Springer Tracts in Modern Physics
Volume 145

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Springer Tracts in Modern Physics

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Handbook of Feynman Path Integrals

Springer

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Physics and Astronomy Classification Scheme (PACS):
03.65-W, 31.15.Kb, 03.65Db

ISSN 0081-3869
ISBN 3-540-57135-3 Springer-Verlag Berlin Heidelberg New York

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Grosche, Christian: Handbook of Feynman path integrals / C. Grosche; F. Steiner. – Berlin; Heidelberg; New York; Barcelona; Budapest; Hong Kong; London; Milan; Paris; Singapore; Tokyo: Springer, 1998
(Springer tracts in modern physics; Vol. 145)
ISBN 3-540-57135-3

Typesetting: Camera-ready copy by the authors using a Springer TEX macro package
Cover design: *design & production* GmbH, Heidelberg
SPIN: 10717112 56/3012 - 5 4 3 2 1 – Printed on acid-free paper

To the memory of Isabel Steiner

Preface

Our *Handbook of Feynman Path Integrals* appears just fifty years after Richard Feynman published his pioneering paper in 1948 entitled "Space-Time Approach to Non-Relativistic Quantum Mechanics". As it is the case with many books, its origin goes back to a course first given by one of us (F.S.) on Feynman path integrals at the University of Hamburg during the summer semester of 1983. The other author was one of the students attending these lectures and who eventually decided to work on this subject for his diploma thesis. This was the starting point of our collaboration during the 1980s. At that time our main common interest was in the question of how to solve non-Gaussian path integrals (like the one for the hydrogen atom) and, more generally, path integrals in arbitrary curvilinear coordinates. It was in 1983, too, that one of us (F.S.) began to collect papers and preprints on path integrals, and to set up a comprehensive list of references on this subject. Eventually a systematic literature search was carried out (by C.G.). While we were working in various fields, above all in quantum chromodynamics, string theory, and quantum chaos, we conceived the idea of writing a *Handbook* on Feynman path integrals which would, on the one hand, serve the reader as a thorough *introduction* to the theory of path integrals, but would, on the other hand, also establish for the first time a comprehensive *table of Feynman path integrals* together with an extensive list of *references.* The whole enterprise was, however, delayed by various circumstances for several years. Here we put forward our *Handbook* to the gentle reader!

The book follows the general idea as originally conceived. Chapters 1–5 have the character of a *textbook* and give a self-contained, and up-to-date introduction to the theory of path integrals for those readers who have not yet studied path integrals, but have a good knowledge of the fundamentals of quantum mechanics as covered by standard courses in theoretical physics. Chapter 6 makes up the largest part of this Handbook and contains a rather complete *table of path integrals* in non-relativistic quantum mechanics, including supersymmetric quantum mechanics, and statistical mechanics. To each path integral listed in the table we attach a comprehensive list of *references* which altogether make up almost 1000 references. The *Introduction* in Chap. 1 is mainly of a historical nature and gives the reader some insight into the remarkable development of Feynman's path integral approach. Since some of the historical facts are not so well known we thought it would be worthwhile to present them in Chap. 1.

Large parts of the material presented in Chaps. 1–5 have been used and tried out first by one of us (F.S.) in various courses given over the last 15 years at the Universities of Hamburg and Ulm, and at the University of Lausanne and the ETH Lausanne, respectively, in 1985 and 1995, as part of the Troisième Cycle de la Physique en Suisse Romande. We are grateful to all students and colleagues who have attended these lectures and who have contributed by their questions and remarks to the clarification and improvement of our presentation.

We are indebted for help and criticism to many friends and colleagues, including Sergio Albeverio, Jens Bolte, Philippe Choquard, Ludwig Dąbrowski, Gianfausto Dell'Antonio, Josef Devreese, Cécile DeWitt-Morette, Ismael Duru, Klaus Fredenhagen, Martin Gutzwiller, Urs Hugentobler, Akira Inomata, Chris Isham, Georg Junker, John Klauder, Hagen Kleinert, Hajo Leschke, Gerhard Mack, Dieter Mayer, Peter Minkowski, Holger Ninnemann, David Olive, George Papadopoulos, Axel Pelster, George Pogosyan, Cesare Reina, Martin Reuter, Oliver Rudolph, Virulh Sa-yakanit, Larry Schulman, Alexei Sissakian, Oleg Smolyanov, Wichit Sritrakool, Ulrich Weiss, Frederik Wiegel, Pavel Winternitz, Kurt Bernardo Wolf, and Arne Wunderlin. Important soft-ware advise was provided by Michael Behrens, Otto Hell, Phillip Kent, Dennis Moore, Jan Hendrik Peters, Peter Schilling, Thomas Sippel-Dau, and Katherine Wipf. We are also grateful to our secretaries Ingrid Gruhler, Doris Laudahn, Bärbel Lossa, Graziella Negadi, Alexandra Poretti, and Galina Sandukovskaya.

Furthermore we thank Springer–Verlag, in particular Urda and Wolf Beiglböck, Hans Kölsch, and Jacqueline Lenz for their editorial guidance.

Financial support by the Deutsche Forschungsgemeinschaft DFG is gratefully acknowledged.

Last but not least C.G. is deeply indebted to Gertrude Huber and Diana Paris for their love, understanding and support at a critical time.

It so happens that Feynman would have celebrated his 80th birthday on May 11 this year, and it seems therefore that the publication of our handbook is quite well timed.

Hamburg and Ulm, May 1998

Christian Grosche
Frank Steiner

Table of Contents

1 Introduction

The conventional formulation of quantum mechanics in terms of operators in Hilbert space is a *Hamiltonian approach.*[1] It was invented and developed by Bohr, Born, Dirac, Heisenberg, Jordan, Pauli, Schrödinger, and others in the years 1925–26. The basic quantity in quantum mechanics is a certain complex function Ψ called a *probability amplitude* or *wave function* associated with every quantum mechanical (pure) state. In the simplest case of a single particle the wave function $\Psi(\mathbf{x},t)$ is the total amplitude for the particle to arrive at a particular point $(\mathbf{x},t)$ in space and time from the past in some (perhaps unspecified) situation. The probability (density) of finding the particle at the point $\mathbf{x}$ and at the time t is $|\Psi(\mathbf{x},t)|^2$. In the usual approach to quantum mechanics the wave function Ψ is calculated by solving a differential equation, which for non-relativistic systems, i.e., for particles of low velocity, is the *Schrödinger equation*

$$\mathrm{i}\,\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \underline{\mathrm{H}}_{\mathbf{x}}\Psi(\mathbf{x},t) \ . \tag{1.1.1}$$

Here $\underline{\mathrm{H}}_{\mathbf{x}}$ is a differential operator called the *Hamiltonian* or *Schrödinger operator*, which is derived from the *classical Hamiltonian* $H(\mathbf{p},\mathbf{x})$ of the associated classical system.[2] The Schrödinger equation (1.1.1) is a kind of wave equation,[3] and this explains why the probability amplitude $\Psi(\mathbf{x},t)$ is commonly called the (Schrödinger) wave function. Obviously, the Schrödinger equation (1.1.1) is a *deterministic equation*, since knowledge of Ψ at $t = t'$

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[1] In the following discussion we shall not distinguish between Heisenberg's *matrix mechanics* discovered in June 1925 and Schrödinger's *undulatory mechanics* discovered during the winter 1925–26, since the two, apparently dissimilar approaches, were proved to be mathematically equivalent by Schrödinger, and independently by Dirac, already in 1926.

[2] For details, the reader is referred to Sect. 2.

[3] The crucial point that the factor $\mathrm{i} = \sqrt{-1}$ in (1.1.1) is unavoidable, took Schrödinger several months to finally accept. See the discussions in [867, 938] on this point. It was recognized by Ehrenfest in 1927 that an analytic continuation in time to "Euclidean time", $t \to -\mathrm{i}\,t$, transforms the Schrödinger equation into the *heat or diffusion equation* [295], see also the remarks on the *Feynman–Kac formula* on p.18 and in Sect. 2.2.

implies its knowledge at all subsequent times $t'' > t'$. However, the interpretation of $|\Psi|^2$ as the probability of an event is an *indeterministic interpretation.*[4]

Introducing the Green function $K(\mathbf{x}'',\mathbf{x}';t'',t')$ of the time-dependent Schrödinger equation (1.1.1), the *quantum mechanical time evolution* of the wave function $\Psi(\mathbf{x},t)$ is explicitly given by the integral relation

$$\Psi(\mathbf{x}'',t'') = \int d\mathbf{x}' \, K(\mathbf{x}'',\mathbf{x}';t'',t')\Psi(\mathbf{x}',t') \ , \tag{1.1.2}$$

which determines the probability amplitude at a final point $\mathbf{x}''$ at time t'' in terms of the probability amplitude $\Psi(\mathbf{x}',t')$ at an initial point $\mathbf{x}'$ at time t'. Equation (1.1.2) shows that the Green function K plays the rôle of an *integral kernel.* In fact, K is identical to the kernel of the quantum mechanical time-evolution operator ($T = t'' - t' > 0$)

$$K(\mathbf{x}'',\mathbf{x}';t'',t') = \left\langle \mathbf{x}'' \middle| \exp\left(-\frac{\mathrm{i}}{\hbar} T \, \underline{\mathrm{H}}_{\mathbf{x}} \right) \middle| \mathbf{x}' \right\rangle \ , \tag{1.1.3}$$

see also (2.1.19). Since the integral relation (1.1.2) is completely equivalent to the Schrödinger equation(1.1.1), it offers the possibility of considering (1.1.2) as the basic time-evolution equation in quantum mechanics and thus as an *alternative* to the operator Schrödinger equation. This is exactly Feynman's approach in his *path integral formulation of quantum mechanics* [326]. In this approach the integral kernel K is the primary object, and that is the reason why the time-dependent Green function K is in this context commonly called the *Feynman kernel.*[5] "A quantum mechanical system is described equally well by specifying the function K, or by specifying the Hamiltonian $\underline{\mathrm{H}}_{\mathbf{x}}$ from which it results. For some purposes the specification in terms of K is easier to use and visualize." [328]. It is clear from (1.1.2) and (1.1.3) that the Feynman kernel $K(\mathbf{x}'',\mathbf{x}';t'',t')$ has the meaning of a transition-probability amplitude to get from the point $(\mathbf{x}',t')$ to the point $(\mathbf{x}'',t'')$, or in Feynman's words: "A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time" [326].

It is a remarkable fact that by taking the Feynman kernel K as the primary object, one is led to a novel formulation of quantum mechanics (though mathematically equivalent to the more usual Hamiltonian approach) which turns out to be a *Lagrangian formulation.* Furthermore, instead of operators in Hilbert space and differential equations one has to deal with *Feynman path integrals* called *functional integrals* in mathematics. Although the path integral formulation to quantum mechanics, as a complete theory, is solely

[4] For a very lucid exposition of the fundamental concepts of probability and probability amplitudes in quantum mechanics, see Feynman's original paper [326], the Feynman–Hibbs book [340], and [669].

[5] This is the terminology which we shall adopt throughout this book. Note, however, that the Feynman kernel K is also called the *propagator* by some authors.

the work of Feynman [325, 326], the important discovery of the close analogy between the Feynman kernel $K(\mathbf{x}'', \mathbf{x}'; t' + \mathrm{d}t, t')$ associated with an infinitesimal displacement from time t' to time $t' + \mathrm{d}t$ and the phase factor $\exp[\frac{\mathrm{i}}{\hbar}\mathcal{L}\, \mathrm{d}t]$ with $\mathcal{L}$ being the classical Lagrangian is due to Dirac [254].

By 1927 Dirac had worked out his *transformation theory* [252] by studying so-called quantum mechanical transformation functions which depend on pairs of conjugate variables which, at the classical level, are connected via canonical transformations.[6] However, the canonical transformations considered by Dirac in this paper come from a generating function which is of type 2 (following the traditional labeling of canonical transformations), i.e., it connects an initial momentum P with a final position q. Thus the associated transformation function called $(q|P)$ by Dirac [252] cannot be identified with the Feynman kernel K which reads in Dirac's notation of 1932 $(q_{t''}|q_{t'})$. (In our notation we have $K(\mathbf{x}'', \mathbf{x}'; t'', t') = \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle$ with $|\mathbf{x}, t\rangle := \exp(+\frac{\mathrm{i}}{\hbar} t\, \underline{\mathrm{H}}_{\mathbf{x}})|\mathbf{x}\rangle$, see (1.1.3).)[7] The time transformation function $(q_{t''}|q_{t'})$ appears for the first time in Dirac's paper [254] entitled "The Lagrangian in Quantum Mechanics" received by the *Physikalische Zeitschrift der Sowjetunion* on November 19, 1932. Let us quote from the introduction of this paper [254, pp. 64, 65]: "Quantum mechanics was built up on a foundation of analogy with the Hamiltonian theory of classical mechanics. This is because the classical notion of canonical coordinates and momenta was found to be one with a very simple quantum analogue, as a result of which the whole of the classical Hamiltonian theory, which is just a structure built up on this notion, could be taken over in all its details into quantum mechanics. – Now there is an alternative formulation for classical dynamics, provided by the Lagrangian. This requires one to work in terms of coordinates and velocities instead of coordinates and momenta. The two formulations are, of course, closely related, but there are reasons for believing that the Lagrangian one is the more fundamental. ... For these reasons it would seem desirable to take up the question of what corresponds in the quantum theory to the Lagrangian method of the classical theory. A little consideration shows, however, that one cannot expect to be able to take over the classical Lagrangian equations in any very direct way. These equations involve partial derivatives of the Lagrangian with respect to the coordinates and velocities and no meaning can be given to such derivatives in quantum mechanics. The only differentiation process that can be carried out with respect to the dynamical variables of quantum mechanics is that of forming Poisson brackets and this process leads to the Hamiltonian theory. ... We must therefore seek our quantum Lagrangian theory in an indirect way. We must try to take over the *ideas* of

6 The earlier work of Jordan [546] concerns time-independent canonical transformations which are not relevant for our discussion.

7 The transformation function $(q|P)$ is the subject of Van Vleck's famous paper [906]. For a discussion of this paper, its relation to Dirac's papers [252, 254] and the story of Van Vleck's determinant, see [193].

the classical Lagrangian theory, not the *equations* of the classical Lagrangian theory."

Dirac considers [254] two sets of conjugate variables (p, q) and (P, Q) but supposes now that, at the classical level, the independent variables of the generating function are q and Q. Let R be the corresponding *generating function.*[8] Then the corresponding dependent variables are given by

$$p = \frac{\partial R}{\partial q} , \qquad P = -\frac{\partial R}{\partial Q} , \tag{1.1.4}$$

where $R = R(q, Q; t)$. Notice that this *generating function of canonical transformations* is of type 1.

In the section entitled "The Lagrangian and the Action Principle" Dirac [l.c., p.67] continues: "The equations of motion of the classical theory cause the dynamical variables to vary in such a way that their values q_t, p_t at any time t are connected with their values q_T, p_T at any other time T by a contact transformation, which may be put into the form (1.1.4) with $q, p = q_t, p_t$; $Q, P = q_T, p_T$ and R equal to the time integral of the Lagrangian over the range T to t. In the quantum theory the q_t, p_t will still be connected with the q_T, p_T by a contact transformation and there will be a transformation function $(q_t|q_T)$ connecting the two representations in which the q_t and the q_T are diagonal respectively. The work of the preceding section now shows that

$$(q_t|q_T) \qquad \text{corresponds to} \qquad \exp\left(\frac{\mathrm{i}}{\hbar}\int_T^t \mathcal{L}\,\mathrm{d}t\right) , \tag{1.1.5}$$

where $\mathcal{L}$ is the Lagrangian. If we take T to differ only infinitesimally from t, we get the result

$$(q_{t+\mathrm{d}t}|q_t) \qquad \text{corresponds to} \qquad \exp\left(\frac{\mathrm{i}}{\hbar}\mathcal{L}\,\mathrm{d}t\right) . \tag{1.1.6}$$

The transformation functions in (1.1.5) and (1.1.6) are very fundamental things in the quantum theory and it is satisfactory to find that they have their classical analogues, expressible simply in terms of the Lagrangian."

The above citations show quite clearly that Dirac had carried the programme of formulating a Lagrangian approach to quantum mechanics quite far. What is strange, however, is that the "very fundamental things" appear in (1.1.5) and (1.1.6) not in the form of equations, but rather Dirac uses the vague formulation *corresponds to.*[9] We may thus ask with Schwinger [842]: "Why, then, did Dirac not make a more precise, if less general, statement?

[8] Dirac uses S instead of R.

[9] Already in the second edition of his famous book *The Principles of Quantum Mechanics*, which appeared in 1935 [256], Dirac included these results in Sect. 33, but neither in this nor in later editions did he quantify the correspondence expressed in (1.1.5) and (1.1.6). See also Dirac's paper from 1945 [255].

Because he was interested only in a general question: What, in quantum mechanics, corresponds to the classical principle of stationary action?"

In order to answer this *general question*, Dirac considered the basic *composition law* for $(q_t|q_T)$ in the form

$$(q_t|q_T) = \int (q_t|q_m)\,\mathrm{d}q_m(q_m|q_{m-1})\,\mathrm{d}q_{m-1}\ldots(q_2|q_1)\,\mathrm{d}q_1(q_1|q_T) \ , \qquad (1.1.7)$$

where the time interval $T \to t$ has been divided up "into a large number of small sections $T \to t_1, t_1 \to t_2, \ldots, t_{m-1} \to t_m, t_m \to t$ by the introduction of a sequence of intermediate times $t_1, t_2, \ldots, t_m$", and "where q_k denotes q at the intermediate time t_k, $(k = 1, 2, \ldots, m)$." He then compared the composition law (1.1.7) with the product

$$A(tT) = A(tt_m)A(t_mt_{m-1})\ldots A(t_2t_1)A(t_1T) \ , \qquad (1.1.8)$$

where

$$\exp\left(\frac{\mathrm{i}}{\hbar}\int_T^t \mathcal{L}\,\mathrm{d}t\right) = A(tT) \ ,$$

"so that $A(tT)$ is the classical analogue of $(q_t|q_T)$."

"Equation (1.1.7) at first sight does not seem to correspond to equation (1.1.8), since on the right-hand side of (1.1.7) we must integrate after doing the multiplication while on the right-hand side of (1.1.8) there is no integration.

"Let us examine the discrepancy by seeing what becomes of (1.1.7) when we regard t as extremely small. From the results (1.1.5) and (1.1.6) we see that the integrand in (1.1.7) must be of the form $\mathrm{e}^{\mathrm{i}F/\hbar}$ where F is a function of $q_T, q_1, q_2, \ldots, q_m, q_t$ which remains finite as $\hbar$ tends to zero. Let us now picture one of the intermediate q's, say q_k, as varying continuously while the others are fixed. Owing to the smallness of $\hbar$, we shall then in general have $F/\hbar$ varying extremely rapidly. This means that $\mathrm{e}^{\mathrm{i}F/\hbar}$ will vary periodically with a very high frequency about the value zero, as a result of which its integral will be practically zero. The only important part in the domain of integration of q_k is thus that for which a comparatively large variation in q_k produces only a very small variation in F. This part is the neighbourhood of a point for which F is stationary with respect to small variations in q_k.

"We can apply this argument to each of the variables of integration in the right-hand side of (1.1.7) and obtain the result that the only important part in the domain of integration is that for which F is stationary for small variations in all the intermediate q's. But, by applying (1.1.5) to each of the small time sections, we see that F has for its classical analogue

$$\int_{t_m}^t \mathcal{L}\,\mathrm{d}t + \int_{t_{m-1}}^{t_m} \mathcal{L}\,\mathrm{d}t + \cdots + \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t + \int_T^{t_1} \mathcal{L}\,\mathrm{d}t = \int_T^t \mathcal{L}\,\mathrm{d}t \ , \qquad (1.1.9)$$

which is just the action function which classical mechanics requires to be stationary for small variations in all the intermediate q's. This shows the way in which (1.1.7) goes over into classical results when $\hbar$ becomes extremely small."

Thus Dirac showed by considering the *semiclassical limit* $\hbar \to 0$ that the multiple-integral construction (1.1.7) of the time transformation function contains the quantum analogue of the classical action principle, a fundamental result, indeed.

"Why, in the decade that followed, didn't someone pick up the computational possibilities offered by this integral approach to the time transformation function? To answer this question bluntly, perhaps no one needed it – until Feynman came along." [842].

Feynman[10] was working as a research assistant at Princeton during 1940–41. In the course of his graduate studies he discovered together with Wheeler an action principle using half advanced and half retarded potentials [920]. The problem was the infinite self-energy of the electron, and it turned out that the new action "principle could deal successfully with the infinity arising in the application of classical electrodynamics. – The problem then became one of applying this action principle to quantum mechanics in such a way that classical mechanics could arise naturally as a special case of quantum mechanics when h was allowed to go to zero. – Feynman searched for any ideas which might have been previously worked out in connecting quantum-mechanical behaviour with such classical ideas as the Lagrangian or, in particular, Hamilton's principal function R, the indefinite integral of the Lagrangian." [340]. At a Princeton beer party Feynman learned from Herbert Jehle, a former student of Schrödinger in Berlin, who had newly arrived from Europe, of Dirac's paper [254]. The natural question that then arose was what Dirac had meant by the phrase "corresponds to", see (1.1.5) and (1.1.6). Feynman found that Dirac's statement actually means "proportional to" such that (1.1.6) is to within a constant factor an equality. Based on this result and the composition law (1.1.7) in the limit $m \to \infty$, Feynman interpreted the multiple-integral construction (1.1.7) as an "integral over all paths" and wrote this down for the first time in his Ph D thesis [325] presented to the Faculty of Princeton University on May 4, 1942. During the war Feynman worked at Los Alamos, and after the war his primary direction of work was towards quantum electrodynamics. So it happened that a complete theory of the *path integral approach to quantum mechanics* was worked out only in 1947. Feynman submitted his paper to *The Physical Review*, but the editors rejected it! Thus he rewrote it and sent it to *Reviews of Modern Physics*, where it finally appeared in spring 1948 under the title "Space-Time Approach to Non-Relativistic Quantum Mechanics" [326].

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[10]Concerning the work and life of Richard P. Feynman, the reader may consult the following sources: Feynman's Nobel lecture [336]; Feynman's two autobiographies [338, 339]; the excellent biography by Gleick [396], including "a Feynman bibliography"; the special issue "Richard Feynman" in *Physics Today* **42** (February 1989); Dyson's autobiography [287], and Schweber's book [836].

Feynman's paper [326] is one of the most beautiful and most influential papers in physics written during the last fifty years. While at first sight the path integral formulation appears to be "merely a reformulation of quantum mechanics, equivalent to the usual formulation" [377], there are now some indications that "the path integral formulation of quantum mechanics may be more fundamental than the conventional one, in that there is a crucial domain where it may apply and the conventional formulation may fail. That is the domain of quantum cosmology." [377]

Let us briefly sketch how Feynman arrived [326] at his path integral.[11] First he considered the limit $m \to \infty$ of the composition law (1.1.7), which is equivalent to the limit $\epsilon \to 0$, if the intermediate times t_k are, for simplicity, chosen to be equidistant, i.e., $t_k = t' + k\epsilon$, $(k = 0, 1, \ldots, N-1)$, with $N := m+1$, $t' := T = t_0$, $t'' := t = t_N$, and $\epsilon = (t'' - t')/N$. Then (1.1.7) becomes

$$(q_{t''}|q_{t'}) = \lim_{\epsilon\to 0} \int \mathrm{d}q_1 \ldots \mathrm{d}q_{N-1} (q_{t''}|q_{N-1})(q_{N-1}|q_{N-2}) \ldots (q_2|q_1)(q_1|q_{t'}) \tag{1.1.10}$$

assuming that the limit exists, of course. Converting from Dirac's notation to the notation introduced in (1.1.2) and (1.1.3), one obtains

$$K(x'', x'; t'', t') = \lim_{\epsilon\to 0} \prod_{k=1}^{N-1} \int \mathrm{d}x_k \prod_{j=0}^{N-1} K(x_{j+1}, x_j; t_j + \epsilon, t_j) \ . \tag{1.1.11}$$

This multiple-integral representation of the Feynman kernel is built up by the *short-time kernels* $K(x_{j+1}, x_j; t_j + \epsilon, t_j)$, $(j = 0, 1, \ldots, N-1)$, for which Feynman [326] writes, in the limit $\epsilon \to 0$,

$$K(x_{j+1}, x_j; t_j + \epsilon, t_j) = \frac{1}{A} \exp\left[\frac{\mathrm{i}}{\hbar}\epsilon\mathcal{L}\left(x_{j+1}, \frac{x_{j+1} - x_j}{\epsilon}\right)\right] \tag{1.1.12}$$

thus replacing Dirac's vague correspondence (1.1.6) by a precise statement involving some normalization factor $A = A(\epsilon)$ for each instant of time, suitably adjusted. (Remember that the Lagrangian depends on the trajectory $x(t)$ and the velocity $\dot{x}(t)$, i.e., $\mathcal{L} = \mathcal{L}(x(t), \dot{x}(t))$.) Inserting (1.1.12) into (1.1.11), Feynman obtained

$$K(x'', x'; t'', t') = \lim_{\epsilon\to 0} A^{-N} \prod_{k=1}^{N-1} \int \mathrm{d}x_k \exp\left[\frac{\mathrm{i}}{\hbar} \sum_{j=0}^{N-1} \epsilon\mathcal{L}\left(x_{j+1}, \frac{x_{j+1} - x_j}{\epsilon}\right)\right] \ . \tag{1.1.13}$$

Since the sum in the exponent becomes in the limit $\epsilon \to 0$ just the *action* R,

$$\lim_{\epsilon\to 0} \sum_{j=0}^{N-1} \epsilon\mathcal{L}\left(x_{j+1}, \frac{x_{j+1} - x_j}{\epsilon}\right) = \int_{t'}^{t''} \mathcal{L}(x(t), \dot{x}(t))\mathrm{d}t =: R[x(t)] \ , \tag{1.1.14}$$

[11] A detailed account of the general theory will be given in Chap. 2.

one arrives at the *Feynman path integral*

$$K(x'',x';t'',t') = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left(\frac{\mathrm{i}}{\hbar}R[x(t)]\right) . \tag{1.1.15}$$

Here the symbol $\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)$ is defined[12] by (1.1.13) and represents (some kind of) an integration over the space of functions $x(t)$, i.e., all possible paths, connecting the points (x',t') and (x'',t''). For a particle of mass m moving in a one-dimensional potential $V(x)$, i.e., $\mathcal{L} = \frac{m}{2}\dot{x}^2 - V(x)$, Feynman derived for the normalization factor A

$$A = \left(\frac{2\pi\,\mathrm{i}\,\hbar\epsilon}{m}\right)^{1/2} . \tag{1.1.16}$$

Many years later, in his *Nobel lecture*, Feynman described his discovery as follows [336]: "In that way I found myself thinking of a large number of integrals, one after the other in sequence. In the integrand was the product of the exponentials, which, of course, was the exponential of the sum of terms like $\epsilon\mathcal{L}$. Now $\mathcal{L}$ is the Lagrangian and ϵ is like the time interval $\mathrm{d}t$, so that if you took a sum of such terms, that's exactly like an integral. That's like Riemann's formula for the integral $\int \mathcal{L}\,\mathrm{d}t$; you just take the value at each point and add them together. We are to take the limit as $\epsilon \to 0$, of course. Therefore, the connection between the wave function of one instant and the wave function of another instant a finite time later could be obtained by an infinite number of integrals (because ϵ goes to zero, of course) of exponential $(\mathrm{i}\,R/\hbar)$, where R is the action expression (1.1.14). At least, I had succeeded in representing quantum mechanics directly in terms of the action R. ... This led later on to the idea of the amplitude of the path; that for each possible way that the particle can go from one point to another in space-time, there's an amplitude. That amplitude is e to the $\mathrm{i}\,/\hbar$ times the action for the path. Amplitudes from various paths superpose by addition. This then is another, a third way, of describing quantum mechanics, which looks quite different than that of Schrödinger or Heisenberg, but is equivalent to them."

