A Screw Motion Approach to Uniqueness Analysis of Head-Eye Geometry

Homer H. Chen

AT&T Bell Laboratories, 4E-630
Crawfords Corner Road
Holmdel, NJ 07733
homer@vax135.att.com

Abstract

This paper employs the screw motion theory to solve a class of pose determination problems that can be characterized by a homogeneous transform equation of the form $\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are known motions and $\mathbf{X}$ is an unknown coordinate transformation. Unlike existing methods, our method gives rise to a sound geometric interpretation that takes both rotation and translation into consideration. We derive a screw congruence theorem and show that the problem is to find a rigid transformation which will bring one group of lines to overlap another. We also provide a complete analysis of the conditions under which the solution can be uniquely determined.

1. Introduction

A general rigid body displacement can be described as a rotation about a *unique* axis and a translation along the *same* axis. Such a combination of rotation and translation is called *screw motion* (Fig.1). The concept of screw motion was developed more than one and half centuries ago. Curiously, its significance for computer vision problems has not been well recognized. In this paper, we elaborate on the screw motion description and illustrate its strengths through an example known as head-eye, or hand-eye, calibration.

To control camera motion for data acquisition, it is convenient to mount the camera on a positioning device such as a pan-tilt table or a robot. In computer vision, such a head-eye setup greatly facilitates motion stereo, continuous object tracking, and active perception. In robotics, the hand-eye configuration has been widely used to enable intelligent interaction of a robot with its environment or to enhance the performance of a work cell [2]. The robot motion is typically expressed in a local coordinate frame attached to the robot, while the image data are described in the camera coordinate frame. To translate the image interpretation to robot action, both image data and robot motions need to be referred to the same base. Therefore, the transformation between the robot and the camera coordinate frames—the head-eye geometry—must be determined.

The head-eye calibration is an important problem [13]. The head-eye geometry can be determined by activating a sequence of robot movements and measuring the simultaneous motion of the camera by observing a number of control points at known positions in space [16]. This calibration technique has the advantage that it avoids direct measurement and is more accurate than other methods. An important issue is the uniqueness of solution. That is, given a number of corresponding robot and camera motions, can the head-eye geometry be uniquely determined? The uniqueness condition is of practical significance because it provides a guideline for activating the robot during calibration. In this paper, we address the uniqueness issue by studying the following questions:

- How many robot movements are required for solving the problem?
- What motion patterns can give rise to a unique solution for the relative orientation?
- Does the uniqueness of the relative orientation guarantee a unique solution for the relative position?
- What motion patterns can lead to a complete solution for both relative orientation and position?

Although some of these questions have been studied in the robot vision literature, no paper, to the best of our knowledge, has been able to address all of them correctly and completely. Most previous work attacked the problem by decoupling the rotation and translation parameters. We will show the flaws of such an approach.

We describe a new approach using screw motion decomposition. The fact that the rotation axis and the translation can be made parallel allows us to consider the entire motion as one single geometric entity—a line in space associated with two scalar quantities—as opposed to two independent vectors. This appealing feature of screw motion makes possible a unified treatment of the motion parameters. The resulting algorithm has a number of advantages: it is simple to implement, it is fast, it has a sound geometrical interpretation, and, most importantly, it is general (e.g., it can handle parallel rotation axes which previous methods fail to do).

2. Calibration by Head Movements

2.1 Notations

By convention, symbols representing matrices will be written in upper-case bold, vectors in lower-case bold, and scalars in lower case. A rigid displacement described as a rotation $\mathbf{R}_A$ followed by a translation $\mathbf{t}_A$ will be denoted by $(\mathbf{R}_A, \mathbf{t}_A)$ or by a 4×4 homogeneous transform matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_A & \mathbf{t}_A \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where T denotes matrix transposition. A screw motion, defined in Section 3.1, will be denoted by (d, θ, L), where L represents the screw axis, θ is the rotation angle, and d is the translation along the screw axis.

2.2 The Problem

The operations involved in the calibration technique are to move the robot and to measure the induced camera motion. The robot motion can be computed directly from the encoder values at the joints of the robot. Since we are only concerned with discrete movement, the term "motion" is used to mean finite displacement throughout the paper. As the robot moves to a new location, the position and orientation of the camera relative to the fixed world can be determined by using control points or lines at known positions in space. This is the well-known camera location determination problem and many techniques have been developed to solve the problem [10], [14], [12], [20], [15], [7], [18]. Given the robot and camera motions, we want to find the position and orientation of the camera relative to the robot.

