Eight-Space Quaternion Approach for Robotic Hand-eye Calibration

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Abstract— In this paper, normalized quaternions (Euler parameters) are used to transform the kinematic equation into two simple and well-structured linear systems; where the kinematic equation, $\boldsymbol{H}_l\boldsymbol{H}_x = \boldsymbol{H}_x\boldsymbol{H}_c$, is formulated for the problem of finding the relative position and orientation between the reference frames of a link-mounted sensor and the link. Two distinct robot movements are required to obtain a unique solution. This leads to an overdetermined system of equations, and the least-squares solution are obtained. Further, least-squares closed-form solutions to these systems are derived using the Gaussian elimination and Schur decomposition analysis. The selection criterion and the solution formulae can be easily incorporated in application programs which require the calculation of the relative position and orientation of the sensor.

1 Introduction

In order to locate an object in the base coordinate frame of a robot manipulator, the relative position and orientation between the coordinate frame of the link on which a camera is rigidly mounted and the coordinate frame of the camera has to be known by the robot controller. The task of finding the position and orientation of the camera relative to the robot link on which the camera is installed is referred to as the robot hand-to-eye calibration by Tsai and Lenz (1989) [1], for the camera (i.e., eye) is usually mounted rigidly on the last link of the robot (i.e., hand or gripper).

In related publications [1, 2, 3, 4, 5, 6, 7, 8], there is a mathematical approach used to solve above problem by formulating it as a equation of the form $\boldsymbol{H}_l\boldsymbol{H}_x = \boldsymbol{H}_x\boldsymbol{H}_c$; where $\boldsymbol{H}_l$ and $\boldsymbol{H}_c$ are known as relative coordinate transformations of link and that of camera, respectively, and $\boldsymbol{H}_x$ is unknown coordinate transformation. It is important to be specific about the location of the sensor with respect to the link on which the sensor is mounted.

In this study, we are interested in establishing a totally different, more compact, quaternion formulation than before for the hand/eye calibration. At first, two 4 by 4 linear systems, which highly facilitate the solutions of least-squares methods, are formulated with two quaternion unknowns: $\mathbf{p}_x$ and $\mathbf{r}_x$. The 4 by 1 quaternion vector $\mathbf{p}_x$ specifies the unknown rotation matrix $\boldsymbol{R}_x$, and $\mathbf{r}_x$ gives the unknown translation vector.

In order to obtain unique solutions for rotation and translation, two distinct movements have to be made to obtain two sets of data. This leads to an overdetermined system of equations. Typically this overdetermined system can be solved by least-squares methods as proposed by Shiu and Ahmad (1987). However, we develop simple criteria for choosing an independent set of equations automatically. Furthermore, we combine equations of rotation and translation to be a overdetermined homogeneous system when given n data sets. The least-squares solutions of the overdetermined system can minimize the error with noise existing. This involving finding the exact solution of the normal equation of the original overdetermined system. Method developed in this direction is referred to as *eight space method*.

Closed-form least-squares solutions to the system of the normal equation are derived using the Gaussian elimination analysis and Schur decomposition. A criterion for selecting the independent set of equations is developed. A set of closed-form formulae for the solution of these equations are derived. The selection criterion and the solution formulae can be easily incorporated in application programs which require the calculation of the relative position and orientation of the sensor.

The objective of this paper is to employ quaternions approach to reformulate the kinematic equation of hand-eye calibration problem. Consequently, the entire simplified process, non-iterative, of attaining least squares solutions is speedy and accurate.

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Further, the closed-form least-squares solution based on quaternions has been presented.