One of the first physicists who understood Feynman's "intuitive method"[13] was Dyson. He gave the following description of Feynman in those days [287]: "Dick was also a profoundly original scientist. He refused to take anybody's

[12]The identifying notation $\mathcal{D}x(t)$ was not yet used in [326]. It was introduced by Feynman in [330].

[13]"As a result", Feynman said [336], "the work was criticized, I don't know whether favorably or unfavorably, and the 'method' was called the 'intuitive method'. For those who do not realize it, however, I should like to emphasize that there is a lot of work involved in using this 'intuitive method' successfully .... Nevertheless, a very great deal more truth can become known than can be proven."

word for anything. This meant that he was forced to rediscover or reinvent for himself almost the whole of physics. It took him five years of concentrated work to reinvent quantum mechanics. He said that he couldn't understand the official version of quantum mechanics that was taught in the textbooks, and so he had to begin afresh from the beginning. This was a heroic enterprise. He worked harder during those years than anybody else I ever knew. At the end he had his version of quantum mechanics that he could understand. He then went on to calculate with his version of quantum mechanics how an electron should behave. He was able to reproduce the result that Hans [Bethe] had calculated using orthodox theories a little earlier. But Dick could go much further. He calculated with his own theory fine details of the electron's behaviour that Hans's method could not touch. Dick could calculate these things far more accurately, and far more easily, than anybody else could. The calculation that I did for Hans, using the orthodox theory, took me several months of work and several hundred sheets of paper. Dick could get the same answer, calculating on a blackboard, in half an hour." In his last remarks on an electron's behaviour Dyson refers to Feynman's famous work in *quantum electrodynamics*, culminating in the *Feynman rules* [328–330] and *Feynman diagrams* [329], which Feynman first derived[14] using the path integral method, and which nowadays can be found in every textbook on quantum field theory and elementary particle physics. Feynman describes this work as follows [336]: "The rest of my work was simply to improve the techniques then available for calculations, making diagrams to help analyze perturbation theory quicker. Most of this was first worked out by guessing – you see .... I included diagrams for the various terms of the perturbation series, improved notations to be used, worked out easy ways to evaluate integrals which occurred in these problems, and so on, and made a kind of handbook on how to do quantum electrodynamics. ... At this stage, I was urged to publish this because everybody said it looks like an easy way to make calculations and wanted to know how to do it. I had to publish it missing two things; one was proof of every statement in a mathematically conventional sense. Often, even in a physicist's sense, I did not have a demonstration of how to get all of these rules and equations from conventional electrodynamics."

The path integral (1.1.15) is the *fundamental quantum-mechanical rule* in Feynman's third way of describing quantum mechanics. The rule tells us [340] "how much each trajectory contributes to the total amplitude to go from (x',t') to (x'',t''). It is not that just the particular path of extreme action contributes; rather, it is that all the paths contribute. They contribute equal amounts to the total amplitude, but contribute at different phases. The phase of the contribution from a given path is the action R for that path in units of the quantum of action $\hbar$. That is, to summarize: The probability $P(x'',x';t'',t')$ to go from a point x' at the time t' to the point x'' at t'' is the absolute square $P(x'',x';t'',t') = |K(x'',x';t'',t')|^2$ of an amplitude

[14]See Feynman's remark in [330].

$K(x'', x'; t'', t')$ to go from (x', t') to (x'', t''). This amplitude is the sum of contributions $\Phi[x(t)]$ from each path

$$K(x'', x'; t'', t') = \sum_{\substack{\text{over all paths} \\ \text{from } x' \text{ to } x''}} \Phi[x(t)] \ . \tag{1.1.17}$$

The contribution of a path has a phase proportional to the action R:

$$\Phi[x(t)] = \text{const}\, e^{(i/\hbar)R[x(t)]} \ . \tag{1.1.18}$$

The action is that for the corresponding classical system, see (1.1.14)."

The only purpose of rewriting the path integral (1.1.15) in the (even more symbolic) sum form (1.1.17) is to illustrate its interpretation as a *sum over all paths* or *sum over all histories*. Equation (1.1.17) makes it particularly clear that the total amplitude K depends on the whole space-time history, i.e., all paths, and that it is obtained by a superposition of the amplitudes $\Phi[x(t)]$ from all paths $x(t)$ which connect the space-time points (x', t') and (x'', t'').[15]

In this book we will mainly use the *lattice definition* (1.1.13) of the path integral (1.1.15). Almost all path integral solutions presented here have been obtained by using this definition (and its generalizations to many degrees of freedom, curvilinear coordinates, etc.), i.e., have been worked out using the subdivision and limiting processes involved in (1.1.13). Feynman was fully aware of the mathematical problems associated with the *integration in functional spaces*. Already in his 1948 paper he wrote in a footnote [326]: "There are very interesting mathematical problems involved in the attempts to avoid the subdivision and limiting processes. Some sort of complex measure is being associated with the space of functions $x(t)$. Finite results can be obtained under unexpected circumstances because the measure is not positive everywhere, but the contributions from most of the paths largely cancel out. These curious mathematical problems are sidestepped by the subdivision process. However, one feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of calculus." In writing the path integral (1.1.13) in the "less restrictive notation" (1.1.15), Feynman expressed his strong belief that "the concept of the sum over all paths, like the concept of an ordinary integral, is independent of a special definition and valid in spite of the failure of such definitions" [340]. At this point we will not discuss the mathematical aspects of path integrals, but rather we will focus our attention on the question of whether the new formulation of quantum mechanics had a favorable reception.

It is not too surprising to learn that Feynman's ideas were not appreciated in the beginning among the physicists of the older generation who had laid

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[15] Actually, it turns out that the Feynman "measure" $\mathcal{D}x(t)$ is concentrated on the class of continuous but nowhere differentiable functions, see the remarks on p.18 and in Chap. 2.

the foundations of quantum mechanics – with the notable exception of Pauli, see below.[16] In the orthodox formulation and interpretation of quantum mechanics the idea of an electron's orbit had been completely abandoned by 1925, and thus it appeared that Feynman's path integral approach, which is based in an essential way on the notion of paths, is a regression to improper classical ideas.

In a talk given on the occasion of Schwinger's 60th birthday celebration in 1978, Feynman narrated the reaction of Niels Bohr at the famous Pocono conference in 1948 [337]: "That was chaos, and then, all the time I was pushed back, away from the mathematics into my so-called physical ideas until I was driven to the point of describing quantum mechanics as an amplitude for every path, for every trajectory that a particle can take there's an amplitude and Professor Bohr got up and explained to me that already in 1920 they realized that the concept of a path in quantum mechanics – that you could specify the position as a function of time – that was not a legitimate idea and I gave up at this point.

As already mentioned, one of the first physicists of the younger generation who immediately appreciated Feynman's approach to quantum mechanics and quantum electrodynamics, was Dyson. Although Dyson [285] did not work with path integrals, he thoroughly understood Feynman's method which permitted him to see the relationships among the conventional operator formulations of quantum electrodynamics, that of Schwinger [838] and Tomonaga [895], and that of Feynman [328–330].

One of the most fundamental aspects of path integrals is that they offer a very transparent method to systematically derive the *semiclassical limit* $\hbar \to 0$ of quantum mechanics. To study this limit let us consider the D-dimensional generalization

$$K(\mathbf{x}'',\mathbf{x}';t'',t') = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}}{\hbar} R[\mathbf{x}(t)]\right) \tag{1.1.19}$$

of the path integral (1.1.15) where we restrict ourselves to Cartesian coordinates, $\mathbf{x}(t) = (x^1(t),\ldots,x^D(t))$. Here $\mathcal{D}\mathbf{x}(t)$ denotes $\prod_{a=1}^{D} \mathcal{D}x^a(t)$ and $R[\mathbf{x}(t)]$ is the corresponding classical action.

At this point the reader is reminded that Dirac [254] had already shown that the multiple-integral construction (1.1.7) contains the quantum analogue of the classical action principle. Feynman remarked [326]: "The points he [Dirac] makes concerning the passage to the classical limit $\hbar \to 0$ are very beautiful ... ", and he translated Dirac's argument into the path integral language. If $\hbar$ is small, the integrand $\exp\left((\mathrm{i}/\hbar)R[\mathbf{x}(t)]\right)$ will be a rapidly

[16]Here we do not include Bethe, who was most likely the first to judge rightly the value of Feynman's space-time view, since he started his career in the late 1920s when the principles of quantum mechanics were already invented.

varying functional of the path $\mathbf{x}(t)$, and thus the region in functional space at which $\mathbf{x}(t)$ contributes most strongly is that at which the phase of the exponent, i.e., the classical action R, varies least rapidly with $\mathbf{x}(t)$ (method of stationary phase). "We see then that the classical path is that for which the integral

$$R[\mathbf{x}(t)] = \int_{t'}^{t''} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(t))\mathrm{d}t \tag{1.1.20}$$

suffers no first-order change on varying the path. This is Hamilton's principle and leads directly to the Lagrangian equation of motion." [326]

In order to apply the *functional analogue* of the *method of stationary phase*[17] to the path integral (1.1.19), we expand the functional $R[\mathbf{x}(t)]$ about the classical trajectory $\mathbf{x}_{\mathrm{Cl}}(t)$ in a sort of *functional Taylor series*

$$R[\mathbf{x}(t)] = R_{\mathrm{Cl}} + \frac{1}{2}\delta^2 R[\mathbf{x}_{\mathrm{Cl}}(t)] + \frac{1}{6}\delta^3 R[\mathbf{x}_{\mathrm{Cl}}(t)] + \dots\ , \tag{1.1.21}$$

where R_{Cl} is the classical action evaluated along an actual path $\mathbf{x}_{\mathrm{Cl}}(t)$ of the system,

$$R_{\mathrm{Cl}} = R_{\mathrm{Cl}}(\mathbf{x}'', \mathbf{x}'; t'', t') := R[\mathbf{x}_{\mathrm{Cl}}(t)]\ . \tag{1.1.22}$$

Here $\mathbf{x}_{\mathrm{Cl}}(t)$ is the solution of Hamilton's principle, $\delta R = 0$, with the end-point conditions $\mathbf{x}(t') = \mathbf{x}'$, $\mathbf{x}(t'') = \mathbf{x}''$.[18] In (1.1.21) $\delta^2 R[\mathbf{x}_{\mathrm{Cl}}(t)]$ is a quadratic functional with regard to the *quantum fluctuation* $\mathbf{q}(t) = \mathbf{x}(t) - \mathbf{x}_{\mathrm{Cl}}(t)$ $(\mathbf{q}(t') = \mathbf{q}(t'') = 0)$

$$\begin{aligned}\delta^2 R[\mathbf{x}_{\mathrm{Cl}}(t)] = \int_{t'}^{t''} \Bigg\{ & \left.\frac{\delta^2 \mathcal{L}}{\delta x^a \delta x^b}\right|_{\mathbf{x}=\mathbf{x}_{\mathrm{Cl}}} q^a(t) q^b(t) + 2 \left.\frac{\delta^2 \mathcal{L}}{\delta x^a \delta \dot{x}^b}\right|_{\mathbf{x}=\mathbf{x}_{\mathrm{Cl}}} q^a(t)\dot{q}^b(t) \\ & + \left.\frac{\delta^2 \mathcal{L}}{\delta \dot{x}^a \delta \dot{x}^b}\right|_{\mathbf{x}=\mathbf{x}_{\mathrm{Cl}}} \dot{q}^a(t)\dot{q}^b(t) \Bigg\} \mathrm{d}t\ .\end{aligned} \tag{1.1.23}$$

Inserting the "Taylor series" (1.1.21) into the path integral (1.1.19), we obtain

$$\begin{aligned}&K(\mathbf{x}'', \mathbf{x}'; t'', t') \\ &= \exp\left(\frac{\mathrm{i}}{\hbar} R_{\mathrm{Cl}}\right) \int\limits_{\mathbf{q}(t')=0}^{\mathbf{q}(t'')=0} \mathcal{D}\mathbf{q}(t) \exp\left[\frac{1}{2}\left(\frac{\mathrm{i}}{\hbar}\right)\delta^2 R[\mathbf{x}_{\mathrm{Cl}}(t)] + \dots\right]\ .\end{aligned} \tag{1.1.24}$$

Here we have used the fact that the Feynman "measure" transforms as $\mathcal{D}\mathbf{x}(t) \to \mathcal{D}\mathbf{q}(t)$ under the translation $\mathbf{x}(t) \to \mathbf{x}_{\mathrm{Cl}}(t) + \mathbf{q}(t)$, since $\mathbf{x}_{\mathrm{Cl}}(t)$ is a fixed function. This is obvious from the lattice definition (1.1.13), since $\mathrm{d}x_k^a = \mathrm{d}x^a(t_k) \to \mathrm{d}(x_{\mathrm{Cl}}^a(t_k) + q^a(t_k)) = \mathrm{d}q_k^a$.

[17] See Sect. 5.2.

[18] In general, there will exist many solutions to the variational problem, see the discussion on p.17 and in Sect. 5.2. Here we ignore this problem.

Retaining only the quadratic functional (1.1.23) in the path integral (1.1.24), the remaining path integration over $\mathcal{D}\mathbf{q}(t)$ can be carried out since it is quadratic in $\mathbf{q}(t)$ (Gaussian path integral), and one obtains for small time intervals $t'' - t'$ the semiclassical (sc) formula[19]

$$K_{\mathrm{sc}}(\mathbf{x}'',\mathbf{x}';t'',t') = \frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{D/2}}\left[\det\left(-\frac{\partial^2 R_{\mathrm{Cl}}}{\partial x''^a \partial x'^b}\right)\right]^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}';t'',t')\right] . \tag{1.1.25}$$

Throughout this book we shall call the formula (1.1.25) *Pauli's formula* and the determinant

$$D := D(\mathbf{x}'',\mathbf{x}';t'',t') := \det\left(-\frac{\partial^2 R_{\mathrm{Cl}}}{\partial x''^a \partial x'^b}\right) \tag{1.1.26}$$

the *Morette–Van Hove determinant* for reasons which will become clear below. Here $\partial^2 R_{\mathrm{Cl}}/\partial x''^a \partial x'^b$ is a $D \times D$ matrix $(a, b = 1, \ldots, D)$.

We would like to make several remarks:

i) It can already be seen from (1.1.24), which is still exact, that the Feynman kernel K can be reduced to a product of two functions, where one of these functions is exactly given by the phase factor $\exp\left[(\mathrm{i}/\hbar)R_{\mathrm{Cl}}\right]$ and depends upon the classical path, while the remaining function is the Feynman kernel for a system to proceed from $\mathbf{q} = 0$ at $t = t'$ to $\mathbf{q} = 0$ at $t = t''$ and does not therefore depend on $\mathbf{x}', \mathbf{x}''$, or $\mathbf{x}_{\mathrm{Cl}}$, being only a function of t', t''.
ii) If the Lagrangian is quadratic to begin with, like in the case of a forced harmonic oscillator, the action functional cannot depend on $\mathbf{q}$ more than quadratically, and hence Pauli's formula (1.1.25) is exact. This was already observed by Feynman [326, 330], and a method to compute such *Gaussian path integrals* was developed by him in his thesis [325].
iii) To the best of our knowledge, the first paper on path integrals, apart from Feynman's, written by a physicist was submitted by Cécile Morette[20] in 1950 [710]. In this paper we find for the first time the general method of the functional Taylor expansion (1.1.21) applied to path integrals. Pauli's formula (1.1.25) was not directly derived, but rather Morette started from Feynman's ansatz (1.1.12) for the short-time kernel and determined the normalization factor by a *unitarity condition*. The formula obtained for this factor was, however, not yet expressed in terms of the determinant

[19]The exact formula for finite time intervals $t'' - t' > 0$ is given in equation (5.2.10).
[20]Morette married DeWitt [234–236] in 1951 and is identical to Morette–DeWitt [711, 712, 629] *and* DeWitt–Morette [147, 237–248].

(1.1.26). It so happened that Morette, Pauli and Van Hove were simultaneously at the Institute for Advanced Studies in Princeton during the fall of 1949 and the following winter.[21] Morette discussed her work with Van Hove who immediately saw the intimate relationship to the classical *Hamilton–Jacobi theory* and the *calculus of variation* (see Sect. 5.2). As a result the prefactor was (up to a sign factor) found to have the final form as given in (1.1.25). This result was then included in Morette's paper [710], with an acknowledgment to Van Hove, and also published by Van Hove in [905]. In both papers the minus sign in the determinant (1.1.26) is missing, and thus the semiclassical formula for K presented in the two papers is not correct since it is lacking a factor $(-1)^{D/2}$. It should be remarked that Van Hove's work was not on path integrals.

iv) Pauli was well-known for having been very critical and sometimes even very harsh in seminars, say. "Pauli could be ruthless in dismissing work he considered shallow or flimsy: *ganz falsch* (utterly false) – or worse, *nicht einmal falsch* (not even false)" [396, p.115].[22] It is therefore quite

[21] Historical details can be found in [193].

[22] While Feynman was still working at Princeton with Wheeler on their time-symmetric electrodynamics [920], Wheeler asked him to prepare a seminar on that. Feynman remembers [338]: "So it was to be my first technical talk, and Wheeler made arrangements with Eugene Wigner to put it on the regular seminar schedule.

"A day or two before the talk I saw Wigner in the hall. 'Feynman', he said, 'I think that work you're doing with Wheeler is very interesting, so I've invited Russel to the seminar.' Henry Norris Russel, the famous, great astronomer of the day, was coming to the lecture!

"Wigner went on. 'I think Professor von Neumann could also be interested.' Johnny von Neumann was the greatest mathematician around. 'And Professor Pauli is visiting from Switzerland, it so happens, so I've invited Professor Pauli to come' – Pauli was a very famous physicist – and by this time I'm turning yellow. Finally, Wigner said, 'Professor Einstein only rarely comes to our weekly seminars, but your work is so interesting that I've invited him specially, so he's coming, too.' " ... "Then the time came to give the talk, and here are these *monster minds* in front of me, waiting!" ... "But then a miracle occurred, as it has occurred again and again in my life, and it's very lucky for me: the moment I start to think about the physics, and have to concentrate on what I'm explaining, nothing else occupies my mind – I'm completely immune to being nervous. So after I started to go, I just didn't know who was in the room. I was only explaining this idea, that's all.

"But then the end of the seminar came, and it was time for questions. First off, Pauli, who was sitting next to Einstein, gets up and says, 'I do not sink dis teory can be right, because of dis, and dis, and dis,' and he turns to Einstein and says, 'Don't you agree, Professor Einstein?' Einstein says, 'Nooooooooooooo,' a nice, German-sounding 'No,' – very polite. 'I find only that it would be very difficult to make a corresponding theory for gravitational interaction.' ...

"I wish I had remembered what Pauli said, because I discovered years later remarkable that Pauli was, to the best of our knowledge, the first among the physicists of the older generation, having laid the foundations of quantum mechanics, who fully appreciated the new approach developed by Feynman. From a letter [766, letter no. 997] dated January 8, 1949, which Pauli sent from Zürich to Dyson, we can quite precisely infer *when* it happened that Pauli got interested in Feynman's approach to quantum electrodynamics. In this letter Pauli writes: "I thank you very much for sending your paper. It was not easy to read for us because the 'Feynman theory', which you compare with the Schwinger–Tomonaga formalism was entirely unknown here and we had to reconstruct it from your paper. Obviously, Pauli refers to Dyson's first paper [285][23] which was received on October 6, 1948 by *The Physical Review* and which was sent to him personally by Dyson. On May 10, 1949 Pauli's famous paper with Villars [768] was received by *Reviews of Modern Physics*, which contains what is nowadays known as the *Pauli–Villars regularization*, and in this paper the two papers by Dyson [285], and Feynman's talk at the Pocono Conference as well as his paper [327] are cited. Thus we can almost be sure that Pauli had read the three papers [326–329] of Feynman's when he arrived in Princeton on November 29, 1949, where he stayed until the end of April 1950.[24]

During his stay in Princeton Morette and Van Hove presented to Pauli at the occasion of an appointment with him the semiclassical formula (1.1.25). As a result of the discussion with Morette and Van Hove, Pauli wrote a couple of research notes entitled *Feynman's Methode der Lagrangefunktion* (PN 8/121–123),[25] *Van Hove* (PN 8/150) and *Diskutiere Van Hove's Formel* (PN 8/154–159). In these notes Pauli corrected the sign factor mentioned under iii) and then considered the semiclassical formula (1.1.25) as an *ansatz* for small but finite time intervals $t'' - t'$. After some calculations, which are very similar to those worked out by him in his article for the *Handbuch der Physik* published in 1933 [764], he obtained the important (exact) result that K_{sc}, (1.1.25), satisfies the Schrödinger equation up to terms of order $\hbar^2$, called "wrong terms", proportional to $D^{-1/2}\nabla^2_{\mathbf{x}''}D^{1/2}$, the coefficient of order $\hbar^0$ being the Hamilton–Jacobi equation, and that of order $\hbar^1$ the continuity equation satisfied by the probability density $D(\mathbf{x}'', \mathbf{x}'; t'', t')$, the square of the amplitude of K_{sc}.

that the theory was not satisfactory when it came to making the quantum theory. It's possible that that great man noticed the difficulty immediately and explained it to me in the question ... "

[23]The second one was received only on February 24, 1949 by *The Physical Review*.

[24]The dates are taken from [766, p. 711 and p. 915].

[25]These notes are in the Pauli Archives at CERN, Geneva. The meaning of, e.g., PN 8/121 is "Pauli Nachlass" Box 8, p. 121.

During the winter semester 1950–51 Pauli gave a course at the ETH Zürich on "Ausgewählte Kapitel aus der Feldquantisierung" [765]. The lecture notes contain an *Appendix* entitled *"Der Feynman'sche Zugang zur Quantenelektrodynamik"* (Feynman's approach to quantum electrodynamics). There one finds in equation (172) precisely the semiclassical formula (1.1.25) and the proof that K_{sc} satisfies the Schrödinger equation up to terms of order $\hbar^2$. Furthermore, Pauli shows that if K_{sc} is inserted for the short-time kernel in the D-dimensional generalization of the multiple-integral representation (1.1.11), one obtains the exact Feynman kernel. From reading the lecture notes it becomes clear that Pauli understood and appreciated Feynman's path integral approach completely. However, it is interesting to observe that he did not quote Van Hove nor Morette. It seems [193] that one of the reasons why Pauli did not react to Morette's functional approach is that at that time it was not known that the saddle point approximation to Feynman's path integral yields the same result as the time-dependent WKB approximation. One can almost be sure that Pauli's opinion at that time is adequately expressed in a comment which was made several years later by Gel'fand and Yaglom in their famous review [376][26] on integration in functional spaces. Commenting in footnote 21 on Morette's paper [710], they wrote: "We note, however, that the strictness of the quoted proof is substantially lowered due to the fact that the question of the precise meaning of functional integrals studied was not discussed." Since Pauli had checked directly that K_{sc}, (1.1.25), satisfies the Schrödinger equation up to terms of order $\hbar^2$, there was no doubt about his proof. This explains Pauli's remark at the end of a letter of April 1951 [767, letter no. 1230], congratulating Bryce Seligman-DeWitt and Cécile Morette on their marriage:[27] "By the way, Cécile may be interested in the way I have treated the Feynman-action principle in my mimeographed lectures. It is a kind of generalization of the WBK method to time-dependent solutions."

v) We do not know of any other papers written by Pauli in which he treats Feynman path integrals. There is, however, another clear indication showing that Pauli considered Feynman's Lagrangian approach to quantum mechanics as an important alternative to the conventional operator approach: in the fall of 1951 Pauli accepted Choquard as a Ph D student and asked him to study the higher order terms in the semiclassical expansion of the Feynman kernel for small but finite time intervals, in particular for Lagrangians which depend on $\mathbf{x}$ more than quadratically. Choquard received his Ph D in December 1953 and published his thesis in *Helvetica Physica Acta* [192]. Choquard's paper contains a very thorough and systematic study of the semiclassical approximation to the Feynman kernel K generalizing Pauli's formula (1.1.25). As

[26] In the following we quote from the English translation published in 1960.
[27] See also footnote 20.

specific examples he considered two-dimensional *quantum billiards* (La "boule de billard") and *confinement potentials* (i.e., anharmonic oscillators with $V(x) \sim x^{2k}$, $k > 1$), which in recent years play an important rôle in various fields of modern physics. Among several new results which he obtained, he made the important observation that for such systems Pauli's formula has to be modified in an essential way by replacing it by an *infinite sum* of the form $(T := t'' - t' > 0)$[28]

$$K_{sc}(\mathbf{x}'', \mathbf{x}'; T) = \frac{1}{(2\pi \mathrm{i} \hbar)^{D/2}} \sum_{n=0}^{\infty} D_n^{1/2} \mathrm{e}^{\mathrm{i} R_n/\hbar} \quad . \tag{1.1.27}$$

Here the nth term with action R_n is the contribution of the nth member of an *infinity of classical trajectories* passing through $\mathbf{x}'$ at time t' and $\mathbf{x}''$ at time t'' for given $\mathbf{x}'$ and $\mathbf{x}''$ and fixed time interval T. This reflects the important fact that, since the time T is fixed, but not the energy of the classical paths, there exist infinitely many solutions to Hamilton's principle, $\delta R = 0$, where R_n denotes the classical action evaluated along the nth path. Furthermore, he could show that there exits a *minimal time* t_m such that his semiclassical formula (1.1.27) holds for $0 < T < t_m$. The time t_m is determined by the so-called *conjugate points* of the classical trajectories (in the sense of Jacobi), which are the points at which the Morette–Van Hove determinants D_n become singular.[29] The singularities of D_n have been investigated, in the context of semiclassical quantum mechanics, for the first time by Choquard [192].

vi) The time-evolution kernel K is the primary object in Feynman's path integral approach and contains the complete information about a given quantum mechanical system, i.e., wave functions and energy levels. But in order to extract this information from K, one needs to have a (semiclassical) formula for it which is valid for finite times, i.e., *beyond the conjugate points.*[30] This is a difficult problem and requires among other things non-trivial results from Hamilton–Jacobi theory and the calculus of variations in the large. Going beyond the conjugate points was first achieved by Gutzwiller[31] in 1967 [479] who found the correct generalization of Choquard's formula (1.1.27) valid for arbitrary times T. Gutzwiller made this formula as the starting point for the derivation of the by now famous *Gutzwiller trace formula*, which is the basic *semiclas-*

[28]The systems considered in [192] are time-independent and therefore K depends on T only, i.e., $K(\boldsymbol{x}'', \boldsymbol{x}'; t'', t') = K(\boldsymbol{x}'', \boldsymbol{x}'; t'' - t', 0) =: K(\boldsymbol{x}'', \boldsymbol{x}'; T)$.

