**} = (b_M^{**} = b_R^{**} = \frac{1}{2},\ b_l^{**} = b_r^{**} = \frac{1}{2})$.
  - Player 1 is sequentially rational because

$$\mathbb{E}[u_1(M, \boldsymbol{b}_2)] = \mathbb{E}[u_1(R, \boldsymbol{b}_2)] = 2 \geq 0 = \mathbb{E}[u_1(L, \boldsymbol{b}_2)]$$

  - Since player $2'$s information set is reached, and her beliefs must be defined by Bayes rule, thus

$$\mu^{**}(x_1) = \frac{b_M^*}{b_M^* + b_R^*} = \frac{1}{2} \quad \text{and} \quad 1 - \mu^{**}(x_1) = \frac{b_R^*}{b_M^* + b_R^*} = \frac{1}{2}$$

    The sequential rationality for player 2 requires

$$\underbrace{\mathbb{E}[u_2(l; \boldsymbol{\mu}^{**})]}_{=2} = \underbrace{\mathbb{E}[u_2(r; \boldsymbol{\mu}^{**})]}_{=2} \geq \underbrace{\mathbb{E}[u_2(m; \boldsymbol{\mu}^{**})]}_{=2}$$

    and this is clearly satisfied, and hence, $(\boldsymbol{b}^{**}, \boldsymbol{\mu}^{**})$ is a PBE.
  - To show that $(\boldsymbol{b}^{**}, \boldsymbol{\mu}^{**})$ is an indeed SE, consider a completely mixed behavioral strategy profile, $(\boldsymbol{b}_1^n, \boldsymbol{b}_2^n) = (b_1^n(L) = \frac{2}{n}, b_1^n(M) = \frac{1}{2} - \frac{1}{n}, b_1^n(R) = \frac{1}{2} - \frac{1}{n}, b_2^n(l) = \frac{1}{2} - \frac{1}{n}, b_2^n(m) = \frac{2}{n}, b_2^n(r) = \frac{1}{2} - \frac{1}{n})$.

Then,

$$\mu^n(x_1) = \frac{b_1^n(M)}{b_1^n(M) + b_1^n(R)} = \frac{\frac{1}{2} - \frac{1}{n}}{1 - \frac{2}{n}} = \frac{1}{2} \to \mu^{**}(x_1) = \frac{1}{2}$$

4.4 Signaling Games

4.4.1 Model Description

- Two players, the sender who is informed, and the receiver who is not.
- $\Theta = \{\theta_1, \dots, \theta_k\}$ denotes the set of possible types of the sender. $p \in \Delta\Theta$ denotes the common prior probability distribution over the type space.
- $M = \{m_1, m_2, \dots, m_h\}$ denotes the set of message available to the sender.
- $A = \{a_1, \dots, a_l\}$ denotes the set of actions available to the receiver.
- The payoffs is a mapping such that $u_i : \Theta \times M \times A \to \mathbb{R}$ where $i \in \{sender, receiver\}$, so that $u_i(\theta, m, a)$.
- The signaling games follow in the order of
  - Nature chooses θ according to $p \in \Delta(\Theta)$.
  - The sender privately observes $\theta \in \Theta$, then sends $m \in M$.
  - The receiver observes m and chooses an action $a \in A$,
  - The payoffs at the corresponding terminal node determines $u_s(\theta, m, a)$ and $u_r(\theta, m, a)$.
- A **behavioral (mixed) strategy for the sender** is defined by $\boldsymbol{\sigma} : \Theta \to \Delta(M)$. Let $\sigma(m \mid \theta)$ denote the probability that the sender of type θ sends message m.
- A **behavioral (mixed) strategy for the receiver** is defined by $\boldsymbol{\rho} : M \to \Delta(A)$. Let $\rho(a \mid m)$ denote the probability that the receiver chooses a for given message m.
- A **system of belief** specifies a probability distribution over the set of the sender's possible types Θ at every information set of the receiver. That is, for every possible message $m \in M$, $\boldsymbol{\mu} = \{\mu(\theta_1 \mid m), \dots, \mu(\theta_k \mid m)\}$ such that $\sum_{j=1}^{k} \mu(\theta_j \mid m) = 1$.

Definition 4.8. A perfect Bayesian equilibrium of a signaling game is an assessment $(\boldsymbol{\sigma}^*, \boldsymbol{\rho}^*, \boldsymbol{\mu}^*)$ such that

1) **Sequential Rationality** for the sender: For all $\theta \in \Theta$

$$\boldsymbol{\sigma}^*(\cdot \mid \theta) \in \underset{\boldsymbol{\sigma} \in \Delta(M)}{\operatorname{argmax}}\; u_s(\theta, \boldsymbol{\sigma}, \boldsymbol{\rho}^*)$$

2) **Sequential Rationality** for the receiver: For all $m \in M$

$$\boldsymbol{\rho}^*(\cdot \mid m) \in \underset{\boldsymbol{\rho} \in \Delta(A)}{\operatorname{argmax}} \sum_{\theta \in \Theta} \mu^*(\theta \mid m) u_r(\theta, m, \boldsymbol{\rho})$$

3) **Bayes Rule**: For all $\theta \in \Theta$ and for all $m \in M$, if $\sum_{\theta' \in \Theta} p(\theta')\sigma^*(m \mid \theta') > 0$, then

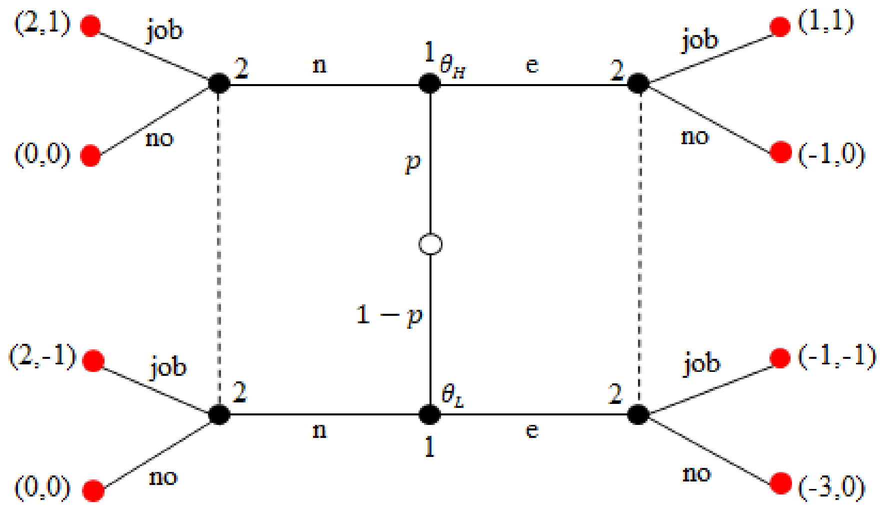
$$\mu^*(\theta \mid m) = \frac{p(\theta)\sigma^*(m \mid \theta)}{\sum_{\theta' \in \Theta} p(\theta')\sigma^*(m \mid \theta')}$$

and otherwise $\mu^*(\cdot \mid m)$ is any probability distribution over Θ.

Theorem 4.1. **[Fudenberg and Tirole, 1991]** *Consider an extensive form game of incomplete information with independent types. If either each player has at most two possible types or there are two periods, then the sets of PBE and SE coincide.*

4.4.2 Spence (1973): Job Market Game

- $\Theta = \{\theta_H, \theta_L\}$ with $p(\theta_H) = p$, $M = \{n, e\}$ and $A = \{job, no\}$.
- Assume that
  - Education has no effect on productivity.
  - Education is costly for both types but costlier for θ_L. (e.g. -1 for θ_H and -3 for θ_L)
  - Both types prefer being employed (2) to being unemployed (0).
  - A firm prefers a high productivity worker (1) to no worker at all (0) to a low productivity (-1).



(c) Job market signaling

- **Separating equilibria**
  - $\sigma^*(n \mid \theta_L) = 1$ and $\sigma^*(n \mid \theta_H) = 1$: **YES**
    * Given that both information sets are reached,

$$\mu^*(\theta_H \mid e) = 1 \text{ and } \mu^*(\theta_L \mid n) = 1$$

    * The firm's best response is

$$\rho^*(job \mid e) = 1 \text{ and } \rho^*(no \mid n) = 1$$

    * Type θ_H has no profitable deviation, because $u_s^*(\theta_H, e, job) = 1 \geq u_s(\theta_H, n, no) = 0$.
    * Type θ_L has no profitable deviation, because $u_s^*(\theta_L, n, no) = 0 \geq u_s(\theta_L, e, job) = -1$.
  - $\sigma(e \mid \theta_L) = 1$ and $\sigma(n \mid \theta_H) = 1$: **NO**

– Given that both information sets are reached,

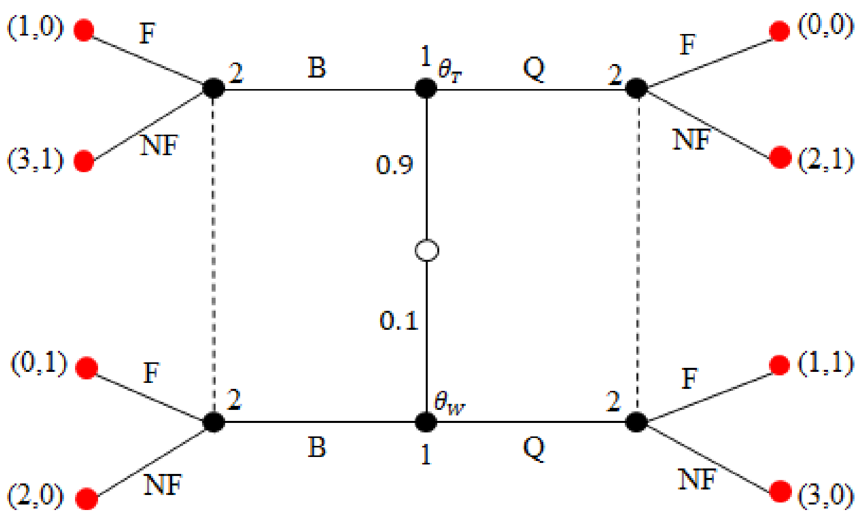
$$\mu(\theta_H \mid n) = 1 \text{ and } \mu(\theta_L \mid e) = 1$$

– The firm's best response is

$$\rho^*(job \mid n) = 1 \text{ and } \rho^*(no \mid e) = 1$$

– However, type θ_L has a profitable deviation, because $u_s(\theta_L, e, no) = -3 < u_s(\theta_L, n, job) = 2$.

4.4.3 Cho and Kreps (1987): Beer and Quiche



(d) Beer and Quiche

- **Pooling equilibria**
  - $\sigma^*(B \mid \theta_T) = 1$ and $\sigma^*(B \mid \theta_W) = 1$: **YES**
    * Given that only one information set is reached

$$\mu^*(\theta_T \mid B) = 0.9 \text{ and } \mu^*(\theta_W \mid B) = 0.1$$

    * The firm's payoffs from playing F and NF at information set (B) are

$$0.1 = 0.9 \times 0 + 0.1 \times 1 = F < NF = 0.9 \times 1 + 0.1 \times 0 = 0.9$$

      Thus, $\rho^*(NF \mid B) = 1$.
    * Type θ_T has no profitable deviation, because $u_s^*(\theta_T, B, NF) = 3 \geq u_s(\theta_T, Q, \rho^*(F \mid Q))$ for all $\rho^*(F \mid Q)$.
    * Type θ_W has no profitable deviation if $u_s^*(\theta_W, B, NF) = 2 \geq u_s(\theta_W, Q, \rho^*(F \mid Q)) = 3 - 2\rho^*(F \mid Q) \Leftrightarrow \rho^*(F \mid Q) \geq \frac{1}{2}$, and this holds only if $\mu^*(\theta_T \mid Q) \leq \frac{1}{2}$.

* Hence, $\sigma^*(B \mid \theta_T) = 1, \sigma^*(B \mid \theta_W) = 1, \mu^*(\theta_T \mid B) = 0.9, \mu^*(\theta_T \mid Q) \leq \frac{1}{2}, \rho^*(NF \mid B) = 1, \rho^*(F \mid Q) \geq \frac{1}{2})$ is a PBE.

- $\sigma^*(Q \mid \theta_T) = 1$ and $\sigma^*(Q \mid \theta_W) = 1$: **YES**
  * Given that only one information set is reached

$$\mu^*(\theta_T \mid Q) = 0.9 \text{ and } \mu^*(\theta_W \mid Q) = 0.1$$

  * The firm's payoffs from playing F and NF are at information set (Q) are

$$0.1 = 0.9 \times 0 + 0.1 \times 1 = F < NF = 0.9 \times 1 + 0.1 \times 0 = 0.9$$

    Thus, $\rho^*(NF \mid Q) = 1$.
  * Type θ_W has no profitable deviation, because $u_s^*(\theta_W, Q, NF) = 3 \geq u_s(\theta_W, B, \rho^*(F \mid B))$ for all $\rho^*(F \mid Q)$.
  * Type θ_T has no profitable deviation if $u_s^*(\theta_T, Q, NF) = 2 \geq u(\theta_T, B, \rho^*(F \mid B)) = 3 - 2\rho^*(F \mid B) \Leftrightarrow \rho^*(F \mid B) \geq \frac{1}{2}$, and this holds only if $\mu^*(\theta_T \mid B) \leq \frac{1}{2}$.
  * Hence, $\sigma^*(Q \mid \theta_T) = 1, \sigma^*(Q \mid \theta_W) = 1, \mu^*(\theta_T \mid Q) = 0.9, \mu^*(\theta_T \mid B) \leq \frac{1}{2}), \rho^*(NF \mid Q) = 1, \rho^*(F \mid B) \geq \frac{1}{2})$ is a PBE.

4.4.4 Refining Off Path Beliefs and Intuitive Criterion

Consider a pooling equilibrium, $\sigma(\theta_T) = \sigma(\theta_W) = Q$ in Beer and Quiche game. In this pooling equilibrium, we have shown that we need $\mu(\theta_T \mid B) \leq \frac{1}{2}$. However, this belief is somewhat strange because this assumes that mistakes are more likely to come from θ_W, who has nothing to gain regardless of player 2's response, than by θ_T, who may gain if player 2 responds with NF. In fact, this equilibrium fails to satisfy Cho and Kreps's **intuitive criterion**. In the note that follows, notice that we discuss a pooling equilibrium where $\sigma(\theta_T) = \sigma(\theta_W) = Q$.

Definition 4.9. Consider a non-empty set of T of Θ, a message $m \in M$ and a belief $\mu(\cdot \mid m) \in \Delta(T)$. Let $BR(\boldsymbol{\mu}, m)$ be the set of actions that are best responses to m given the belief $\mu(\cdot \mid m) \in \Delta(\Theta)$,

$$BR(\boldsymbol{\mu}, m) = \underset{a \in A}{\operatorname{argmax}} \sum_{\theta \in \Theta} \mu(\theta \mid m) u_r(\theta, m, a)$$

and $BR(T, m)$ be the set of best responses by the receiver to beliefs concentrated on the subset of T of Θ,

$$BR(T, m) = \bigcup_{\mu(\cdot \mid m) \in \Delta(T)} BR(\boldsymbol{\mu}, m)$$

Example 4.8. In Beer and Quiche game, $BR(\Theta, Q) = \{F, NF\}$ because F is a best response to Q for every $\mu(\theta_T \mid Q) \leq \frac{1}{2}$ and NF is a best response to Q for every $\mu(\theta_T \mid Q) \geq \frac{1}{2}$. By the same argument, it is easy to check $BR(\Theta, B) = \{F, NF\}$.

Definition 4.10. For each $m \in M$, let $J(m)$ be the subset of Θ such that

$$\max_{a \in BR(\Theta, m)} u_s(\theta, m, a) < u_s^*(\theta)$$

Example 4.9. In Beer and Quiche game, $J(B) = \{\theta_W\}$ can be found as follows.

- $\theta_T \notin J(B)$ since

$$\max_{a \in BR(\Theta, B)} u_s(\theta_T, B, a) = \max\{u_s(\theta_T, B, F), u_s(\theta_T, B, NF)\} = 3 > u_s^*(\theta_T) = 2$$

- $\theta_W \in J(B)$ since

$$\max_{a \in BR(\Theta, B)} u_s(\theta_W, B, a) = \max\{u_s(\theta_W, B, F), u_s(\theta_W, B, NF)\} = 2 < u_s^*(\theta_W) = 3$$

Definition 4.11. **[Intuitive criterion]** Let an equilibrium [26] be given. If there exists an **off path** message $m \in M$ and $\theta \in \Theta$ such that

$$\max_{a \in BR(\Theta \setminus J(m), m)} u_s(\theta, m, a) > u_s^*(\theta)$$

Then, we say that this equilibrium fails to satisfy the intuitive criterion.