2 Quaternion Formulations

Writing the homogeneous transformation equation in its rotational and translational transformations gives

$$\begin{bmatrix} \boldsymbol{R}_l & \mathbf{r}_l \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_x & \mathbf{r}_x \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_x & \mathbf{r}_x \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_c & \mathbf{r}_c \\ 0^T & 1 \end{bmatrix}.$$

Expanding it and equating both sides yields the following two sets of equations:

$$\boldsymbol{R}_l \boldsymbol{R}_x = \boldsymbol{R}_x \boldsymbol{R}_c \tag{1}$$

and

$$(\boldsymbol{R}_l - \boldsymbol{I})\mathbf{r}_x = \boldsymbol{R}_x \mathbf{r}_c - \mathbf{r}_l \tag{2}$$

where $\boldsymbol{I}$ is a 3×3 identity matrix, and the homogeneous transformation matrix $\boldsymbol{H}$ includes a rotational transformation in terms of a 3×3 rotation matrix $\boldsymbol{R}$ and a translational transformation in terms of a 3×1 spatial vector $\mathbf{r}$. The first equation which happens to be a constraint equation among the rotation matrices is called *the equation of rotation.* The second equation which blends physical quantities related to rotations and translations is called *the equation of translation.*

2.1 The Equation of Rotation

As we know, the equation of rotation is given in terms of finite rotation matrices. The manipulation of equations with successive rotation matrices is difficult, especially when we intend to apply a least-squares method to the problem. An equivalent equation of rotation in the form of a linear system is preferable. This can be achieved, when we formulate the equation of rotation in quaternion space.

The quaternion $\mathbf{p}$, or $[\cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\mathbf{u}]^T$, is used to specify the rotation matrix $\boldsymbol{R}$ hereafter, where θ and $\mathbf{u}$ are the angle and axis of rotation of $\boldsymbol{R}$ respectively. In quaternions, we have

$$\overset{\pm}{p} = \begin{bmatrix} e_0 & -\mathbf{e}^T \\ \mathbf{e} & e_0\boldsymbol{I} - \pm\tilde{\mathbf{e}} \end{bmatrix}, \quad \tilde{\mathbf{e}} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix};$$

where $\tilde{\mathbf{e}}$ denotes a skew symmetric matrix corresponding to $\mathbf{e}$. The more fundamentals of quaternions algebra can be found in [9].

The given equation of rotation can be written in terms of quaternion matrices as

$$\begin{aligned} \boldsymbol{R}_c &= \boldsymbol{R}_x^T \boldsymbol{R}_l \boldsymbol{R}_x \\ &= \left(\bar{p}_x \overset{+}{p}_x{}^T\right)\left(\overset{+}{p}_l \bar{p}_l{}^T\right)\left(\overset{+}{p}_x \bar{p}_x{}^T\right) = \overset{+}{p}_x{}^T \overset{+}{p}_l \bar{p}_x \bar{p}_l{}^T \bar{p}_x{}^T \overset{+}{p}_x \\ &= (\overset{+}{p}_x{}^T \overset{+}{p}_l \overset{+}{p}_x)(\bar{p}_x \bar{p}_l \bar{p}_x{}^T)^T = \overset{+}{p}_c \bar{p}_c{}^T, \end{aligned} \tag{3}$$

since

$$\begin{cases} \left(\overset{+}{\overbrace{\overset{+}{p}_x{}^T \overset{+}{p}_l\, \mathbf{p}_x}}\right) = \overset{+}{p}_x{}^T \overset{+}{p}_l \overset{+}{p}_x \\ \left(\overset{-}{\overbrace{\overset{+}{p}_x{}^T \overset{+}{p}_l\, \mathbf{p}_x}}\right) = \bar{p}_x \bar{p}_l \bar{p}_x{}^T. \end{cases}$$

From (3), we get $\overset{+}{p}_x{}^T \overset{+}{p}_l\, \mathbf{p}_x = \mathbf{p}_c$ or $\overset{+}{p}_l\, \mathbf{p}_x = \overset{+}{p}_x\, \mathbf{p}_c$ or

$$\overset{+}{p}_l\, \mathbf{p}_x = \bar{p}_c\, \mathbf{p}_x. \tag{4}$$

It is important to explicate some general constraints when there is one equation of rotation. Since $\boldsymbol{R}_l$ and $\boldsymbol{R}_c$ are similar, the angles of those are identical; that is, $\theta_l = \theta_c \equiv \theta$. Applying a rotation transformation or its inverse transformation to its quaternion results in the same quaternion. That is,