[29]See Sect. 5.2 for a definition of conjugate points and their geometrical meaning.

[30]For example, if one wants to calculate the Green function G, see equation (2.1.25), one has to integrate K over the whole time interval, $T \in (0, \infty)$.

[31]It is, presumably, not by chance that Gutzwiller is also a former student of Pauli. He wrote his diploma thesis (on meson theory and the anomalous magnetic moment of the proton!) under the direction of Pauli in 1949.

sical quantization rule for *strongly chaotic systems* [483, 869]. This will be discussed in Chap. 5.

In answering the question of whether the new (third) formulation of quantum mechanics had a favourable reception, we have so far paid attention only to the physicists. Since the path integral is, after all, a mathematical object, it is interesting to enquire about the reaction among the mathematicians. It is quite remarkable to learn that there appeared already in 1949 an interesting paper by the mathematician Mark Kac [555] which was written, as the author indicates, under the strong influence of Feynman's work.[32] Kac had worked in probability theory [554], in particular on the extension of Wiener's work [930] on *Brownian motion.* In this work there had already appeared a special measure in the space of continuous functions, called *Wiener measure.* Kac realized that if the path integral (1.1.15) is analytically continued to purely imaginary time ("Euclidean time"), $t \to -\mathrm{i}t$, see footnote 3, it can be rewritten in terms of the well-defined conditional Wiener measure. In fact, the Feynman path integral can then be interpreted as the mean value (expectation value) of the real functional $\exp\left[-\frac{1}{\hbar}\int_{t'}^{t''} V(x(t))\mathrm{d}t\right]$ over the trajectories of a Brownian particle, also called a *diffusion* or *Wiener process.* Thus Kac was able to show that Feynman's path integral, considered in Euclidean time, is a well-defined *functional integral.* Following Kac's article a lot of papers appeared in the mathematical literature developing these same ideas further.[33] In quantum mechanics, the reformulation of Feynman's path integral expression for the kernel K in terms of the Wiener measure is well known today as the *Feynman-Kac Formula*, see e.g. [397, 706, 854]. Kac later felt that he was better known as the K in F–K than for anything else in his career [556, p.115–116].

Although we do not know whether Feynman was aware of Wiener's work in probability theory, it is quite clear that he had realized the *stochastic nature* of the dominant paths in his path integral. Already in 1942 Feynman wrote in his Ph D thesis [325]: "Although the average value of the displacement of a particle in the time $\mathrm{d}t$ is $v\mathrm{d}t$, where v is the mean velocity, the mean value of the square of this displacement is not of order $\mathrm{d}t^2$, but only of order $\mathrm{d}t$." And in his 1948 paper he even refers to *Brownian motion:* "The 'velocities' $(x_{j-1} - x_j)/\epsilon$ which are important are very high, being of order $(\hbar/m\epsilon)^{1/2}$ which diverges as $\epsilon \to 0$. The paths involved are, therefore, continuous but possess no derivative. They are of a type familiar from study of Brownian motion." [326]. That Feynman was "familiar", indeed, with the theory of Brownian motion and, more generally, with the theory of diffusion processes is well known. During the war Feynman worked at Los Alamos in the theoretical

[32] Kac heard Feynman describe his path integral at Cornell, see Gleick [396, p. 249].

[33] For a rather complete mathematical review on *Integration in Functional Spaces and its Applications in Quantum Physics*, covering the years until 1955, the reader should consult the famous paper by Gel'fand and Yaglom [376].

division, and in 1944 Bethe, who was in charge of this division, decided to make Feynman a group leader. The official name of the group was *T-4, Diffusion Problems* [396, p. 171]! We also know that Feynman was during these years very close to the great mathematician John von Neumann who served as a travelling consultant and helped Feynman and his group with the numerical computations on the first computers available then. It is hard to believe that von Neumann did not tell Feynman about Wiener's work, knowing that Feynman was working on diffusion problems!

Coming back to the Feynman–Kac formula, we would like to make another remark. By reading the papers which appeared on this subject during the last fifty years, we cannot avoid getting the impression that some authors consider Feynman's original work on the path integral as a minor contribution relative to the rigorous work of Kac and other mathematicians. We will not comment on this, but rather cite Kac [556] himself who certainly knew how to judge Feynman's contribution: "There are two kinds of geniuses, the 'ordinary' and the 'magicians'. An ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what they have done, we feel certain that we, too, could have done it. It is different with the magicians. They are, to use mathematical jargon, in the orthogonal complement of where we are and the working of their minds is for all intents and purposes incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark. They seldom, if ever, have students because they cannot be emulated and it must be terribly frustrating for a brilliant young mind to cope with the mysterious ways in which the magician's mind works. Richard Feynman is a magician of the highest caliber."

Our intention in this *Introduction* was to give the reader some historical insights into the remarkable development of Feynman's path integral approach, and to enable him or her to see things in their right perspective. Since some of the facts which we have touched upon are not so well known, we thought it would be worthwhile to present them here.

The decades since the early 1950s, have seen a triumphal success of Feynman's path integral method. The applications cover many different areas, notably in physics, chemistry and mathematics. In this *Handbook* we shall mention and/or list a large number of these applications, i.e., a large number of path integrals together with an extensive list of almost 1000 references.

Our book is organized as follows. In Chap. 2 we give an introduction to the *General Theory of Path Integrals*. This chapter is self-contained and is written for those readers who have not yet studied path integrals, but have a good knowledge of the fundamentals of quantum mechanics as covered by standard courses in theoretical physics. Sections 2.1 and 2.2 contain the basic definitions and properties of path integrals, while Sections 2.3 and 2.4 provide some rules for how to compute simple path integrals. Sections 2.5–2.11 are written on a more advanced level, with an increasing degree of difficulty. The techniques described there have been developed only recently, and it is only with these new techniques that, for example, it has been possible to compute the path integral for the hydrogen atom, which is *the* prototype example of quantum mechanics, see Sect. 2.10.

In Chap. 3 we discuss and compute in detail the path integrals which we have called *basic path integrals.* It turns out that practically all path integrals, that can be calculated in closed form, can be in some way or another reduced to these basic path integrals.

Chapter 4 contains an introduction to *perturbation theory*, which is most elegantly derived from the path integral. This is the quantum mechanical analogue of Feynman's original derivation of the Feynman rules in quantum electrodynamics. However, in the non-relativistic case, treated in Chap. 4, we do not rephrase the formulæ in terms of graphs (although this is possible). In Sect. 4.2 we discuss an example for which the perturbation series can be summed up exactly. Section 4.3 deals with an application of path integrals to statistical mechanics, in particular with the partition function and the so-called effective potentials. In Sect. 4.4 we discuss the semiclassical expansion of the path integral about the harmonic approximation.

In Chap. 5 we give a short introduction to the *semiclassical theory* and its recent applications in the field of quantum chaos. The basic formula is Gutzwiller's expression for the Feynman kernel which is derived in Sect. 5.2. This completes the work started by Feynman, Morette, Van Hove, Pauli and Choquard, as described in the foregoing introduction. In Sect. 5.3 we derive the corresponding semiclassical formula for the Green function and, finally, Sect. 5.4 contains a discussion of the *Gutzwiller trace formula* which is the basic relation in the theory of quantum chaos.

Our final Chap. 6, which makes up the largest part of this *Handbook*, contains a rather complete *table of path integrals* in non-relativistic quantum mechanics, including supersymmetric quantum mechanics, and statistical mechanics. The path integrals in this table are classified according to our *basic path integrals* introduced in Chap. 3.[34] A comparison of the table with the known exact solutions of the Schrödinger equation shows that it is possible nowadays, with the modern techniques described in Chap. 2, to solve all path integrals for which the Schrödinger equation can be solved. To each path integral listed in the *table* we attach a comprehensive list of *references* which provides for the reader easy access to the original literature and thus offers the possibility of having a closer look at the derivation of the various path integrals and their applications in different fields.

For the sake of completeness we include some references corresponding to relativistic path integral solutions, i.e., for the Klein–Gordon [70, 82, 117, 118, 212, 226, 324, 331, 394, 510, 613, 653, 654, 810, 828, 912] and the Dirac equation [17, 24, 55, 98, 117, 118, 203, 212, 226, 340, 243, 372, 370, 393, 508,

[34] A brief outline of our classification of path integrals was presented in our previous papers [469, 470].

536–538, 718, 731, 738, 754, 769, 789, 796, 803, 806, 879], which are, however, incomplete; we do not dwell on the mathematical definitions, problems and ambiguities of these path integral representations. For p-adic path integrals, see e.g. [154, 693, 759, 846, 907, 944].

For the interested reader who wishes to study certain fields or applications in more detail, we give the following list of *textbooks* on Feynman path integrals:

Feynman and Hibbs *Quantum Mechanics and Path Integrals* [340]
Feynman *Statistical Mechanics* [334]
Schulman *Techniques and Applications of Path Integration* [828]
Simon *Functional Integration and Quantum Physics* [854]
Glimm and Jaffe *Quantum Physics: A Functional Point of View* [397]
Albeverio and Høegh-Krohn *Mathematical Theory of Feynman Path Integrals* [18]
Antoine and Tirapegui *Functional Integration: Theory and Applications* [27]
Dittrich and Reuter *Classical and Quantum Dynamics* [257]
Exner *Open Quantum Systems and Feynman Integrals* [306]
Inomata, Kuratsuji and Gerry *Path Integrals and Coherent States of* SU(2) *and* SU(1, 1) [528]
Junker *Supersymmetric Methods in Quantum and Statistical Physics* [550]
Kac, Uhlenbeck, Hibbs and van der Pol *Probability and Related Topics in Physical Sciences* [557]
Khandekar, Lawande and Bhagwat *Path-Integral Methods and Their Applications* [587]
Kleinert *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics* [613]
Langouche, Roekaerts and Tirapegui *Functional Integration and Semiclassical Expansions* [637]
Roepstorff *Path Integral Approach to Quantum Physics* [801]
Smolyanov and Shavgulidze *Continual Integrals* [857]
Weiss *Quantum Dissipative Systems* [915]
Wiegel *Introduction to Path-Integral Methods in Physics and Polymer Science* [927].

In addition to these textbooks the reader may consult the Special Issue on Functional Integration in *Journal of Mathematical Physics* edited by DeWitt–Morette [240], and the following *conference proceedings*:

Arthurs *Functional Integration and Its Applications* [35]
Papadopoulos and Devreese *Path Integrals and Their Applications in Quantum, Statistical, and Solid State Physics* [755]
Swanson *Path Integrals and Quantum Processes* [884]

The proceedings of the conference series *Path Integrals from meV to MeV* beginning in 1985 [484, 662, 819, 153, 414, 817, 940].

Although, in this book, we shall not discuss the application of path integrals to *quantum field theory*, we would like to give the following list of textbooks in which these matters are discussed:

Becher, Böhm and Joos *Gauge Theories of Strong and Electroweak Interactions* [66]
Bogoliubov and Shirkov *Introduction to the Theory of Quantized Fields* [107]
Creutz *Quarks, Gluons and Lattices* [210]
Das *Field Theory. A Path Integral Approach* [217]
Faddeev *Introduction to Functional Methods* [311]
Faddeev and Slavnov *Gauge Fields: Introduction to Quantum Theory* [313]
Feynman *Quantum Electrodynamics* [335]
Itzykson and Drouffe *Statistical Field Theory* [533]
Itzykson and Zuber *Quantum Field Theory* [534]
Kugo *Eichtheorie* [626]
Lee *Particle Physics and Introduction to Field Theory* [645]
Montvay and Münster *Quantum Fields on a Lattice* [707]
Popov *Functional Integrals in Quantum Field Theory and Statistical Physics* [782]
Ramond *Field Theory* [790]
Rebbi *Lattice Gauge Theories and Monte Carlo Simulations* [793]
Rivers *Path Integral Methods in Quantum Field Theory* [799]
Roepstorff *Path Integral Approach to Quantum Physics* [801]
Rothe *Lattice Gauge Theories* [808]
Swanson *Path Integrals and Quantum Processes* [884]
Weinberg *The Quantum Theory of Fields* [916].

Several of Feynman's original papers [326, 328–330] as well as Dirac's paper [254], and some other seminal papers on quantum electrodynamics are reprinted in *Quantum Electrodynamics* [840] edited by Schwinger.

2 General Theory

2.1 The Feynman Kernel and the Green Function

Let us start with the simplest case, i.e., with the one-dimensional motion of a particle of mass m under the influence of the time-independent force $F(x) = -\mathrm{d}V(x)/\mathrm{d}x$, where $V(x)$ denotes the potential, and $x = x(t) \in \mathbb{R}$ the classical trajectory as a function of time $t \in \mathbb{R}$. The classical dynamics can be completely formulated in terms of the *classical Lagrangian*

$$\mathcal{L} = \mathcal{L}(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x) \ , \tag{2.1.1}$$

or, equivalently, in terms of the *classical Hamiltonian*

$$H = H(p, x) := p\dot{x} - \mathcal{L} = \frac{1}{2m}p^2 + V(x) \ . \tag{2.1.2}$$

Here $\dot{x} = \mathrm{d}x(t)/\mathrm{d}t$ is the velocity, and $p := \partial\mathcal{L}/\partial\dot{x} = m\dot{x}$ the generalized momentum conjugate to x.

The standard formulation of quantum mechanics starts from the Hamiltonian (2.1.2). Working in the Schrödinger picture, the canonical variables (x, p) are replaced by the time-independent Hermitian operators $(\underline{\mathrm{x}}, \underline{\mathrm{p}})$ which act on time-dependent state vectors $|\Psi(t)\rangle \in \mathcal{H}$ of a separable Hilbert space $\mathcal{H}$. The algebra of the operators is fixed by the *Heisenberg commutation relation*

$$[\underline{\mathrm{x}}, \underline{\mathrm{p}}] := \underline{\mathrm{x}}\underline{\mathrm{p}} - \underline{\mathrm{p}}\underline{\mathrm{x}} = \mathrm{i}\,\hbar \ , \tag{2.1.3}$$

where $\hbar$ denotes Planck's constant divided by 2π. At a given time t, the physical state of the quantum mechanical system is completely described by the vector $|\Psi(t)\rangle \in \mathcal{H}$. Replacing the canonical variables in the classical Hamiltonian (2.1.2) by the corresponding operators leads to the well-defined *quantum Hamiltonian (operator)*

$$\underline{\mathrm{H}} := H(\underline{\mathrm{p}}, \underline{\mathrm{x}}) = \frac{1}{2m}\underline{\mathrm{p}}^2 + V(\underline{\mathrm{x}}) \ . \tag{2.1.4}$$

(In the general case one encounters operator-ordering problems which will be discussed in subsequent sections.) The quantum mechanical *time-evolution* is governed by the *Schrödinger equation*

$$\mathrm{i}\,\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \underline{\mathrm{H}}|\Psi(t)\rangle \ . \tag{2.1.5}$$

Knowing the state $|\Psi(t')\rangle \in \mathcal{H}$ at the initial time t', the problem of quantum mechanics consists in computing the state of the system at an arbitrary final time $t'' > t'$. The general solution of (2.1.5) can be written as

$$|\Psi(t'')\rangle = \underset{\sim}{\mathrm{U}}(t'',t')|\Psi(t')\rangle \ , \tag{2.1.6}$$

where $\underset{\sim}{\mathrm{U}}$ denotes the unitary *time-evolution operator* satisfying the operator equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t''}\underset{\sim}{\mathrm{U}}(t'',t') = \underset{\sim}{\mathrm{H}}\underset{\sim}{\mathrm{U}}(t'',t') \tag{2.1.7}$$

with the initial condition $\underset{\sim}{\mathrm{U}}(t',t') = \mathbb{1}$. For the time-independent Hamiltonian (2.1.4) one immediately obtains the explicit solution ($T := t'' - t'$)

$$\underset{\sim}{\mathrm{U}}(t'',t') = \exp\left(- \frac{\mathrm{i}}{\hbar}\underset{\sim}{\mathrm{H}}\,T\right) \tag{2.1.8}$$

which fulfils the *composition law*

$$\underset{\sim}{\mathrm{U}}(t'',t') = \underset{\sim}{\mathrm{U}}(t'',t)\underset{\sim}{\mathrm{U}}(t,t') \tag{2.1.9}$$

for arbitrary times t', t, t''.

In almost all practical calculations one does not work in the abstract Hilbert space, but rather in the so-called *x-representation*, respectively *coordinate space representation*. Consider the eigenvectors $|x\rangle$ of the position operator $\underset{\sim}{\mathrm{x}}$ satisfying

$$\underset{\sim}{\mathrm{x}}|x\rangle = x|x\rangle \tag{2.1.10}$$

with the continuous spectrum $x \in \mathbb{R}$. (Here we restrict ourselves to systems where the one-dimensional motion of the particle takes place on the whole real line without additional topological constraints. Systems where the motion is confined to smaller regions, e.g., the half-space $x \geq 0$, will be discussed in later sections.) Then we have the (continuum-) *normalization*

$$\langle x'|x\rangle = \delta(x' - x) \tag{2.1.11}$$

and the *completeness relation*

$$\int_{\mathbb{R}} \mathrm{d}x\, |x\rangle\langle x| = \mathbb{1} \ . \tag{2.1.12}$$

$\delta(x' - x)$ denotes the Dirac delta-function. Using (2.1.12) we get the following *transformation formula* from the abstract Hilbert space $\mathcal{H}$ to the *x-representation*

$$|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x\, |x\rangle\langle x|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x\, \Psi(x,t)|x\rangle \ , \tag{2.1.13}$$

where $\Psi(x,t)$ denotes the (complex-valued) *Schrödinger wave function* corresponding to the state vector $|\Psi(t)\rangle$ defined by

$$\Psi(x,t) := \langle x|\Psi(t)\rangle = \langle \Psi(t)|x\rangle^* \ . \tag{2.1.14}$$

With the help of (2.1.11) one immediately derives from (2.1.13) the *normalization*

$$\langle \Psi(t)|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}x\, |\Psi(x,t)|^2 = 1 \tag{2.1.15}$$

which shows that $\Psi(x,t) \in \mathcal{L}^2(\mathbb{R})$. In the x-representation there exists an explicit realization of the Hermitian operators $(\underline{x}, \underline{p})$ satisfying the commutation relation (2.1.3): $\underline{x}$ acts on the wave function Ψ as a multiplication operator, while $\underline{p}$ acts as the differential operator $-\mathrm{i}\hbar\partial/\partial x$. Then the *Schrödinger equation* (2.1.5) takes the standard form

$$\mathrm{i}\hbar\frac{\partial \Psi(x,t)}{\partial t} = \underline{H}_x \Psi(x,t) \tag{2.1.16}$$

with the differential operator (Schrödinger operator) $\underline{H}_x$ defined by

$$\underline{H}_x = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \ . \tag{2.1.17}$$

Using the definition (2.1.14) and the relation (2.1.12), one immediately derives the *time-evolution equation* for the Schrödinger wave function $(t'' > t')$

$$\Psi(x'',t'') = \int_{\mathbb{R}} \mathrm{d}x'\, K(x'',x';t'',t')\Psi(x',t') \tag{2.1.18}$$

with the (retarded) *Feynman kernel* $(T = t'' - t')$

$$\begin{aligned} K(x'',x';t'',t') &:= \langle x''|\underline{U}(t'',t')|x'\rangle \Theta(t''-t') \\ &= \left\langle x'' \middle| \exp\left(-\frac{\mathrm{i}}{\hbar}\underline{H}_x T\right) \middle| x' \right\rangle \Theta(T) \ . \end{aligned} \tag{2.1.19}$$

Here $\Theta(T)$ denotes the Heaviside step function defined by[1]

$$\Theta(T) := \begin{cases} 1 & T \geq 0 \\ 0 & T < 0 \ . \end{cases} \tag{2.1.20}$$

Sometimes the kernel K is called "the propagator" since it is nothing else than the (complex) transition-probability amplitude which "propagates" the initial wave function (probability amplitude) at time t' to the final wave function at the final time t'' as can be seen from (2.1.18). Note, however, that K is not the same as the so-called "Feynman propagator" (see Sect. 2.5.3, and, e.g. [534, 916]) which plays an important rôle in quantum field theory.

In the special case of time-independent potentials, as considered in this section, the kernel K does not depend on the time variables t', t'' separately,

[1] Often $\Theta(T)$ is omitted in $\langle x''|\underline{U}|x'\rangle$, and we tacitly assume $T \geq 0$.

but is homogeneous in time, see (2.1.19). We therefore usually write in the time-independent case

$$K(x'',x';T) := K(x'',x';t'',t') = K(x'',x';T,0) \ . \tag{2.1.21}$$

Using $\Theta'(T) = \delta(T)$ it is easy to see that K satisfies the inhomogeneous Schrödinger equation

$$\left(\mathrm{i}\,\hbar\frac{\partial}{\partial t''} - \underset{\sim}{\mathrm{H}}_{x''}\right)K(x'',x';T) = \mathrm{i}\,\hbar\delta(x''-x')\delta(T) \tag{2.1.22}$$

with the initial condition (see (2.1.11) and (2.1.19))

$$\lim_{T\to 0+} K(x'',x';T) = \delta(x''-x') \ . \tag{2.1.23}$$

The composition law (2.1.9) implies for the kernel K $(t' < t_1 < t'')$

$$K(x'',x';t'',t') = \int_{\mathbb{R}} \mathrm{d}x_1\, K(x'',x_1;t'',t_1)K(x_1,x';t_1,t') \tag{2.1.24}$$

which is an important *law for the composition of amplitudes* for events which occur successively in time [326, 340].

There are many quantum mechanical systems for which the time-dependent kernel K cannot be given in explicit form, but instead its Fourier transform with respect to time can be explicitly written down. We are thus led to define the energy-dependent (outgoing) *Green function*

$$G(x'',x';E) := \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}(E+\mathrm{i}\,\epsilon)T/\hbar}\,K(x'',x';T) \ , \tag{2.1.25}$$

where a small positive imaginary part $(\epsilon > 0)$ has been added to the energy E.[2] From (2.1.19) we obtain

$$G(x'',x';E) = \left\langle x''\left|\frac{1}{\underset{\sim}{\mathrm{H}}_x - E - \mathrm{i}\,\epsilon}\right|x'\right\rangle \ . \tag{2.1.26}$$

In mathematics, the operator $(\underset{\sim}{\mathrm{A}}-z)^{-1}$, $z \in \mathbb{C}\backslash\mathrm{spec}(\underset{\sim}{\mathrm{A}})$, is called the resolvent of a given operator $\underset{\sim}{\mathrm{A}}$, and thus G is the *resolvent kernel* of the Hamiltonian $\underset{\sim}{\mathrm{H}}_x$ in the coordinate representation. Knowing the Green function, we can recover the Feynman kernel via the inverse Fourier transform

$$K(x'',x';T) = \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}\,ET/\hbar}\,G(x'',x';E) \ . \tag{2.1.27}$$

The Green function satisfies the inhomogeneous Schrödinger equation

$$(\underset{\sim}{\mathrm{H}}_{x''} - E)G(x'',x';E) = \delta(x''-x') \ . \tag{2.1.28}$$

[2] Usually we shall not explicitly write the $\mathrm{i}\,\epsilon$, but tacitly assume that the various expressions are regularized according to this rule.

In the generic case, the kernels K and G will decompose into two terms corresponding to the contributions from the bound states (discrete spectrum) and the scattering states (continuous spectrum) of a given quantum system.

Let us briefly discuss the simplest case for which both kernels can be explicitly given, i.e., the free particle ($V(x) \equiv 0$) which is described by the *free Hamiltonian*

$$\underset{\sim}{H}_0 := \frac{1}{2m}\underset{\sim}{p}^2 \ . \tag{2.1.29}$$

Since $\underset{\sim}{H}_0$ depends on $\underset{\sim}{p}$ only, it is natural to go to the *p-representation* and consider the eigenvectors $|p\rangle$ of the momentum operator $\underset{\sim}{p}$

$$\underset{\sim}{p}|p\rangle = p|p\rangle \ , \quad \langle p'|p\rangle = \delta(p'-p) \ , \quad \int_{\mathbb{R}} \mathrm{d}p\, |p\rangle\langle p| = \mathbb{1} \tag{2.1.30}$$

with $p \in \mathbb{R}$. Then the most general solution of the free Schrödinger equation reads ("wave packet")

$$|\Psi(t)\rangle = \int_{\mathbb{R}} \mathrm{d}p\, \Phi(p)\, \mathrm{e}^{-\mathrm{i}E(p)t/\hbar}\, |p\rangle \tag{2.1.31}$$

with $\Phi(p) \in \mathcal{L}^2(\mathbb{R})$ and

$$\underset{\sim}{H}_0|p\rangle = E(p)|p\rangle \ , \qquad E(p) = \frac{p^2}{2m} \ . \tag{2.1.32}$$

Obviously, in this case the energy spectrum is *continuous*, $E(p) \geq 0$, and the corresponding wave functions are *plane waves* (free scattering solutions)

$$\Psi_p(x) := \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, \mathrm{e}^{\mathrm{i}px/\hbar} \tag{2.1.33}$$

satisfying the *orthogonality relation*

$$\int_{\mathbb{R}} \mathrm{d}x\, \Psi^*_{p'}(x)\Psi_p(x) = \delta(p'-p) \ . \tag{2.1.34}$$

(Notice that while the plane wave (2.1.33) is *not* normalizable, the wave packet (2.1.31) is square integrable since $\Phi(p) \in \mathcal{L}^2(\mathbb{R})$, which implies for the corresponding wave function $\Psi(x,t) = \langle x|\Psi(t)\rangle \in \mathcal{L}^2(\mathbb{R})$.)

