Macroeconomics 2 (387D #34455)

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May 14, 2019

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This note is based on lectures by Saroj Bhattarai at UT Austin in 2019 Spring.

Contents

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1 Part 1: Partial Equilibrium

In this part, we will study problems where components of the state, especially prices, which are not under the control of the agent are taken as exogenously given.

1.1 Preliminaries

Definition 1.1. **[No Ponzi scheme]** A ponzi scheme refers to the case where a person or agent borrows and postpones k payment indefinitely (or roll over indefinitely). We usually do not allow a ponzi scheme regardless of the periodic budget constraint being satisfied. There are two reasons to do so. First, in institutional sense it is fraud. Second, in a technical perspective, a ponzi scheme may incur no optimal solution. No ponzi condition can be written as $\lim_{T\to\infty} \frac{a_{T+1}}{R^T} \geq 0$, [1] which requires the present value of saving (or debt) in the limit (or at the terminal period if finite) must be non-negative.

Definition 1.2. **[Transversality condition]** The transversality condition is an necessary condition for optimality which makes it possible to pin down a unique solution path among many candidate solutions that satisfy FOCs. [2]

$$\lim_{T\to\infty} \underbrace{\underbrace{\beta^T u'(c_T^*)}_{\text{PV utility of consumption}} \underbrace{a_{T+1}^*}_{\text{asset at } T}}_{\text{PV of shadow price of asset}} = 0$$

- The no ponzi condition and the TVC are similar. However, the former is a constraint on the choice set of a agent while the TVC is a necessary condition of optimality, given constraints.

- The FOCs and the TVC, when combined, are both necessary and sufficient for interior optimality. Moreover, no ponzi condition is met if the TVC and the Euler equations hold. By Euler equations, we have $u'(c_T^*) = (\beta R)^{-T} u'(c_0^*)$, and hence $\lim_{T\to\infty} \beta^T u'(c_T^*) a_{T+1}^* = 0 \Leftrightarrow \lim_{T\to\infty} \beta^T (\beta R)^{-T} u'(c_0^*) a_{T+1}^* = 0 \Leftrightarrow u'(c_0^*) \lim_{T\to\infty} \frac{a_{T+1}^*}{R^T} = 0 \Leftrightarrow \lim_{T\to\infty} \frac{a_{T+1}^*}{R^T} = 0$.

- Analogously, if we are under finite time horizon T, no ponzi condition can be understood as $a_{T+1} \geq 0$ or $(\frac{a_{T+1}}{R^T} \geq 0)$. Similarly, the finite time horizon version for the TVC can be written as $\beta^T u'(c_T^*) a_{T+1}^* = 0$. In general finite horizon problems, the FOCs (the budget constraints and the Euler equations) and the TVC, when combined, are also both necessary and sufficient for interior optimality. You can also notice that the TVC $\underbrace{\beta^T u'(c_T^*)}_{>0} a_{T+1}^* = 0$ implies that $a_{T+1}^* = 0$. That is, the TVC guarantees that no ponzi condition is satisfied with equality.

- In conclusion, we only need to consider the FOCs and TVC in order to characterize solutions since no Ponzi condition is automatically met. In other words, you do not need to include no ponzi condition constraint when you construct the Lagrangian function given that the TVC is considered as an additional optimality condition.

[1] Note that we use the present value of assets (chosen) evaluated in terms of consumption goods at period T. Thus, if we use the budget constraint $c_t + qb_{t+1} = b_t$, then no ponzi condition is $\lim_{T\to\infty} \frac{qb_{T+1}}{(\frac{1}{q})^T} = \lim_{T\to\infty} q^{T+1} b_{T+1} \geq 0$.

[2] If use a_T^* instead of a_{T+1}^*, it should be multiplied by R for investment return to be taken into. For example, $\lim_{T\to\infty} \beta^T u'(c_T^*) f'(k_T^*) k_T^* = 0$.

1.2 Saving Problem

Each household's desire to smooth his or her consumption streams over time can be implemented only when they solve savings problems. So, consumption models involve savings problems. Consider the following saving problem. A consumer seeks to solve

$$\max_{\{c_{t+j},a_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t\Big[\sum_{j=0}^{\infty}\beta^j u(c_{t+j})\Big]$$

where $\beta \in (0,1)$, and $u \in C^2$ such that $u' > 0$, and $u'' < 0$, and $\mathbb{E}_t$ denotes a mathematical expectation operator conditioned on time t information. The consumer faces a sequence of budget constraints, for each $j \geq 0$,

$$c_{t+j} + a_{t+j} = R_{t+j}a_{t+j-1} + y_{t+j} \tag{1.1}$$

where a_{t+j} is the consumer's holdings of an asset at the end of period $t+j$ (or at the beginning of period $t+j+1$). [3] y_{t+j} is a random income sequence, and R_{t+j} is also a random gross rate of return on the asset between $t+j-1$ and $t+j$. If we rewrite $R_{t+j} = 1 + r_{t+j}$, we can interpret r_{t+j} as a net return rate. Notice that we are assuming sequences of R_{t+j} and y_{t+j} as exogenous. We also assume that

$$a_{t-1} \text{ is given} \tag{1.2}$$

$$\lim_{j\to\infty} \mathbb{E}_t\Bigg[\frac{a_{t+j}}{\prod\limits_{l=t+1}^{t+j} R_l}\Bigg] \geq 0 \tag{1.3}$$

Note that 1.3 is called No-ponzi condition, denoted by, ${}_nP_g$, which represents an environment or restriction that the expected value of present value of the saving in the limit should be non-negative.
In order to characterize the solution, consider the following Lagrangian function.

$$\mathcal{L}_t = \mathbb{E}_t\Big[\sum_{j=0}^{\infty}\beta^j\big\{u(c_{t+j}) + \lambda_{t+j}(R_{t+j}a_{t-1+j} + y_{t+j} - c_{t+j} - a_{t+j})\big\}\Big]$$

where λ_{t+j} is the current value Lagrangian multiplier. The first order conditions for c_t and a_t are [4]

$$u'(c_t) = \lambda_t \text{ and } \lambda_t = \beta\mathbb{E}_t[\lambda_{t+1}R_{t+1}]$$

which can be augmented to derive the Euler equation.

$$u'(c_t) = \beta\mathbb{E}_t[R_{t+1}u'(c_{t+1})] \tag{1.4}$$

Notice that we also need the transversality condition as an optimality condition.

$$\lim_{j\to\infty}\mathbb{E}_t\Big[\beta^j u'(c_{t+j})a_{t+j}\Big] = 0 \tag{1.5}$$

We will assume that first-order conditions with TVC are necessary and sufficient conditions for optimality. Then, we can solve this problem using Euler equation, TVC, together with budget constraints.

[3] You may define a_{t+j+1} as an asset at the beginning of period $t+j+1$.
[4] To be more rigorous, we need to find the first order conditions for c_{t+j} and a_{t+j}.

1.2.1 Linear Quadratic Permanent Income Theory

To obtain a version of the permanent income theory of Friedman (1955) and Hall (1978), set $R_{t+j} = R$ for all $j \geq 0$, and $\beta R = 1$. Under these assumptions, 1.4 can be rewritten as

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})] \tag{1.6}$$

What does this relation tell us?

1. Expected marginal utility is a martingale. [5]
2. The current or ex-ante marginal utility is the best predictor of next period's marginal utility. Ex-post, it can change only if the expectation is not realized.
3. The changes in marginal utility are unpredictable based on past information.

Then, we can define

$$u'(c_{t+1}) = \underbrace{u'(c_t)}_{\text{predicted part}} + \underbrace{\epsilon_{t+1}}_{\text{unpredicted part}}$$

where ϵ_{t+1} are shocks that make $u'(c_{t+1})$ unpredictable. Thus, we can rewrite

$$\epsilon_{t+1} = u'(c_{t+1}) - \mathbb{E}_t[u'(c_{t+1})]$$

where $\mathbb{E}_t[\epsilon_{t+1}] = 0$.
If we further assume that $u(c_t) = -\frac{1}{2}(c_t - \gamma)^2$ where γ is a bliss point, then

$$u'(c_t) = \mathbb{E}_t[u'(c_{t+1})] \;\Leftrightarrow\; c_t = \mathbb{E}_t[c_{t+1}] \tag{1.7}$$

Thus, in this case, consumption is a martingale, and hence changes in consumption are not predictable as well. As above, if we put

$$c_{t+1} = c_t + \epsilon_{t+1}$$

Then, we also have

$$\epsilon_{t+1} = c_{t+1} - \mathbb{E}_t[c_{t+1}] \text{ and } \mathbb{E}_t[\epsilon_{t+1}] = 0$$

Given that $\triangle c_{t+1} = c_{t+1} - c_t$ is not predictable from the information at period t, $\triangle c_{t+1}$ cannot be correlated with them at all. That is, no variables known in period t are correlated with $\triangle c_{t+1}$. Thus, if we regress lagged variables obtained before or at period t on $\triangle c_{t+1}$, their coefficient must be zero. That is, running the following regression gives us $\beta_j = 0$ for all j.

$$\triangle c_{t+1} = \sum_{0 \leq j \leq t} \beta_j X_{t-j} + \varepsilon_{t+1}$$

By the same reason, we also have $\alpha = 0$ if we run the following regression.

$$\triangle c_{t+1} = \alpha \mathbb{E}_t[\triangle y_{t+1}] + \varepsilon_{t+1}$$

This is because, $\mathbb{E}_t[\triangle y_{t+1}]$ is also something that we expect at period t. Thus, we can conclude that consumption does not response to expected income changes. What about responses to unexpected income changes that could be

[5] A martingale is a stochastic process (i.e., a sequence of random variables) $\{x_t\}$ which satisfies, at every, $\mathbb{E}_t[x_{t+j}] = x_t$ for any $j > 0$.

either permanent or transitory? To see, let us solve this model first. What I mean by solving a model is that we would like to represent endogenous variables as functions of a history of shocks or state variables.
From a period budget constraints 1.1, for all $j \geq 0$, we have

$$(1+r)a_{t-1+j} = c_{t+j} + a_{t+j} - y_{t+j}$$

Writing a flow of period budget constraint and doing recursively substitution for a_{t+j} terms yields

$$\begin{aligned}
(1+r)a_{t-1} &= c_t + \cancel{a_t} - y_t \\
\cancel{a_t} &= \frac{1}{(1+r)}[c_{t+1} + \cancel{a_{t+1}} - y_{t+1}] \\
\cancel{\frac{1}{(1+r)}a_{t+1}} &= \frac{1}{(1+r)^2}[c_{t+2} + \cancel{a_{t+2}} - y_{t+2}] \\
&\vdots \\
\cancel{\frac{1}{(1+r)^{-2+j}}a_{t-2+j}} &= \frac{1}{(1+r)^{j-1}}[c_{t+j-1} + \cancel{a_{t-1+j}} - y_{t-1+j}] \\
\cancel{\frac{1}{(1+r)^{-1+j}}a_{t-1+j}} &= \frac{1}{(1+r)^{j}}[c_{t+j} + a_{t+j} - y_{t+j}]
\end{aligned}$$

We have

$$(1+r)a_{t-1} = -\sum_{l=0}^{j} \frac{1}{(1+r)^l}[y_{t+l} - c_{t+l}] + \frac{1}{(1+r)^j}a_{t+j}$$

Taking expectation at period t and letting $j \to \infty$

$$(1+r)a_{t-1} = -\mathbb{E}_t\Big[\sum_{j=0}^{\infty} \frac{1}{(1+r)^j}[y_{t+j} - c_{t+j}]\Big] + \underbrace{\lim_{j\to\infty} \mathbb{E}_t[\frac{a_{t+j}}{(1+r)^j}]}_{=0 \text{ by TVC}}$$

which gives us a present value or intertemproal budget constraint as below. [6]

$$(1+r)a_{t-1} = -\mathbb{E}_t\Big[\sum_{j=0}^{\infty} \frac{1}{(1+r)^j}[y_{t+j} - c_{t+j}]\Big] \tag{1.8}$$

Recall Euler equation 1.7, using Law of iterated expectation, we can derive

$$\begin{aligned}
c_t &= \mathbb{E}_t[c_{t+1}] \\
&= \mathbb{E}_t[\mathbb{E}_{t+1}[c_{t+2}]] \\
&= \mathbb{E}_t[c_{t+2}]
\end{aligned}$$

By the same manner, we have

$$\mathbb{E}_t[c_{t+j}] = c_t \text{ for all } j \geq 1 \tag{1.9}$$

[6] It should be noted that TVC makes it possible to convert a flow of period budget constraints into the present value budget constraint. Thus, the present value budget constraint can be only justified under optimality.

Plugging 1.9 into 1.8, we have

$$(1+r)a_{t-1} = -\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] + \underbrace{\sum_{j=0}^{\infty}\frac{c_t}{(1+r)^j}}_{=\frac{(1+r)}{r}c_t}$$

Thus,

$$\begin{aligned}
c_t &= \frac{r}{1+r}\Bigg\{\underbrace{\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big]}_{\text{PV of all future income}} + (1+r)a_{t-1}\Bigg\} \\
&= \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] + ra_{t-1}
\end{aligned}$$

This closed form solution tells the followings.

1. Consumption at period t depends on expectation of all future income onward.
2. The consumer seeks consumption smoothing over time. $c_t \neq b(y_t - \tau_t)$ where b is the marginal propensity to consume. Here, we may think $\frac{r}{1+r}$ as the MPC.
3. As long as the PV of all future income is constant, c_t does not change. Thus, timing of taxation does not matter; Ricardian Equivalence.
4. With quadratic utility, uncertainty level (variance or volatility) in income does not affect decisions; This is a property known as **certainty equivalence**. Decisions are the same as if y_t took on its expected value with certainty. This property comes from linearity in marginal utilities and seems to be problematic in a sense that people usually tend to save more under bigger uncertainty. In this quadratic utility model, no **precautionary saving** exists.

To analyze saving behavior, we first need to characterize a_t. From a period budget constraint together with c_t, it is easy to derive a_t as follow.

$$\begin{aligned}
a_t &= (1+r)a_{t-1} + y_t - c_t \\
&= (1+r)a_{t-1} + y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] - ra_{t-1} \\
&= a_{t-1} + y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big]
\end{aligned}$$

Notice that a_t has a unit root, and so it may not be stationary even though c_t and y_t are so. Moreover, transitory changes can make permanent changes. Given that

$$\begin{aligned}
s_t &\equiv a_t - a_{t-1} \\
&= y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] \text{ or equivalently } -\sum_{j=1}^{\infty}\frac{\mathbb{E}_t[\triangle y_{t+j}]}{(1+r)^j}
\end{aligned} \tag{1.10}$$

where $\triangle y_{t+1} = y_{t+1} - y_t$. Note that the PV of all future income at period t is negatively correlated with the saving at period t. That is, if a stream of future income is expected to increase, then saving decreases.
Now, we want to analyze consumption dynamics. At period t, what we can expect in changes in consumption is

$$\begin{aligned}
\triangle c_{t+1} &= c_{t+1} - c_t \\
&= c_{t+1} - \mathbb{E}_t[c_{t+1}] \\
&= \frac{r}{1+r}\mathbb{E}_{t+1}\Big[\sum_{j=0}^{\infty}\frac{y_{t+1+j}}{(1+r)^j}\Big] + ra_t - \mathbb{E}_t\Big[\frac{r}{1+r}\mathbb{E}_{t+1}\Big[\sum_{j=0}^{\infty}\frac{y_{t+1+j}}{(1+r)^j}\Big] + ra_t\Big] \\
&= \frac{r}{1+r}\mathbb{E}_{t+1}\Big[\sum_{j=0}^{\infty}\frac{y_{t+1+j}}{(1+r)^j}\Big] - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+1+j}}{(1+r)^j}\Big] \\
&= \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}]}{(1+r)^j} \qquad (1.11)
\end{aligned}$$

Note that consumption changes between time t and $t+1$ is proportional to the revision in expected earnings due to the new information accruing in that same time interval.

1.2.2 Example with a Specific Income Process

At this point, to make further progress, we need to make some assumptions on the statistical properties of income process. Suppose that y_t follows AR(1) process. That is,

$$y_t = \rho y_{t-1} + \epsilon_t$$

where $\rho \in [0, 1]$ and $\epsilon_t \sim \mathcal{N}(0, 1^2)$. Then,

$$\mathbb{E}_t[y_{t+1}] = \rho y_t,\ \mathbb{E}_t[y_{t+2}] = \mathbb{E}_t[\mathbb{E}_{t+1}[y_{t+2}]] = \mathbb{E}_t[\rho y_{t+1}] = \rho^2 y_t, \ldots,$$

Thus, in general, we have

$$\mathbb{E}_t[y_{t+j}] = \rho^j y_t \text{ for all } j \geq 1$$

Thus,

[Case 1] If $\rho = 0$ (pure transitory)

$$\begin{aligned}
c_t &= \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] + ra_{t-1} = \frac{r}{1+r}y_t + ra_{t-1} \\
a_t &= a_{t-1} + y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = a_{t-1} + \frac{1}{1+r}y_t \\
s_t &= y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = \frac{1}{1+r}y_t \\
\triangle c_{t+1} &= \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}]}{(1+r)^j} = \frac{r}{1+r}\epsilon_{t+1}
\end{aligned}$$

[Case 2] If $\rho = 1$ (fully permanent)

$$c_t = \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] + ra_{t-1} = y_t + ra_{t-1}$$

$$a_t = a_{t-1} + y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = a_{t-1}$$

$$s_t = y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = 0$$

$$\triangle c_{t+1} = \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}]}{(1+r)^j} = \epsilon_{t+1}$$

[Case 3] If $0 < \rho < 1$,

$$c_t = \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] + ra_{t-1} = \frac{r}{1+r-\rho}y_t + ra_{t-1}$$

$$a_t = a_{t-1} + y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = a_{t-1} + \frac{1-\rho}{1+r-\rho}y_t$$

$$s_t = y_t - \frac{r}{1+r}\mathbb{E}_t\Big[\sum_{j=0}^{\infty}\frac{y_{t+j}}{(1+r)^j}\Big] = \frac{1-\rho}{1+r-\rho}y_t$$

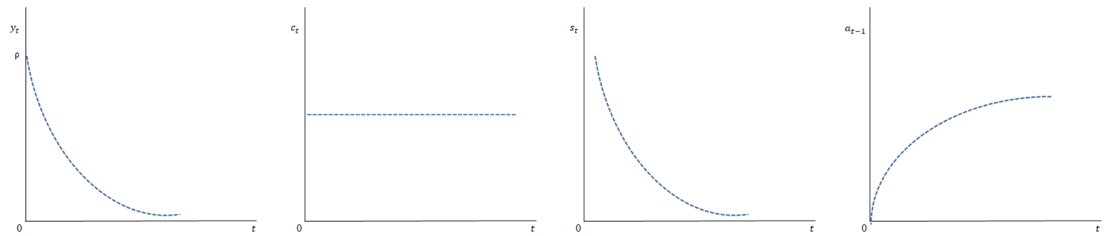
$$\triangle c_{t+1} = \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}]}{(1+r)^j} = \frac{r}{1+r-\rho}\epsilon_{t+1}$$

Note that

$$\begin{aligned}
[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}] &\equiv \mathbb{E}_{t+1}[y_{t+1+j}] - \mathbb{E}_t[y_{t+1+j}] \\
&= \rho^j y_{t+1} - \rho^{j+1}y_t \\
&= \rho^j[\rho y_t + \epsilon_{t+1}] - \rho^j \rho y_t \\
&= \rho^j \epsilon_{t+1}
\end{aligned}$$

This expression is very intuitive and it tells you how much of the innovation to your income process you are going to consume. If $\rho = 1$, $\triangle c_{t+1} = \epsilon_{t+1}$ that is you are going to consume it all, as you expect it to be fully permanent. On the other hand if $\rho = 0$ you expect it to be temporary so you only consume its annuity value, i.e. $\triangle c_{t+1} = \frac{r}{1+r}\epsilon_{t+1}$. Regarding case 3, we can draw impulsive functions as below.

Figure 1.1: Impulse functions



1.2.3 Prudence and Precautionary Savings

In this subsubsection, we depart from quadratic utility and work with preferences where the marginal utility is nonlinear in order to establish how consumption and saving react to income uncertainty. Unfortunately departing from quadratic utility reduces a lot our ability of obtaining analytical characterization of the consumption function. Consider a two-period version of the income fluctuations problem discussed above. A consumer seeks to solve

$$\max_{\{c_0, c_1, a_0\}} u(c_0) + \beta \mathbb{E}[u(c_1)]$$

subject to

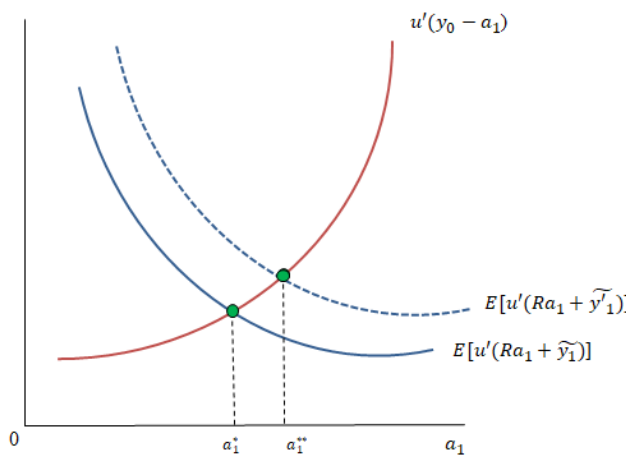
$$c_0 + a_1 = y_0 \text{ and } c_1 = Ra_1 + \widetilde{y}_1$$

where y_0 is given, and income next period $\widetilde{y}_1$ is also exogenous but stochastic. If we retain the assumption $\beta R = 1$, the corresponding Euler equation is

$$u'(c_0) = \mathbb{E}[u'(c_1)] \;\Leftrightarrow\; u'(y_0 - a_1) = \mathbb{E}[u'(Ra_1 + \widetilde{y}_1)]$$

Notice that both sides are functions of a_1. The LHS is increasing in a_1 and the RHS is decreasing in a_1 since $u'' < 0$, and hence a_1^* is uniquely determined as the figure below.

Figure 1.2: Two-period model saving



What happens to optimal consumption at $t = 0$ if the uncertainty over income next period $\widetilde{y}_1$ rises, i.e. as future income becomes more risky? Consider a mean-preserving spread of $\widetilde{y}_1$. Define ϵ to be a random variable with zero mean and positive variance such that $\widetilde{y}'_1 = \widetilde{y}_1 + \epsilon$ where $\mathbb{E}[\epsilon \mid \widetilde{y}_1] = 0$ and $Var(\epsilon) = \sigma^2$. Now, Consider the right hand side of the Euler equation

$$\begin{aligned}
\mathbb{E}[u'(Ra_1 + \widetilde{y}'_1)] &= \mathbb{E}[u'(Ra_1 + \widetilde{y}_1 + \epsilon)] \\
&= \mathbb{E}[\mathbb{E}[u'(Ra_1 + \widetilde{y}_1 + \epsilon) \mid \widetilde{y}_1]] \\
&\geq \mathbb{E}[u'(\mathbb{E}[Ra_1 + \widetilde{y}_1 + \epsilon \mid \widetilde{y}_1])] \text{ (Only if } u' \text{ is convex, then by Jensen's inequality)} \\
&= \mathbb{E}[u'(Ra_1 + \widetilde{y}_1)]
\end{aligned}$$

where the first equality uses the law of iterated expectations, the weak inequality follows from the fact that u' is convex and from Jensen's inequality and the last equality simply from the definition of conditional mean and from the fact that ϵ has 0 mean. This shows that a mean-preserving spread of $\widetilde{y}_1$ will increase the value of the RHS, for all possible values of a_1 i.e it will increase the marginal value of resources tomorrow, which shifts upward the RHS, inducing a rise in a_1 and a fall in c_0.
The convexity of the marginal utility (or $u''' > 0$) is called **prudence** and is a property of preferences, like risk aversion; risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function. Prudence is a motive for additional savings in order to take precaution against possible negative realizations of the income shock next period. In this sense, savings induced by prudence are called **precautionary savings** or **self-insurance**. Recall that quadratic utility functions have $u''' = 0$, and hence we cannot observe precautionary saving incentives in the model.

1.2.4 Permanent and Temporary Components

Now, suppose that y_t follows such that

$$y_t = p_t + v_t \text{ and } p_t = p_{t-1} + u_t$$

where v_t and u_t are i.i.d over time and $v_t \perp u_t$. Notice the income process is the sum of two orthogonal components, a permanent component p_t which follows a martingale and a transitory component v_t. First, notice that

$$\begin{aligned}
\triangle y_{t+1} &= y_{t+1} - y_t \\
&= (p_{t+1} + v_{t+1}) - (p_t + v_t) \\
&= (p_t + u_{t+1} + v_{t+1}) - (p_t + v_t) \\
&= u_{t+1} + \triangle v_{t+1}
\end{aligned}$$

Under the quadratic utility function with $\beta R = 1$, using equations 1.10 and 1.11, we have

$$\begin{aligned}
s_t &= -\sum_{j=1}^{\infty} \frac{\mathbb{E}_t[\triangle y_{t+j}]}{(1+r)^j} \\
&= -\sum_{j=1}^{\infty} \frac{\mathbb{E}_t[u_{t+j} + \triangle v_{t+j}]}{(1+r)^j} \\
&= -\frac{\mathbb{E}_t[u_{t+1} + v_{t+1} - v_t]}{(1+r)} - \sum_{j=2}^{\infty} \frac{\mathbb{E}_t[u_{t+j} + \triangle v_{t+j}]}{(1+r)^j} \\
&= \frac{v_t}{1+r}
\end{aligned}$$

and

$$\begin{aligned}
\triangle c_{t+1} &= \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{[\mathbb{E}_{t+1} - \mathbb{E}_t][y_{t+1+j}]}{(1+r)^j} \\
&= \frac{r}{1+r} \underbrace{v_{t+1}}_{\text{transitory}} + \underbrace{u_{t+1}}_{\text{permanent}}
\end{aligned}$$

given that $\mathbb{E}_{t+1}[y_{t+1+j}] = p_{t+1}$ if $j \geq 0$, $\mathbb{E}_t[y_{t+1+j}] = p_t$ if $j \geq 0$, $\mathbb{E}_{t+1}[y_{t+1}] = p_{t+1} + v_{t+1}$, and $p_{t+1} - p_t = u_{t+1}$.

Notice that households adjust their consumption responding to the annuitized change in income. This means that they will respond only weakly to pure transitory shocks (v_{t+1}), whereas one for one to permanent shocks. Also notice that saving responds almost one for one to the transitory shocks and not at all to permanent shocks.

1.2.5 Relevant Empirical Research

In this subsubsection, we will discuss empirical papers trying to test the validity of permanent income hypothesis.

- **Hall and Mishikin(1982, ECMA)**: Using panel data, for each household $i \in \{1, 2, \ldots, n\}$, and each period $t \in \{0, 1, \ldots, T\}$, they calculate $\triangle c_{it}$ and $\triangle y_{it}$. According to the PIH,

$$\triangle c_{it} = \frac{r}{1+r} v_{it} + u_{it} \text{ and } \triangle y_{it} = u_{it} + \triangle v_{it}$$

  Thus,

$$\begin{aligned}
Var(\triangle c_{it}) &= (\frac{r}{1+r})^2 \sigma_v^2 + \sigma_u^2 \\
Var(\triangle y_{it}) &= \sigma_u^2 + Var(\triangle v_{it}) \\
&= \sigma_u^2 + Var(v_{it} - v_{it-1}) \\
&= \sigma_u^2 + 2\sigma_v^2 \\
Cov(\triangle c_{it}, \triangle y_{it}) &= (\frac{r}{1+r}) \sigma_v^2 + \sigma_u^2
\end{aligned}$$

  where σ_v^2 and σ_u^2 are cross-sectional variances. Given that we have three unknowns and three restrictions, we can identify parameters. Their estimation tells that $(\frac{r}{1+r}) \simeq 0.29 \Leftrightarrow r \simeq 0.41$, which implies we can hardly support PIH theory.

- **Hall(1978, JPE) Section IV**: The simplest testable implication of PIH is that only the first lagged value of consumption helps predict current consumption. He regresses c_t on lagged consumption $c_{t-1}, c_{t-2}, \ldots$. That is,

$$c_t = \alpha + \beta_1 c_{t-1} + \beta_2 c_{t-2} + \beta_3 c_{t-3} + \beta_4 c_{t-4} + \epsilon_t$$

  According to his estimations, we have $\beta_1^* = 1.130(0.092)$, $\beta_2 = -0.040(0.142)$, $\beta_3 = 0.030(0.142)$, and $\beta_4 = -0.113(0.093)$. Moreover, $H_0 : \beta_2 = \beta_3 = \beta_4 = 0$ with F statistic 1.7 where the critical point of the F distribution of 2.7 at the 5% level. In conclusion, we found very weak evidence against PIH.

- **Hall(1978, JPE) Section V-1**: If lagged income has substantial predictive power beyond that of lagged consumption, then the life cycle-permanent income hypothesis is refuted. He regresses c_t on c_{t-1} and y_{t-1}. That is,

$$c_t = \alpha + \gamma_1 c_{t-1} + \gamma_2 y_{t-1} + \epsilon_t$$

  According to his estimations, $\gamma_1^{**} = 1.024(0.044)$ and $\gamma_2 = -0.010(0.032)$ are found, which means that a single lagged level of disposable income has essentially no predictive value at all. The F statistic for the exclusion of all but the constant and c_{t-1} is 0.1, far below the critical F of 3.9.

- **Hall(1978, JPE) Section V-2**: Theory and prevailing practice agree that contemporaneous wealth has a strong influence on consumption, so lagged wealth is a logical variable to test. He regresses c_t on lagged stock prices, $s_{t-1}, s_{t-2}, \ldots$. That is,

$$c_t = \alpha + \delta_1 c_{t-1} + \delta_2 s_{t-1} + \delta_3 s_{t-2} + \delta_4 s_{t-3} + \delta_5 s_{t-4} + \epsilon_t$$

and he found that $\delta_1^{***} = 1.012(0.004)$ and $\delta_2^{***} = 0.223(0.051)$. This regression results also supports PIH as well.