$$\boldsymbol{R}^T \mathbf{p} = \boldsymbol{R}\mathbf{p} = \mathbf{p}; \tag{5}$$

where the subscript could be l, c or x. From equation (1), the quaternions for $\boldsymbol{R}_l$ and $\boldsymbol{R}_c$ are related by

$$\mathbf{p}_l = \boldsymbol{R}_x \mathbf{p}_c, \tag{6}$$

and the quaternion matrices are related by

$$\overset{+}{p}_l = \overset{+}{p}_x \overset{+}{p}_c \overset{+}{p}_x{}^T \tag{7}$$

and

$$\bar{p}_l = \bar{p}_x{}^T \bar{p}_c \bar{p}_x. \tag{8}$$

In contrast, there are similar one-data constraints in three space [1]. Comparing the constraint equations in three space and four space, we realize that there is one-to-one correspondence between three space and four space; however, the constraints in four space govern more physical quantities in each equation than those in three space. For instance, equation (7) includes spatial quantities $\theta_l = \theta_c$, $\mathbf{u}_l = \boldsymbol{R}_x \mathbf{u}_c$, and $\tilde{\mathbf{u}}_l = \boldsymbol{R}_x\, \tilde{\mathbf{u}}_c \boldsymbol{R}_x^T$ in one equation.

Given two existent equations of rotation and their corresponding quaternions $\mathbf{p}_{l_i}$ and $\mathbf{p}_{c_i}$ (i =1,2). There is a two-data constraint in quaternions as follows:

$$\mathbf{p}_{l_1} \cdot \mathbf{p}_{l_2} = \mathbf{p}_{c_1} \cdot \mathbf{p}_{c_2}. \tag{9}$$

2.2 The Equation of Translation

In quaternion space, the equation of translation, $(\boldsymbol{R}_l - \boldsymbol{I})\mathbf{r}_x = \boldsymbol{R}_x \mathbf{r}_c - \mathbf{r}_l \equiv \boldsymbol{\omega}$, is equivalent to the following equation:

$$\left(\overset{+}{p}_l - \bar{p}_l\right)\mathbf{r}_x \equiv \boldsymbol{\omega}'. \tag{10}$$

If the equation of translation exists, (10) can be further expressed in terms of spatial quantities as

$$\left(\overset{+}{\mathbf{u}}_l - \bar{\mathbf{u}}_l\right)\mathbf{r}_x = \frac{1}{\sin\frac{\theta}{2}}\boldsymbol{\omega}' \tag{11}$$

where $\mathbf{r}_x$ and ω' are vector quaternions, and

$$\omega' = \bar{\boldsymbol{p}}_l\,\omega = \left(\sin\frac{\theta}{2}\bar{\mathbf{u}}_l + \cos\frac{\theta}{2}\boldsymbol{I}\right)\omega. \tag{12}$$

Given an existent equation of translation and an existent equation of rotation, and their corresponding quaternions $\mathbf{p}_l$, $\mathbf{p}_c$, $\mathbf{r}_l$, and $\mathbf{r}_c$, the dot product of rotation quaternion and its corresponding translation quaternion satisfies the following relationship:

$$\mathbf{p}_l^T\mathbf{r}_l = \mathbf{p}_c^T\mathbf{r}_c. \tag{13}$$

If $\bar{\boldsymbol{p}}_x$ or $\bar{\boldsymbol{p}}_l^T$ is substituted, the left-hand side can be transformed to be

$$(\bar{\boldsymbol{p}}_x\overset{+}{\boldsymbol{p}}_l\bar{\boldsymbol{p}}_l^T - \bar{\boldsymbol{p}}_x)\mathbf{r}_x = (\overset{+}{\boldsymbol{p}}_l\bar{\boldsymbol{p}}_c^T - \boldsymbol{I})\,\bar{\boldsymbol{p}}_x\,\mathbf{r}_x,$$

and can be expressed as

$$(\overset{+}{\boldsymbol{p}}_l\bar{\boldsymbol{p}}_c^T - \bar{\boldsymbol{p}}_c^T\bar{\boldsymbol{p}}_c)\,\bar{\boldsymbol{p}}_x\,\mathbf{r}_x = \bar{\boldsymbol{p}}_c^T\,(\overset{+}{\boldsymbol{p}}_l - \bar{\boldsymbol{p}}_c)\,\bar{\boldsymbol{p}}_x\,\mathbf{r}_x.$$