Example 4.10. In Beer and Quiche game, since $\Theta \setminus J(B) = \{\theta_T\}$, then $BR(\Theta \setminus J(B), B)$ only allows beliefs $\mu(\theta_T \mid B) = 1$. Then, the receiver's unique best response after B is NF, i.e. $BR(\Theta \setminus J(B), B) = \{NF\}$.

- For (B, θ_T),

$$\max_{a \in BR(\Theta \setminus J(B), B)} u_s(\theta, m, a) = u_s(\theta_T, B, NF) = 3 > u_s^*(\theta_T) = 2$$

- For (B, θ_W),

$$\max_{a \in BR(\Theta \setminus J(B), B)} u_s(\theta, m, a) = u_s(\theta_W, B, NF) = 2 < u_s^*(\theta_W) = 3$$

In conclusion, a pooling equilibrium $\sigma(\theta_T) = \sigma(\theta_W) = Q$ fails to satisfy the intuitive criterion.

[26] Need not be a pooling equilibrium.

5 Part 5: Information Economics

5.1 Adverse Selection

5.1.1 Model Description

We will use mainly the insurance setting, but the analysis applied much more widely.

- There are $i = 1, 2, \dots, m$ agents who have the same VNM preference with a Bernoulli utility index function, $u(\cdot)$ such that u is continuous, $u' > 0$ and $u'' < 0$. [27]
- Suppose that each agent has an identical initial wealth denoted by ω.
- There are two states of the world: One where an agent does not have an accident, $\omega_g = \omega$ ("good" state) and one where an agent suffers an loss L and so, $\omega_b = \omega - L$ ("bad state").
- Each agent has their own probability of the accident, π_i, which is assigned by nature and hence exogenous.
- We consider an competitive market where there are many firms which are risk-neutral and want to maximize their profits.
- There are no costs associated with providing insurance.

5.1.2 Full Information as Benchmark

Further assume that

- Firms can somehow observe each agent's probability, π_i, without any costs.
- From each agent, firms receive p_i as a premium and pays B_i in return only if the bad state occurs. Thus, (if insurance purchased), each agent wealth level is $\omega_g = \omega - p_i$ when in the good state, and $\omega_b = \omega - p_i - L + B_i$.

In a competitive market equilibrium, firms compete to provide agent i with the best deal (p_i, B_i) subject to the firms (expected) breaking even. (otherwise, their profits must be zero.) Thus, firms seek to solve

$$\max_{\{p_i, B_i\}} \pi_i u(\omega - p_i - L + B_i) + (1 - \pi_i) u(\omega - p_i) \quad \text{s.t} \quad p_i - \pi_i B_i \geq 0$$

Setting the Lagrangian function

$$\mathcal{L} = \pi_i u(\omega - p_i - L + B_i) + (1 - \pi_i) u(\omega - p_i) + \lambda_i [p_i - \pi_i B_i]$$

Then, the two first order conditions are

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0 \;\Leftrightarrow\; \pi_i u'(\omega - p_i - L + B_i) + (1 - \pi_i) u'(\omega - p_i) = \lambda_i$$
$$\frac{\partial \mathcal{L}}{\partial B_i} = 0 \;\Leftrightarrow\; \pi_i u'(\omega - p_i - L + B_i) = \lambda_i \pi_i$$

Thus, we have

$$u'(\omega - p_i - L + B_i) = \pi_i u'(\omega - p_i - L + B_i) + (1 - \pi_i) u'(\omega - p_i) \;\Leftrightarrow\; u'(\omega - p_i - L + B_i) = u'(\omega - p_i)$$

[27] Notice that $u'' < 0$ implies that agents are risk-averse.

Therefore, $B_i^* = L$ and $p_i^* = \pi_i L$.

Definition 5.1. [Fully insured] If an agent's wealth (or payment) is independent of the realized state of the world, we say that an agent is fully insured.

Definition 5.2. [Actuarially fair] If an insurance premium, p_i, is determined at which firms' (expected) profit is zero, we say that provided insurance is actuarially fair.

Remark 5.1. In full information benchmark case, each agent is fully insured, and provided insurance is actuarially fair.

Remark 5.2. The risk-neutral firms carry all the risks, and the risk-averse agents bear nothing. This is an efficient situation in the sense that risks do not affect risk-neutral firms (agents) profits (utilities).

5.1.3 Asymmetric Information

Now, we want to analyze the case where firms cannot observe each agent's probability, π_i. Further assume that

- Firms know the distribution of $\pi_i \sim i.i.d.\ F(\pi)$ on the support $[\underline{\pi}, \overline{\pi}]$.
- Suppose that firms can only offer full-insurance contracts $B = L$ in return for some premium p. [28]
- Given that we consider a competitive market, $(p, B = L)$ must be unique in the equilibrium because all agents to want to buy insurance from the firm offering the lowest p.

Can p^* be determined at $p^* = \mathbb{E}[\pi] \times L$? That is, the premium just covers the average risk of agents. However, this cannot be an equilibrium in general. Because, if offered so, any agents with $\pi_i < \mathbb{E}[\pi]$ would not buy this insurance, and only (and all) agents with $\pi_j \geq \mathbb{E}[\pi]$ will buy it, so that firms make negative profits. [29]
Now, let us derive p^*. Consider a π-type agent. This agent will buy insurance at p if

$$u(\omega - p) \geq (1-\pi)u(\omega - L) + \pi u(\omega) \;\Leftrightarrow\; \pi \geq \frac{u(\omega) - u(\omega - p)}{u(\omega) - u(\omega - L)} \equiv h(p)$$

Notice that

$$h'(p) = \frac{u'(\omega - p)}{u(\omega) - u(\omega - L)} > 0 \;\text{ and }\; h''(p) = -\frac{u''(\omega - p)}{u(\omega) - u(\omega - L)} > 0$$

Then, p^* should be a equilibrium price if

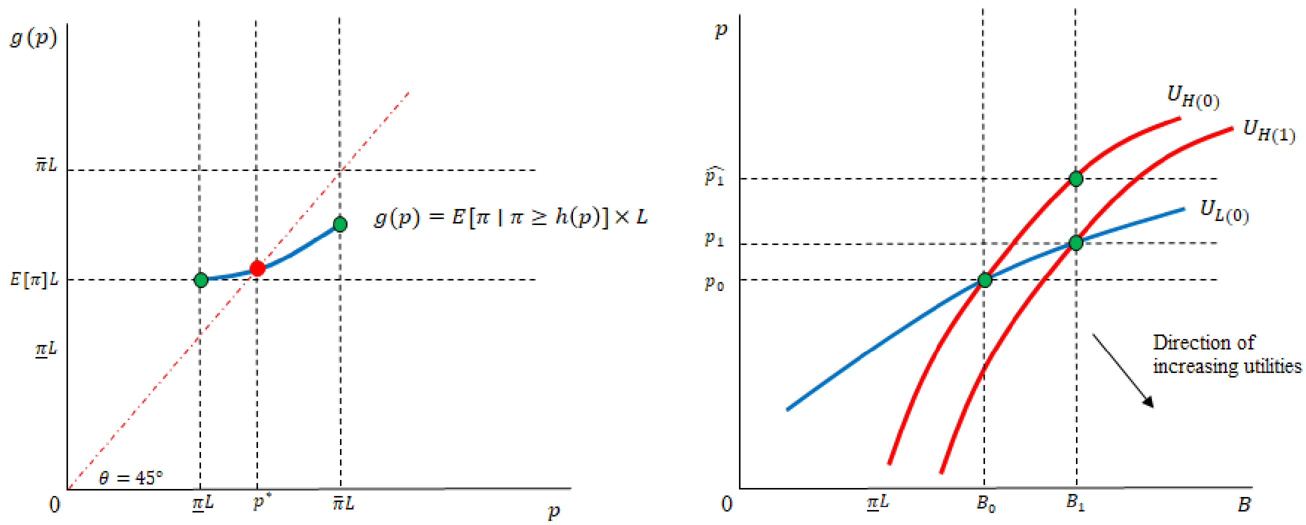
$$p^* = \underbrace{\mathbb{E}[\pi \mid \underbrace{\pi \geq h(p^*)}_{\text{buyer}}]}_{\text{buyers' avg. riskness}} \times L$$

To show this p^* always exists, define $g(p) \equiv \mathbb{E}[\pi \mid \pi \geq h(p)] \times L$. First, notice as p increases, $g(p)$ also increases, thus $g(p)$ is an increasing function with respect to p. Second, observe that $g(\underline{\pi}L) = \mathbb{E}[\pi]L \leq \overline{\pi}L$, and $g(\overline{\pi}L) \leq \overline{\pi}L$. In sum, $g(p)$ is an increasing function from $[\underline{\pi}L, \overline{\pi}L]$ to $[\underline{\pi}L, \overline{\pi}L]$ with $g(\underline{\pi}L) = \mathbb{E}[\pi]L$. Thus, there exists $p^* \in [\underline{\pi}L, \overline{\pi}L]$ such that $p^* = g(p^*)$ as a fixed point. Notice that p^* may not be unique and not be efficient. [30]

[28] The reason we assume this is that we want to compare p with p_i in full information case where $B_i = L$. That is, restricting $B = L$ gives us how p should be different from p_i. Also notice that if assumed so, p (and B) cannot depend on agent i, because π_i is not observable. Lastly, p is the only choice variables that firms need to choose.

[29] Those agents taking insurance will on average have higher risk than firms assumed, $\mathbb{E}[\pi]$.

[30] For example, if $\pi \sim U[0,1]$, then $p^* = L$ is an equilibrium. In that case, $h(p^*) = 1$, and there is total market failure in the sense that only the most risky type $\pi = 1$ would buy insurance coverage $B^* = L$ at price $p^* = L$. In effect, the agent with $\pi = 1$ pays exactly what she gets back after the certain accident. In this equilibrium, the risk is fully borne by risk-averse agents, which is inefficient.



(a) The existence of p^* as a fixed point

(b) Single crossing property

5.2 Signaling

5.2.1 Model Description

In this subsection, we will analyze the situation in which agents are able to signal their types. For simplicity, let us assume that we have only two types, low type π_L and high type π_H and for every agent i, π_i is i.i.d. on $\{\pi_L, \pi_H\}$ with $P(\pi_L) = \alpha$.

Proposition 5.1. [Single crossing] *The indifference curves for π_L type and π_H type in (B, p) space cross only once.*

Proof. Let (B, p) be given. Then, each type's expected utility can be written as

$$U_i(B, p) = \pi_i u(\omega - p - L + B) + (1 - \pi_i)u(\omega - p)$$

where $i \in \{H, L\}$. First, notice that it is clear that both types have the upward sloping indifference curves in (B, p) space and moving toward south-east direction yields higher level of utilities.

Now, fix any (B_0, p_0) in (B, p) space. Consider both types' indifference curves crossing (B_0, p_0). Given that the marginal rate of substitution at (B_0, p_0) is

$$\frac{dp}{dB} = \frac{\pi_i u'(\omega - p_0 - L + B_0)}{(1 - \pi_i)u'(\omega - p_0)} \text{ and } \frac{\pi_H}{1 - \pi_H} > \frac{\pi_L}{1 - \pi_L}$$

Therefore, at any point in (B, P), the π_L type has a flatter indifference curve.

Lastly, fix an indifference curve crossing (B_0, p_0) and pick any point (B_1, p_1) on this indifference curve such that $(B_1, p_1) \gg (B_0, p_0)$. Since the π_H type's indifference curve which crosses (B_1, p_1) must be located at more northern and eastern position than her indifference curve crossing (B_0, p_0), we can know that $\widehat{p}_1 > p_1$ if $(B_1, \widehat{p}_1)$ is on the π_H type's difference curve crossing (B_0, p_0). This lets us know that on the right hand side of (B_0, p_0), both types' indifference curves crossing (B_0, p_0) never meet again. By the same argument, they cannot cross on the left hand side of (B_0, p_0). $\square$

This signaling game follows

- Any agent i makes a contract offer $(B,p) \in \mathbb{R}^2$ to firms.
- A pure strategy for any agent i is a function from $\{\pi_H, \pi_L\}$ to $(B,p) \in \mathbb{R}^2$ and we will denote it by $\sigma^*((B,p) \mid \pi_H) = 1 \equiv (B_H, p_H)$ and $\sigma^*((B,p) \mid \pi_L) = 1 \equiv (B_L, p_L)$.
- Without observing the riskiness of the agent, π_i, firm decide to accept or reject the offer. $\rho(\mathit{Accept} \mid (B,p)) = 1$ or $\rho(reject \mid (B,p) = 1$. (We only consider a pure strategy for firms.)

Definition 5.3. A perfect Bayesian equilibrium [31]of this signaling game is an assessment $(\boldsymbol{\sigma}^*, \boldsymbol{\rho}^*, \beta^*(B,p))$ such that

1) **Sequential Rationality** for agents: Given the firm's strategy $\boldsymbol{\rho}^*$, each type of agent proposes a contract to maximize her expected utility subject to the offer being accepted.

2) **Sequential Rationality** for firms: Given its beliefs, $\boldsymbol{\rho}^*$ maximizes firms' expected profits.

3) **Bayes Rule**: Firms are Bayesian. Let $Pr(\pi_L \mid (B,p))$ be denoted by $\beta(B,p)$, the firms' posterior belief that the offer (B,p) was made by a low-risk agent.

In this context, we can define separating and pooling equilibria as follow.

4) **Separating equilibria**: $(B_H, p_H) \neq (B_L, p_L)$. Then, every on-path reveals the agent's type, so $\beta^*(B_L, p_L) = 1$ and $\beta^*(B_H, p_H) = 0$.

5) **Pooling equilibria**: $(B_H, p_H) = (B_L, p_L)$. Then, on-path offers convey no information, so $\beta^*(B_L, p_L) = \alpha$.

5.2.2 Firm's Profit Maximization and Reservation Utilities

In (B,p) spaces, a firm's decision whether to accept or reject can be summarized as follows.

- A contract (B,p) is profitable with both types of agents if $p \geq \pi_H B$.
- A contract (B,p) is never profitable with both types of agent if $p < \pi_L B$.
- A contract (B,p) is profitable only with $\pi_L$type if $\pi_L B \leq p < \pi_H B$.

Now, let us analyze each type agent's reservation utilities.

- Type π_H : Since firms always accept any contract such that $p \geq \pi_H L$, a high-risk type agent seeks to solve

$$\max_{(B,p)\in\mathbb{R}^2} U_H(B,p) \quad \text{s.t} \quad p \geq \pi_H B$$

  In this case, a high-risk type agent can ensure the full information contract where $p^* = \pi_H L$ and $L = B$. Let us denote this level of utility (reservation utility for π_H) by U_H^{FI}.

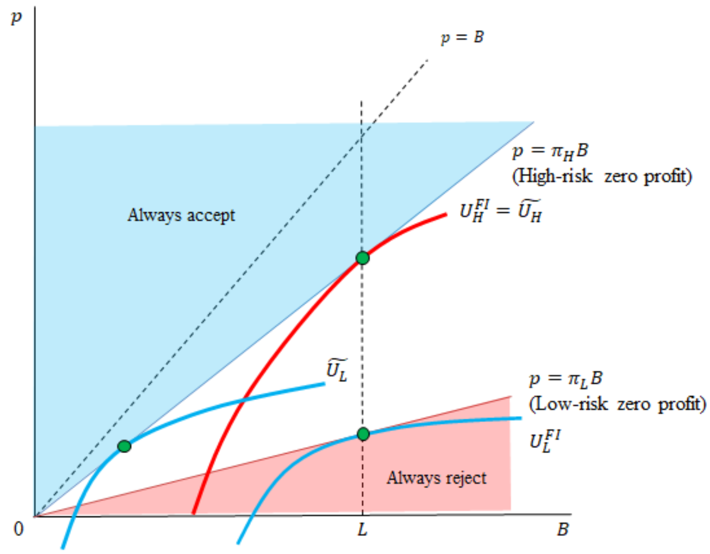
- Type π_L : Similarly, a low-risk type agent also seeks to solve

$$\max_{(B,p)\in\mathbb{R}^2} U_L(B,p) \quad \text{s.t} \quad p \geq \pi_H B$$

  Let us denote this reservation utility for π_L by $\widetilde{U}_L$.

[31]By 4.1, PBE are actually SE in this game.

What we discussed can be represented as below.



(c) Decision area for firms and reservation utilities

5.2.3 Incentive Compatibility and Individually Rationality

Definition 5.4. [Incentive compatibility] For each type, a pair of contract (B_L, p_L) and (B_H, p_H) is said to be incentive-compatible if

$$U_L(B_L, p_L) \geq U_L(B_H, p_H) \text{ and } U_H(B_H, p_H) \geq U_H(B_L, p_L)$$

Notice that incentive compatibility is a weaker condition than sequential rationality.

