Correspondence

Comments on "Calibration of Wrist-Mounted Robotic Sensors by Solving Homogeneous Transform Equations of the Form $AX = XB$"

Hanqi Zhuang and Zvi S. Roth

The paper by Shiu and Ahmad [1] presents a closed-form solution to the homogeneous transform equation $AX = XB$. The result is of great value for sensor fusion applications. The derivation of a unique solution for the rotation equation $R_A R_X = R_X R_B$ (where R_A, R_X, and R_B are the rotation parts of A, X, and B, respectively), although containing many useful ideas, is somewhat lengthy and can be presented much more compactly using quaternion algebra. There are two direct benefits of such an approach. First, the condition that angles of rotations of R_{A1} and R_{A2} (in [1, theorem 4]) are not equal to π is unnecessary. This condition came from a particular choice of a particular solution [1, eqs. (37)-(39)]. Second, the overall computational complexity of the final solution is much reduced (comparing [1, eqs. (25), (37)-(39), (44)] with (19), (20), and (25) below).

R_A and R_B must have the same angle of rotation by [1, lemma 4]. Let $R_A = \mathrm{Rot}\,(\boldsymbol{k}_A, \theta)$, $R_B = \mathrm{Rot}\,(\boldsymbol{k}_B, \theta)$, and $R_X = \mathrm{Rot}\,(\boldsymbol{k}_X, \omega)$.

Lemma 1: $R_A R_X = R_X R_B$ is equivalent to the following:

$$\sin(\theta/2)\sin(\omega/2)(\boldsymbol{k}_A + \boldsymbol{k}_B) \times \boldsymbol{k}_X = \sin(\theta/2)\cos(\omega/2)(\boldsymbol{k}_B - \boldsymbol{k}_A) \quad (1)$$

where $\times$ denotes a cross product operation.

Proof: Let $\mathbf{a} \equiv (a, \boldsymbol{a})$, $\mathbf{b} \equiv (b, \boldsymbol{b})$, and $\mathbf{x} \equiv (x, \boldsymbol{x})$ be quaternions corresponding to the rotation matrices R_A, R_B, and R_X, where a, b, and x are scalars and $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{x}$ are 3×1 vectors [2]. Let $\boldsymbol{u}_i$ and $\boldsymbol{b}_i$ be the ith column vector of the identity matrix and B, respectively. $\boldsymbol{u}_i$ and $\boldsymbol{b}_i$ may be treated as pure quaternions (that is, quaternions with zero scalars). $R_A R_X = R_X R_B$ can be written as

$$R_A R_X \boldsymbol{u}_i = R_X \boldsymbol{b}_i, \qquad i = 1, 2, 3. \quad (2)$$

However according to [2],

$$R_X \boldsymbol{b}_i = \mathbf{x} \circ \boldsymbol{b}_i \circ \mathbf{x}^*, \qquad i = 1, 2, 3 \quad (3)$$

where the superscript * denotes the Hamiltonian conjugate of the quaternion and $\circ$ denotes a quaternion product operation. Also

$$R_A R_X \boldsymbol{u}_i = \mathbf{a} \circ \mathbf{x} \circ \boldsymbol{u}_i \circ \mathbf{x}^* \circ \mathbf{a}^*, \qquad i = 1, 2, 3. \quad (4)$$

Substituting (3)-(4) into (2) yields

$$\mathbf{a} \circ \mathbf{x} \circ \boldsymbol{u}_i \circ \mathbf{x}^* \circ \mathbf{a}^* = \mathbf{x} \circ \mathbf{b}_i \circ \mathbf{x}^*, \qquad i = 1, 2, 3. \quad (5)$$

Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{x}$ are all unit quaternions, (5) may be rearranged into

$$\boldsymbol{u}_i \circ \mathbf{y} = \mathbf{y} \circ \boldsymbol{b}_i, \qquad i = 1, 2, 3 \quad (6)$$

where

$$\mathbf{y} \equiv (y, \boldsymbol{y}) \equiv \mathbf{x}^* \circ \mathbf{a}^* \circ \mathbf{x}. \quad (7)$$

From (6)

$$\mathbf{y}^* \circ \boldsymbol{u}_i \circ \mathbf{y} = \boldsymbol{b}_i, \qquad i = 1, 2, 3. \quad (8)$$