One then obtains for the *free Feynman kernel*

$$\begin{aligned} K_0(x'',x';T) &:= \left\langle x'' \middle| \exp\left(-\frac{\mathrm{i}}{\hbar}\underset{\sim}{H}_0 T\right) \middle| x' \right\rangle \Theta(T) \\ &= \int_{\mathbb{R}} \mathrm{d}p'' \int_{\mathbb{R}} \mathrm{d}p'\, \langle x''|p''\rangle \left\langle p'' \middle| \exp\left(-\frac{\mathrm{i}}{\hbar}\underset{\sim}{H}_0 T\right) \middle| p' \right\rangle \langle p'|x'\rangle \Theta(T) \\ &= \int_{\mathbb{R}} \mathrm{d}p\, \Psi_p(x'')\Psi^*_p(x')\, \mathrm{e}^{-\mathrm{i}E(p)T/\hbar}\, \Theta(T) \ , \end{aligned} \tag{2.1.35}$$

which has the typical form of a *spectral representation* corresponding to a continuous spectrum. The last expression can be rewritten as

$$K_0(x'',x';T) = \frac{\Theta(T)}{2\pi\hbar}\mathcal{F}\left(\frac{\mathrm{i}\,T}{2m\hbar}, \frac{\mathrm{i}}{\hbar}(x''-x')\right) , \tag{2.1.36}$$

where $\mathcal{F}(z,w)$ ($z,w \in \mathbb{C}$, $z \neq 0$) denotes the *Gaussian integral*

$$\begin{aligned}\mathcal{F}(z,w) &:= \int_{\mathbb{R}} \mathrm{d}x\ \mathrm{e}^{-zx^2+wx} \\ &= \sqrt{\frac{\pi}{z}}\exp\left(\frac{w^2}{4z}\right) =: \sqrt{\frac{\pi}{|z|}}\exp\left(-\frac{\mathrm{i}}{2}\arg(z)+\frac{w^2}{4z}\right) .\end{aligned} \tag{2.1.37}$$

Using (2.1.37) in (2.1.36) gives the explicit expression for the *free Feynman kernel*

$$K_0(x'',x';T) = \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}}\exp\left[\mathrm{i}\,\frac{m}{2\hbar T}(x''-x')^2\right]\Theta(T) . \tag{2.1.38}$$

For the *free Green function* one obtains (see (2.1.25))

$$G_0(x'',x';E) = \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\exp\left(-\frac{1}{\hbar}\sqrt{-2mE}\,|x''-x'|\right) , \tag{2.1.39}$$

which exhibits the correct branch cut on the positive energy axis. (Remember the $\mathrm{i}\,\epsilon$-rule discussed before (2.1.26).) In the physical region ($E > 0$) one can rewrite (2.1.39) as

$$G_0(x'',x';E) = \frac{\mathrm{i}\,m}{\hbar^2 k}\,\mathrm{e}^{\mathrm{i}\,k|x''-x'|} \tag{2.1.40}$$

in terms of the wavenumber $k := \sqrt{2mE}\,/\hbar > 0$.

Now let us consider a Hamiltonian of the form (2.1.4) but having only a *discrete energy spectrum* $E_0 \leq E_1 \leq E_2 \leq \ldots$:

$$\underset{\sim}{\mathrm{H}}|n\rangle = E_n|n\rangle , \qquad n = 0,1,2,\ldots \tag{2.1.41}$$

$$\langle m|n\rangle = \delta_{mn} , \qquad \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1} . \tag{2.1.42}$$

Then the most general solution of the Schrödinger equation (2.1.5) reads

$$|\Psi(t'')\rangle = \sum_{n=0}^{\infty} a_n\,\mathrm{e}^{-\mathrm{i}\,E_nT/\hbar}\,|n\rangle , \tag{2.1.43}$$

where the complex coefficients a_n are uniquely defined in terms of the initial state, $a_n = \langle n|\Psi(t')\rangle$, and satisfy the normalization condition

$$\langle\Psi(t')|\Psi(t')\rangle = \sum_{n=0}^{\infty}|a_n|^2 = 1 . \tag{2.1.44}$$

The Schrödinger wave function corresponding to a given bound state with energy E_n is defined in analogy to (2.1.14)

$$\Psi_n(x) := \langle x|n\rangle \ , \tag{2.1.45}$$

and satisfies the *time-independent Schrödinger equation*

$$\underline{\mathrm{H}}_x\Psi_n(x) = E_n\Psi_n(x) \ , \tag{2.1.46}$$

the *orthonormality relation*

$$\int_{\mathbb{R}} \mathrm{d}x\, \Psi_m^*(x)\Psi_n(x) = \delta_{mn} \ , \tag{2.1.47}$$

and the *completeness relation*

$$\sum_{n=0}^{\infty} \Psi_n(x'')\Psi_n^*(x') = \delta(x''-x') \ . \tag{2.1.48}$$

For the Feynman kernel and the Green function, respectively, one obtains the *spectral representations*

$$K(x'',x';T) = \sum_{n=0}^{\infty} \Psi_n(x'')\Psi_n^*(x')\,\mathrm{e}^{-\mathrm{i}E_nT/\hbar}\,\Theta(T) \tag{2.1.49}$$

$$G(x'',x';E) = \sum_{n=0}^{\infty} \frac{\Psi_n(x'')\Psi_n^*(x')}{E_n - E - \mathrm{i}\epsilon} \ . \tag{2.1.50}$$

For a generic system having both bound and scattering states, the kernels K and G, respectively, consist of two terms, one of the form (2.1.49) and (2.1.50), respectively, and one of the form:

$$K(x'',x';T) = \int_{\mathbb{R}} \mathrm{d}p\, \Psi_p(x'')\Psi_p^*(x')\,\mathrm{e}^{-\mathrm{i}E(p)T/\hbar}\,\Theta(T) \ , \tag{2.1.51}$$

$$G(x'',x';E) = \int_{\mathbb{R}} \mathrm{d}p\, \frac{\Psi_p(x'')\Psi_p^*(x')}{E(p) - E - \mathrm{i}\epsilon} \ , \tag{2.1.52}$$

respectively. Here $\Psi_p(x)$ and $E(p)$ are the corresponding solutions of the Schrödinger equation for the continuous part of the spectrum and are in general different from the free solutions. The *orthonormality* and *completeness relations* for the continuous spectrum are then given by

$$\int_{\mathbb{R}} \mathrm{d}x\, \Psi_p^*(x)\Psi_{p'}(x) = \delta(p-p') \ , \tag{2.1.53}$$

$$\int_{\mathbb{R}} \mathrm{d}p\, \Psi_p(x'')\Psi_p^*(x') = \delta(x''-x') \ . \tag{2.1.54}$$

Finally notice that all formulæ of this section can easily be generalized to quantum systems having D degrees of freedom if we interpret $(\mathbf{x}, \mathbf{p})$ and the Hermitian operators $(\underline{\mathbf{x}}, \underline{\mathbf{p}})$ as D-dimensional vectors in Cartesian coordinates, i.e., $\mathbf{x} = (x^1, x^2, \ldots, x^D) \in \mathbb{R}^D$, etc. The *commutation relation* (2.1.3) has to be replaced by

$$[\underline{x}^k, \underline{p}_l] = \mathrm{i}\,\hbar\delta_l^k \ , \qquad [\underline{x}^k, \underline{x}^l] = 0 = [\underline{p}_k, \underline{p}_l] \ , \tag{2.1.55}$$

and the *Schrödinger operator* (2.1.17) generalizes to

$$\underline{H}_x = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{x}) \ , \tag{2.1.56}$$

where $\Delta := \partial_1^2 + \partial_2^2 + \ldots + \partial_D^2$ denotes the *Laplacian* in $\mathbb{R}^D$ with $\partial_k := \partial/\partial x^k$. The *time-evolution equation* (2.1.18) now reads

$$\Psi(\mathbf{x}'', t'') = \int_{\mathbb{R}^D} d\mathbf{x}'\, K(\mathbf{x}'', \mathbf{x}'; t'', t')\Psi(\mathbf{x}', t') \ . \tag{2.1.57}$$

In (2.1.11,2.1.22, 2.1.23, 2.1.28) and (2.1.48) the one-dimensional delta-function has to be replaced by $\delta^D(\mathbf{x}'' - \mathbf{x}') = \delta(x''^1 - x'^1)\ldots\delta(x''^D - x'^D)$.

It is easy to see that the *D-dimensional free Feynman kernel* is given by a product of the one-dimensional kernel (2.1.38) evaluated at the D Cartesian coordinates, i.e.,

$$K_0(\mathbf{x}'', \mathbf{x}'; T) = \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{D/2} \exp\left(\mathrm{i}\frac{m}{2\hbar T}|\mathbf{x}'' - \mathbf{x}'|^2\right)\Theta(T) \tag{2.1.58}$$

with $|\mathbf{y} - \mathbf{x}|^2 = (y^1 - x^1)^2 + \ldots + (y^D - x^D)^2$. The *D-dimensional free Green function* can be found in Sect. 6.2.1.2. Notice that the free kernel (2.1.58) can be rewritten as

$$K_0(\mathbf{x}'', \mathbf{x}'; T) = \frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{D/2}}\sqrt{\det\left(-\frac{\partial^2 R_{0,\mathrm{Cl}}}{\partial x''^a \partial x'^b}\right)} \exp\left(\frac{\mathrm{i}}{\hbar}R_{0,\mathrm{Cl}}\right) \ , \tag{2.1.59}$$

where $R_{0,\mathrm{Cl}}$ denotes the *classical action* evaluated along the actual path of the system (*Hamilton's principal function* for the path)

$$\begin{aligned} R_{0,\mathrm{Cl}} = R_{0,\mathrm{Cl}}(\mathbf{x}'', \mathbf{x}'; T) &:= \int_{t'}^{t''} dt\, \mathcal{L}_0(\mathbf{x}_{\mathrm{Cl}}, \dot{\mathbf{x}}_{\mathrm{Cl}}) \\ &= \frac{m}{2T}|\mathbf{x}'' - \mathbf{x}'|^2 \end{aligned} \tag{2.1.60}$$

i.e., the action of the free particle evaluated along the *classical path* $\mathbf{x}_{\mathrm{Cl}}(t)$ of a free particle starting at $\mathbf{x}'$ at time $t = t'$ and arriving at $\mathbf{x}''$ at time $t = t''$, i.e., $\mathbf{x}_{\mathrm{Cl}}(t') = \mathbf{x}'$ and $\mathbf{x}_{\mathrm{Cl}}(t'') = \mathbf{x}''$. (Notice that by integrating the Lagrangian along the actual path, the action becomes just a function of the end points.) The determinant in (2.1.59) is a special case of the so-called *Morette–Van*

Hove determinant [193, 710, 765, 905], see Chap. 1 and Sect. 5.2. For the free particle, the $D \times D$ matrix entering (2.1.59) is diagonal and has the simple form $(-\partial^2 R_{0,\text{Cl}}/\partial x''^a \partial x'^b) = (m/T)\delta^{ab}$, and $\det((m/T)\delta^{ab}) = (m/T)^D$.

2.2 The Path Integral in Cartesian Coordinates

The Feynman path integral can be viewed as a generalized integral representation for the time-evolution kernel, i.e., for the Feynman kernel $K(x'', x'; t'', t')$ defined in (2.1.19). To derive the path integral for a one-dimensional system whose classical dynamics is governed by the Lagrangian (2.1.1), we start from the composition law (2.1.24) which can be generalized to ($N \geq 2$)

$$\begin{aligned} K(x'', x'; t'', t') = &\int_{\mathbb{R}} dx_1 \int_{\mathbb{R}} dx_2 \dots \int_{\mathbb{R}} dx_{N-1} \\ &\times K(x'', x_{N-1}; t'', t_{N-1}) \dots K(x_2, x_1; t_2, t_1) K(x_1, x'; t_1, t') \end{aligned} \tag{2.2.1}$$

with $t' < t_1 < t_2 < \ldots < t_{N-1} < t''$. Equation (2.2.1) is the law for the composition of amplitudes for events which occur successively in time. Following Feynman, we can interpret the transition $(x', t') \to (x_1, t_1) \to \ldots \to (x'', t'')$ as a (virtual or real) "path" $x(t)$ of our particle with $x(t') = x'$, $x(t_1) = x_1$, $\ldots, x(t'') = x''$. To simplify the discussion, let us consider an equidistant *time lattice* with *lattice constant* $\epsilon := (t'' - t')/N = T/N > 0$ and define $x_0 := x'$, $t_0 := t'$, $x_N := x''$, $t_N := t''$, i.e., $t_j = t' + \epsilon j$, $x(t_j) = x_j$, $j = 0, 1, \ldots, N$. Using the definition (2.1.21) we can rewrite (2.2.1) as

$$K(x'', x'; T) = \prod_{k=1}^{N-1} \int_{\mathbb{R}} dx_k \prod_{j=0}^{N-1} K(x_{j+1}, x_j; \epsilon) \ . \tag{2.2.2}$$

This relation does not seem to be very useful since the unknown kernel K occurs on both sides. The crucial point is to consider the limit of an infinitesimally fine lattice, i.e., $N \to \infty$, T fixed, which is equivalent to the limit $\epsilon \to 0$. In this limit the r.h.s. of (2.2.2) depends only on the *short-time kernel*

$$\begin{aligned} K(x_{j+1}, x_j; \epsilon) &= \left\langle x_{j+1} \middle| \exp\left(-\frac{\mathrm{i}}{\hbar} \epsilon \underset{\sim}{H} \right) \middle| x_j \right\rangle \\ &= \left\langle x_{j+1} \middle| \exp\left(-\frac{\mathrm{i}}{\hbar} \epsilon \underset{\sim}{H}_0 - \frac{\mathrm{i}}{\hbar} \epsilon V(\underset{\sim}{x}) \right) \middle| x_j \right\rangle \end{aligned} \tag{2.2.3}$$

which can be exactly calculated up to terms of $O(\epsilon^2)$. With the help of the *Zassenhaus formula* [717, 881, 932]

$$\begin{aligned} \exp\left[\epsilon(\underset{\sim}{A} + \underset{\sim}{B})\right] &= \exp\left(\epsilon \underset{\sim}{A}\right) \exp\left(\epsilon \underset{\sim}{B}\right) \\ &\times \exp\left(-\frac{\epsilon^2}{2} [\underset{\sim}{A}, \underset{\sim}{B}] \right) \exp\left\{ \epsilon^3 \left(\frac{1}{3} [\underset{\sim}{B}, [\underset{\sim}{A}, \underset{\sim}{B}]] + \frac{1}{6} [\underset{\sim}{A}, [\underset{\sim}{A}, \underset{\sim}{B}]] \right\} \ldots \right. \end{aligned} \tag{2.2.4}$$

one obtains

$$
\begin{aligned}
K(x_{j+1},x_j;\epsilon) &= \left\langle x_{j+1}\left|\exp\left(-\frac{\mathrm{i}}{\hbar}\epsilon \underset{\sim}{\mathrm{H}}_0-\frac{\mathrm{i}}{\hbar}\epsilon V(\underset{\sim}{\mathrm{x}})\right)\right|x_j\right\rangle \\
&\doteq \left\langle x_{j+1}\left|\exp\left(-\frac{\mathrm{i}}{\hbar}\epsilon \underset{\sim}{\mathrm{H}}_0\right)\right|x_j\right\rangle \exp\left(-\frac{\mathrm{i}}{\hbar}\epsilon V(x_j)\right) \\
&= K_0(x_{j+1},x_j;\epsilon)\exp\left(-\frac{\mathrm{i}}{\hbar}\epsilon V(x_j)\right) \ .
\end{aligned}
\tag{2.2.5}
$$

(Following DeWitt [235], we use the symbol $\doteq$ to denote "equivalence as far as use in the path integral is concerned".) Using the result (2.1.38) for the free kernel, we arrive at the *lattice definition of the Feynman path integral*

$$
\begin{aligned}
K(x'',x';T) &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_{\mathbb{R}}\mathrm{d}x_k \\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2\epsilon}(x_{j+1}-x_j)^2-\epsilon V(x_j)\right)\right] \\
&\equiv \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right] \ .
\end{aligned}
\tag{2.2.6}
$$

Here several remarks are in order:

i) The integrand in (2.2.6) is, for finite N, a complex number $\Phi(x_1,\dots,x_{N-1})$ which is a function of the variables x_j defining the path $x(t)$. Here a path is first defined only by the positions x_j through which it goes at a sequence of equally spaced times, $t_{j+1}=t_j+\epsilon$. Eventually, we imagine that the spacing ϵ approaches zero ($N\to\infty$) so that Φ essentially depends on the entire path $x(t)$ rather than only on just the values of x_j at the particular times t_j, $x_j=x(t_j)$. Following Feynman [326] we might call $\Phi=\Phi[x(t)]$ the probability amplitude functional of paths $x(t)$.
ii) Since $\Phi(x_1,\dots,x_{N-1})$ has to be integrated over all values of x_j, $x_j\in\mathbb{R}$, the path integral (2.2.6) can be interpreted as a "sum over all paths" or a "sum over all histories"

$$
K(x'',x';T)=\sum_{\substack{\text{over all paths}\\ \text{from } x' \text{ to } x''}}\Phi[x(t)] \ . \tag{2.2.7}
$$

iii) The path integral (2.2.6) gives a prescription of how to compute the important quantity Φ for each path: "The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$); ... That is to say, the contribution $\Phi[x(t)]$ from a given path $x(t)$ is proportional to $\exp\left(\frac{\mathrm{i}}{\hbar}R[x(t)]\right)$, where the *action* R is the time integral of the classical Lagrangian taking along the path in question" [326]

$$R = R[x(t)] := \int_{t'}^{t''} \mathcal{L}(x(t), \dot{x}(t))\, \mathrm{d}t$$
$$= \int_{t'}^{t''} \left[\frac{m}{2}\dot{x}^2(t) - V(x(t))\right] \mathrm{d}t \ . \qquad (2.2.8)$$

Thus Feynman's formulation of quantum mechanics in terms of path integrals can be called a "Lagrangian formulation of quantum mechanics" in contrast to the standard formulation discussed in Sect. 2.1 which is a Hamiltonian formulation.

vi) The close analogy between the time-evolution kernel ("transformation function") and the quantity $\exp\left(\frac{\mathrm{i}}{\hbar}R\right)$ was first pointed out by Dirac [254, 256] who asked the question "of what corresponds in the quantum theory to the Lagrangian method of the classical theory" [254]. Dirac's remark played an important rôle in Feynman's development of the space-time view of quantum electrodynamics (see the remarks in [326, 336]).

v) In the above derivation of the path integral (2.2.6) it was necessary and sufficient to keep all terms including $O(\epsilon)$. This has to be kept in mind in all calculations, transformation of variables, etc., by using the lattice definition of the path integral. The relevance of this remark becomes clear if one notices that "The 'velocities' $(x_{j+1} - x_j)/\epsilon$ which are important are very high, being of order $(\hbar/m\epsilon)^{1/2}$ which diverges as $\epsilon \to 0$. The paths involved are, therefore, continuous but possess no derivative. They are of a type familiar from study of Brownian motion" [326]. The stochastic nature of the Feynman paths implies that the action integral (2.2.8) has to be properly treated as a stochastic integral which is automatically ensured by the lattice definition (2.2.6). This was already realized in 1942 by Feynman in his thesis where he wrote: "Although the average value of the displacement of a particle in the time $\mathrm{d}t$ is $v\mathrm{d}t$, where v is the mean velocity, the mean value of the square of this displacement is not of order $\mathrm{d}t^2$, but only of order $\mathrm{d}t$" [325].

vi) Obviously the "measure-term" $\mathcal{D}x(t)$ defined by (2.2.6) is not a measure in the usual mathematical sense since it is complex-valued due to the presence of the factor $\mathrm{i} = \sqrt{-1}$. Moreover, each term $A_N = (m/2\pi\mathrm{i}\epsilon\hbar)^{N/2}$, $B_N = \int\prod \mathrm{d}x_j$ and $C_N = \exp(\mathrm{i}m(x_j - x_{j-1})^2/2\epsilon\hbar)$ is not defined as $N \to \infty$, respectively $\epsilon \to 0$. Only to the combination $A_N B_N C_N$ can a meaningful definition of "measure" ("pseudomeasure", "Feynman measure") be given, e.g., Albeverio et al. [11, 13, 14, 16, 18–20], Exner [306], Kac [554], Morette–DeWitt [239, 710–712], Wiener [930]. A convenient way to obtain a mathematically rigorous path integral is to make an analytic continuation in time ("Wick rotation", "Euclidean time") using the fact that $\mathrm{e}^{-T\underline{\mathrm{H}}/\hbar}$ is obtained from $\mathrm{e}^{-\mathrm{i}T\underline{\mathrm{H}}/\hbar}$ by an analytic continuation which replaces T by $-\mathrm{i}T$. Using the substitutions $\epsilon \to -\mathrm{i}\epsilon, \mathrm{d}t \to -\mathrm{i}\mathrm{d}t$, $\dot{x}^2 = (\mathrm{d}x/\mathrm{d}t)^2 \to -\dot{x}^2$, one is led to the *Euclidean path integral* $(T > 0)$

$$
\begin{aligned}
K_E(x'',x';T) &:= K(x'',x';-\mathrm{i}T) = \left\langle x'' \middle| \exp\left[-\frac{T}{\hbar}\Big(\underline{\mathrm{H}}_0 + V(\underline{\mathrm{x}})\Big)\right] \middle| x' \right\rangle \\
&= \lim_{N\to\infty}\left(\frac{m}{2\pi\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_{\mathbb{R}} \mathrm{d}x_k \\
&\qquad \times \exp\left[-\frac{1}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2\epsilon}(x_{j+1}-x_j)^2 + \epsilon V(x_j)\right)\right] \\
&\equiv \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_E x(t) \exp\left[-\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + V(x)\right)\mathrm{d}t\right] \\
&=: \int \mathcal{D}W[x] \exp\left(-\frac{1}{\hbar}\int_{t'}^{t''} V(x(t))\,\mathrm{d}t\right) \ ,
\end{aligned}
\tag{2.2.9}
$$

where $\mathcal{D}W[x]$ denotes integration with respect to the conditional Wiener measure [397]. In the mathematical literature the representation (2.2.9) for the *Euclidean Feynman kernel* $K_E(T)$ is called the *Feynman-Kac formula* [554, 794, 854], and instead of path integration one speaks of *functional integration.*

In the path integral (2.2.9) the contribution from a given path $x(t)$ is proportional to $\exp(-R_E[x(t)]/\hbar)$ and thus positive definite, where R_E denotes the *Euclidean action*

$$
\begin{aligned}
R_E[x(t)] &:= \int_{t'}^{t''} \mathcal{L}_E\big(x(t),\dot{x}(t)\big)\mathrm{d}t \\
&:= \int_{t'}^{t''} \left[\frac{m}{2}\dot{x}^2(t) + V(x(t))\right]\mathrm{d}t \ .
\end{aligned}
\tag{2.2.10}
$$

A comparison of the Euclidean action (2.2.10) with the standard action (2.2.8) shows that the Euclidean path integral can be interpreted as describing a particle moving in a potential *minus* V, i.e., in the *inverted potential* $-V(x)$. This observation lies at the heart of the so-called *instanton approximation*, see e.g. [200, 638], which gives in the semiclassical limit ($\hbar \to 0$) the dominant non-perturbative contribution to the amplitude for transmission through a potential barrier (barrier penetration is not seen in any order of perturbation theory in $\hbar$).

vii) If the time T in the Euclidean path integral (2.2.9) is redefined to be $\beta\hbar$, one obtains the path integral formulation of the *density matrix* ρ *in statistical mechanics* [334, 340]

$$
\begin{aligned}
\rho(x'',x';\beta) &:= \langle x'' | \,\mathrm{e}^{-\beta\underline{\mathrm{H}}}\, | x' \rangle \\
&= \int\limits_{x(0)=x'}^{x(\beta\hbar)=x''} \mathcal{D}_E x(s) \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}\left(\frac{m}{2}\dot{x}^2(s) + V(x(s))\right)\mathrm{d}s\right] \ ,
\end{aligned}
\tag{2.2.11}
$$

with $\beta = (k_B \cdot \text{temperature})^{-1}$, and k_B Boltzmann's constant. From (2.2.11) one easily derives a very powerful representation for the *partition function* Z (or the free energy F) in statistical mechanics by taking the trace

$$
\begin{aligned}
Z &= \mathrm{e}^{-\beta F} = \operatorname{Tr} \mathrm{e}^{-\beta \underset{\sim}{H}} = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n} = \int_{\mathbb{R}} \mathrm{d}x \, \rho(x, x; \beta) \\
&= \int_{\mathbb{R}} \mathrm{d}x \int\limits_{x(0)=x}^{x(\beta\hbar)=x} \mathcal{D}_E x(s) \exp\left[-\frac{1}{\hbar} \int_0^{\beta\hbar} \left(\frac{m}{2} \dot{x}^2(s) + V(x(s)) \right) \mathrm{d}s \right] .
\end{aligned} \tag{2.2.12}
$$

Here we have assumed that $\underset{\sim}{H}$ has a purely discrete spectrum, see (2.1.41), otherwise the contribution of the continuous spectrum must be properly treated.

viii) The potential $V(x)$ appearing in the action may also be complex valued. The imaginary part of the potential can be understood as a source, respectively a sink, for particles [936]. A complex potential can also appear from a transformation of a time-independent Hamiltonian to a time-dependent one [440, 773], which has the consequence that the new Hamiltonian does not conserve the energy, which is exactly balanced by the imaginary part of the potential to guarantee energy conservation of the entire system. The corresponding term can also be interpreted as a "path-dependent measure" [773].

ix) The derivation of the path integral (2.2.6) can be put on a rigorous mathematical basis by starting not from the composition law (2.2.1) but instead from the definition (2.1.19)

$$
K(x'', x'; T) = \left\langle x'' \middle| \exp\left[-\frac{\mathrm{i}}{\hbar} T \left(\underset{\sim}{H}_0 + V(\underset{\sim}{x}) \right) \right] \middle| x' \right\rangle \Theta(T) \tag{2.2.13}
$$

and employing *Trotter's formula* [569, 854, 896]:
Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be self-adjoint operators on a separable Hilbert space so that $\underset{\sim}{A} + \underset{\sim}{B}$ is self-adjoint. Then

$$
\exp[\mathrm{i}t(\underset{\sim}{A} + \underset{\sim}{B})] = \operatorname*{s-lim}_{N\to\infty} \left[\exp(\mathrm{i}t\underset{\sim}{A}/N) \exp(\mathrm{i}t\underset{\sim}{B}/N) \right]^N . \tag{2.2.14}
$$

If furthermore, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are bounded from below, then $(t > 0)$

$$
\exp[-t(\underset{\sim}{A} + \underset{\sim}{B})] = \operatorname*{s-lim}_{N\to\infty} \left[\exp(-t\underset{\sim}{A}/N) \exp(-t\underset{\sim}{B}/N) \right]^N . \tag{2.2.15}
$$

x) If the Hamiltonian $\underset{\sim}{H}$ is time dependent, the solution (2.1.8) for the time-evolution operator has to be replaced by the *Feynman–Dyson formula*

$$
\underset{\sim}{U}(t'', t') = T \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \underset{\sim}{H}(t) \, \mathrm{d}t \right) \tag{2.2.16}
$$

$$:= \mathbb{1} - \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \mathrm{d}t_1 \underline{\mathrm{H}}(t_1) + \left(-\frac{\mathrm{i}}{\hbar}\right)^2 \int_{t'}^{t''} \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2\, \underline{\mathrm{H}}(t_1)\underline{\mathrm{H}}(t_2) + \dots \tag{2.2.17}$$

and the Feynman kernel is, in an obvious generalization of (2.1.19), defined by

$$K(x'', x'; t'', t') := \left\langle x'' \middle| T \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \underline{\mathrm{H}}_x(t)\,\mathrm{d}t\right) \middle| x' \right\rangle \Theta(t'' - t') \ . \tag{2.2.18}$$

(Here T denotes the time-ordered product.) In this case the lattice definition of the Feynman path integral reads

$$\begin{aligned} K(x'', x'; t'', t') &= \lim_{N\to\infty} \left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{N/2} \prod_{k=1}^{N-1} \int_{\mathbb{R}} \mathrm{d}x_k \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2\epsilon}(x_{j+1} - x_j)^2 - \epsilon V(x_j, t_j)\right)\right] \\ &\equiv \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x,t)\right)\mathrm{d}t\right] \ . \end{aligned} \tag{2.2.19}$$

xi) It is not difficult to see that the *D-dimensional path integral in Cartesian coordinates* ($\mathbf{x} \in \mathbb{R}^D$) is given by

$$\begin{aligned} K(\mathbf{x}'', \mathbf{x}'; t'', t') &= \lim_{N\to\infty} \left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{DN/2} \prod_{k=1}^{N-1} \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}_k \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2\epsilon}|\mathbf{x}_{j+1} - \mathbf{x}_j|^2 - \epsilon V(\mathbf{x}_j, t_j)\right)\right] \\ &\equiv \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x},t)\right)\mathrm{d}t\right] \ , \end{aligned} \tag{2.2.20}$$

with $\mathbf{x}_j := \mathbf{x}(t_j) = (x^1(t_j), x^2(t_j), \dots, x^D(t_j))$. Here $\mathcal{D}\mathbf{x}(t)$ has the (formal) meaning $\mathcal{D}\mathbf{x}(t) = \mathcal{D}x^1(t)\mathcal{D}x^2(t)\dots\mathcal{D}x^D(t)$, where $\mathcal{D}x^k(t)$ denotes the one-dimensional "differential" appearing in the one-dimensional path integral (2.2.6).

