New Batch Method

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Abstract to be added.

We begin by defining a Gaussian probability distribution on $SE(3)$ (assuming the norm $\|\Sigma\|$ is small) as

$$\rho(H;M,\Sigma) = \frac{1}{(2\pi)^3|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}F(M^{-1}H)}$$

where $\|\Sigma\|$ denotes the determinant of Σ and

$$F(H) = [\log^{\vee}(H)]^T \Sigma^{-1} [\log^{\vee}(H)].$$

Previously, in order to determine the mean of the convolution of two PDFs, Baker-Campbell-Hausdorff formula is used given the assumption that function f_1 and f_2 are both highly focused. If $X,Y \in se(3)$,

$$log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \ldots \quad (1)$$

If X and Y are further constrained to be small so that $\|X\| \ll 1$ and $\|Y\| \ll 1$, then the first approximation of Eq.(**??**) can be written as:

$$\log(e^X e^Y) = X + Y \quad (2)$$

As shown in [**?**], the mean of covariance of the convolution of two highly focused functions are:

$$M_{1*2} = M_1 M_2 \text{ and } \Sigma_{1*2} = Ad(M_2^{-1})\Sigma_1 Ad^T(M_2^{-1}) + \Sigma_2 \quad (3)$$

where

$$Ad(H) = \begin{pmatrix} R & \mathbb{O} \\ \hat{\mathbf{x}}R & R \end{pmatrix} \quad (4)$$

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1 First Order Approximation of M

Though this approximation works well when the distribution of X is treated as a Delta function, it fails to extend to the case where its distribution is a general PDF $f(X)$. In an alternative to the first order approximation using Baker-Campbell-Hausdorff formula, it is possible to only assume $M^{-1}H$ is small so that $\|M^{-1}H - \mathbb{I}\| \ll 1$. Given the Taylor expansion of the matrix logarithm described as:

$$\log(\mathbb{I} + X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \ldots \quad (5)$$

Then it is straight forward to have:

$$\begin{aligned} &\log(M^{-1}H) \\ =&\log(\mathbb{I} + (M^{-1}H - \mathbb{I})) \\ =&\left(M^{-1}H - \mathbb{I}\right) - \left(M^{-1}H - \mathbb{I}\right)^2/2 + \left(M^{-1}H - \mathbb{I}\right)^3/3 - \ldots \end{aligned} \quad (6)$$

Given the definition of the mean M of a probability density $f(H)$ as:

$$\int_{SE(3)} \log(M^{-1}H) f(H) dH = \mathbb{O} \quad (7)$$

The first order approximation of Eq.(**??**) is:

$$\int_{SE(3)} (M^{-1}H - \mathbb{I}) f(H) dH \approx \mathbb{O} \quad (8)$$

$$M^{-1} \int_{SE(3)} H f(H) dH \approx \mathbb{I} \quad (9)$$

Define the first order approximation of M as $\widehat{M}$:

$$\widehat{M} \doteq \int_{SE(3)} Hf(H)dH \tag{10}$$

$$f_A(H) = \frac{1}{n}\sum_{i=1}^{n} \delta(A_i^{-1}H) \text{ and } f_B(H) = \frac{1}{n}\sum_{i=1}^{n} \delta(B_i^{-1}H).$$

If $f(H)$ is of the form of $f_A(H)$ given above, then

$$\begin{aligned} &\sum_{i=1}^{n} \log(M_A^{-1}A_i) = \mathbb{O} \text{ and} \\ &\Sigma_A = \frac{1}{n}\sum_{i=1}^{n} \log^\vee(M_A^{-1}A_i)[\log^\vee(M_A^{-1}A_i)]^T. \end{aligned} \tag{11}$$

Discrete version will be:

$$\widehat{M_A} \doteq \sum_{i=1}^{n} A_i \left(\frac{1}{n}\sum_{j=1}^{n} \delta(A_j^{-1}A_i)dH\right) = \frac{1}{n}\sum_{i=1}^{n} A_i \tag{12}$$

Note that $\widehat{M}$ is generally not a group element in $SE(3)$, and the corresponding $SE(3)$ version can be obtained by projecting $\widehat{M}$ into $SE(3)$ using singular value decomposition (SVD) technique.:

$$R_{\widehat{M}} = U\Sigma V^T \tag{13}$$

The rotation part of the projected $\widehat{M}$ (named as $\widehat{M}_{proj}$) is:

$$R_{\widehat{M}_{proj}} = UV^T \tag{14}$$

More details are needed to actually project $\widehat{M}$ into $SE(3)$.

2 Second Order Approximation of M

Take the second order approximation in Eq.(**??**):

$$\int_{SE(3)} \left((M^{-1}H - \mathbb{I}) - \frac{1}{2}(M^{-1}H - \mathbb{I})^2\right) f(H)dH \approx \mathbb{O} \tag{15}$$

$$\int_{SE(3)} \left(2M^{-1}H - \frac{1}{2}HM^{-1}H - \frac{3}{2}\mathbb{I}\right) f(H)dH \approx \mathbb{O} \tag{16}$$

Multiply M on both side of Eq.(**??**):

$$\int_{SE(3)} \left(2H - \frac{1}{2}HM^{-1}H - \frac{3}{2}M\right) f(H)dH \approx \mathbb{O} \tag{17}$$

Substituting Eq.(**??**) into Eq.(**??**), we have:

$$2\widehat{M} - \frac{1}{2}\int_{SE(3)} HM^{-1}Hf(H)dH - \frac{3}{2}M \approx \mathbb{O} \tag{18}$$

