Exercises-1

Course: Matrix Analysis and Applications
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TA: TBD

I hereby certify that all the work in these exercises is mine alone. I have neither received assistance from another person or group, nor have I given assistance to another person.

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1. Read the introductory section "Deep Learning and Neural Nets" of the book, Gilbert Strang, *Linear Algebra and Learning from Data*, Wellesley-Cambridge University Press, 2019, pp. iii-v.

2. For each of the subsets below, verify whether it is a subspace or not.

   (a) $\mathbb{V}_1 \cap \mathbb{V}_2$, where $\mathbb{V}_1 = \{\boldsymbol{x} = \alpha \boldsymbol{a}_1 \,|\, \alpha \in \mathbb{R}\}$, $\mathbb{V}_2 = \{\boldsymbol{x} = \alpha \boldsymbol{a}_2 \,|\, \alpha \in \mathbb{R}\}$, $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^m$ and $\boldsymbol{a}_1 \neq \boldsymbol{a}_2 \neq \boldsymbol{0}$.

   (b) $\mathbb{V}_1 \cup \mathbb{V}_2$, where $\mathbb{V}_1$ and $\mathbb{V}_2$ are the same as defined above.

   (c) $\mathbb{X} \oplus \mathbb{Y}$, where $\mathbb{X}, \mathbb{Y} \subseteq \mathbb{R}^m$ are subspaces and $\oplus$ denotes the direct sum, i.e., $\mathbb{X} \oplus \mathbb{Y} = \{\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} \,|\, \boldsymbol{x} \in \mathbb{X}, \boldsymbol{y} \in \mathbb{Y}\}$.

   (d) $\{\boldsymbol{a}\}$, where $\boldsymbol{a} \neq \boldsymbol{0}$.

   (e) $\mathbb{S}_\perp = \{\boldsymbol{y} \in \mathbb{R}^m \,|\, \boldsymbol{y}^T \boldsymbol{x} = 0, \text{ for all } \boldsymbol{x} \in \mathbb{S}\}$, where $\mathbb{S} \subseteq \mathbb{R}^m$ is a nonempty subset.

   (f) $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$, where $\boldsymbol{A}$ is given.

3. Given $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove that $\mathcal{R}(\boldsymbol{A})_\perp = \mathcal{N}(\boldsymbol{A}^T)$.

4. Let $\boldsymbol{M} = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \cdots, \boldsymbol{m}_r\}$ and $\boldsymbol{N} = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \cdots, \boldsymbol{m}_r, \boldsymbol{v}\}$ be two sets of vectors from the same vector space. Prove that $\text{span}(\boldsymbol{M}) = \text{span}(\boldsymbol{N})$ if and only if $\boldsymbol{v} \in \text{span}(\boldsymbol{M})$.

5. If $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix}$ is a square matrix such that $\mathcal{N}(\boldsymbol{A}_1) = \mathcal{R}(\boldsymbol{A}_2^T)$, prove that $\boldsymbol{A}$ must be nonsingular.

6. Given $\boldsymbol{x} \in \mathbb{R}^n$, for each of the functions below, verify whether it is a norm.

   (a) $\|\boldsymbol{x}\|_2 \triangleq \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_{n-1}|^2 + |x_n|}$.

   (b) $\|\boldsymbol{x}\|_1 \triangleq |x_1| + |x_2| + \cdots + |x_n|$.

   (c) $\|\boldsymbol{x}\|_\infty \triangleq \max\limits_{i=1,\cdots,n} |x_i|$.

   (d) $\text{card}(\boldsymbol{x}) \triangleq \sum\limits_{i=1}^{n} \mathbf{1}(x_i \neq 0)$, where the indicator function is defined as $\mathbf{1}(x_i \neq 0) = \begin{cases} 1, & x_i \neq 0, \\ 0, & x_i = 0. \end{cases}$

7. Einstein needed a new definition of length and distance in 4-dimensional space-time. Lorentz proposed this one, which Einstein accepted ($c =$ speed of light):

$$\boldsymbol{v} = (x, y, z, t), \tag{1}$$
$$\|\boldsymbol{v}\|^2 = x^2 + y^2 + z^2 - c^2t^2. \tag{2}$$

Is this a true norm in $\mathbb{R}^4$?

8. Prove the Cauchy-Schwartz inequality for the complex case, i.e., for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$,

$$|\boldsymbol{x}^H \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \, \|\boldsymbol{y}\|_2,$$

and equality holds if and only if $\boldsymbol{x} = \alpha \boldsymbol{y}$ for some $\alpha \in \mathbb{C}$.