In equation (10), multiplying both sides by a $\bar{\boldsymbol{p}}_x$ gives

$$\bar{\boldsymbol{p}}_x\,(\boldsymbol{R}_l - \boldsymbol{I})\mathbf{r}_x = -(\overset{+}{\mathbf{r}}_l - \bar{\mathbf{r}}_c)\mathbf{p}_x. \tag{14}$$

By multiplying both sides of equation (14) by a $\bar{\boldsymbol{p}}_c$, the equation of translation can be represented as follows:

$$(\overset{+}{\boldsymbol{p}}_l - \bar{\boldsymbol{p}}_c)\mathbf{r}'_x + \bar{\boldsymbol{p}}_c\,(\overset{+}{\mathbf{r}}_l - \bar{\mathbf{r}}_c)\mathbf{p}_x = 0; \tag{15}$$

where $\mathbf{r}'_x = \bar{\boldsymbol{p}}_x\,\mathbf{r}_x$.

3 The Mathematical Model

In this study, there is a new proposal that we combine the equations of rotation and translation into a linear system in matrix form applying quaternion algebra. By this, the entire algorithm is simplified. The least-squares solutions and closed-form least-squares solutions are also obtained by [1, 4, 6, 7, 8].

Based on the quaternion formulation, we combine the equations of rotation and translation and obtain the homogeneous system as follows:

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p}_x \\ \mathbf{r}'_x \end{bmatrix} \equiv \boldsymbol{C}\mathbf{x} = \mathbf{0}; \tag{16}$$

where

$$\begin{cases} \boldsymbol{A} = \bar{\boldsymbol{p}}_c\left(\overset{+}{\mathbf{r}}_l - \bar{\mathbf{r}}_c\right) \\ \boldsymbol{B} = \left(\overset{+}{\boldsymbol{p}}_l - \bar{\boldsymbol{p}}_c\right) \end{cases}.$$

The coefficient of equation (16) is symmetric. The system must be solved with two constraints. The first constraint is the normality constraint of Euler parameters:

$$\mathbf{p}_x^T\mathbf{p}_x = 1. \tag{17}$$

The second constraint is

$$\mathbf{p}_x^T\mathbf{r}'_x = \mathbf{0}. \tag{18}$$

3.1 Coefficient Matrices

It is necessary to analyze the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for obtaining the solution of the equation (16). The study of $\boldsymbol{A}$ requires only the study of $\boldsymbol{A}' \equiv (\overset{+}{\mathbf{r}}_l - \bar{\mathbf{r}}_c)$ because $\bar{\boldsymbol{p}}_c$ is orthonormal.

Lemma 3.1 *The matrix $\boldsymbol{A}' \equiv (\overset{+}{\mathbf{r}}_l - \bar{\mathbf{r}}_c)$ has the following properties:*

1. *It is skew-symmetric, and the rank is either 2 or 4.*
2. *The eigenvalues are always two pairs of pure complex numbers, given in (20), if $k_{l_i} \neq k_{c_i}$.*
3. *The eigenvalues are one pairs of pure complex numbers and two zeros, given in (20), if $k_{l_i} = k_{c_i}$.*

Proof. Let $k_l \equiv \mathbf{r}_l^T\mathbf{r}_l$ and $k_c \equiv \mathbf{r}_c^T\mathbf{r}_c$. The characteristic equation of $\boldsymbol{A}'$ can be found as

$$\lambda^4 + 2(k_l + k_c)\lambda^2 + (k_l - k_c)^2 = 0, \tag{19}$$

and the eigenvalues of $\boldsymbol{A}'$ are

$$\lambda = \pm\sqrt{-(k_l + k_c) + \pm 2\sqrt{k_l k_c}}. \tag{20}$$

In mathematics, $(k_l + k_c)$ is always greater than or equal to $2\sqrt{k_l k_c}$. The eigenvalues of $\boldsymbol{A}'$ are always two pairs of pure complex numbers under the condition that $(k_l + k_c) > 2\sqrt{k_l k_c}$. Two of the eigenvalues of $\boldsymbol{A}'$ are zero, and the other two are a pair of pure complex numbers under the condition that $k_l = k_c$. □

