Exercises-2

Course: Matrix Analysis and Applications
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I hereby certify that all the work in these exercises is mine alone. I have neither received assistance from another person or group, nor have I given assistance to another person.

Name:__________ **Student ID:**__________ **Signature:**__________ **Date:**__________

1. Find the eigenvalues and eignevectors of the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -3 & 1 & -3 \\ 20 & 3 & 10 \\ 2 & -2 & 4 \end{bmatrix}.$$

2. Determine if either of the following matrices is diagonalizable

$$\boldsymbol{A} = \begin{bmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{bmatrix}.$$

3. Estimate the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 6 & 1 \\ 1 & 0 & -5 \end{bmatrix}$.

4. Determine the induced norm $\|\boldsymbol{A}\|_2$ as well as $\|\boldsymbol{A}^{-1}\|_2$ for the nonsingular matrix

$$\boldsymbol{A} = \frac{1}{\sqrt{3}} \begin{bmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{bmatrix}.$$

5. Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$. Please prove the following statements.

   (a) $\|\boldsymbol{A}\|_p = \max\limits_{\|\boldsymbol{x}\|_p=1} \|\boldsymbol{A}\boldsymbol{x}\|_p$ is a norm for any $p \geq 1$;

   (b) $\|\boldsymbol{A}\boldsymbol{x}\|_p \leq \|\boldsymbol{A}\|_p \|\boldsymbol{x}\|_p$ holds for any $p \geq 1$;

   (c) $\|\boldsymbol{A}\boldsymbol{B}\|_p \leq \|\boldsymbol{A}\|_p \|\boldsymbol{B}\|_p$ holds for any $p \geq 1$;

   (d) $\|\boldsymbol{Q}\boldsymbol{A}\boldsymbol{U}\|_F = \|\boldsymbol{A}\|_F$ holds for any unitary matrices $\boldsymbol{Q} \in \mathbb{C}^{m\times m}$ and $\boldsymbol{U} \in \mathbb{C}^{n\times n}$;

   (e) $\|\boldsymbol{Q}\boldsymbol{A}\boldsymbol{U}\|_2 = \|\boldsymbol{A}\|_2$ holds for any unitary matrices $\boldsymbol{Q} \in \mathbb{C}^{m\times m}$ and $\boldsymbol{U} \in \mathbb{C}^{n\times n}$.

6. Using the induced matrix norm, prove that if $\boldsymbol{A}$ is nonsingular, then

$$\min_{\|x\|=1} \|\boldsymbol{A}\boldsymbol{x}\| = \frac{1}{\|\boldsymbol{A}^{-1}\|}.$$

7. Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ is a Vandemonde matrix with distinct roots. Please verify that any collection of r columns of $\boldsymbol{A}$, with $r \leq m$, is linearly independent.

8. Suppose $\boldsymbol{A}$ has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   a) Give a basis for the nullspace and a basis for the column space;

   b) Find a particular solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$;

   c) Find all solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$;

   d) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then ________ would be in the column space.