Definition 5.5. [Individual Rationality] For each type, a contract (B_i, p_i) is said to be individually rational if

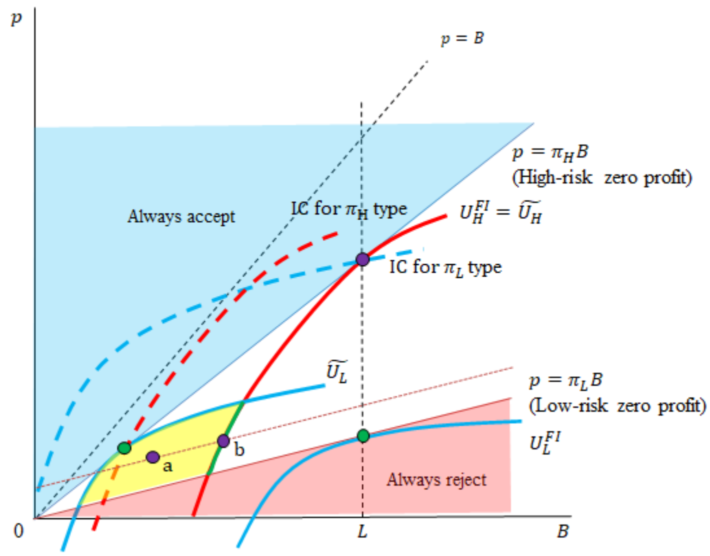
$$U_i(B_i, p_i) \geq \widetilde{U}_i$$

That is, (B_i, p_i) needs to achieve at least agent i's reservation utility.

5.2.4 Separating Equilibria

The following pairs of contracts describe the set of possible separating equilibrium contracts.

- $p_H = \pi_H L$, $B_H = L$: Since $\beta^*(B_H, p_H) = 0$, firms reject the offer (B_H, p_H) if and only if $p_H < \pi_H B_H$. Therefore, π_H type is facing the full-information problem, and hence we have $p_H = \pi_H L$ and $B_H = L$. (In this case, IR condition is trivially satisfied for type π_H.)
- For π_H type, IC condition is $U_H(B_L, p_L) \leq U_H^{FI}$.
- For π_L type, firms accept only if $p_L \geq \pi_L B_L$, and IR condition is $U_L(B_L, p_L) \geq \widetilde{U}_L$. (Notice that whenever IR condition is satisfied, then IC condition is trivially satisfied for type π_L.)



(d) Separating equilibria

- These equilibria always exist, and there are many of them as yellow area with (B_H, p_H) depicted above.
- The high-risk agent gets the efficient contract, while the low-risk agent is under-insured relative to the full information benchmark which is not efficient.
- **Pareto-undominated contract**: In yellow area, it is easy to check that contract $\boldsymbol{b}$ (with (B_H, p_H)) Pareto dominates $\boldsymbol{a}$ (with (B_H, p_H)) because it gives the same utility to high-type but higher utility to low-type. The set of Pareto-undominated contract equilibria illustrated as green line.
- Let $\beta(B, p)$ denote the firm's posterior belief that is facing a low-type agent. Then,

$$\beta^*(B,p) = \begin{cases} 1 & \text{if } \ (B,p) = (B_L, p_L) \text{ (any point in yellow area)} \\ 0 & \text{if } \ (B,p) \neq (B_L, p_L) \text{ (excpet that specific point)} \end{cases}$$

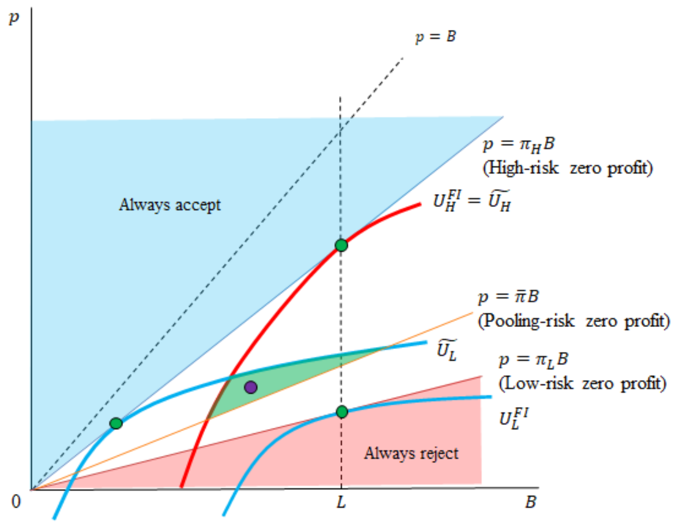
  Therefore, it only accepts such a contract if

$$\begin{aligned} &\rho^*(accept \mid (B,p)) = 1 \ \text{ if } (B,p) \in \{(B_L,p_L),(B_H,p_H)\} \text{ or } p \geq \pi_H B \\ &\rho^*(reject \mid (B,p)) = 1 \ \text{ otherwise} \end{aligned}$$

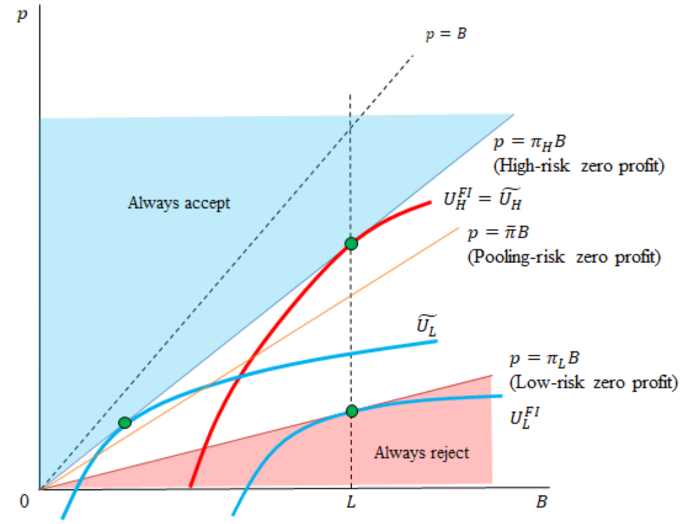
5.2.5 Pooling Equilibria

First, notice that pooling equilibria may not exist. If exist

- $\beta^*(B,p) = \alpha$.
- Let $\overline{\pi} = \alpha\pi_L + (1-\alpha)\pi_H$ be the average riskness. Then, firms' expected profits would be non-negative if $p \geq \overline{\pi} B$.
- Since IC conditions are trivially satisfied, we only need IR conditions for both types, i.e. $u_L(B,p) \geq \widetilde{U}_L$ and $u_H(B,p) \geq U_H^{FI}$.



(e) Pooling equilibria (existence)



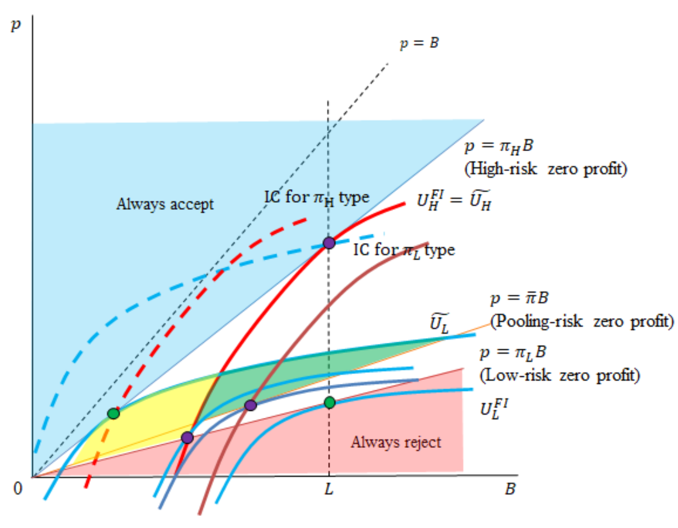
(f) Pooling equilibria (Non-existence)

- If α is large enough, then pooling equilibrium contract could be better than the outcome for both types of agents in all separating equilibria as depicted below.
- Let $\beta(B, p)$ denote the firm's posterior belief that is facing a low-type agent. Then,

$$\beta^*(B,p) = \begin{cases} \alpha & \text{if } (B,p) = (B^*,p^*) \text{ (any point in green area)} \\ 0 & \text{if } (B,p) \neq (B^*,p^*) \text{ (excpet that specific point)} \end{cases}$$

  Therefore, it only accepts such a contract if

$$\rho^*(accept \mid (B,p)) = 1 \text{ if } (B,p) \text{ in green area or } p \geq \pi_H B$$
$$\rho^*(reject \mid (B,p)) = 1 \text{ otherwise}$$



(g) When pooling equilibrium is better

5.2.6 Equilibrium Selection: Intuitive Criterion (Cho and Kreps (1987))

As we have seen in previous subsubsections, the characteristics of separating and pooling equilibria (when they exist) are multiplicity and beliefs on off-path contracts offered by high-risk types. In the context of this game, a sequential equilibrium with contracts (B_L, p_L) and (B_H, p_H) satisfies the intuitive criterion if [32]

- For any $(B, p) \notin \{(B_L, p_L), (B_H, p_H)\}$ such that for $i, j \in \{L, H\}$, $i \neq j$,

$$U_i(B, p) > U_i(B_i, p_i) \ \text{ and } \ U_j(B, p) < U_j(B_j, p_j)$$

  Then, firms believe that the agent offering (B, p) has risk π_i for sure.

Proposition 5.2. *There is only one sequential equilibrium that satisfies the intuitive criterion, the separating equilibrium that is the best for the low-risk agents.*

Proof. First, let us show that there are no pooling equilibria satisfying the intuitive criterion. Consider any pooling equilibrium. Then, the any point (B, p) in purple area satisfies

$$U_L(B, p) > U_L(B^*, p^*) \ \text{ and } \ U_H(B, p) < U_H(B^*, p^*)$$

But, we assigns $\beta^*(B, p) = 0$ not 1.

Second, consider any separating equilibrium which is not the best for the low-risk types. Then, the any point (B, p) in purple area satisfies

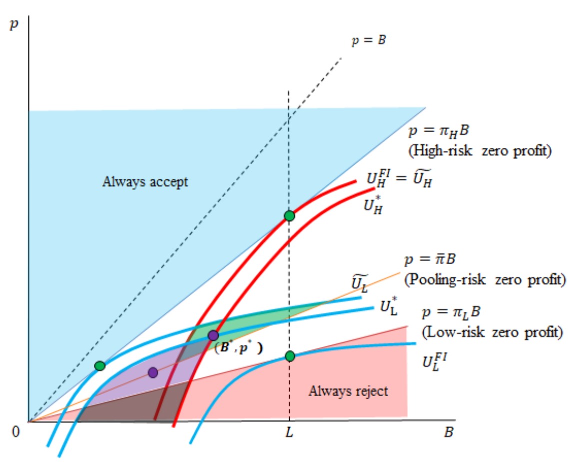
$$U_L(B, p) > U_L(B^*, p^*) \ \text{ and } \ U_H(B, p) < U_H(B^*, p^*)$$

But, we assigns $\beta^*(B, p) = 0$ not 1.

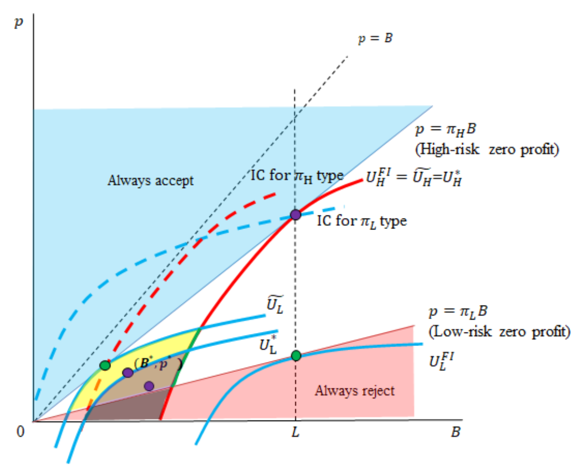
Lastly, consider the separating equilibrium which is the best for the low-risk types. In this case, for any point (B, p) in gray area, we have

$$U_L(B, p) < U_L(B^*, p^*) \ \text{ and } \ U_H(B, p) < U_H(B^*, p^*)$$

because firms reject the offer (B, p) in that area due to $p < \pi_L B$. Thus, assigning $\beta^*(B, p) = 0$ does not matter. Therefore, this is the only sequential equilibrium satisfying the intuitive criterion.

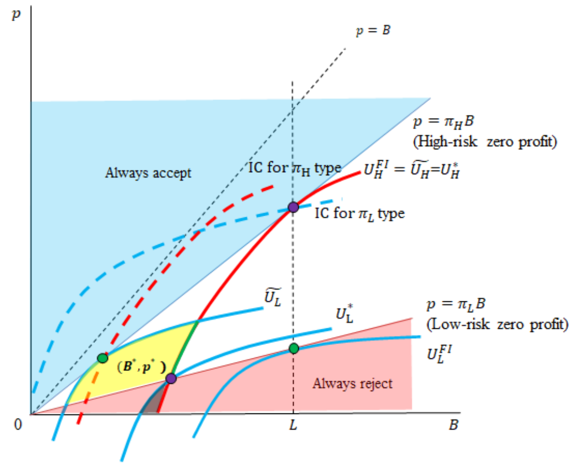


(h) Pooling equilibria fail the intuitive criterion



(i) Separating equilibria fail the intuitive criterion

[32] Notice that this is a sufficient condition for the intuitive criterion.



(j) The separating equilibrium satisfies the intuitive criterion

□

5.3 Competitive Screening

Now, suppose that there are a number of insurance firms (two for simplicity) that proposes contracts.

- A Firm $j \in \{A, B\}$ can offer a pair of contracts, (B_L^j, p_L^j) and (B_H^j, p_H^j) aimed at two types of agent.
- An agent can choose any of the contracts on offer or choose not to be insured. (equivalently choose the null contract $(0, 0)$)

Notice that in this game, there does not exist any beliefs, and hence SPE (NOT SE) is an equilibrium concept that we need. However, we borrow the terminologies, pooling and separating equilibria from the signaling game and continue to use them.

Lemma 5.1. *In equilibrium, firms make zero profits.*

Proof.
[**Step 1**] First, notice that firms' profits cannot be negative since they are guaranteed zero profits by not serving the market.

[**Step 2**] By way of contradiction, suppose that there is a firm which makes strictly positive profits. Let (B_L^*, p_L^*) and (B_H^*, p_H^*) denote the contracts chosen by the low- and high-risk consumers, respectively, in equilibrium. Then, the total (expected) profits for two firms can be written as

$$\Pi^* \equiv \Pi^{A*} + \Pi^{B*} = \alpha(p_L^* - \pi_L B_L^*) + (1 - \alpha)(p_H^* - \pi_H B_H^*) > 0$$

WLOG, assume that $\Pi^{A*} \geq \Pi^{B*}$, so that we have $\Pi^{B*} < \Pi^*$.

[**Step 3**] Suppose that $(B_L^*, p_L^*) = (B_H^*, p_H^*) = (B^*, p^*)$. Consider the following deviation by firm B such that firm B offers $(B^* + \epsilon, p^*)$ where $\epsilon > 0$, and hence captures the entire market. Letting $\epsilon \to 0$, we have $\Pi^{B**} \to \Pi^*$, and hence $\Pi^{B**} > \Pi^{B*}$, which contradicts the equilibrium hypothesis.

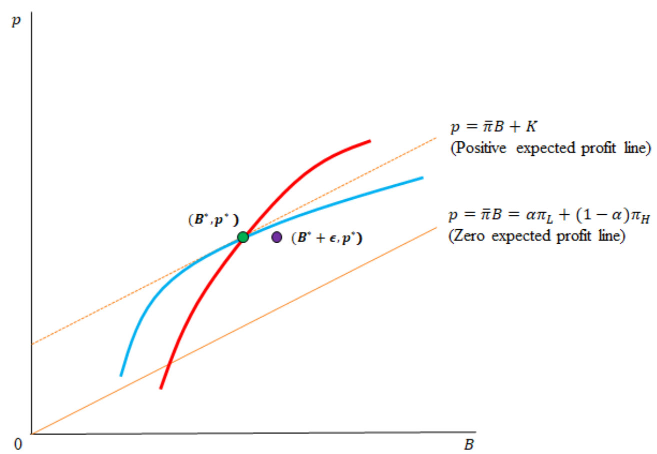
[**Step 4**] Suppose that $(B_L^*, p_L^*) \neq (B_H^*, p_H^*)$. From the IC constraints, we have

$$U_H(B_H^*, p_H^*) \geq U_H(B_L^*, p_L^*) \text{ and } U_L(B_L^*, p_L^*) \geq U_L(B_H^*, p_H^*)$$

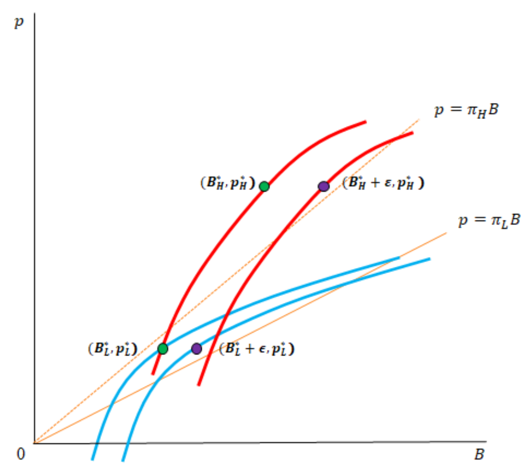
Consider the following deviation by firm B such that firm B offers $(B_L^* + \epsilon, p_L^*)$ and $(B_H^* + \varepsilon, p_H^*)$ where $\epsilon, \varepsilon > 0$ so as to guarantee the separation such that

$$U_H(B_H^* + \varepsilon, p_H^*) > U_H(B_L^* + \epsilon, p_L^*) \;\text{ and }\; U_L(B_L^* + \epsilon, p_L^*) > U_L(B_H^* + \varepsilon, p_H^*)$$

Then, firm B captures the entire market again. Letting $\epsilon, \varepsilon \to 0$, we have $\Pi^{B***} \to \Pi^*$, and hence $\Pi^{B***} > \Pi^{B*}$, which contradicts the equilibrium hypothesis.