- **Johnson et al(2006, AER)**: PIH argues that as long as income change is predictable in advance, timing does not matter. They exploit quasi-random experiment of pre-announced tax policy change where the timing of rebates mailed are randomized. This helps deal with endogeniety issues. Their regression equation is

$$c_{i,t+1} - c_{i,t} = \sum_s \beta_{0s} \times \text{month}_{s,i} + \boldsymbol{\beta}_1' \boldsymbol{X}_{i,t} + \beta_2 R_{i,t+1} (\text{or } \beta_2 I(R_{i,t+1} > 0)) + u_{i,t+1}$$

where $\text{month}_{s,i}$ is a complete set of indicator variables for every period in the sample, used to absorb the seasonal variation in consumption as well as all other concurrent aggregate factors; and $\boldsymbol{X}_{i,t}$ represents control variables such as age or changes in family composition. $R_{i,t+1}$ represents key rebate variables, which can be a dollar amount or indicator whether received.

Figure 1.3: Table 2 in Johnson et al (2006, AER)

TABLE 2—THE CONTEMPORANEOUS RESPONSE OF EXPENDITURES TO THE TAX REBATE

	Panel A. Dependent variable: dollar change in expenditures on:					
	Food	Strictly nondurable goods	Nondurable goods	Food	Strictly nondurable goods	Nondurable goods
Estimation method	OLS	OLS	OLS	OLS	OLS	OLS
Rebate	0.109	0.239	0.373			
	(0.056)	(0.115)	(0.135)			
I(Rebate > 0)				51.5	96.2	178.8
				(27.6)	(53.6)	(65.0)
Age	0.570	0.449	1.165	0.552	0.391	1.106
	(0.320)	(0.550)	(0.673)	(0.318)	(0.548)	(0.670)
Change in adults	130.3	285.8	415.8	131.1	287.7	418.6
	(57.8)	(90.0)	(102.8)	(57.8)	(90.2)	(102.9)
Change in children	73.7	98.3	178.4	74.0	98.7	179.2
	(45.3)	(82.4)	(98.3)	(45.3)	(82.5)	(98.3)
RMSE	934	1680	2047	934	1680	2047
R^2 (percent)	0.6	0.6	0.6	0.6	0.6	0.6

- PIH predicts that $\beta_2 = 0$, however, as you can see the baseline result from table above, we have $\beta_2 = 0.239(0.115)$ for strictly nondurable goods, and $\beta_2 = 0.373(0.135)$ for nondurable goods when $R_{i,t+1}$ is used. Even if we use an indicator variable $I(R_{i,t+1} > 0)$, we still have $\beta_2 = 96.2(53.6)$ and $\beta_2 = 178.8(65.0)$ for strictly nondurable goods and nondurable goods, respectively. This results refute PIH.

- In order to derive dynamic response of expenditure to the tax rebate, they also include lagged rebate variable. They show that the coefficient for $R_{i,t+1}$ is 0.386(0.135) and that for $R_{i,t}$ is $-0.082(0.115)$, which implies that cumulative fraction of rebate spent over both three month periods is $0.386 + (0.386 - 0.082) = 0.691$. This result also rejects PIH.

2 Part 2: General Equilibrium

2.1 Endowment Economy without Uncertainty

2.1.1 Model Description

Time is discrete and indexed by $t = 0, 1, 2, \dots$. There are I agents that live forever in this pure exchange economy. There are no firms, and the government is absent as well. In each period the each agent consumes a non-storable consumption good. Hence there are (countably) infinite number of commodities, namely consumption in periods $t = 0, 1, 2, \dots$.

Definition 2.1. [**Allocation**] An allocation is a sequence $\{(c_t^1, c_t^2, \dots, c_t^I)\}_{t=0}^{\infty}$ of consumption in each period for each agent.

Agents have preferences over consumption allocations that can be represented by the utility function

$$U^i(\{c_t^i\}) = \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{2.1}$$

where $\beta \in (0, 1)$. Note that every agent is assumed to have the same discount factor β. (Of course, it need not be so in general.) We also assume that each agent has deterministic endowment stream $\{e_t^i\}_{t=0}^{\infty}$ of the consumption goods. There is no risk in this model and all agents know their endowment pattern perfectly in advance. All information is public, i.e, all households know everything. At period 0, before endowments are received and consumption takes place, the all agents meet at central market place and trade all commodities, i.e. trade consumption for all future dates. Let p_t [7] denote the price, in period 0, of one unit of consumption to be delivered in period t, in terms of an abstract unit of account. We will see later that prices are only determined up to a constant, so we can always normalize the price of one commodity to 1 and make it the numeraire. All agents are assumed to behave competitively in that they take the sequence of prices $\{p_t\}_{t=0}^{\infty}$ as given and beyond their control when making their consumption decisions.

Definition 2.2. [**Date-0 trade or Arrow-Debreu market structure**] The market opens at date 0 and the trade takes place once and for all. From then on, agents simply carry out their trading arrangements. Thus, agents face the inter-temporal (lifetime) budget constraint (only one budget constraint exists).

2.1.2 Date-0 Trade Competitive Equilibrium

Definition 2.3. [**Arrow-Debreu competitive equilibrium**] An Arrow-Debreu competitive equilibrium (of the endowment economy without uncertainty) consists of prices $\{p_t^*\}_{t=0}^{\infty}$ and allocations $\{(c_t^{1*}, c_t^{2*}, \dots, c_t^{I*})\}_{t=0}^{\infty}$ such that

1) Given a sequence of prices $\{p_t^*\}_{t=0}^{\infty}$, $\{(c_t^{1*}, c_t^{2*}, \dots, c_t^{I*})\}_{t=0}^{\infty}$ solves each agent's utility maximization problem, i.e.

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t } \sum_{t=0}^{\infty} p_t^* c_t^i \leq \sum_{t=0}^{\infty} p_t^* e_t^i \text{ and } c_t^i \geq 0, \ \forall t \geq 0$$

2) The market clears

$$\sum_{i=1}^{I} c_t^{i*} = \sum_{i=1}^{I} e_t^i, \ \forall t \geq 0$$

[7] We may use Q_t^0 following the notation used by Ljungqvist and Sargent.

It should be clearly noted that the elements of an equilibrium are “prices” and “allocations”. For arbitrary prices, $\{p_t\}_{t=0}^{\infty}$ it may be the case that total consumption in the economy desired by all agents, $\sum_{i=1}^{I} c_t^{i*}$ at these prices does not equal total endowment $\sum_{i=1}^{I} e_t^{i*}$. We will call equilibrium a situation in which prices are “right” in the sense that they induce agents to choose consumption so that total consumption equals total endowment in each period.

2.1.3 Solving for the Equilibrium

For simplicity, assume that we have only two agents who have endowment stream $\{e_t^i\}_{t=0}^{\infty}$ of the consumption goods given by

$$e_t^1 = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \quad \text{and} \quad e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$$

Assuming Inada condition being satisfied, consider the following Lagrangian function for agent i.

$$\mathcal{L}^i = \sum_{t=0}^{\infty} \beta^t u(c_t^i) + \lambda^i \Big[\sum_{t=0}^{\infty} p_t^* e_t^i - \sum_{t=0}^{\infty} p_t^* c_t^i \Big]$$

The first order condition for c_t^i is

$$\beta^t u'(c_t^i) = \lambda^i p_t, \ \forall t \geq 0$$

Thus, we have

$$\lambda^1 p_t^* = \beta^t u'(c_t^{1*}), \ \text{and} \ \lambda^2 p_t^* = \beta^t u'(c_t^{2*})$$

Notice that augmenting these equation yields

$$p_t^* = \frac{\beta^t u'(c_t^{1*})}{\lambda^1} = \frac{\beta^t u'(c_t^{2*})}{\lambda^2} \Rightarrow \frac{u'(c_t^{1*})}{u'(c_t^{2*})} = \frac{\lambda^1}{\lambda^2}$$

By the market clearing condition, we know that $c_t^{1*} + c_t^{2*} = e_t^1 + e_t^2 = 2, \ \forall t \geq 0$, and therefore

$$\frac{u'(c_t^{1*})}{u'(2 - c_t^{1*})} = \underbrace{\frac{\lambda^1}{\lambda^2}}_{\text{fixed}}$$

which implies that

$$c_t^{1*} = c^{1*} \quad \text{and} \quad c_t^{2*} = c^{2*}$$

Given that the first order conditions for c_t^i and c_{t+1}^i are

$$\beta^t u'(c_t^i) = \lambda^i p_t^* \quad \text{and} \quad \beta^{t+1} u'(c_{t+1}^i) = \lambda^i p_{t+1}^*$$

Since $c_t^{i*} = c_{t+1}^{i*}$, we have

$$\beta p_t^* = p_{t+1}^*$$

Let us normalize $p_0^* = 1$, then

$$p_t^* = \beta^t$$

Finally, using the budget constraint for each agent,

$$\sum_{t=0}^{\infty} p_t^* e_t^i = \sum_{t=0}^{\infty} \beta^t e_t^{i*} = \begin{cases} \frac{2}{1-\beta^2} & \text{if } i=1 \\ \frac{2\beta}{1-\beta^2} & \text{if } i=2 \end{cases}$$

with the fact that $c_t^{i*} = c^{i*}$

$$\sum_{t=0}^{\infty} p_t^* c_t^{i*} = \frac{c^{i*}}{1-\beta}$$

We have

$$c^{1*} = \frac{2}{1+\beta} > 1 \quad \text{and} \quad c^{2*} = \frac{2\beta}{1+\beta} < 1$$

2.1.4 Pareto Optimality and the First Welfare Theorem

In this subsection, we will demonstrate that for this economy a competitive equilibrium is socially optimal. Our notion of optimality will be Pareto efficiency. [8]

Definition 2.4. [Feasible allocation] An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is feasible if

1) $c_t^i \geq 0, \ \forall t \geq 0, \ i = 1, 2$
2) $c_t^1 + c_t^2 = e_t^1 + e_t^2 \ \forall t \geq 0$

Feasibility requires that consumption is non-negative and satisfies the resource constraint for $\forall t \geq 0$.

Definition 2.5. [Pareto Efficiency] An allocation $\{(c_t^1, c_t^2)\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ such that

$$U^i(\{\tilde{c}_t^i\}) \geq U^i(\{\tilde{c}_t^i\}), \ \forall i = 1, 2$$
$$U^i(\{\tilde{c}_t^i\}) > U^i(\{\tilde{c}_t^i\}), \ \text{for some } i = 1, 2$$

Theorem 2.1. [The First Welfare Theorem] *Let $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ be a competitive equilibrium allocation (or ADE). Then $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ is Pareto efficient.*

Proof. The proof will be by contradiction; we will assume $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ is not Pareto efficient and derive a contradiction to this assumption.

Suppose that $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ is not Pareto efficient. Then, by the definition of Pareto efficiency, there must exist another feasible allocation $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ such that

$$U^i(\{\tilde{c}_t^i\}) \geq U^i(\{\tilde{c}_t^i\}), \ \forall i = 1, 2$$
$$U^i(\{\tilde{c}_t^i\}) > U^i(\{\tilde{c}_t^i\}), \ \text{for some } i = 1, 2$$

Without loss of generality, assume that strict inequality holds for $i = 1$.

[Step 1] Show that

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^1 > \sum_{t=0}^{\infty} p_t^* c_t^{1*}$$

[8] For simplicity, we consider two agents case here, however, notice that the first welfare theorem 2.1 holds for I agents in general.

where $\{p_t^*\}_{t=0}^{\infty}$ are the competitive equilibrium prices associated with $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$. Suppose not,

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^1 \leq \sum_{t=0}^{\infty} p_t^* c_t^{1*}$$

then for household 1, $\{(\tilde{c}_t^1)\}_{t=0}^{\infty}$ is better and not more expensive, which cannot be the case since $\{(c_t^{1*})\}_{t=0}^{\infty}$ is part of a competitive equilibrium. Therefore,

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^1 > \sum_{t=0}^{\infty} p_t^* c_t^{1*} \tag{2.2}$$

[Step 2] Show that

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} p_t^* c_t^{2*}$$

Suppose not.

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^2 < \sum_{t=0}^{\infty} p_t^* c_t^{2*}$$

But then there exists a $\delta > 0$ such that

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} p_t^* c_t^{2*}$$

Remember that we normalized $p_0^* = 1$. Now, define an alternative allocation for agent 2 as such that

$$\hat{c}_t^2 = \tilde{c}_t^2 + \delta \text{ if } t = 0 \quad \text{and} \quad \hat{c}_t^2 = \tilde{c}_t^2, \ \forall t \geq 1$$

Obviously

$$\sum_{t=0}^{\infty} p_t^* \hat{c}_t^2 = \sum_{t=0}^{\infty} p_t^* \tilde{c}_t^2 + \delta \leq \sum_{t=0}^{\infty} p_t^* c_t^{2*} \quad \text{and} \quad U^2(\{\hat{c}_t{}^2\}) > U^2(\{\tilde{c}_t{}^2\}) \geq U(\{c_t^{2*}\})$$

which cannot be the case since $\{(c_t^{2*})\}_{t=0}^{\infty}$ is part of a competitive equilibrium. Therefore,

$$\sum_{t=0}^{\infty} p_t^* \tilde{c}_t^2 \geq \sum_{t=0}^{\infty} p_t^* c_t^{2*} \tag{2.3}$$

[Step 3] Now sum equations 2.2 and 2.3 to obtain

$$\sum_{t=0}^{\infty} p_t^* (\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_{t=0}^{\infty} p_t^* (c_t^{1*} + c_t^{2*})$$

Then, there must exist t such that

$$\tilde{c}_t^1 + \tilde{c}_t^2 > c_t^{1*} + c_t^{2*} = e_t^1 + e_t^2$$

which contradict with the fact that $\{(\tilde{c}_t^1, \tilde{c}_t^2)\}_{t=0}^{\infty}$ is a feasible allocation. Thus, $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ is Pareto efficient. $\square$

2.1.5 Sequential Markets Equilibrium

The market structure of Arrow-Debreu equilibrium in which all agents meet only once, at the beginning of time, to trade claims to future consumption may seem empirically implausible. We will call a market structure in which markets for consumption and assets open in each period sequential markets and the corresponding equilibrium sequential markets equilibrium.

Definition 2.6. [One-period asset] An one-period asset (or bond), a_{t+1}, is a promise to pay 1 unit of the consumption good in $t+1$ in change for $\frac{1}{1+r_{t+1}}(=q_t)$ units of the consumption good in period t where r_{t+1} denotes the net interest rate on one period assets from period t to period $t+1$. In this sense, we can interpret q_t as the relative price of one unit of the consumption good in period $t+1$ in terms of the period t. (Note that p_t in ADE is calculated relative to period 0.)

We would to make some comments on q_t.

- With regard to p_t in AD market, we always have $q_t = \frac{p_{t+1}}{p_t}$.
- Let $\{x_t\}_{t=0}^{\infty}$ be a stream of consumption goods. The present value of $\{x_t\}_{t=0}^{\infty}$ can be computed by using q_t as follows.

$$\begin{aligned} PV\Big[\{x_t\}_{t=0}^{\infty}\Big] &= x_0 + q_0 x_1 + q_0 q_1 x_2 + q_0 q_1 q_2 x_3 \dots \\ &= \sum_{t=0}^{\infty} \Big[\underbrace{\prod_{i=0}^{t-1} q_i}_{t \text{ terms}}\Big] x_t \end{aligned}$$

  with the convention of $\prod\limits_{i=0}^{-1} q_i = 1$.

- For agent i, their present value of lifetime consumption must be equal to that of lifetime endowment. Given that

$$PV\Big[\{c_t^i\}_{t=0}^{\infty}\Big] = \sum_{t=0}^{\infty} \Big[\prod_{l=0}^{t-1} q_l\Big] c_t^i \quad \text{and} \quad PV\Big[\{e_t^i\}_{t=0}^{\infty}\Big] = \sum_{t=0}^{\infty} \Big[\prod_{l=0}^{t-1} q_l\Big] e_t^i$$

  If $\prod\limits_{l=0}^{t-1} q_l = p_t$, then we have

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i$$

  which we use as the budget constraint in AD market structure.

Agents start out their life with initial assets holding a_0^i. Mostly we will focus on the situation in which $a_0^i = 0$ for all i, but sometimes we want to start a household off with initial wealth ($a_0^i > 0$) or initial debt ($a_0^i < 0$).

Definition 2.7. [Sequential Markets Equilibrium] A sequential markets equilibrium (of the endowment economy without uncertainty) consists of initial assets $\{\hat{a}_0^i\}_{i=1}^{I}$ borrowing limits $\{(\overline{A_t^1}, \overline{A_t^2}, \dots, \overline{A_t^I})\}_{t=0}^{\infty}$, asset prices $\{\hat{q}_t\}_{t=0}^{\infty}$ and allocations $\{(\hat{c}_t^1, \hat{a}_{t+1}^1), (\hat{c}_t^2, \hat{a}_{t+1}^2), \dots, (\hat{c}_t^I, \hat{a}_{t+1}^I)\}_{t=0}^{\infty}$ such that

1) Given a sequence of asset prices $\{\hat{q}_t\}_{t=0}^{\infty}$, $\{(\hat{c}_t^1, \hat{a}_{t+1}^1), (\hat{c}_t^2, \hat{a}_{t+1}^2), \dots, (\hat{c}_t^I, \hat{a}_{t+1}^I)\}_{t=0}^{\infty}$ solves each agent's utility maximization problem,

$$\max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \text{ s.t } c_t^i + \hat{q}_t a_{t+1}^i \leq e_t^i + a_t^i, \ c_t^i \geq 0, \text{ and } \underbrace{a_{t+1}^i > -\overline{A_t^i}}_{\text{nPg condition}}, \ \forall t \geq 0,$$

2) The markets clear

$$\sum_{i=1}^{I} \hat{c}_t^i = \sum_{i=1}^{I} e_t^i, \ \forall t \geq 0$$

$$\sum_{i=1}^{I} \hat{a}_{t+1}^i = 0, \ \forall t \geq 0$$

Note that no ponzi condition is necessary to guarantee existence of equilibrium. [9] We are now ready to state the equivalence theorem relating AD equilibria and SM equilibria. Again, for simplicity, we only consider two agents model. Further to assume that $a_0^1 = a_0^2 = 0$ and $\sup_t \{e_t^1\}_{t=0}^{\infty} = \sup_t \{e_t^2\}_{t=0}^{\infty} < \infty$. Also assume that we set $\overline{A^1} = \overline{A_t^1}$ and $\overline{A^2} = \overline{A_t^2}$ as the present value of lifetime endowment, so that households are never constrained in the amount they can borrow.

2.1.6 Equivalence between ADE and SME

Theorem 2.2. [Equivalence between ADE and SME] The set of equilibrium allocations under the AD and SM market structures coincide. In other words,

1) Let $\{p_t^*\}_{t=0}^{\infty}$ and $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ be an ADE such that

$$\frac{p_{t+1}^*}{p_t^*} \leq \xi < 1, \ \forall t \geq 0 \tag{2.4}$$

Then there exist $(\overline{A^1}, \overline{A^2})$ and a corresponding SME, $\{\hat{q}_t\}_{t=0}^{\infty}$ and $\{(\hat{c}_t^1, \hat{a}_{t+1}^1), (\hat{c}_t^2, \hat{a}_{t+1}^2)\}_{t=0}^{\infty}$ such that

$$\hat{c}_t^i = c_t^{i*}, \ \forall t \geq 0 \quad and \quad i = 1, 2$$

2) Conversely, let $\{\hat{q}_t\}_{t=0}^{\infty}$ and $\{(\hat{c}_t^1, \hat{a}_{t+1}^1), (\hat{c}_t^2, \hat{a}_{t+1}^2)\}_{t=0}^{\infty}$ be a SME such that

$$\begin{aligned} a_{t+1}^i &> -\overline{A^i}, \ \forall t \geq 0 \quad and \quad i = 1, 2 \\ q_t &\leq \epsilon < 1, \ \forall t \geq 0 \end{aligned} \tag{2.5}$$

for some $\epsilon > 0$. Then there exists a corresponding ADE, $\{p_t^*\}_{t=0}^{\infty}$ and $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ such that

$$c_t^{i*} = \hat{c}_t^i, \ \forall t \geq 0 \quad and \quad i = 1, 2$$

Proof. The key to the proof is to show the equivalence of the budget sets for the AD and SM market structures.

[Step 1] Show that the SM budget constraints and no Ponzi condition also satisfies the AD budget constraint. Normalize $p_0^* = 1$. By definitions of q_t and p_t, we can always relate

$$q_t = \frac{p_{t+1}}{p_t} \tag{2.6}$$

[9] Suppose that the constraint on borrowing were absent. Assume that there would exist a SME $\{(\hat{c}_t^i, \hat{a}_{t+1}^i)_{i=1,2}\}_{t=0}^{\infty}$ and $\{\hat{q}_t\}_{t=0}^{\infty}$. WLOG, agent 1 could always do better by borrowing more in period 0, consuming it and then rolling over the additional debt forever, by borrowing more and more. That is, $c_0^1 = \hat{c}_0^1 + \epsilon$, $c_t^1 = \hat{c}_t^1$ if $t \geq 1$ and $a_1^1 = \hat{a}_1 - \hat{q}_0^{-1}\epsilon$ and $a_{t+1}^1 = \hat{a}_{t+1} - \prod_{i=0}^{t} \hat{q}_i^{-1}\epsilon$. (Note that $\hat{q}_i^{-1} > 1$, thus $\lim_{t\to\infty} \prod_{i=0}^{t} \hat{q}_i^{-1} = \infty$, which implies that household 1's borrowing limit is not bounded above.) Hence without a limit on borrowing no SME can exist because any household i would run Ponzi schemes and augment their consumption without bound.

For each period, the budget constraint for agent i in a sequential market

$$
\begin{aligned}
c_0^i + q_0 a_1^i &= e_0^i \\
c_1^i + q_1 a_2^i &= e_1^i + a_1^i \\
&\vdots \\
c_T^i + q_T a_{T+1}^i &= e_T^i + a_T^i
\end{aligned}
$$

For $k(k \geq 1)$th row, multiply $\prod\limits_{l=0}^{k-1} q_l$ for both sides, then we have

$$
\begin{aligned}
c_0^i + q_0 a_1^i &= e_0^i \\
q_0 c_1^i + q_0 q_1 a_2^i &= q_0 e_1^i + q_0 a_1^i \\
&\vdots \\
\prod_{l=0}^{T-1} q_l c_T^i + \prod_{l=0}^{T-1} q_l a_{T+1}^i &= \prod_{l=0}^{T-1} q_l e_T^i + \prod_{l=0}^{T-1} q_l a_T^i
\end{aligned}
$$

Summing up both sides yields

$$
\begin{aligned}
c_0^i + \cancel{q_0 a_1^i} &= e_0^i \\
q_0 c_1^i + \cancel{q_0 q_1 a_2^i} &= q_0 e_1^i + \cancel{q_0 a_1^i} \\
&\vdots \\
\prod_{l=0}^{T-1} q_l c_T^i + \prod_{l=0}^{T-1} q_l a_{T+1}^i &= \prod_{l=0}^{T-1} q_l e_T^i + \cancel{\prod_{l=0}^{T-1} q_l a_T^i}
\end{aligned}
$$

and hence we eventually have

$$
\sum_{t=0}^{T} \left[\prod_{l=0}^{t-1} q_l\right] c_t^i + \prod_{l=0}^{T-1} q_l a_{T+1}^i = \sum_{t=0}^{T} \left[\prod_{l=0}^{t-1} q_l\right] e_t^i
$$

Using pricing equating 2.6 and $p_0 = 1$, we have

$$
\prod_{l=0}^{t-1} q_l = q_0 q_1 \ldots q_{t-1} = \frac{p_1}{p_0} \times \frac{p_2}{p_1} \times \ldots \times \frac{p_t}{p_{t-1}} = \frac{p_t}{p_0} = p_t \;\Leftrightarrow\; p_t = \underbrace{q_0 q_1 \ldots q_{t-1}}_{t \text{ elements}}
$$

Thus, we have

$$
\sum_{t=0}^{T} p_t c_t^i + p_T a_{T+1}^i = \sum_{t=0}^{T} p_t e_t^i
$$

Taking the limits with respect to T on both sides gives us

$$
\sum_{t=0}^{\infty} p_t c_t^i + \lim_{T \to \infty} p_T a_{T+1}^i = \sum_{t=0}^{\infty} p_t e_t^i
$$

Since $\lim_{T\to\infty} p_T = 0$ by 2.4 and $a^i_{T+1} > -\overline{A^i}$ is bounded, we have

$$\sum_{t=0}^{\infty} p_t c^i_t = \sum_{t=0}^{\infty} p_t e^i_t$$

Therefore, any allocation that satisfies the SM budget constraints and no Ponzi condition also satisfies the AD budget constraint.

[**Step 2**] Now, suppose that we have an ADE, $\{p^*_t\}_{t=0}^{\infty}$ and $\{(c^{1*}_t, c^{2*}_t)\}_{t=0}^{\infty}$. We want to show that there exists a SME with the same consumption allocation, i.e.

$$\hat{c}^i_t = c^{i*}_t, \ \forall t \geq 0 \text{ and } t = 1, 2$$

Define asset holdings for household i as

$$\hat{a}^i_{t+1} = \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}(c^{i*}_{t+\tau} - e^i_{t+\tau})}{p^*_{t+1}} \tag{2.7}$$

First note that $\{(\hat{c}^1_t, \hat{a}^1_{t+1}), (\hat{c}^2_t, \hat{a}^2_{t+1})\}_{t=0}^{\infty}$ certainly satisfy market clearing conditions. To see this, observe that $\forall t \geq 0$

$$\begin{aligned}
\hat{c}^1_t + \hat{c}^2_t &= c^{1*}_t + c^{2*}_t = e^1_t + e^2_t \\
\hat{a}^1_{t+1} + \hat{a}^2_{t+1} &= \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}\{(c^{1*}_{t+\tau} - e^1_{t+\tau}) + (c^{2*}_{t+\tau} - e^2_{t+\tau})\}}{p^*_{t+1}} = \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}(c^{1*}_{t+\tau} + c^{2*}_{t+\tau} - e^2_{t+\tau} - e^2_{t+\tau})}{p^*_{t+1}} = 0
\end{aligned}$$

It should be also noted that the consumption and asset holdings so constructed satisfies the SM budget constraint. To show this, we just need to check the following equality $\forall t \geq 0$.

$$\begin{aligned}
& \hat{c}^i_t + \hat{q}_t \hat{a}^i_{t+1} = e^i_t + \hat{a}^i_t \\
\Leftrightarrow \ & c^{i*}_t + \hat{q}_t \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}(c^{i*}_{t+\tau} - e^i_{t+\tau})}{p^*_{t+1}} = e^i_t + \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}(c^{i*}_{t+\tau} - e^i_{t+\tau})}{p^*_{t+1}} \\
\Leftrightarrow \ & c^{i*}_t + \hat{q}_t \sum_{\tau=1}^{\infty} \frac{p^*_{t+\tau}}{p^*_{t+1}}(c^{i*}_{t+\tau} - e^i_{t+\tau}) = e^i_t + \sum_{\tau=1}^{\infty} \frac{p^*_{t-1+\tau}}{p^*_t}(c^{i*}_{t-1+\tau} - e^i_{t-1+\tau}) \\
\Leftrightarrow \ & c^{i*}_t + \hat{q}_t \Big[(c^*_{t+1} - e^i_{t+\tau}) + \hat{q}_{t+1}(c^*_{t+2} - e^i_{t+2}) + \hat{q}_{t+1}\hat{q}_{t+2}(c^*_{t+3} - e^i_{t+3}) + \dots\Big] \\
& \quad = e^i_t + (c^{i*}_t - e^i_t) + \Big[\hat{q}_t(c^*_{t+1} - e^i_{t+\tau}) + \hat{q}_t\hat{q}_{t+1}(c^*_{t+2} - e^i_{t+2}) + \hat{q}_t\hat{q}_{t+1}\hat{q}_{t+2}(c^*_{t+3} - e^i_{t+3}) + \dots\Big] \\
\Leftrightarrow \ & c^{i*}_t + \hat{q}_t \Big[(c^*_{t+1} - e^i_{t+\tau}) + \hat{q}_{t+1}(c^*_{t+2} - e^i_{t+2}) + \hat{q}_{t+1}\hat{q}_{t+2}(c^*_{t+3} - e^i_{t+3}) + \dots\Big] \\
& \quad = e^i_t + (c^{i*}_t - e^i_t) + \hat{q}_t\Big[(c^*_{t+1} - e^i_{t+\tau}) + \hat{q}_{t+1}(c^*_{t+2} - e^i_{t+2}) + \hat{q}_{t+1}\hat{q}_{t+2}(c^*_{t+3} - e^i_{t+3}) + \dots\Big] \\
\Leftrightarrow \ & c^{i*}_t = c^{i*}_t
\end{aligned}$$

Next, we want to show that we can find a borrowing limit $\overline{A^i}$ large enough so that no Ponzi condition is never binding with asset levels given by 2.7. Recall 2.4 and $\sup_t \{e_t^1\}_{t=0}^{\infty} = \sup_t \{e_t^2\}_{t=0}^{\infty} < \infty$, and observe that

$$\hat{a}_{t+1}^i \geq -\sum_{\tau=1}^{\infty} \frac{p_{t+\tau}^* e_{t+\tau}^i}{p_{t+1}^*} \geq -\sum_{\tau=1}^{\infty} \xi^{\tau-1} e_{t+\tau}^i > -\infty$$

So that we can take

$$\overline{A^i} = 1 + \sup_t \left\{ \sum_{\tau=1}^{\infty} \xi^{\tau-1} e_{t+\tau}^i \right\} < \infty$$

This borrowing limit $\overline{A^i}$ is so high that household i, knowing that she can't run a Ponzi scheme, will never hit it.

It remains to show that $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ maximizes lifetime utility, subject to the sequential market budget constraints and the borrowing constraints defined by $\overline{A^i}$. Take any other allocation satisfying the SM budget constraints under equation 2.6. In step 1, we show that then this allocation would also satisfy the AD budget constraint and thus could have been chosen at ADE prices. If this alternative allocation would yield higher lifetime utility than the allocation $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty} = \{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$, it must have been chosen as part of an ADE, which it was not. Hence $\{(\hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ must be optimal within the set of allocations satisfying the SM budget constraints at equation 2.6.