Because the camera is mounted rigidly on the robot, the transformation from coordinate frame F_A through F'_A to F'_B is the same as the transformation from F_A through F_B to F'_B, as shown in Fig. 2. Therefore,

$$\mathbf{A}\,\mathbf{X} = \mathbf{X}\,\mathbf{B} \tag{1}$$

or

$$\mathbf{R_X} = \mathbf{R_A}\,\mathbf{R_X}\,\mathbf{R_B^T} \tag{2}$$

$$\mathbf{t_X} = \mathbf{R_A}\,\mathbf{t_X} + \mathbf{t_A} - \mathbf{R_X}\,\mathbf{t_B} \tag{3}$$

where $\mathbf{A}$ is the robot motion, $\mathbf{B}$ is the induced camera motion, and $\mathbf{X}$ is the robot-to-camera coordinate transformation to be determined.

2.3 Previous Work

Several researchers have investigated the solution of (1). Chen [5] investigated the relationship between great circles on a quaternion sphere and proposed a linear method for solving a binocular motion problem (with different unknowns). In the context of head-eye calibration, Shiu and Ahmad [16] were apparently the first to attack the problem and to report a mathematical solution. They proved by an algebraic approach that (2) and (3) are both rank deficient, and so one needs to make at least two robot movements in order to determine $\mathbf{R_X}$ and $\mathbf{t_X}$. Their method, however, generates more unknowns than necessary. To improve it, Tsai and Lenz [19] used a variation of quaternions to represent the rotation. A similar attempt was made by Chou and Kamel [8]. These methods solve (2) and (3) separately and assume that $\mathbf{t_X}$ can be determined if the solution for $\mathbf{R_X}$ is found.

Shiu and Ahmad noticed that the robot motions should be constrained if a unique solution is sought [16]. They claimed that the condition for a unique solution for $\mathbf{R_X}$ is that the robot cannot rotate around parallel or antiparallel axes. The uniqueness condition for $\mathbf{t_X}$ is the same as that for $\mathbf{R_X}$. Tsai and Lenz [19] also came to the same conclusion. But a disagreement exists about whether this condition should be necessary and sufficient [19] or necessary only [16].

Before attempting to resolve the disagreement, we ask a fundamental question: Is the decoupling a justified approach for uniqueness analysis? In other words, can we consider rotation and translation independently? The decoupling approach has long been used in the computer vision community to attack pose/motion determination problems. The popularity of this approach is perhaps because that rigid displacement has traditionally been described as a rotation followed by a translation. Under this decomposition, there seems no apparent relationship between the rotation axis and the translation, leading one to think that the rotation and translation can be treated separately. We argue that such an approach is inadequate for uniqueness analysis of the current problem. A new approach that takes both rotation and translation into account will be described.

3. Screw Motion Decomposition

3.1 Chasles' Theorem

Chasles' theorem [4] states that a general rigid body displacement can be accomplished by means of a translation along a unique axis and a rotation about the same axis. Such a description of rigid body displacement is called a *screw*, and the unique axis is called the *screw axis*. The screw motion was discovered at the beginning of the 18th century, and since then it has emerged as one of the most convenient means of describing spatial displacement [1], [9], [3].

The screw axis need not be fixed to pass through the origin, but instead can be displaced from the origin. Associated with the screw axis are two quantities: the rotation angle and the translation. The screw axis is a line in space and hence requires four parameters to describe its location. Thus, we need six independent parameters to describe a screw—the same as the number of degrees of freedom of a 3-D rigid object.

A screw is denoted by (d, θ, L), where d is the translation along the screw axis, θ is the rotation angle, $0 < \theta < 2\pi$, and L is the screw axis. The values of θ and d are confined to positive numbers in this paper. The screw axis in parametric form is described by

$$L\colon\ \mathbf{p} = \mathbf{c} + \kappa\mathbf{n}, \quad -\infty < \kappa < \infty$$

where $\mathbf{n}$ is a unit direction vector representing the axis of rotation and $\mathbf{c}$ is the position of the line to the origin, $\mathbf{c}\cdot\mathbf{n} = 0$. There is an inherent two-way sign ambiguity with this line representation. To fix it, our approach is to make $\mathbf{n}$ point to the same direction as the screw translation so that the sense of translation is consistent with that of rotation (right hand rule). If $\mathbf{n}$ is in the opposite direction of the screw translation, we flip the sign of $\mathbf{n}$ and replace θ by $2\pi - \theta$. If $d = 0$ (no translation along the screw axis), the sign ambiguity of $\mathbf{n}$

can be resolved by confining the range of rotation angle to $0 < \theta < \pi$. Note, however, the sign of $\mathbf{n}$ remains ambiguous when $d = 0$ and $\theta = \pi$. Also, when $\theta = 0$ (pure translation), the screw axis is undefined [3].