However

$$\boldsymbol{b}_i = \mathbf{b} \circ \boldsymbol{u}_i \circ \mathbf{b}^*, \qquad i = 1, 2, 3. \quad (9)$$

Comparing (8) and (9), one obtains

$$\mathbf{y} = \mathbf{b}^*. \quad (10)$$

By substituting (10) into (7), the following equation for $\mathbf{x}$ is obtained:

$$\mathbf{x} \circ \mathbf{b}^* = \mathbf{a}^* \circ \mathbf{x}. \quad (11)$$

Expanding (11) provides

$$(\boldsymbol{b} \cdot \boldsymbol{x},\ -x\boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{x}) = (\boldsymbol{a} \cdot \boldsymbol{x},\ -x\boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{x}) \quad (12)$$

where $\cdot$ denotes a dot product operation. Equation (12) is equivalent to

$$(\boldsymbol{b} + \boldsymbol{a}) \times \boldsymbol{x} = x(\boldsymbol{b} - \boldsymbol{a}) \quad (13)$$

since the scalar part $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{x} = 0$ is implied by (13). However, for R_A, R_B, and R_X being rotation matrices [2]

$$\boldsymbol{a} = \sin(\theta/2)\boldsymbol{k}_A \quad (14a)$$

$$\boldsymbol{b} = \sin(\theta/2)\boldsymbol{k}_B \quad (14b)$$

$$\boldsymbol{x} = \sin(\omega/2)\boldsymbol{k}_X \quad (14c)$$

$$x = \cos(\omega/2) \quad (14d)$$

Substituting (14) into (13) yields (1). ■

If $\theta \neq 0$, then from (1)

$$\tan(\omega/2)(\boldsymbol{k}_A + \boldsymbol{k}_B) \times \boldsymbol{k}_X = \boldsymbol{k}_B - \boldsymbol{k}_A. \quad (15)$$

Let $\Omega(\boldsymbol{v})$ denote a skew symmetric matrix that corresponds to $\boldsymbol{v} \equiv [v_x, v_y, v_z]^T$

$$\Omega(\boldsymbol{v}) \equiv \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}. \quad (16)$$

Define $G_1 \equiv \Omega(\boldsymbol{k}_A + \boldsymbol{k}_B)$, $\boldsymbol{h}_1 \equiv \boldsymbol{k}_B - \boldsymbol{k}_A$, and

$$\boldsymbol{z} \equiv \tan(\omega/2)\boldsymbol{k}_X. \quad (17)$$

Equation (15) can then be written as

$$G_1 \boldsymbol{z} = \boldsymbol{h}_1. \quad (18)$$

Since $\mathrm{Rank}\,(G_1) = 2$, one needs two matrix equations to obtain a unique solution, as argued in [1]. Let $R_{Ai} R_X = R_X R_{Bi}$, $i = 1, 2$, be two given matrix equations. $R_{Ai} = \mathrm{Rot}\,(\boldsymbol{k}_{Ai}, \theta_i)$ and $R_{Bi} = \mathrm{Rot}\,(\boldsymbol{k}_{Bi}, \theta_i)$. Then if $\theta_i \neq 0$, one can stack the equations of the form (18) into

$$G\boldsymbol{z} = \boldsymbol{h} \quad (19)$$

where

$$G \equiv \begin{bmatrix} \Omega(\boldsymbol{k}_{A1} + \boldsymbol{k}_{B1}) \\ \Omega(\boldsymbol{k}_{A2} + \boldsymbol{k}_{B2}) \end{bmatrix} \quad (20a)$$

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The authors are with the Robotics Center and the Department of Electrical Engineering, Florida Atlantic University, Boca Raton, FL 33431.
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TABLE I
A Unique Solution Produced with Rotation Angles of R_{A1} and R_{A2} Equal to π

	Rotation axis			Rotation angle (rad)
R_{A1}	0.851	0.441	0.286	π
R_{A2}	0.430	0.901	0.062	π
R_{B1}	0.802	0.383	0.459	π
R_{B2}	0.752	0.643	−0.149	π
R_X	0.718	0.573	0.396	0.90745

$$h \equiv \begin{bmatrix} k_{B1} - k_{A1} \\ k_{B2} - k_{A2} \end{bmatrix} \tag{20b}$$

and z is given in (17).