[Step 3] Now, suppose $\{\hat{q}_t\}_{t=0}^{\infty}$ and $\{(\hat{c}_t^1, \hat{a}_{t+1}^1), (\hat{c}_t^2, \hat{a}_{t+1}^2)\}_{t=0}^{\infty}$ be a SME such that

$$a_{t+1}^i > -\overline{A^i}, \ \forall t \geq 0 \ \text{ and } \ i = 1, 2$$
$$q_t \leq \epsilon < 1, \ \forall t \geq 0$$

We want to show that there exists a corresponding ADE, $\{p_t^*\}_{t=0}^{\infty}$ and $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ such that

$$c_t^{i*} = \hat{c}_t^i, \ \forall t \geq 0 \text{ and } t = 1, 2$$

Again it is obvious that $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ satisfies market clearing and, as well as, the AD budget constraint. The only thing that remains to show is that it maximizes utility within the set of allocations satisfying the AD budget constraints, for prices $p_0^* = 1$ and $p_{t+1}^* = p_t^* q_t$. For any other allocation satisfying the AD budget constraint we could construct asset holdings such that this allocation together with the asset holdings satisfies the SM budget constraints. The only complication is that in the SM agent maximization problem there is an additional constraint, no Ponzi constraints. Thus the set over which we maximize in the AD case is larger, since the borrowing constraints are absent in the AD formulation, and we need to rule out that allocations that would violate SM no Ponzi conditions are optimal choices in the AD agent problem, at the equilibrium prices. However, by assumption the no Ponzi conditions are not binding at the SM equilibrium allocation, that is $a_{t+1}^i > -\overline{A^i}$ for all t. But for maximization problems with concave objective and convex constraint set (such as the SM agent maximization problem) if in the presence of the additional constraints $a_{t+1}^i > -\overline{A^i}$ for a maximizing choice these constraints are not binding, then this maximizer is also a maximizer of the relaxed problem with the constraint removed. Hence $\{(c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ is optimal for agent i within the set of allocations satisfying only the AD budget constraint. □

2.2 Endowment Economy with Uncertainty

2.2.1 Representation of Uncertainty

We start with the notion of an event.

Definition 2.8. [Event or State] Suppose that there are N possible events (or states) each period, which is drawn by "nature". We write

$$s_t \in S = \{1, 2, \dots, N\}$$

Note that S is assumed to be finite (need NOT be so in general), and S is independent on time.

Definition 2.9. [Event History] An event history s^t (from period 0 to period t) is a vector of length $t+1$ summarizing the realization of all events up to period t. Formally, we write

$$s^t = \underbrace{[s_0, s_1, \dots, s_t]}_{(t+1)} \in S^t = \underbrace{S \times S \cdots \times S}_{(t+1)} = \prod_{i=0}^{t} S_i$$

Thus, any event history s^t lies in S^t, the set of all possible event histories.

Definition 2.10. For each particular event history $s^t \in S^t$, we denote the probability of that event history by $\pi_t(s^t)$. It can be also understood as a function

$$\pi_t(s^t) : S^t \to [0, 1]$$

such that for each $t \in \{0, 1, 2, \dots\}$

$$\sum_{s^t \in S^t} \pi_t(s^t) = 1$$

Definition 2.11. The conditional probability of s^t given s^τ with $t > \tau$ is denoted by

$$\pi_t(s^t \mid s^\tau)$$

If we do not impose any transition probability among states across time [10], our notation allows the particular cases;

$$\pi_{t+1}(s_{t+1}) = \pi_{t+1}(s_{t+1} \mid s^t) = \pi_{t+1}(s_{t+1} \mid s_t)$$

Definition 2.12. [Consumption space] In general, consumption space in infinite discrete time horizon can be identified by restricted l^∞ space such that

$$C = \Big\{ \{c_t\}_{t=0}^{\infty} : 0 \le c_t < \infty, \ \forall t \ge 0 \Big\}$$

If we have uncertainty in economy, for each t, consumption stream should be indexed by event history. Thus, $c_t(s^t)$.

We now want to specify agent's preferences on C, consumption space. Basically, we assume that agents seek to maximize their expected lifetime utility where $\mathbb{E}_0$ is the usual expectation operator at period 0, prior to any realization of risk (in particular the risk with respect to s_0). Assuming that preferences admit a von-Neumann Morgenstern utility function representation (or expected utility framework), we can represent agents' preferences

[10] Most of the time, we assume a particular structure of the stochastic process for s_t. Often it is assumed to follow a Markov chain, which we will discuss in details later.

by [11]

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)u(c_t(s^t)) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(\underbrace{c_t(s^t)}_{\text{r.v.}})\right]$$

Also notice that we usually assume u is twice continuously differentiable, $u'(c) > 0$, $u''(c) < 0$, and Inada condition unless otherwise stated.

2.2.2 Markov Process

Most of the time, we assume a particular structure of the stochastic process for s_t. Often it is assumed to follow a Markov process.

Definition 2.13. [**Markov process**] A stochastic process $\{x_t\}$ is said to have the **Markov property** if for all k such that $1 \le k \le t$ and for all $t \ge 0$,

$$P(x_{t+1} \mid x_t, x_{t-1}, \dots, x_{t-k}) = P(x_{t+1} \mid x_t)$$

A stochastic process that has Markov property is called a **Markov process**. [12]

Suppose that there are N states $S = \{1, 2, \dots, N\}$. Define the **transition probability** such that

$$p_{ij} = P(s_{t+1} = j \mid s_t = i)$$

And assume time homogeneity, that is, p_{ij} does not change over time. (Not dependent on t)
Then, **transition matrix** which represents every transition probability of all possible $N \times N$ cases are

$$P_{(N\times N)} = (p_{ij}) = \begin{bmatrix} p_{11} & p_{12} & \cdots & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & \cdots & p_{NN} \end{bmatrix}$$

Note that for each ith row,

$$\sum_{j=1}^{N} p_{ij} = 1 \;\Leftrightarrow\; \sum_{j=1}^{N} P(s_{t+1} = j \mid s_t = i) = 1$$

Let probability distribution over states in period t, $\pi_t^i = P(s_t = i)$ with $\pi_t = [\pi_t^1, \pi_t^2, \dots, \pi_t^N]'$ such that $\sum_{i=1}^N \pi_t^i = 1$.
Note that [13]

$$\pi_{t+1}^j = \sum_{i=1}^{N} p_{ij}\pi_t^i$$

[11] In calculating expected lifetime utility level, we have to consider all possible event histories $s^t \in S^t$ at every period t. Thus, double sigma notation with index of time t and event history s^t is necessary.

[12] In short, only the last period information matters.

[13] $P(s_{t+1} = j) = \pi_{t+1}^j = \sum_{i=1}^{N} p_{ij}\pi_t^i = \sum_{i=1}^{N} P(s_{t+1} = j \mid s_t = i)P(s_t = i) = \sum_{i=1}^{N} P(s_{t+1} = j, s_t = i)$

and

$$\pi_{t+1}=\begin{bmatrix}\pi_{t+1}^1\\ \pi_{t+1}^2\\ \vdots\\ \vdots\\ \pi_{t+1}^N\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^N p_{i1}\pi_t^i\\ \sum_{i=1}^N p_{i2}\pi_t^i\\ \vdots\\ \vdots\\ \sum_{i=1}^N p_{iN}\pi_t^i\end{bmatrix}=P^T\pi_t$$

Also, $\pi_t=(P^T)^t\pi_0$.

Suppose that we start a Markov chain at $\tilde{\pi}_0$. Then

$$\pi_{t+1}(s^{t+1})=p(s_{t+1}\mid s_t)\times p(s_t\mid s_{t-1})\times\ldots\times p(s_1\mid s_0)\times\tilde{\pi}(s_0)$$

Now, define **stationary distribution** of $\bar{\pi}$ of a Markov chain $\{s_t\}$ such that

$$\overline{\pi}=P^T\overline{\pi}$$

Then, $\overline{\pi}$ is the eigenvector of P^T **(normalized to length 1)** associated with the eigenvalue 1.

Example 2.1. Let $S=\{1,2\}$ and $p_{11}=\frac{3}{4},\ p_{12}=\frac{1}{4},\ p_{21}=\frac{1}{4}$ and $p_{22}=\frac{3}{4}$. Then, [14]

$$P=(p_{ij})=\begin{bmatrix}\frac{3}{4}&\frac{1}{4}\\ \frac{1}{4}&\frac{3}{4}\end{bmatrix}$$

Since we are looking for the eigenvector associated with the eigenvalue 1, the $\bar{\pi}$ must satisfy

$$1\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{3}{4}&\frac{1}{4}\\ \frac{1}{4}&\frac{3}{4}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Thus, $x_1=x_2$ and since $x_1+x_2=1$, and therefore $\overline{\pi}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$.

2.2.3 Social Planner (Pareto) Problem

Let us assume that each agent has a stream of endowments $e_t^i(s^t)$. An allocation mechanism can be understood as a mapping from an economy into allocations for each agent i, where an allocation is denoted $\{(c_t^1(s^t),c_t^2(s^t),\ldots,c_t^I(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ and requires that the aggregate allocation is never greater than the aggregate endowment at any $t\geq 0$ and $s^t\in S^t$.
In this subsubsection, we will study an alternative allocation mechanism, which is performed by the social planner. Basically, the social planner seeks to solve

$$\max_{\{(c_t^1(s^t),c_t^2(s^t),\ldots,c_t^I(s^t))\}_{t=0,s^t\in S^t}^{\infty}}\sum_{i=1}^{I}\lambda_i\left[\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)u(c_t^i(s^t))\right]\text{ s.t }\sum_{i=0}^{I}c_t^i(s^t)\leq\sum_{i=0}^{I}e_t^i(s^t)=e_t(s^t)$$
$$\text{and}\ \ c_t^i(s^t)\geq 0,\ \forall i,\ \forall t\geq 0,\ \forall s^t\in S^t$$

where λ_i is a Pareto weight for agent i and $e_t(s^t)$ is an aggregate endowment at history s^t in period t. Notice that the social planner only has a feasibility constraint, and this social planner problem only cares about allocations, not price.

[14] Note that $P=P'$.

Assuming Inada condition being satisfied, the Lagrange function for this social planner problem can be written as

$$\mathcal{L} = \sum_{i=1}^{I} \lambda_i \left[\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)) \right] + \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \theta_t(s^t) \left[\sum_{i=0}^{I} e_t^i(s^t) - \sum_{i=0}^{I} c_t^i(s^t) \right]$$

Then, the first order condition for $c_t^i(s^t)$ is

$$\lambda_i \beta^t \pi_t(s^t) u'(c_t^i(s^t)) = \theta_t(s^t)$$

And hence, for any two agents i and j, we have

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \underbrace{\frac{\lambda_j}{\lambda_i}}_{\text{fixed}} \tag{2.8}$$

Notice that this is a key equation, which tells that marginal utilities of agents i and j move together given that λ_i and λ_j are time-invariant. This also can be interpreted that growth rates of marginal utilities of agent i and j should be equalized.

Moreover, since $u'(c)$ is strictly decreasing [15], we know that there exists an inverse function, $(u')^{-1}$. Thus, the equation 2.8 can be rewritten

$$c_t^i(s^t) = (u')^{-1} \left[\frac{\lambda_j}{\lambda_i} u'(c_t^j(s^t)) \right]$$

Plugging this into the feasibility condition yields, for any $t \geq 0$ and $s^t \in S^t$,

$$\sum_{i=0}^{I} (u')^{-1} \left[\frac{\lambda_j}{\lambda_i} u'(c_t^j(s^t)) \right] = e_t(s^t)$$

which implies that for any agent $j \in \{1, 2, \ldots . I\}$, the agent j's consumption level at history s^t in period only depends on aggregate endowment (or a function of aggregate endowment). In other words, individual level idiosyncratic risks are completely pulled out.

2.2.4 Date-0 Trade Competitive Equilibrium

Definition 2.14. [Arrow-Debreu securities] An Arrow–Debreu security is a history contingent claims to consumption traded in time 0. If one unit of an Arrow-Debreu security contingent on history s^t is purchased at $t = 0$, then one unit of consumption goods is delivered only if a particular history s^t is realized. Let us denote the price of Arrow-Debreu security contingent on history s^t by $p_t(s^t)$. [16]

Definition 2.15. [Arrow-Debreu competitive equilibrium] An Arrow-Debreu competitive equilibrium (of the endowment economy with uncertainty) consists of prices $\{p_t^*(s^t)\}_{t=0, s^t \in S^t}^{\infty}$ and allocations $\{(c_t^{1*}(s^t), c_t^{2*}(s^t), \ldots, c_t^{I*}(s^t))\}_{t=0, s^t \in S^t}^{\infty}$ such that

1) Given a sequence of prices $\{p_t^*(s^t)\}_{t=0, s^t \in S^t}^{\infty}$, $\{(c_t^{1*}(s^t), c_t^{2*}(s^t), \ldots, c_t^{I*}(s^t))\}_{t=0, s^t \in S^t}^{\infty}$ solves each agent's utility

[15] Because $u''(c) < 0$.

[16] We may use $q_t^0(s^t)$ following the notation used by Ljungqvist and Sargent.

maximization problem, i.e.

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t u(c_t^i(s^t))c_t^i(s^t) \geq 0, \text{ s.t } \sum_{t=0}^{\infty}\sum_{s^t\in S^t} p_t^* c_t^i \leq \sum_{t=0}^{\infty}\sum_{s^t\in S^t} p_t^* e_t^i \text{ and } \forall t \geq 0, \ \forall s^t \in S^t$$

2) The market clears

$$\sum_{i=1}^{I} c_t^{i*}(s^t) = \sum_{i=1}^{I} e_t^i(s^t) = e_t(s^t), \ \forall t \geq 0, \ \forall s^t \in S^t$$

Note that the budget constraint does not include probabilities $\pi_t(s^t)$ by itself. This is because, $p_t(s^t)$ is a function of probability, and hence already reflected in the budget constraint. Also notice that each agent has only a consolidated time-zero budget constraint.

Let us solve the general problem first. Let μ_i be the Lagrange multiplier for agent i's budget constraint. Assuming Inada condition being satisfied, the Lagrange function for for agent i can be written

$$\mathcal{L}^i = \sum_{t=0}^{\infty}\sum_{s^t\in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)) + \mu_i \left[\sum_{t=0}^{\infty}\sum_{s^t\in S^t} p_t^*(s^t) e_t^i(s^t) - \sum_{t=0}^{\infty}\sum_{s^t\in S^t} p_t^*(s^t) c_t^i(s^t) \right]$$

The first order condition for $c_t^i(s^t)$ is

$$\mu_i p_t^*(s^t) = \beta^t \pi_t(s^t) u'(c_t^i(s^t))$$

And hence, for any two agents i and j, we have

$$\frac{u'(c_t^j(s^t))}{u'(c_t^i(s^t))} = \underbrace{\frac{\mu_j}{\mu_i}}_{\text{fixed}} \tag{2.9}$$

Notice that the equation 2.9 is very similar to the equation 2.8 in the sense that it gives us the same prediction. That is, in this complete AD market, we can expect that marginal utilities of agents i and j move together, and each agent's consumption level at history s^t in period only depends on aggregate endowment.

The property that the solution sets of social planner problems and Arrow-Debreu competitive equilibrium are identical comes from the fact that the asset market allows agent to completely share risks between them by offering a full set of history contingent assets. [17] If not, this cannot be the case.

On the other hand, the price $p_t^*(s^t)$ is given by

$$p_t^*(s^t) = \frac{1}{\mu_i} \beta^t \pi_t(s^t) u'(c_t^i(s^t))$$

and we usually normalize $p_0^*(s^0) = 1$, so that the price $p_t^*(s^t)$ can be interpreted as a relative price of time zero goods. In addition, an ADE and social planner problem will have the same allocation if $\lambda_i = \frac{1}{\mu_i}$ for all i. In this case, we have

$$\begin{aligned} p_t^*(s^t) &= \lambda_i \beta^t \pi_t(s^t) u'(c_t^i(s^t)) \\ &= \theta_t(s^t) \end{aligned}$$

[17] This is also true in sequential market if all state dependent Arrow securities are traded. We will discuss this in the next subsubsection.

where $\theta_t(s^t)$ is a Lagrange multiplier for the social planner problem. This shows that the competitive equilibrium is equal to the shadow price in the social planner problem.

Example 2.2. [No Aggregate Uncertainty under CRRA utility function] Let $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ where $\gamma > 0$. Suppose that $s_t \in [0, 1]$ and endowment process is given by

$$e_t^1(s^t) = s_t \;\text{ and }\; e_t^2(s^t) = 1 - s_t$$

Note that we have no aggregate uncertainty, that is,

$$e_t(s^t) = 1, \;\forall t \geq 0, \forall s^t \in S^t$$

Using the equation 2.9, we have

$$\frac{u'(c_t^j(s^t))}{u'(c_t^i(s^t))} = \frac{\mu_j}{\mu_i} \;\Leftrightarrow\; \left(\frac{c_t^j(s^t)}{c_t^i(s^t)}\right)^{-\gamma} = \frac{\mu_j}{\mu_i} \;\Leftrightarrow\; c_t^i(s^t) = c_t^j(s^t)\left(\frac{\mu_j}{\mu_i}\right)^{\frac{1}{\gamma}}$$

which shows that consumption allocations are constant fraction of each other regardless of history s^t and period t. Using the market clearing, for any agent j, we can show that

$$\begin{aligned}
\sum_{i=1}^{I} c_t^{i*}(s^t) &= c_t^j(s^t)\left(\frac{\mu_j}{\mu_1}\right)^{\frac{1}{\gamma}} + \ldots + c_t^j(s^t)\left(\frac{\mu_j}{\mu_{j-1}}\right)^{\frac{1}{\gamma}} + c_t^j(s^t) + \ldots + c_t^j(s^t)\left(\frac{\mu_j}{\mu_I}\right)^{\frac{1}{\gamma}} \\
&= \underbrace{\left(\left(\frac{\mu_j}{\mu_1}\right)^{\frac{1}{\gamma}} + \ldots + \left(\frac{\mu_j}{\mu_{j-1}}\right)^{\frac{1}{\gamma}} + 1 + \ldots + \left(\frac{\mu_j}{\mu_I}\right)^{\frac{1}{\gamma}}\right)}_{\equiv s_j} c_t^j(s^t) \\
&= e_t(s^t)
\end{aligned}$$

Given that $\left(\left(\frac{\mu_j}{\mu_1}\right)^{\frac{1}{\gamma}} + \ldots + \left(\frac{\mu_j}{\mu_{j-1}}\right)^{\frac{1}{\gamma}} + 1 + \ldots + \left(\frac{\mu_j}{\mu_I}\right)^{\frac{1}{\gamma}}\right)$ is a constant, we can see that $c_t^j(s^t)$ is just a constant fraction of aggregate endowment at history s^t in period t.
Since their is no aggregate uncertainty in this economy, this directly implies that

$$c_t^{j*}(s^t) = s_j \underbrace{e_t(s^t)}_{=1} = s_j \equiv c^{j*}, \;\forall t \geq 0, \forall s^t \in S^t$$

Using the agent j's budget constraint with the price $p_t^*(s^t) = \frac{1}{\mu_i}\beta^t \pi_t(s^t) u'(c_t^i(s^t))$, we have

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) e_t^j(s^t) = c^{j*} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t)$$

and hence

$$c^{j*} = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) e_t^j(s^t)$$

2.2.5 Sequential Markets Equilibrium

Definition 2.16. [Arrow securities] Let $a_{t+1}^i(s^t, s_{t+1})$ denote the quantities of the Arrow security purchased in period t when history s^t is realized. It pays out $a_{t+1}^i(s^t, s_{t+1})$ units of the consumption good in period $t+1$, if s_{t+1}

is realized. Let $q_t(s^t, s_{t+1})$ denote the period t price of a unit of the Arrow security described.

Definition 2.17. [Sequential Markets Equilibrium] A sequential markets equilibrium (of the endowment economy with uncertainty) consists of initial assets $\{\hat{a}_0^i(s^0)\}_{i=1,s^0\in S^0}^I$, borrowing limits $\{\overline{A_t^i}(s^t, s_{t+1})\}_{i=1,t\geq 0,s^t\in S^t,s_{t+1}\in S}^I$, Arrow security prices $\{\hat{q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^\infty$ and allocations $\{\hat{c}_t^i(s^t), \hat{a}_{t+1}^i(s^t, s_{t+1})\}_{i=1,t\geq 0,s^t\in S^t,s_{t+1}\in S}^I$ such that

1) Given a sequence of Arrow security prices $\{\hat{q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^\infty$, $\{\hat{c}_t^i(s^t), \hat{a}_{t+1}^i(s^t, s_{t+1})\}_{i=1,t\geq 0,s^t\in S^t,s_{t+1}\in S}^I$ solves each agent's utility maximization problem,

$$
\begin{aligned}
&\max_{\{c_t^i(s^t), a_{t+1}^i(s^t,s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}}^\infty} \sum_{t=0}^\infty \sum_{s^t\in S^t} \beta^t \pi_t(s^t) u(c_t^i(s^t)) \text{ s.t} \\
c_t^i(s^t) + \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) &\leq e_t^i(s^t) + a_t^i(s^t),\ \forall t\geq 0,\ \forall s^t\in S^t \\
c_t^i(s^t) &\geq 0,\ \ \forall t\geq 0,\ \forall s^t \in S^t \\
a_{t+1}^i(s^t, s_{t+1}) &> -\overline{A^i}_{t+1}(s^t, s_{t+1}),\ \forall t\geq 0,\ \forall s^t\in S^t \text{ and } \forall s_{t+1}\in S
\end{aligned}
$$

2) The markets clear

$$
\begin{aligned}
\sum_{i=1}^I \hat{c}_t^i(s^t) =& \sum_{i=1}^I e_t^i(s^t),\ \ \forall t\geq 0,\ \forall s^t\in S^t \\
\sum_{i=1}^I \hat{a}_{t+1}^i(s^t) =& \ 0,\ \ \forall t\geq 0,\ \forall s^t \in S^t
\end{aligned}
$$

To solve this problem, consider the Lagrange function for agent i.

$$
\begin{aligned}
\mathcal{L}^i =& \sum_{t=0}^\infty \beta^t \sum_{s^t\in S^t} \pi_t(s^t) u(c_t^i(s^t)) + \sum_{t=0}^\infty \sum_{s^t\in S^t} \lambda_t^i(s^t) \Big(e_t^i(s^t) + a_t^i(s^{t-1}, s_t) - c_t^i(s^t) - \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \Big) \\
=& \sum_{t=0}^\infty \beta^t \sum_{s^t\in S^t} \Big[\pi_t(s^t) u(c_t^i(s^t)) + \tilde{\lambda}_t^i(s^t) \Big(e_t^i(s^t) + a_t^i(s^{t-1}, s_t) - c_t^i(s^t) - \sum_{s_{t+1}\in S} q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \Big) \Big]
\end{aligned}
$$

The first order condition for $c_t^i(s^t)$ and $a_{t+1}^i(s^t, s_{t+1})$ are

$$
\tilde{\lambda}_t^i(s^t) = \pi_t(s^t) u'(c_t^i(s^t)) \ \text{ and } \ \tilde{\lambda}_t^i(s^t) q_t(s^t, s_{t+1}) = \beta \tilde{\lambda}_{t+1}^i(s^{t+1})
$$

Therefore,

$$
\begin{aligned}
u'(c_t^i(s^t)) &= \frac{1}{\pi_t(s^t)} \cdot \frac{\beta}{q_t(s^t, s_{t+1})} \cdot \tilde{\lambda}_{t+1}^i(s^{t+1}) \\
&= \frac{1}{\pi_t(s^t)} \cdot \frac{\beta}{q_t(s^t, s_{t+1})} \cdot \pi_{t+1}(s^{t+1}) \cdot u'(c_{t+1}^i(s^t, s_{t+1})) \\
&= \beta \cdot \pi_{t+1}(s^{t+1} \mid s^t) \cdot \frac{u'(c_{t+1}^i(s^t, s_{t+1}))}{q_t(s^t, s_{t+1})}
\end{aligned}
$$

and hence

$$q_t(s^t, s_{t+1}) = \beta \cdot \pi_{t+1}(s^{t+1} \mid s^t) \cdot \frac{u'(c^i_{t+1}(s^t, s_{t+1}))}{u'(c^i_t(s^t))}$$

Using transversality condition with the budget constraint

$$\lim_{t\to\infty} \beta^{t+1} \sum_{s_{t+1}|s^t} \pi_{t+1}(s^{t+1} \mid s^t) u'(c^i_{t+1}(s^t, s_{t+1})) \cdot 1 \cdot a^i_{t+1}(s^t, s_{t+1}) = 0$$

and market clearing conditions yields the desired solution.

2.2.6 Relevant Empirical Research

In this subsubsection, we will discuss empirical papers trying to test whether market is indeed complete or not.

- **Mace(1991, JPE)**: Under the assumption $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, [18] we know that $c^j_t = \theta_j e_t$ if market is complete. Mace tries to test this argument in the following way. By taking the log for both sides,

  $$\log c^j_t = \log\theta_j + \log e_t$$

  Then, taking difference over time allows us to eliminate an individual fixed effect θ_j, so we have

  $$\underbrace{\triangle\log c^j_t}_{\text{growth rate of individual consumption}} = \underbrace{\triangle\log e_t}_{\text{growth rate of aggregate endowment}}$$

  Thus, we can expect that the growth rate of consumption should be equal to that of aggregate endowment, and it does not depend on personal idiosyncratic shock.

- Notice that at least two periods observation is needed for this regression. So, he uses CEX(1980-84) panel data, and does pooled OLS regression. His first regression equation is as follow. [19]

  $$\triangle\log c^j_t = \beta_1 \triangle\log y_t + \beta_2 \triangle\log y^j_t + \epsilon^j_t$$

  According to complete market framework, prefect risk sharing is possible and we should expect $\beta_1 = 1$ and $\beta_2 = 0$. For this regression,

- His second regression is

  $$\triangle\log c^j_t = \alpha_1 \triangle\log y_t + \sum_{k=1}^{n} \gamma_k E^j_{k,t} + \epsilon^j_t$$

  where $E^j_{k,t}$ is an employment state dummy [20] which also captures some personal idiosyncratic shock. Again, if market is complete, we should have $\alpha_1 = 1$ and $\gamma_k = 0$ for all k.

- His estimations show that $\beta_1^{***} = 1.06(0.08)$, $\beta_2^{***} = 0.04(0.007)$ with F statistics 14.12^* under $H_0 : \beta_1 = 1$ and $\beta_2 = 0$ in the first regression. In the second regression, he also shows that $\alpha_1^{***} = 1.05(0.08)$ with F statistics of 1.85^* under $H_0 : \alpha_1 = 1$ and $\gamma_k = 0$ for all k. In conclusion, full insurance or complete risk sharing prediction is hard to be supported, although there is also a mixed result as shown in the first regression.

[18] Mace also uses an exponential utility function.

[19] This OLS model can be messed up if there exist endogeneity. The key issue here is that ϵ^j_t may be correlated with y^j_t and c^j_t simultaneously. The preference shock also matters.

[20] He uses 1) employed as reference person and 2) employed as reference person and spouse.

- **Cochrane(1991, JPE)**: From the first order condition in the social planner problem, $\lambda_i \beta^t u'(c_t^i) = \frac{\theta_t}{\pi_t} = \theta_t'$, we can show that

$$\frac{\beta u'(c_{t+1}^i)}{u'(c_t^i)} = \underbrace{\frac{\theta_{t+1}'}{\theta_t'}}_{\text{common}}, \quad \forall i$$

  That is, stochastic discount factor $\frac{\beta u'(c_{t+1}^i)}{u'(c_t^i)}$ is unique in complete market, and hence must be constant across households.

- Under the assumption $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, he drives the following equation first.

$$\log\Big(\frac{c_{t+1}^i}{c_t^i}\Big) = \text{constant } - \frac{1}{\gamma}\log\Big(\frac{\theta_{t+1}'}{\theta_t'}\Big) + \epsilon_{t+1}^i$$

  which tells that the growth rate of consumption is not dependent on personal idiosyncratic shock. To check this argument, he considers the following regression equation.

$$\log\Big(\frac{c_{t+1}^i}{c_t^i}\Big) = \alpha + \beta' \underbrace{X_{t+1}^i} + \epsilon_{t+1}$$

  where X_{t+1}^i are variables such as days of work loss because of illness, job loss dummy variable, and the number of weeks spent looking for employment.

- Again, if complete market framework works here, $\boldsymbol{\beta} = \mathbf{0}$. Using PSID(1980-83) data, [21] he runs cross-sectional regression and find that $H_0: \boldsymbol{\beta} = \mathbf{0}$ should be rejected as shown in the following table.

Figure 2.1: Table 2 in Cochrane (1991, JPE)

TABLE 2

CROSS-SECTIONAL REGRESSIONS OF CONSUMPTION GROWTH ON IDIOSYNCRATIC VARIABLES AND TESTS FOR INDEPENDENCE

Composition Changes in Sample?	Regression Coefficient			p-Value of χ^2 Test (%)*	Number of Households in Category or Other Statistic		
1. Illness							
	Days	Days > 0	Days ≥ 100		Total	Days > 0	Days ≥ 100
No	−.05 (−2.36)	−.18 (−.78)	$\boxed{-14.22\ (-2.89)}$	43.04	1,738	870	99
Yes	−.06 (−3.74)	.70 (.40)	−11.27 (3.29)	53.16	4,614	2,689	333
2. Involuntary Job Loss							
					Total	Lost Jobs	
No	$\boxed{-24.03\ (-4.95)}$			−.065	1,173	76	
Yes	−26.74 (−7.81)			1.1E-10	3,373	291	
3. Weeks Job Search Given Involuntary Job Loss							
	Weeks	Weeks > 0			Weeks > 0	Mean Weeks If Weeks > 0	
No	−.34 (−1.10)	$\boxed{-5.65\ (-.58)}$		99.9	18	22.56	
Yes	−.73 (−3.83)	−15.92 (−1.90)		10.4	46	29.52	

[21] LHS growth rate in consumption from 1980-83, and RHS X_{t+1}^i cumulatively measured.

3 Part 3: Asset Pricing

3.1 Primers on Asset Pricing

Many asset pricing models assume complete markets and price an asset by breaking it into a sequence of history contingent claims, evaluating each component of that sequence with the relevant "state price deflator" $q_t^0(s^t)$, then adding up those values. The asset is redundant, in the sense that it offers a bundle of history contingent dated claims, each component of which has already been priced by the market. In this subsection, we will use $q_t^0(s^t)$ instead of $p_t(s^t)$ to denote Arrow-Debreu security prices following Ljungqvist and Sargent chapter 8 and 13.