3.2 Conversion between $(\mathbf{R}, \mathbf{t})$ and (d, θ, L)

The motion description we are most familiar with is probably the one that decomposes a rigid body displacement into a rotation followed by a translation. This section discusses the conversion between this conventional motion description, denoted by $(\mathbf{R}, \mathbf{t})$, and the corresponding screw description (d, θ, L). We will concentrate on the representation which describes $\mathbf{R}$ as a rotation through an angle ϕ and around an axis $\mathbf{r}$ (a unit vector) that passes through the origin.

Denote the parallel and perpendicular translation components by $\mathbf{t}_{\parallel}$ and $\mathbf{t}_{\perp}$, respectively,

$$\mathbf{t}_{\parallel} = (\mathbf{t} \cdot \mathbf{r})\, \mathbf{r} \tag{4}$$

$$\mathbf{t}_{\perp} = \mathbf{t} - \mathbf{t}_{\parallel}. \tag{5}$$

Then the motion of an object can be described by

$$\mathbf{p}' = \mathbf{R}\, \mathbf{p} + \mathbf{t}_{\perp} + \mathbf{t}_{\parallel}. \tag{6}$$

We mentioned earlier that the screw axis does not have to pass through the origin. This gives us an important clue that, if we can make the translation parallel to the rotation axis, it must be done by displacing the rotation axis from the origin so that the perpendicular translation component is completely absorbed into rotation. Furthermore, to effect the absorption, the displacement should be perpendicular to the rotation axis. Let the displacement vector be denoted by $\mathbf{c}$, $\mathbf{c} \cdot \mathbf{r} = 0$. Since displacing the rotation axis can be thought of as a translational coordinate transformation, we have

$$(\mathbf{p}' - \mathbf{c}) = \mathbf{R}\,(\mathbf{p} - \mathbf{c}) + a\, \mathbf{r}$$

where a is a number. The term $a\,\mathbf{r}$ represents a translation along the rotation axis. Rewriting the equation as

$$\mathbf{p}' = \mathbf{R}\, \mathbf{p} + (\mathbf{I} - \mathbf{R})\, \mathbf{c} + a\, \mathbf{r} \tag{7}$$

and equating (7) to (6), we obtain

$$\mathbf{t}_{\parallel} = a\, \mathbf{r} \tag{8}$$

$$\mathbf{t}_{\perp} = (\mathbf{I} - \mathbf{R})\, \mathbf{c} \tag{9}$$

where $\mathbf{I}$ is an identity matrix. We see from (8) that the screw translation is equal to the parallel translation component. The importance of (9) is illustrated in Fig. 3, where $\vec{OD} = \mathbf{t}_{\perp}$, $\vec{DC} = \mathbf{R}\mathbf{c}$, and $\vec{OC} = \mathbf{c}$. Since $\vec{DC} = \mathbf{R}\,\vec{OC}$, we have $OC = DC$ (hence $\Delta\, OCD$ is an isosceles triangle) and $\angle OCD = \phi$. Clearly, O can be rotated to D by $\mathbf{R}$ if the new rotation axis passes through C. Thus, $\mathbf{t}_{\perp}$ is completely absorbed into rotation.

The above argument also gives rise to a simple method for computing the displacement vector $\mathbf{c}$: In the plane containing $\overline{OD}$ and perpendicular to $\mathbf{r}$, find a point C on the bisector of $\overline{OD}$ so that $\angle OCD = \phi$. There are two such points, one on each side of $\overline{OD}$, but only one of them has the same sense of rotation as $\mathbf{R}$. This explains why the screw axis is unique. From Fig. 3, the vector $\mathbf{c}$, which in fact describes the position of the screw axis, is given by

$$\mathbf{c} = (\mathbf{t}_{\perp} + \mathbf{r} \times \mathbf{t}_{\perp} \cot\frac{\phi}{2})/2. \tag{10}$$

To summarize, the conversion of (d, θ, L) from $(\mathbf{R}, \mathbf{t})$ can be done by first determining $\mathbf{t}_{\parallel}$ from (4) and $\mathbf{c}$ from (10). Then, the screw translation d, the rotation angle θ, and the direction vector $\mathbf{n}$ of the screw axis are determined as follows:

$$\begin{cases} d = a,\ \theta = \phi,\ \mathbf{n} = \mathbf{r} & \text{if } a \geq 0 \\ d = -a,\ \theta = 2\pi - \phi,\ \mathbf{n} = -\mathbf{r} & \text{if } a < 0 \end{cases}$$

where $a = \mathbf{t} \cdot \mathbf{r}$. The conversion from (d, θ, L) to $(\mathbf{R}, \mathbf{t})$ is straightforward. Since θ and $\mathbf{n}$ are given, $\mathbf{R}$ is readily available and $\mathbf{t} = \mathbf{t}_{\parallel} + \mathbf{t}_{\perp} = d\mathbf{n} + (\mathbf{I} - \mathbf{R})\mathbf{c}$.