Lemma 3.2 *The rank of matrix $\left(\overset{+}{\boldsymbol{p}}_l \pm \bar{\boldsymbol{p}}_c\right)$ is two, and the eigenvalues (denoted by λ) are*

$$\lambda = \pm j\left(\sqrt{\mathbf{e}_l^T\mathbf{e}_l} \pm \sqrt{\mathbf{e}_c^T\mathbf{e}_c}\right) \tag{21}$$

or

$$\lambda = \left\{0,\ 0,\ \pm j2\sin\frac{\theta}{2}\right\}. \tag{22}$$

Proof. The characteristic equation of $\left(\overset{+}{\boldsymbol{p}}_l \pm \bar{\boldsymbol{p}}_c\right)$ can be derived as

$$\lambda^4 + 2(\mathbf{e}_l^T\mathbf{e}_l + \mathbf{e}_c^T\mathbf{e}_c)\lambda^2 + (\mathbf{e}_l^T\mathbf{e}_l - \mathbf{e}_c^T\mathbf{e}_c)^2 = 0, \tag{23}$$

then the eigenvalues are derived as given in (21). Since $\theta_l = \theta_c = \theta$, equation (21) can be further simplified to $\pm j\left(\sin\frac{\theta}{2} \pm \sin\frac{\theta}{2}\right)$. This yields equation (22). □

Lemma 3.3 *The matrix*

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & 0 \end{bmatrix}$$

is singular (or rank deficient).

Proof.

$$\begin{aligned} detC &= det(\boldsymbol{A})det(\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}) \quad (24)\\ &= det(\boldsymbol{A})\det(\boldsymbol{B})\det(\boldsymbol{A}^{-1})\det(\boldsymbol{B}) = 0 \end{aligned}$$

since the $\det\boldsymbol{B} = 0$. □

The coefficient matrix $\boldsymbol{C}$ is singular or rank-deficient. Therefore, the homogeneous system (16) has non-trivial solutions.

4 Least-Squares Solutions

When the data sets are extended to n, the least-squares solutions have to be obtained in the presence of noise. The homogeneous system for n sets of data is represented as

$$\begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_2 \\ \vdots \\ \boldsymbol{C}_n \end{bmatrix} \mathbf{x} = \mathbf{0}. \quad (25)$$

The least-squares solutions of (25) can minimize the error when noise exists. The solution of the normal equation $\boldsymbol{N}\mathbf{x} = \mathbf{0}$ are achieved for the solutions of (25); where $\boldsymbol{N} = \sum_{i=1}^{n} \boldsymbol{C}_i^T \boldsymbol{C}_i$. The least-squares solution for the linear system is the eigenvector corresponding to the smallest eigenvalue of the coefficient matrix of the normal equation. The eigenvector has to be normalized once the eigenvector is obtained. let the length of the first four components be $l = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$ where x_i represents ith component of $\mathbf{x}$, then we get

$$\begin{bmatrix} \mathbf{p}_x \\ \mathbf{r}'_x \end{bmatrix} = \frac{1}{l}\mathbf{x}. \quad (26)$$

Recall that $\mathbf{r}'_x = \bar{\boldsymbol{p}}_x\ \mathbf{r}_x$ where $\mathbf{r}_x$ is presented in quaternion space. In three-space, the equation becomes

$$\mathbf{r}'_x = \boldsymbol{E}^T \mathbf{r}_x; \quad (27)$$

where $\boldsymbol{E} = [-\mathrm{e}_x \quad \mathrm{e}_{0_x}\boldsymbol{I} + \tilde{\mathrm{e}}_x]$. Therefore, we can achieve $\mathbf{r}_x$ in three-space by

$$\mathbf{r}_x = \boldsymbol{E}\mathbf{r}'_x$$

since $\boldsymbol{E}^T\boldsymbol{E} = \boldsymbol{I}$.