The 2nd order approximation of M is denoted by $\overline{M}$ defined as:

$$2\widehat{M} - \frac{1}{2}\int_{SE(3)} H\overline{M}^{-1}Hf(H)dH - \frac{3}{2}\overline{M} = \mathbb{O} \tag{19}$$

$$2\widehat{M} - \overline{M}\frac{1}{2}\int_{SE(3)} \overline{M}^{-1}H\overline{M}^{-1}Hf(H)dH - \frac{3}{2}\overline{M} = \mathbb{O} \tag{20}$$

$$2\widehat{M} - \frac{1}{2n}\sum_{i=1}^{n} A_i\overline{M}^{-1}A_i - \frac{3}{2}\overline{M} = \mathbb{O} \tag{21}$$

The second term of Eq.(**??**) is very similar to the definition of the covariance of $f(H)$, and maybe $\overline{M}$ can be updated using the information of the covariance. Also, take a look at the cubness of variance in the first volume. The same technique as in Eq.(**??**) can be employed to project $\widehat{M}$ into $SE(3)$.

3 First Order Approximation of Σ

Given the definition of the covariance Σ of a PDF $f(H)$ as:

$$\Sigma = \int_{SE(3)} \log^\vee(M^{-1}H)[\log^\vee(M^{-1}H)]^T f(H)dH \tag{22}$$

Its first order approximation $\widehat{\Sigma}$ can be written as:

$$\widehat{\Sigma} \doteq \int_{SE(3)} (M^{-1}H - \mathbb{I})^\vee[(M^{-1}H - \mathbb{I})^\vee]^T f(H)dH \tag{23}$$

The discrete version for Σ_A will be:

$$\Sigma_A = \frac{1}{n}\sum_{i=1}^{n} (M_A^{-1}A_i - \mathbb{I})^\vee[(M_A^{-1}A_i - \mathbb{I})^\vee]^T. \tag{24}$$

By defining $Q = M^{-1}H$, Eq.(**??**) can be written as:

$$\widehat{\Sigma} \doteq \int_{SE(3)} (Q - \mathbb{I})^\vee[(Q - \mathbb{I})^\vee]^T f(Q)dQ \tag{25}$$

If $\|G - \mathbb{I}\| \ll 1$, then $\Sigma = \widehat{\Sigma}$

$$(Q - \mathbb{I})^\vee = \begin{pmatrix} \frac{1}{2}(R - R^T) \\ \mathbf{t} \end{pmatrix} \tag{26}$$

4 Second Order Approximation of Σ

5 First Order Approximation of M_{1*2}

Given two functions, $f_1, f_2 \in (L^1 \cap L^2)(SE(3))$, the convolution is defined as:

$$(f_1 * f_2)(H) \doteq \int_{SE(3)} f_1(K) f_2(K^{-1}H) dK. \tag{27}$$

The corresponding mean M_{1*2} will be given as:

$$\int_{SE(3)} \log(M_{1*2}^{-1}H)(f_1 * f_2)(H) dH = \mathbb{O} \tag{28}$$

If both Σ_1 and Σ_2 are very small, then $M_{1*2} \approx M_1 M_2$. Without using this approximation, take the assumption that $M^{-1}H$ is small and we will have:

$$\int_{SE(3)} \int_{SE(3)} \left(M_{1*2}^{-1}H - \mathbb{I}\right) f_1(K) f_2(K^{-1}H) dK dH \approx \mathbb{O} \tag{29}$$

Define $L = K^{-1}H$,

$$\int_{SE(3)} \int_{SE(3)} \left(M_{1*2}^{-1}KL - \mathbb{I}\right) f_1(K) f_2(L) dK dL \approx \mathbb{O} \tag{30}$$

By using Eq.(**??**) twice,

$$M_{1*2}^{-1} \widehat{M_1} \widehat{M_2} \approx \mathbb{I} \tag{31}$$

Define $\widehat{M}_{1*2}$ as:

$$\widehat{M}_{1*2} = \widehat{M_1} \widehat{M_2} \tag{32}$$

There are two ways to solve for M_{1*2}^{proj}.

$$M_{1*2}^{proj} = \begin{cases} \left(\widehat{M_1}\widehat{M_2}\right)_{proj} \\ \widehat{M_1}_{proj} \widehat{M_2}_{proj} \end{cases} \tag{33}$$

6 Second Order Approximation of M_{1*2}

For simplicity, we drop the domain of integral $SE(3)$,

$$\int\int \left(2M_{1*2}^{-1}H - \frac{1}{2}M_{1*2}^{-1}HM_{1*2}^{-1}H - \frac{3}{2}\mathbb{I}\right) f_1(K) f_2(K^{-1}H) dK dH \approx \mathbb{O} \tag{34}$$

Substitute $L = K^{-1}H$,

$$\int\int \left(2M_{1*2}^{-1}KL - \frac{1}{2}M_{1*2}^{-1}KLM_{1*2}^{-1}KL - \frac{3}{2}\mathbb{I}\right) f_1(K) f_2(L) dK dL \approx \mathbb{O} \tag{35}$$

Employing the definition of $\widehat{M}$,

$$M_{1*2}^{-1} \widehat{M_1} \widehat{M_2} \approx \frac{1}{4} \int\int \left(\frac{1}{2}M_{1*2}^{-1}KLM_{1*2}^{-1}KL\right) f_1(K) f_2(L) dK dL + \frac{3}{4}\mathbb{I} \tag{36}$$