4. Screw Congruence

Since the robot motion and the camera motion are the same motion but expressed in different coordinate frames, there must be a certain geometrical relationship more explicit than (1) between these two motions. The relationship will be investigated by using screws to represent robot and camera motions.

4.1 Invariants

We first prove that the rotation angle and the screw translation are invariant with respect to coordinate transformations. Let (d_A, θ_A, L_A) be a robot motion and (d_B, θ_B, L_B) be the corresponding camera motion. We want to show that $\theta_A = \theta_B$ and $d_A = d_B$.

An interesting property of a rotation matrix is that one of its eigenvalues is always equal to $+1$ and the other two are a pair of complex conjugates $e^{j\phi}$ and $e^{-j\phi}$, where ϕ is the angle of rotation [11]. For any non-identity rotation, there is one fixed vector that remains unmoved by the rotation. This vector is the rotation axis, which corresponds to the eigenvector associated with the eigenvalue $+1$ of the rotation matrix.

The angular invariant can be proved by noting that

$$\mathbf{R}_A = \mathbf{R}_X \mathbf{R}_B \mathbf{R}_X^T$$

which is a similarity transformation since $\mathbf{R}_X$ is an orthogonal matrix. Hence, $\mathbf{R}_A$ and $\mathbf{R}_B$ have the same eigenvalues. Furthermore, an eigenvector $\mathbf{b}$ of $\mathbf{R}_B$ corresponds to an eigenvector $\mathbf{R}_X\mathbf{b}$ of $\mathbf{R}_A$ [17]. Taking the sign of the rotation axis into account, we have

$$\begin{cases} \phi_A = \phi_B \\ \mathbf{n}_A = \mathbf{R}_X\, \mathbf{n}_B \end{cases} \tag{11a}$$

or

$$\begin{cases}\phi_A = 2\pi - \phi_B \\ \mathbf{n}_A = -\mathbf{R}_X \mathbf{n}_B.\end{cases} \tag{11b}$$

We will come back and show that only the first solution is valid.

We now prove the second invariant. Rewriting (3) as

$$\mathbf{R}_A \mathbf{t}_X + \mathbf{t}_{A\perp} + \mathbf{t}_{A\parallel} = \mathbf{R}_X(\mathbf{t}_{B\perp} + \mathbf{t}_{B\parallel}) + \mathbf{t}_X \tag{12}$$

and substituting

$$\mathbf{t}_{A\perp} = (\mathbf{I} - \mathbf{R}_A)\mathbf{c}_A \tag{13}$$

$$\mathbf{t}_{B\perp} = (\mathbf{I} - \mathbf{R}_B)\mathbf{c}_B \tag{14}$$

$$\mathbf{t}_{A\parallel} = d_A \mathbf{n}_A \tag{15}$$

$$\mathbf{t}_{B\parallel} = d_B \mathbf{n}_B \tag{16}$$

into (12), yields

$$(\mathbf{I} - \mathbf{R}_A)(\mathbf{R}_X\mathbf{c}_B - \mathbf{c}_A + \mathbf{t}_X) = (d_A - d_B)\mathbf{n}_A \tag{17}$$

where we have also substituted $\mathbf{R}_X\mathbf{R}_B$ by $\mathbf{R}_A\mathbf{R}_X$ and $\mathbf{n}_B$ by $\mathbf{R}_X^T \mathbf{n}_A$. Letting

$$\mathbf{v} = \mathbf{R}_X \mathbf{c}_B - \mathbf{c}_A + \mathbf{t}_X \tag{18}$$

and premultiplying both sides of (17) by $\mathbf{n}_A^T$, we have

$$\mathbf{n}_A^T \mathbf{v} - \mathbf{n}_A^T \mathbf{R}_A \mathbf{v} = d_A - d_B.$$