5 Closed-Form Least-Squares Solutions

The normal equation that need to be solved for a least-squares solution can be partitioned into four sub-matrices as follows:

$$\begin{bmatrix} \boldsymbol{U} & \boldsymbol{V} \\ \boldsymbol{V}^T & \boldsymbol{W} \end{bmatrix} \begin{bmatrix} \mathbf{p}_x \\ \mathbf{r}'_x \end{bmatrix} = 0; \quad (28)$$

where

$$\begin{cases} \boldsymbol{U} = \sum \boldsymbol{A}_i^T \boldsymbol{A}_i + \boldsymbol{B}_i^T \boldsymbol{B}_i \\ \boldsymbol{V} = \sum \boldsymbol{A}_i^T \boldsymbol{B}_i \\ \boldsymbol{W} = \sum \boldsymbol{B}_i^T \boldsymbol{B}_i \end{cases}.$$

The homogeneous linear system possesses non-trivial solutions if it has more unknowns than equations. In other words, the systems has non-trivial solutions when the rank of the coefficient matrix is not full.

5.1 The coefficient Matrices

Let's examine the matrix $\boldsymbol{U}$ now. Since $\boldsymbol{A}'$ and $\boldsymbol{B}$ are skew-symmetric matrix and

$$\begin{aligned} \boldsymbol{A}^T\boldsymbol{A} &= \left(\overset{+}{\mathbf{r}}_l - \overset{-}{\mathbf{r}}_c\right)^T \bar{\boldsymbol{p}}_c^T \bar{\boldsymbol{p}}_c \left(\overset{+}{\mathbf{r}}_l - \overset{-}{\mathbf{r}}_c\right) \quad (29)\\ &= \left(\overset{+}{\mathbf{r}}_l - \overset{-}{\mathbf{r}}_c\right)^T \left(\overset{+}{\mathbf{r}}_l - \overset{-}{\mathbf{r}}_c\right) \\ &\equiv \boldsymbol{A}'^T\boldsymbol{A}, \end{aligned}$$

$\boldsymbol{U}$ can be written as

$$\boldsymbol{U} = \sum_{i=1} \boldsymbol{A}_i'^T \boldsymbol{A}_i' + \sum_{i=1} \boldsymbol{B}_i^T \boldsymbol{B}_i = -\sum \boldsymbol{A}_i'^2 - \sum \boldsymbol{B}_i^2. \quad (30)$$

Lemma 5.1 *The matrix $-\boldsymbol{A}'^2$ has the following properties:*

1. *The eigenvalues of $-\boldsymbol{A}_i'^T$ are always positive and real.*
2. *The matrix $-\boldsymbol{A}_i'^T$ is positive definite and satisfies $\mathbf{x}^T(-\boldsymbol{A}_i'^T)\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, if $k_{l_i} \neq k_{c_i}$.*
3. *The matrix $-\boldsymbol{A}_i'^T$ is positive semi-definite and satisfies $\mathbf{x}^T(-\boldsymbol{A}_i'^T)\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$, if $k_{l_i} = k_{c_i}$.*

Proof. The first property can be proved by showing that $-\boldsymbol{A}_i'^2\mathbf{X} = -\lambda_i'^2\mathbf{X}$ is true when $-\boldsymbol{A}_i'^T\mathbf{X} = -\lambda_i'\mathbf{X}$: Since the eigenvalues of $-\boldsymbol{A}_i'^2$ is $-\lambda_i^2$, and λ_i is a pure complex number, λ_i^2 is a positive real number. Since $-\boldsymbol{A}_i'^2$ has four positive and non-zero real eigenvalues, when $k_{l_i} \neq k_{c_i}$, the matrix is positive definite and satisfies $\mathbf{x}^T(-\boldsymbol{A}_i'^T)\mathbf{x} > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$. The same argument goes for the third property.

Lemma 5.2 *The matrix $-\boldsymbol{B}'^2$ has the following properties:*

1. *The eigenvalues of $-\boldsymbol{B}_i'^T$ are always two zero and two positive real numbers.*
2. *The matrix $-\boldsymbol{A}_i'^T$ is positive semi-definite and satisfies $\mathbf{x}^T(-\boldsymbol{A}_i'^T)\mathbf{x} \geq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$.*

Proof. We skip the proof of this lemma because it is very similar to the proof of Lemma 5.1.