1. Pricing redundant assets: Let $\{d_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ be a stream of claims on time t and history s^t consumption, then the time zero price of this asset is

$$P_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t) d_t(s^t)$$

2. Riskless consol: Suppose that $d_t(s^t) = 1$ for all $t \geq 0$ and $s^t \in S^t$, then the time zero price of this asset is

$$P_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} q_t^0(s^t)$$

3. One period return (or one period pricing kernel): The price of asset which pays one unit of consumption at history $s^{\tau+1}$ in period $\tau+1$ in terms of history s^τ in period τ is

$$\begin{aligned} \underbrace{q_{\tau+1}^{\tau}(s^{\tau+1})}_{\substack{\text{one period}\\ \text{pricing kernel at } \tau}} &\equiv \frac{q_{\tau+1}^0(s^{\tau+1})}{q_\tau^0(s^\tau)} \\ &= \frac{\beta^{\tau+1} u'(c_{\tau+1}^i(s^{\tau+1}))\pi_{\tau+1}(s^{\tau+1})}{\beta^\tau u'(c_\tau^i(s^\tau))\pi_\tau(s^\tau)} \\ &= \beta \frac{u'(c_{\tau+1}^i(s^{\tau+1}))}{u'(c_\tau^i(s^\tau))} \pi_{\tau+1}(s^{\tau+1} \mid s^\tau) \end{aligned}$$

4. If we want to find the price at time τ at history s^τ on a claim to a random payoff $\omega(s_{\tau+1})$, we use

$$\begin{aligned} p_\tau^\tau(s^\tau) &= \sum_{s_{\tau+1}} q_{\tau+1}^\tau(s^{t+1})\omega(s_{\tau+1}) \\ &= \sum_{s_{\tau+1}} \beta \frac{u'(c_{\tau+1}^i(s^{\tau+1}))}{u'(c_\tau^i(s^\tau))} \pi_{\tau+1}(s^{\tau+1} \mid s^\tau)\omega(s_{\tau+1}) \\ &= \sum_{s_{\tau+1}} \underbrace{\beta \frac{u'(c_{\tau+1}(s^{\tau+1}))}{u'(c_\tau(s^\tau))}}_{\text{common across } i} \pi_{\tau+1}(s^{\tau+1} \mid s^\tau)\omega(s_{\tau+1}) \\ &= \mathbb{E}_\tau\left[\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)} \omega(s_{\tau+1})\right] \end{aligned} \tag{3.1}$$

   Notice that we usually denote a stochastic discount factor $\beta \frac{u'(c_{\tau+1})}{u'(c_\tau)}$ by $m_{\tau+1}$.

5. Let $R_{\tau+1} \equiv \frac{\omega(s_{\tau+1})}{p_\tau^\tau(s^\tau)}$ be the one-period gross return on the asset. Then, for any asset, the equation 3.1 can be

rewritten

$$p_\tau^\tau(s^\tau) = \mathbb{E}_\tau\left[\beta\frac{u'(c_{\tau+1})}{u'(c_\tau)}\omega(s_{\tau+1})\right] \;\Leftrightarrow\; 1 = \mathbb{E}_\tau\left[m_{\tau+1}R_{\tau+1}\right] \tag{3.2}$$

3.2 Lucas Tree Pricing Model

3.2.1 Model Description

Now, we consider a representative agent model. In this economy, we have two types of assets, one is risk free bond, and the other one is Lucas tree share.

- Let B_t be an agent's one period risk free bond holdings between t and $t+1$.
- Let R_t be a gross interest rate of one period risk free bond, measured in units of time $t+1$ consumption goods per time t consumption goods. [22]
- Let N_t be an agent's holdings of equity share of Lucas tree between t and $t+1$.
- Let p_t be an unit equity share price of Lucas tree in period t.
- Let y_t be a dividend entitled to unit equity share of Lucas tree in period t.

Then, a representative agent seeks to solve

$$\max_{\{c_{t+j},N_{t+j},B_{t+j}\}_{j=0}^\infty} \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j u(c_{t+j})\right] \quad \text{s.t} \tag{3.3}$$

$$\begin{aligned} c_{t+j} + B_{t+j} + p_{t+j}N_{t+j} =& R_{t+j-1}B_{t+j-1} + (y_{t+j}+p_{t+j})N_{t+j-1},\ \forall j \geq 0 \\ N_{t+j-1} = 1,&\ B_{t+j-1} = 0 \ \text{ and no ponzi condition} \end{aligned}$$

where $0 < \beta < 1$, u is twice continuously differentiable, $u' > 0$, and $u'' < 0$.

Definition 3.1. [Competitive equilibrium in Lucas Tree Model] [23] [24] A competitive equilibrium (of the Lucas tree model) consists of prices $\{p_t^*, R_t^*\}_{t=0}^\infty$ and allocations $\{c_t^*, N_t^*, B_t^*\}_{t=0}^\infty$ such that

1) Given a sequence of prices $\{p_t^*, R_t^*\}_{t=0}^\infty$, $\{c_t^*, N_t^*, B_t^*\}_{t=0}^\infty$ solves a representative agent's utility maximization problem, 3.3

2) Given an exogenous process of $\{y_t\}_{t=0}^\infty$ and initial asset conditions, the markets clear, i.e.

$$c_t^* = y_t,\ N_t^* = 1, \quad \text{and} \quad B_t^* = 0,\ \forall t \geq 0$$

Before deriving the first order conditions, we first note that an optimal solution to the agent's maximization problem must also satisfy the following transversality conditions. [25]

$$\lim_{k\to\infty} \mathbb{E}_t[\beta^k u'(c_{t+k})B_{t+k}] = 0$$
$$\lim_{k\to\infty} \mathbb{E}_t[\beta^k u'(c_{t+k})p_{t+k}N_{t+k}] = 0$$

[22] R_t is known in period t.

[23] Here, we implicitly assume that we start from period 0 not t.

[24] Note that every variable is a function of y_t, and hence they should be written as $p_t(y_t)$, $R_t(y_t)$, $c_t(y_t)$, $N_t(y_t)$ and $B_t(y_t)$ rigorously. I omit (y_t) just for simpler notation.

[25] Notice that $c_{t+k} = y_{t+k}$ and $N_{t+k} = 1$ in competitive equilibrium.

Assuming an interior solution, the first order conditions for c_t, B_t, and N_t are

$$u'(c_t) = \lambda_t,\ \lambda_t = R_t\beta\mathbb{E}_t[\lambda_{t+1}], \quad \text{and} \quad \lambda_t p_t = \beta\mathbb{E}_t[(p_{t+1} + y_{t+1})\lambda_{t+1}]$$

which gives us two Euler equations associated with B_t and N_t

$$u'(c_t) = \beta R_t \mathbb{E}_t[u'(c_{t+1})] \ \Leftrightarrow\ 1 = \mathbb{E}_t[\beta\frac{u'(c_{t+1})}{u'(c_t)}R_t] \tag{3.4}$$

$$u'(c_t) = \beta\mathbb{E}_t[u'(c_{t+1})(\frac{p_{t+1} + y_{t+1}}{p_t})] \ \Leftrightarrow\ 1 = \mathbb{E}_t[\beta\frac{u'(c_{t+1})}{u'(c_t)}(\frac{p_{t+1} + y_{t+1}}{p_t})] \tag{3.5}$$

where R_t is the gross return on risk free bond whereas $(\frac{p_{t+1}+y_{t+1}}{p_t})$ is the gross return on Lucas tree. As you may notice, two equations 3.4 and 3.5 have exactly the same structure to the equation 3.2 that we found in complete market. Notice that Lucas tree economy is not complete in the sense that a full set of Arrow-Debreu securities is not provided, however, we get to the same conclusion.
Now, let us derive a competitive price p_t more explicitly from the equation 3.5. Using market clearing condition, it can be rewritten

$$\begin{aligned} u'(y_t)p_t &= \beta\mathbb{E}_t[u'(y_{t+1})(p_{t+1} + y_{t+1})] \\ &= \beta\mathbb{E}_t[u'(y_{t+1})p_{t+1}] + \beta\mathbb{E}_t[u'(y_{t+1})y_{t+1}] \end{aligned}$$

Since $u'(y_{t+1})p_{t+1} = \beta\mathbb{E}_{t+1}[u'(y_{t+2})(p_{t+2} + y_{t+2})]$, we have

$$\begin{aligned} u'(y_t)p_t &= \beta\mathbb{E}_t[u'(y_{t+1})y_{t+1}] + \beta\mathbb{E}_t[\beta\mathbb{E}_{t+1}[u'(y_{t+2})(p_{t+2} + y_{t+2})]] \\ &= \beta\mathbb{E}_t[u'(y_{t+1})y_{t+1}] + \beta^2\mathbb{E}_t[u'(y_{t+2})p_{t+2}] + \beta^2\mathbb{E}_t[u'(y_{t+2})y_{t+2}] \end{aligned}$$

Recursion on this infinitely yields

$$u'(y_t)p_t = \sum_{k=1}^{\infty} \beta^k\mathbb{E}_t[u'(y_{t+k})y_{t+k}] + \underbrace{\lim_{k\to\infty} \beta^k\mathbb{E}_t[u'(y_{t+k})p_{t+k}]}_{=0 \text{ by TVC}}$$

Thus,

$$p_t^* = \sum_{k=1}^{\infty} \mathbb{E}_t[\beta^k\frac{u'(y_{t+k})}{u'(y_t)}y_{t+k}] = \sum_{k=1}^{\infty} \mathbb{E}_t[m_{t,t+k}y_{t+k}]$$

which shows that the share price is an expected discounted value of a stream of dividends but with time-varying stochastic discount rates.

Example 3.1. Suppose that we have a linear utility function, so that a representative agent is risk-neutral. In this case, $\frac{u'(y_{t+j})}{u'(y_t)} = 1$, and hence $p_t^* = \sum_{k=1}^{\infty} \mathbb{E}_t[\beta^k y_{t+k}]$. Notice that only β is used as a discount factor, not stochastic one.

Example 3.2. Suppose that $u(c) = \log c$. In this case, $p_t^* = \sum_{k=1}^{\infty} [\beta^k\frac{u'(y_{t+k})}{u'(y_t)}y_{t+k}] = \sum_{k=1}^{\infty} [\beta^k\frac{y_t}{\cancel{y_{t+k}}}\cancel{y_{t+k}}]$. Thus, $p_t^* = \frac{\beta}{1-\beta}y_t$. Notice that p_t^* only depends on the current dividend y_t, i.e. a representative agent is myopic in some sense.

Example 3.3. Consider a deterministic economy. In this case, we do not need any expectation operator. Thus,

$p_t^* = \sum_{k=1}^{\infty} [\beta^k \frac{u'(y_{t+k})}{u'(y_t)} y_{t+k}]$. Given that $\beta^k \frac{u'(y_{t+k})}{u'(y_t)} = R_{t+k-1}^{-1}$, it can be also written as $p_t^* = \sum_{k=1}^{\infty} \frac{y_{t+k}}{R_{t+k-1}}$.

3.2.2 Extension of Model

- Let $B_{1,t}$ be a risk-free bond which pays one unit of consumption good after one period, and the corresponding price of this bond be $Q_{1,t}$.
- Let $B_{2,t}$ be a risk-free bond which pays one unit of consumption good after two periods, and the corresponding price of this bond be $Q_{2,t}$.
- The remaining set up is identical to the basic model.

Now, a representative agent seeks to solve

$$\max_{\{c_{t+j}, N_{t+j}, B_{1,t+j}, B_{2,t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right] \text{ s.t}$$

$$c_{t+j} + Q_{1,t+j} B_{1,t+j} + Q_{2,t+j} B_{2,t+j} + p_{t+j} N_{t+j} = B_{1,t+j-1} + Q_{1,t+j} B_{2,t+j-1} + (y_{t+j} + p_{t+j}) N_{t+j-1}, \ \forall j \geq 0$$
$$N_{t+j-1} = 1, \ B_{1,t+j-1} = B_{2,t+j-1} = 0 \ \text{ and no ponzi condition}$$

Then, the first order conditions for c_t, $B_{1,t}, B_{2,t}$ and N_t are

$$u'(c_t) = \lambda_t, \ \lambda_t Q_{1,t} = \beta \mathbb{E}_t[\lambda_{t+1}], \ \lambda_t Q_{2,t} = \beta \mathbb{E}_t[\lambda_{t+1} Q_{1,t+1}], \text{ and } \lambda_t p_t = \beta \mathbb{E}_t[\lambda_{t+1}(p_{t+1} + y_{t+1})]$$

and similarly we have the following transversality conditions.

$$\lim_{k \to \infty} \mathbb{E}_t[\beta^k u'(c_{t+k}) Q_{1,t+k} B_{1,t+k}] = 0$$
$$\lim_{k \to \infty} \mathbb{E}_t[\beta^k u'(c_{t+k}) Q_{2,t+k} B_{2,t+k}] = 0$$
$$\lim_{k \to \infty} \mathbb{E}_t[\beta^k u'(c_{t+k}) p_{t+k} N_{t+k}] = 0$$

With respect to $Q_{1,t}$ and $Q_{2,t}$, augmenting two first order conditions yield

$$Q_{1,t} = \mathbb{E}_t[\beta \frac{\lambda_{t+1}}{\lambda_t}] = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} \times 1]$$
$$Q_{2,t} = \mathbb{E}_t[\beta \frac{\lambda_{t+1}}{\lambda_t} Q_{1,t+1}] = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} Q_{1,t+1}]$$

Given that $Q_{1,t+1} = \mathbb{E}_{t+1}[\beta \frac{u'(c_{t+2})}{u'(c_{t+1})}]$, we have

$$Q_{2,t} = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} \mathbb{E}_{t+1}[\beta \frac{u'(c_{t+2})}{u'(c_{t+1})}]]$$
$$= \mathbb{E}_t[\beta^2 \frac{u'(c_{t+2})}{u'(c_t)}] \text{ or } \mathbb{E}_t[\beta^2 \frac{\lambda_{t+2}}{\lambda_t}]$$

According to pure expectation hypothesis, we need to have

$$Q_{2,t} = Q_{1,t} \mathbb{E}_t[Q_{1,t+1}] = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}] \mathbb{E}_t[Q_{1,t+1}]$$

But what we have is only

$$Q_{2,t} = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} Q_{1,t+1}]$$

Thus, in general, $Q_{2,t} = Q_{1,t}\mathbb{E}_t[Q_{1,t+1}]$ is not true. However, if a representative agent is risk neutral, or in a deterministic case, it can be the case.

For any $j \geq 1$, once we find $Q_{j,t}$, then we can calculate an average gross return on risk-free bond having j terms, i.e. $\widetilde{R}_{j,t} = (\frac{1}{Q_{j,t}})^{\frac{1}{j}}$, so that we can explore the term structure of interest rates.

Under Lucas tree economy framework, we have shown that

$$R_t^{-1} = \beta \mathbb{E}_t[\frac{u'(c_{t+1})}{u'(c_t)}]$$

From this equation, we can find that R_t and β is negatively correlated. In other words, when a representative agent is very patient (with high β), then low interest rate R_t induces $B_t = 0$ for market clearing.

Let us further assume that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ where $\gamma > 0$. Recall that a parameter γ represents two perspectives, risk aversion and inter-temporal elasticity of substitution. Given that IES for CRRA utility function is $\frac{1}{\gamma}$, we can interpret that higher γ means lower willingness to substitute between periods. In other words, consumption smoothing is more preferred when γ is high. Thus, if we have a CRRA utility function, we can rewrite

$$R_t = \beta^{-1}\mathbb{E}_t[(\frac{c_{t+1}}{c_t})^{\gamma}]$$

and hence find the followings.

- Expectation of consumption growth is positively correlated with R_t.
- Higher γ induces higher R_t. For consumption smoothing higher R_t is required.

3.2.3 Return Comparison between Several Assets

Now, let R_{t+1}^i be the gross return of any asset i between period t and $t+1$. Then, for any asset i, we can write

$$1 = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}^i] = \mathbb{E}_t[m_{t+1} R_{t+1}^i]$$

In the case of risk free bond, we write [26]

$$1 = \mathbb{E}_t[m_{t+1} R_t^f] \Leftrightarrow \frac{1}{R_t^f} = \mathbb{E}_t[m_{t+1}]$$

Let us define excess returns between asset i and j as $R_{t+1}^e = R_{t+1}^i - R_{t+1}^j$. Is $\mathbb{E}_t[R_{t+1}^e] = 0$? Intuitively, it sound plausible, however, what theory implies is not $\mathbb{E}_t[R_{t+1}^e] = 0$ but $\mathbb{E}_t[m_{t+1}R_{t+1}^e] = 0$. For further investigation, let us rewrite the first equation as below.

$$1 = \mathbb{E}_t[m_{t+1} R_{t+1}^i] \Leftrightarrow 1 = \mathbb{E}_t[m_{t+1}]\mathbb{E}_t[R_{t+1}^i] + Cov_t(m_{t+1}, R_{t+1}^i)$$

[26] Note that return on risk free bond is known in period t, so unlike any other asset i, we use subscript t instead of $t+1$.

Then, using the fact that $\frac{1}{R_t^f} = \mathbb{E}_t[m_{t+1}]$, we have

$$1 = \frac{1}{R_t^f}\mathbb{E}_t[R_{t+1}^i] + Cov_t(m_{t+1}, R_{t+1}^i)$$

Therefore,

$$\begin{aligned}
\mathbb{E}_t[R_{t+1}^i] - R_t^f &= -R_t^f Cov_t(m_{t+1}, R_{t+1}^i) \\
&= -R_t^f Cov_t(\beta\frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}^i) \\
&= \underbrace{-\frac{Cov_t(u'(c_{t+1}), R_{t+1}^i)}{\mathbb{E}_t[c_{t+1}]}}_{\text{risk adjustment term}}
\end{aligned} \tag{3.6}$$

Suppose that $u'(c_{t+1})$ and R_{t+1}^i are positively correlated. Then, we have $\mathbb{E}_t[R_{t+1}^i] < R_t^f$. Since higher $u'(c_{t+1})$ means that low level of consumption s_{t+1}, we may treat an asset i as something like insurance. Now, suppose that $u'(c_{t+1})$ and R_{t+1}^i is negatively correlated. Then, an asset i can be interpreted as an usual risk stock with higher premium than risk free bond. For empirical studies, the equation 3.6 can be also written as

$$\begin{aligned}
\mathbb{E}_t[R_{t+1}^i] - R_t^f &= -R_t^f Cov_t(m_{t+1}, R_{t+1}^i) \\
&= -\underbrace{\frac{Cov_t(m_{t+1}, R_{t+1}^i)}{Var_t(m_{t+1})}}_{\text{regression coefficient}}\frac{Var_t(m_{t+1})}{\mathbb{E}_t[m_{t+1}]}
\end{aligned}$$

It should be very important to notice that we never use variances of assets in asset pricing. Then, where does variance of assets come from? For this end, let us recall the **Sharpe Ratio** which is defined by

$$\frac{\mathbb{E}_t[R_{t+1}^i] - R_t^f}{\sigma(R_{t+1}^i)} \tag{3.7}$$

In terms of long-run, we drop subscript t. Then, $1 = \mathbb{E}[m]\mathbb{E}[R^i] + Cov(m, R^i)$ with Sharpe ratio gives us

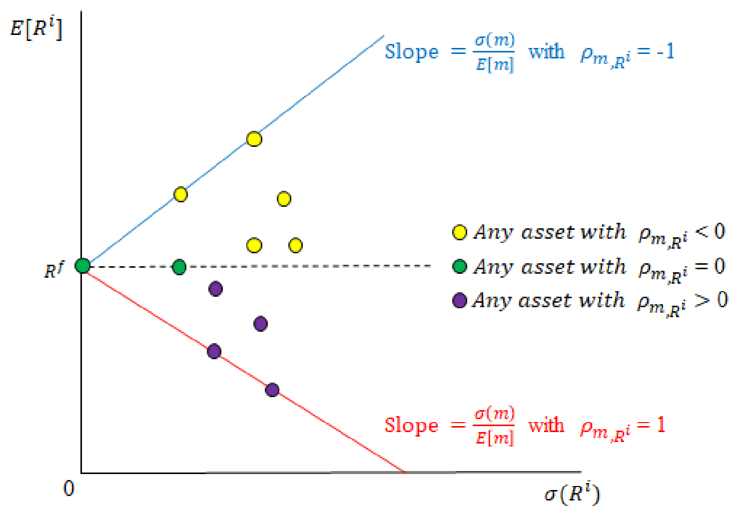
$$\begin{aligned}
1 = \mathbb{E}[m]\mathbb{E}[R^i] + Cov(m, R^i) \;&\Leftrightarrow\; 1 = \mathbb{E}[m]\mathbb{E}[R^i] + \rho_{m,R^i}\sigma(R^i)\sigma(m) \\
&\Leftrightarrow\; \frac{1}{\mathbb{E}[m]} = \mathbb{E}[R^i] + \rho_{m,R^i}\frac{\sigma(m)}{\mathbb{E}[m]}\sigma(R^i) \\
&\Leftrightarrow\; \frac{\mathbb{E}[R^i] - R_f}{\sigma(R^i)} = -\frac{\rho_{m,R^i}\sigma(m)}{\mathbb{E}[m]}
\end{aligned}$$

Thus,

$$\left|\frac{\mathbb{E}[R^i] - R_f}{\sigma(R^i)}\right| \leq \frac{\sigma(m)}{\mathbb{E}[m]} \;\Leftrightarrow\; \left|\mathbb{E}[R^i] - R_f\right| \leq \frac{\sigma(m)}{\mathbb{E}[m]}\sigma(R^i)$$

The figure below shows what mean-variance frontier looks like. Note that all assets should be located in the wedge. The assets lie on the frontier if and only if $\rho_{m,R^i} = \pm 1$, i.e. m and R^i are perfectly correlated. Although we use $\sigma(R^i)$ term, asset pricing has nothing to do with variances of assets as explained before. Only covariance matters.

Figure 3.1: Mean-variance frontier



3.2.4 Equity Premium Puzzle

We choose to proceed in the fashion of Hansen and Singleton (1983) to illuminate the equity premium puzzle. Let the real rates of return on stocks and bonds between periods t and $t+1$ be denoted $1 + r^s_{t+1}$ and $1 + r^b_{t+1}$, respectively. Then, we have

$$1 = \mathbb{E}_t[\beta \frac{u'(c_{t+1})}{u'(c_t)}(1 + r^i_{t+1})], \ i = s, b \tag{3.8}$$

We posit exogenous stochastic processes for both endowments (consumption) and rates of return,

$$\frac{c_{t+1}}{c_t} = \bar{c}_\triangle \exp(\epsilon_{c,t+1} - \frac{\sigma_c^2}{2}) \ \text{ and } \ 1 + r^i_{t+1} = (1 + \bar{r}^i)\exp(\epsilon_{i,t+1} - \frac{\sigma_i^2}{2}), \ i = s, b$$

where $\{\epsilon_{c,t+1}, \epsilon_{s,t+1}, \epsilon_{b,t+1}\}$ are jointly normally distributed with zero means and variances $\{\sigma_c^2, \sigma_s^2, \sigma_b^2\}$. Further assume $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, and take unconditional expectation of the equation 3.8

$$\begin{aligned} 1 &= \beta\mathbb{E}[(\frac{c_{t+1}}{c_t})^{-\gamma}(1 + r^i_{t+1})] \\ &= \beta\mathbb{E}\left[\left\{\bar{c}_\triangle \exp(\epsilon_{c,t+1} - \frac{\sigma_c^2}{2})\right\}^{-\gamma}(1 + \bar{r}^i)\exp(\epsilon_{i,t+1} - \frac{\sigma_i^2}{2})\right] \\ &= \beta(1 + \bar{r}^i)\bar{c}_\triangle^{-\gamma}\mathbb{E}\left[\exp(-\gamma\{\epsilon_{c,t+1} - \frac{\sigma_c^2}{2}\} + \epsilon_{i,t+1} - \frac{\sigma_i^2}{2})\right] \end{aligned}$$

Notice that $\exp(-\gamma\{\epsilon_{c,t+1} - \frac{\sigma_c^2}{2}\} + \epsilon_{i,t+1} - \frac{\sigma_i^2}{2})$ follows a log normal distribution with mean of $-\frac{\sigma_i^2}{2} + \frac{\gamma\sigma_c^2}{2}$ and variances of $\gamma^2\sigma_c^2 + \sigma_i^2 - 2\gamma Cov(\epsilon_c, \epsilon_i)$. Therefore,

$$\begin{aligned} \mathbb{E}\left[\exp(-\gamma\{\epsilon_{c,t+1} - \frac{\sigma_c^2}{2}\} + \epsilon_{i,t+1} - \frac{\sigma_i^2}{2})\right] &= \exp(-\frac{\sigma_i^2}{2} + \frac{\gamma\sigma_c^2}{2} + \frac{1}{2}\{\gamma^2\sigma_c^2 + \sigma_i^2 - 2\gamma Cov(\epsilon_c, \epsilon_i)\}) \\ &= \exp(\frac{1}{2}\gamma(\gamma+1)\sigma_c^2 - \gamma Cov(\epsilon_c, \epsilon_i)) \end{aligned}$$

Thus,

$$1 = \beta(1+\overline{r}^i)\overline{c}_{\triangle}^{-\gamma}\exp(\frac{1}{2}\gamma(\gamma+1)\sigma_c^2 - \gamma Cov(\epsilon_c, \epsilon_i))$$

Taking the log and rearranging terms yields

$$\log(1+\overline{r}^i) = -\log\beta + \gamma\log\overline{c}_{\triangle} - \frac{\gamma(\gamma+1)\sigma_c^2}{2} + \gamma Cov(\epsilon_c, \epsilon_i) \tag{3.9}$$

It is informative to interpret the equation 3.9 in terms of risk free bond case where $Cov(\epsilon_c, \epsilon_i) = 0$.

- In the case of risk neutral agents, $\gamma = 0$, we have $(1+\overline{r}^i) = \frac{1}{\beta}$, which has the familiar implication that the interest rate is equal to the inverse of the subjective discount factor.

- In the case of deterministic growth where $\sigma_c^2 = 0$, we have $\log(1+\overline{r}^i) = -\log\beta + \gamma\log\overline{c}_{\triangle}$. The second term says that the safe interest rate is positively related to γ.

- In the case of uncertainty, the third term, $-\frac{\gamma(\gamma+1)\sigma_c^2}{2}$, says that higher consumption volatility induces the downward pressure on the interest rate as precautionary saving incentives work.

We now turn to the equity premium by taking the difference between the expressions 3.9 for $i = s, b$. That is,

$$\log(1+\overline{r}^s) - \log(1+\overline{r}^b) = \gamma[Cov(\epsilon_c, \epsilon_s) - \underbrace{Cov(\epsilon_c, \epsilon_b)}_{=0}]$$

which can be written approximately

$$\overline{r}^s - \overline{r}^b \simeq \gamma Cov(\epsilon_c, \epsilon_s)$$

After approximating with the covariance between consumption growth and real yields on stocks, an equity premium 6% would require a $\gamma = 27$, which is unacceptable in economics.
What if we accept $\gamma = 27$? The answer is that we face the risk free rate puzzle. As you can see below, higher γ cannot explain very small risk free rate $\overline{r}^b$ in the market.

$$\log(1+\overline{r}^b) = \underbrace{-\log\beta}_{>0} + \underbrace{\gamma\log\overline{c}_{\triangle}}_{\gg 0} - \underbrace{\frac{\gamma(\gamma+1)\sigma_c^2}{2}}_{\simeq 0}$$

4 Part 4: Real Business Cycle Model

4.1 Solving Linear Expectation Models

In general, **solving a model** means deriving each endogenous variable as a function of exogenous variables or finding a (time invariant) policy function of (endogenous and exogenous) state variables.

4.1.1 Simple Cases

Consider the following AR1 process.

$$y_{t+1} = \rho y_t + \epsilon_{t+1} \text{ where } \mid \rho \mid < 1 \text{ and } y_0 \text{ is given} \tag{4.1}$$

Now, we want to express y_{t+1} as a function of exogenous variables $\{\epsilon_t\}$ and y_0. Using backward substitution, for each $t \geq 0$, we have

$$\begin{aligned}
y_{t+1} &= \rho[\rho y_{t-1} + \epsilon_t] + \epsilon_{t+1} = \rho^2 y_{t-1} + \rho\epsilon_t + \epsilon_{t+1} \\
&= \rho^2[\rho y_{t-2} + \epsilon_{t-1}] + \rho\epsilon_t + \epsilon_{t+1} = \rho^3 y_{t-2} + \rho^2\epsilon_{t-1} + \rho\epsilon_t + \epsilon_{t+1} \\
&= \qquad \vdots \\
&= \sum_{j=0}^{t} \rho^j \epsilon_{t+1-j} + \rho^{t+1} y_0
\end{aligned}$$

Now, consider the following process.

$$\mathbb{E}_t[y_{t+1}] = b y_t + \epsilon_t \tag{4.2}$$

Assume that $\mid b \mid > 1$. Using forward substitution

$$\begin{aligned}
y_t &= -\frac{1}{b}\epsilon_t + \frac{1}{b}\mathbb{E}_t[y_{t+1}] = -\frac{1}{b}\epsilon_t + \frac{1}{b}\mathbb{E}_t\Big[-\frac{1}{b}\epsilon_{t+1} + \frac{1}{b}\mathbb{E}_{t+1}[y_{t+2}]\Big] \\
&= -\frac{1}{b}\epsilon_t - (\frac{1}{b})^2\mathbb{E}_t[\epsilon_{t+1}] + \frac{1}{b^2}\mathbb{E}_t[y_{t+2}] \\
&= \qquad \vdots \\
&= -\frac{1}{b}\sum_{j=0}^{\infty}(\frac{1}{b})^j\mathbb{E}_t[\epsilon_{t+j}] + \lim_{j\to\infty}(\frac{1}{b})^j\underbrace{\mathbb{E}_t[y_{t+j}]}_{<\infty}
\end{aligned}$$

Thus, for each $t \geq 0$, y_t is uniquely determined and bounded. For example, let $\epsilon_t = \rho\epsilon_{t-1} + v_t$ where $v_t \sim (0, \sigma_v^2)$. Then, $\mathbb{E}_t[\epsilon_{t+j}] = \rho^j\epsilon_t$, and therefore

$$\begin{aligned}
y_t &= -\frac{1}{b}\sum_{j=0}^{\infty}(\frac{1}{b})^j\mathbb{E}_t[\epsilon_{t+j}] = -\frac{1}{b}\sum_{j=0}^{\infty}(\frac{1}{b})^j\mathbb{E}_t[\rho^j\epsilon_t] \\
&= -\frac{1}{b}\frac{\epsilon_t}{1-\frac{\rho}{b}} = -\frac{\epsilon_t}{b-\rho}
\end{aligned}$$

Consider the same process with 4.2, but assume that $\mid b \mid < 1$. Then, it is easy to see that any bounded solution such that $y_{t+1} = by_t + \epsilon_t + \eta_{t+1}$ with $\mathbb{E}_t[\eta_{t+1}] = 0$ satisfies the process, and hence multiple solutions exist.