Rewriting $\mathbf{n}_A^T \mathbf{R}_A \mathbf{v}$ as $\mathbf{v}^T \mathbf{R}_A^T \mathbf{n}_A$ and noting that $\mathbf{R}_A^T \mathbf{n}_A = \mathbf{n}_A$, we arrive at

$$0 = d_A - d_B.$$

Therefore, the screw translation is also an invariant. If we replace $\mathbf{n}_B$ by $-\mathbf{R}_X^T\mathbf{n}_A$ in the derivation (that is, assuming (11b) is true), we obtain $d_A + d_B = 0$. So, d_A and d_B must have opposite signs. If d_B is negative, we change the signs of d_B and $\mathbf{n}_B$ and replace ϕ_B by $2\pi - \phi_B$. If d_A is negative, we do the same thing for the robot motion. In either case, the result is that only (11a) is valid, as we claimed earlier. This completes the proof.

It is interesting to note that we actually experience these two invariants in our daily life: Picture yourself watching a person tightening a real screw; the number of turns of the screw counted by this person must be equal to the number counted by you. The number obtained represents a combination of the rotation angle and the distance that the screw has traveled.

4.2 Transformation of a Line

Before we show the relationship between screw axes of the robot and camera motions, it is worthwhile discussing the transformation of a line in space.

Consider a line L: $\mathbf{p} = \mathbf{c} + \kappa\mathbf{n}$, $-\infty < \kappa < \infty$, and a rigid body displacement $(\mathbf{R}, \mathbf{t})$. Denote the line after the displacement by L': $\mathbf{q} = \mathbf{c}' + \kappa'\mathbf{n}'$, $-\infty < \kappa' < \infty$. Then it can be shown that

$$\mathbf{n}' = \mathbf{R}\mathbf{n} \tag{19}$$

$$\mathbf{c}' = \mathbf{R}\,\mathbf{c} + \mathbf{t} - (\mathbf{n}' \cdot \mathbf{t})\,\mathbf{n}'. \tag{20}$$

4.3 Relationship between Screw Axes

Since the screw translation is invariant, (17) becomes

$$\mathbf{R}_A \mathbf{v} = \mathbf{v} \tag{21}$$

and we want to determine $\mathbf{v}$. This is an eigen problem and the solution is the eigenvector associated with the eigenvalue $+1$ of $\mathbf{R}_A$. Thus

$$\mathbf{R}_X \mathbf{c}_B - \mathbf{c}_A + \mathbf{t}_X = \lambda\,\mathbf{n}_A \tag{22}$$

where λ is a real number representing the magnitude of the eigenvector. Premultiplying both sides of (22) by $\mathbf{n}_A^T$ yields

$$\lambda = \mathbf{n}_A^T \mathbf{R}_X \mathbf{c}_B - \mathbf{n}_A^T \mathbf{c}_A + \mathbf{n}_A^T \mathbf{t}_X.$$

Since $\mathbf{c}_A^T\mathbf{n}_A = 0$ and $\mathbf{n}_A^T\mathbf{R}_X\mathbf{c}_B = \mathbf{n}_B^T\mathbf{R}_X^T\mathbf{R}_X\mathbf{c}_B = \mathbf{n}_B^T\mathbf{c}_B = 0$, we have

$$\lambda = \mathbf{n}_A^T \mathbf{t}_X.$$

Substituting this equation back into (22) yields

$$\mathbf{c}_A = \mathbf{R}_X \mathbf{c}_B + \mathbf{t}_X - (\mathbf{n}_A^T \mathbf{t}_X)\,\mathbf{n}_A. \tag{23}$$

Comparing (23) with (20) and (11a) with (19), we thus prove that L_A is the transformed line of L_B and the transformation is $(\mathbf{R}_X, \mathbf{t}_X)$.

To recap, we have shown that under a coordinate transformation the rotation angle and the screw translation are invariant, and the screw axis undergoes the same transformation. This is true for each pair of corresponding screws in the two coordinate frames; therefore, there is a congruent relationship between the two sets of screws. We summarize the results as a theorem.

Screw Congruence Theorem: *Let $(\mathbf{R}_X, \mathbf{t}_X)$ be the transformation that takes a Cartesian frame F_A into exact alignment with another Cartesian frame F_B. Also let (d_A, θ_A, L_A) and (d_B, θ_B, L_B) be two screw motion descriptions of an object obtained in F_A and F_B, respectively. Then*

- $d_A = d_B$
- $\theta_A = \theta_B$
- $\mathbf{n}_A = \mathbf{R}_X \mathbf{n}_B$
- $\mathbf{c}_A = \mathbf{R}_X \mathbf{c}_B + \mathbf{t}_X - (\mathbf{n}_A^T \mathbf{t}_X)\,\mathbf{n}_A$

where $\mathbf{c}_A$ and $\mathbf{c}_B$ are the position vectors of L_A and L_B, and $\mathbf{n}_A$ and $\mathbf{n}_B$ are the direction vectors of L_A and L_B, respectively.