Lemma 2: A necessary and sufficient condition for (19) to have a unique solution z is

$$k_{A1} \times k_{A2} \neq 0. \tag{21}$$

Proof: A necessary and sufficient condition for z to be a unique solution is that rank $(G) = 3$ since (19) is a consistent system of linear equations. By elementary matrix transformations on G, an equivalent condition is

$$(k_{A1} + k_{B1}) \times (k_{A2} + k_{B2}) \neq 0. \tag{22}$$

There exists a rotation matrix R such that

$$k_{Ai} = R k_{Bi} \tag{23}$$

according to [1, theorem 3]. Substituting (23) into (22) yields

$$((I + R) k_{A1}) \times ((I + R) k_{A2}) \neq 0. \tag{24}$$

$(I + R) k_{Ai}$, $i = 1, 2$, can be treated as the scaled, rotated, and translated version of k_{Ai}. Since k_{A1} and k_{A2} undergo the same rotation and translation, they will not become parallel after the rotation and translation if they are not parallel before the operation. Equation (24) is thus equivalent to (21). ■

Equation (19) may be extended to accommodate more measurement equations to allow a least squares solution of z. Finally, if $\|z\| \neq 0$, k_X and ω can be obtained from (17) by

$$k_X = z / \|z\| \tag{25a}$$

$$\omega = 2 \operatorname{atan}(z_{\max} / k_{X,\max}) \tag{25b}$$

where $z_{\max}$ and $k_{X,\max}$ are the elements having the maximum absolute values among the three components of z and k_X, respectively. If $\|z\| = 0$, from (17) $\omega = 0$ and k_X is arbitrarily chosen as $[0, 0, 1]^T$.

We summarize with the following.

Theorem 1: A consistent system of two rotation matrix equations $R_{Ai} R_X = R_X R_{Bi}$, $i = 1, 2$, has a unique solution if the axes of rotation for R_{A1} and R_{A2} are neither parallel nor antiparallel one to another and the rotation angles of R_{A1} and R_{A2} are both nonzero.

Table I presents a numerical example chosen from a number of simulation results.

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Y. C. Shiu and Shaheen Ahmad

In their comment [1], Zhuang and Roth extended and simplified part of the results in our paper [2] by reformulating the solutions for the rotational part of the homogeneous transform equation $AX = XB$ as a quaternion equation $\mathbf{x} \circ \mathbf{b}^* = \mathbf{a}^* \circ \mathbf{x}$ [1, eq. (11)]. This quaternion equation can actually be written as $\mathbf{ax} = \mathbf{xb}$, a form parallel to $AX = XB$, after premultiplication by $\mathbf{a}$ and postmultiplication by $\mathbf{b}$. Using their new approach, the solutions for the angle and the axis of rotation of X are simplified, mainly because their method eliminates the need to compute particular solutions for $A_i X = X B_i$, where i is the index to the ith motion.

The main contribution of Zhuang and Roth's approach is that it allows A and B to have a 180° angle of rotation. The generalization of the solution method to the 180° case is important for practical reasons explained below. Recall that A is the relative robot motion and B is the resulting sensor motion [2]. For robotic applications that require sensor-wrist calibration (see [3]–[5]), one should choose the robot motions A_i or the camera motions B_i such that the resulting solution is as accurate and noise insensitive as possible. One approach to assess noise sensitivities for a given set of A_i and B_i is by perturbing A_i and B_i and observing the resulting errors in X. Since the rotation angle ranges from −180° to 180°, it is important to be able to study the noise sensitivities at ±180°.

Although Zhuang and Roth's method does have significant advantages over our original solution for the rotational part, their method does not provide the geometric insight that the solution for $AX = XB$ has a rotational degree of freedom about $\mathbf{k}_A$ (the axes of rotation of A), as stated in [2, theorem 1]. We would also like to point out that the solution to the translational part of X discussed in our paper is not affected by this discussion since its computation is not dependent on how the rotational part is computed.

References

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Y. C. Shiu is with the Department of Electrical Engineering, Wright State University, Dayton, OH 45435.

S. Ahmad is with the School of Electrical Engineering, Real-Time Robot Control Laboratory, Purdue University, West Lafayette, IN 47907.

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