(k) Profitable deviation in pooling



(l) Profitable deviation in separating

□

Proposition 5.3. *There does not exist a pure strategy pooling equilibrium in the insurance screening game.*

Proof. Let (B^*, p^*) be a pooling equilibrium. By the previous lemma, we know that

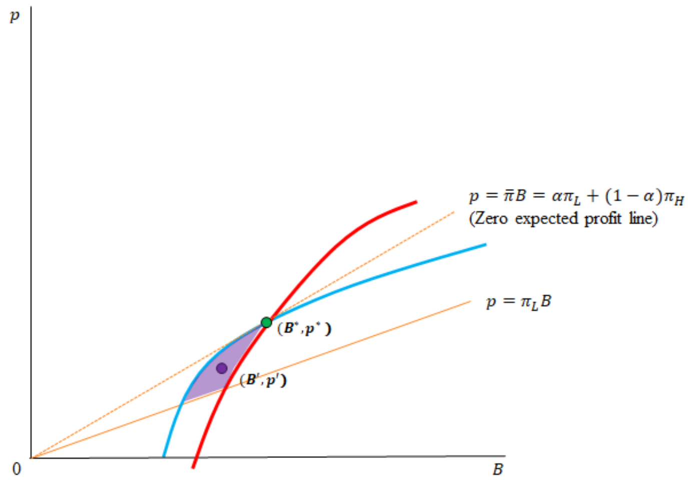
$$\Pi^* = \alpha(p^* - \pi_L B^*) + (1-\alpha)(p^* - \pi_H B^*) = 0$$

Thus, $p^* - \pi_L B^* > 0$ and $p^* - \pi_H B^* < 0$. Suppose that firm A is offering this contract, then firm B can offer a contract (B', p'), which is very close to the pooling contract but only attractive to the low-risk agents, so that obtain strictly positive profits. [33] Specifically, consider (B', p') such that

$$U_H(B', p') < U_H(B^*, p^*) \;\text{ and }\; U_L(B', p') > U_L(B^*, p^*)$$

Letting $(B', p') \to (B^*, p^*)$, the payoff from this deviation tends to $p^* - \pi_L B^* > 0$.

[33] This is called **creaming skimming**.



(m) No pooling equilibrium

□

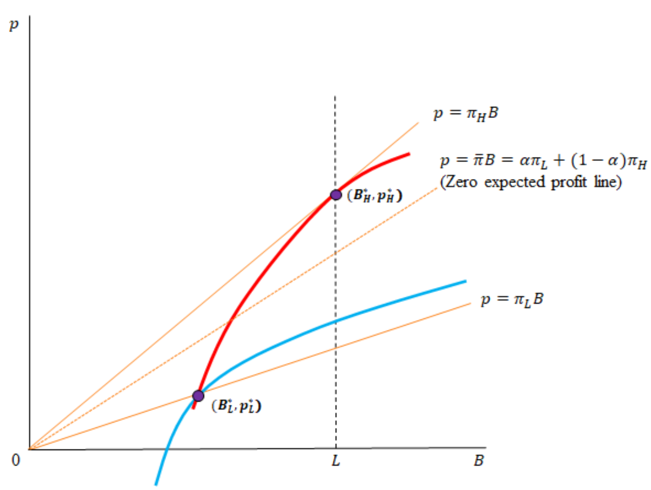
Proposition 5.4. *The only possible separating equilibrium involves*

1) High-risk agents are offered the full insurance contract $(B_H, p_H) = (L, \pi_H L)$, *yielding full information benchmark utility.*

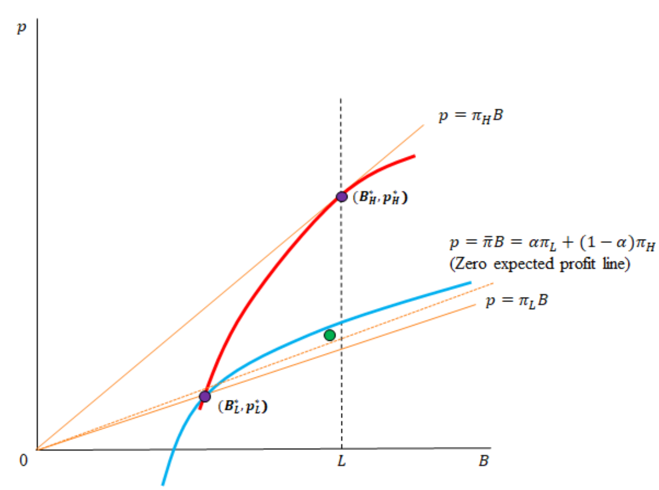
2) Low-risk agents are offered an contract that does not tempt the high-risk agents.

3) This pair of contracts correspond to the contracts offered by the consumers in the signaling game that satisfy the intuitive criterion.

Proof. Given an isoprofit line, the firm offering the contract closest to the agents' preferred contract wins all consumers. Thus, both firms will only offer the preferred contract subject to IC holding for both types. Therefore, high-risk agents are offered the full insurance contract $(L, \pi_H L)$, and low-risk agents are offered the contract which is on the low-type zero profit contract $(B_L, \pi_L L)$ such that high-risk agents are indifferent between $(L, \pi_H L)$ and $(B_L, \pi_L L)$, i.e. $U_H(B_L, \pi_L L) = U_H(L, \pi_H L)$ as shown below.



(n) Separating equilibrium exists



(o) No separating equilibrium exists

.

Lastly, notice that if α is close enough to 1 as on the right figure above, then there might be a profitable deviation. Thus, the screening model admits no pure strategy separating equilibrium. In other words, the separating equilibrium exists if α is low enough, so that the proportion of high-risk agents are sufficiently large. □

5.4 Principal-Agent Problem: Hidden Information

5.4.1 Model Description

- The principal's utility is defined by $S(q) - t$ where $S(q)$ is the principal's valuation of q units such that $S'(q) > 0, S''(q) < 0$ and $S(0) = 0$, and t is the transfer to the agent.
- The agent's marginal cost is $\theta \in \{\underline{\theta}, \overline{\theta}\}$, which is privately known to her. However, $\Pr(\theta = \underline{\theta}) = \nu$ is common knowledge.
- Given that the total cost function is $C(q, \theta) = \theta q$ for type θ, the agent's utility function is defined by $t - \theta q$.
- Define $\triangle\theta \equiv \overline{\theta} - \underline{\theta} > 0$ as a measure of the uncertainty regarding the agent's marginal cost.
- The agent's outside option (or reservation utility, $\overline{u}$) is normalized to zero.
- This game follows
  - Nature chooses $\theta \in \{\underline{\theta}, \overline{\theta}\}$ with the prior $\Pr(\theta = \underline{\theta}) = \nu$. Then, the agent observes her realized type, but the principal does not.
  - The principal offers a menu of contracts aimed for each type. [34]
  - The agent accepts or rejects the contract, and if agent accepts a contract in the menu, that contract contract is executed. [35]

5.4.2 First-Best Benchmark: Full Information

Further assume that

- The principal can observe the agent's type, and can therefore require two production levels, corresponding to the agent's type.

For each of type $\theta \in \{\underline{\theta}, \overline{\theta}\}$, the efficient level can be found by solving

$$\max_{q \geq 0} \{S(q) - t + t - \theta q\}$$

Thus, the first-best output levels satisfy

$$S'(\underline{q}^{FI}) = \underline{\theta} \quad \text{and} \quad S'(\overline{q}^{FI}) = \overline{\theta}$$

How this first-best benchmark can be implemented is depicted as below. First, notice that the each type of agent has an individually rationality constraint such that

$$\underline{t} - \underline{\theta}\underline{q}^{FI} \geq 0 \quad \text{and} \quad \bar{t} - \overline{\theta}\overline{q}^{FI} \geq 0$$

[34] A menu of contracts is offered at interim stage (Not at the ex-ante stage) where there is asymmetric information.

[35] The principal has full commitment to the offered contract, that is, renegotiation about the contract after observing the type is not allowed. We also assume that q is verifiable.

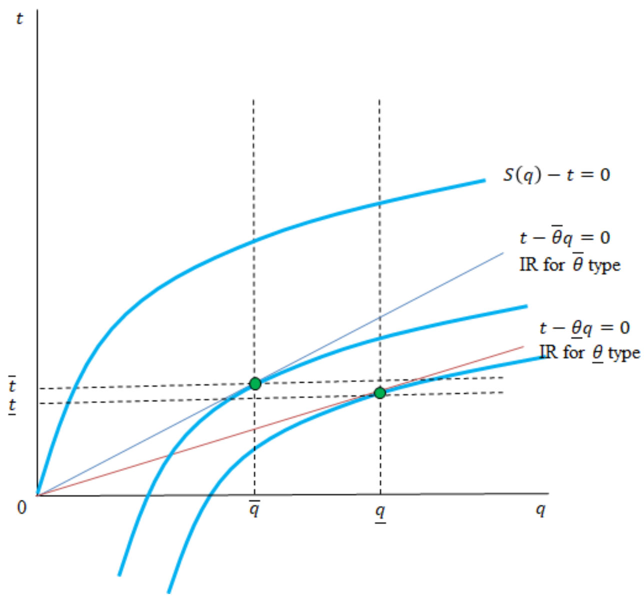
Therefore, the profit maximizing offers are

$$\underline{t}^{FI} = \underline{\theta}\underline{q}^{FI} \quad \text{and} \quad \overline{t}^{FI} = \overline{\theta}\overline{q}^{FI}$$

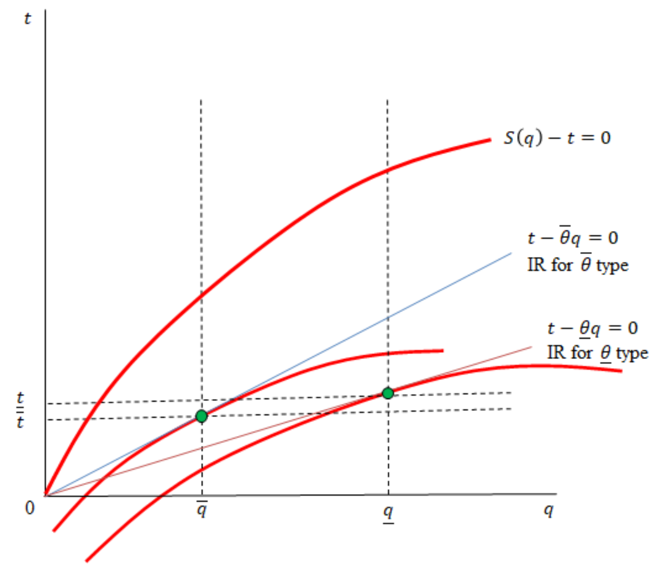
Also notice that in the first-best benchmark, the principal extracts the whole surplus, and hence the agent's utility in equilibrium are

$$\underline{U}^{FI} = \underline{t}^{FI} - \underline{\theta}\underline{q}^{FI} = 0 \quad \text{and} \quad \overline{U}^{FI} = \overline{t}^{FI} - \overline{\theta}\overline{q}^{FI} = 0$$

Observe that we always have $\overline{q}^{FI} < \underline{q}^{FI}$ but both $\underline{t}^{FI} \geq \overline{t}^{FI}$ and $\underline{t}^{FI} \leq \overline{t}^{FI}$ should be possible depending the curvature of $S(q)$.



(p) $\overline{t}^* > \underline{t}^*$



(q) $\overline{t}^* < \underline{t}^*$

5.4.3 Second-Best Case: Asymmetric Information

Going back to the original model, let $(\underline{q}, \underline{t}; \overline{q}, \overline{t})$ denote a menu of contract. Then, IC and IR constraints for both types are

$$IR_L : \ \underline{t} - \underline{\theta}\underline{q} \geq 0 \quad \text{and} \quad IR_H : \ \overline{t} - \overline{\theta}\overline{q} \geq 0$$
$$IC_L : \ \underline{t} - \underline{\theta}\underline{q} \geq \overline{t} - \underline{\theta}\overline{q} \quad \text{and} \quad IC_H : \ \overline{t} - \overline{\theta}\overline{q} \geq \underline{t} - \overline{\theta}\underline{q}$$

Notice that the first-best contracts are not incentive compatible and hence cannot be implemented because the low-cost type strictly prefers to the high-cost type's contract. (i.e. IC_L must be violated.)
By $IC_L + IC_H$, we have

$$(\underline{t} - \underline{\theta}\underline{q}) + (\overline{t} - \overline{\theta}\overline{q}) \geq (\overline{t} - \underline{\theta}\overline{q}) + (\underline{t} - \overline{\theta}\underline{q}) \quad \Leftrightarrow \quad \underbrace{(\underline{\theta} - \overline{\theta})}_{<0}\underbrace{(\overline{q} - \underline{q})}_{\leq 0} \geq 0 \quad \Leftrightarrow \quad \overline{q} \leq \underline{q}$$

In words, it is necessary that the low-cost type is required to produce more in order to satisfy IC, and conversely, for any pair $\overline{q} \leq \underline{q}$, we can also find transfers to satisfy IC_H and IC_L.

The principal's expected payoff maximization problem can be written as

$$\max_{(\underline{q},\underline{t},\overline{q},\overline{t})} \Big[\nu(S(\underline{q})-\underline{t})+(1-\nu)(S(\overline{q})-\overline{t})\Big] \quad \text{s.t} \quad IR_L, IR_H, IC_L \text{ and } IC_H$$

Let us define information rent for each type $\overline{U}=\overline{t}-\overline{\theta}\overline{q}$ and $\underline{U}=\underline{t}-\underline{\theta}\underline{q}$, then we can rewrite the problem above as

$$\max_{(\underline{q},\underline{t},\overline{q},\overline{t})} \Big[\overbrace{\underbrace{\nu(S(\underline{q})-\underline{\theta}\underline{q})+(1-\nu)(S(\overline{q})-\overline{\theta}\overline{q})}_{\text{the expected total social value}}}^{\text{want to maximize}} - \overbrace{\underbrace{\{\nu\underline{U}+(1-\nu)\overline{U}\}}_{\text{expected info. rent}}}^{\text{want to minimize}}\Big] \quad \text{s.t} \quad IR_L, IR_H, IC_L \text{ and } IC_H$$

Using the notation of $\overline{U}$, $\underline{U}$ and $\triangle\theta$, four constraints also can be rewritten as

$$IR_L: \ \underline{U}\geq 0 \quad \text{and} \quad IR_H: \ \overline{U}\geq 0$$
$$IC_L: \ \underline{U}\geq \overline{U}+\triangle\theta\overline{q} \quad \text{and} \quad IC_H: \ \overline{U}\geq \underline{U}-\triangle\theta\underline{q}$$

Let $(\underline{q}^*,\underline{t}^*,\overline{q}^*,\overline{t}^*)$ be an optimal menu of contracts satisfying all four constraints. Then,

1. IR_L is slack. [36]
2. IC_L is tight. [37]
3. IR_H is tight. [38]
4. IC_H is satisfied if IC_L is tight and monotonicity holds. [39]

Therefore, the principal just needs to solve (and check whether monotonicity holds later)

$$\max_{(\underline{q},\overline{q})} \Big[\nu(S(\underline{q})-\underline{\theta}\underline{q})+(1-\nu)(S(\overline{q})-\overline{\theta}\overline{q})-\nu\triangle\theta\overline{q}\Big]$$

The first order conditions for $\underline{q}$ and $\overline{q}$ are

$$S'(\underline{q}^*)=\underline{\theta} \quad \text{and} \quad S'(\overline{q}^*)=\frac{\nu}{(1-\nu)}\triangle\theta+\overline{\theta}$$

Finally notice that

1. No output distortion for the low-cost type.
2. The information rents for each type are $\underline{U}=\underline{t}^*-\underline{\theta}\underline{q}^*=\overline{t}^*-\underline{\theta}\overline{q}^*=\overline{t}^*-\underline{\theta}\overline{q}^*-\overline{U}=\overline{t}^*-\underline{\theta}\overline{q}^*-(\overline{t}^*-\overline{\theta}\overline{q}^*)=\triangle\theta\overline{q}^*$ and $\overline{U}=0$.