4.1.2 Second-Order Difference Equation Using Lag Operator

Consider the following second-order difference equation.

$$\mathbb{E}_t[X_{t+1}] = aX_t + bX_{t-1} + \epsilon_t \tag{4.3}$$

Now, we want to find a (time-invariant) policy function of state (endogenous and exogenous) variables. Define the lag operator such that

$$LX_t = X_{t-1} \quad \text{and} \quad L^{-1}X_t = X_{t+1}$$

Then, we can rewrite the equation 4.3 as

$$\begin{aligned}
\mathbb{E}_t[L^{-1}X_t] = aX_t + bLX_t + \epsilon_t \quad &\Leftrightarrow \quad \mathbb{E}_t[(L^{-1} - a - bL)X_t] = \epsilon_t \\
&\Leftrightarrow \quad \mathbb{E}_t[(L^{-2} - aL^{-1} - b)LX_t] = \epsilon_t
\end{aligned}$$

Consider $L^{-2} - aL^{-1} - b$ as a quadratic equation with respect to L^{-1}, then there must be two roots (Need not be real numbers) for this equation, say μ_1 and μ_2. Assume that $| \mu_1 | < 1$ and $| \mu_2 | > 1$, then

$$\begin{aligned}
\mathbb{E}_t[(L^{-2} - aL^{-1} - b)LX_t] = \epsilon_t \quad &\Leftrightarrow \quad \mathbb{E}_t[(L^{-1} - \mu_1)(L^{-1} - \mu_2)LX_t] = \epsilon_t \\
&\Leftrightarrow \quad \underbrace{\mathbb{E}_t[(L^{-1} - \mu_1)LX_t]}_{=X_t - \mu_1 X_{t-1}} = \mathbb{E}_t\left[\frac{\epsilon_t}{L^{-1} - \mu_2}\right]
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\epsilon_t}{L^{-1} - \mu_2}\right] &= -\mathbb{E}_t\left[\frac{\mu_2^{-1}\epsilon_t}{1 - L^{-1}\mu_2^{-1}}\right] \\
&= -\mu_2^{-1}\mathbb{E}_t\left[\sum_{j=0}^{\infty} (L\mu_2)^{-j}\epsilon_t\right] \\
&= -\mu_2^{-1}\mathbb{E}_t\left[\sum_{j=0}^{\infty} \mu_2^{-j}\underbrace{L^{-j}\epsilon_t}_{=\epsilon_{t+j}}\right] \\
&= -\mu_2^{-1}\sum_{j=0}^{\infty} \mu_2^{-j}\mathbb{E}_t[\epsilon_{t+j}]
\end{aligned}$$

Therefore,

$$X_t = \mu_1 X_{t-1} - \mu_2^{-1}\sum_{j=0}^{\infty} \mu_2^{-j}\mathbb{E}_t[\epsilon_{t+j}]$$

If ϵ_t follows AR1 process such that $\epsilon_t = \rho\epsilon_{t-1} + v_t$ where $\mathbb{E}_{t-1}[v_t] = 0$, then we have

$$X_t = \mu_1 X_{t-1} - \frac{\epsilon_t}{\mu_2 - \rho}$$

4.1.3 The Method of Undetermined Coefficients

Consider the same second-order difference equation with the equation 4.3.

$$\mathbb{E}_t[X_{t+1}] = aX_t + bX_{t-1} + \epsilon_t \tag{4.4}$$

where $\epsilon_t = \rho\epsilon_{t-1} + \eta_t$ such that $\mathbb{E}_{t-1}[\eta_t] = 0$. Guess $X_t = GX_{t-1} + H\epsilon_t$, and determine G and H by comparing the coefficients of two different representations. That is,

$$\begin{aligned}
\mathbb{E}_t[X_{t+1}] &= \mathbb{E}_t[GX_t + H\epsilon_{t+1}] \\
&= GX_t + H\rho\epsilon_t \\
&= G[GX_{t-1} + H\epsilon_t] + H\rho\epsilon_t \\
&= G^2 X_{t-1} + [GH + H\rho]\epsilon_t
\end{aligned}$$

On the other hand

$$\begin{aligned}
aX_t + bX_{t-1} + \epsilon_t &= a[GX_{t-1} + H\epsilon_t] + bX_{t-1} + \epsilon_t \\
&= [aG + b]X_{t-1} + [aH + 1]\epsilon_t
\end{aligned}$$

Then, we have the following two equations

$$G^2 = aG + b \quad \text{and} \quad GH + H\rho = aH + 1$$

Solve $G^2 = aG + b$ with respect to G and take G^* such that $\mid G^* \mid < 1$, (for stationary solution) and plug G^* into the second equation to obtain H^*.

4.1.4 Blanchard-Kahn Method

Blanchard and Kahn use an eigenvalue decomposition to decouple a system of expectational difference equations for its stable and unstable block. In order to understand this method, we first need to distinguish between predetermined variables and forward-looking variables.

Definition 4.1. [Predetermined and forward-looking variable] A variable x_t is said to be **predetermined** if it depends on information at $t-1$, and **forward-looking** if it depends on information at t.

Now, consider a model in the following form

$$\underbrace{A_{(n\times n)} \begin{bmatrix} \mathbb{E}_t[\boldsymbol{x}_{1,t+1}] \\ \boldsymbol{x}_{2,t+1} \end{bmatrix} = B_{(n\times n)} \begin{bmatrix} \boldsymbol{x}_{1,t} \\ \boldsymbol{x}_{2,t} \end{bmatrix} + C_{(n\times m)}\boldsymbol{\epsilon}_t}_{\text{state space form representation}} \;\Leftrightarrow\; A\mathbb{E}_t[\boldsymbol{x}_{t+1}] = B\boldsymbol{x}_t + C\boldsymbol{\epsilon}_t \tag{4.5}$$

where $\boldsymbol{x}_0$ is given, $\boldsymbol{x}_{1,t}$ is an $n_1 \times 1$ vector of endogenous forward-looking variables, $\boldsymbol{x}_{2,t}$ is an $n_2 \times 1$ vector of endogenous predetermined variables, and $\boldsymbol{\epsilon}_t$ is an $m \times 1$ vector of exogenous shocks such that $\boldsymbol{\epsilon}_t = R\boldsymbol{\epsilon}_{t-1} + M\boldsymbol{u}_t$. Let $n_1 + n_2 = n$. The goal is to solve the equation and get an $n \times 1$ vector such that

$$\boldsymbol{x}_{t+1} = F\boldsymbol{x}_t + G_1\boldsymbol{\epsilon}_{t+1} + G_0\boldsymbol{\epsilon}_t$$

Further assume that $A_{(n\times n)}$ is invertible, then we can rewrite the equation 4.5 as

$$\begin{aligned}
\mathbb{E}_t[\boldsymbol{x}_{t+1}] &= \underbrace{A^{-1}B}_{\equiv D}\boldsymbol{x}_t + \underbrace{A^{-1}C}_{\equiv E}\boldsymbol{\epsilon}_t \\
&= D_{(n\times n)}\boldsymbol{x}_t + E_{(n\times m)}\boldsymbol{\epsilon}_t
\end{aligned}$$

Now, we apply the spectral (eigenvalue) decomposition (or more generally Jordan decomposition) to the matrix D and get the following.

$$D = P\Lambda P^{-1}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and each λ_i is an eigenvalue for D, and $P = (\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n)$ where $\boldsymbol{e}_i$ is the eigenvector corresponding to λ_i. Notice that if λ_i are distinct, then we exactly have $D = P\Lambda P^{-1}$, and if some of the eigenvalues of D are repeated, then we need to apply Jordan decomposition of D. For simplicity, assume that we have distinct λ_is. Then,

$$\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_{(n_1\times n_1)} & \mathbf{0}_{(n_1\times n_2)} \\ \mathbf{0}_{(n_2\times n_1)} & \boldsymbol{\lambda}_{(n_2\times n_2)} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_{(n_1+n_2)} \end{bmatrix}$$

where $\mid \lambda_1 \mid > \mid \lambda_2 \mid > \ldots > \mid \lambda_{n_1+n_2} \mid$ in the descending order down the diagonal. Let the number of eigenvalues outside the unit circle as n_U, and the number of eigenvalues inside the unit circle as $n_S = n - n_U$.

Theorem 4.1. [Blanchard Kahn]

1) If $n_U = n_1$, then the system is saddle-path stable and a unique solution exists.

2) If $n_U > n_1$, no boundary solution exists.

3) If $n_U < n_1$, multiple solution exist.

where n_1 is the number of forward-looking variables.

4.1.5 The Principles of Log Linearization

Let $F(x_1, x_2, \ldots, x_n) = 0$ and $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]'$. Define $\overline{\boldsymbol{x}} = [\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}]'$ as the non-stochastic steady state. Using the first-order Taylor expansion around the steady state, we have

$$F(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} + \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} (x_i - \overline{x_i})$$

Now, we can rewrite the first-order Taylor expansion above as

$$F(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} + \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \left(\frac{x_i - \overline{x_i}}{\overline{x_i}}\right)\overline{x_i} \tag{4.6}$$

Notice that

$$\log x_i - \log \overline{x_i} = \log\left(\frac{x_i}{\overline{x_i}}\right) = \log\left(1 + \frac{x_i - \overline{x_i}}{\overline{x_i}}\right) \approx \frac{x_i - \overline{x_i}}{\overline{x_i}}$$

Let us denote the log deviation of a variable from its corresponding steady state value by

$$\widehat{x}_t \equiv \log x_i - \log \overline{x_i} \approx \frac{x_i - \overline{x_i}}{\overline{x_i}}$$

Then, we can write the equation 4.6 as

$$F(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} + \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n) \mid_{\boldsymbol{x}=\overline{\boldsymbol{x}}} \widehat{x}_t\overline{x_i}$$

Example 4.1. Let $x_t y_t = z_t$. Then,

$$\begin{aligned}
\overline{xy} + \overline{y}(x_t - \overline{x}) + \overline{x}(y_t - \overline{y}) = \overline{z} + (z_t - \overline{z}) \quad &\Leftrightarrow \quad \overline{y}(x_t - \overline{x}) + \overline{x}(y_t - \overline{y}) = (z_t - \overline{z}) \\
&\Leftrightarrow \quad \overline{yx}\frac{(x_t - \overline{x})}{\overline{x}} + \overline{yx}\frac{(y_t - \overline{y})}{\overline{y}} = \overline{z}\frac{(z_t - \overline{z})}{\overline{z}} \\
&\Leftrightarrow \quad \frac{(x_t - \overline{x})}{\overline{x}} + \frac{(y_t - \overline{y})}{\overline{y}} = \frac{(z_t - \overline{z})}{\overline{z}} \\
&\Leftrightarrow \quad \widehat{x}_t + \widehat{y}_t = \widehat{z}_t
\end{aligned}$$

Example 4.2. Let $x_t + y_t = z_t$. Then,

$$\begin{aligned}
\overline{x} + (x_t - \overline{x}) + \overline{y} + (y_t - \overline{y}) = \overline{z} + (z_t - \overline{z}) \quad &\Leftrightarrow \quad (x_t - \overline{x}) + (y_t - \overline{y}) = (z_t - \overline{z}) \\
&\Leftrightarrow \quad \overline{x}\frac{(x_t - \overline{x})}{\overline{x}} + \overline{y}\frac{(y_t - \overline{y})}{\overline{y}} = \overline{z}\frac{(z_t - \overline{z})}{\overline{z}} \\
&\Leftrightarrow \quad \overline{x}\widehat{x}_t + \overline{y}\widehat{y}_t = \overline{z}\widehat{z}_t \\
&\Leftrightarrow \quad \frac{\overline{x}}{\overline{x} + \overline{y}}\widehat{x}_t + \frac{\overline{y}}{\overline{x} + \overline{y}}\widehat{y}_t = \widehat{z}_t
\end{aligned}$$

Whenever log linearizations are well-defined, you can use the following useful formula.

- $\widehat{xy} = \widehat{x} + \widehat{y}$, $\widehat{\alpha x} = \widehat{x}$, $\widehat{x^\alpha} = \alpha\widehat{x}$, $\widehat{x/y} = \widehat{x} - \widehat{y}$ and $\widehat{x+y} = \frac{\overline{x}}{\overline{x}+\overline{y}}\widehat{x} + \frac{\overline{y}}{\overline{x}+\overline{y}}\widehat{y}$.
- $\widehat{x-y} = \frac{\overline{x}}{\overline{x}-\overline{y}}\widehat{x} - \frac{\overline{y}}{\overline{x}-\overline{y}}\widehat{y}$, and $\widehat{x+\alpha} = \frac{\overline{x}}{\overline{x}+\overline{\alpha}}\widehat{x} + \frac{\overline{\alpha}}{\overline{x}+\overline{\alpha}}\underbrace{\widehat{\alpha}}_{=0} = \frac{\overline{x}}{\overline{x}+\overline{\alpha}}\widehat{x}$.
- $\widehat{xy+z} = \frac{\overline{xy}}{\overline{xy}+\overline{z}}\widehat{xy} + \frac{\overline{z}}{\overline{xy}+\overline{z}}\widehat{z} = \frac{\overline{xy}}{\overline{xy}+\overline{z}}(\widehat{x} + \widehat{y}) + \frac{\overline{z}}{\overline{xy}+\overline{z}}\widehat{z}$.

Another issue is an expectation operator. How can we log linearize an equation with an expectation operator? Suppose that there are two states $j = 1, 2$ with probabilities π_1 and π_2 and a random variable x takes on x_1 and x_2 in state 1 and 2, respectively. Then

$$\mathbb{E}(x) = \pi_1 x_1 + \pi_2 x_2$$

In the steady state, $x_1 = x_2 = \overline{x}$, and observe that

$$\begin{aligned}
\widehat{\mathbb{E}(x)} &= \pi_1 \widehat{x_1 + \pi_2} x_2 \\
&= \frac{\overline{\pi_1 x_1}}{\overline{\pi_1 x_1} + \overline{\pi_2 x_2}}\widehat{\pi_1 x_1} + \frac{\overline{\pi_2 x_2}}{\overline{\pi_1 x_1} + \overline{\pi_2 x_2}}\widehat{\pi_2 x_2} = \frac{\pi_1 \overline{x_1}}{\pi_1 \overline{x_1} + \pi_2 \overline{x_2}}\widehat{x_1} + \frac{\pi_1 \overline{x_2}}{\pi_1 \overline{x_1} + \pi_2 \overline{x_2}}\widehat{x_2} \\
&= \pi_1 \widehat{x_1} + \pi_2 \widehat{x_2} \quad (\because \pi_1 + \pi_2 = 1) \\
&= \mathbb{E}(\widehat{x})
\end{aligned}$$

In short, we can change the order of log linearization and expectation operator if necessary. For example, the Euler equation $\frac{1}{c_t} = \beta \mathbb{E}_t[R_{t+1}\frac{1}{c_{t+1}}]$ can be log linearized as follows.

$$\begin{aligned}
\widehat{\frac{1}{c_t}} = \beta \mathbb{E}_t[\widehat{R_{t+1}\frac{1}{c_{t+1}}}] \quad &\Leftrightarrow \quad -\widehat{c}_t = \mathbb{E}_t[\widehat{R_{t+1}\frac{1}{c_{t+1}}}] \\
&\Leftrightarrow \quad -\widehat{c}_t = \mathbb{E}_t[\widehat{R_{t+1}}] - \mathbb{E}_t[\widehat{c_{t+1}}]
\end{aligned}$$

4.2 Simple RBC Model without Growth (Social Planner Problem)

4.2.1 Model Description

- Preferences:

$$\max_{\{c_t,i_t,k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \log(c_t)\right]$$

- Capital accumulation process:

$$i_t = k_{t+1} - k_t,\ \forall t \geq 0$$

- Production technology:

$$y_t = a_t k_t^{\alpha},\ \forall t \geq 0 \ \text{ with } \ \text{given} \ \ k_0$$

- Resource constraint:

$$c_t + i_t = y_t,\ \forall t \geq 0$$

- Shock process:

$$\log a_{t+1} = \rho \log a_t + e_{t+1}$$

  where $e_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

4.2.2 Deriving the First Order Conditions

Consider the following Lagrangian function where capital accumulation process is plugged into the resource constraint. [27]

$$\mathcal{L} = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\Big\{\log(c_t) + \lambda_t(a_t k_t^{\alpha} + k_t - c_t - k_{t+1})\Big\}\right]$$

The first order conditions for c_t and k_{t+1} are

$$\frac{1}{c_t} = \lambda_t \ \text{ and } \ \lambda_t = \beta\mathbb{E}_t[\lambda_{t+1}(a_{t+1}\alpha k_{t+1}^{\alpha-1} + 1)]$$

The TVC is

$$\lim_{t\to\infty}\mathbb{E}_0[\beta^t u'(c_t)k_{t+1}] = \lim_{t\to\infty}\mathbb{E}_0[\beta^t \lambda_t k_{t+1}] = 0$$

From the first order conditions for c_t and k_{t+1}, we can derive the following Euler equation.

$$1 = \mathbb{E}_t[\underbrace{\beta\frac{c_t}{c_{t+1}}}_{\substack{=m_{t+1} \\ \text{s.d.c}}}\underbrace{(a_{t+1}\alpha k_{t+1}^{\alpha-1} + 1)}_{\substack{=R_{t+1} \\ \text{gross return from capital}}}]$$

Let $\bar{c}, \bar{k}, \bar{i}$ and $\bar{y}$ denote the steady state values for consumption, capital, investment and production. Notice that we need 4 equations; the Euler equation, the capital accumulation process, the resource constraint, and the production technology to find those 4 values. From the Euler equation, we have

$$1 = \beta(\alpha\bar{k}^{\alpha-1} + 1)$$

[27] Notice that k_0 is given, and no ponzi constraint should be satisfied in this problem.

Thus,

$$\overline{k} = \left[\frac{1-\beta}{\alpha\beta}\right]^{\frac{1}{\alpha-1}} \text{ and } \; \overline{i} = 0$$

Then, we have

$$\overline{y} = \left[\frac{1-\beta}{\alpha\beta}\right]^{\frac{\alpha}{\alpha-1}}$$

From the resource constraint, we have

$$\overline{c} = \overline{y}$$

4.2.3 Log Linearization and State Space Form Representation

The log linearization of the resource constraint is

$$\begin{aligned}
\widehat{k}_{t+1} &= \widehat{a_t k_t^\alpha + k_t - c_t} \\
&= \frac{\overline{k}^\alpha}{\overline{k}^\alpha + \overline{k} - \overline{c}}(\widehat{a}_t + \alpha\widehat{k}_t) + \frac{\overline{k}}{\overline{k}^\alpha + \overline{k} - \overline{c}}\widehat{k}_t - \frac{\overline{c}}{\overline{k}^\alpha + \overline{k} - \overline{c}}\widehat{c}_t \\
&= \frac{1}{\overline{k}^\alpha + \overline{k} - \overline{c}}\left[\overline{k}^\alpha[\widehat{a}_t + \alpha\widehat{k}_t] + \overline{k}\widehat{k}_t - \overline{c}\widehat{c}_t\right] \\
&= \overline{k}^{\alpha-1}\widehat{a}_t + (\alpha\overline{k}^{\alpha-1} + 1)\widehat{k}_t - \frac{\overline{c}}{\overline{k}}\widehat{c}_t \\
&= -\left[\frac{1-\beta}{\alpha\beta}\right]\widehat{c}_t + \frac{1}{\beta}\widehat{k}_t + \left[\frac{1-\beta}{\alpha\beta}\right]\widehat{a}_t
\end{aligned}$$

And that of the Euler equation is

$$\begin{aligned}
-\widehat{c}_t &= -\mathbb{E}_t[\widehat{c}_{t+1}] + (1-\beta)\mathbb{E}_t[\widehat{a}_{t+1}] + (1-\beta)(\alpha-1)\widehat{k}_{t+1} \\
&= -\mathbb{E}_t[\widehat{c}_{t+1}] - (1-\beta)(1-\alpha)\widehat{k}_{t+1} + (1-\beta)\rho\widehat{a}_t \\
\Leftrightarrow \;\; \widehat{c}_t &= \mathbb{E}_t[\widehat{c}_{t+1}] + (1-\beta)(1-\alpha)\widehat{k}_{t+1} - (1-\beta)\rho\widehat{a}_t
\end{aligned}$$

Now, we can represent the results of log linearization in the state space form as below.

$$A_{(2\times 2)}\begin{bmatrix}\mathbb{E}_t[\widehat{c}_{t+1}] \\ \widehat{k}_{t+1}\end{bmatrix} = B_{(2\times 2)}\begin{bmatrix}\widehat{c}_t \\ \widehat{k}_t\end{bmatrix} + C_{(2\times 1)}a_t$$

which is

$$\begin{bmatrix}1 & (1-\alpha)(1-\beta) \\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbb{E}_t[\widehat{c}_{t+1}] \\ \widehat{k}_{t+1}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ -\frac{(1-\beta)}{\alpha\beta} & \frac{1}{\beta}\end{bmatrix}\begin{bmatrix}\widehat{c}_t \\ \widehat{k}_t\end{bmatrix} + \begin{bmatrix}(1-\beta)\rho \\ \frac{1-\beta}{\alpha\beta}\end{bmatrix}\widehat{a}_t$$

4.3 Balanced Growth RBC Model (Social Planner Problem)

4.3.1 Model Description

- Preferences:

$$\max_{\{c_t, N_t, L_t, i_t, k_{t+1}\}_{t=0}^\infty} \mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t u(c_t, L_t)\right]$$

- Capital accumulation process:

$$\gamma k_{t+1} = i_t + (1-\delta)k_t, \;\; \forall t \geq 0 \tag{4.7}$$

- Time constraint:

$$N_t + L_t = 1, \quad \forall t \geq 0 \tag{4.8}$$

- Production technology:

$$y_t = A_t F(k_t, N_t), \ \forall t \geq 0 \ \text{ with } \ \text{given} \ \ k_0 \tag{4.9}$$

- Resource constraint:

$$c_t + i_t = y_t, \ \forall t \geq 0 \tag{4.10}$$

- Shock process:

$$\log A_{t+1} = \rho \log A_t + e_{t+1} \tag{4.11}$$

where $e_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

4.3.2 Deriving the First Order Conditions

Consider the following Lagrangian function where the capital accumulation process and the time constraint are plugged into the objective function and the resource constraint.

$$\mathcal{L} = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t, 1 - N_t) + \lambda_t \big(A_t F(k_t, N_t) + (1-\delta)k_t - c_t - \gamma k_{t+1} \big) \Big\} \right]$$

The first order conditions for c_t, N_t and k_{t+1} are

$$\underbrace{u_c(c_t, 1 - N_t)}_{\text{MU in consumption}} = \underbrace{\lambda_t}_{\text{shadow price of output}} \tag{4.12}$$

and

$$\underbrace{u_L(c_t, 1 - N_t)}_{\text{MU in leisure}} = \underbrace{\lambda_t \underbrace{A_t F_N(k_t, N_t)}_{\text{MP of Labor}}}_{\text{in utility terms}} \tag{4.13}$$

Lastly,

$$\underbrace{\gamma \lambda_t}_{\text{current utility cost of capital}} = \underbrace{\beta \mathbb{E}_t \big[\lambda_{t+1} [A_{t+1} F_k(k_{t+1}, N_{t+1}) + (1-\delta)] \big]}_{\text{expected PV of future product of capital}} \tag{4.14}$$

The TVC is

$$\lim_{t \to \infty} \mathbb{E}_0 [\beta^t u_c(c_t, 1 - N_t) k_{t+1}] = 0$$

Let us further assume that $y_t = A_t k_t^{1-\alpha} N_t^{\alpha}$ and $u(c_t, L_t) = \log c_t + \theta \log L_t$. Then, as we did in simple RBC model, we can find the steady state values. In general, it is convenient to use the ratio variables such as the steady state capital to labor ratio, $\frac{\overline{k}}{\overline{N}}$. From the inter-temporal Euler equation, we have

$$\gamma = \beta [(1-\alpha) \left(\frac{\overline{k}}{\overline{N}} \right)^{-\alpha} + (1-\delta)] \quad \Leftrightarrow \quad \underbrace{\underbrace{\frac{\gamma}{\beta} - 1}_{\text{net real interest rate}} + \delta}_{\text{rental price of captial}} = \underbrace{(1-\alpha) \left(\frac{\overline{k}}{\overline{N}} \right)^{-\alpha}}_{= F_k(\overline{k}, \overline{N})}$$

Let $\frac{\gamma}{\beta} - 1 = r$. Then, we have

$$\frac{\overline{k}}{\overline{N}} = \left[\frac{r+\delta}{1-\alpha}\right]^{-\frac{1}{\alpha}}$$

Then, it is easy to check that

$$\frac{\overline{y}}{\overline{N}} = \left(\frac{\overline{k}}{\overline{N}}\right)^{1-\alpha}, \quad \frac{\overline{c}}{\overline{N}} = \left(\frac{\overline{k}}{\overline{N}}\right)^{1-\alpha} - [\gamma - (1-\delta)]\left(\frac{\overline{k}}{\overline{N}}\right), \quad \frac{\overline{i}}{\overline{N}} = [\gamma - (1-\delta)]\frac{\overline{k}}{\overline{N}} \text{ and } \overline{N} + \overline{L} = 1$$

From the intra-temporal Euler equation, we have

$$\frac{\theta\overline{c}}{1-\overline{N}} = \alpha\left(\frac{\overline{k}}{\overline{N}}\right)^{1-\alpha} \quad \Leftrightarrow \quad \frac{\theta\frac{\overline{c}}{\overline{N}}}{\frac{1}{\overline{N}} - 1} = \alpha\left(\frac{\overline{k}}{\overline{N}}\right)^{1-\alpha}$$

Thus,

$$\overline{N} = \frac{1}{\frac{\theta}{\alpha}\left[1 - [\gamma - (1-\delta)]\frac{(1-\alpha)}{(r+\delta)}\right] + 1}$$

Given that $\overline{N}$ is found as a function of parameters, we can also find the other five steady state values $\overline{k}$, $\overline{y}$, $\overline{c}$, $\overline{i}$ and $\overline{L}$ as well.

From the **long run average data**, we usually obtain the parameters for RBC model. For example, if we use quarterly data model, then

- $\gamma = 1.004$ ($\because$ 1.6% growth per year), $\delta = 0.025$ ($\because$ 10% depreication per year).
- $\beta = 0.98$ ($\because$ $\beta = b\gamma^{1-\sigma}$, 6.5% real interest rate per year and $\sigma = 1$ if log utility).
- $\alpha = \frac{\omega N}{Y} = \frac{2}{3}$ ($\because$ labor share)
- For θ, we usually target $\overline{N} = 0.2$, for that $\theta = 3.78$.

However, the shock process parameter ρ comes from so called **Solow Residual** estimation.

- From the non-normalized model, we have

  $$Y_t = A_t K_t^{1-\alpha}(X_t N_t)^{\alpha} \text{ where } \frac{X_t}{X_{t-1}} = \gamma$$

  Taking the log for both sides yields

  $$\log Y_t = \log A_t + (1-\alpha)\log K_t + \alpha\log N_t + \alpha\log X_t$$

- Since Solow residual can be written as

  $$\begin{aligned}\log SR_t &= \log Y_t - (1-\alpha)\log K_t - \alpha\log N_t \\ &= \log A_t + \alpha\log X_t\end{aligned}$$

  Filtering out a linear trend $\alpha\log X_t$ gives us $\log A_t$, and fitting these data into AR(1) process yields

  $$\log A_t = \rho\log A_{t-1} + e_t$$

where $\rho = 0.979$ and $\sigma_e = 0.0072$. Notice that ρ is very close to 1, which shows that the shock is very persistent.

4.3.3 Log Linearization

Under the assumption $u(c_t, L_t) = \log c_t + \theta \log L_t$ and $y_t = A_t k_t^{1-\alpha} N_t^{\alpha}$, log linearization of the first order conditions 4.12, 4.13 and 4.14 are

$$-\widehat{c}_t = \widehat{\lambda}_t \tag{4.15}$$

$$-\widehat{L}_t = \widehat{\lambda}_t + \widehat{A}_t + (1-\alpha)(\widehat{k}_t - \widehat{N}_t) \tag{4.16}$$

$$-\mathbb{E}_t[\widehat{\lambda}_{t+1}] + \widehat{\lambda}_t = \beta\Big[\frac{1}{\beta} - \frac{(1-\delta)}{\gamma}\Big]\mathbb{E}_t\Big[\widehat{A}_{t+1} + \alpha(\widehat{N}_{t+1} - \widehat{k}_{t+1})\Big] \tag{4.17}$$

From the equations 4.15 and 4.16, we have

$$-\widehat{L}_t = -\widehat{c}_t + \widehat{A}_t + (1-\alpha)(\widehat{k}_t - \widehat{N}_t) \tag{4.18}$$

and from the equations 4.15 and 4.17, we also have

$$\mathbb{E}_t[\widehat{c}_{t+1}] - \widehat{c}_t = \beta\Big[\frac{1}{\beta} - \frac{(1-\delta)}{\gamma}\Big]\mathbb{E}_t\Big[\widehat{A}_{t+1} + \alpha(\widehat{N}_{t+1} - \widehat{k}_{t+1})\Big] \tag{4.19}$$

Since in the competitive equilibrium, $\widehat{\omega}_t = \widehat{A}_t + (1-\alpha)(\widehat{k}_t - \widehat{N}_t)$, thus the equation 4.18 can be written as

$$-\widehat{L}_t = -\widehat{c}_t + \underbrace{\widehat{A}_t + (1-\alpha)(\widehat{k}_t - \widehat{N}_t)}_{\substack{=\widehat{\omega}_t \\ \text{labor demand curve}}}$$

Using the result of 4.21, we have $\widehat{N}_t = -\frac{\overline{L}}{\overline{N}}\widehat{L}_t$, and hence

$$\begin{aligned}\widehat{N}_t &= \frac{\overline{L}}{\overline{N}}\Big[-\widehat{c}_t + \widehat{A}_t + (1-\alpha)[\widehat{k}_t - \widehat{N}_t]\Big] \\ &= \underbrace{\frac{1-\overline{N}}{\overline{N}}}_{\text{elasticity}}\Big[-\widehat{c}_t + \widehat{\omega}_t\Big]\end{aligned}$$

Then, holding $-\widehat{c}_t$ fixed, $\frac{1-\overline{N}}{\overline{N}}$ can be interpreted as elasticity of labor supply.
Also notice that in the competitive equilibrium, $\widehat{r}_{t+1} = \widehat{A}_{t+1} + \alpha(\widehat{N}_{t+1} - \widehat{k}_{t+1})$, thus the equation 4.17 can be written as

$$\begin{aligned}\mathbb{E}_t[\widehat{c}_{t+1}] - \widehat{c}_t &= \beta\Big[\frac{1}{\beta} - \frac{(1-\delta)}{\gamma}\Big]\mathbb{E}_t\Big[\underbrace{\widehat{A}_{t+1} + \alpha(\widehat{N}_{t+1} - \widehat{k}_{t+1})}_{=\widehat{r}_{t+1}}\Big] \\ &= \beta\Big[\frac{1}{\beta} - \frac{(1-\delta)}{\gamma}\Big]\mathbb{E}_t[r_{t+1}]\end{aligned}$$

Here, we can also find that $\widehat{r}_{t+1}$ is affected by persistence of shock, $\widehat{A}_{t+1}$, and the changes in labor/capital ratio, $\widehat{N}_{t+1} - \widehat{k}_{t+1}$.
Lastly, log linearization of capital accumulation process 4.7, time constraint 4.8, production technology 4.9, resource

constraint 4.10 and shock process 4.11 are

$$\frac{1}{\gamma}\frac{\bar{i}}{\bar{k}}\widehat{i}_t + \frac{(1-\delta)}{\gamma}\widehat{k}_t = \widehat{k}_{t+1} \tag{4.20}$$

$$\overline{N}_t\widehat{N}_t + L_t\widehat{L}_t = 0 \tag{4.21}$$

$$\widehat{y}_t = \widehat{A}_t + (1-\alpha)\widehat{k}_t + \alpha\widehat{N}_t \tag{4.22}$$

$$\widehat{y}_t = \frac{\bar{c}}{\bar{y}}\widehat{c}_t + \frac{\bar{i}}{\bar{y}}\widehat{i}_t \tag{4.23}$$

$$\widehat{A}_t = \rho\widehat{A}_{t-1} + e_t \tag{4.24}$$

4.4 Balanced Growth RBC Model (Decentralized Economy 1)

In this decentralized economy, we assume that a representative household makes an investment decision.