2.3 Gaussian Path Integrals and Zeta Function Regularization

Gaussian path integrals are functional generalizations of the *Gaussian integral* (2.1.37) ($z \in \mathbb{C}, z \neq 0$)

$$\mathcal{F}(z) := \mathcal{F}(z,0) = \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-zx^2} = \sqrt{\frac{\pi}{z}} \ , \tag{2.3.1}$$

and its generalization to D-dimensional Cartesian coordinates

$$\begin{aligned} \mathcal{F}_D(\mathbf{M}) &:= \int_{\mathbb{R}} \mathrm{d}x^1 \ldots \int_{\mathbb{R}} \mathrm{d}x^D \exp\left(- \sum_{i,j=1}^{D} x^i M_{ij} x^j \right) \\ &\equiv \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-\mathbf{x}^t \mathbf{M} \mathbf{x}} \ , \end{aligned} \tag{2.3.2}$$

where $\mathbf{M}$ is a real symmetric $D \times D$ matrix with matrix elements $\{M_{ij}\}$. $\mathbf{M}$ can be diagonalized, $\mathbf{M} = \mathbf{N}^t \mathbf{M}_D \mathbf{N}$, where $\mathbf{N}$ is an orthogonal matrix ($\mathbf{N}^t = \mathbf{N}^{-1}$, $\det \mathbf{N} = 1$) and $\mathbf{M}_D$ denotes a diagonal matrix with real eigenvalues $\lambda_1, \ldots, \lambda_D$. With $\mathbf{y} = \mathbf{N}\mathbf{x}$ one obtains

$$\begin{aligned} \mathcal{F}_D(\mathbf{M}) &= \int_{\mathbb{R}^D} \mathrm{d}\mathbf{y}\, \mathrm{e}^{-\mathbf{y}^t \mathbf{M}_D \mathbf{y}} = \prod_{k=1}^{D} \int_{\mathbb{R}} \mathrm{d}y_k\, \mathrm{e}^{-\lambda_k (y^k)^2} \\ &= \pi^{D/2} (\lambda_1 \times \cdots \times \lambda_D)^{-1/2} = \pi^{D/2} (\det \mathbf{M}_D)^{-1/2} \\ &= \pi^{D/2} (\det \mathbf{M})^{-1/2} \ . \end{aligned} \tag{2.3.3}$$

(Here we have assumed that all eigenvalues of $\mathbf{M}$ are non-zero.) We thus have for the *D-dimensional Gaussian integral*

$$\int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-\mathbf{x}^t \mathbf{M} \mathbf{x}} = \pi^{D/2} (\det \mathbf{M})^{-1/2} \ . \tag{2.3.4}$$

We note also the more general result

$$\int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-\mathbf{x}^t \mathbf{M} \mathbf{x} + \mathbf{u}^t \mathbf{x} + \mathbf{x}^t \mathbf{u}} = \pi^{D/2} (\det \mathbf{M})^{-1/2}\, \mathrm{e}^{\mathbf{u}^t \mathbf{M}^{-1} \mathbf{u}} \ . \tag{2.3.5}$$

The simplest *Gaussian path integral* occurs if one considers the *Feynman path integral* (2.2.6) *for a free particle*

$$\begin{aligned} K_0(x'', x'; T) &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left(\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \frac{m}{2} \dot{x}^2 \mathrm{d}t \right) \\ &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\, \mathrm{e}^{\mathrm{i}\, R_0[x]/\hbar} \end{aligned}$$

$$= \int_{q(t')=0}^{q(t'')=0} \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i}\, R_0[x_{\mathrm{Cl}}+q]/\hbar} \ , \tag{2.3.6}$$

where in the last step the paths $x(t), t' \le t \le t''$, $x(t') = x'$, $x(t'') = x''$, have been decomposed into the *classical path* $x_{\mathrm{Cl}}(t)$, $x_{\mathrm{Cl}}(t') = x'$, $x_{\mathrm{Cl}}(t'') = x''$, and a *quantum fluctuation* $q(t)$ satisfying at the end points $q(t') = q(t'') = 0$, i.e., $x(t) = x_{\mathrm{Cl}}(t) + q(t)$. Since $x_{\mathrm{Cl}}(t)$ satisfies the classical equation of motion with the prescribed boundary conditions, one obtains

$$R_0[x_{\mathrm{Cl}} + q] = R_0[x_{\mathrm{Cl}}] + R_0[q] = R_{0,\mathrm{Cl}} + R_0[q]$$

and thus

$$K_0(x'', x'; T) = F_0(T)\, \mathrm{e}^{\mathrm{i}\, R_{0,\mathrm{Cl}}/\hbar} \tag{2.3.7}$$

with

$$F_0(T) := \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i}\, R_0[q]/\hbar} = K_0(0,0;T) \ . \tag{2.3.8}$$

(Here we have set $t' = 0$ and $t'' = T$). It remains to compute the special *Gaussian path integral*

$$\begin{aligned} F_0(T) &= \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(t) \exp\left(\frac{\mathrm{i}}{\hbar} \int_0^T \frac{m}{2} \dot{q}^2 \mathrm{d}t \right) \\ &= \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(t) \exp\left[-\frac{1}{2} \left(\frac{m}{\mathrm{i}\,\hbar} \right) \int_0^T q(t) \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \right) q(t)\, \mathrm{d}t \right] \\ &= \sqrt{\frac{m}{\pi\, \mathrm{i}\, \hbar}} \left[\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \right) \right]^{-1/2} \end{aligned} \tag{2.3.9}$$

which can be considered as the infinite-dimensional (functional) generalization of the D-dimensional Gaussian integral (2.3.4). In the last step we have introduced the *functional determinant* of the operator $\underline{\mathrm{A}} := -\mathrm{d}^2/\mathrm{d}t^2$ which acts on square integrable functions $q(t)$ with *Dirichlet boundary conditions*

$$\underline{\mathrm{A}} q_n(t) = \lambda_n q_n(t) \ , \qquad 0 \le t \le T,\ q_n(0) = 0, q_n(T) = 0 \ . \tag{2.3.10}$$

To define in general the functional determinant of a positive elliptic operator $\underline{\mathrm{A}}$, we define the *MP zeta function associated to* $\underline{\mathrm{A}}$ (the zeta function of Minakshisundaram and Pleijel [778])

$$\zeta_{\underline{\mathrm{A}}}(s) := \mathrm{Tr}\, \underline{\mathrm{A}}^{-s} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \ , \tag{2.3.11}$$

where $s \in \mathbb{C}$ with $\Re s > \sigma_a$, and σ_a denotes the abscissa of absolute convergence of the Dirichlet series (2.3.11). It follows from general arguments [488, 792] that $\zeta_{\underset{\sim}{A}}(s)$ possesses a meromorphic analytic continuation into the whole complex s-plane with $s = 0$ being a regular point and, in particular, that its first derivative at $s = 0$, $\zeta'_{\underset{\sim}{A}}(0)$, is well defined. Then the *functional determinant of the operator* $\underset{\sim}{A}$ is defined by

$$\det \underset{\sim}{A} := \mathrm{e}^{-\zeta'_{\underset{\sim}{A}}(0)} \quad . \tag{2.3.12}$$

Coming back to the Gaussian path integral (2.3.9), we have to solve the Dirichlet eigenvalue problem (2.3.10) for $\underset{\sim}{A} := -\,\mathrm{d}^2/\mathrm{d}t^2$. The result is $\lambda_n = (\pi n/T)^2$ and $q_n(t) = c_n \sin(\pi n t/T)$, $n \in \mathbb{N}$, and thus the MP zeta function of $-\,\mathrm{d}^2/\mathrm{d}t^2$ reads ($\Re s > 1/2$)

$$\zeta_{-\,\mathrm{d}^2/\mathrm{d}t^2}(s) = \left(\frac{T}{\pi}\right)^{2s} \zeta(2s) \ , \tag{2.3.13}$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$, $\Re s > 1$, is the Riemann zeta function. With $\zeta(0) = -1/2$, $\zeta'(0) = -\log(2\pi)/2$ one obtains $\zeta'_{-\,\mathrm{d}^2/\mathrm{d}t^2}(0) = -\log(2T)$, and thus

$$\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) = 2T \ , \tag{2.3.14}$$

which leads with (2.3.9) to the final result

$$\int\limits_{q(0)=0}^{q(T)=0} \mathcal{D}q(t) \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T \frac{m}{2}\dot{q}^2 \mathrm{d}t\right) = \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}} \ . \tag{2.3.15}$$

This yields with (2.3.7) and (2.1.60) the correct expression (2.1.38) for the free Feynman kernel. In general we have the *Gaussian path integral*

$$\int\limits_{q(0)=0}^{q(T)=0} \mathcal{D}q(t) \exp\left[-\frac{1}{2}\left(\frac{m}{\mathrm{i}\,\hbar}\right)\int_0^T q(t)\,\underset{\sim}{A}\,q(t)\,\mathrm{d}t\right] = \sqrt{\frac{m}{\pi\,\mathrm{i}\,\hbar}}\,(\det \underset{\sim}{A})^{-1/2} \ . \tag{2.3.16}$$

2.4 Evaluation of Path Integrals by Fourier Series

A useful method to solve path integrals is to expand the paths $x(t)$ into a complete orthogonal system of functions $\{f_n(t)\}$

$$x(t) = \sum_{n=1}^{\infty} a_n f_n(t) \ , \qquad t' \le t \le t'' \ . \tag{2.4.1}$$

Instead of discretizing the time t, we approximate the paths by the first N terms of the series (2.4.1)

$$x^{(N)}(t) := \sum_{n=1}^{N} a_n f_n(t) \ , \tag{2.4.2}$$

and consider the approximate paths $x^{(N)}(t)$ as functions of the coefficients a_n. We thus have a transformation from the Feynman paths, defined by the positions $\{x_j\}$ through which they go at a sequence of equally spaced times, to the coefficients $\{a_n\}$. (Set $x_j = x^{(N)}(t_j)$; $j = 1, \ldots, N$.) The main step to be carried out is then to transform the Feynman "measure" $\mathcal{D}x(t)$ to a new "measure" $\mathcal{D}a$ which stands for the integration over the coefficients a_n. In the N^{th}-approximation one has $\mathcal{D}^{(N)}a \sim \prod_{n=1}^{N} \mathrm{d}a_n$.

As an example, we discuss the path integral for a time-independent Hamiltonian and choose $t' = 0, t'' = T$ and $x(0) = x(T) = 0$, i.e., we consider

$$F(T) := \int\limits_{x(0)=0}^{x(T)=0} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right] \ . \tag{2.4.3}$$

We have already seen in Sect. 2.3 that the path integral for the free particle can be reduced to a product of two functions, where one of these two functions depends upon the classical path and the remaining function depends on the time interval T only and is given by a path integral over closed orbits as in (2.4.3). As another example for such a factorization we will discuss below the harmonic oscillator.

Since all paths $x(t)$ in the path integral (2.4.3) go from $x(0) = 0$ to $x(T) = 0$, i.e, they are closed, it is convenient to choose basis functions $\{f_n(t)\}$, $0 \le t \le T$, which satisfy the same *Dirichlet boundary conditions*, i.e., $f_n(0) = f_n(T) = 0$ for all $n \in \mathbb{N}$. We are thus led [340] to study the finite sine-Fourier series

$$x^{(N)}(t) = \sum_{n=1}^{N} a_n \sin\left(\frac{n\pi t}{T}\right) \tag{2.4.4}$$

with a fundamental period of T. Then the transformation of the path integral (2.4.3) reads as follows

$$\begin{aligned}
&\int\limits_{x(0)=0}^{x(T)=0} \mathcal{D}x(t) \exp\left(\frac{\mathrm{i}}{\hbar}R[x(t)]\right) \\
&= \int \mathcal{D}a \exp\left\{\frac{\mathrm{i}}{\hbar}R\left[\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right)\right]\right\} \\
&:= \lim_{N\to\infty}\left[\left(\frac{\pi}{\sqrt{2}}\right)^N N!\left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{(N+1)/2}\right]\prod_{n=1}^{N}\int_{\mathbb{R}}\mathrm{d}a_n\, \mathrm{e}^{\mathrm{i}R[x^{(N)}(t)]/\hbar} \ .
\end{aligned} \tag{2.4.5}$$

It should be pointed out that the transformation of the path integral from the original Feynman paths to the above "Fourier paths", $x_j = x^{(N)}(t_j) \to a_n$, is not a simple coordinate transformation as suggested in [340]. Indeed, if the prefactor in (2.4.5) is calculated using the Jacobian $J^{(N)} := \det(\partial x^{(N)}(t_j)/\partial a_n) = [(N+1)/2]^{N/2}$, one does not obtain the correct expression.

As an application of the evaluation of path integrals by Fourier series, let us consider the *one-dimensional harmonic oscillator* given by the classical Lagrangian

$$\mathcal{L}_{\text{osc}}(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \ . \tag{2.4.6}$$

The corresponding Feynman path integral reads

$$K_{\text{osc}}(x'', x'; T) = \int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)\, \mathrm{e}^{\mathrm{i}\, R_{\text{osc}}[x(t)]/\hbar} \tag{2.4.7}$$

with the classical action given by

$$\begin{aligned} R_{\text{osc}}[x(t)] &= \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,\mathrm{d}t \\ &= \frac{m}{2}\int_0^T x(t)\underset{\sim}{A}_{\text{osc}} x(t)\,\mathrm{d}t \ . \end{aligned} \tag{2.4.8}$$

Here we have introduced the positive elliptic operator[3] (see Sect. 2.3)

$$\underset{\sim}{A}_{\text{osc}} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega^2 \ . \tag{2.4.9}$$

Expanding the Feynman paths $x(t)$ around the classical path $x_{\text{Cl}}(t)$, $x(t) = x_{\text{Cl}}(t) + q(t)$, with the end point conditions $x_{\text{Cl}}(0) = x'$, $x_{\text{Cl}}(T) = x''$, $q(0) = q(T) = 0$ (see also Sect. 2.3 and Sect. 3.2), we obtain the following factorization of the path integral (2.4.7)

$$K_{\text{osc}}(x'', x'; T) = \exp\left(\tfrac{\mathrm{i}}{\hbar} R_{\text{osc}}[x_{\text{Cl}}]\right) F_{\text{osc}}(T) \ . \tag{2.4.10}$$

Here the coordinate-independent "amplitude" $F_{\text{osc}}(T)$ is given by a simple path integral over all closed quantum fluctuations $q(t)$

$$F_{\text{osc}}(T) := K_{\text{osc}}(0, 0; T) = \int\limits_{q(0)=0}^{q(T)=0} \mathcal{D}q(t)\, \mathrm{e}^{\mathrm{i}\, R_{\text{osc}}[q(t)]/\hbar} \ . \tag{2.4.11}$$

With the Fourier paths (2.4.4) for the quantum fluctuations $q(t)$ we obtain for the action

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[3] $\underset{\sim}{A}_{\text{osc}}$ is positive, indeed, if $T < \pi/\omega$.

$$R_{\mathrm{osc}}[q^{(N)}(t)] = \frac{m}{2}\sum_{m,n=1}^{N} a_m a_n \int_0^T \left[\left(\frac{m\pi}{T}\right)\left(\frac{n\pi}{T}\right)\cos\left(\frac{m\pi t}{T}\right)\cos\left(\frac{n\pi t}{T}\right) - \omega^2 \sin\left(\frac{m\pi t}{T}\right)\sin\left(\frac{n\pi t}{T}\right)\right]\mathrm{d}t$$

$$= \frac{mT}{4}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2 - \omega^2\right]a_n^2 \ . \tag{2.4.12}$$

Inserting this expression for the action into the path integral (2.4.11), we get

$$F_{\mathrm{osc}}(T) = \lim_{N\to\infty}\left\{\left(\frac{\pi}{\sqrt{2}}\right)^N N!\left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{(N+1)/2} F_{\mathrm{osc}}^{(N)}(T)\right\} \tag{2.4.13}$$

with

$$F_{\mathrm{osc}}^{(N)}(T) := \prod_{n=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}a_n \,\exp\left\{-\frac{mT}{4\,\mathrm{i}\,\hbar}\sum_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2-\omega^2\right]a_n^2\right\}$$

$$= \pi^{N/2}\left(\frac{mT}{4\,\mathrm{i}\,\hbar}\right)^{-N/2}\left\{\prod_{n=1}^{N}\left[\left(\frac{n\pi}{T}\right)^2-\omega^2\right]\right\}^{-1/2} , \tag{2.4.14}$$

where in the last step we have used the Gaussian integral (2.3.1) (see also (2.3.4)). The last expression can be rewritten as follows

$$F_{\mathrm{osc}}^{(N)}(T) = \left(\frac{\pi}{\sqrt{2}}\right)^{-N}\frac{1}{N!}\left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{-N/2}\left\{\prod_{n=1}^{N}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]\right\}^{-1/2} \tag{2.4.15}$$

and thus we obtain with (2.4.13)

$$F_{\mathrm{osc}}(T) = \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}}\left\{\prod_{n=1}^{\infty}\left[1-\left(\frac{\omega T}{n\pi}\right)^2\right]\right\}^{-1/2} . \tag{2.4.16}$$

Here we have already performed the limit $N \to \infty$ which is allowed since we have the convergent infinite product

$$\prod_{n=1}^{\infty}\left[1-\left(\frac{x}{n\pi}\right)^2\right] = \frac{\sin x}{x} \tag{2.4.17}$$

which leads to the final result

$$F_{\mathrm{osc}}(T) = \sqrt{\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sin\omega T}} \ . \tag{2.4.18}$$

This is the correct expression for the harmonic oscillator, see Sect. 3.2.

Obviously, the evaluation of path integrals by Fourier series is closely related to the zeta function regularization discussed in Sect. 2.3. Indeed, in terms of the operator (2.4.9) we obtain with the Gaussian path integral (2.3.16) immediately

$$\begin{aligned} F_{\mathrm{osc}}(T) &= \int\limits_{q(0)=0}^{q(T)=0} \mathcal{D}q(t) \exp\left[-\frac{1}{2}\left(\frac{m}{\mathrm{i}\hbar}\right)\int_0^T x(t)\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}x(t)\,\mathrm{d}t\right] \\ &= \sqrt{\frac{m}{\pi\mathrm{i}\hbar}}\left(\det \underset{\sim}{\mathrm{A}}_{\mathrm{osc}}\right)^{-1/2} . \end{aligned} \tag{2.4.19}$$

Here the functional determinant is given by (2.3.12), i.e,

$$\det \underset{\sim}{\mathrm{A}}_{\mathrm{osc}} = \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\omega^2\right) = \mathrm{e}^{-\zeta'_{\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}}(0)} \tag{2.4.20}$$

in terms of the corresponding MP zeta function (2.3.11)

$$\zeta_{\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}}(s) = \mathrm{Tr}\,\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}^{-s} = \sum_{n=1}^{\infty}\frac{1}{\lambda_n^s} , \tag{2.4.21}$$

where λ_n are the eigenvalues of the eigenvalue problem

$$\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}q_n(t) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\omega^2\right)q_n(t) = \lambda_n q_n(t) , \tag{2.4.22}$$

$0 \leq t \leq T$, $q_n(0) = q_n(T) = 0$. The eigenvalues of the operator $\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}$ are given by $\lambda_n = (n\pi/T)^2 - \omega^2$, $n \in \mathbb{N}$, and we obtain for the MP zeta function ($\Re s > 1/2$)

$$\zeta_{\underset{\sim}{\mathrm{A}}_{\mathrm{osc}}}(s) = \left(\frac{T}{\pi}\right)^{2s}\sum_{n=1}^{\infty}\frac{1}{(n^2-(\omega T/\pi)^2)^s} . \tag{2.4.23}$$

As a result one eventually finds

$$\det \underset{\sim}{\mathrm{A}}_{\mathrm{osc}} = \frac{2}{\omega}\sin\omega T \tag{2.4.24}$$

which gives with (2.4.19) the same result (2.4.18) as before. Observe that (2.4.18) approaches in the limit $\omega \to 0$ the correct result (2.3.15) for the free particle case.

A simple way to obtain (2.4.24) goes as follows [868]

$$\begin{aligned} \det \underset{\sim}{\mathrm{A}}_{\mathrm{osc}} &= \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\omega^2\right) = \det\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\mu_n}\right) \\ &= 2T\,\frac{\sin\omega T}{\omega T} = \frac{2}{\omega}\sin\omega T \end{aligned} \tag{2.4.25}$$

where we have used (2.3.14) for the free particle determinant, and $\mu_n = (n\pi/T)^2$ for the eigenvalues of the operator $-\mathrm{d}^2/\mathrm{d}t^2$.

2.5 Path Integration Over Coherent States

2.5.1 Coherent States and the Bargmann Representation. In 1926, Schrödinger [823, 867] constructed "minimum uncertainty wave packets" for the harmonic oscillator, later popularized as "coherent states". The systematic use for Bose systems of states based on non-Hermitian operators is due to Schwinger [839]. Nowadays coherent states play an important rôle in many branches of physics, for instance in quantum field theory and quantum optics. In his paper [823], Schrödinger wanted to illustrate by the example of "Planck's linear oscillator" that it is always possible to find solutions of his "undulatory mechanics" in the form of well-localized wave packets whose centre of gravity oscillates without change of shape with the period of the corresponding classical trajectory of a point particle like, for example, the Kepler orbits of the electron in the H-atom.