The theorem can be applied as a motion constraint or as a rigidity constraint. For the former application, the theorem indicates that the unknown coordinate transformation can be determined by computing the motion of the screw axes, and that the transformation is independent of the rotation angle and the screw translation. When used as a rigidity constraint, the theorem indicates that the corresponding mutual angles and distances between screw axes should be the same. In other words, the angles and distance between screw axes are preserved under coordinate transformation. This is a powerful constraint for correspondence identification.

5. Uniqueness Analysis

We are now ready to attack the four questions posed in Section 1. We will see how the use of screw motion description makes our analysis easier. All these questions are answered by simple geometrical arguments.

5.1 Well-Defined Screws

We need only consider the screw axes when analyzing the uniqueness of head-eye geometry. For each well-defined screw, the screw axis is a directed line in space.

It is well known that two directed lines determine a rigid body displacement; therefore, two robot motions are required in order to determine the head-eye geometry. One robot movement is insufficient because once the pair of corresponding screw axes are made to overlap, they can still rotate arbitrarily around, and translate arbitrarily along, themselves. There remains one degree of freedom in both rotation and translation. This agrees with the algebraic result described in [16]. Adding a second robot motion with its screw axis not parallel to the first one would freeze the two degrees of freedom. Therefore, two robot movements are sufficient for determining the head-eye geometry. It is obvious that the solution is unique when the two screw axes of robot motions are skew lines or intersect at a point.

Two lines in space can be skew, intersecting, parallel, or coincident. We have just discussed the first two cases. Now, let us consider the remaining two cases. When two robot motions have parallel or antiparallel screw axes, any motion except translations along the screws would take corresponding screw axes apart, once corresponding screw axes are made to overlap. Therefore, $\mathbf{R}_X$ can be uniquely determined but $\mathbf{t}_X$ has one degree of freedom. Note that in this case five of the six motion parameters can be determined. This disproves the result obtained by previous methods. The solution for $\mathbf{R}_X$ can be found by first constructing a triad from the two robot screw axes, and then computing the transformation that brings the triad into exact alignment with its corresponding triad constructed from the camera motions. Each triad is composed of three vectors: the direction of the two parallel screw axes, the common perpendicular between them, and the cross product of the first two vectors (Fig. 4). The result obtained here clearly indicates that the uniqueness of $\mathbf{R}_X$ does not necessarily guarantee the uniqueness of $\mathbf{t}_X$.

In the last case, we have coaxial robot motions. Since the two screw axes coincide, the second robot motion does not provide any additional information for solving the problem. Therefore, both $\mathbf{R}_X$ and $\mathbf{t}_X$ cannot be determined. Clearly, this is the only case in which $\mathbf{R}_X$ cannot be solved.

For well-defined screws, we conclude that the head-eye geometry can be uniquely determined from two robot motions if and only if the two screw axes are skew or intersecting lines. If the two screw axes are parallel or antiparallel, five of the six motion parameters can be determined; the only one that is left free is the translation along the screw axes. If the two screw lines coincide, the problem cannot be solved.

5.2 Ambiguous or Undefined Screws

Screws belonging to this category are either undefined or ambiguous; that is, the motions have either a zero rotation ($\theta = 0$) or a zero translation along the screw axis plus a half-turn rotation ($d = 0$, $\theta = \pi$) (see Section 3.1). Unlike previous methods, which completely fail to deal with motions of this type, our method is able to find a partial, or even a complete, solution in most cases.

An important point to note is that the effect of two motions is a composite motion which gives rise to another screw that is generally different from the first two [6]. Therefore, we can actually construct three screws from two robot motions. The composite motion (denoted by $\mathbf{A}_c$) of $\mathbf{A}_1$ and $\mathbf{A}_2$ is given by

$$\mathbf{R}_{Ac} = \mathbf{R}_{A2}\,\mathbf{R}_{A1}$$

$$\mathbf{t}_{Ac} = \mathbf{R}_{A2}\,\mathbf{t}_{A1} + \mathbf{t}_{A2}.$$

The same subscripts are used for the screws. We distinguish five cases and discuss them separately.