4.4.1 Model Description

Definition 4.2. [Competitive equilibrium] A competitive equilibrium consists of prices $\{\omega_t^*, r_t^*\}_{t=0}^{\infty}$ and allocations $\{c_t^*, N_t^{s*}, L_t^*, i_t^*, k_{t+1}^{s*}\}_{t=0}^{\infty}$ and $\{k_t^{d*}, N_t^{d*}, \pi_t^*\}_{t=0}^{\infty}$ such that

1) Household side: Given a sequence of prices $\{\omega_t^*, r_t^*\}_{t=0}^{\infty}$ and $\{\pi_t^*\}_{t=0}^{\infty}$, $\{c_t^*, N_t^{s*}, L_t^*, i_t^*, k_{t+1}^{s*}\}_{t=0}^{\infty}$ solves a representative household's utility maximization problem

$$\max_{\{c_t, N_t^s, L_t, i_t, k_{t+1}^s\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t, L_t)\right] \quad \text{s.t} \tag{4.25}$$

$$c_t + i_t = \omega_t^* N_t^s + r_t^* k_t^s + \pi_t^*, \quad \forall t \geq 0 \tag{4.26}$$

$$i_t = \gamma k_{t+1}^s - (1-\delta)k_t^s, \quad \forall t \geq 0 \tag{4.27}$$

$$N_t^s + L_t = 1, \quad \forall t \geq 0 \tag{4.28}$$

k_0 is given and No ponzi constraint

2) Firm side: Given a sequence of prices $\{\omega_t^*, r_t^*\}_{t=0}^{\infty}$, for each $t \geq 0$, $\{k_t^{d*}, N_t^{d*}\}_{t=0}^{\infty}$ solves a representative firm's **(static)** profit maximization problem

$$\max_{\{k_t^d, N_t^d\}} A_t F(k_t^d, N_t^d) - r_t^* k_t^d - \omega_t^* N_t^d$$

where $F = (k_t^d)^{1-\alpha}(N_t^d)^{\alpha}$.

3) The markets clear, for each $t \geq 0$

$$c_t^* + i_t^* = A_t F(k_t^{d*}, N_t^{d*}),\ k_t^{s*} = k_t^{d*} \quad \text{and} \quad N_t^{s*} = N_t^{d*}$$

given initial condition k_0 and exogenous process for shocks $\{A_t\}_{t=0}^{\infty}$.

4.4.2 Deriving the First Order Conditions

For a representative household side, we can get the following Lagrangian function by plugging two constraints 4.27 and 4.28 into the objective function and the budget constraint.

$$\mathcal{L} = \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\Big\{u(c_t, 1-N_t^s) + \lambda_t\big(\omega_t^* N_t^s + [r_t^* + (1-\delta)]k_t^s + \pi_t^* - c_t - \gamma k_{t+1}^s\big)\Big\}\Big]$$

Thus, the first order conditions for c_t, N_t^s and k_{t+1}^s are

$$\begin{aligned} u_c(c_t, 1-N_t^s) &= \lambda_t \\ u_L(c_t, 1-N_t^s) &= \lambda_t \omega_t^* \\ \gamma\lambda_t &= \beta\mathbb{E}_t\Big[\lambda_{t+1}[r_{t+1}^* + (1-\delta)]\Big] \end{aligned}$$

The TVC is

$$\lim_{t\to\infty}\mathbb{E}_0[\beta^t u_c(c_t, 1-N_t^s)k_{t+1}^s] = 0$$

From the first order conditions for c_t and N_t^s, we can derive the intra-temporal Euler equation

$$\frac{u_L(c_t, 1-N_t^s)}{u_c(c_t, 1-N_t^s)} = \omega_t^*$$

and from the first order conditions for c_t and k_{t+1}^s, we can also derive the inter-temporal Euler equation

$$u_c(c_t, 1-N_t^s) = \frac{\beta}{\gamma}\mathbb{E}_t\Big[u_c(c_{t+1}, 1-N_{t+1}^s)[r_{t+1}^* + (1-\delta)]\Big]$$

For a representative firm side, it is easy to check that the first order conditions for N_t^d and k_t^d are

$$\omega_t^* = A_t(k_t^d)^{1-\alpha}\alpha(N_t^d)^{\alpha-1} \quad \text{and} \quad r_t^* = A_t(1-\alpha)(k_t^d)^{-\alpha}(N_t^d)^{\alpha}$$

Given that $F(k_t^d, n_t^d)$ is CRS and the input markets are competitive, we can show that for each $t \geq 0$, $\pi_t^* = 0$.

$$\begin{aligned} \pi_t^* &= A_t F(k_t^{d*}, N_t^{d*}) - \omega_t^* N_t^{d*} - r_t^* k_t^{d*} \\ &= A_t F(k_t^{d*}, N_t^{d*}) - A_t F_N(k_t^{d*}, N_t^{d*})N_t^{d*} - A_t F_k(k_t^{d*}, N_t^{d*})k_t^{d*} \\ &= A_t\Big[F(k_t^{d*}, N_t^{d*}) - \{\underbrace{F_N(k_t^{d*}, N_t^{d*})N_t^{d*} + F_k(k_t^{d*}, N_t^{d*})k_t^{d*}}_{=F(k_t^{d*}, N_t^{d*})}\}\Big] \\ &= 0 \end{aligned}$$

Lastly, notice that plugging the firm's first order conditions with the market clearing conditions into the first order conditions for the household problem gives us the same first order conditions that we derived in the social planner's problem. Thus, the allocations must coincide.

4.5 Balanced Growth RBC Model (Decentralized Economy 2)

Unlike the previous subsection, now we assume that a representative firm makes an investment decision.

4.5.1 Model Description

Definition 4.3. [Competitive equilibrium] A competitive equilibrium consists of prices $\{\omega^*_{t+j}, p^*_{t+j}, m^*_{t,t+j}\}_{j=0}^{\infty}$ and allocations $\{c^*_{t+j}, N^{s*}_{t+j}, L^*_{t+j}, s^*_{t+j}\}_{j=0}^{\infty}$ and $\{k^*_{t+j+1}, i^*_{t+j}, N^{d*}_{t+j}, \pi^*_{t+j}\}_{j=0}^{\infty}$ such that

1) Household side: Given a sequence of prices $\{\omega^*_{t+j}, p^*_{t+j}\}_{j=0}^{\infty}$ and $\{\pi^*_{t+j}\}_{j=0}^{\infty}$, $\{c^*_{t+j}, N^{s*}_{t+j}, L^*_{t+j}, s^*_{t+j}\}_{j=0}^{\infty}$ solves a representative household's utility maximization problem

$$\max_{\{c_{t+j}, N^s_{t+j}, L_{t+j}, s_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j u(c_{t+j}, L_{t+j})\right] \text{ s.t} \tag{4.29}$$

$$c_{t+j} + p^*_{t+j}s_{t+j} = \omega^*_{t+j}N^s_{t+j} + [p^*_{t+j} + \pi^*_{t+j}]s_{t+j-1}, \quad \forall j \geq 0 \tag{4.30}$$

$$N^s_{t+j} + L_{t+j} = 1, \quad \forall j \geq 0 \tag{4.31}$$

$$s_{t-1}(=1) \text{ is given and No ponzi constraint}$$

2) Firm side: Given a sequence of price $\{\omega^*_{t+j}\}_{j=0}^{\infty}$ and stochastic discount factor $\{m^*_{t,t+j}\}_{j=0}^{\infty}$, $\{k^*_{t+j+1}, i^*_{t+j}, N^{d*}_{t+j}\}_{j=0}^{\infty}$ solves a representative firm's **(dynamic)** profit maximization problem [28]

$$\max_{\{k_{t+j+1}, i_{t+j}, N^d_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t\left[\sum_{j=0}^{\infty} m_{t,t+j}\Big(A_{t+j}F(k_{t+j}, N^d_{t+j}) - \omega^*_{t+j}N^d_{t+j} - i_{t+j}\Big)\right] \text{ s.t}$$

$$i_{t+j} = \gamma k_{t+j+1} - (1-\delta)k_{t+j}, \quad \forall j \geq 0$$

$$k_{t-1} \text{ is given and No ponzi constraint}$$

where $m_{t,t+j} = \beta^j \frac{u_c(c_{t+j}, 1-N_{t+j})}{u_c(c_t, 1-N_t)}$ and $F = (k_{t+j})^{1-\alpha}(N^d_{t+j})^{\alpha}$.

3) The markets clear, for each $j \geq 0$

$$c^*_{t+j} + i^*_{t+j} = A_{t+j}F(k^*_{t+j}, N^{d*}_{t+j}),\ s^*_{t+j} = 1 \text{ and } N^{s*}_{t+j} = N^{d*}_{t+j}$$

given initial conditions k_{t-1} and s_{t-1} and exogenous process for shocks $\{A_{t+j}\}_{j=0}^{\infty}$.

4.5.2 Deriving the First Order Conditions

For a representative household side, we can get the following Lagrangian function by plugging the time constraint into the objective function.

$$\mathcal{L} = \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^{t+j}\Big\{u(c_{t+j}, 1-N^s_{t+j}) + \lambda_{t+j}\big(\omega^*_{t+j}N^s_{t+j} + [p^*_{t+j} + \pi^*_{t+j}]s_{t+j-1} - c_{t+j} - p^*_{t+j}s_{t+j}\big)\Big\}\right]$$

[28] A representative firm's profit maximization problem is not static anymore because dynamics of capital should be taken into account. Given that it is a dynamic problem, a firm needs to maximize the expected sum of profits, then one important issue arises here. How can we properly discount profits? The answer is that we need to use a unique stochastic discount factor of a representative household.

Thus, the first order conditions for c_t, N_t^s and s_t are

$$\begin{aligned} u_c(c_t, 1 - N_t^s) &= \lambda_t \\ u_L(c_t, 1 - N_t^s) &= \lambda_t \omega_t^* \\ \lambda_t p_t^* &= \mathbb{E}_t\Big[\beta \lambda_{t+1}(p_{t+1}^* + \pi_{t+1}^*)\Big] \end{aligned}$$

The TVC is

$$\lim_{j\to\infty} \mathbb{E}_t[\beta^j u_c(c_{t+j}, 1 - N_{t+j}^s) p_{t+j} s_{t+j}] = 0$$

From the first order condition for c_t and N_t^s, we can derive the intra-temporal Euler equation

$$\frac{u_L(c_t, 1 - N_t^s)}{u_c(c_t, 1 - N_t^s)} = \omega_t^*$$

and from the first order conditions for c_t and s_t, we can also derive the inter-temporal Euler equation

$$1 = \mathbb{E}_t\Big[\underbrace{\beta \frac{u_c(c_{t+1}, 1 - N_{t+1}^s)}{u_c(c_t, 1 - N_t^s)}}_{=m_{t,t+1}} \frac{(p_{t+1}^* + \pi_{t+1}^*)}{p_t^*}\Big]$$

For a representative firm side, the corresponding Lagrangian function can be written as

$$\mathcal{L} = \mathbb{E}_t\Big[\sum_{j=0}^{\infty} m_{t,t+j}^* \Big\{A_{t+j} F(k_{t+j}, N_{t+j}^d) + (1-\delta)k_{t+j} - \omega_{t+j}^* N_{t+j}^d - \gamma k_{t+j+1}\Big\}\Big]$$

Then, the first order conditions for N_t^d and k_{t+1} are

$$\omega_t^* = A_t (k_t)^{1-\alpha} \alpha (N_t^d)^{\alpha-1}$$

and

$$\mathbb{E}_t\Big[m_{t,t+1}^*[A_{t+1}(1-\alpha)(k_{t+1})^{-\alpha}(N_{t+1}^d)^{\alpha} + (1-\delta)]\Big] = \gamma$$

which is equivalent to

$$\gamma = \mathbb{E}_t\Big[\beta \frac{u_c(c_{t+1}, 1 - N_{t+1}^s)}{u_c(c_t, 1 - N_t^s)} [A_{t+1}(1-\alpha)(k_{t+1})^{-\alpha}(N_{t+1}^d)^{\alpha} + (1-\delta)]\Big]$$

Again, notice that we have the same first order conditions that we derived in the social planner's problem. Thus, the allocations must be the same.

4.6 Challenges against RBC Model

4.6.1 Labor Supply Elasticity

Consider the following social planner problem.

$$\max_{\{c_t, N_t, k_{t+1}\}_{t=1}^{\infty}} \mathbb{E}_0\Big[\sum_{t=0}^{\infty} \beta^t \Big\{\log c_t + \frac{\theta}{1-\eta}(1-N_t)^{1-\eta} + \lambda_t\big(A_t F(k_t, N_t) + (1-\delta)k_t - c_t - \gamma k_{t+1}\big)\Big\}\Big]$$

As we derived in subsubsection 4.3.3, we can derive

$$\widehat{N}_t = \frac{1-\overline{N}}{\eta\overline{N}}\left[-\widehat{c}_t + \widehat{\omega}_t\right]$$

Then, $\frac{1-\overline{N}}{\eta\overline{N}}$ can be understood as the elasticity of labor supply holding $-\widehat{c}_t$ fixed. In subsubsection 4.3.3, we implicitly set $\eta = 1$ and target $\overline{N} = 0.2$ from the long run average data, which leads to assuming $\frac{1-\overline{N}}{\eta\overline{N}} = 4$. Although the RBC models calibrated in that way have the decent prediction power, some economists argue that assuming the elasticity of labor supply as 4 is absurd, because it should be at most 1 according to the micro estimates. Conversely, in order to impose $\frac{1-\overline{N}}{\eta\overline{N}} = 1$, we may set $\eta = 4$, however, then the prediction power of RBC models decreases too much.

4.6.2 Shock Process Estimation

Let $Y_t = A_t(Z_tK_t)^{1-\alpha}(N_tX_t)^{\alpha}$ where Z_t is the utilization rate. Suppose that $K_{t+1} = I_t + [1-\delta(Z_t)]k_t$ such that $\delta'(Z_t) > 0$. That is, the depreciation rate is the function Z_t, and using more K_t yields higher depreciation. By taking the log for both sides, we have

$$\underbrace{\log Y_t - \log K_t^{1-\alpha} - \alpha\log N_t}_{=\log SR_t} = \alpha\log X_t + \log A_t + (1-\alpha)\log Z_t$$

Thus, using Solow residual as an estimator for shock process (after filtering out a linear trend $\alpha\log X_t$) always yields measurement error

$$d_t = (1-\alpha)\log Z_t$$

5 Part 5: Monetary Policy and New Keynesian Model

5.1 Classical Dichotomy Model

In this subsection, we will consider the model in which real variables and nominal variables are determined separately.

5.1.1 Model Description

Definition 5.1. [Competitive equilibrium] A competitive equilibrium consists of (nominal) prices $\{P_t^*, Q_t^*, W_t^*\}_{t=0}^{\infty}$ and allocations $\{c_t^*, N_t^{s*}, B_t^*\}_{t=0}^{\infty}$ and $\{N_t^{d*}, \pi_t^*\}_{t=0}^{\infty}$ such that

1) Household side: Given a sequence of prices $\{P_t^*, Q_t^*, W_t^*\}_{t=0}^{\infty}$, $\{c_t^*, N_t^{s*}, B_t^*\}_{t=0}^{\infty}$ solves a representative household's utility maximization problem

$$\max_{\{c_t, N_t^s, B_t\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t, N_t^s)\right] \text{ s.t} \tag{5.1}$$

$$P_t^* c_t + Q_t^* B_t = B_{t-1} + W_t^* N_t^s + \pi_t^*, \quad \forall t \geq 0 \tag{5.2}$$

$$B_{-1}(=0) \text{ is given and No ponzi constraint}$$

2) Firm side: Given a sequence of prices $\{P_t^*, W_t^*\}_{t=0}^{\infty}$, for each $t \geq 0$, $\{N_t^{d*}\}_{t=0}^{\infty}$ solves a representative firm's **(static)** profit maximization problem

$$\max_{\{N_t^d\}} P_t^* A_t (N_t^d)^{1-\alpha} - W_t^* N_t^d$$

3) The markets clear, for each $t \geq 0$

$$c_t^* = A_t (N_t^{s*})^{1-\alpha}, \ B_t^* = 0 \ \text{ and } \ N_t^{s*} = N_t^{d*}$$

given initial condition $B_{-1}(=0)$ and exogenous process for shocks $\{A_t\}_{t=0}^{\infty}$.

5.1.2 Deriving the First Order Conditions

For a representative household side, we have the following Lagrangian function.

$$\mathcal{L} = \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \Big\{u(c_t, N_t^s) + \lambda_t\big(B_{t-1} + W_t^* N_t^s + \pi_t^* - P_t^* c_t - Q_t^* B_t\big)\Big\}\right]$$

Thus, the first order conditions for c_t and N_t are

$$u_c(c_t, N_t^s) = \lambda_t P_t^* \ \text{ and } \ u_N(c_t, N_t^s) = -\lambda_t W_t^*$$

Therefore, we have

$$\frac{u_N(c_t, N_t^s)}{u_c(c_t, N_t^s)} = -\frac{W_t^*}{P_t^*} \tag{5.3}$$

The first order condition for B_t is

$$\lambda_t Q_t^* = \mathbb{E}_t\Big[\beta \lambda_{t+1}\Big] \ \Leftrightarrow \ Q_t^* = \mathbb{E}_t\Big[\beta \frac{u_c(c_{t+1}, N_{t+1}^s)}{u_c(c_t, N_t^s)} \times \underbrace{\frac{P_t^*}{P_{t+1}^*}}_{\text{inverse inflation}}\Big] \tag{5.4}$$

Notice that the first order condition for B_t is irrelevant to characterize the allocations. If we further assume that

$$u(c_t, N_t^s) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{(N_t^s)^{1+\phi}}{1+\phi}$$

Then, the two Euler equations 5.3 and 5.4 can be written as

$$\frac{W_t^*}{P_t^*} = c_t^\sigma (N_t^s)^\phi \tag{5.5}$$

$$Q_t^* = \mathbb{E}_t\Big[\beta\Big(\frac{c_{t+1}}{c_t}\Big)^{-\sigma}\frac{P_t^*}{P_{t+1}^*}\Big] \tag{5.6}$$

For a representative firm side, the first order condition for N_t^d is

$$P_t^* A_t(1-\alpha)(N_t^d)^{-\alpha} = W_t^* \tag{5.7}$$

Notice that

$$P_t^* = \underbrace{\frac{W_t^*}{A_t(1-\alpha)(N_t^d)^{-\alpha}}}_{=\text{marginal cost}}$$

To see this more clearly, consider the following cost minimization problem for a representative firm.

$$\min_{\{N_t^d\}} W_t^* N_t^d \text{ s.t } \; A_t(N_t^d)^{1-\alpha} \geq y_t$$

Then, the first order condition for N_t^d is

$$\mu_t = \frac{W_t^*}{A_t(1-\alpha)(N_t^d)^{-\alpha}}$$

Given that the cost function is

$$TC_t(y_t, W_t^*) = W_t^* N_t^d \; + \mu_t[\; y_t - A_t(N_t^d)^{1-\alpha}]$$

By the envelope theorem, we know that

$$\underbrace{\frac{\partial C(y_t, W_t)}{\partial y_t}}_{=\text{marginal cost}} = \mu_t$$

Now, we want to find the solution $\{\underbrace{c_t^*, N_t^{s*}(= N_t^{d*}), y_t^*, \frac{W_t^*}{P_t^*}}_{=\text{real}}, \underbrace{Q_t^*, \frac{P_{t+1}^*}{P_t^*}}_{=\text{nominal}}\}$. Using the market clearing conditions, we can derive $c_t^* = A_t(N_t^{s*})^{1-\alpha}$ and $y_t^* = A_t(N_t^{s*})^{1-\alpha}$, and hence the intra-temporal Euler equation 5.5 and the first order condition 5.7 with these two conditions determine the solution for four real variables. For example, y_t^* is

$$(y_t^*)^\sigma \Big(\frac{y_t^*}{A_t}\Big)^{\frac{\phi}{1-\alpha}} = A_t(1-\alpha)\Big(\frac{y_t^*}{A_t}\Big)^{\frac{-\alpha}{1-\alpha}}$$

Thus,

$$y_t^* = (1-\alpha)^{\frac{1-\alpha}{(1-\alpha)\sigma+\phi+\alpha}} A_t^{\frac{1+\phi}{(1-\alpha)\sigma+\phi+\alpha}}$$

Taking the log for both sides yields

$$\underbrace{\log y_t}_{\equiv \mathsf{y}_t} = \underbrace{\frac{1-\alpha}{(1-\alpha)\sigma+\phi+\alpha}\log(1-\alpha)}_{\equiv v_y} + \underbrace{\frac{1+\phi}{(1-\alpha)\sigma+\phi+\alpha}}_{\equiv \chi_{ya}}\log A_t$$

5.1.3 Monetary Policy

What about two nominal variables, $\{Q_t^*, \frac{P_{t+1}^*}{P_t^*}\}$? Given that we have two unknowns but only one equation 5.6, generally we cannot solve this equation for Q_t^* and $\frac{P_{t+1}^*}{P_t^*}$. Thus, any exogenous monetary policy should be involved here as an additional condition. Before exploring various monetary policies, let us examine the equation 5.6 first. Notice that taking the log for both sides gives us an approximation of the equation 5.6 as follows.

$$\log Q_t^* = \log\beta + \mathbb{E}_t\Big[-\sigma\Big(\log c_{t+1}^* - \log c_t^*\Big) + \Big(\log P_t^* - \log P_{t+1}^*\Big)\Big]$$

Using some more approximations $-\log Q_t^* = i_t^*$, $\rho = -\log\beta$ and $\Pi_{t+1}^* = \log P_{t+1}^* - \log P_t^*$ [29], we have

$$-i_t^* = -\rho - \sigma\mathbb{E}_t[\mathsf{c}_{t+1}] + \sigma\mathbb{E}_t[\mathsf{c}_t] - \mathbb{E}_t[\Pi_{t+1}^*]$$

which can be written as

$$\mathsf{c}_t^* = \mathbb{E}_t[\mathsf{c}_{t+1}^*] - \frac{1}{\sigma}\Big(\underbrace{\underbrace{I_t^* - \mathbb{E}_t[\Pi_{t+1}^*]}_{\substack{\text{ex-ante real interest}\\ \equiv r_t^*}} - \rho}_{\equiv \widehat{r}_t}\Big) \tag{5.8}$$

It should be noticed that we already determine c_t^* and c_{t+1}^*, and as a result r_t^* also has been determined from the real sector side. Recall that we have $\mathsf{c}_t^* = \mathsf{y}_t^*$ by the market clearing condition, and hence the equation 5.8 can be written as

$$\mathsf{y}_t^* = \mathbb{E}_t[\mathsf{y}_{t+1}^*] - \frac{1}{\sigma}(r_t^* - \rho) \;\Leftrightarrow\; r_t^* = \rho + \sigma\mathbb{E}_t[\mathsf{y}_{t+1}^* - \mathsf{y}_t^*]$$

1. **Exogenous interest rate rule**: Suppose that i_t^* is given. Can this uniquely determine Π_{t+1}^*? No, notice that both $\Pi_{t+1} = i_t^* - r_t^* + \xi_{t+1}$ and $\Pi_{t+1} = i_t^* - r_t^* + \xi_{t+1} + \varsigma_{t+1}$ such that $\mathbb{E}_t[\xi_{t+1}] = \mathbb{E}_t[\varsigma_{t+1}] = 0$ satisfy $r_t^* = i_t^* - \mathbb{E}_t[\Pi_{t+1}^*]$, thus there are infinitely many solutions with sunspot shocks.

2. **Feedback interest rate rule**: Suppose that i_t^* is determined by the rule $i_t^* = \rho + \phi_\pi\Pi_t^*$ where $\phi_\pi > 0$. [30] Then,

$$r_t^* = \rho + \phi_\pi\Pi_t^* - \mathbb{E}_t[\Pi_{t+1}^*] \quad\Leftrightarrow\quad \Pi_t^* = \frac{1}{\phi_\pi}\widehat{r}_t + \frac{1}{\phi_\pi}\mathbb{E}_t[\Pi_{t+1}^*]$$

   Thus, if $\phi_\pi > 1$, solving forward gives us a unique bounded solution

$$\Pi_t^* = \sum_{k=0}^{\infty}\left(\frac{1}{\phi_\pi}\right)^{k+1}\mathbb{E}_t[\widehat{r}_{t+k}]$$

   However, if $\phi_\pi < 1$, we cannot solve in this way. Moreover, any bounded solution such that $\Pi_{t+1}^* = \phi_\pi\Pi_t^* - \widehat{r}_t + \xi_{t+1}$ with $\mathbb{E}_t[\xi_{t+1}] = 0$ satisfies the given condition. Thus, we still have the indeterminacy as exogenous

[29] Notice that $Q_t^* = \frac{1}{1+i_t^*}$ and $\beta = \frac{1}{1+\rho}$.

[30] For simplicity, we use contemporaneous inflation Π_t^* but may use Π_{t-1}^*.

interest rate rule.

3. **Money supply rule**: Instead of pinning down Π_t, we can directly determine $\mathtt{p}_t$ under the money supply rule. Suppose that money demand and supply are given by $\mathtt{M}_t^d - \mathtt{p}_t = \mathtt{y}_t - \eta i_t$ and $\mathtt{M}_t^s$ where $\mathtt{M}_t^d = \log M_t^d$, $\mathtt{M}_t^s = \log M_t^s$, $\mathtt{p}_t = \log P_t$ and $\eta > 0$. In equilibrium in which $\mathtt{M}_t^{s*} = \mathtt{M}_t^{d*} = \mathtt{M}_t^*$, we have

$$i_t^* = \frac{\mathtt{y}_t^* - (\mathtt{M}_t^* - \mathtt{p}_t^*)}{\eta} = r_t^* + \mathbb{E}_t[\Pi_{t+1}^*] = r_t^* + \mathbb{E}_t[\mathtt{p}_{t+1}^*] - \mathtt{p}_t^*$$

Rearranging terms yields

$$\mathtt{p}_t = \underbrace{\Big(\frac{\eta}{1+\eta}\Big)}_{<1}\mathbb{E}_t[\mathtt{p}_{t+1}] + \Big(\frac{1}{1+\eta}\Big)\mathtt{M}_t^* + \underbrace{\Big(\frac{\eta r_t^* - \mathtt{y}_t^*}{\eta}\Big)}_{\equiv u_t}$$

Solving forward yields

$$\mathtt{p}_t = \frac{1}{1+\eta}\sum_{k=0}^{\infty}\Big(\frac{\eta}{1+\eta}\Big)^k \mathbb{E}_t[\mathtt{M}_{t+k}^*] + \underbrace{\sum_{k=0}^{\infty}\Big(\frac{\eta}{1+\eta}\Big)^k \mathbb{E}_t[u_{t+k}]}_{\equiv u_t'}$$

which is also written as

$$\mathtt{p}_t = \mathtt{M}_t^* + \sum_{k=1}^{\infty}\Big(\frac{\eta}{1+\eta}\Big)^k \mathbb{E}_t[\triangle \mathtt{M}_{t+k}^*] + u_t'$$

where $\triangle \mathtt{M}_{t+k}^* = \mathtt{M}_{t+k}^* - \mathtt{M}_{t+k-1}^*$. So, if any exogenous money supply rule such as $\triangle \mathtt{M}_t^* = \rho_m \triangle \mathtt{M}_{t-1}^*$ is given, we can find $\mathtt{p}_t$. Lastly, notice that exogenous money supply rules are robust policies than interest rate rules.

5.2 Money in Utility Model

5.2.1 Model Description

Definition 5.2. [Competitive equilibrium] A competitive equilibrium consists of prices $\{P_t^*, Q_t^*, W_t^*\}_{t=0}^{\infty}$ and allocations $\{c_t^*, N_t^{s*}, B_t^*, \frac{M_t^{d*}}{P_t^*}\}_{t=0}^{\infty}$ and $\{N_t^{d*}, \pi_t^*\}_{t=0}^{\infty}$ such that

1) Household side: Given a sequence of prices $\{P_t^*, Q_t^*, W_t^*\}_{t=0}^{\infty}$, $\{c_t^*, N_t^{s*}, B_t^*, \frac{M_t^{d*}}{P_t^*}\}_{t=0}^{\infty}$ solves a representative household's utility maximization problem

$$\max_{\{c_t, N_t^s, \frac{M_t^d}{P_t^*}\}_{t=0}^{\infty}} \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t u(c_t, \frac{M_t^d}{P_t^*}, N_t^s)\Big] \quad \text{s.t} \tag{5.9}$$

$$P_t^* c_t + Q_t^* B_t + M_t^d = B_{t-1} + M_{t-1}^d + W_t^* N_t^s + \pi_t^* + \tau_t P_t^*, \quad \forall t \geq 0 \tag{5.10}$$

$B_{-1}(=0)$ and M_{-1} are given and No ponzi constraint

where $u_{\frac{M}{P}} \geq 0$ and $u_{\frac{M}{P}} > 0$ only up to $\overline{\frac{M}{P}}$ (satiation point) and $\tau_t P_t^*$ is the lump-sum transfer (seigniorage) from the government.