Theorem 5.1 *The matrix* $U = \sum_{i=1} A_i'^T A_i' + \sum_{i=1} B_i^T B_i$ *is*

1. *positive definite when* $k_{l_i} \neq k_{c_i}$ *for any* i, *or*
2. *The matrix is positive semi-definite when* $k_{l_i} = k_{c_i}$ *for all* i.

Proof. Since

$$\begin{aligned} \mathbf{x}^T U \mathbf{x} &= \mathbf{x}^T[-\sum A_i'^2 - \sum B_i^2]\mathbf{x} \quad (31) \\ &= \mathbf{x}^T[-\sum A_i'^2]\mathbf{x} + \mathbf{x}^T[-\sum B_i^2]\mathbf{x} \\ &= \mathbf{x}^T[-A_1'^2]\mathbf{x} + \mathbf{x}^T[-A_2'^2]\mathbf{x} + \ldots \\ &\quad + \mathbf{x}^T[-B_1^2]\mathbf{x} + \mathbf{x}^T[-B_2^2]\mathbf{x} + \ldots . \end{aligned}$$

From the Lemma 5.1 and Lemma 5.2, we assure $\mathbf{x}^T[-U]\mathbf{x} > 0$ as long as $k_{l_i} \neq k_{c_i}$, and $\mathbf{x}^T[-U]\mathbf{x} \geq 0$ as long as $k_{l_i} = k_{c_i}$.

5.2 Solutions

Since U is non-singular (if $k_l \neq k_c$), after applying generalized Gaussian elimination, the normal equation becomes

$$\begin{bmatrix} U_{4\times4} & V_{4\times4} \\ 0_{4\times4} & W'_{4\times4} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{x_{4\times1}} \\ \mathbf{r}'_{x_{4\times1}} \end{bmatrix} = \mathbf{0}; \quad (32)$$

where $W' = W - V^T U^{-1} V$ is the Schur complement of V in N [10]. Since there are eight scalar unknowns in ten scalar equations, there must be six independent equations in (32). Because U is full rank, there are only two independent equations in the second set of (32). Since W' is symmetric and rank of it is two, we use *singular value decomposition* or *Gaussian elimination* to analyze the matrix W'. This matrix can be transformed to be the matrix form with the third and fourth zero row vectors. Therefore, we choose two equation from $W'\mathbf{r}'_x = \mathbf{0}$ to solve simultaneously with the constraints (17) and (18).

The dependent set of 2 variables, denoted by $\mathbf{r}_1$, in $\mathbf{r}'_x$ can be solved in terms of the independent set $\mathbf{r}_2$ as

$$\mathbf{r}_1 = \phi\, \mathbf{r}_2. \quad (33)$$

Since

$$W'\mathbf{r}'_x = \begin{bmatrix} W'_{11} & W'_{12} \\ W'_{21} & W'_{22} \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \mathbf{0},$$

we will obtain

$$\phi = (W'_{11} - W'_{21})^{-1}(W'_{22} - W'_{12}). \quad (34)$$

Now, we define

$$\phi \equiv \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix},$$

and

$$\mathbf{r}'_x \equiv \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \end{bmatrix}.$$

From the equation (28), we get

$$\mathbf{p}_x = \alpha^T \mathbf{r}'_x, \quad \alpha = -(U^{-1}V)^T. \quad (35)$$