[36]Suppose not. Then, $\underline{t}^*-\underline{\theta}\underline{q}^*=\overline{u}$, since IC_L is satisfied, we have $\overline{u}\geq\overline{t}^*-\underline{\theta}\overline{q}^*$, and then $\overline{u}\geq\overline{t}^*-\underline{\theta}\overline{q}^*>\overline{t}^*-\overline{\theta}\overline{q}^*$ by the assumption of $\overline{\theta}>\underline{\theta}$, which shows that this menu of contracts violates IR_H. Thus, IR_L should be slack in an optimal menu of contracts.
[37]Suppose not. Then, $\underline{t}^*-\underline{\theta}\underline{q}^*>\overline{t}^*-\underline{\theta}\overline{q}^*$. Since IR_L is slack, we can take $\epsilon>0$ such that $(\underline{t}^*-\epsilon)-\underline{\theta}\underline{q}^*\geq\overline{u}$ and $(\underline{t}^*-\epsilon)-\underline{\theta}\underline{q}^*>\overline{t}^*-\underline{\theta}\overline{q}^*$. That is, we can lower $\underline{t}^*$ slightly but IR_L and IC_L still hold. Notice that this does not affect the validity of IR_H and IC_H, however, strictly increases the objective function, and so yields a contradiction.
[38]Suppose not. Then, $\underline{t}^*-\overline{\theta}\overline{q}^*>\overline{u}$. Now, lower slightly $\underline{t}^*$ and $\overline{t}^*$ by the same amount, so that IR_L and IR_H still hold. Notice that this does not affect the validity of IC_L and and IC_H, however, strictly increases the objective function, and hence yields a contradiction.
[39]Observe that $\overline{t}^*-\overline{\theta}\overline{q}^*=\overline{t}^*-\underline{\theta}\overline{q}^*+\underline{\theta}\overline{q}^*-\overline{\theta}\overline{q}^*=\overline{t}^*-\underline{\theta}\overline{q}^*-\overline{q}^*(\overline{\theta}-\underline{\theta})=\underline{t}^*-\underline{\theta}\underline{q}^*-\overline{q}^*(\overline{\theta}-\underline{\theta})$ ($\because$ IC_L is tight) $>\underline{t}^*-\underline{\theta}\underline{q}^*-\underline{q}^*(\overline{\theta}-\underline{\theta})=\underline{t}^*-\overline{\theta}\underline{q}^*$ ($\because$ assuming $\overline{q}^*<\underline{q}^*$). Therefore, we have $\overline{t}^*-\overline{\theta}\overline{q}^*>\underline{t}^*-\overline{\theta}\underline{q}^*$ as desired.

3. The downward output distortion for the high-cost type: $S'(\overline{q}^*) = \overline{\theta} + \frac{\nu}{(1-\nu)} \triangle\theta > \overline{\theta} \;\Leftrightarrow\; \overline{q}^* < \overline{q}^{FI}$. Increasing $\overline{q}^*$ increases total surplus but also increases information rent to the low-cost type. Also notice that the higher ν and $\triangle\theta$ yields the higher distortion.

5.4.4 Model Extension: Continuum of Types

- Let $\theta \in [\underline{\theta}, \overline{\theta}] \equiv \Theta$ with a CDF $F(\theta)$ and a PDF $f(\theta) > 0$ on $[\underline{\theta}, \overline{\theta}]$.
- The principal offers a menu of contracts $\{q(\cdot), t(\cdot)\}$ where $q : \Theta \to \mathbb{R}^+$ and $t : \Theta \to \mathbb{R}^+$.
- If the agent accepts the offer, she announces her type (free to lie), [40] and the corresponding contract is executed.

For a menu of contracts to be incentive compatible, we need

$$t(\theta) - \theta q(\theta) \geq t(\widetilde{\theta}) - \theta q(\widetilde{\theta}),\ \forall (\theta, \widetilde{\theta}) \in \Theta^2$$

which implies that

$$t(\theta) - \theta q(\theta) \geq t(\widetilde{\theta}) - \theta q(\widetilde{\theta}) \;\text{ and }\; t(\widetilde{\theta}) - \widetilde{\theta} q(\widetilde{\theta}) \geq t(\theta) - \widetilde{\theta} q(\theta)$$

adding and we obtain

$$(\theta - \widetilde{\theta})[q(\theta) - q(\widetilde{\theta})] \leq 0 \quad \Leftrightarrow \quad \theta > \widetilde{\theta} \;\text{ then }\; q(\theta) \leq q(\widetilde{\theta})$$

Thus, $q(\theta)$ is **monotonic**, more specifically, it is weakly decreasing.

- Further assume that both $q(\theta)$ and $t(\theta)$ are differentiable.

Define

$$u(\widehat{\theta}; \theta) = t(\widehat{\theta}) - \theta q(\widehat{\theta})$$

For the truth telling to be locally optimal (or a menu of contract to be locally incentive compatible), the first order condition requires

$$\left.\frac{\partial u(\widehat{\theta}; \theta)}{\partial \widehat{\theta}}\right|_{\widehat{\theta}=\theta} = 0 \quad \Leftrightarrow \quad t'(\theta) - \theta q'(\theta) = 0$$

and the second order condition is

$$\left.\frac{\partial^2 u(\widehat{\theta}; \theta)}{\partial \widehat{\theta}^2}\right|_{\widehat{\theta}=\theta} = 0 \quad \Leftrightarrow \quad t''(\theta) - \theta q''(\theta) \leq 0$$

Differentiating the first order condition with respect to θ, we have

$$t''(\theta) - \theta q''(\theta) - q'(\theta) = 0 \quad \Rightarrow \quad q'(\theta) \leq 0$$

In other words, the (local) second order condition is always satisfied if both the first order condition and monotonicity are satisfied.
Since the local IC holds for all $\theta \in \Theta$, we have

$$t'(\theta) - \theta q'(\theta) = 0$$

[40] Such a contract is sometimes called a direct revelation mechanism.

Integrating from $\widetilde{\theta}$ to θ yields

$$\int_{\widetilde{\theta}}^{\theta} [t'(x) - xq'(x)]dx = 0 \quad \Leftrightarrow \quad \int_{\widetilde{\theta}}^{\theta} t'(x)dx = \int_{\widetilde{\theta}}^{\theta} xq'(x)dx$$

Notice that integrating $\int_{\widetilde{\theta}}^{\theta} xq'(x)dx$ by parts yields

$$\int_{\widetilde{\theta}}^{\theta} xq'(x)dx = [xq(x)]_{\widetilde{\theta}}^{\theta} - \int_{\widetilde{\theta}}^{\theta} q(x)dx$$

thus, we obtain

$$t(\theta) - \theta q(\theta) = t(\widetilde{\theta}) - \widetilde{\theta} q(\widetilde{\theta}) - \int_{\widetilde{\theta}}^{\theta} q(x)dx$$

Define $U(\theta) = t(\theta) - \theta q(\theta)$ as the informational rent for all $\theta \in \Theta$, then we can write

$$U(\theta) = U(\widetilde{\theta}) - \int_{\widetilde{\theta}}^{\theta} q(x)dx$$

which implies that if a menu of contracts $\{q(\cdot), t(\cdot)\}$ is locally incentive compatible, then $\frac{\partial^2 U(\theta)}{\partial \theta^2} = -q'(\theta) \geq 0$, then the surplus $U(\cdot)$ is convex in θ. In particular, choosing $\widetilde{\theta} = \underline{\theta}$ and $\widetilde{\theta} = \overline{\theta}$, we have

$$U(\theta) = U(\underline{\theta}) - \underbrace{\int_{\underline{\theta}}^{\theta} q(x)dx}_{>0} \quad \text{or} \quad U(\theta) = U(\overline{\theta}) + \underbrace{\int_{\theta}^{\overline{\theta}} q(x)dx}_{>0}$$

The above result is sometimes called **payoff equivalence** in mechanism design: expected transfer to type θ can be calculated by using the payment $t(\underline{\theta})$ (or $t(\overline{\theta})$) to the lowest cost type (or the highest cost type) and the function $q(\cdot)$.

Moreover, also notice that

$$U'(\theta) = \underbrace{t'(\theta) - \theta q'(\theta)}_{=0 \text{ by local IC}} - q(\theta)$$

Lastly, we want to argue that the agents do not want to lie globally either. Observe that

$$t(\theta) - \theta q(\theta) = t(\widetilde{\theta}) - \widetilde{\theta} q(\widetilde{\theta}) - \int_{\widetilde{\theta}}^{\theta} q(x)dx \quad \Leftrightarrow \quad t(\theta) - \theta q(\theta) = t(\widetilde{\theta}) - \theta q(\widetilde{\theta}) + \underbrace{\theta q(\widetilde{\theta}) - \widetilde{\theta} q(\widetilde{\theta})}_{(\theta - \widetilde{\theta}) q(\widetilde{\theta})} - \int_{\widetilde{\theta}}^{\theta} q(x)dx$$

Given that $q(\cdot)$ is weakly decreasing, $(\theta - \widetilde{\theta}) q(\widetilde{\theta}) > \int_{\widetilde{\theta}}^{\theta} q(x)dx$, thus we have

$$t(\theta) - \theta q(\theta) \geq t(\widetilde{\theta}) - \theta q(\widetilde{\theta}), \quad \forall \theta, \widetilde{\theta} \in \Theta$$

In conclusion, the first order condition for local IC and monotonicity are sufficient to obtain global IC.

The optimization problem of the principal can then be written

$$\max_{\{q(\theta),t(\theta)\}} \int_{\underline{\theta}}^{\overline{\theta}} [S(q(\theta)) - t(\theta)]f(\theta)d\theta = \max_{\{q(\theta),U(\theta)\}} \int_{\underline{\theta}}^{\overline{\theta}} [S(q(\theta)) - \theta q(\theta) - U(\theta)]f(\theta)d\theta \text{ s.t}$$

$$U'(\theta) = -q(\theta), \ \forall\theta \in \Theta \text{ (Local IC)}$$

$$q'(\theta) \leq 0, \ \forall\theta \in \Theta \text{ (Monotonicity)}$$

$$U(\theta) \geq 0, \ \forall\theta \in \Theta \text{ (IR)}$$

First, notice that $U'(\theta) \leq 0$ from local IC, and hence $U(\theta) \geq 0, \ \forall\theta \in \Theta$ (IR) is equivalent to $U(\overline{\theta}) \geq 0$. Furthermore, it is clear that $U(\overline{\theta}) = 0$ as in the binary case where IR is binding only for the most inefficient type. Thus, we only need to consider local IC and monotonicity constraints. Let us ignore the monotonicity constraint temporary. From the local IC, we have

$$\int_{\theta}^{\overline{\theta}} U'(x)dx = -\int_{\theta}^{\overline{\theta}} q(x)dx \quad \Leftrightarrow \ U(\theta) = \underbrace{U(\overline{\theta})}_{=0} + \int_{\theta}^{\overline{\theta}} q(x)dx$$

Plugging this into the principal's objective function, we have

$$\max_{\{q(\theta)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left[S(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(x)dx\right] f(\theta)d\theta$$

Changing the order of integration yields

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[\int_{\theta}^{\overline{\theta}} q(x)dx\Big] f(\theta)d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{x} q(x)f(\theta)d\theta dx = \int_{\underline{\theta}}^{\overline{\theta}} q(x)F(x)dx = f(\theta) \int_{\underline{\theta}}^{\overline{\theta}} q(\theta)\frac{F(\theta)}{f(\theta)}d\theta$$

Thus, the principal's problem becomes

$$\max_{\{q(\theta)\}} \int_{\underline{\theta}}^{\overline{\theta}} \left[S(q(\theta)) - \left(\theta + \frac{F(\theta)}{f(\theta)}\right) q(\theta)\right] f(\theta)d\theta$$

In order to maximize the integrand pointwise, we need

$$S'(q^*(\theta)) = \underbrace{\theta + \frac{F(\theta)}{f(\theta)}}_{\text{virtual type}}$$

Lastly, we need to check whether the $q(\theta)$ is indeed weakly decreasing. This requires $\frac{d}{d\theta}(\theta + \frac{F(\theta)}{f(\theta)}) \geq 0$, and notice that if $\frac{f(\theta)}{F(\theta)}$ is decreasing in θ, then $q^*(\theta)$ is clearly decreasing in θ as desired.
Finally notice that the under asymmetric information, the optimal menu of contracts has the following properties.

1. No output distortion for the most productive type: $S'(q^*(\underline{\theta})) = \underline{\theta}$ as in FI benchmark.

2. The downward output distortion for all other types:

$$S'(q^*(\theta)) = \theta + \frac{F(\theta)}{f(\theta)} > \theta$$

3. The information rents for all types:

$$U(\theta) = \underbrace{U(\overline{\theta})}_{=0} + \int_{\theta}^{\overline{\theta}} q(x)dx, \quad \forall \theta \in \Theta$$

5.5 Principal-Agent Problem: Hidden Action

5.5.1 Model Description

- The principal's (insurer) profits are determined at least in part by the agent's (driver) effort level, which is not observable by the principal.
- The principal is risk-neutral and would like the agent to exert (profit-maximizing) effort level while the agent is risk-averse and dislikes effort. Let ω be an initial wealth level and $u(\cdot)$ be a concave Bernoulli utility function for the agent.
- For now, assume effort $a \in \{0, 1\}$ and if accident occurs, the principal observes losses of $l \in \mathbb{L} = \{0, 1, \ldots, L\}$, which is stochastically dependent on the agent's effort level, a, and therefore partially informative about it.
- Let $d(a)$ denote the agent's disutility from effort such that $d(0) < d(1)$.
- The dependence is described by the conditional probability $\pi_l(a) > 0$ for all $l \in \mathbb{L}$, and for all $a \in \{0, 1\}$. [41]
- The principal offers insurance policy $(p, B_0, \ldots, B_L)$ where p denotes the price and B_l the insurance payment in case of loss l.

Definition 5.6. [Monotone likelihood ratio property; MLRP] The likelihood ratio $\frac{\pi_l(0)}{\pi_l(1)}$ is strictly increasing in l. That is, higher effort is more likely to result in lower losses.

The implications of MLRP are

- Conditioning on observation of l, the relative probability that low effort was exerted rather than high effort increases with l.
- The CDF over l conditional on $a = 0$ first-order stochastically dominates the CDF over l conditional on $a = 1$.

5.5.2 First-Best Benchmark: Full Information

Further assume that

- The principal can observe the agent's effort level.

Then, the principal can make benefit payment conditional on the observed effort level. In other words, the principal can choose the effort level that she would like the agent to exert and observe deviations. An insurance policy will therefore specify the premium $p^{FI}(a)$ and payouts $B_l^{FI}(a)$ under an effort level a required by the firm. If it observes that the agent incurred a loss of l and exerted effort level a, it will pay $B_l^{FI}(a)$, however the firm will not pay out anything if the agent exerted effort level $a' \neq a$. Thus, the principal seeks to solve

$$\max_{\{a,p,B_1,\ldots,B_L\}} \left[p - \sum_{l=0}^{L} \pi_l(a) B_l \right] \quad \text{s.t.} \quad \sum_{l=0}^{L} \pi_l(a) u(\omega - p - l + B_l) - d(a) \geq \overline{u}$$

[41] Every realization of l could have come from either effort level.

Let us solve this problem in 2 steps. First, fix $a \in \{0, 1\}$, and consider the Lagrangian function.

$$\mathcal{L} = p - \sum_{l=0}^{L} \pi_l(a) B_l + \lambda \left[\sum_{l=0}^{L} \pi_l(a) u(\omega - p - l + B_l) - d(a) - \overline{u} \right]$$

The first order conditions for p, B_l and λ are

$$1 - \lambda \sum_{l=0}^{L} \pi_l(a) u'(\omega - p - l + B_l) = 0 \tag{5.1}$$

$$-\pi_l(a) + \lambda \pi_l(a) u'(\omega - p - l + B_l) = 0, \quad \forall l \in \mathbb{L} \tag{5.2}$$

$$\sum_{l=0}^{L} \pi_l(a) u(\omega - p - l + B_l) - d(a) - \overline{u} \geq 0 \tag{5.3}$$

Summing the first order condition 5.2 over l makes the first order condition 5.1 redundant. Also it is easy to argue that $\lambda > 0$ and hence the constraint binds with equality. From the first order condition 5.2, we obtain the **Borch optimal risk-sharing condition** for the case where the principal is risk-neutral

$$\frac{1}{u'(\omega - p - l + B_l)} = \lambda, \quad \forall l \in \mathbb{L} \tag{5.4}$$

which implies that $-l + B_l (\equiv \alpha)$ must be the same for all $l \in \mathbb{L}$. That is, the agent's consumption is constant across states, so that she is fully insured. Also notice that this is true for all $a \in \{0, 1\}$. In words, when the agent is risk-averse and the principal risk-neutral, the principal bears all the risk and provides the agent full insurance in the first-best outcome.