2) Firm side: Given a sequence of prices $\{P_t^*, Q_t^*, W_t^*\}_{t=0}^{\infty}$, for each $t \geq 0$, $\{N_t^{d*}\}_{t=0}^{\infty}$ solves a representative firm's **(static)** profit maximization problem

$$\max_{\{N_t^d\}} P_t^* A_t (N_t^d)^{1-\alpha} - W_t^* N_t^d$$

3) The markets clear, for each $t \geq 0$

$$c_t^* = A_t(N_t^{s*})^{1-\alpha},\ B_t^* = 0,\ M_t^{s*} = M_t^{d*} \ \text{ and } \ N_t^{s*} = N_t^{d*}$$

given initial conditions $B_{-1}(= 0)$ and M_{-1} and exogenous process for shocks $\{A_t\}_{t=0}^{\infty}$.

4) The government budget constraint is satisfied, for each $t \geq 0$

$$\tau_t P_t^* = M_t^{s*} - M_{t-1}^{s*}$$

5.2.2 Optimal Monetary Policy

For a representative household side, the Lagrangian function is

$$\mathcal{L} = \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\Big\{u(c_t, \frac{M_t^d}{P_t}, N_t^s) + \lambda_t\big(B_{t-1} + \frac{M_{t-1}^d}{P_{t-1}^*}P_{t-1}^* + W_t^* N_t^s + \pi_t^* + \tau_t P_t^* - P_t^* c_t - Q_t^* B_t - \frac{M_t^d}{P_t^*}P_t^*\big)\Big\}\Big]$$

The first order conditions for c_t, N_t^s, B_t and $\frac{M_t^d}{P_t^*}$ are

$$u_c(c_t, \frac{M_t^d}{P_t^*}, N_t^s) = \lambda_t P_t^* \ \text{ and } \ u_N(c_t, \frac{M_t^d}{P_t^*}, N_t^s) = -\lambda_t W_t^*$$
$$\lambda_t Q_t^* = \mathbb{E}_t[\beta\lambda_{t+1}] \ \text{ and } \ u_{\frac{M}{P}}(c_t, \frac{M_t^d}{P_t^*}, N_t^s) = \lambda_t P_t^* - \beta\mathbb{E}_t[\lambda_{t+1}P_t^*]$$

From the first, third and fourth conditions, we have

$$\frac{u_{\frac{M}{P}}(c_t, \frac{M_t^d}{P_t^*}, N_t)}{u_c(c_t, \frac{M_t^d}{P_t^*}, N_t)} = 1 - \mathbb{E}_t[\frac{\lambda_{t+1}}{\lambda_t}] = 1 - Q_t = 1 - \exp(-i_t) \tag{5.11}$$

where $1 - Q_t$ can be interpreted as an opportunity cost of holding money.
To determine what is the optimal monetary policy, consider the following **(static)** social planner's problem. [31]

$$\max_{\{c_t, N_t, \frac{M_t}{P_t}\}} u(c_t, \frac{M_t}{P_t}, N_t) \ \text{ s.t } \ c_t \leq A_t N_t^{1-\alpha}$$

Then, the first order conditions for c_t, N_t and $\frac{M_t}{P_t}$ are

$$u_c(c_t, \frac{M_t}{P_t}, N_t) = \lambda_t,\ u_N(c_t, \frac{M_t}{P_t}, N_t) = -\lambda_t(1-\alpha)A_t N_t^{-\alpha} \ \text{ and } \ u_{\frac{M}{P}}(c_t, \frac{M_t}{P_t}, N_t) = 0$$

Notice that the resource constraint does not include M_t, which means that the social cost (in terms of output) of producing money is zero as we can see from the first order condition for $\frac{M_t}{P_t}$. From the first and third conditions, we have

$$\frac{u_{\frac{M}{P}}(c_t, \frac{M_t}{P_t}, N_t)}{u_c(c_t, \frac{M_t}{P_t}, N_t)} = 0$$

[31] Since there is no capital, the social planner problem is also static.

In the competitive economy, we know that the equation 5.11 holds. Thus, the most efficient allocation requires

$$\frac{u_{\frac{M}{P}}(c_t, \frac{M_t^d}{P_t^*}, N_t^s)}{u_c(c_t, \frac{M_t^d}{P_t^*}, N_t^s)} = 1 - \exp(-i_t) = 0 \;\Leftrightarrow\; i_t^* = 0$$

In other words, M_t^{s*} should be injected into the economy as much as $u_{\frac{M}{P}}(c_t, \frac{M_t^d}{P_t^*}, N_t^s)$ reaches the satiation point at which $u_{\frac{M}{P}}(c_t, \frac{M_t^d}{P_t^*}, N_t^s) = 0$. This is called **Friedman's rule**.

5.2.3 Implementation Issue

Suppose that the central bank implements $i_t^* = 0$ for each $t \geq 0$. If they follow the rule $i_t = 0$ for each $t \geq 0$, can they actually achieve what they intended at equilibrium? The answer is NO as we have shown in the subsubsection 5.1.3. Instead, consider the following feedback rule

$$i_t = \phi(r_{t-1} + \Pi_t) \;\text{ where }\; \phi > 1 \quad \Leftrightarrow \quad \Pi_{t+1} = \frac{1}{\phi} i_{t+1} - r_t$$

Given that $i_t = r_t + \mathbb{E}_t[\Pi_{t+1}]$, [32] solving forward yields

$$i_t = \mathbb{E}_t[\frac{1}{\phi} i_{t+1} - r_t] + r_t = \frac{1}{\phi}\mathbb{E}_t[i_{t+1}] = \ldots = \lim_{k\to\infty} \frac{1}{\phi^k}\mathbb{E}_t[i_{t+k}]$$

Thus, we have $i_t^* = 0$ for each $t \geq 0$. It should be noticed that either $\phi = 1.01$ or $\phi = 10,000$ yields the same results, which means that implementation to achieve $i_t^* = 0$ is not unique.

5.3 Monopolistic Competition Model with Flexible Prices

The basic model of monopolistic competition is drawn from Dixit and Stiglitz (1977).

5.3.1 Household Side Problem

The preferences of the representative household are defined over a composite consumption good C_t and the time devoted to market employment N_t, which can be summarized as

$$\mathbb{E}_t\Big[\sum_{i=0}^{\infty} \beta^i \Big(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta}\Big)\Big]$$

where

$$C_t = \Big[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj\Big]^{\frac{\theta}{\theta-1}} \quad \text{where} \;\; \theta > 1 \tag{5.12}$$

We assume that there is a continuum of firms of measure 1, and each firm j produces differentiated product c_{jt}. Now, we want to deal with the household's problem in two steps. As the first step, for given aggregate level C_t, we would like to find the expenditure (or cost) minimizing combination of individual goods. To this end, we need to solve the following static problem.

$$\min_{\{c_{jt}\}} \int_0^1 p_{jt} c_{jt} dj \;\;\text{s.t}\;\; \Big[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj\Big]^{\frac{\theta}{\theta-1}} \geq C_t$$

[32] Notice that r_t is (already) determined from the real sector side, and hence it is not endogenous in terms of monetary policy.

Since the corresponding Lagrangian function can be written as

$$\mathcal{L} = -\int_0^1 p_{jt} c_{jt} dj + \psi_t \left[\left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} - C_t \right]$$

The first order condition for c_{jt} is

$$p_{jt} = \psi_t \underbrace{\left[\int_0^1 c_{jt}^{\frac{1-\theta}{\theta}} dj \right]^{\frac{1}{\theta-1}}}_{=C_t^{\frac{1}{\theta}}} c_{jt}^{\frac{-1}{\theta}} \quad \Leftrightarrow \quad p_{jt} = \psi_t C_t^{\frac{1}{\theta}} c_{jt}^{\frac{-1}{\theta}} \quad \Leftrightarrow \quad c_{jt} = \left(\frac{p_{jt}}{\psi_t} \right)^{-\theta} C_t \tag{5.13}$$

Plugging $c_{jt} = \left(\frac{p_{jt}}{\psi_t} \right)^{-\theta} C_t$ into the equation 5.12, we have

$$C_t = \left[\int_0^1 \left(\left(\frac{p_{jt}}{\psi_t} \right)^{-\theta} C_t \right)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} = \left[\int_0^1 (p_{ij})^{1-\theta} dj \right]^{\frac{\theta}{\theta-1}} \left(\frac{1}{\psi_t} \right)^{-\theta} C_t$$

which gives us

$$\psi_t = \left[\int_0^1 (p_{jt})^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \equiv P_t \tag{5.14}$$

Notice that the Lagrangian multiplier is the appropriately aggregated price index for consumption. Then, using the equation 5.13, the demand for good j can be written as [33]

$$c_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\theta} C_t \tag{5.15}$$

Now, as the second step, given the cost of achieving any given level of C_t, the household needs to choose C_t and N_t optimally. That is,

$$\begin{aligned} &\max_{\{C_{t+i}, B_{t+i}, N_{t+i}\}_{i=0}^{\infty}} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} \right) \right] \text{ s.t} \\ &P_{t+i} C_{t+i} + B_{t+i} = (1 + i_{t+i-1}) B_{t+i-1} + W_{t+i} N_{t+i} + \pi_{t+i}^*, \quad \forall i \geq 0 \\ &\quad B_{t-1} (= 0) \text{ is given and No ponzi constraint} \end{aligned} \tag{5.16}$$

Dividing by P_{t+i} for both sides of the periodic budget constraint, we can get

$$C_{t+i} + \frac{B_{t+i}}{P_{t+i}} = (1 + i_{t+i-1}) \frac{B_{t+i-1}}{P_{t+i}} + \frac{W_{t+i}}{P_{t+i}} N_{t+i} + \widehat{\pi}_{t+i}^*, \quad \forall i \geq 0 \tag{5.17}$$

where $\widehat{\pi}_{t+i}^* = \frac{\pi_{t+i}^*}{P_{t+i}}$ is the real profits.

For this problem, the Lagrangian function can be written as

$$\mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta} + \lambda_{t+i} \left\{ (1 + i_{t+i-1}) \frac{B_{t+i-1}}{P_{t+i}} + \frac{W_{t+i}}{P_{t+i}} N_{t+i} + \widehat{\pi}_{t+i}^* - C_{t+i} - \frac{B_{t+i}}{P_{t+i}} \right\} \right) \right]$$

[33] The price elasticity of demand for good j is equal to θ. As $\theta \to \infty$, the individual firms will have less market power.

The first order conditions for C_t, N_t and B_t are

$$C_t^{-\sigma} = \lambda_t,\ \chi N_t^{\eta} = \lambda_t \frac{W_t}{P_t} \ \text{ and } \ \frac{\lambda_t}{P_t} = \mathbb{E}_t\left[\beta\lambda_{t+1}\frac{(1+i_t)}{P_{t+1}}\right]$$

Thus, the intra-temporal and inter-temporal Euler equations can be summarized as

$$\frac{\chi N_t^{\eta}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \ \text{ and } \ C_t^{-\sigma} = \beta(1+i_t)\mathbb{E}_t\left[\frac{P_t}{P_{t+1}}C_{t+1}^{-\sigma}\right]$$

The TVC is

$$\lim_{i\to\infty}\mathbb{E}_t[\beta^i C_{t+i}^{-\sigma}\frac{B_{t+i}}{P_{t+i}}] = 0$$

5.3.2 Firm Side Problem

Let $y_{jt} = Z_t N_{jt}$ be the firm j's production function where Z_t is the aggregate TFP shock. As similar to the household problem, we would like to take two steps to solve firms' profit maximization problem. As the first step, consider the firm j's cost minimization problem which can be written as below.

$$\min_{\{N_{jt}\}}\Big(\frac{W_t}{P_t}\Big)N_{jt} \ \text{ s.t } \ Z_t N_{jt} \geq y_{jt}$$

Given that the Lagrangian function for this problem is

$$\mathcal{L} = -\Big(\frac{W_t}{P_t}\Big)N_{jt} + \varphi_t(Z_t N_{jt} - y_{jt})$$

The first order condition for N_{jt} is

$$\frac{W_t}{P_t} = \varphi_t Z_t \quad \Leftrightarrow \quad \frac{W_t}{Z_t P_t} = \varphi_t$$

where $\frac{W_t}{Z_t P_t}$ can be interpreted as the real marginal cost. It should be also noticed that $\frac{W_t}{Z_t P_t}$ is the same across firms.

As the second step, consider the firm j's pricing decision problem for profit maximization as below.

$$\max_{\{p_{jt}\}}\Big(\frac{p_{jt}}{P_t} - \varphi_t\Big)c_{jt} \ \text{ s.t } \ c_{jt} = \Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}C_t$$

where $\frac{p_{jt}}{P_t}$ is the real price for good j and φ_t is the real average cost. [34] Plugging the constraint into the objective function, we have

$$\widehat{\pi}_{jt} = \Big(\frac{p_{jt}}{P_t} - \varphi_t\Big)\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}C_t = \Big(\frac{p_{jt}}{P_t}\Big)^{-\theta+1}C_t - \varphi_t\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}C_t$$

The profit maximizing p_{jt} should satisfy

$$(1-\theta)\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}\frac{C_t}{P_t} + \theta\varphi_t\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta-1}\frac{C_t}{P_t} = 0 \quad \Leftrightarrow \quad (1-\theta)\frac{p_{jt}}{P_t} + \theta\varphi_t = 0$$

Therefore,

$$p_{jt}^* = \underbrace{\frac{\theta}{\theta-1}}_{=\text{markup}} \times \ \varphi_t P_t = \frac{\theta}{\theta-1}\frac{W_t}{Z_t} = p_t^*,\ \forall j \text{ (symmetric equilibrium)}$$

[34]Since the production function is CRS, the marginal cost is equivalent to the average cost.

Notice that $\varphi_t P_t$ is the nominal marginal cost, and $\frac{\theta}{\theta-1} > 1$. Thus, we can find that the price is not identical to the marginal cost in this model. Also note that the markup is constant over time.

5.3.3 Market Clearing Conditions and Aggregation

The market clearing conditions are

$$c_{jt}^* = y_{jt}^*,\ B_t^* = 0 \text{ and } \int_0^1 N_{jt}^* dj = N_t^*$$

In this flexible price model, each firm sets the same price $p_{jt}^* = p_t^*$ as shown above. Thus,

$$P_t^* = \left[\int_0^1 (p_{jt}^*)^{1-\theta} dj\right]^{\frac{1}{1-\theta}} = \left[(p_t^*)^{1-\theta}\int_0^1 1dj\right]^{\frac{1}{1-\theta}} = p_t^* = p_{jt}^*$$

If then, we can also find that

$$N_{jt}^* = N_t^* \quad \text{and} \quad Y_t^* = y_{jt}^*$$

where $Y_t^* \equiv \left[\int_0^1 (y_{jt}^*)^{\frac{1-\theta}{\theta}} dj\right]^{\frac{\theta}{1-\theta}}$ is the aggregation output defined by CES aggregation.
On the other hand, from each firm j's production function $y_{jt} = Z_t N_{jt},\ \forall j,$ we can obtain the following aggregation production function by linear aggregation.

$$\int_0^1 y_{jt} dj = \int_0^1 Z_t N_{jt} dj \quad \Leftrightarrow \quad Y_t^{'} = Z_t N_t$$

Therefore, the aggregated real variables $\{Y_t^*, C_t^*, N_t^*, \frac{W_t^*}{P_t^*}\}$ can be characterized by

$$Y_t^{'*} = Z_t N_t^*,\ Y_t^*(= Y_t^{'*}) = C_t^*,\ \frac{\chi (N_t^*)^\eta}{(C_t^*)^{-\sigma}} = \frac{W_t^*}{P_t^*} \text{ and } \Big(\frac{\theta-1}{\theta}\Big) Z_t = \frac{W_t^*}{P_t^*} \tag{5.18}$$

and we can solve this system of equations as classical dichotomy model. Here, it is very important to recognize that in general, we are not able to write $Y_t^{'*} = Y_t^*$ because they are using the different aggregators.

5.3.4 Flexible Price Output and Efficient Output

Using the first order conditions 5.18, we can find Y_t^* which we will denote by Y_t^{f*} [35]

$$\frac{\chi (\frac{Y_t^*}{Z_t})^\eta}{(Y_t^*)^{-\sigma}} = \Big(\frac{\theta-1}{\theta}\Big) Z_t \quad \Leftrightarrow \quad Y_t^* \equiv Y_t^{f*} = \Big(\frac{1}{\chi\mu}\Big)^{\frac{1}{\sigma+\eta}} Z_t^{\frac{1+\eta}{\sigma+\eta}} \quad \text{where} \quad \mu = \Big(\frac{\theta}{\theta-1}\Big) > 1$$

Given that the market is not perfectly competitive, we know that there exists some inefficiency in this economy. To find the efficient output level, consider the following aggregation (static) social planner's problem.

$$\max_{\{C_t, N_t\}} \frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \quad \text{s.t} \quad C_t = Y_t \quad \text{and} \quad Y_t = Z_t N_t$$

[35] When prices are flexible, output is a function of the aggregate productivity shock, reflecting the fact that in the absence of sticky prices, the new Keynesian model reduces to a real business cycle model.

It is easy to find

$$Y_t^{e*} = \left(\frac{1}{\chi}\right)^{\frac{1}{\sigma+\eta}} Z_t^{\frac{1+\eta}{\sigma+\eta}}$$

and hence $Y_t^{e*} > Y_t^{f*}$. Note that

- The wedge between SPP and flexible price economy is constant over the time.
- After log-linearization for both Y_t^e and Y_t^f, we have $\widehat{Y}_t^e = \widehat{Y}_t^f$. In other words, there is only level distortion in the steady state.

5.4 Monopolistic Competition Model with Sticky Prices

5.4.1 Household Side Problem

The household problem is identical to the problem in flexible price economy, which can be rewritten (in aggregation terms) as

$$\begin{aligned}
\max_{\{C_{t+i}, \frac{B_{t+i}}{P_{t+i}}, N_{t+i}\}_{i=0}^{\infty}} \quad & \mathbb{E}_t\left[\sum_{i=0}^{\infty}\beta^i\left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \chi\frac{N_{t+i}^{1+\eta}}{1+\eta}\right)\right] \quad \text{s.t} \\
C_{t+i} + \frac{B_{t+i}}{P_{t+i}} &= \frac{(1+i_{t+i-1})}{P_{t+i}}B_{t+i-1} + \frac{W_{t+i}}{P_{t+i}}N_{t+i} + \widehat{\pi}^*_{t+i}, \quad \forall i \geq 0 \qquad (5.19) \\
& B_{t-1}(=0) \text{ is given and No ponzi constraint}
\end{aligned}$$

where

$$C_t = \left[\int_0^1 c_{jt}^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}, \quad P_t = \left[\int_0^1 (p_{jt})^{1-\theta} dj\right]^{\frac{1}{1-\theta}} \quad \text{and} \quad c_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\theta} C_t$$

Thus, the intra-temporal and inter-temporal Euler equations can be summarized as

$$\frac{\chi N_t^\eta}{C_t^{-\sigma}} = \frac{W_t}{P_t} \quad \text{and} \quad C_t^{-\sigma} = \beta(1+i_t)\mathbb{E}_t\left[\frac{P_t}{P_{t+1}}C_{t+1}^{-\sigma}\right]$$

5.4.2 Firm Side Problem

The specific model of price stickiness used here is due to Calvo (1983). Each period, the firms that adjust their price are randomly selected, and a fraction $1-\omega$ of all firms adjust while the remaining ω fraction do not adjust. Profits at some future date $t+s$ are affected by the choice of price at time t only if the firm has not received another opportunity to adjust between t and $t+s$. The probability of this is ω^s. When solving the firm's problem, the first step is identical to the case of flexible price economy, so we will begin with the second step.

$$\max_{\{p_{jt}\}} \mathbb{E}_t\left[\sum_{i=0}^{\infty}\omega^i \triangle_{i,t+i}\left(\frac{p_{jt}}{P_{t+i}} - \varphi_{t+i}\right)c_{j,t+i}\right] \quad \text{s.t.} \quad c_{j,t+i} = \left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta} C_{t+i} \qquad (5.20)$$

where $\triangle_{i,t+i} = \frac{\beta^i C_{t+i}^{-\sigma}}{C_t^{-\sigma}}$. Since the corresponding Lagrangian function can be written as

$$\mathcal{L} = \mathbb{E}_t\left[\sum_{i=0}^{\infty}\omega^i \triangle_{i,t+i}\left\{\left(\frac{p_{jt}}{P_{t+i}}\right)^{1-\theta} - \varphi_{t+i}\left(\frac{p_{jt}}{P_{t+i}}\right)^{-\theta}\right\}C_{t+i}\right]$$

The first order condition for p_{jt} is

$$\mathbb{E}_t\Bigg[\sum_{i=0}^{\infty}\omega^i\triangle_{i,t+i}\Big\{(1-\theta)\Big(\frac{p_{jt}}{P_{t+i}}\Big)+\theta\varphi_{t+i}\Big\}\Big(\frac{1}{p_{jt}}\Big)\Big(\frac{p_{jt}}{P_{t+i}}\Big)^{-\theta}C_{t+i}\Bigg]=0$$

However, all firms adjusting in period t face the same problem, so all adjusting firms will set the same price. Let p_t^* be the optimal price chosen by all firms adjusting at time t. Then, we have

$$\mathbb{E}_t\Bigg[\sum_{i=0}^{\infty}\omega^i\triangle_{i,t+i}\Big\{(1-\theta)\Big(\frac{p_t^*}{P_{t+i}}\Big)+\theta\varphi_{t+i}\Big\}\Big(\frac{1}{p_t^*}\Big)\Big(\frac{p_t^*}{P_{t+i}}\Big)^{-\theta}C_{t+i}\Bigg]=0 \tag{5.21}$$

The first order condition 5.21 shows that a firm must take into account expected future marginal cost as well as current marginal cost. For sanity check, let $\omega=0$ (i.e. flexible price economy), then we have (as we expect)

$$(1-\theta)\Big(\frac{p_t^*}{P_t}\Big)+\theta\varphi_t=0 \quad\Leftrightarrow\quad p_t^*=\frac{\theta}{\theta-1}P_t\varphi_t$$

The aggregate price index is an average of the price charged by the fraction of $1-\omega$ of firms setting their price in period t and the average of the remaining fraction of ω of all firms that do not change their price in period t. However, because the adjusting firms were selected randomly from among all firms, the average price of the non-adjusters is just the average price of all firms that prevailed in period $t-1$.

$$P_t=\Bigg[\int_0^1(p_{jt})^{1-\theta}dj\Bigg]^{\frac{1}{1-\theta}} \;\Leftrightarrow\; P_t^{1-\theta}=\int_0^1(p_{jt})^{1-\theta}dj=(1-\omega)(p_t^*)^{1-\theta}+\omega(P_{t-1})^{1-\theta}$$

5.4.3 Market Clearing Conditions and Aggregation

The market clearing conditions are the same as before.

$$c_{jt}^*=y_{jt}^*,\ B_t^*=0 \text{ and } \int_0^1 N_{jt}^*dj=N_t^*$$

However, in this sticky price model, an equilibrium is not symmetric anymore, and hence we should be careful in aggregation. Notice that

$$\Bigg[\int_0^1(y_{jt}^*)^{\frac{1-\theta}{\theta}}dj\Bigg]^{\frac{\theta}{1-\theta}}=Y_t^*=C_t^*=\Bigg[\int_0^1(c_{jt}^*)^{\frac{1-\theta}{\theta}}dj\Bigg]^{\frac{\theta}{1-\theta}}$$

but

$$\int_0^1 y_{jt}dj=\int_0^1 Z_tN_{jt}dj \quad\Leftrightarrow\quad Y_t^{'}=Z_tN_t \quad\Leftrightarrow\quad Y^{'}=\int_0^1 y_{jt}dj=\int_0^1\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}Y_tdj=Y_t\int_0^1\Big(\frac{p_{jt}}{P_t}\Big)^{-\theta}dj$$

which shows that $Y_t^{'*}\neq Y_t^*$.

5.4.4 Log Linearization

Recall that we have the following 7 conditions in aggregated variables which characterize an equilibrium in this sticky price model.

$$\frac{\chi N_t^\eta}{C_t^{-\sigma}} = \frac{W_t}{P_t} \quad \text{and} \quad C_t^{-\sigma} = \beta(1+i_t)\mathbb{E}_t\left[\frac{P_t}{P_{t+1}}C_{t+1}^{-\sigma}\right]$$
$$\varphi_t = \frac{W_t}{Z_t P_t} \quad \text{and} \quad \mathbb{E}_t\left[\sum_{i=0}^{\infty}\omega^i \triangle_{i,t+i}\left\{(1-\theta)\left(\frac{p_t^*}{P_{t+i}}\right)+\theta\varphi_{t+i}\right\}\left(\frac{1}{p_t^*}\right)\left(\frac{p_t^*}{P_{t+i}}\right)^{-\theta}C_{t+i}\right]=0$$
$$C_t = Y_t,\ Y_t' = Z_t N_t \quad \text{and} \quad P_t^{1-\theta} = (1-\omega)(p_t^*)^{1-\theta} + \omega(P_{t-1})^{1-\theta}$$

We will do log linearization of each condition (and Y_t^f for later reference) around the steady state of "**flexible price economy**" where zero-inflation holds, i.e. $\overline{\Pi} = 0 \ \Leftrightarrow\ 1+\overline{\Pi} = 1$. In the steady state of flexible price economy, we have

$$\frac{\overline{W}}{\overline{P}} = \overline{\varphi} = \frac{\theta-1}{\theta},\ \overline{c}_j = \overline{C} = \overline{y}_j = \overline{Y},\ \overline{p}_j = \overline{p}^* = \overline{P} \quad \text{and} \quad \overline{N}_j = \overline{N}$$

Doing log linearization around this steady state yields

$$\eta\widehat{N}_t + \sigma\widehat{C}_t = \widehat{W}_t - \widehat{P}_t \tag{5.22}$$
$$-\sigma\widehat{C}_t = \widehat{i}_t + \mathbb{E}_t[-\Pi_{t+1} - \sigma\widehat{C}_{t+1}] \quad \text{where } \widehat{i}_t = \widehat{(1+i_t)} \tag{5.23}$$
$$\widehat{\varphi}_t = \widehat{W}_t - \widehat{Z}_t - \widehat{P}_t \tag{5.24}$$
$$\widehat{C}_t = \widehat{Y}_t \quad \text{and} \quad \widehat{Y}_t' = \widehat{Z}_t + \widehat{N}_t \tag{5.25}$$
$$\widehat{P}_t = (1-\omega)\widehat{p}_t^* + \omega\widehat{P}_{t-1} \tag{5.26}$$
$$\widehat{p}_t^* = (1-\beta\omega)\mathbb{E}_t\left[\sum_{i=0}^{\infty}(\beta\omega)^i(\widehat{P}_{t+i} + \widehat{\varphi}_{t+i})\right] \tag{5.27}$$
$$\widehat{Y}_t^f = \left(\frac{1+\eta}{\sigma+\eta}\right)\widehat{Z}_t \tag{5.28}$$

For the equation 5.26, from $(P_t)^{1-\theta} = (1-\omega)(p_t^*)^{1-\theta} + \omega(P_{t-1})^{1-\theta}$,

$$\begin{aligned}(1-\theta)\widehat{P}_t &= \frac{(1-\omega)(\overline{p}^*)^{1-\theta}}{(1-\omega)(\overline{p}^*)^{1-\theta}+\omega(\overline{P})^{1-\theta}}\widehat{(p_t^*)^{1-\theta}} + \frac{\omega(\overline{P})^{1-\theta}}{(1-\omega)(\overline{p}^*)^{1-\theta}+\omega(\overline{P})^{1-\theta}}\widehat{(P_{t-1})^{1-\theta}} \\ &= (1-\omega)(1-\theta)\widehat{p}_t^* + \omega(1-\theta)\widehat{P}_{t-1}\end{aligned}$$

which gives us

$$\widehat{P}_t = (1-\omega)\widehat{p}_t^* + \omega\widehat{P}_{t-1}$$

For the equation 5.27, let us rearranging terms in $\mathbb{E}_t\left[\sum_{i=0}^{\infty}\omega^i \triangle_{i,t+i}\left\{(1-\theta)\left(\frac{p_t^*}{P_{t+i}}\right)+\theta\varphi_{t+i}\right\}\left(\frac{1}{p_t^*}\right)\left(\frac{p_t^*}{P_{t+i}}\right)^{-\theta}C_{t+i}\right] = 0$, so that we have

$$\mathbb{E}_t\left[\sum_{i=0}^{\infty}(\beta\omega)^i\Phi_{i,t+i}(\theta-1)(p_t^*)^{-\theta}(P_{t+i})^{\theta-1}C_{t+i}\right] = \mathbb{E}_t\left[\sum_{i=0}^{\infty}(\beta\omega)^i\Phi_{i,t+i}\theta\varphi_{t+i}(p_t^*)^{-1-\theta}(P_{t+i})^{\theta}C_{t+i}\right] \tag{5.29}$$

where $\Phi_{i,t+i} = \frac{C_{t+i}^{-\sigma}}{C_t^{-\sigma}}$. Now, notice that

$$\widehat{\sum \alpha^t x_t} = x_0 + \alpha \widehat{x_1 + \alpha^2} x_2 + \ldots = \frac{1}{\overline{x} + \alpha\overline{x} + \alpha^2\overline{x}\ldots}\sum_{t=0}^{\infty}\overline{x}\alpha^t\widehat{\alpha^t x_t} = \frac{1}{\frac{1}{1-\alpha}}\sum_{t=0}^{\infty}\alpha^t\widehat{x}_t = (1-\alpha)\sum_{t=0}^{\infty}\alpha^t\widehat{x}_t$$

Using this principle, the log linearization of the LHS of the equation 5.29 can be written as

$$(1-\beta\omega)\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i\Big\{\widehat{\Phi}_{i,t+i} - \theta\widehat{p}_t^* + (\theta-1)\widehat{P}_{t+i} + \widehat{C}_{t+i}\Big\}\Big]$$

By the same manner, the log linearization of the RHS of the equation 5.29 is

$$(1-\beta\omega)\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i\Big\{\widehat{\Phi}_{i,t+i} + \widehat{\varphi}_{t+i} - (1+\theta)\widehat{p}_t^* + \theta\widehat{P}_{t+i} + \widehat{C}_{t+i}\Big]$$

Thus, we have

$$(1-\beta\omega)\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i\Big\{\cancel{\widehat{\Phi}_{i,t+i}} - \cancel{\theta\widehat{p}_t^*} + (\cancel{\theta}-1)\widehat{P}_{t+i} + \cancel{\widehat{C}_{t+i}}\Big\}\Big] = (1-\beta\omega)\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i\Big\{\cancel{\widehat{\Phi}_{i,t+i}} + \widehat{\varphi}_{t+i} - (1+\cancel{\theta})\widehat{p}_t^* + \cancel{\theta\widehat{P}_{t+i}} + \cancel{\widehat{C}_{t+i}}\Big\}\Big]$$

which can be rearranged as

$$\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i(\widehat{P}_{t+i} + \widehat{\varphi}_{t+i} - \widehat{p}_t^*)\Big] = 0 \;\Leftrightarrow\; \widehat{p}_t^* = \underbrace{(1-\beta\omega)\mathbb{E}_t\Big[\sum_{i=0}^{\infty}(\beta\omega)^i(\widehat{P}_{t+i} + \widehat{\varphi}_{t+i})\Big]}_{\text{expected PV of nominal MC}}$$

Lemma 5.1. $\widehat{Y}_t' = \widehat{Y}_t$.