Let us consider a one-dimensional bosonic system described by creation and annihilation operators $\underset{\sim}{a}^\dagger$ and $\underset{\sim}{a}$, respectively, obeying the *bosonic commutation relation*

$$[\underset{\sim}{a}, \underset{\sim}{a}^\dagger] = \mathbb{1} \ . \tag{2.5.1}$$

Of course, the prototype example of such a system is the one-dimensional harmonic oscillator with Hamiltonian

$$\underset{\sim}{H}_{\text{osc}} = \frac{\underset{\sim}{p}^2}{2m} + \frac{m}{2}\omega^2 \underset{\sim}{q}^2 \ . \tag{2.5.2}$$

Introducing for this system the non-Hermitian operators

$$\underset{\sim}{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\underset{\sim}{q} + \mathrm{i}\,\frac{\underset{\sim}{p}}{m\omega}\right) \ , \qquad \underset{\sim}{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\underset{\sim}{q} - \mathrm{i}\,\frac{\underset{\sim}{p}}{m\omega}\right) \ , \tag{2.5.3}$$

one derives the commutation relation (2.5.1) from the Heisenberg commutation relation $[\underset{\sim}{q}, \underset{\sim}{p}] = \mathrm{i}\,\hbar$. In terms of $\underset{\sim}{a}$ and $\underset{\sim}{a}^\dagger$ the Hamiltonian (2.5.2) takes on the simple form

$$\underset{\sim}{H}_{\text{osc}} = \hbar\omega \underset{\sim}{a}^\dagger \underset{\sim}{a} + \frac{1}{2}\hbar\omega\, \mathbb{1} \ . \tag{2.5.4}$$

Under the assumption that our general bosonic system has only a discrete spectrum

$$\underset{\sim}{H}|n\rangle = E_n|n\rangle \ , \quad n \in \mathbb{N}_0 \ , \quad \langle m|n\rangle = \delta_{mn} \ , \tag{2.5.5}$$

we construct the *Fock space* of states by

$$|n\rangle = \frac{1}{\sqrt{n!}}(\underset{\sim}{a}^\dagger)^n|0\rangle \ , \tag{2.5.6}$$

where the ground state $|0\rangle$, the Fock vacuum, is defined as the normalized eigenstate of $\underset{\sim}{a}$ with eigenvalue zero, i.e.,

$$\underset{\sim}{a}|0\rangle = 0 \ , \qquad \langle 0|0\rangle = 1 \ . \tag{2.5.7}$$

An arbitrary normalized Fock space vector $|\psi\rangle$, $||\,|\psi\rangle||^2 = \langle\psi|\psi\rangle = 1$, can be expanded into the orthonormal Fock basis $\{|n\rangle\}$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} (\underline{a}^\dagger)^n |0\rangle \tag{2.5.8}$$

with complex coefficients $\{c_n = \langle n|\psi\rangle\}$ satisfying

$$\langle\psi|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 = 1 \ . \tag{2.5.9}$$

Introducing the functions $\psi(z^*)$ of a complex variable $z \in \mathbb{C}$,

$$\psi(z^*) := \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n!}} (z^*)^n \ , \tag{2.5.10}$$

we can rewrite (2.5.8) as

$$|\psi\rangle = \psi(\underline{a}^\dagger)|0\rangle \ . \tag{2.5.11}$$

Since (2.5.9) implies $|c_n| \leq 1$, we infer that the series (2.5.10) is absolutely convergent for all $z \in \mathbb{C}$

$$\sum_{n=0}^{\infty} \frac{|c_n|}{\sqrt{n!}} |z|^n \leq \sum_{n=0}^{\infty} \frac{|z|^n}{\sqrt{n!}} < \infty \tag{2.5.12}$$

and thus the function $\psi(z^*)$ is an entire function of z^*. Therefore (2.5.10) and (2.5.11) define a mapping from the quantum mechanical Fock space $\{|\psi\rangle\}$ to the space of entire functions $\{\psi(z^*)\}$ which is known as the *Bargmann representation* [53] of Hilbert space. As an orthonormal basis in this space we choose

$$f_n(z^*) := \frac{(z^*)^n}{\sqrt{n!}} \ , \qquad n \in \mathbb{N}_0 \ , \tag{2.5.13}$$

with the orthonormality relation

$$(f_m|f_n) := \int_{\mathbb{C}} \mathrm{d}\mu(z) f_m^*(z^*) f_n(z^*) = \delta_{mn} \ . \tag{2.5.14}$$

Here we have introduced the measure

$$\mathrm{d}\mu(z) := \mathrm{e}^{-|z|^2} \frac{\mathrm{d}^2 z}{\pi} \ , \tag{2.5.15}$$

where $\mathrm{d}^2 z = \mathrm{d}(\Re z)\,\mathrm{d}(\Im z) = \mathrm{d}z\mathrm{d}z^*/2\mathrm{i}$, and the integration in (2.5.14) takes place over the whole complex z-plane. Thus in the Bargmann representation the Fock state (2.5.8) is represented by the entire function

$$\psi(z^*) = \sum_{n=0}^{\infty} c_n f_n(z^*) \tag{2.5.16}$$

with scalar product

$$(\psi_1|\psi_2) := \int_{\mathbb{C}} \mathrm{d}\mu(z)\psi_1^*(z^*)\psi_2(z^*) \ . \tag{2.5.17}$$

Introducing for any $z \in \mathbb{C}$ the *coherent state*

$$|z\rangle := \sum_{n=0}^{\infty} f_n(z)|n\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \tag{2.5.18}$$

with the associated adjoint state

$$\langle z| = \sum_{n=0}^{\infty} f_n(z^*)\langle n| \ , \tag{2.5.19}$$

we can identify the Bargmann representation $\psi(z^*)$ of a Fock state $|\psi\rangle$ as the *coherent state representation* of $|\psi\rangle$, i.e.,

$$\psi(z^*) = \langle z|\psi\rangle \ . \tag{2.5.20}$$

Indeed, we have with (2.5.8)

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} c_n \langle z|n\rangle \ , \tag{2.5.21}$$

which gives with $\langle z|n\rangle = f_n(z^*)$, see (2.5.19),

$$\langle z|\psi\rangle = \sum_{n=0}^{\infty} c_n f_n(z^*) = \psi(z^*) \ . \tag{2.5.22}$$

The coherent state (2.5.18) can be rewritten as

$$|z\rangle = \sum_{n=0}^{\infty} \frac{\left(z\underline{a}^\dagger\right)^n}{n!}|0\rangle = \mathrm{e}^{z\underline{a}^\dagger}\,|0\rangle \ . \tag{2.5.23}$$

(Notice that (2.5.23) differs by the factor $\mathrm{e}^{-|z|^2/2}$ from the alternative definition of coherent states obtained by acting with the unitary operator $\mathrm{e}^{z\underline{a}^\dagger - z^*\underline{a}} = \mathrm{e}^{-|z|^2/2}\,\mathrm{e}^{z\underline{a}^\dagger}\,\mathrm{e}^{-z^*\underline{a}}$ on the vacuum state $|0\rangle$. It turns out that the definition (2.5.18) is more convenient for the construction of the path integral over coherent states.)

From (2.5.23) one derives the important property that the coherent state $|z\rangle$ is an eigenstate of the (non-Hermitian) annihilation operator $\underline{a}$ with complex eigenvalue $z \in \mathbb{C}$. Indeed, using the commutation relation (2.5.1) one easily shows $[\underset{\sim}{a}, (\underset{\sim}{a}^\dagger)^n] = n(\underset{\sim}{a}^\dagger)^{n-1} = \mathrm{d}(\underset{\sim}{a}^\dagger)^n/\mathrm{d}\underset{\sim}{a}^\dagger$ ($n \in \mathbb{N}$) and, more generally, $[\underset{\sim}{a}, g(\underset{\sim}{a}^\dagger)] = \mathrm{d}g(\underset{\sim}{a}^\dagger)/\mathrm{d}\underset{\sim}{a}^\dagger$ for any nice function $g(x)$, and thus we obtain

$$\begin{aligned}\underset{\sim}{a}|z\rangle &= \underset{\sim}{a}\,\mathrm{e}^{z\underset{\sim}{a}^\dagger}|0\rangle = [\underset{\sim}{a}, \mathrm{e}^{z\underset{\sim}{a}^\dagger}]|0\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}\underset{\sim}{a}^\dagger}\mathrm{e}^{z\underset{\sim}{a}^\dagger}|0\rangle = z\,\mathrm{e}^{z\underset{\sim}{a}^\dagger}|0\rangle = z|z\rangle \ . \end{aligned} \tag{2.5.24}$$

In particular, the coherent state with $z = 0$ is identical to the Fock vacuum $|0\rangle$.

Summarizing, we observe that the operator $\underset{\sim}{a}$ acts on coherent states by multiplication by z, while $\underset{\sim}{a}^\dagger$ is represented by $\partial/\partial z$. Furthermore, we have the completeness relation

$$\int_{\mathbb{C}} \mathrm{d}\mu(z)|z\rangle\langle z| = \mathbb{1} \tag{2.5.25}$$

and the scalar product

$$\langle z|z'\rangle = \mathrm{e}^{z^* z'} \ . \tag{2.5.26}$$

From (2.5.25) and (2.5.26) it follows that $\langle z|z'\rangle$ acts in the Bargmann representation like a Dirac delta distribution (reproducing kernel)

$$\begin{aligned}\psi(z^*) = \langle z|\psi\rangle &= \int \mathrm{d}\mu(z')\langle z|z'\rangle\langle z'|\psi\rangle \\ &= \int \mathrm{d}\mu(z')\,\mathrm{e}^{z^* z'}\,\psi(z'^*) \ , \end{aligned} \tag{2.5.27}$$

while the operators $\underset{\sim}{a}^\dagger$ and $\underset{\sim}{a}$ act as

$$(\underset{\sim}{a}^\dagger\psi)(z^*) := \langle z|\underset{\sim}{a}^\dagger|\psi\rangle = z^*\psi(z^*) \ , \tag{2.5.28}$$

$$(\underset{\sim}{a}\psi)(z^*) := \langle z|\underset{\sim}{a}|\psi\rangle = \frac{\mathrm{d}}{\mathrm{d}z^*}\psi(z^*) \ . \tag{2.5.29}$$

An arbitrary operator $\underset{\sim}{A} = A(\underset{\sim}{a}, \underset{\sim}{a}^\dagger)$ in Fock space can be written as

$$\underset{\sim}{A} = \sum_{m,n=0}^{\infty} |m\rangle A_{mn}\langle n| \ , \qquad A_{mn} := \langle m|\underset{\sim}{A}|n\rangle \ . \tag{2.5.30}$$

Its matrix representation is given by

$$\begin{aligned}\langle z|\underset{\sim}{A}|z'\rangle &= \sum_{m,n=0}^{\infty} f_m(z^*)A_{mn}f_n(z') \\ &=: A(z^*, z') \ , \end{aligned} \tag{2.5.31}$$

where the analytical function $A(z^*, z')$ plays the rôle of an integral kernel which represents the action of the operator $\underset{\sim}{A}$ on a state $\psi(z^*)$ in the Bargmann representation

$$(\underset{\sim}{A}\psi)(z^*) = \int d\mu(z')A(z^*, z')\psi(z'^*) \ . \tag{2.5.32}$$

The product of two operators $\underset{\sim}{A}$ and $\underset{\sim}{B}$ possesses the matrix representation

$$\begin{aligned}(\underset{\sim}{A}\underset{\sim}{B})(z^*, z') &= \langle z|\underset{\sim}{A}\underset{\sim}{B}|z'\rangle \\ &= \int d\mu(z'')A(z^*, z'')B(z''^*, z') \ . \end{aligned} \tag{2.5.33}$$

Using the commutation relation (2.5.1) it is always possible to bring an operator $\underset{\sim}{A}$ into *normal ordered form* with all the operators $\underset{\sim}{a}^\dagger$ standing on the left of the operators $\underset{\sim}{a}$, i.e.,

$$\underset{\sim}{A} = \sum_{k,l=0}^{\infty} c_{kl}(\underset{\sim}{a}^\dagger)^k(\underset{\sim}{a})^l \ . \tag{2.5.34}$$

(This should not be confused with "normal ordering" or Wick's ordering denoted by a double-dot symbol, e.g., $:\underset{\sim}{a}\underset{\sim}{a}^\dagger := \underset{\sim}{a}^\dagger\underset{\sim}{a}$. The normal ordered form of the operator $\underset{\sim}{A} = \underset{\sim}{a}\underset{\sim}{a}^\dagger$ is $\underset{\sim}{A} = \underset{\sim}{a}^\dagger\underset{\sim}{a} + \mathbb{1} \neq : \underset{\sim}{A} :$.) The *normal symbol* of the operator (2.5.34) denoted by $A^N(z^*, z')$ is defined by [311, 313]

$$A^N(z^*, z') := \sum_{k,l=0}^{\infty} c_{kl} z^{*k} z'^l \ . \tag{2.5.35}$$

The relation between the kernel (2.5.31) and its corresponding normal symbol A^N is given by

$$\begin{aligned}A(z^*, z') = \langle z|\underset{\sim}{A}|z'\rangle &= \sum_{k,l=0}^{\infty} c_{kl} z^{*k} z'^l \langle z|z'\rangle \\ &= e^{z^* z'} A^N(z^*, z') \ . \end{aligned} \tag{2.5.36}$$

Thus to obtain the matrix representation of an arbitrary operator $\underset{\sim}{A}$ in the Bargmann representation, one first brings $\underset{\sim}{A}$ into its normal ordered form (2.5.34), then forms the normal symbol (2.5.35) and finally just multiplies A^N by $e^{z^* z'}$. For more details on coherent states, the reader is referred to Faddeev [311], Faddeev and Slavnov [313], and Klauder und Skagerstam [602].

2.5.2 The Path Integral. For the one-dimensional Feynman kernel we have

$$\begin{aligned}K(x'', x'; t'', t') &= \langle x''|\underset{\sim}{U}(t'', t')|x'\rangle\Theta(t'' - t') \\ &= \int d\mu(z'') \int d\mu(z')\langle x''|z''\rangle U(z''^*, z'; t'', t')\Theta(t'' - t')\langle z'|x'\rangle \\ &= \sum_{m,n=0}^{\infty} \psi_m(x'')K_{mn}(t'', t')\psi_n^*(x') \end{aligned} \tag{2.5.37}$$

with the *time-evolution kernel*

$$U(z''^*, z'; t'', t') := \langle z''|\mathrm{U}(t'', t')|z'\rangle \tag{2.5.38}$$

and the *transition-matrix element* [340, p.144]

$$\begin{aligned}K_{mn}(t'', t') &:= \langle m|\mathrm{U}(t'', t')|n\rangle\Theta(t'' - t')\\ &= \int_{\mathbb{R}} \mathrm{d}x'' \int_{\mathbb{R}} \mathrm{d}x' \psi_m^*(x'')K(x'', x'; t'', t')\psi_n(x')\\ &= \int \mathrm{d}\mu(z'') \int \mathrm{d}\mu(z') f_m(z'')U(z''^*, z'; t'', t')\Theta(t'' - t')f_n(z'^*) \ .\end{aligned} \tag{2.5.39}$$

Here we have used ($\psi_n(x) := \langle x|n\rangle$)

$$\langle x|z\rangle = \sum_{n=0}^{\infty} f_n(z)\psi_n(x) \ . \tag{2.5.40}$$

For $t'' > t'$ we have the expansion

$$U(z''^*, z'; t'', t') = \sum_{m,n=0}^{\infty} f_m(z''^*)K_{mn}(t'', t')f_n(z') \ , \tag{2.5.41}$$

which shows that the time-evolution kernel $U(z''^*, z'; t'', t')$ is the generating function of the transition amplitudes K_{mn}

$$K_{mn}(t'', t') = \frac{1}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial(z''^*)^m \partial z'^n} U(z''^*, z'; t'', t')\bigg|_{z''^*=z'=0} \ . \tag{2.5.42}$$

Notice that $|K_{mn}(t'', t')|^2$ is the probability for the transition during the time interval $t'' - t' > 0$ from the initial state $|n\rangle$ at time t' to the final state $|m\rangle$ at time t''. In particular

$$K_{00}(t'', t') = U(z''^*, z'; t'', t')\bigg|_{z''^*=z'=0} \tag{2.5.43}$$

is the vacuum-vacuum transition amplitude.

From (2.5.39) we derive $\lim_{t''\to t'} K_{mn}(t'', t') = \delta_{mn}$, and thus obtain from (2.5.41)

$$\lim_{t''\to t'} U(z''^*, z'; t'', t') = \sum_{n=0}^{\infty} f_n(z''^*)f_n(z') = \mathrm{e}^{z''^* z'} \ . \tag{2.5.44}$$

This shows together with (2.5.36) that for the normal symbol U^N we have the initial condition

$$\lim_{t''\to t'} U^N(z''^*, z'; t'', t') = 1 \ . \tag{2.5.45}$$

For the harmonic oscillator (2.5.2) we get

$$e^{-\mathrm{i}T\underline{\mathrm{H}}_{\mathrm{osc}}/\hbar}\,|z\rangle = e^{-\mathrm{i}\omega T/2}\,|z\,e^{-\mathrm{i}\omega T}\rangle \tag{2.5.46}$$

and thus

$$\begin{aligned} U_{\mathrm{osc}}(z''^*, z'; t'', t') &= \langle z''|\,e^{-\mathrm{i}T\underline{\mathrm{H}}_{\mathrm{osc}}/\hbar}\,|z'\rangle \\ &= e^{-\mathrm{i}\omega T/2}\langle z''|z'\,e^{-\mathrm{i}\omega T}\rangle = e^{-\mathrm{i}\omega T/2}\exp\left(z''^*\,e^{-\mathrm{i}\omega T}\,z'\right) \ . \end{aligned} \tag{2.5.47}$$

Expanding the last result yields

$$U_{\mathrm{osc}}(z''^*, z'; t'', t') = \sum_{n=0}^{\infty} f_n(z''^*)\,e^{-\mathrm{i}E_nT/\hbar}\,f_n(z') \tag{2.5.48}$$

with the correct spectrum $E_n = \hbar\omega(n+\frac{1}{2})$, and a comparison with (2.5.41) gives the expected result

$$K_{mn}^{\mathrm{osc}}(T) = e^{-\mathrm{i}E_nT/\hbar}\,\delta_{mn} \ . \tag{2.5.49}$$

To derive the path integral in the coherent state representation, we start from the semigroup property of the time-evolution operator $(t'' > t > t')$

$$\underline{\mathrm{U}}(t'', t') = \underline{\mathrm{U}}(t'', t)\,\underline{\mathrm{U}}(t, t') \tag{2.5.50}$$

and obtain

$$\begin{aligned} U(z''^*, z'; t'', t') &= \langle z''|\underline{\mathrm{U}}(t'', t')|z'\rangle \\ &= \left\langle z''\left|\prod_{k=0}^{N}\underline{\mathrm{U}}(t_{k+1}, t_k)\right|z'\right\rangle \\ &= \prod_{j=1}^{N}\int d\mu(z_j)\prod_{k=0}^{N} U(z_{k+1}^*, z_k; t_{k+1}, t_k) \end{aligned} \tag{2.5.51}$$

which is exact for any $N \geq 1$ with $\epsilon = t_{k+1} - t_k := T/(N+1), T = t'' - t' > 0$ fixed, $t_0 = t', t_{N+1} = t''$, $z_0 = z'$ and $z_{N+1}^* = z''^*$. For the time-evolution operator we have for small ϵ

$$\underline{\mathrm{U}}(t_{k+1}, t_k) = \mathbb{1} - \frac{\mathrm{i}}{\hbar}\epsilon H(\underline{\mathrm{a}}^\dagger, \underline{\mathrm{a}}; t_k) + O(\epsilon^2) \ . \tag{2.5.52}$$

Let us assume, without loss of generality, that the Hamiltonian in Fock space is already given in normal ordered form, so that we obtain for the normal symbol of $\underline{\mathrm{U}}$

$$U^N(z_{k+1}^*, z_k; t_{k+1}, t_k) = 1 - \frac{\mathrm{i}}{\hbar}\epsilon H(z_{k+1}^*, z_k; t_k) + O(\epsilon^2)$$

$$= \exp\left[-\frac{\mathrm{i}}{\hbar}\epsilon H(z_{k+1}^*, z_k; t_k) + O(\epsilon^2)\right] \tag{2.5.53}$$

and thus for the kernel of $\underset{\sim}{U}$ in the space of coherent states

$$U(z_{k+1}^*, z_k; t_{k+1}, t_k) = \exp\left[z_{k+1}^* z_k - \frac{\mathrm{i}}{\hbar}\epsilon H(z_{k+1}^*, z_k; t_k) + O(\epsilon^2)\right] . \tag{2.5.54}$$

Inserting the last expression into (2.5.51) we obtain with $\mathrm{d}\mu(z) = \mathrm{e}^{-|z|^2}\frac{\mathrm{d}z\mathrm{d}z^*}{2\pi\mathrm{i}}$ the lattice definition of the *path integral in the coherent state representation*

$$U(z''^*, z'; t'', t') = \lim_{N\to\infty}\prod_{j=1}^{N}\int_{\mathbb{C}}\frac{\mathrm{d}z_j\mathrm{d}z_j^*}{2\pi\mathrm{i}}$$
$$\times \exp\left[-\sum_{k=1}^{N}|z_k|^2 + \sum_{k=0}^{N} z_{k+1}^* z_k - \frac{\mathrm{i}}{\hbar}\sum_{k=0}^{N}\epsilon H(z_{k+1}^*, z_k; t_k)\right] . \tag{2.5.55}$$

Here we have assumed, as usual, that the terms $O(\epsilon^2)$ do not contribute in the limit $\epsilon \to 0$ and, of course, that the limit $N \to \infty$ exists. To interpret (2.5.55) as a path integral, we consider *independent* complex paths $z(t)$ and $z^*(t)$ with $z(t_k) = z_k$, $z^*(t_k) = z_k^*$ and the *boundary conditions*

$$z_0 = z(t') = z' \ , \qquad z_{N+1}^* = z^*(t'') = z''^* \ . \tag{2.5.56}$$

Notice that we do *not* require at the end points $z(t'') = z''$ and $z^*(t') = z'^*$. Indeed, only the values z' and z''^* are fixed in the kernel (2.5.55).

The first two sums in the exponent of (2.5.55) can be rewritten as

$$-\sum_{k=1}^{N} z_k^* z_k + \sum_{k=1}^{N} z_k^* z_{k-1} + z_{N+1}^* z_N = z_{N+1}^* z_N - \sum_{k=1}^{N}\epsilon z_k^*\frac{z_k - z_{k-1}}{\epsilon} \ , \tag{2.5.57}$$

which gives in the limit $N \to \infty, \epsilon \to 0$

$$z''^* z(t'') - \int_{t'}^{t''} z^*(t)\dot{z}(t)\,\mathrm{d}t \ ,$$

and thus we obtain the following *path integral over coherent states*

$$U(z''^*, z'; t'', t') = \int_{z(t')=z'}^{z^*(t'')=z''^*} \mathcal{D}z(t)\mathcal{D}z^*(t)$$
$$\times \exp\left[z''^* z(t'') + \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Big(\mathrm{i}\,\hbar z^*(t)\dot{z}(t) - H(z^*(t), z(t); t)\Big)\mathrm{d}t\right] . \tag{2.5.58}$$

Here we have introduced the "path differentials"

$$\mathcal{D}z(t)\mathcal{D}z^*(t) \doteq \lim_{N\to\infty} \prod_{j=1}^{N} \int_{\mathbb{C}} \frac{dz_j dz_j^*}{2\pi i} . \qquad (2.5.59)$$

Here the following remark is in order: instead of (2.5.57), the first two sums in the exponent in (2.5.55) can also be rewritten as

$$\frac{1}{2}\left(|z_{N+1}|^2 + |z_0|^2\right) + \frac{1}{2}\sum_{k=0}^{N} \epsilon\left(\frac{z_{k+1}^* - z_k^*}{\epsilon}\right) z_k - \frac{1}{2}\sum_{k=0}^{N} \epsilon z_{k+1}^* \left(\frac{z_{k+1} - z_k}{\epsilon}\right)$$

$$\underset{\left(\substack{N\to\infty \\ \epsilon\to 0}\right)}{\to} \frac{1}{2}\left(z''^* z(t'') + z^*(t')z'\right) + \frac{1}{2}\int_{t'}^{t''} \left(\dot{z}^*(t)z(t) - z^*(t)\dot{z}(t)\right) dt \qquad (2.5.60)$$

which would seem to lead to a symmetrized version of the path integral (2.5.58). (In fact, this symmetrized version of the path integral (2.5.58) is often used in the literature.) However, this symmetrization is in general not correct within the path integral, since the above integrals have to be considered as *stochastic integrals*! Only if in the above integral $\dot{z}^*$ is interpreted as the *forward derivative*, $\dot{z}^*(t) = \lim_{\epsilon\to 0}\left(z^*(t+\epsilon) - z^*(t)\right)/\epsilon$, and $\dot{z}(t)$ as the *backward derivative*, $\dot{z}(t) = \lim_{\epsilon\to 0}\left(z(t) - z(t-\epsilon)\right)/\epsilon$, do the above expressions lead to the correct path integral, i.e., the correct lattice definition (2.5.55).