Substituting $\mathbf{p}_x$ into the constraint $\mathbf{p}_x^T \mathbf{r}'_x = 0$, we can get

$$\mathbf{r}'^T_x \alpha \mathbf{r}'_x = 0.$$

Further partitioning $\mathbf{r}'_x$ into $\mathbf{r}_1$ and $\mathbf{r}_2$ and then substituting $\mathbf{r}_1$ into it gives

$$\mathbf{r}_2^T \Omega \mathbf{r}_2 = 0; \quad (36)$$

where $\Omega = \phi^T \alpha_{11} \phi + \alpha_{21}\phi + \phi^T \alpha_{12} + \alpha_{22}$, and

$$\Omega \equiv \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}.$$

Substituting $\mathbf{p}_x$ into $\mathbf{p}_x^T \mathbf{p}_x = 1$, we get

$$\mathbf{r}'^T_x \beta \mathbf{r}'_x = 1;$$

where $\beta = \alpha\alpha^T$. Further partitioning $\mathbf{r}'_x$ and β and substituting $\mathbf{r}_1$ gives

$$\mathbf{r}_2^T \Psi \mathbf{r}_2 = 1; \quad (37)$$

where $\Psi = \phi^T \beta_{11}\phi + \beta_{21}\phi + \phi^T \beta_{12} + \beta_{22}$, and

$$\Psi \equiv \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}.$$

If the equations (36) and (37) are solved simultaneously, the relationships are as follows:

$$\begin{cases} \Omega_{11} r_{21}^2 + (\Omega_{12} + \Omega_{21}) r_{21} r_{22} + \Omega_{22} r_{22}^2 = 0 \\ \Psi_{11} r_{21}^2 + (\Psi_{12} + \Psi_{21}) r_{21} r_{22} + \Psi_{22} r_{22}^2 = 1. \end{cases} \quad (38)$$

Solving the first equation in terms of r_{22} gives

$$r_{21} = h_1 r_{22}; \quad (39)$$

where $h_1 = \frac{-(\Omega_{12}+\Omega_{21}) \pm \sqrt{(\Omega_{12}+\Omega_{21})^2 - 4\Omega_{11}\Omega_{22}}}{2\Omega_{11}}$. Substituting r_{21} into the second equation and solving for r_{22} gives

$$r_{22} = \pm\sqrt{\frac{1}{h_2}}; \quad (40)$$

where $h_2 = \Psi_{22} + (\Psi_{12} + \Psi_{21}) h_1 + \Psi_{11} h_1^2$, and the negative h_2 should be discarded. From r_{22}, the vector $\mathbf{r}'_x$ can be achieved by

$$\mathbf{r}'_x = \pm \begin{bmatrix} \phi_{11} h_1 + \phi_{12} \\ \phi_{21} h_1 + \phi_{22} \\ h_1 \\ 1 \end{bmatrix} \sqrt{\frac{1}{h_2}}. \quad (41)$$

The unknown quaternion $\mathbf{p}_x = \boldsymbol{\alpha}^T \mathbf{r}'_x$ and $\mathbf{r}_x = \bar{p}_x^T \mathbf{r}'_x$ can be directly computed. The positive or negative sign for $\mathbf{r}'_x$ accounts for the fact that $\mathbf{p}_x$ can be positive or negative; however, there is only one possibility for $\mathbf{r}_x$ after transformation $\bar{p}_x^T \mathbf{r}'_x$, and $\mathbf{r}_x$ can be uniquely determined.

6 Conclusion

In this research, we have derived the eight-space formulation based on quaternions to obtain the least-squares solution for the hand-eye calibration problem. We also have established the least-aquares closed-form solutions for these systems in the presence of noise. The whole algorithm is distinguished from previous quaternion methods by the fact that it is used to solve for the unknowns $\boldsymbol{R}_x$ and $\mathbf{r}_x$ simultaneously.

From equations of rotation and translation in quaternion space, we have derived two 4 by 4 linear systems in terms of two quaternion unknowns: $\mathbf{p}_x$ and $\mathbf{r}_x$, which define the unknown rotation and translation, respectively. We then have manipulated these linear systems to become an overdetermined system. This linear system facilitates the application of least-squares methods to the solution of this calibration problem because the number of unknowns does not increase with the number of equations, and the solution to the problem is simply the solution to the normal equations of the linear system. After that, we have derived the closed-form least-squares solution using Schur decomposition and Gaussian elimination. This solution is indispensable in studying the case of existence of noise.

The proposed method provided a compact, and speedy process for attaining the required solution. Based on the constraints (7), (8), (9), and (13) presented in this proposal, we will study the uniqueness condition for hand-eye calibration problem in future researches. We expect that the eight-space formulation possess the least singularity.

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