Proof. Observe that

$$\widehat{Y}_t = \Big[\widehat{\int_0^1 (y_{jt})^{\frac{1-\theta}{\theta}}}dj\Big]^{\frac{\theta}{1-\theta}} = \frac{\theta}{1-\theta}\Big[\int_0^1 \widehat{(y_{jt})^{\frac{1-\theta}{\theta}}}dj\Big] = \int_0^1 \widehat{y}_{jt}dj$$

and

$$\widehat{Y}_t' = [\widehat{\int_0^1 y_{jt}dj}] = \int_0^1 \widehat{y}_{jt}dj$$

∎

5.4.5 New Keynesian IS and Phillips Curve

From the linearized version of the model, we can derive the New Keynesian IS and Phillips Curve. Combining the equations 5.23 and 5.25, we have

$$-\sigma\widehat{C}_t = \widehat{i}_t + \mathbb{E}_t[-\Pi_{t+1} - \sigma\widehat{C}_{t+1}] \;\Leftrightarrow\; \widehat{Y}_t = \mathbb{E}_t[\widehat{Y}_{t+1}] - \frac{1}{\sigma}\Big\{\underbrace{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}]}_{\equiv r_t}\Big\} \tag{5.30}$$

By the same manner, we can also obtain the equivalent representation from the flexible price economy, [36]

$$\widehat{Y}_t^f = \mathbb{E}_t[\widehat{Y}_{t+1}^f] - \frac{1}{\sigma} r_t^f \tag{5.31}$$

Subtracting the equation 5.31 from 5.30 yields

$$\begin{aligned}
\widehat{Y}_t - \widehat{Y}_t^f &= \mathbb{E}_t[\widehat{Y}_{t+1} - \widehat{Y}_{t+1}^f] - \frac{1}{\sigma}[r_t - r_t^f] \\
&= \mathbb{E}_t[\widehat{Y}_{t+1} - \widehat{Y}_{t+1}^f] - \frac{1}{\sigma}\left\{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}] - r_t^f\right\}
\end{aligned}$$

Define output gap $x_t \equiv \widehat{Y}_t - \widehat{Y}_t^f$, we get the New Keynesian (or Dynamic) IS curve as follows.

$$x_t = \mathbb{E}_t[x_{t+1}] - \frac{1}{\sigma}\left\{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}] - r_t^f\right\} \tag{5.32}$$

Notice that (holding $\mathbb{E}_t[x_{t+1}]$ fixed) the real interest rate gap $r_t - r_t^f$ (equivalently, $\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}] - r_t^f$) is negatively correlated with the output gap.
Now, from the equation 5.26, we have

$$\widehat{P}_t - \omega\widehat{P}_t = (1-\omega)\widehat{p}_t^* + \omega\widehat{P}_{t-1} - \omega\widehat{P}_t \quad \Leftrightarrow \quad (1-\omega)(\widehat{p}_t^* - \widehat{P}_t) = \omega\underbrace{(\widehat{P}_t - \widehat{P}_{t-1})}_{\equiv \Pi_t} \quad \Leftrightarrow \quad \widehat{p}_t^* = \frac{\omega}{(1-\omega)}\Pi_t + \widehat{P}_t \tag{5.33}$$

In addition, representing the equation 5.27 in a recursive formation yields

$$\widehat{p}_t^* = (1-\beta\omega)\mathbb{E}_t\left[\sum_{i=0}^{\infty}(\beta\omega)^i(\widehat{P}_{t+i} + \widehat{\varphi}_{t+i})\right] \quad \Leftrightarrow \quad \widehat{p}_t^* = (1-\beta\omega)(\widehat{P}_t + \widehat{\varphi}_t) + \beta\omega\mathbb{E}_t[\widehat{p}_{t+1}^*] \tag{5.34}$$

Combining the equations 5.33 and 5.34 gives us

$$(1-\beta\omega)(\widehat{P}_t+\widehat{\varphi}_t)+\beta\omega\mathbb{E}_t[\widehat{p}_{t+1}^*] = \frac{\omega}{(1-\omega)}\Pi_t+\widehat{P}_t \quad \Leftrightarrow \quad (1-\beta\omega)(\widehat{P}_t+\widehat{\varphi}_t)+\beta\omega\mathbb{E}_t[\frac{\omega}{(1-\omega)}\Pi_{t+1}+\widehat{P}_{t+1}] = \frac{\omega}{(1-\omega)}\Pi_t+\widehat{P}_t$$

Rearranging terms for Π_t yields

$$\begin{aligned}
\frac{\omega}{(1-\omega)}\Pi_t &= -\widehat{P}_t + (1-\beta\omega)(\widehat{P}_t + \widehat{\varphi}_t) + \beta\omega\mathbb{E}_t[\frac{\omega}{(1-\omega)}\Pi_{t+1} + \widehat{P}_{t+1}] \\
&= (1-\beta\omega)\widehat{\varphi}_t - \beta\omega\widehat{P}_t + \beta\omega\mathbb{E}_t[\frac{\omega}{(1-\omega)}\Pi_{t+1}] + \beta\omega\mathbb{E}_t[\widehat{P}_{t+1}] \\
&= (1-\beta\omega)\widehat{\varphi}_t + \frac{\beta\omega}{(1-\omega)}\mathbb{E}_t[\Pi_{t+1}]
\end{aligned}$$

Thus, we have

$$\Pi_t = \underbrace{\frac{(1-\omega)(1-\beta\omega)}{\omega}}_{\equiv\widetilde{\kappa}}\widehat{\varphi}_t + \beta\mathbb{E}_t[\Pi_{t+1}] \tag{5.35}$$

Notice that $\widehat{\varphi}_t$ is real marginal cost, but interconnected with inflation (or monetary sector), and hence we can know that the traditional dichotomy does not work here. Conversely, letting $\omega \to 0$ yields the flexible price economy where we have the vertical Phillips curve, which means that the real sector is determined independently from Π_t.

[36] Notice that r_t^f and r_t^n are interchangeable.

Now, combining the equations 5.22, 5.24 and 5.25, we have

$$\widehat{\varphi}_t = \eta(\widehat{Y}_t - \widehat{Z}_t) + \sigma\widehat{Y}_t - \widehat{Z}_t = (\eta+\sigma)\widehat{Y}_t - (1+\eta)\widehat{Z}_t$$

From the equation 5.28, we obtain

$$\widehat{Y}_t^f = \frac{1+\eta}{\sigma+\eta}\widehat{Z}_t \;\Leftrightarrow\; 0 = (\eta+\sigma)\widehat{Y}_t^f - (1+\eta)\widehat{Z}_t$$

Therefore, $\widehat{\varphi}_t = (\sigma+\eta)x_t$. Plugging this into the equation 5.35, we finally get the Phillips curve as follows.

$$\Pi_t = \underbrace{\widetilde{\kappa}(\sigma+\eta)}_{\equiv\kappa} x_t + \beta\mathbb{E}_t[\Pi_{t+1}] = \beta\mathbb{E}_t[\Pi_{t+1}] + \kappa x_t \tag{5.36}$$

In conclusion, we summarize all the first order conditions of private sector as two equations, NK IS curve and Phillips curve.

$$\Pi_t = \beta\mathbb{E}_t[\Pi_{t+1}] + \kappa x_t \;\text{ and }\; x_t = \mathbb{E}_t[x_{t+1}] - \frac{1}{\sigma}\Big\{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}] - r_t^f\Big\}$$

As you can see from the equations, the real side x_t and the monetary side Π_t are not independently but jointly determined, and hence the monetary policy matters in the sense that it affects the real side. For example, if $x_t > 0$ (i.e. economy is overheated), we see that Π_t also increases.

5.4.6 Solving a Model with Monetary Policy

Notice that in IS curve and Phillips curve, we have three unknowns but only two equations. To close the model, we need to introduce the monetary policy as we did in classical dichotomy model. Consider the following feedback rule (or Taylor rule)

$$\widehat{i}_t = \phi_\pi\Pi_t + \phi_x x_t + \epsilon_t$$

where ϵ_t is a monetary policy shock. First, notice that for unique equilibrium path, ϕ_π and ϕ_x should satisfy

- Assume $\phi_x = 0$, then $\phi_\pi > 1$ ensures the long-run response of interest rate to inflation is greater than 1.
- Assume $\phi_x \neq 0$, in the long-run $\Pi = \beta\Pi + \kappa x \;\Leftrightarrow\; \frac{(1-\beta)}{\kappa}\Pi = x$, thus $i_t = \phi_\pi\Pi_t + \phi_x x_t \rightarrow [\phi_\pi + \frac{(1-\beta)}{\kappa}\phi_x] > 1$ is needed.

Now, suppose that there is no technology shock, then we have $r_t^f = 0$. Further assume that $\epsilon_t = \rho\epsilon_{t-1} + v_t$ where $v_t \sim (0, \sigma_v^2)$. Then, we can solve the following equations by the method of undetermined coefficients.

$$\Pi_t = \beta\mathbb{E}_t[\Pi_{t+1}] + \kappa x_t, \quad x_t = \mathbb{E}_t[x_{t+1}] - \frac{1}{\sigma}\Big\{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}]\Big\} \;\text{ and }\; \widehat{i}_t = \phi_\pi\Pi_t + \phi_x x_t + \epsilon_t$$

Notice that all three variables are forward looking, there is no endogenous state variable, and the model is linear. Guess

$$\Pi_t = \delta_\pi\epsilon_t, \quad x_t = \delta_x\epsilon_t \;\text{ and }\; i_t = \delta_i\epsilon_t$$

Since

$$\mathbb{E}_t[\Pi_{t+1}] = \delta_\pi\mathbb{E}_t[\epsilon_{t+1}] = \delta_\pi\mathbb{E}_t[\rho\epsilon_t + v_{t+1}] = \delta_\pi\rho\epsilon_t$$
$$\mathbb{E}_t[x_{t+1}] = \delta_x\mathbb{E}_t[\epsilon_{t+1}] = \delta_x\mathbb{E}_t[\rho\epsilon_t + v_{t+1}] = \delta_x\rho\epsilon_t$$

Therefore, we have

$$\begin{aligned}
&\Pi_t = \beta \mathbb{E}_t[\Pi_{t+1}] + \kappa x_t \quad \Leftrightarrow \quad \delta_\pi \epsilon_t = (\beta \delta_\pi \rho + \kappa \delta_x)\epsilon_t \quad \Leftrightarrow \quad \delta_\pi = \beta \delta_\pi \rho + \kappa \delta_x \\
&x_t = \mathbb{E}_t[x_{t+1}] - \frac{1}{\sigma}\Big\{\widehat{i}_t - \mathbb{E}_t[\Pi_{t+1}]\Big\} \quad \Leftrightarrow \quad \delta_x \epsilon_t = \left[\delta_x \rho - \frac{1}{\sigma}(\delta_i - \delta_\pi \rho)\right]\epsilon_t \quad \Leftrightarrow \quad \delta_x = \delta_x \rho - \frac{1}{\sigma}(\delta_i - \delta_\pi \rho) \\
&\widehat{i}_t = \phi_\pi \Pi_t + \phi_x x_t + \epsilon_t \quad \Leftrightarrow \quad \delta_i \epsilon_t = (\phi_\pi \delta_\pi + \phi_x \delta_x + 1)\epsilon_t \quad \Leftrightarrow \quad \delta_i = \phi_\pi \delta_\pi + \phi_x \delta_x + 1
\end{aligned}$$

Solving this system of equations for δ_π, δ_x and δ_i gives us the solution. For example, assume that $\rho = 0$, (i.e. shock is i.i.d), $\phi_x = 0$ and $\sigma = 1$, then we will have

$$\delta_\pi = -\frac{1}{\phi_\pi + \frac{1}{\kappa}}, \quad \delta_x = \frac{1}{\kappa}\delta_\pi \quad \text{and} \quad \delta_i = \phi_\pi \delta_\pi + 1$$

5.4.7 Optimal Monetary Policy

Woodford demonstrated that deviations of the expected discounted utility of the representative agent around the level of steady-state utility can be approximated by [37]

$$EDU \approx -\Omega \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\Pi_{t+i}^2 + \lambda (x_{t+i} - x^*)^2\right)\right] + \text{t.i.p.}$$

where t.i.p. indicates terms independent of policy and $x^* \equiv \log(\frac{\overline{y}^e}{\overline{y}})$. In order to remove markup wedge, assume that the firms are subsidized by lump-sum transfers to produce the efficient output level, so that $x^* = 0$. [38] Then, the central bank seeks to solve

$$\begin{gathered}
\min_{\{\Pi_{t+i}, x_{t+i}, \widehat{i}_{t+i}\}_{i=0}^{\infty}} \frac{1}{2}\mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\Pi_{t+i}^2 + \lambda x_{t+i}^2\right)\right] \text{ s.t.} \\
x_{t+i} = \mathbb{E}_{t+i}[x_{t+i+1}] - \frac{1}{\sigma}\Big\{\widehat{i}_{t+i} - \mathbb{E}_{t+i}[\Pi_{t+i+1}] - r_{t+i}^f\Big\} \text{ and } \Pi_{t+i} = \beta \mathbb{E}_{t+i}[\Pi_{t+i+1}] + \kappa x_{t+i}, \ \forall i \geq 0
\end{gathered}$$

First, it is very important to recognize that the **IS constraint is slack** in the sense that $\widehat{i}_{t+i}$ is not included in objective function, and hence we can always satisfy that constraint by properly choosing $\widehat{i}_{t+i}$.
It is obvious that the first best case is to achieve $\Pi_{t+i}^* = 0$ and $x_{t+i}^* = 0$ for all $i \geq 0$. For this, the central bank just need to achieve $\widehat{i}_{t+i} = r_{t+i}^f$ for all $i \geq 0$, which means that whenever there is a productivity shock, nominal interest rate moves to track and follow r_{t+i}^f, which is implementable by using the following Taylor rule

$$\widehat{i}_{t+i} = \underbrace{r_{t+i}^f}_{\text{shock}} + \phi_\pi \Pi_{t+i} \quad \text{where} \quad \phi_\pi > 1$$

Notice that any $\phi_\pi > 1$ works, and since $\Pi_{t+i}^* = 0$ in equilibrium, $\phi_\pi \Pi_{t+i}$ plays a role of off-path commitment.
Now, let us consider the case where there is a cost shock in the Phillips curve.

$$\Pi_t = \beta \mathbb{E}_t[\Pi_{t+1}] + \kappa x_t + \epsilon_t$$

[37] Inflation generates the dispersion of price, which leads to the dispersion of output and hence consumption. Since the marginal utility is diminishing, this dispersion affects the representative household negatively.

[38] For welfare analysis, we need to define $x_t \equiv \widehat{Y}_t - \widehat{Y}_t^e$ not $\widehat{Y}_t - \widehat{Y}_t^f$. By subsidizing, we can make the monopolistic flexible price economy identical to the perfect competition economy.

In this case, $\Pi^*_{t+i} = 0$ and $x^*_{t+i} = 0$ for all $i \geq 0$ is still desirable but not feasible because of ϵ_t as shown below. That is, setting $x^*_{t+i} = 0$ for all $i \geq 0$ does not guarantee $\Pi^*_t = 0$ anymore even under i.i.d. shock

$$\begin{aligned}\Pi_t &= \beta \mathbb{E}_t \left[\beta \mathbb{E}_{t+1}[\Pi_{t+2}] + \kappa x_{t+1} + \epsilon_{t+1}\right] + \kappa x_t + \epsilon_t \\ &= \beta^2 \mathbb{E}_t[\Pi_{t+2}] + \beta\kappa \mathbb{E}_t[x_{t+1}] + \kappa x_t + \beta \mathbb{E}_t[\epsilon_{t+1}] + \epsilon_t \\ &= \underbrace{\kappa \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t[x_{t+i}]}_{=0} + \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t[\epsilon_{t+i}]\end{aligned}$$

In this situation, we want to find the second best outcome depending on whether commitment is possible or not.

5.4.8 Commitment and Time Inconsistency

Under the commitment, the central bank can choose (or directly control future variables), and hence they seek to solve

$$\min_{\{\Pi_{t+i}, x_{t+i}\}_{i=0}^{\infty}} \frac{1}{2} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\Pi^2_{t+i} + \lambda x^2_{t+i}\right)\right] \text{ s.t. } \Pi_{t+i} = \beta \mathbb{E}_{t+i}[\Pi_{t+i+1}] + \kappa x_{t+i} + \epsilon_{t+i}, \ \forall i \geq 0$$

Then, the corresponding Lagrangian function is [39]

$$\mathcal{L}_t = \frac{1}{2} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(\Pi^2_{t+i} + \lambda x^2_{t+i} + \psi_{t+i}\left[\Pi_{t+i} - \beta \Pi_{t+i+1} - \kappa x_{t+i} - \epsilon_{t+i}\right]\right)\right]$$

The first order conditions for Π_{t+i} and x_{t+i} are

$$\Pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} = 0, \quad \forall i \geq 0 \text{ where } \psi_{t-1} = 0 \tag{5.37}$$

and

$$\lambda x_{t+i} - \kappa \psi_{t+i} = 0, \ , \ \forall i \geq 0 \tag{5.38}$$

Rewriting the first order condition 5.37, we have

$$\Pi_t + \psi_t = 0 \text{ and } \Pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} = 0, \quad \forall i \geq 1$$

which reveals the dynamic inconsistency that characterizes the optimal commitment policy. At time t, the central bank sets $\Pi_t = -\psi_t$ and promises to set $\Pi_{t+1} = -(\psi_{t+1} - \psi_t)$ in the future. But when period $t+1$ arrives, the central bank re-optimization will again obtain $\Pi_{t+1} = -\psi_{t+1}$ as its optimal setting for inflation.
Alternative definition of an optimal commitment policy requires that the central bank satisfy conditions

$$\Pi_{t+i} + \psi_{t+i} - \psi_{t+i-1} = 0 \text{ and } \lambda x_{t+i} - \kappa \psi_{t+i} = 0, \quad \forall i \geq 0$$

which is labeled as the **timeless perspective** approach by Woodford in the sense that time t is not special. Combining these two condition yields the following **targeting rule**

$$\Pi_{t+i} = -\frac{\lambda}{\kappa}(x_{t+i} - x_{t+i-1}) \tag{5.39}$$

[39]By law of iterated expectation, we can write $\mathbb{E}_t[\mathbb{E}_{t+i}[\Pi_{t+i+1}]]= \mathbb{E}_t[\Pi_{t+i+1}]$ in the constraint.

5.4.9 Discretion

When the central bank operates with discretion, it acts each period to minimize the loss function subject to Phillips curve. Since the decisions at date t does not bind in any future dates, the central bank is unable to affect the private sector's expectation about future inflation. More precisely, period t and $t+1$ are seemingly linked through $\mathbb{E}_t[\Pi_{t+1}]$, however, $\mathbb{E}_t[\Pi_{t+1}]$ is actually a function of exogenous shock process $\{\epsilon_t\}$ because it is the only state variables. In other words, if there is no endogenous state variable, $\mathbb{E}_t[\Pi_{t+1}]$ is just exogenous to the central bank. Thus, the central bank seeks to solve

$$\min_{\{\Pi_t, x_t\}} \frac{1}{2}\left(\Pi_t^2 + \lambda x_t^2\right) \text{ s.t. } \Pi_t = \beta \mathbb{E}_t[\Pi_{t+1}] + \kappa x_t + \epsilon_t$$

The first order conditions for Π_t and x_t are

$$\Pi_t + \psi_t = 0 \text{ and } \lambda x_{t+1} - \kappa \psi_t = 0$$

Thus, we have the following targeting rule.

$$\Pi_t = -\frac{\lambda}{\kappa} x_t$$

Notice that the targeting rule is the same under discretion or under the commitment only at period t.
Now, we want to find the equilibrium expressions for Π_t and x_t under the discretion. Plugging the targeting rule into Phillips curve gives us

$$\left(1 + \frac{\kappa^2}{\lambda} x_t\right) = \beta \mathbb{E}_t[x_{t+1}] - \left(\frac{\kappa}{\lambda}\right) \epsilon_t$$

Guessing a solution of the form $x_t = \delta_x \epsilon_t$, we obtain δ_x by the method of undetermined coefficients

$$\delta_x = -\left[\frac{\kappa}{\lambda(1-\beta\rho) + \kappa^2}\right]$$

If $\rho = 0$, then we get

$$x_t = -\frac{\kappa}{(\lambda + \kappa^2)} \epsilon_t \text{ and } \Pi_t = \frac{\lambda}{(\lambda + \kappa^2)} \epsilon_t$$

Lastly, we can back out $\widehat{i}_t$ from the IS. Given that $\rho = 0$ and there is no productivity shock, we have

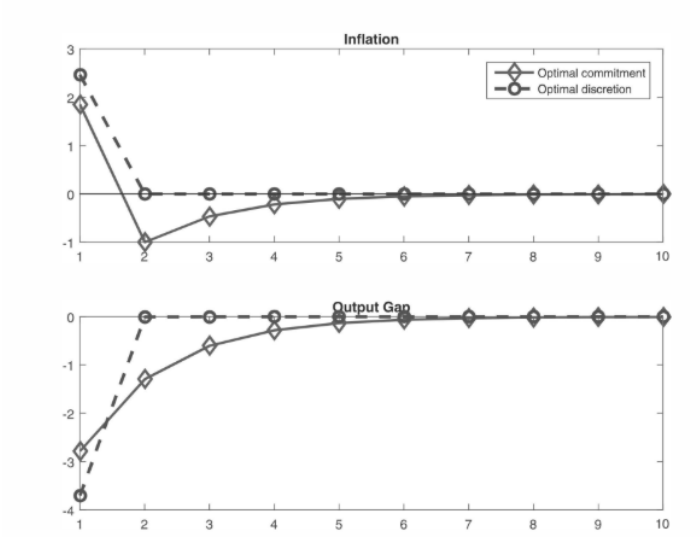
$$\widehat{i}_t = -\sigma x_t = \frac{\sigma\kappa}{(\lambda + \kappa^2)} \epsilon_t$$

To achieve this equilibrium, the monetary policy should follow

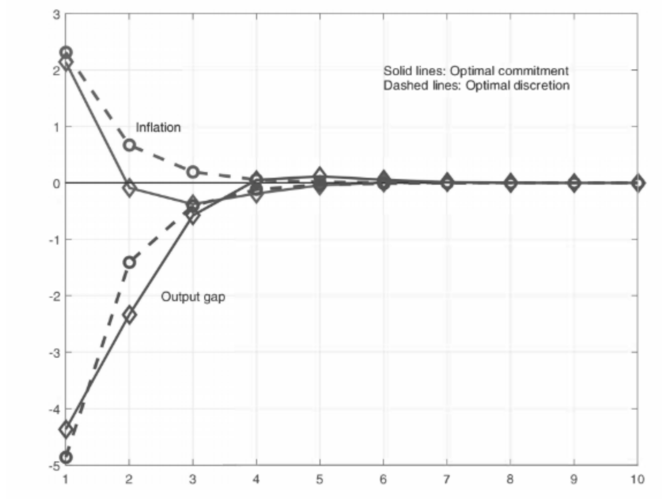
$$\begin{aligned}
\widehat{i}_t &= \frac{\sigma\kappa}{(\lambda + \kappa^2)} \epsilon_t + \phi_\pi [\Pi_t - \Pi_t] \\
&= \frac{(\sigma\kappa - \lambda\phi_\pi)}{(\lambda + \kappa^2)} \epsilon_t + \phi_\pi \Pi_t \quad \text{where} \quad \phi_\pi > 1
\end{aligned}$$

5.4.10 Commitment versus Discretion

Under the commitment, the central bank has more tools to achieve their goal in the sense that they can directly affect future variables. Thus, the central bank wants to absorb the cost shock smoothly over time as depicted in sub-figure(a) below.



(a) Commitment versus Discretion without endogenous state variable



(b) Commitment versus Discretion with endogenous state variable

5.4.11 Extension of Commitment and Discretion with Endogenous State Variable

Suppose that [40]

$$\Pi_t = (1-\phi)\beta\mathbb{E}_t[\Pi_{t+1}] + \phi\Pi_{t-1} + \kappa x_t + \epsilon_t$$

Under the commitment, the Lagrangian function can be written as

$$\mathcal{L}_t = \frac{1}{2}\mathbb{E}_t\left[\sum_{i=0}^{\infty}\beta^i\left(\Pi_{t+i}^2 + \lambda x_{t+i}^2 + \psi_{t+i}\left[\Pi_{t+i} - (1-\phi)\beta\Pi_{t+i+1} - \phi\Pi_{t+i-1} - \kappa x_{t+i} - \epsilon_{t+i}\right]\right)\right]$$

The first order conditions for Π_{t+i} and x_{t+i} are

$$\Pi_{t+i} + \psi_{t+i} - (1-\phi)\psi_{t+i-1} - \beta\phi\mathbb{E}_t[\psi_{t+i+1}] = 0, \quad \forall i \geq 0 \quad \text{where} \quad \psi_{t-1} = 0$$

and

$$\lambda x_{t+i} - \kappa\psi_{t+i} = 0, \quad \forall i \geq 0$$

Notice that we still have time inconsistency in this model unless $\phi = 1$.
Under the discretion, it should be notice that the problem is not static because we have an endogenous state variable Π_{t-1}. Adopting the Markov Perfect equilibrium as a solution concept, we can write the corresponding bellman equation as below.

$$V(\Pi_{t-1}, \epsilon_t) = \min_{\{\Pi_t, x_t\}} \frac{1}{2}\left(\Pi_t^2 + \lambda x_t^2\right) + \beta\mathbb{E}_t[V(\Pi_t, \epsilon_{t+1})] \text{ s.t. } \Pi_t = (1-\phi)\beta\mathbb{E}_t[\Pi_{t+1}] + \phi\Pi_{t-1} + \kappa x_t + \epsilon_t \tag{5.40}$$

First, guess that

$$\Pi_t = b_1\epsilon_t + b_2\Pi_{t-1} \quad \text{and} \quad x_t = b_3\epsilon_t + b_4\Pi_{t-1}$$

which yields

$$\mathbb{E}_t[\Pi_{t+1}] = b_2\Pi_t \quad \text{and} \quad \mathbb{E}_t[x_{t+1}] = b_4[b_1\epsilon_t + b_2\Pi_{t-1}]$$

[40] If $\phi = 0$, then we get back to the standard case.

Plugging our guess in the Phillips curve, we obtain

$$\Pi_t = (1-\phi)\beta b_2 \Pi_t + \phi \Pi_{t-1} + \kappa x_t + \epsilon_t \tag{5.41}$$

Then, the RHS of the equation 5.40 can be written as

$$\mathcal{L}_t = \frac{1}{2}\left(\Pi_t^2 + \lambda x_t^2\right) + \beta \mathbb{E}_t[V(\Pi_t, \epsilon_{t+1})] + \theta_t \left[\Pi_t - (1-\phi)\beta b_2 \Pi_t + \phi \Pi_{t-1} + \kappa x_t + \epsilon_t\right]$$

The first order conditions for Π_t and x_t are

$$\Pi_t + \beta \mathbb{E}_t[V_\pi(\Pi_t, \epsilon_{t+1})] + \theta_t[1-(1-\phi)\beta b_2] = 0 \quad \text{and} \quad \lambda x_t - \theta_t \kappa = 0 \tag{5.42}$$

Applying the envelope theorem on the LHS of the equation 5.40, we get

$$V_\pi(\Pi_{t-1}, \epsilon_t) = -\phi\theta_t \;\Rightarrow\; \mathbb{E}_t[V_\pi(\Pi_t, \epsilon_{t+1})] = -\phi \mathbb{E}_t[\theta_{t+1}] \tag{5.43}$$

Combining two conditions 5.42 and 5.43 yields

$$\Pi_t = \frac{\beta\phi\lambda}{\kappa}\mathbb{E}_t[x_{t+1}] - [1-(1-\phi)\beta b_2]\frac{\lambda x_t}{\kappa} \tag{5.44}$$

Plugging our guess into the equation 5.44 gives us

$$b_1\epsilon_t + b_2\Pi_{t-1} = \frac{\beta\phi\lambda}{\kappa} b_4[b_1\epsilon_t + b_2\Pi_{t-1}] - \frac{\lambda}{\kappa}[1-(1-\phi)\beta b_2][b_3\epsilon_t + b_4\Pi_{t-1}] \tag{5.45}$$

From the given Phillips curve and inter-temporal Euler equation, using the method of undetermined coefficients gives us

$$\begin{aligned} b_1 &= (1-\phi)\beta b_1 b_2 + \kappa b_3 + 1 \quad \text{and} \quad b_2 = (1-\phi)\beta b_2^2 + \phi + \kappa b_4 \\ b_1 &= \frac{\beta\phi\lambda}{\kappa} b_1 b_4 - \frac{\lambda}{\kappa}[1-(1-\phi)\beta b_2] b_3 \quad \text{and} \quad b_2 = \frac{\beta\phi\lambda}{\kappa} b_2 b_4 - \frac{\lambda}{\kappa}[1-(1-\phi)\beta b_2] b_4 \end{aligned}$$

Solving this system of equations for b_1, b_2, b_3 and b_4 gives us the solution, which may not be unique. Moreover, for bounded solution, we also need to check $\mid b_2 \mid < 1$.
Lastly, and more importantly, in this discretion case, we see an inertia in the solution $\Pi_t = b_1\epsilon_t + b_2\Pi_{t-1}$, and therefore discretion and commitment **are not drastically different** as shown in sub-figure(b) above.