Numerical Methods
Lab 5 [Differentiation and Richardson Extrapolation]

i. Open the Colab file shared in BUX.

ii. Create a copy of that shared file in your drive.

iii. Rename the Colab filename using the format **Name-ID-Lab Section**

Part 1: Differentiation: Forward, Backward, And Central

We have already learned about *forward differentiation*, *backward differentiation,* and *central differentiation*. In this part of the assignment, we will write methods to calculate these values and check how they perform.

The equations are as follows:

$$\text{forward differentiation, } f'(x) \simeq \frac{f(x+h)-f(x)}{h} \tag{4.6}$$

$$\text{backward differentiation, } f'(x) \simeq \frac{f(x)-f(x-h)}{h} \tag{4.7}$$

$$\text{central differentiation, } f'(x) \simeq \frac{f(x+h)-f(x-h)}{2h} \tag{4.8}$$

Task 1 - 2 Marks

You need to implement the functions backward_diff (f, h, x), central_diff (f, h, x), error_1(f, f_prime, h, x).

From this portion of the implementation, you will get to know how to calculate the forward differentiation, backward differentiation, and central differentiation. The forward differentiation is done for you.

Part 2: Richardson Extrapolation

We used the central difference method to calculate the derivatives of functions in the task. In this task, we will use Richardson extrapolation to get a more accurate result. Let,

$$D_h = \frac{f(x_1+h) - f(x_1-h)}{2h} \tag{5.1}$$

General Taylor Series formula:

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 + \dots$$

Using Taylor's theorem to expand we get,

$$f(x_1+h) = f(x_1) + f'(x_1)h + \frac{f''(x_1)}{2}h^2 + \frac{f'''(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 + \frac{f^{(5)}(x_1)}{5!}h^5 + O(h^6) \tag{5.2}$$

$$f(x_1-h) = f(x_1) - f'(x_1)h + \frac{f''(x_1)}{2}h^2 - \frac{f'''(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 - \frac{f^{(5)}(x_1)}{5!}h^5 + O(h^6) \tag{5.3}$$

Subtracting 5.3 from 5.2 we get,

$$f(x_1+h) - f(x_1-h) = 2f'(x_1)h + 2\frac{f'''(x_1)}{3!}h^3 + 2\frac{f^{(5)}(x_1)}{5!}h^5 + O(h^7) \tag{5.4}$$

So,

$$\begin{aligned} D_h &= \frac{f(x_1+h) - f(x_1-h)}{2h} \\ &= \frac{1}{2h}\left(2f'(x_1)h + 2\frac{f'''(x_1)}{3!}h^3 + 2\frac{f^{(5)}(x_1)}{5!}h^5 + O(h^7)\right) \\ &= f'(x_1) + \frac{f'''(x_1)}{6}h^2 + \frac{f^{(5)}(x_1)}{120}h^4 + O(h^6) \end{aligned} \tag{5.5}$$

We get our derivative $f'(x)$ plus some error terms of order $>= 2$ Now, we want to bring our error order down to 4.

If we use h, and $\frac{h}{2}$ as step size in 5.5, we get,

$$D_h = f'(x_1) + f'''(x_1)\frac{h^2}{6} + f^{(5)}(x_1)\frac{h^4}{120} + O(h^6) \tag{5.6}$$

$$D_{h/2} = f'(x_1) + f'''(x_1)\frac{h^2}{2^2.6} + f^{(5)}(x_1)\frac{h^4}{2^4.120} + O(h^6) \tag{5.7}$$

Multiplying 5.7 by 4 and subtracting from 5.6 we get,

$$D_h - 4D_{h/2} = -3f'(x) + f^{(5)}(x_1)\frac{h^4}{160} + O(h^6)$$

$$\implies D_h^{(1)} = \frac{4D_{h/2} - D_h}{3} = f'(x) - f^{(5)}(x_1)\frac{h^4}{480} + O(h^6) \tag{5.8}$$

Let's calculate the derivative using 5.8

Task 2 - 2 Marks

You need to implement the functions dh(f, h, x), dh1(f, h, x), error(f, hs, x_i).

- **a.** The function dh(f, h, x) takes **three** parameters as input: a function f, a value h, and a set of values x.
- **b.** The function dh1(f, h, x) takes the same type of values as dh(f, h, x) as input. It calculates the derivative using the previously defined dh(f, h, x) function and using equation 5.8 and returns the values.
- **c.** The error(f, hs, x_i) function takes a function f as input. It also takes a list of different values of h as hs and a specific value as x_i as input. It calculates the derivatives as point x_i using both functions described in **B** and **C**, i.e. dh and dh1.

Daily Evaluation - 4 marks

Students have learned about various differentiation methods such as forward, central, and backward differentiations and Richardson Extrapolation. They are now required to apply this understanding through a set of implementation exercises, which will be provided separately.