1) L_{A1} is well-defined, L_{A2} is undefined. In this case, $\mathbf{R}_{Ac} = \mathbf{R}_{A1}$ and $\mathbf{t}_{Ac} = \mathbf{t}_{A1} + \mathbf{t}_{A2}$. Therefore, L_{Ac} is parallel to L_{A1}. Unless $\theta_{A1} = \pi$ and $\mathbf{t}_{A2}\cdot\mathbf{n}_{A1} + d_{A1} = 0$ (making L_{Ac} ambiguous), L_{Ac} is well-defined; so, as discussed earlier, $\mathbf{R}_X$ has a unique solution but $\mathbf{t}_X$ has one degree of freedom. If L_{Ac} is ambiguous, it can be resolved by applying the angular constraint described in Case 3. Note that $\mathbf{A}_c$ and $\mathbf{A}_1$ are coaxial motions if $\mathbf{t}_{A2}$ is parallel to L_{A1}. When this happens, the head-eye geometry cannot be determined.

2) L_{A1} and L_{A2} are undefined. That is, both $\mathbf{A}_1$ and $\mathbf{A}_2$ are pure translations; thus, $\mathbf{R}_{Ac} = \mathbf{I}$ and $\mathbf{t}_{Ac} = \mathbf{t}_{A1} + \mathbf{t}_{A2}$. In this case, the composite motion is also a pure translation. The head-eye geometry cannot be determined.

3) L_{A1} is well-defined, L_{A2} is ambiguous. Since L_{A1} is well-defined, according to the screw congruence theorem, L_{B1} must also be well-defined. By the same logic, L_{B2} must be ambiguous. Since L_{A1} is well-defined, we may fix the sign of L_{A2} and L_{B2} by imposing the constraint that the angle between L_{A1} and L_{A2} should be equal to the angle between L_{B1} and L_{B2} (Section 4.3). This way, we may still be able to find a unique solution for $\mathbf{R}_X$ and $\mathbf{t}_X$ if L_{A1} is not parallel or antiparallel to L_{A2}. Note, however, this method does not work if L_{A1} is perpendicular to L_{A2}. Our approach is to introduce L_{Ac} into the analysis. In this case, L_{Ac} must be in a plane parallel to both L_{A2} and the common perpendicular P_{12} of L_{A2} and L_{A1}, and that $\theta_{Ac} = \pi$ (Fig. 5) [6]. For the problem to be solved, L_{Ac} must be a well-defined screw not perpendicular to L_{A2}. Since $\theta_{Ac} = \pi$, L_{Ac} is certainly not undefined. Thus, it is either well-defined or ambiguous. For L_{Ac} to be ambiguous, however, either

L_{A1} and L_{A2} intersect at a point or L_{Ac} is parallel to L_{A2}. The latter cannot happen because we know θ_{A1}, which is twice the angle between P_{c1} and P_{12} [6], is nonzero. Therefore, L_{Ac} is well-defined. Finally, L_{Ac} is not perpendicular to L_{A2} as long as $\theta_{A1} \neq \pi$. In summary, for the alternative approach to work, it is required that $\theta_{A1} \neq \pi$ and L_{A1} and L_{A2} do not intersect. When this is not true, there exist two solutions for $\mathbf{R}_X$ but a unique solution for $\mathbf{t}_X$.

4) L_{A1} and L_{A2} are both ambiguous. That is, $d_{A1} = d_{A2} = 0$ and $\theta_{A1} = \theta_{A2} = \pi$. In this case, L_{Ac} is perpendicular to L_{A1} and L_{A2} (Fig. 6). As a result, the sign ambiguities of L_{A1} and L_{A2} cannot be resolved no matter whether L_{Ac} is well-defined or not. Therefore, there exist two solutions for $\mathbf{R}_X$. If L_{A1} is not parallel to L_{A2}, $\mathbf{t}_X$ has a unique solution; otherwise $\mathbf{t}_X$ has one degree of freedom.

5) L_{A1} is ambiguous, L_{A2} is undefined. Similar to Case 1, we have $\mathbf{R}_{Ac} = \mathbf{R}_{A1}$ and $\mathbf{t}_X = \mathbf{t}_{A1} + \mathbf{t}_{A2}$. Thus the screw axis L_{Ac} is parallel to L_{A1} and is well-defined if $\mathbf{t}_{A2} \cdot \mathbf{n}_{A1} \neq 0$. In this case the angular constraint can be applied (as in Case 3) to fix the signs to L_{A1} and L_{B1}; thus, $\mathbf{R}_X$ can be uniquely determined and $\mathbf{t}_X$ has one degree of freedom. When $\mathbf{t}_{A2} \cdot \mathbf{n}_{A1} = 0$, both L_{A1} and L_{Ac} are ambiguous and consequently $\mathbf{R}_X$ has two solutions. When $\mathbf{t}_{A2}$ is parallel to L_{A1}, $\mathbf{A}_c$ becomes coaxial with $\mathbf{A}_1$; the problem cannot be solved.