Since $(-p - l + B_l) \mid_{l=0} = (-p + B_0)$, we can interpret B_0 as a part of the price p and normalize $B_0 (= 0)$, so that we have $\alpha = 0$. Therefore, we have

$$B_l^{FI} = l, \quad \forall l \in \mathbb{L}$$

To determine $p^{FI}(a)$ for given $a \in \{0, 1\}$, use the IR constraint, so that

$$\sum_{l=0}^{L} \pi_l(a) u(\omega - p(a)^{FI}) = d(a) + \overline{u}$$

Notice that since $d(1) > d(0)$, we need to have $p(1)^{FI} < p(0)^{FI}$.

As the second step, find $a^* \in \{0, 1\}$ by comparing

$$V^{FI}(0) = \left[p(0)^{FI} - \sum_{l=0}^{L} \pi_l(0) l \right] \text{ and } V^{FI}(1) = \left[p(1)^{FI} - \sum_{l=0}^{L} \pi_l(1) l \right]$$

5.5.3 Moral Hazard

Under the asymmetric information, the principal has an additional IC constraint such that

$$\sum_{l=0}^{L} \pi_l(a) u(\omega - p - l + B_l) - d(a) \geq \sum_{l=0}^{L} \pi_l(a') u(\omega - p - l + B_l) - d(a'), \quad \forall a' \neq a$$

Fix $a \in \{0, 1\}$, derive the optimal policy.

Let $a = 0$. If first-best contract were incentive-compatible, then the principal would choose it (since it maximizes profits subject to one fewer constraint). To check whether it is so, under the first-best contract, the IC is

$$\sum_{l=0}^{L} \pi_l(0) u(\omega - p(0)^{FI}) - d(0) \geq \sum_{l=0}^{L} \pi_l(1) u(\omega - p(0)^{FI}) - d(1) \quad \Rightarrow \quad d(0) \leq d(1)$$

Thus, the first-best contract is IC since $d(0) < d(1)$ by assumption.
Now, let $a = 1$. Consider the following Lagrangian function.

$$\begin{aligned} \mathcal{L} = p - \sum_{l=0}^{L} \pi_l(1) B_l + \lambda \left[\sum_{l=0}^{L} \pi_l(1) u(\omega - p - l + B_l) - d(1) - \overline{u} \right] \\ + \mu \left[\sum_{l=0}^{L} \pi_l(1) u(\omega - p - l + B_l) - d(1) - \sum_{l=0}^{L} \pi_l(0) u(\omega - p - l + B_l) + d(0) \right] \end{aligned}$$

The first order conditions for p, B_l, λ and μ are

$$1 - \sum_{l=0}^{L} \Big(\lambda \pi_l(1) + \mu(\pi_l(1) - \pi_l(0)) \Big) u'(\omega - p - l + B_l) = 0 \tag{5.5}$$

$$-\pi_l(1) + \Big(\lambda \pi_l(1) + \mu(\pi_l(1) - \pi_l(0)) \Big) u'(\omega - p - l + B_l) = 0, \quad \forall l \in \mathbb{L} \tag{5.6}$$

$$\sum_{l=0}^{L} \pi_l(1) u(\omega - p - l + B_l) - d(1) - \overline{u} \geq 0 \tag{5.7}$$

$$\sum_{l=0}^{L} \pi_l(1) u(\omega - p - l + B_l) - d(1) - \sum_{l=0}^{L} \pi_l(0) u(\omega - p - l + B_l) + d(0) \geq 0 \tag{5.8}$$

By the same manner, it is easy to see that the first order condition 5.5 is redundant, and we can normalize $B_0 = 0$. Thus, from the first order condition 5.6,

$$\frac{1}{u'(\omega - p - l + B_l)} = \lambda + \mu \left[1 - \frac{\pi_l(0)}{\pi_l(1)} \right] \quad \forall l \in \mathbb{L} \tag{5.9}$$

which can be thought of as a **modified Borch rule**: The agent cannot be fully insured since the principal faces a trade-off between insuring the agent and providing her with incentives to exert high effort. Notice that

- $\lambda > 0$. If $\lambda = 0$, then IR does not bind, then the principal could lower payout for every l so that the agent's utility decreases by ϵ for every l. The MLRP assumption implies that there must be some l such that $\pi_l(0) > \pi_l(1)$ and l' such that $\pi_{l'}(0) < \pi_{l'}(1)$. Thus, $1 - \frac{\pi_l(0)}{\pi_l(1)} < 0$ and $1 - \frac{\pi_{l'}(0)}{\pi_{l'}(1)} > 0$, and hence $\lambda > 0$.

- $\mu > 0$. If $\mu = 0$, then IC does not bind, and hence the principal would face the same problem as under FI. And we already know that the agent wants to exert $a = 0$ in this case, and thus IC would be violated. Now, suppose that $\mu < 0$. Since the RHS of 5.9 is strictly increasing in l by MLRP, and hence $u'(\omega - p - l + B_l)$ is strictly decreasing in l. Thus, we need $(-l + B_l)$ is strictly increasing in l. This implies that IC would be violated due to the fact that $\pi_l(0)$ FOSD $\pi_l(1)$.

- Given that $\mu > 0$, we need to have $(-l + B_l)$ is strictly decreasing in l. Moreover, normalizing $B_0 = 0$ yields $(-l + B_l) < 0$ for all $l > 0$. This means that the optimal insurance policy specifies a co-payment that increases with the size of the loss.

Similarly, as the second step, find $a^* \in \{0,1\}$. Given that $V^{FI}(0) = V^{MH}(0)$ and $V^{FI}(1) \geq V^{MH}(1)$, we have

$$V^{FI}(1) - V^{FI}(0) \geq V^{MH}(1) - V^{MH}(0)$$

which implies that if $a = 0$ is Pareto-optimal, then $0 > V^{FI}(1) - V^{FI}(0) \geq V^{MH}(1) - V^{MH}(0)$, and hence it is also optimal for the firm to induce $a^* = 0$. However, if $a = 1$ is Pareto-optimal, then it can be the case that $V^{FI}(1) - V^{FI}(0) > 0 > V^{MH}(1) - V^{MH}(0)$, and hence the firm chooses $a^* = 0$. That is, the information asymmetries may lead to inefficient outcomes.

5.5.4 Continuous Effort Level and The First-Order Approach

- The principal is a firm and the agent is an employee.
- The agent can exert effort level $a \in \mathbb{R}^+$, and receives wage ω from the principal.
- Suppose that the employee's performance (or output) q can take only two values, $q \in \{0,1\}$.
- Assume that $P(a) \equiv P(q = 1 \mid a)$ is strictly increasing in concave in a, and $P(0) = 0$, $P(\infty) = 1$, and $P'(0) > 1$.
- The firm maximizes the expected profits with a Bernoulli utility function $v(q-\omega)$ such that $v' > 0$ and $v'' \leq 0$.
- The agent maximizes the expected utility with a Bernoulli utility function $u(\omega)$ such that $u' > 0$ and $u'' \leq 0$.
- The cost of effort is $\psi(a) = a$.

Under the full information, the firm can observe a and hence can make payment of the wage profile (ω_0, ω_1) contingent on a. Thus, the firm seeks to solve

$$\max_{\{a,\omega_0,\omega_1\}} \; [P(a)v(1-\omega_1) + (1-P(a))v(-\omega_0)] \quad \text{s.t. } P(a)u(\omega_1) + (1-P(a))u(\omega_0) - a \geq \overline{u}$$

Consider the Lagrangian function.

$$\mathcal{L} = [P(a)v(1-\omega_1) + (1-P(a))v(-\omega_0)] + \lambda\left[P(a)u(\omega_1) + (1-P(a))u(\omega_0) - a - \overline{u}\right]$$

The first order conditions for ω_0, ω_1 and a are

$$-(1-P(a))v'(-\omega_0) + \lambda(1-P(a))u'(\omega_0) = 0 \tag{5.10}$$

$$-P(a)v'(1-\omega_1) + \lambda P(a)u'(\omega_1) = 0 \tag{5.11}$$

$$P'(a)\left[v(1-\omega_1) - v(-\omega_0) + \lambda(u(\omega_1) - u(\omega_0))\right] = \lambda \tag{5.12}$$

Combining the first order conditions 5.10 and 5.11 yields the Borch optimal risk-sharing rule

$$\lambda = \frac{v'(-\omega_0)}{u'(\omega_0)} = \frac{v'(1-\omega_1)}{u'(\omega_1)} \tag{5.13}$$

Thus, the following condition with the tight IR constraint

$$\frac{v'(-\omega_0)}{u'(\omega_0)} = \frac{v'(1-\omega_1)}{u'(\omega_1)} = P'(a)\left[v(1-\omega_1) - v(-\omega_0) + \lambda(u(\omega_1) - u(\omega_0))\right]$$

makes it possible to find ω_0^*, ω_1^* and a^*.

- Under the full information, we have
  - $\omega_1^{FI} \geq \omega_0^{FI}$ with strict inequality if $v'' < 0$, and $1 - \omega_1^{FI} \geq 0 - \omega_0^{FI}$ with strict inequality if $u'' < 0$.
  - In words, both the firm and agent get a higher payoff when output is high. If neither party is risk-neutral, both parties bear some risk.

Example 5.1. [Risk-neutral principal] Let the firm be risk-neutral by setting $v(x) = kx$ where $k > 0$. The Borch optimal risk-sharing rule gives

$$\lambda = \frac{k}{u'(\omega_0)} = \frac{k}{u'(\omega_1)}$$

so that the optimal contract fully insures the agent $\omega_0^* = \omega_1^* \equiv \omega^*$. Then, from the first order condition for a, we have

$$P'(a)k = \frac{k}{u'(\omega^*)} \quad \Leftrightarrow \quad P'(a^*) = \frac{1}{u'(\omega^*)}$$

The equation above with the tight IR constraint $u(\omega^*) - a = \overline{u}$ can be solved for a^* and ω^*.

Example 5.2. [Risk-neutral agent] Let the agent be risk-neutral by setting $u(x) = cx$ where $c > 0$. The Borch optimal risk-sharing rule gives

$$\lambda = \frac{v'(-\omega_0)}{c} = \frac{v'(1-\omega_1)}{c}$$

so that the optimal contract fully insures the firm $1 = \omega_1^* - \omega_0^*$. Then, from the first order condition for a, we have

$$P'(a^*)\lambda c = \lambda \quad \Leftrightarrow \quad P'(a^*) = \frac{1}{c}$$

The equation above with the tight IR constraint $P(a^*)u(1+\omega_0^*) + (1-P(a^*))u(\omega_0) - a^* = \overline{u}$ can be solved for ω_0^* and ω_1^*.

Under the asymmetric information, the firm seeks to solve

$$\max_{\{a,\omega_0,\omega_1\}} \quad [P(a)v(1-\omega_1) + (1-P(a))v(-\omega_0)] \quad \text{s.t.}$$
$$P(a)u(\omega_1) + (1-P(a))u(\omega_0) - a \geq \overline{u} \quad \text{and} \quad a \in \operatorname*{argmax}_{\widehat{a}} \{P(\widehat{a})u(\omega_1) + (1-P(\widehat{a}))u(\omega_0) - \widehat{a}\}$$

Given a wage profile (ω_0, ω_1), the first and second order condition for a in the agent's utility maximization problem are

$$P'(a)\left[u(\omega_1) - u(\omega_0)\right] - 1 = 0 \quad \text{and} \quad P''(a)\left[u(\omega_1) - u(\omega_0)\right] < 0 \tag{5.14}$$

Since $P''(a) < 0$, and hence the SOC requires $u(\omega_1) > u(\omega_0) \;\Leftrightarrow\; \omega_1 > \omega_0$, otherwise $a^* = 0$. Now, consider the Lagrangian function for the firm. [42]

$$\mathcal{L} = [P(a)v(1-\omega_1) + (1-P(a))v(-\omega_0)] + \lambda\left[P(a)u(\omega_1) + (1-P(a))u(\omega_0) - a - \overline{u}\right] + \mu\left[P'(a)\left[u(\omega_1) - u(\omega_0)\right] - 1\right]$$

[42] Given our assumptions on P and u, the solution $a^*(\omega_1, \omega_0)$ to the agent's maximization problem is unique. Let us therefore replace the IC constraint by the first-order condition. This method is called the first-order approach.

The first order condition for ω_0, ω_1 and a are

$$-(1-P(a))v'(-\omega_0)+\lambda(1-P(a))u'(\omega_0)-\mu P'(a)u'(\omega_0)=0 \quad\Leftrightarrow\quad \frac{v'(-\omega_0)}{u'(\omega_0)}=\lambda-\mu\frac{P'(a)}{(1-P(a))} \tag{5.15}$$

$$-P(a)v'(1-\omega_1)+\lambda P(a)u'(\omega_1)+\mu P'(a)u'(\omega_1)=0 \quad\Leftrightarrow\quad \frac{v'(1-\omega_1)}{u'(\omega_1)}=\lambda+\mu\frac{P'(a)}{P(a)} \tag{5.16}$$

$$P'(a)\left[v(1-\omega_1)-v(-\omega_0)\right]+\lambda\left[P'(a)\left(u(\omega_1)-u(\omega_0)\right)-1\right]+\mu P''(a)\left[u(\omega_1)-u(\omega_0)\right]=0 \tag{5.17}$$

Whatever values μ takes, the LHS of the conditions 5.15 and 5.16 are positive, and hence $\lambda > 0$.
Now, we want to show that $\mu > 0$ if the agent is risk-averse. Suppose that $\mu < 0$, then from the conditions 5.15, 5.16 and the full information optimal risk-sharing rule,

$$\begin{cases}\omega_1^{FI} > \omega_1 \\ \omega_0^{FI} < \omega_0\end{cases} \quad\Leftrightarrow\quad \begin{cases}1-\omega_1^{FI} < 1-\omega_1 \\ -\omega_0^{FI} > -\omega_0\end{cases}$$

As shown under the full information optimal risk-sharing rule, $1-\omega_1^{FI} > -\omega_0^{FI}$, and hence $1-\omega_1 > -\omega_0$.
Plugging 5.14 into the condition 5.17 yields

$$P'(a)\underbrace{\left[v(1-\omega_1)-v(-\omega_0)\right]}_{>0}+\lambda\underbrace{\left[P'(a)\left(u(\omega_1)-u(\omega_0)\right)-1\right]}_{=0 \text{ by FOC}}+\mu\underbrace{P''(a)\left[u(\omega_1)-u(\omega_0)\right]}_{<0 \text{ by SOC}}=0$$

Thus, the RHS cannot be zero, which is a contradiction.

- At the optimal contract, the agent cannot be fully insured even if the principal (or firm) is risk-neutral. Instead the agent needs to be rewarded for outcomes whose frequency increases with effort. In the binary output case, this condition is simple in that the agent receives a higher wage than under optimal risk-sharing if output is high, and vice-versa if output is low.

6 Part 6: Mechanism Design

6.1 Model Description

- There is only one item for sale, and the seller values it zero.
- There are N potential buyers, and each buyer's payoff is $v_i - p$ if she wins the object and has to pay p, or $-p$ if she does not win the object and has to pay p.
- Assume that buyers are risk-neutral and seek to maximize expected payoff, and their budget constraints do not bind.
- The buyer i's valuation $v_i \in \Theta_i \equiv [0,1]$ is a random variable with CDF F_i and pdf f_i.
- Different buyer's valuations are independent. Each buyer i knows the realization of her own v_i but not that of v_{-i}.
- The seller knows none of the realized valuations.
- The joint distribution of $(v_1, \ldots, v_N)$ is common knowledge among buyers and seller.

6.2 The Revelation Principle

Definition 6.1. [Selling mechanism] A selling mechanism consists of $(S_i, p_i, c_i)_{i=1}^N$ such that

1) For each i, buyer i's action set S_i is a non-empty set

2) For each i, buyer i's winning probability if action profile $(s_1, \ldots, s_N)$ is played, $p_i : \times_{j=1}^N S_j \to [0,1]$ (Assume that $\sum_i p_i \leq 1$ for all $\boldsymbol{s} \in \times_{j=1}^N S_j$)

3) For each i, buyer i's expected payment if action profile $(s_1, \ldots, s_N)$ is played, $c_i : \times_{j=1}^N S_j \to \mathbb{R}$

For every $\boldsymbol{s} \in \times_{j=1}^N S_j$, a selling mechanism prescribes a probability of winning the object and a cost function for each player. Thus, player i' (expected) payoff can be represented as $p_i(\boldsymbol{s})v_i - c_i(\boldsymbol{s})$. Under this **game of incomplete information**, a strategy for player i therefore can be written as $\boldsymbol{\sigma}_i : \Theta_i \to S_i$. [43] Let's consider the Bayes-Nash equilibria of this game.

Definition 6.2. [Direct selling mechanism] A selling mechanism is direct if $S_i = \Theta_i$ for all $i = 1, 2, \ldots, N$. (A strategy for buyer i is to report a type.) Specifically, we can write $p_i : [0,1]^N \to [0,1]$ and $c_i : [0,1]^N \to [0,1]$.