2.5.3 The Forced Harmonic Oscillator and the Feynman Propagator. The Hamiltonian of the *forced harmonic oscillator* reads in normal ordered form as follows

$$H(\underset{\sim}{a}^\dagger, \underset{\sim}{a}; t) = \hbar\omega \underset{\sim}{a}^\dagger \underset{\sim}{a} + \tfrac{1}{2}\hbar\omega\, \mathbb{1} - \sqrt{\frac{\hbar m}{2\omega}}\, J(t)(\underset{\sim}{a}^\dagger + \underset{\sim}{a}) , \qquad (2.5.61)$$

where $J(t)$ is a real c-number source corresponding to the driving force $F(t) = mJ(t)$. (From the definitions (2.5.3) one sees that the driving term $-\sqrt{\hbar m/2\omega}\, J(t)(\underset{\sim}{a}^\dagger + \underset{\sim}{a})$ equals $-mJ(t)\underset{\sim}{q}$.) The kernel U of this system has the following path integral representation

$$U(z''^*, z'; t'', t') = e^{-i\omega T/2} \int_{z(t')=z'}^{z^*(t'')=z''^*} \mathcal{D}z(t)\mathcal{D}z^*(t) \exp\left(z''^* z(t'') + \frac{i}{\hbar} R[z^*, z]\right) , \qquad (2.5.62)$$

where we have introduced the classical "action" ($c := \sqrt{m/2\hbar\omega}$)

$$R[z^*, z] = i\hbar \int_{t'}^{t''} \left[(\dot{z} + i\omega z - icJ)z^* - icJz\right] dt . \qquad (2.5.63)$$

Since the path integral (2.5.62) is of Gaussian type, it can be exactly solved by the method of stationary phase

$$U(z''^*, z'; t'', t') = \mathrm{e}^{-\mathrm{i}\omega T/2} \exp\left(z''^* z_{\mathrm{Cl}}(t'') + \frac{\mathrm{i}}{\hbar} R[z^*, z_{\mathrm{Cl}}] \right) . \tag{2.5.64}$$

Here the "classical path" $z_{\mathrm{Cl}}(t)$ is the solution of the equation of motion

$$\frac{\delta R[z^*, z_{\mathrm{Cl}}]}{\delta z^*(t)} = \mathrm{i}\,\hbar(\dot{z}_{\mathrm{Cl}} + \mathrm{i}\,\omega z_{\mathrm{Cl}} - \mathrm{i}\,cJ) = 0 \tag{2.5.65}$$

with the initial condition $z_{\mathrm{Cl}}(t') = z'$. The solution of this inhomogeneous equation is given by $(t' < t < t'')$

$$z_{\mathrm{Cl}}(t) = \mathrm{e}^{-\mathrm{i}\,\omega(t-t')}\, z' + \mathrm{i}\, c \int_{t'}^{t} \mathrm{e}^{-\mathrm{i}\,\omega(t-s)}\, J(s) \mathrm{d}s \ , \tag{2.5.66}$$

and thus we obtain for the expression in (2.5.64)

$$\begin{aligned} z''^* z_{\mathrm{Cl}}(t'') + \frac{\mathrm{i}}{\hbar} R[z^*, z_{\mathrm{Cl}}] &= z''^* z_{\mathrm{Cl}}(t'') + \mathrm{i}\, c \int_{t'}^{t''} J(t) z_{\mathrm{Cl}}(t)\, \mathrm{d}t \\ = z''^*\, \mathrm{e}^{-\mathrm{i}\,\omega(t''-t')}\, z' &+ \mathrm{i}\, c z''^* \int_{t'}^{t''} \mathrm{e}^{-\mathrm{i}\,\omega(t''-t)}\, J(t)\, \mathrm{d}t \\ &+ \mathrm{i}\, c z' \int_{t'}^{t''} \mathrm{e}^{-\mathrm{i}\,\omega(t-t')}\, J(t)\, \mathrm{d}t - c^2 \int_{t'}^{t''} \mathrm{d}t \int_{t'}^{t} \mathrm{d}s\, J(t)\, \mathrm{e}^{-\mathrm{i}\,\omega(t-s)}\, J(s) \ . \end{aligned} \tag{2.5.67}$$

The last integral can be rewritten as follows:

$$\begin{aligned} &\int_{t'}^{t''} \mathrm{d}t \int_{t'}^{t} \mathrm{d}s\, J(t)\, \mathrm{e}^{-\mathrm{i}\,\omega(t-s)}\, J(s) \\ &= \int_{t'}^{t''} \mathrm{d}t \int_{t'}^{t''} \mathrm{d}s\, J(t)\, \mathrm{e}^{-\mathrm{i}\,\omega(t-s)}\, J(s) \Theta(t-s) \\ &= \frac{1}{2} \int_{t'}^{t''} \mathrm{d}t \int_{t'}^{t''} \mathrm{d}s\, J(t) \Big[\mathrm{e}^{-\mathrm{i}\,\omega(t-s)}\, \Theta(t-s) + \mathrm{e}^{-\mathrm{i}\,\omega(s-t)}\, \Theta(s-t) \Big] J(s) \ . \end{aligned} \tag{2.5.68}$$

Introducing the *Feynman propagator*

$$\begin{aligned} D_F(t) &:= \frac{1}{2\,\mathrm{i}\,\omega} \Big[\mathrm{e}^{-\mathrm{i}\,\omega t}\, \Theta(t) + \mathrm{e}^{\mathrm{i}\,\omega t}\, \Theta(-t) \Big] \\ &= \frac{1}{2\,\mathrm{i}\,\omega}\, \mathrm{e}^{-\mathrm{i}\,\omega |t|} \end{aligned} \tag{2.5.69}$$

we obtain the final result for the *kernel of the forced harmonic oscillator*

$$U(z''^*, z'; t'', t') = U_{\mathrm{osc}}(z''^*, z'; t'', t') Z[J] \ , \tag{2.5.70}$$

where U_{osc} denotes the kernel (2.5.47) of the harmonic oscillator without driving force ($J \equiv 0$), and the *generating functional* $Z[J]$ ($Z[0] = 1$) is given by

$$
\begin{aligned}
Z[J] &\equiv Z(z''^*, z'; t'', t'|J) \\
&:= \exp\left[\mathrm{i}\sqrt{\frac{m}{2\hbar\omega}} \int_{t'}^{t''} \left(\mathrm{e}^{-\mathrm{i}\omega(t''-t)}\, z''^* + \mathrm{e}^{-\mathrm{i}\omega(t-t')}\, z' \right) J(t)\, \mathrm{d}t \right. \\
&\qquad \left. - \frac{\mathrm{i}\, m}{2\hbar} \int_{t'}^{t''} \mathrm{d}t \int_{t'}^{t''} \mathrm{d}s\, J(t) D_F(t-s) J(s) \right] . \qquad (2.5.71)
\end{aligned}
$$

The Feynman propagator satisfies the inhomogeneous wave equation

$$
\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) D_F(t-s) = -\delta(t-s) \qquad (2.5.72)
$$

and has the integral representation

$$
D_F(t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\omega' t}}{\omega'^2 - \omega^2 + \mathrm{i}\epsilon} . \qquad (2.5.73)
$$

Here a few remarks are in order

i) The Feynman propagator D_F, (2.5.69) and (2.5.73), is a simple (non-relativistic) example of the so-called Feynman propagators which play an important rôle in quantum field theory, in particular as building blocks of the Feynman rules, see e.g. [534]. It should not be confused with the Feynman kernel $K(x'', x'; t'', t')$ which is, however, also called the propagator (or Feynman propagator) by some authors.

ii) It is clear from the integral representation (2.5.73) that the specific regularization corresponding to the above "i ϵ-rule" is of crucial importance for obtaining the correct Feynman propagator. It is a remarkable fact that the coherent state path integral leads automatically to the correct regularization. This is in contrast to the usual path integral which due to its oscillatory character in physical time leads to (2.5.73) *without* the i ϵ-regularization. To cure this disease, one can proceed as follows. The regularization (2.5.73) can be interpreted as making the replacement $\omega^2 \to \omega^2 - \mathrm{i}\epsilon$ which leads in the usual path integral for the harmonic oscillator to the replacement

$$
\begin{aligned}
&\exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \frac{m}{2}\omega^2 x^2 \mathrm{d}t \right) \\
&\qquad \to \exp\left(-\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \frac{m}{2}\omega^2 x^2 \mathrm{d}t \right) \exp\left(-\epsilon \frac{m}{2\hbar} \int_{t'}^{t''} x^2 \mathrm{d}t \right)
\end{aligned}
$$

and thus gives for $\epsilon > 0$ a Gaussian damping.

iii) Taking functional derivatives of the path integral (2.5.62), for instance, $\delta U/\delta J(t)|_{J=0}$, one obtains the expectation values of powers of $\underset{\sim}{a} + \underset{\sim}{a}^\dagger \propto \underset{\sim}{q}$, but these are completely determined by the generating functional $Z[J]$, see (2.5.70) and (2.5.71).

Finally let us also note the *Gaussian integration formula for coherent states.* Let $\mathbf{A}$ be a $(d \times d)$-matrix of a non-singular quadratic form whose Hermitian part is positive, and $\mathbf{z}$ and $\mathbf{u}$ stand for vectors of complex numbers. Then

$$\int \frac{\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{z}^*}{(2\pi\mathrm{i})^d}\,\mathrm{e}^{-\mathbf{z}^*\mathbf{A}\mathbf{z}+\mathbf{u}^*\mathbf{z}+\mathbf{u}\mathbf{z}^*} = \frac{\mathrm{e}^{\mathbf{u}^*\mathbf{A}^{-1}\mathbf{u}}}{\det \mathbf{A}} \ . \tag{2.5.74}$$

2.6 Fermionic Path Integrals

2.6.1 Fermionic Coherent States. Since path integrals exhibit in a particularly clear way the close relationship which exists between classical and quantum mechanics, it would seem a priori that we would encounter some difficulties when extending the treatment to fermions. Fortunately the relevant construction in terms of an anticommuting algebra has been devised by Berezin [74–78], Martin [682], and Schwinger [838]. Fermionic path integrals were introduced by Berezin [76], Faddeev [311], Faddeev and Slavnov [313], and other authors. For a review, see Klauder and Skagerstam [602]. In the context of supersymmetric quantum mechanics see also Berezin et al. [77–78], DeWitt [236], and Singh and Steiner [855].

To be specific, we restrict ourselves to a Fermi system with a single spin variable. Generalization to many degrees of freedom is straightforward. Consider a spin-$\frac{1}{2}$ particle which is described by the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ with $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$, $[\sigma_x, \sigma_y] = 2\,\mathrm{i}\,\sigma_z$ and cyclic permutations, $\{\sigma_x, \sigma_y\} := \sigma_x\sigma_y + \sigma_y\sigma_x = 0$ and cyclic permutations. With $\sigma_\pm := (\sigma_x \pm \mathrm{i}\,\sigma_y)/2$ we can define fermionic annihilation and creation operators, respectively, by

$$\underset{\sim}{a} := \sigma_+ \ , \qquad \underset{\sim}{a}^\dagger := \sigma_- \ , \tag{2.6.1}$$

which obey the *fermionic anticommutation relations*

$$\{\underset{\sim}{a}, \underset{\sim}{a}^\dagger\} = \mathbb{1} \ , \qquad \underset{\sim}{a}^2 = (\underset{\sim}{a}^\dagger)^2 = 0 \ . \tag{2.6.2}$$

Notice that we have $[\underset{\sim}{a}, \underset{\sim}{a}^\dagger] = \sigma_z$. As in the bosonic case, we shall construct a Hilbert space of "entire functions" (the related mathematical theory of the corresponding functional analysis is also called *superanalysis*, e.g. [77, 236, 428, 724, 803] and references therein), where, however, the rôle of the complex numbers z and z^* (see Sect. 2.5.1) is now played by anticommuting variables, so-called *Grassmann variables* η and $\bar{\eta}$ satisfying

$$\{\eta, \bar{\eta}\} = 0 \ , \qquad \eta^2 = \bar{\eta}^2 = 0 \ . \tag{2.6.3}$$

The most general function of the two variables η and $\bar{\eta}$ has the polynomial form

$$f(\bar{\eta}, \eta) = c_0 + c_1\bar{\eta} + c_2\eta + c_3\bar{\eta}\eta \tag{2.6.4}$$

(c_i are complex numbers). We define derivatives ∂ and $\bar{\partial}$ with respect to η and $\bar{\eta}$, respectively, by

$$\begin{aligned} \partial f(\bar{\eta},\eta) &= c_2 - c_3\bar{\eta} \ , \\ \bar{\partial} f(\bar{\eta},\eta) &= c_1 + c_3\eta \ , \end{aligned} \tag{2.6.5}$$

i.e., ∂ suppresses η, while $\bar{\partial}$ suppresses $\bar{\eta}$ after having brought the relevant variable to the left. The *coherent state representation of a fermionic state* $|\psi\rangle$ is defined as the "entire function"

$$\psi(\bar{\eta}) = c_0 + c_1\bar{\eta} \ , \tag{2.6.6}$$

while the complex conjugate is given by

$$\bar{\psi}(\eta) = c_0^* + c_1^*\eta \ . \tag{2.6.7}$$

The scalar product in this representation is defined by

$$\langle\phi|\psi\rangle = d_0^* c_0 + d_1^* c_1 \tag{2.6.8}$$

if $\phi(\bar{\eta}) = d_0 + d_1\bar{\eta}$. In order to derive a fermionic path integral, we have to express the scalar product in terms of an integral over Grassmann variables. The *Berezin integrals* [74–78] are introduced as follows

$$\int \mathrm{d}\bar{\eta}\,\bar{\eta} = \int \mathrm{d}\eta\,\eta = 1 \ , \qquad \int \mathrm{d}\bar{\eta}\,1 = \int \mathrm{d}\eta\,1 = 0 \ , \tag{2.6.9}$$

where $\mathrm{d}\eta$ and $\mathrm{d}\,\bar{\eta}$ anticommute. This gives for a general function $f(\bar{\eta},\eta)$

$$\left.\begin{aligned} \int \mathrm{d}\eta\, f(\bar{\eta},\eta) &= \partial f(\bar{\eta},\eta) \ , \\ \int \mathrm{d}\bar{\eta}\, f(\bar{\eta},\eta) &= \bar{\partial} f(\bar{\eta},\eta) \ , \\ \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\, f(\bar{\eta},\eta) &= \bar{\partial}\partial f(\bar{\eta},\eta) \ . \end{aligned}\right\} \tag{2.6.10}$$

With $\mathrm{e}^{-\bar{\eta}\eta} = 1 - \bar{\eta}\eta$ we obtain

$$\begin{aligned} \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{e}^{-\bar{\eta}\eta}\,\bar{\phi}(\eta)\psi(\bar{\eta}) &= \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,(1-\bar{\eta}\eta)(d_0^* + d_1^*\eta)(c_0 + c_1\bar{\eta}) \\ &= \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,(d_1^* c_1 \eta\bar{\eta} - d_0^* c_0 \bar{\eta}\eta) \\ &= d_1^* c_1 + d_0^* c_0 = \langle\phi|\psi\rangle \ . \end{aligned} \tag{2.6.11}$$

The Fock space of our spin-$\frac{1}{2}$ particle consists only of the two states $|0\rangle$ and $|1\rangle$ with $\underset{\sim}{a}|0\rangle = 0$, $|1\rangle = \underset{\sim}{a}^\dagger|0\rangle$ to which we associate in the fermionic coherent state representation the two basis functions

$$f_0(\eta) = 1 \ , \qquad f_1(\eta) = \eta \ , \tag{2.6.12}$$

and the *fermionic coherent state*

$$|\eta\rangle = \sum_{n=0,1} f_n(\eta)|n\rangle = |0\rangle + \eta|1\rangle \tag{2.6.13}$$

such that $\psi(\bar\eta)$ is given for a general Fock state $|\psi\rangle = \sum_{n=0,1} c_n|n\rangle$ by

$$\begin{aligned}\psi(\bar\eta) &:= \langle\eta|\psi\rangle = \sum_{n=0,1} c_n\langle\eta|n\rangle \\ &= \sum_{n=0,1} c_n\bar f_n(\eta) = c_0 + c_1\bar\eta \ .\end{aligned} \tag{2.6.14}$$

With these definitions we find

$$\begin{aligned}(\underset{\sim}{a}^\dagger\psi)(\bar\eta) &:= \langle\eta|\underset{\sim}{a}^\dagger|\psi\rangle = \sum_{n=0,1} c_n\langle\eta|\underset{\sim}{a}^\dagger|n\rangle \\ &= c_0\langle\eta|\underset{\sim}{a}^\dagger|0\rangle = c_0\langle\eta|1\rangle = c_0\bar\eta \\ &= \bar\eta\psi(\bar\eta)\end{aligned} \tag{2.6.15}$$

and similarly

$$\begin{aligned}(\underset{\sim}{a}\psi)(\bar\eta) &:= \langle\eta|\underset{\sim}{a}|\psi\rangle = c_1\langle\eta|\underset{\sim}{a}|1\rangle \\ &= c_1\langle\eta|0\rangle = c_1 = \bar\partial\psi(\bar\eta) \ ,\end{aligned} \tag{2.6.16}$$

which shows that the operator $\underset{\sim}{a}^\dagger$ acts in the space of "entire functions" $\psi(\bar\eta)$ by multiplication by $\bar\eta$, while $\underset{\sim}{a}$ acts by $\bar\partial$ (see the close analogy with (2.5.28) and (2.5.29) in the bosonic case).

An arbitrary operator $\underset{\sim}{A} = A(\underset{\sim}{a}, \underset{\sim}{a}^\dagger)$ in Fock space can be written as

$$\underset{\sim}{A} = \sum_{m,n=0,1} |m\rangle A_{mn}\langle n| \ , \qquad A_{mn} = \langle m|\underset{\sim}{A}|n\rangle \ . \tag{2.6.17}$$

Its matrix representation is given by

$$A(\bar\eta, \eta') := \langle\eta|\underset{\sim}{A}|\eta'\rangle = \sum_{m,n=0,1} \bar\eta^m A_{mn}\eta'^n \tag{2.6.18}$$

and plays the rôle of the integral kernel representing the action of the operator $\underset{\sim}{A}$ on $\psi(\bar\eta)$:

$$(\underset{\sim}{A}\psi)(\bar\eta) = \int \mathrm{d}\bar\eta'\,\mathrm{d}\eta'\,\mathrm{e}^{-\bar\eta'\eta'}\,A(\bar\eta, \eta')\psi(\bar\eta') \ . \tag{2.6.19}$$

The product of two operators $\underset{\sim}{A}$ and $\underset{\sim}{B}$ possesses the matrix representation

$$\begin{aligned}(\underset{\sim}{A}\underset{\sim}{B})(\bar\eta, \eta') &= \langle\eta|\underset{\sim}{A}\underset{\sim}{B}|\eta'\rangle \\ &= \int \mathrm{d}\bar\eta''\,\mathrm{d}\eta''\,\mathrm{e}^{-\bar\eta''\eta''}\,A(\bar\eta, \eta'')B(\bar\eta'', \eta') \ .\end{aligned} \tag{2.6.20}$$

If an operator $\underset{\sim}{A}$ is given in normal ordered form

$$\underset{\sim}{A} = \sum_{k,l=0,1} c_{kl}(\underset{\sim}{a}^\dagger)^k(\underset{\sim}{a})^l \ , \tag{2.6.21}$$

we define its *normal symbol* by

$$A^N(\bar\eta, \eta') := \sum_{k,l=0,1} c_{kl}(\bar\eta)^k(\eta')^l \ . \tag{2.6.22}$$

It is then easy to see that the relation between the kernel (2.6.18) and its corresponding normal symbol is given by

$$A(\bar\eta, \eta') = e^{\bar\eta\eta'} A^N(\bar\eta, \eta') \ . \tag{2.6.23}$$

Finally let us mention the formula for the N-dimensional *fermionic Gaussian integral* for N independent Grassmann variables $\eta_n, \bar\eta_n$, and Grassmann "sources" $J_n, \bar J_n$

$$\int \prod_{k=1}^N d\bar\eta_k\, d\eta_k \exp\left[-\sum_{m,n=1}^N \bar\eta_m A_{mn}\eta_n + \sum_{n=1}^N(\bar\eta_n J_n + \bar J_n \eta_n)\right]$$
$$= \det \mathbf{A}\cdot \exp\left(\sum_{m,n=1}^N \bar J_m A^{-1}_{mn} J_n\right) \ . \tag{2.6.24}$$

This should be compared with the corresponding bosonic Gaussian integral (2.5.74). Notice, in particular, the different powers of $\det \mathbf{A}$ in the bosonic and fermionic formulæ, respectively.

2.6.2 The Fermionic Path Integral. One observes that the properties of the fermionic coherent states discussed in Sect. 2.6.1 are very similar to the bosonic case discussed in Sect. 2.5.1. Based on this close analogy with the bosonic situation, one easily derives the fermionic path integral for the corresponding time-evolution kernel

$$U(\bar\eta'', \eta'; t'', t') := \langle \eta''|\underset{\sim}{U}(t'', t')|\eta'\rangle \ . \tag{2.6.25}$$

The result is the following *fermionic path integral* [311, 855]

$$U(\bar\eta'', \eta'; t'', t') = \int_{\eta(t')=\eta'}^{\bar\eta(t'')=\bar\eta''} \mathcal{D}\bar\eta(t)\mathcal{D}\eta(t)$$
$$\times \exp\left[\bar\eta''\eta(t'') + \frac{i}{\hbar}\int_{t'}^{t''}\Big(i\hbar\bar\eta(t)\dot\eta(t) - H(\bar\eta(t), \eta(t); t)\Big)dt\right]$$
$$= \lim_{N\to\infty}\int d\bar\eta_{N-1}\, d\eta_{N-1}\dots d\bar\eta_1\, d\eta_1$$

$$\times \exp\left[\bar{\eta}_N \eta_N - \sum_{k=1}^{N} \left(\bar{\eta}_k(\eta_k - \eta_{k-1}) + \frac{\mathrm{i}\,\epsilon}{\hbar} H(\bar{\eta}_k, \eta_{k-1}; t_k)\right)\right] . \tag{2.6.26}$$

Here $\bar{\eta}_k$ and η_k denote Grassmann variables satisfying $\{\eta_k, \eta_l\} = \{\bar{\eta}_k, \bar{\eta}_l\} = \{\bar{\eta}_k, \eta_l\} = 0$ for all k, l; $\bar{\eta}_k = \bar{\eta}(t_k)$, etc., $t_k = t' + \epsilon k$, $\epsilon = (t'' - t')/N$. The boundary conditions are imposed by requiring $\eta(t)$ to be fixed at $t = t'$, $\eta_0 = \eta(t') = \eta'$, and $\bar{\eta}(t)$ to be fixed at $t = t''$, $\bar{\eta}_N = \bar{\eta}(t'') = \bar{\eta}''$. Furthermore $H(\bar{\eta}, \eta; t)$ is obtained from the Hamiltonian $H(\underset{\sim}{a}^\dagger, \underset{\sim}{a}; t)$ given in normal ordered form by replacing the fermion creation and annihilation operators as $\underset{\sim}{a}^\dagger \to \bar{\eta}$, $\underset{\sim}{a} \to \eta$.

2.6.3 The Path Integral for a Spin-$\frac{1}{2}$ Particle in a Time-Dependent Magnetic Field Coupled to Two External Sources. Let us consider the motion of a spin-$\frac{1}{2}$ particle in a time-dependent magnetic field $B(t)$ along the z axis coupled to two external time-dependent Grassmann sources $J(t), \bar{J}(t)$ characterized by the Hamiltonian ($\hbar = 1$)

$$\underset{\sim}{H} = -\frac{1}{2} B(t)\sigma_z - \bar{J}(t)\sigma_+ - \sigma_- J(t) . \tag{2.6.27}$$

With $\sigma_+ = \underset{\sim}{a}, \sigma_- = \underset{\sim}{a}^\dagger$ and $\sigma_z = [\underset{\sim}{a}, \underset{\sim}{a}^\dagger]$ one obtains the following normal ordered form of the Hamiltonian (2.6.27)

$$H(\underset{\sim}{a}^\dagger, \underset{\sim}{a}; t) = B(t)(\underset{\sim}{a}^\dagger \underset{\sim}{a} - \tfrac{1}{2}\,\mathbb{1}) - \bar{J}(t)\underset{\sim}{a} - \underset{\sim}{a}^\dagger J(t) \tag{2.6.28}$$

in terms of the fermionic operators $\underset{\sim}{a}, \underset{\sim}{a}^\dagger$. Inserting the corresponding kernel $H(\bar{\eta}, \eta; t)$ into the path integral (2.6.26) and carrying out the integrations at every lattice point we obtain [855]

$$\begin{aligned} U(\bar{\eta}'', \eta'; t'', t') = \lim_{N\to\infty} \exp&\left[\bar{\eta}_N \prod_{k=1}^{N}(1 - \mathrm{i}\,\epsilon B_k)\eta_0\right. \\ &+ \mathrm{i}\,\bar{\eta}_N \sum_{k=1}^{N} \epsilon J_k \prod_{l=k+1}^{N}(1 - \mathrm{i}\,\epsilon B_l) + \mathrm{i}\sum_{k=1}^{N} \epsilon \bar{J}_k \prod_{l=1}^{k-1}(1 - \mathrm{i}\,\epsilon B_l)\eta_0 \\ &\left.- \sum_{k=2}^{N} \epsilon \bar{J}_k \sum_{l=1}^{k-1} \epsilon J_l \prod_{m=l+1}^{k-1}(1 - \mathrm{i}\,\epsilon B_m)\right] \exp\left(\frac{\mathrm{i}}{2}\int_{t'}^{t''} B(t)\,\mathrm{d}t\right) \end{aligned} \tag{2.6.29}$$

just by using the standard Gaussian integration rule (2.6.24) for the Grassmann variables. The continuum limit of (2.6.29) can be easily written down as

$$U(\bar{\eta}'', \eta'; t'', t') = \exp\left\{\bar{\eta}'' \exp\left(-\mathrm{i}\int_{t'}^{t''} B(t)\,\mathrm{d}t\right)\eta'\right.$$

$$+\mathrm{i}\bar{\eta}''\int_{t'}^{t''}\mathrm{d}t\,J(t)\exp\left(-\mathrm{i}\int_{t'}^{t''}\Theta(s-t)B(s)\,\mathrm{d}s\right)$$
$$+\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\bar{J}(t)\exp\left(-\mathrm{i}\int_{t'}^{t''}\Theta(t-s)B(s)\,\mathrm{d}s\right)\eta'$$
$$-\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t''}\mathrm{d}s\,\bar{J}(t)D_F(t,s)J(s)\Bigg\}\exp\left(\frac{\mathrm{i}}{2}\int_{t'}^{t''}B(t)\,\mathrm{d}t\right)\ . \tag{2.6.30}$$

Here we introduced the *fermionic Feynman propagator* in the presence of an external magnetic field $B(t)$

$$D_F(t,s):=\Theta(t-s)\exp\left(\mathrm{i}\int_s^t B(\tau)d\tau\right)\ . \tag{2.6.31}$$

The time-evolution kernel (2.6.30) has the following general decomposition

$$U(\bar{\eta}'',\eta';t'',t')=K_{00}+K_{11}\bar{\eta}''\eta'+\bar{\eta}''K_{10}+K_{01}\eta'\ , \tag{2.6.32}$$

where the coefficients K_{mn} are given by

$$K_{00}=\exp\left(\frac{\mathrm{i}}{2}\int_{t'}^{t''}B(t)\,\mathrm{d}t\right)\exp\left(-\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t''}\mathrm{d}s\,\bar{J}(t)D_F(t,s)J(s)\right)\ , \tag{2.6.33a}$$

$$K_{11}=\exp\left(-\mathrm{i}\int_{t'}^{t''}B(t)\,\mathrm{d}t\right)K_{00}\ , \tag{2.6.33b}$$

$$K_{10}=\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,J(t)\exp\left(-\mathrm{i}\int_{t}^{t''}B(s)\mathrm{d}s\right)K_{00}\ , \tag{2.6.33c}$$

$$K_{01}=\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\bar{J}(t)\exp\left(-\mathrm{i}\int_{t'}^{t}B(s)\mathrm{d}s\right)K_{00}\ . \tag{2.6.33d}$$

The K_{mn} are actually the matrix elements of the time-evolution operator in the two-dimensional Fock space spanned by the two vectors $|0\rangle$ and $|1\rangle$

$$K_{mn}=\left\langle m\left|T\exp\left(-\mathrm{i}\int_{t'}^{t''}H(\underline{a}^\dagger,\underline{a};t)\,\mathrm{d}t\right)\right|n\right\rangle\ . \tag{2.6.34}$$

Notice that the *generating functional* (=vacuum persistence amplitude) for the pure fermionic system is given by

$$Z_F[J,\bar{J}]=\frac{K_{00}}{K_{00}|_{J=\bar{J}=0}}=\exp\left(-\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t''}\mathrm{d}s\,\bar{J}(t)D_F(t,s)J(s)\right)\ . \tag{2.6.35}$$

2.6.4 Supersymmetric Quantum Mechanics. In supersymmetric (SUSY) quantum mechanics [379, 724, 934] one considers the Hamiltonian (units $\hbar = 2m = 1$ are used in this section)

$$\underset{\sim}{H}^{\mathrm{SUSY}} = \frac{1}{2}\underset{\sim}{p}^2 + \frac{1}{2}\left[V(\underset{\sim}{q})\right]^2 + \frac{1}{2}[\underset{\sim}{a}^\dagger, \underset{\sim}{a}]V'(\underset{\sim}{q}) \tag{2.6.36}$$

which corresponds to the "quantum Lagrangian"

$$\mathcal{L}^{\mathrm{SUSY}} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\left[V(q)\right]^2 + \mathrm{i}\,\underset{\sim}{a}^\dagger\dot{\underset{\sim}{a}} - \frac{1}{2}[\underset{\sim}{a}^\dagger, \underset{\sim}{a}]V'(q) \ , \tag{2.6.37}$$

where $\underset{\sim}{a}$ and $\underset{\sim}{a}^\dagger$ are fermionic operators. The potential $V(q)$ is called the superpotential. For the choice $V(q) = q$ the above system describes two non-interacting bosonic and fermionic oscillators. Notice that the fermionic part in the Hamiltonian (2.6.36) can be identified with the Hamiltonian (2.6.27) of a spin-$\frac{1}{2}$ particle in a time-dependent magnetic field $B(t) = V'(q(t))$ without external sources.