6. Conclusion

The aim of this paper was to demonstrate the use of screw theory for motion problems. The screw motion is a unique description of rigid transformations. We have exploited the strengths of the screw motion description for head-eye calibration. We have shown that the rotational and translational parameters should not be decoupled, for otherwise the generality and efficacy of the resulting algorithm would be adversely affected.

We have proved that the rotation angle and the screw translation are invariant with respect to coordinate transformation, and that the problem can be solved by finding the rigid transformation that aligns the camera screw axes with the corresponding robot screw axes. The screw congruence theorem derived here can be used to compute the head-eye transformation, to verify the correspondence between robot and camera motions, and to resolve the sign ambiguity of screw axes.

For well-defined screws, we have shown that the necessary and sufficient condition of a uniqueness solution for the head-eye geometry is that the screw axes of two robot motions are either skew or intersecting lines. For undefined or ambiguous screws, we have shown that a partial solution or even a complete solution may be obtained by investigating the screw geometry of composite motion. The screw motion theory has been considered mathematically elegant but of no practical use [11]. To the contrary, we found it extremely useful. We hope that the work reported here will inspire more thoughts on the use of screw theory for motion research.

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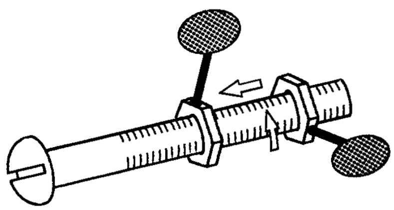

Fig. 1. The displacement of a rigid object from one position to another can be described as a screw motion.

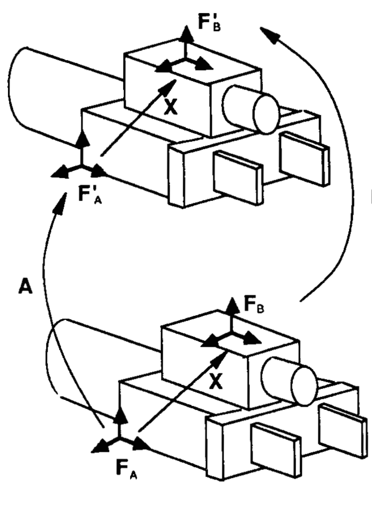

Fig. 2. Transformations between coordinate frames.

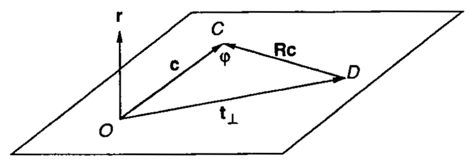

Fig. 3. The three vectors $\mathbf{t}_\perp$, $\mathbf{c}$, and $\mathbf{Rc}$ form an isosceles triangle with $\angle OCD = \phi$ in a plane perpendicular to the rotation axis $\mathbf{r}$.

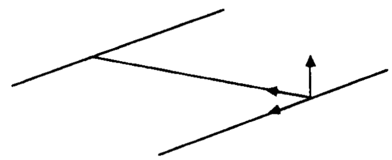

Fig. 4. Constructing a triad for a pair of parallel screw axes.

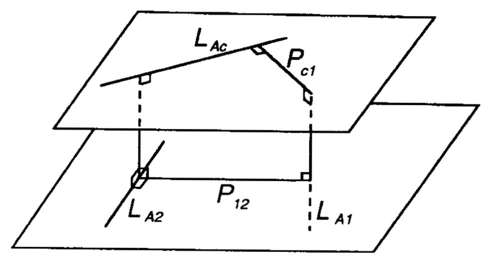

Fig. 5. A screw geometry. L_{A1} is perpendicular to L_{A2}.

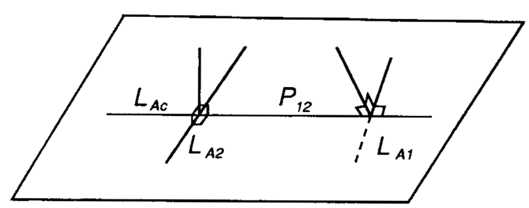

Fig. 6. A screw geometry. L_{Ac} is perpendicular to L_{A1} and L_{A2}.