Definition 6.3. A Bayes-Nash equilibrium $\boldsymbol{\sigma}^*$ of a direct selling mechanism is **truthful** if $\boldsymbol{\sigma}_i^*(v_i) = v_i$ for all $v_i \in \Theta_i$ for all $i = 1, 2, \ldots, N$.

Definition 6.4. [Equivalence of BNE] Consider two selling mechanisms, $(S_i, p_i, c_i)_{i=1}^N$ and $(\widehat{S}_i, \widehat{p}_i, \widehat{c}_i)_{i=1}^N$. Bayes-Nash equilibria of these mechanism, $\boldsymbol{\sigma}^*$ and $\widehat{\boldsymbol{\sigma}}^*$, are called equivalent if for every $\boldsymbol{v} \in \times_{i=1}^N \Theta_i$, and for all $i = 1, 2, \ldots, N$

$$p_i(\boldsymbol{\sigma}_1^*(v_1), \ldots, \boldsymbol{\sigma}_N^*(v_N)) = \widehat{p}_i(\widehat{\boldsymbol{\sigma}}_1^*(v_1), \ldots, \widehat{\boldsymbol{\sigma}}_N^*(v_N)) \quad \text{and} \quad c_i(\boldsymbol{\sigma}_1^*(v_1), \ldots, \boldsymbol{\sigma}_N^*(v_N)) = \widehat{c}_i(\widehat{\boldsymbol{\sigma}}_1^*(v_1), \ldots, \widehat{\boldsymbol{\sigma}}_N^*(v_N))$$

Proposition 6.1. [Revelation principle] *For every BNE of any selling mechanism, there exists an equivalent truthful BNE of a direct selling mechanism.*

[43] We will only consider the pure strategy in this lecture.

Proof. Let $(S_i, p_i, c_i)_{i=1}^N$ be any selling mechanism and $\boldsymbol{\sigma}^*$ be a BNE of this selling mechanism. Construct for every $\boldsymbol{v} \in \times_{i=1}^N \Theta_i$, and for all $i = 1, 2, \ldots, N$

$$\widehat{S}_i = \Theta_i,\ p_i(\boldsymbol{\sigma}_1^*(v_1), \ldots, \boldsymbol{\sigma}_N^*(v_N)) = \widehat{p}_i(v_1, \ldots, v_N) \text{ and } c_i(\boldsymbol{\sigma}_1^*(v_1), \ldots, \boldsymbol{\sigma}_N^*(v_N)) = \widehat{c}_i(v_1, \ldots, v_N)$$

First, we want to show that telling the truth is a BNE of this direct selling mechanism. Since $\boldsymbol{\sigma}^*$ is a BNE of $(S_i, p_i, c_i)_{i=1}^N$, for all $i = 1, 2, \ldots, N$ and for all $v_i, v_i' \in \Theta_i$

$$\begin{aligned} &\int_{\Theta_{-i}} \left[v_i p_i(\boldsymbol{\sigma}_i^*(v_i), \boldsymbol{\sigma}_{-i}^*(v_{-i})) - c_i(\boldsymbol{\sigma}_i^*(v_i), \boldsymbol{\sigma}_{-i}^*(v_{-i}))\right] f_{-i}(v_{-i}) dv_{-i} \\ \geq &\int_{\Theta_{-i}} \left[v_i p_i(\boldsymbol{\sigma}_i^*(v_i'), \boldsymbol{\sigma}_{-i}^*(v_{-i})) - c_i(\boldsymbol{\sigma}_i^*(v_i'), \boldsymbol{\sigma}_{-i}^*(v_{-i}))\right] f_{-i}(v_{-i}) dv_{-i} \end{aligned}$$

which is equivalent to

$$\int_{\Theta_{-i}} [v_i \widehat{p}_i(v_i, v_{-i}) - \widehat{c}_i(v_i, v_{-i})] f_{-i}(v_{-i}) dv_{-i} \geq \int_{\Theta_{-i}} [v_i \widehat{p}_i(v_i', v_{-i}) - \widehat{c}_i(v_i', v_{-i})] f_{-i}(v_{-i}) dv_{-i}$$

Thus, $\widehat{\boldsymbol{\sigma}}_i^*(v_i) = v_i$ for all $v_i \in \Theta_i$ for all $i = 1, 2, \ldots, N$ is a BNE of the direct selling mechanism $(\Theta_i, \widehat{p}_i, \widehat{c}_i)_{i=1}^N$. Second, the truth telling BNE of the direct selling mechanism is equivalent to the BNE $\boldsymbol{\sigma}^*$ of the original selling mechanism by construction. $\square$

Definition 6.5. [Interim allocation, payment, and expected payoff function] Given a direct selling mechanism $(p_i, c_i)_{i=1}^N$, we write, for all $i \in \{1, \ldots, N\}$ and reports $r_i \in [0, 1]$,

$$\overline{p}_i(r_i) = \int_{[0,1]^{N-1}} p_i(r_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i} \text{ and } \overline{c}_i(r_i) = \int_{[0,1]^{N-1}} c_i(r_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i}$$

Then, the player i's expected payoff from reporting r_i when she has valuation v_i is

$$\int_{[0,1]^{N-1}} [p_i(r_i, v_{-i}) v_i - c_i(r_i, v_{-i})] f_{-i}(v_{-i}) dv_{-i} = \overline{p}_i(r_i) v_i - \overline{c}_i(r_i) \equiv u_i(r_i, v_i)$$

Definition 6.6. [Incentive compatible direct selling mechanism; ICDSM] A direct selling mechanism is called incentive compatible if telling the truth is a BNE. i.e. if for all $i = 1, 2, \ldots, N$ and for all $v_i, r_i \in [0, 1]$

$$\overline{p}_i(v_i) v_i - \overline{c}_i(v_i) \geq \overline{p}_i(r_i) v_i - \overline{c}_i(r_i)$$

6.3 Characterizing Incentive Compatible Direct Selling Mechanism

Proposition 6.2. *A direct selling mechanism is incentive compatible if and only if for all $i = 1, 2, \ldots, N$*

1) *$\overline{p}_i(v_i)$ is non-decreasing in v_i*
2) *For every $v_i \in [0, 1]$, $\overline{c}_i(v_i) = \overline{c}_i(0) + \overline{p}_i(v_i) v_i - \int_0^{v_i} \overline{p}_i(x) dx$*

Proof.
[Only if part] Suppose that a direct selling mechanism is incentive compatible. Let $v_i, v_i' \in [0, 1]$ such that $v_i' > v_i$.

From IC for both types,

$$\begin{aligned}
\overline{p}_i(v_i)v_i - \overline{c}_i(v_i) &\geq \overline{p}_i(v_i')v_i - \overline{c}_i(v_i') \quad \Leftrightarrow \quad \overline{c}_i(v_i') - \overline{c}_i(v_i) \geq (\overline{p}_i(v_i') - \overline{p}_i(v_i))v_i \\
\overline{p}_i(v_i')v_i' - \overline{c}_i(v_i') &\geq \overline{p}_i(v_i)v_i' - \overline{c}_i(r_i) \quad \Leftrightarrow \quad (\overline{p}_i(v_i') - \overline{p}_i(v_i))v_i' \geq \overline{c}_i(v_i') - \overline{c}_i(v_i)
\end{aligned}$$

Thus,

$$(\overline{p}_i(v_i') - \overline{p}_i(v_i)v_i' \geq (\overline{p}_i(v_i') - \overline{p}_i(v_i))v_i \quad \Leftrightarrow \quad (\overline{p}_i(v_i') - \overline{p}_i(v_i))(v_i' - v_i) \geq 0 \quad \Leftrightarrow \quad \overline{p}_i(v_i') \geq \overline{p}_i(v_i)$$

From local IC, for all $v_i \in [0,1]$, we have

$$\left.\frac{\partial u_i(r_i, v_i)}{\partial r_i}\right|_{r_i=v_i} = \overline{p}_i'(r_i)v_i - \overline{c}_i'(r_i)\bigg|_{r_i=v_i} = 0 \quad \Leftrightarrow \quad \overline{p}_i'(v_i)v_i = \overline{c}_i'(v_i)$$

Integrating from 0 to v_i for both sides yields

$$\int_0^{v_i} \overline{p}_i'(x)xdx = \overline{c}_i(v_i) - \overline{c}_i(0) \quad \Leftrightarrow \quad \overline{c}_i(v_i) = \overline{c}_i(0) + \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(x)dx$$

[If part] Form 1) and 2), we have

$$u_i(r_i, v_i) = \overline{p}_i(r_i)v_i - \overline{c}_i(r_i) = \overline{p}_i(r_i)v_i - \overline{c}_i(0) - \overline{p}_i(r_i)r_i + \int_0^{r_i} \overline{p}_i(x)dx = -\overline{c}_i(0) + \overline{p}_i(r_i)(v_i - r_i) + \int_0^{r_i} \overline{p}_i(x)dx$$

Differentiating with respect to r_i yields

$$\frac{\partial u_i(r_i, v_i)}{\partial r_i} = \overline{p}'{}_i(r_i)(v_i - r_i) - \overline{p}_i(r_i) + \overline{p}_i(r_i) = \overline{p}'{}_i(r_i)(v_i - r_i)$$

Notice that $r_i = v_i$ satisfies the FOC, and since $\overline{p}'{}_i(r_i) \geq 0$, if $v_i > r_i$ then $\frac{\partial u_i(r_i, v_i)}{\partial r_i} > 0$ and if $v_i < r_i$ then $\frac{\partial u_i(r_i, v_i)}{\partial r_i} < 0$. Thus, $r_i = v_i$ is globally optimal. $\square$

Example 6.1. Suppose $N = 2$, $f_1(v_1) = 2v_1$ and $f_2(v_2) = 2 - 2v_2$. Assume that we wish to allocate the object to the individual who values it highest, and want the interim payment of type 0 to be 0. From the fact that the object should be allocated to the person with higher value,

$$p_i(v_i, v_j) = \begin{cases} 1 & \text{if} \quad v_i \geq v_j \\ 0 & \text{if} \quad v_i < v_j \end{cases}$$

Then,

$$\begin{aligned}
\overline{p}_1(v_1) &= \int_{[0,1]} p_1(v_1, v_2)f_2(v_2)dv_2 = \int_0^{v_1} 1 \times f_2(v_2)dv_2 + \int_{v_1}^1 0 \times f_2(v_2)dv_2 = \int_0^{v_1} (2 - 2v_2)dv_2 = 2v_1 - v_1^2 \\
\overline{p}_2(v_2) &= \int_{[0,1]} p_2(v_1, v_2)f_1(v_1)dv_1 = \int_0^{v_2} 1 \times f_1(v_1)dv_1 + \int_{v_2}^1 0 \times f_1(v_1)dv_1 = \int_0^{v_2} 2v_1dv_2 = v_2^2
\end{aligned}$$

Notice that both $\overline{p}_1(v_1)$ and $\overline{p}_2(v_2)$ are non-decreasing on $[0,1]$.
Therefore

$$\overline{c}_1(v_1) = \overline{c}_1(0) + \overline{p}_1(v_1)v_1 - \int_0^{v_1} \overline{p}_1(x)dx = v_1^2 - \frac{2}{3}v_1^3 \quad \text{and} \quad \overline{c}_2(v_2) = \overline{c}_2(0) + \overline{p}_2(v_2)v_2 - \int_0^{v_2} \overline{p}_2(x)dx = \frac{2}{3}v_2^3$$

6.4 Revenue Equivalence

Proposition 6.3. [Revenue equivalence] *Consider two selling mechanism and one BNE for each mechanism such that both BNE have the same*

1) $\overline{c}_i(0)$ *for all* $i = 1, 2, \ldots, N$

2) $\overline{p}_i(v_i)$ *for all* $i = 1, 2, \ldots, N$ *and for all* $v_i \in [0, 1]$

Then, the ex-ante expected revenue to the seller from the two equilibria is the same.

Proof. The seller's expected revenue is

$$\begin{aligned} R &= \int_{[0,1]^N} \sum_{i=1}^{N} c_i(v_1, \ldots, v_N) f(v_1, \ldots, v_N) d(v_1, \ldots, v_N) = \sum_{i=1}^{N} \int_{[0,1]} \left[\int_{[0,1]^{N-1}} c_i(v_i, v_{-i}) f(v_{-i}) d(v_{-i}) \right] f(v_i) d_i \\ &= \sum_{i=1}^{N} \int_{[0,1]} \overline{c}_i(v_i) f(v_i) d_i = \sum_{i=1}^{N} \int_{[0,1]} \left[\overline{c}_i(0) + \overline{p}_i(v_i) v_i - \int_0^{v_i} \overline{p}_i(x) dx \right] f(v_i) d_i \end{aligned}$$

The above establishes that R is entirely described by $\overline{c}_i(0)$ and $\overline{p}_i(v_i)$, and hence we get the desired result. ☐

Definition 6.7. [Virtual valuation] Consider the inverse demand function $v = F^{-1}(1 - q)$. The seller's revenue is $qF^{-1}(1 - q)$. Differentiating with respect to q gives her marginal revenue [44]

$$F^{-1}(1 - q) - q \frac{d}{dq} F^{-1}(1 - q) = v - (1 - F(v)) \frac{1}{\frac{d}{dv} F(F^{-1}(1 - q))} = v - \frac{(1 - F(v))}{f(v)} \equiv \psi(v)$$

where $\psi(v)$ can be interpreted as the marginal revenue per person of pricing at v rather than slightly higher $v + \epsilon$.

6.5 Expected Revenue Maximization

In searching for the revenue maximizing selling mechanism for the seller, we can restrict to our attention to direct selling mechanism by the revelation principle. Moreover, by the revenue equivalence theorem, the seller has effectively only the following control variables, $\{\overline{c}_i(0)\}_{i=1}^N$ and $\{\overline{p}_i(v_i)\}_{i=1}^N$. To simplify our analysis, we will use ex-post allocation probabilities $\{p_i(v_i)\}_{i=1}^N$ instead of $\{\overline{p}_i(v_i)\}_{i=1}^N$.

- Assume that each $\psi_i(v_i) = v_i - \frac{(1-F_i(v_i))}{f_i(v_i)}$ is strictly increasing in v_i on $[0, 1]$. The sufficient condition for this assumption is that the hazard rate $\frac{f_i(v_i)}{(1-F_i)}$ is strictly increasing in v_i.
- Further assume that the auctioneer must choose a direct selling mechanism that satisfies individual rationality, i.e. $\overline{p}_i(v_i)v_i - \overline{c}_i(v_i) \geq 0$ for all $i = 1, 2, \ldots, N$ and for all $v_i, \in [0, 1]$ where we have normalized the buyer's reservation utility to zero.

Proposition 6.4. *An incentive compatible direct selling mechanism satisfies the IR for types if and only if it satisfies the IR for the lowest type.*

Proof. Since only if part is obvious, we will prove if part here. The interim expected utility for type v_i is

$$\begin{aligned} U_i(v_i) \equiv \overline{p}_i(v_i)v_i - \overline{c}_i(v_i) &= \overline{p}_i(v_i)v_i - \left[\overline{c}_i(0) + \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(x) dx \right] \\ &= -\overline{c}_i(0) + \int_0^{v_i} \overline{p}_i(x) dx \end{aligned}$$

[44] Use the following expression for the derivatives of the inverse, $(\phi^{-1})'(x) = \frac{1}{\phi'(\phi^{-1}(x))}$.

Differentiating with respect to v_i for both sides yields

$$\frac{d}{dv_i}U_i(v_i) = \frac{d}{dv_i}\left[\overline{p}_i(v_i)v_i - \overline{c}_i(v_i)\right] \geq \overline{p}_i(v_i) \geq 0$$

Thus, the surplus $U_i(v_i)$ is non-decreasing in v_i and (moreover it is convex since $\overline{p}_i'(v_i) \geq 0$). Therefore, IR satisfies for all $v_i \in [0,1]$ if $U_i(0) \geq 0$. Subject to the IR being satisfied, the seller needs to satisfy $\overline{p}_i(0)0 - \overline{c}_i(0) \geq 0 \;\Leftrightarrow\; -\overline{c}_i(0) \geq 0 \;\Leftrightarrow\; \overline{c}_i(0) = 0$ for revenue maximization. In other words, in a revenue maximizing auction, the lowest type gets no surplus, $U_i(0) = 0$. □

Proposition 6.5. [Myerson 1981] *An incentive compatible direct selling mechanism maximizes the seller's expected revenue if and only if for all $i = 1, 2, \ldots, N$ and for all $v_i, \in [0,1]$,*

1) $\overline{c}_i(0) = 0$

2) $p_i(v_i, \ldots, v_N) = \begin{cases} 1 & \text{if } \;\psi_i(v_i) > \max\limits_{j \neq i}\psi_j(v_j) \;\vee\; 0 \\ 0 & \text{otherwise} \end{cases}$

The optimal direct selling mechanism assigns the object to the buyer with the largest virtual valuation, provided that this valuation is positive. If no buyer has a positive virtual valuation, the seller keeps the object.

Proof. Under the assumption of IR, we have $\overline{c}_i(0) = 0$. Consider the seller's expected revenue from each buyer i in an ICDSM with this property

$$\begin{aligned}
\int_0^1 \left[\overline{c}_i(v_i)\right] f_i(v_i)dv_i &= \int_0^1 \Big[\underbrace{\overline{c}_i(0)}_{=0} + \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(x)dx\Big] f_i(v_i)dv_i = \int_0^1 \overline{p}_i(v_i)v_i f_i(v_i)dv_i - \int_0^1 \int_0^{v_i} \overline{p}_i(x)dx f_i(v_i)dv_i \\
&= \int_0^1 \overline{p}_i(v_i)v_i f_i(v_i)dv_i - \int_0^1 \int_x^1 \overline{p}_i(x)f_i(v_i)dv_i dx = \int_0^1 \overline{p}_i(v_i)v_i f_i(v_i)dv_i - \int_0^1 \overline{p}_i(x)[F_i(1) - F_i(x)]dx \\
&= \int_0^1 \overline{p}_i(v_i)v_i f_i(v_i)dv_i - \int_0^1 \overline{p}_i(x)[1 - F_i(x)]dx = \int_0^1 \overline{p}_i(v_i)\left(v_i - \frac{[1 - F_i(v_i)]}{f_i(v_i)}\right) f_i(v_i)dv_i \\
&= \int_0^1 \overline{p}_i(v_i)\psi_i(v_i)f_i(v_i)dv_i
\end{aligned}$$

Using $\overline{p}_i(v_i) = \int_{[0,1]^{N-1}} p_i(v_i, v_{-i})f_{-i}(v_{-i})dv_{-i}$, we have

$$\begin{aligned}
\int_0^1 \left[\overline{c}_i(v_i)\right] f_i(v_i)dv_i &= \int_0^1 \left[\int_{[0,1]^{N-1}} p_i(v_i, v_{-i})f_{-i}(v_{-i})dv_{-i}\right] \psi_i(v_i)f_i(v_i)dv_i \\
&= \int_{[0,1]^N} p_i(\boldsymbol{v})\psi_i(v_i)f(\boldsymbol{v})d\boldsymbol{v}
\end{aligned}$$

Thus, the seller's total expected revenue is

$$R = \sum_{i=1}^N \int_0^1 \left[\overline{c}_i(v)\right] f_i(v_i)dv_i = \sum_{i=1}^N \int_{[0,1]^N} p_i(\boldsymbol{v})\psi_i(v_i)f(\boldsymbol{v})d\boldsymbol{v} = \int_{[0,1]^N} \left(\sum_{i=1}^N p_i(\boldsymbol{v})\psi_i(v_i)\right) f(\boldsymbol{v})d\boldsymbol{v}$$

The revenue is just integration of the weighted sum of virtual valuations. Thus, the seller can maximizes the revenue if she puts all the weight on the highest element.

Lastly, we need to check whether the ex-post allocation functions are incentive compatible. To see this, observe that

$$\overline{p}_i(v_i) = \begin{cases} 0 & \text{if } \ \psi_i(v_i) < 0 \\ \Pr(\psi_i(v_i) > \max\limits_{j \neq i} \psi_j(v_j)) & \text{if } \ \psi_i(v_i) \geq 0 \end{cases}$$

By the assumption $\psi_i'(v_i) > 0$, $\overline{p}_i(v_i)$ is indeed non-decreasing in v_i, and thus thus given mechanism is IC. □

6.6 Payment Rules

Recall that buyer i's interim expected payment in the optimal auction must be

$$\overline{c}_i(v_i) = \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(v_i)dx$$

Definition 6.8. [Constant payment rule]

$$c_i(v_i, v_{-i}) = \overline{c}_i(v_i), \quad \forall v_{-i} \in [0,1]^{N-1}$$

Definition 6.9. [Canonical payment rule]

$$c_i(v_i, v_{-i}) = p_i(v_i, v_{-i})v_i - \int_0^{v_i} p_i(x, v_{-i})dx$$

Define $\psi_{-i} = \max\limits_{j \neq i} \psi_j(v_j)$. If $\psi_i(v_i) < \psi_{-i} \ \Leftrightarrow \ v_i < \psi_i^{-1}(\psi_{-i})$, then the buyer i's loses the auction. Under the canonical payment rule, then $c_i(v_i, v_{-i}) = 0$. If $\psi_i(v_i) > \psi_{-i} \ \Leftrightarrow \ v_i > \psi_i^{-1}(\psi_{-i})$, then the buyer i's wins the auction, then $c_i(v_i, v_{-i}) = v_i - \int_0^{\psi_i^{-1}(\psi_{-i})} 0dx - \int_{\psi_i^{-1}(\psi_{-i})}^1 1dx = v_i - (v_i - \psi_i^{-1}(\psi_{-i})) = \psi_i^{-1}(\psi_{-i})$. In words, under the canonical payment rule, only the winner pays, and she pays the lowest (non-virtual) valuation she could have had and still win the auction.

6.7 Example of Optimal Auctions

6.7.1 Symmetric Case

Suppose $F_i(\cdot) = F(\cdot)$ for all $i = 1, 2, \ldots, N$. In this case, we have $\psi_i(\cdot) = \psi(\cdot)$ for all $i = 1, 2, \ldots, N$. Thus, $\psi_i(v_i) > \psi_j(v_j) \Leftrightarrow v_i > v_j$. In this case, the bidder with the highest value wins the auction, provided that $v_i > \psi^{-1}(0)$. Thus, any of the four standard auctions with reserve price $\psi^{-1}(0)$ is optimal, i.e. maximizes the expected revenue for the seller. Also notice that the revenue maximizing reserve price is independent of the number of bidders. For example, $F \sim U[1, 2]$. Then,

$$\psi(v) = v - \frac{1 - F(v)}{f(v)} = v - \frac{1 - (v-1)}{1} = 2(v-1)$$

Thus,

$$\psi(v) > 0 \quad \Leftrightarrow \quad 2(v-1) > 0 \quad \Leftrightarrow \quad v > 1$$

Any of the four standard auctions with reservation price 1 will be revenue maximizing. Under the constant payment rule, we have

$$c_i(v_i, v_{-i}) = \overline{c}_i(v_i) = \overline{p}_i(v_i)v_i - \int_1^{v_i} \overline{p}_i(v_i)dx, \quad \forall v_{-i} \in [1,2]^{N-1}$$

Since

$$\overline{p}_i(v_i) = \int_{[0,1]^{N-1}} p_i(v_i, v_{-i}) f_{-i}(v_{-i}) dv_{-i} = \Pr(v_i > \max_{j \neq i} v_j) = G(v_i)$$

Then,

$$\begin{aligned} c_i(v_i, v_{-i}) &= G(v_i)v_i - \int_1^{v_i} G(v_i)dx = \int_1^{v_i} g(x)xdx \\ &= G(v_i) \times \int_1^{v_i} \frac{g(x)}{G(v_i)} xdx = \underbrace{G(v_i)}_{\text{win Pr}} \times \mathbb{E}\Big[\max_{j \neq i} v_j \mid \max_{j \neq i} v_j < v_i\Big] \end{aligned}$$

Therefore, the constant payment rule amounts to using a FPA with reserve price 1.
Under the canonical payment rule, if $v_i > \max_{j \neq i} v_j$

$$c_i(v_i, v_{-i}) = p_i(v_i, v_{-i})v_i - \int_1^{v_i} p_i(x, v_{-i})dx = v_i - \int_{\max_{j \neq i} v_j}^{v_i} 1dx = \underbrace{1}_{\text{win Pr}} \times \max_{j \neq i} v_j$$

If $v_i < \max_{j \neq i} v_j$, simply $c_i(v_i, v_{-i}) = 0$.
Therefore, the canonical payment rule amounts to using a SPA with reserve price 1.

6.7.2 Asymmetric Case

In this case, any auction in which the bidder with the highest value always wins the is not optimal. For example, let $f_1(v_1) = 2v_1$ and $f_2(v_2) = 2 - 2v_2$. Then,

$$\psi_1(v_1) = \frac{3v_1}{2} - \frac{1}{2v_1} \quad \text{and} \quad \psi_2(v_2) = \frac{3v_2}{2} - \frac{1}{2}$$

Buyer 2 has the higher virtual valuation if

$$\psi_2(v_2) > \psi_1(v_1) \quad \Leftrightarrow \quad v_2 > v_1 + \frac{1}{3} - \frac{1}{3v_1} \equiv \xi(v_1)$$

Both players' valuations are positive if

$$\psi_1(v_1) > 0 \quad \Leftrightarrow \quad v_1 > \frac{1}{\sqrt{3}} \quad \text{and} \quad \psi_2(v_2) > 0 \quad \Leftrightarrow \quad v_2 > \frac{1}{3}$$

Observe that this auction is biased in favor of buyer 2 in the sense that if the two buyers had the identical valuations, the seller would either not sell to anyone, or sell to buyer 2. Intuitively, favoring the weaker buyer amounts to making the stronger buyer face steeper competition, driving up the stronger buyer's bid. Under the constant payment rule, we have

$$c_i(v_i, v_j) = \overline{p}_i(v_i)v_i - \int_0^{v_i} \overline{p}_i(v_i)dx, \quad \forall v_j \in [0,1]$$

Since

$$\overline{p}_1(v_1) = \begin{cases} 0 & \text{if } v_1 < \frac{1}{\sqrt{3}} \\ \Pr(v_2 < \xi(v_1)) = F_2(\xi(v_1)) & \text{if } v_1 > \frac{1}{\sqrt{3}} \end{cases} \text{ and } \overline{p}_2(v_2) = \begin{cases} 0 & \text{if } v_2 < \frac{1}{3} \\ \Pr(v_2 > \xi(v_1)) = F_1(\xi^{-1}(v_2)) & \text{if } v_2 > \frac{1}{3} \end{cases}$$

Then,

$$c_1(v_1,v_2) = \begin{cases} 0 & \text{if } v_1 < \frac{1}{\sqrt{3}} \\ F_2(\xi(v_1))v_1 - \int_{\frac{1}{\sqrt{3}}}^{v_1} F_2(\xi(x))dx & \text{if } v_1 > \frac{1}{\sqrt{3}} \end{cases} \quad \text{and} \quad c_2(v_2,v_1) = \begin{cases} 0 & \text{if } v_2 < \frac{1}{3} \\ F_1(\xi^{-1}(v_2))v_2 - \int_{\frac{1}{3}}^{v_2} F_1(\xi^{-1}(x))dx & \text{if } v_2 > \frac{1}{3} \end{cases}$$

Under the canonical payment rule, if $v_2 > \max\{\frac{1}{3}, \xi(v_1)\}$, then $p_2(v_2,v_1) = 1$ and $c_2(v_2,v_1) = \{\frac{1}{3}, \xi(v_1)\}$ and if $v_1 > \max\{\frac{1}{\sqrt{3}}, \xi^{-1}(v_2)\}$, then $p_1(v_2,v_2) = 1$ and $c_1(v_1,v_2) = \max\{\frac{1}{\sqrt{3}}, \xi^{-1}(v_2)\}$.

6.8 VCG Mechanism

- Let $\theta_i \in \Theta_i$ be an agent i's type, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N) \in \times_{i=1}^N \Theta_i \equiv \boldsymbol{\Theta}$ be a type profile.
- Let $\boldsymbol{x} = (k, t_1, \ldots, t_N)$ be an alternative where $k \in K$ is a collective decision, K a finite set, and $t_i \in \mathbb{R}$ money transfer paid by agent i.
- When an alternative $\boldsymbol{x}$ is chosen, the agent i's utility is defined by $u_i(\boldsymbol{x}, \theta_i) \equiv v_i(k, \theta_i) - t_i$.
- A direct mechanism or social choice function is a vector valued function such that $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_N(\cdot))$ where $k : \boldsymbol{\Theta} \to K$ and $t_i : \boldsymbol{\Theta} \to \mathbb{R}$ for every i.

Definition 6.10. If the social choice function f is **ex-post efficient**, then for every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, the allocation $k^*(\boldsymbol{\theta})$ is chosen, where $k^*(\boldsymbol{\theta})$ maximizes the joint utilities [45]

$$k^*(\boldsymbol{\theta}) \in \underset{k(\boldsymbol{\theta}) \in K}{\operatorname{argmax}} \sum_{i=1}^{N} v_i(k(\boldsymbol{\theta}), \theta_i)$$

Let $k^*_{-i}(\boldsymbol{\theta})$ denote the efficient allocation in the absence of agent i. That is,

$$k^*_{-i}(\boldsymbol{\theta}) \in \underset{k(\boldsymbol{\theta}) \in K}{\operatorname{argmax}} \sum_{j \neq i} v_j(k(\boldsymbol{\theta}), \theta_j)$$

Definition 6.11. [VCG mechanism] A direct mechanism $f(\cdot) = (k(\cdot), t_1(\cdot), \ldots, t_N(\cdot))$ is called a Vickrey-Clarke-Groves (VCG) mechanism if $k(\boldsymbol{\theta})$ is an efficient allocation function, and the transfer of agent i

$$t_i(\theta_i) = -\sum_{j \neq i} v_j(k^*(\boldsymbol{\theta}), \theta_j) + \sum_{j \neq i} v_j(k^*_{-i}(\boldsymbol{\theta}), \theta_j)$$

where $\sum_{j \neq i} v_j(k^*(\boldsymbol{\theta}), \theta_j)$ is the surplus of others in the presence of i under allocation $k^*(\boldsymbol{\theta})$ and $\sum_{j \neq i} v_j(k^*_{-i}(\boldsymbol{\theta}), \theta_j)$ is the surplus of others in the absence of i under allocation $k^*_{-i}(\boldsymbol{\theta})$.

- The Vickrey auction (SPA) falls into this category. (**V**ickrey mechanism)
- If the agent i's presence alters the collective decision k^*, then she needs to pay a tax equal to her effect on the other agents, and pay zero otherwise. (**C**larke mechanism)
- The agent's transfer depends on her reported type solely thorough her impact on the collective decision k. (**G**loves mechanism)

Proposition 6.6. *Truth telling is a weakly dominant strategy in a VCG mechanism.*

[45] We do not include transfers in the definition.

Proof. Suppose not. Then, there must exist $r \neq \theta_i$ and a type profile $\boldsymbol{\theta}_{-i}$ such that the agent i of type θ_i is strictly better off by reporting r. That is,

$$v_i(k^*(r, \boldsymbol{\theta}_{-i}), \theta_i) - t_i(r, \boldsymbol{\theta}_{-i}) > v_i(k^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_i) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \tag{6.1}$$

By the transfer rule t_i in a VCG mechanism, the equation 6.1 can be written as

$$v_i(k^*(r, \boldsymbol{\theta}_{-i}), \theta_i) + \sum_{j \neq i} v_j(k^*(r, \boldsymbol{\theta}_{-i}), \theta_j) - \cancel{\sum_{j \neq i} v_j(k^*_{-i}(r, \boldsymbol{\theta}_{-i}), \theta_j)} > v_i(k^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_i) + \sum_{j \neq i} v_j(k^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j) - \cancel{\sum_{j \neq i} v_j(k^*_{-i}(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j)}$$

so that we have

$$\sum_{j=1}^{N} v_j(k^*(r, \boldsymbol{\theta}_{-i}), \theta_j) > \sum_{j=1}^{N} v_j(k^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j)$$

which contradicts with the ex-post efficiency of $k^*(\boldsymbol{\theta})$. $\square$

Example 6.2. The utilities derived by agent $i \in \{1, 2, 3\}$ are in the following table. Define the VCG mechanism for this problem.

	A	B	AB
1	7	5	8
2	4	1	6
3	2	9	3

Solution. First, notice that the efficient allocation is to give A to the agent 1 and B to the agent 2. Then, the VCG transfers are

$$t_1 = -9 + \max\{4 + 9, 2 + 1, 6, 3\} = 4, \quad t_2 = -16 + \max\{7 + 9, 5 + 2, 8, 3\} = 0 \quad \text{and} \quad t_3 = -7 + \max\{7 + 1, 5 + 4, 8, 6\} = 2$$

Example 6.3. The utilities derived by agent $i \in \{1, 2, 3\}$ are in the following table. Define the VCG mechanism for this problem if $x > y + z$.

	A	B	AB
1	0	0	x
2	y	0	y
3	0	z	z

Solution. First, notice that the efficient allocation is to give AB to the agent 1. Then, the VCG transfers are

$$t_1 = -0 + \max\{y + z, 0 + 0, y, z\} = y + z, \quad t_2 = -x + \max\{0 + z, 0 + 0, x, z\} = 0 \quad \text{and} \quad t_3 = -x + \max\{0 + 0, 0 + y, x, y\} = 0$$