If one wants to calculate the path integral for SUSY quantum mechanics, one first has to compute the fermionic path integral for the path-dependent "magnetic field" $V'(q(t))$ and then in the second step to integrate the result over the bosonic degrees of freedom. In such calculations, the commutator in the Lagrangian (2.6.37) is usually replaced by $\frac{1}{2}[\underset{\sim}{a}^\dagger, \underset{\sim}{a}] \to \bar{\psi}\psi$ treating $\psi, \bar{\psi}$ as "classical" Grassmann variables, and the fermionic path integral is set equal to the *fermion determinant* (see (2.6.24))

$$\int \mathcal{D}\bar{\psi}(t)\mathcal{D}\psi(t)\exp\left[\mathrm{i}\int_{t'}^{t''}\bar{\psi}\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V'(q(t))\right)\psi\,\mathrm{d}t\right]$$
$$\equiv \det\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V'(q(t))\right) \ . \tag{2.6.38}$$

However, the correct replacement [855] of the commutator to be used in the fermionic path integral is $\frac{1}{2}[\underset{\sim}{a}^\dagger, \underset{\sim}{a}] \to \bar{\psi}\psi - \frac{1}{2}$. In the literature, (2.6.38) is essentially used as the defining equation of the fermion determinant which is later evaluated *not* from the path integral but by solving an eigenvalue problem with appropriate boundary conditions. Since the determinant is finally normalized by hand, one arrives at the correct kernel even though the correct quantum Lagrangian has not been used. Actually the path integral (2.6.38) does not stand well defined without specifying the boundary conditions. Depending on the applications one is interested in, one obtains different expressions for the fermion determinant which, however, can be exactly obtained if one uses the correct path integral (2.6.30).

If one is interested in the trace of the time-evolution operator [205, 391], then we easily obtain [855] from the path integral (2.6.30) putting $J = \bar{J} = 0$ and $B(t) = V'(q(t))$

$$\begin{aligned}
\det\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V'(q(t))\right)_{\text{trace}} &:= \mathrm{Tr}\left[T\exp\left(-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\underset{\sim}{\mathrm{H}}_F\right)\right] \\
&= [K_{00}+K_{11}]_{J=\bar{J}=0} \\
&= \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{e}^{-\bar{\eta}\eta}\,U(-\bar{\eta},\eta;t'',t')|_{J=\bar{J}=0} \\
&= 2\cos\left[\frac{1}{2}\int_{t'}^{t''}\mathrm{d}t\,V'(q(t))\right] ,
\end{aligned} \tag{2.6.39}$$

where $\underset{\sim}{\mathrm{H}}_F$ is the fermionic part of the Hamiltonian (2.6.36). Notice that the argument $-\bar{\eta}$ of U in (2.6.39) is the origin of taking *antiperiodic boundary conditions* in the evaluation of the fermionic determinant in the earlier works. The Euclidean version of (2.6.39) is identical to the result obtained in [205, 391].

The trace of the pure fermionic system in the case of a constant magnetic field $B(t) = 2\omega > 0$ can be easily obtained from (2.6.39)

$$\mathrm{Tr}\left[T\exp\left(-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\underset{\sim}{\mathrm{H}}_F\right)\right] = 2\cos(\omega T) = \mathrm{e}^{\mathrm{i}\,\omega T} + \mathrm{e}^{-\mathrm{i}\,\omega T} \tag{2.6.40}$$

leading to the identification of two energy levels with energy $E_0 = -\omega$ and $E_1 = \omega$ as is expected for a spin-$\frac{1}{2}$ particle in the presence of a constant magnetic field.

If the path integral (2.6.38) is used to define the generating functional as is the case with the *Nicolai map* [724, 855], it must be interpreted as the vacuum-vacuum transition amplitude, and thus we obtain

$$\begin{aligned}
\det\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - V'(q(t))\right)_{\text{vacuum}} &:= K_{00}|_{J=\bar{J}=0} \\
&= -\int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{e}^{-\bar{\eta}\eta}\,\bar{\eta}U(\bar{\eta},\eta;t'',t')|_{J=\bar{J}=0}\eta \\
&= \exp\left[\frac{\mathrm{i}}{2}\int_{t'}^{t''}\mathrm{d}t\,V'(q(t))\right] .
\end{aligned} \tag{2.6.41}$$

This result for the fermion determinant turns out to be exactly the inverse of the bosonic Jacobian for the Nicolai map as evaluated by Ezawa and Klauder [308, 309] using the Stratonovich prescription. One therefore obtains exactly the cancelation of the fermion determinant and the bosonic Jacobian under the Nicolai map.

Finally, let us discuss the *Witten index* [935] $\Delta := \mathrm{Tr}(-\mathbb{1})^{\underset{\sim}{\mathrm{F}}}$, where $(-\mathbb{1})^{\underset{\sim}{\mathrm{F}}} = \mathbb{1} - 2\underset{\sim}{a}^\dagger\underset{\sim}{a}$ is the fermion-number operator, and which has been introduced as a measure of supersymmetry breaking. The regularized version is defined as follows ($\beta > 0$)

$$\Delta(\beta) := \mathrm{Tr}\left[(-\mathbb{1})^{\underset{\sim}{\mathrm{F}}}\mathrm{e}^{-\beta\underset{\sim}{\mathrm{H}}^{\mathrm{SUSY}}}\right]$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x \int_{q(0)=x}^{q(\beta)=x} \mathcal{D}q(s) \exp\left\{ -\frac{1}{2}\int_0^\beta \left[\dot{q}^2 + V^2(q(s))\right]\mathrm{d}s \right\} \operatorname{Tr}\left[(-\mathbb{1})^{\underline{\mathrm{F}}} \mathrm{e}^{-\beta \underline{\mathrm{H}}_F}\right], \tag{2.6.42}$$

where the trace over the bosonic degrees of freedom has been converted to the path integral form. The remaining trace over the fermionic degrees of freedom is immediately obtained from (2.6.33) as

$$\begin{aligned}
\operatorname{Tr}\left[(-\mathbb{1})^{\underline{\mathrm{F}}} \mathrm{e}^{-\beta \underline{\mathrm{H}}_F}\right] &\equiv \det\left(\frac{\mathrm{d}}{\mathrm{d}s} + V'(q(s))\right)_{\text{S-trace}} \\
&= [K_{00}^E - K_{11}^E]_{J=\bar{J}=0} \\
&= \int \mathrm{d}\bar{\eta}\,\mathrm{d}\eta\,\mathrm{e}^{-\bar{\eta}\eta}\, U(\bar{\eta},\eta;-\mathrm{i}\beta,0)|_{J=\bar{J}=0} \\
&= 2\sinh\left[\frac{1}{2}\int_0^\beta \mathrm{d}s\, V'(q(s))\right] .
\end{aligned} \tag{2.6.43}$$

Here the superscript E stands for Euclidean form and S-trace for supertrace. It is to be noted that in this case we have kept the arguments of U unchanged which implements *periodic boundary conditions* in calculating the determinant. Inserting (2.6.43) into (2.6.42) the complete expression for the Witten index is obtained.

2.7 The Path Integral in Spherical Coordinates

We consider the D-dimensional path integral (2.2.20) ($\mathbf{x} \in \mathbb{R}^D$)

$$\begin{aligned}
K(\mathbf{x}'',\mathbf{x}';T) &= \int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\mathrm{d}t\right] \\
&= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2} \prod_{k=1}^{N-1}\int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}_k \\
&\quad \times \prod_{j=1}^{N} \exp\left[\frac{\mathrm{i}m}{2\epsilon\hbar}\left(\mathbf{x}_j^2 + \mathbf{x}_{j-1}^2 - 2\mathbf{x}_j\cdot\mathbf{x}_{j-1}\right) - \frac{\mathrm{i}\epsilon}{\hbar}V(\mathbf{x}_j)\right] .
\end{aligned} \tag{2.7.1}$$

(Notice that in the second product j has been replaced by $j-1$, but that instead of $V(\mathbf{x}_{j-1})$ we still write $V(\mathbf{x}_j)$ which makes, however, no difference under the path integral; indeed one could also use a more symmetric formulation where $V(\mathbf{x}_j)$ is replaced by $(V(\mathbf{x}_j) + V(\mathbf{x}_{j-1}))/2$.)

Let $V(\mathbf{x})$ be a function of $|\mathbf{x}|$ only, $V(\mathbf{x}) = V(|\mathbf{x}|)$, and introduce D-dimensional spherical coordinates

$$\left.\begin{aligned} x^1 &= r\cos\vartheta^1 \\ x^2 &= r\sin\vartheta^1\cos\vartheta^2 \\ x^3 &= r\sin\vartheta^1\sin\vartheta^2\cos\vartheta^3 \\ &\dots \\ x^{D-1} &= r\sin\vartheta^1\sin\vartheta^2\dots\sin\vartheta^{D-2}\cos\varphi \\ x^D &= r\sin\vartheta^1\sin\vartheta^2\dots\sin\vartheta^{D-2}\sin\varphi\ , \end{aligned}\right\} \tag{2.7.2}$$

where $0 \leq \vartheta^\nu \leq \pi$ $(\nu = 1,\dots,D-2)$, $0 \leq \varphi \equiv \vartheta^{D-1} \leq 2\pi$, $r = \big(\sum_{\nu=1}^D (x^\nu)^2\big)^{1/2} \geq 0$, thus $V(\mathbf{x}) = V(r)$. We use the addition theorem

$$\begin{aligned} \cos\psi_{(1,2)} &= \cos\vartheta_1^1\cos\vartheta_2^1 \\ &+ \sum_{m=1}^{D-2}\cos\vartheta_1^{m+1}\cos\vartheta_2^{m+1}\prod_{n=1}^{m}\sin\vartheta_1^n\sin\vartheta_2^n + \prod_{n=1}^{D-1}\sin\vartheta_1^n\sin\vartheta_2^n\ , \end{aligned} \tag{2.7.3}$$

where $\psi_{(1,2)}$ is the angle between two D-dimensional vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ so that $\mathbf{x}_1\cdot\mathbf{x}_2 = r_1r_2\cos\psi_{(1,2)}$. The metric tensor in spherical coordinates is

$$(g_{ab}) = \operatorname{diag}(1, r^2, r^2\sin^2\vartheta^1, \dots, r^2\sin^2\vartheta^1\dots\sin^2\vartheta^{D-2})\ . \tag{2.7.4}$$

If $D = 3$, (2.7.3) reduces to:

$$\cos\psi_{(1,2)} = \cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\varphi_1-\varphi_2)\ . \tag{2.7.5}$$

The D-dimensional measure $d\mathbf{x}$ expressed in spherical coordinates reads

$$d\mathbf{x} = r^{D-1}\,dr\,d\Omega\ , \qquad d\Omega = \prod_{k=1}^{D-1}(\sin\vartheta^k)^{D-1-k}\,d\vartheta^k\ . \tag{2.7.6}$$

$d\Omega$ is the $(D-1)$-dimensional surface element on the unit sphere S^{D-1} and $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$ is the volume of the unit S^{D-1}-sphere. The determinant of the metric tensor is given by

$$g := \det(g_{ab}) = \left(r^{D-1}\prod_{k=1}^{D-1}(\sin\vartheta^k)^{D-1-k}\right)^2\ . \tag{2.7.7}$$

The path integral (2.7.1) can be rewritten in spherical coordinates as follows

$$\begin{aligned} K(\mathbf{x}'',\mathbf{x}';T) &= K(r'',\Omega'',r',\Omega';T) \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int_0^\infty r_k^{D-1}dr_k\int d\Omega_k \\ &\quad\times\prod_{j=1}^{N}\exp\left[\frac{\mathrm{i}m}{2\epsilon\hbar}\big(r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos\psi_{(j,j-1)}\big)-\frac{\mathrm{i}\epsilon}{\hbar}V(r_j)\right]\ . \end{aligned} \tag{2.7.8}$$

For an explicit evaluation of the angular integrations we need the expansion formula for plane waves [413]

$$e^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\sum_{l=0}^{\infty}(l+\nu)I_{l+\nu}(z)C_l^{\nu}(\cos\psi) \ , \tag{2.7.9}$$

(valid for any $\nu \neq 0,-1,-2,\ldots$), where C_l^{ν} are Gegenbauer polynomials and I_{μ} modified Bessel functions. The addition theorem for the M linearly independent real surface (or hyperspherical) harmonics S_l^{μ} of degree l on the S^{D-1}-sphere has the form [303, Vol. II, Chap. IX]:

$$\sum_{\mu=1}^{M} S_l^{\mu}(\boldsymbol{\Omega}_1)S_l^{\mu}(\boldsymbol{\Omega}_2) = \frac{1}{\Omega(D)}\frac{2l+D-2}{D-2}C_l^{\frac{D-2}{2}}(\cos\psi_{(1,2)}) \ , \tag{2.7.10}$$

$M = (2l+D-2)(l+D-3)!/l!(D-3)!$, with the unit vector $\boldsymbol{\Omega} = \mathbf{x}/r$ in $\mathbb{R}^D$. The orthonormality relation reads

$$\int d\boldsymbol{\Omega}\, S_l^{\mu}(\boldsymbol{\Omega})S_{l'}^{\mu'}(\boldsymbol{\Omega}) = \delta_{ll'}\delta_{\mu\mu'} \ . \tag{2.7.11}$$

Combining (2.7.9) and (2.7.10) we get the expansion formula

$$e^{z(\boldsymbol{\Omega}_1\cdot\boldsymbol{\Omega}_2)} = e^{z\cos\psi_{(1,2)}} = 2\pi\left(\frac{2\pi}{z}\right)^{\frac{D-2}{2}}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M} S_l^{\mu}(\boldsymbol{\Omega}_1)S_l^{\mu}(\boldsymbol{\Omega}_2)I_{l+\frac{D-2}{2}}(z) \ . \tag{2.7.12}$$

The angular integrations can then be carried out and the path integral (2.7.8) in spherical coordinates becomes

$$\begin{aligned} &K(r'',\boldsymbol{\Omega}'',r',\boldsymbol{\Omega}';T) \\ &= (r'r'')^{-\frac{D-2}{2}}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M} S_l^{\mu}(\boldsymbol{\Omega}')S_l^{\mu}(\boldsymbol{\Omega}'')\lim_{N\to\infty}\left(\frac{m}{\mathrm{i}\epsilon\hbar}\right)^N\prod_{k=1}^{N-1}\int_0^{\infty} r_k dr_k \\ &\qquad\times\prod_{j=1}^{N}\exp\left[\frac{\mathrm{i}m}{2\epsilon\hbar}(r_j^2+r_{j-1}^2)-\frac{\mathrm{i}\epsilon}{\hbar}V(r_j)\right]I_{l+\frac{D-2}{2}}\left(\frac{m}{\mathrm{i}\epsilon\hbar}r_jr_{j-1}\right) \ . \end{aligned} \tag{2.7.13}$$

Therefore we can separate the radial part of the path integral (partial wave expansion)

$$K(r'',\boldsymbol{\Omega}'',r',\boldsymbol{\Omega}';T) = \frac{(r'r'')^{\frac{1-D}{2}}}{\Omega(D)}\sum_{l=0}^{\infty}\frac{2l+D-2}{D-2}C_l^{\frac{D-2}{2}}(\cos\psi_{(',")})K_l(r'',r';T) \tag{2.7.14}$$

with the angular-momentum dependent *radial kernel* K_l given by the *radial path integral* [865]

$$
\begin{aligned}
K_l(r'',r';T) &= \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{N/2}\prod_{n=1}^{N-1}\int_0^\infty \mathrm{d}r_n \\
&\times \mu^N_{l+\frac{D-2}{2}}[r^2]\cdot\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}(r_j-r_{j-1})^2-\epsilon V(r_j)\right)\right] \\
&\equiv \int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-V(r)\right)\mathrm{d}t\right]\ .
\end{aligned}
\tag{2.7.15}
$$

Here the non-trivial *functional weight* $\mu_\lambda[r^2]$ is defined on the lattice by

$$
\mu_\lambda^N[r^2] := \prod_{j=1}^{N}\sqrt{\frac{2\pi m r_j r_{j-1}}{\mathrm{i}\,\epsilon\hbar}}\;\mathrm{e}^{-m r_j r_{j-1}/\mathrm{i}\,\epsilon\hbar}\,I_\lambda\left(\frac{m r_j r_{j-1}}{\mathrm{i}\,\epsilon\hbar}\right)\ , \tag{2.7.16}
$$

while $\mathcal{D}r(t)$ denotes the usual one-dimensional path integration "measure" as introduced in (2.2.6), but with $r(t)$ restricted, of course, to the half-line $r(t)\in\mathbb{R}_+$.

It should be noted that the dependence of the radial kernel K_l on the angular momentum l is completely governed by the functional weight $\mu_{l+(D-2)/2}$, i.e., there is no centrifugal potential $\hbar^2(l+\frac{D-3}{2})(l+\frac{D-1}{2})/2mr^2$ in the radial action (contrary to the naive expectation), since only the *radial action*

$$
R[r(t)] = \int_{t'}^{t''}\left(\frac{m}{2}\dot r^2(t)-V(r(t))\right)\mathrm{d}t
$$

without centrifugal potential enters the path integral (2.7.15). (For more details on this point, we refer to [865].)

It is seen from (2.7.15) that also the dependence on the dimension D is determined by the functional weight, where D enters only in the combination $l+(D-2)/2$. Thus it follows that it is sufficient to consider the radial kernel for $D=3$ only, since the general case is simply obtained from the three-dimensional kernel by the substitution $l\to l+(D-3)/2$.

Radial path integrals have been studied by Arthurs [33-34], Böhm and Junker [104], Duru [274], Duru and Ünal [282], Edwards and Gulyaev [291], Gerry and Singh [388] (scattering theory), Grosche and Steiner [464], Kapoor [563], Kleinert [609], Langguth and Inomata [633], Peak and Inomata [771], and Steiner [865]. Edwards and Gulyaev discussed the two- and three-dimensional cases, whereas Arthurs concentrated on $D=2$, and Peak and Inomata calculated the path integral for the radial harmonic oscillator including some simple applications. The above derivation of the D-dimensional path integral has been given in [865] (see also [464]). A formulation of path integrals in spherical coordinates due to LaChapelle [628] which is based on an axiomatic formulation of path integrals, cf. Cartier and DeWitt–Morette [147], is, however, not discussed here.

2.8 The Path Integral in General Coordinates

2.8.1 Ordering Prescriptions. In the previous sections we have seen how to construct the Feynman path integral in Cartesian and spherical coordinates, respectively. Naturally, the question arises of how to define and construct a path integral representation in a Riemannian space $\mathbb{M}$ with a coordinate-dependent metric, i.e., for a physical system with the generic classical Lagrangian $\mathcal{L}(\mathbf{q},\dot{\mathbf{q}}) = (m/2)g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q})$ (sums over repeated indices understood). It is well known that a simple replacement of the coordinates q^a and generalized momenta $p_a = \partial\mathcal{L}/\partial\dot{q}^a$ in the corresponding classical Hamiltonian $H(\mathbf{p},\mathbf{q}) = g^{ab}(\mathbf{q})p_a p_b/2m + V(\mathbf{q})$ by their quantum mechanical counterparts is in general not correct since ordering ambiguities exist which must be resolved. (Here g^{ab} denotes the inverse of g_{ab}, i.e., $g^{ac}g_{cb} = \delta^a_b$.)

Let us start with the generic case, where the time-dependent Schrödinger equation on some D-dimensional curved Riemannian manifold $\mathbb{M}$ with metric g_{ab} and line element $\mathrm{d}s^2 = g_{ab}(\mathbf{q})\,\mathrm{d}q^a\,\mathrm{d}q^b$ is given by

$$\underline{\mathrm{H}}\Psi(\mathbf{q},t) := \left[-\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}} + V(\mathbf{q})\right]\Psi(\mathbf{q},t) = \mathrm{i}\,\hbar\frac{\partial}{\partial t}\Psi(\mathbf{q},t) \ , \tag{2.8.1}$$

and where we take (2.8.1) as the *definition* of the quantum theory in the curved space $\mathbb{M}$ [779]. Here m denotes the mass of the particle which moves in the potential $V(\mathbf{q})$. Ψ is the wave function defined in the Hilbert space $\mathcal{L}^2(\mathbb{M})$ – the space of all square integrable functions with respect to the scalar product $(f_1, f_2) = \int_{\mathbb{M}}\sqrt{g}\,f_1^*(\mathbf{q})f_2(\mathbf{q})\,\mathrm{d}\mathbf{q}$ [$g := \det(g_{ab})$, $f_1, f_2 \in \mathcal{L}^2(\mathbb{M})$], and Δ_{LB} is the *Laplace–Beltrami operator* on $\mathbb{M}$

$$\Delta_{\mathrm{LB}} := g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b = g^{ab}\partial_a\partial_b + (g^{ab}\Gamma_a + g^{ab}{}_{,a})\partial_b \ . \tag{2.8.2}$$

It is well known that in general the corresponding momentum operators are **not** given by the simple expressions $\underline{\mathrm{p}}_a = (\hbar/\mathrm{i})\partial/\partial q^a = (\hbar/\mathrm{i})\partial_a$, but instead one has to introduce [464, 736, 764]

$$\underline{\mathrm{p}}_a = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2}\right) \ , \qquad \Gamma_a := \frac{\partial\ln\sqrt{g}}{\partial q^a} \ , \tag{2.8.3}$$

which are the correct Hermitian operators with respect to the above scalar product in the Hilbert space $\mathcal{L}^2(\mathbb{M})$. It is obvious that the relation between the Laplace–Beltrami operator (2.8.2) expressed in terms of the Hermitian operators (2.8.3) and the classical Hamiltonian is in general not so simple because of the operator ordering problem.

This problem can be formulated in a systematic way, e.g. [637], as will be discussed below. Let us consider first, however, the simplest and most discussed ordering prescription, namely the so-called *Weyl ordering*. For a product of powers of coordinate and momentum operators it is defined as

$$(\underset{\sim}{q}^n \underset{\sim}{p}^m)_{\rm Weyl} = \left(\frac{1}{2}\right)^n \sum_{l=0}^{n} \binom{n}{l} \underset{\sim}{q}^{n-l} \underset{\sim}{p}^m \underset{\sim}{q}^l \ , \tag{2.8.4}$$

with the obvious generalization

$$(g^{ab}(\underset{\sim}{q})\underset{\sim}{p}_a\underset{\sim}{p}_b)_{\rm Weyl} = \frac{1}{4}[g^{ab}(\underset{\sim}{q})\,\underset{\sim}{p}_a\underset{\sim}{p}_b + 2\underset{\sim}{p}_a\, g^{ab}(\underset{\sim}{q})\,\underset{\sim}{p}_b + \underset{\sim}{p}_a\underset{\sim}{p}_b\, g^{ab}(\underset{\sim}{q})] \ . \tag{2.8.5}$$

If we use the Weyl ordering (2.8.5) to define a corresponding quantum Hamiltonian $\underset{\sim}{H}$ for our classical Hamiltonian $H(\mathbf{p},\mathbf{q}) = g^{ab}(\mathbf{q})p_a p_b/2m + V(\mathbf{q})$, we find, however, that the operator obtained in this way is not equal to our definition (2.8.1) but rather differs from it by a so-called quantum potential $\Delta V_{\rm Weyl}(\mathbf{q}) \propto \hbar^2$. Explicitly, we find

$$\begin{aligned} \underset{\sim}{H} &:= -\frac{\hbar^2}{2m}\Delta_{\rm LB} + V(\mathbf{q}) \\ &= \frac{1}{8m}[g^{ab}(\mathbf{q})\,\underset{\sim}{p}_a\underset{\sim}{p}_b + 2\underset{\sim}{p}_a\, g^{ab}(\mathbf{q})\,\underset{\sim}{p}_b + \underset{\sim}{p}_a\underset{\sim}{p}_b\, g^{ab}(\mathbf{q})] + V(\mathbf{q}) + \Delta V_{\rm Weyl}(\mathbf{q}) \ , \end{aligned} \tag{2.8.6}$$

where the *quantum potential* turns out to be given by [464]

$$\Delta V_{\rm Weyl} = \frac{\hbar^2}{8m}(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd} - R) = \frac{\hbar^2}{8m}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + 2g^{ab}{}_{,ab}\right] \ . \tag{2.8.7}$$

(It should be pointed out that here we have used the momentum operators $\underset{\sim}{p}_a$ given in (2.8.3).) Here R denotes the scalar curvature, and the Γ^i_{jk} are the Christoffel symbols

$$R = g^{ij}\left(\Gamma^l_{lj,l} - \Gamma^l_{ij,l} + \Gamma^l_{mj}\Gamma^m_{il} - \Gamma^l_{ij}\Gamma^m_{ml}\right) \ , \tag{2.8.8}$$

$$\Gamma^i_{jk} = \frac{1}{2}g^{ia}\left(g_{ja,k} + g_{ka,j} - g_{ik,a}\right) \ . \tag{2.8.9}$$

For a general discussion of the operator-ordering problem, let us consider a monomial in coordinates and momenta (classical quantities)

$$M(n,m) = q^{\mu_1}\dots q^{\mu_n} p_{\nu_1}\dots p_{\nu_m} \ , \tag{2.8.10}$$

and *define* a *correspondence rule* according to

$$\exp\Big[\mathrm{i}(\mathbf{u}\cdot\mathbf{q}+\mathbf{v}\cdot\mathbf{p})\Big] \to D_\Omega(\mathbf{u},\mathbf{v};\underset{\sim}{\mathbf{q}},\underset{\sim}{\mathbf{p}}) \equiv \Omega(\mathbf{u},\mathbf{v})\exp\Big[\mathrm{i}(\mathbf{u}\cdot\underset{\sim}{\mathbf{q}}+\mathbf{v}\cdot\underset{\sim}{\mathbf{p}})\Big] \tag{2.8.11}$$

to generate operators $\underset{\sim}{\mathbf{q}}, \underset{\sim}{\mathbf{p}}$ from coordinates and momenta $\mathbf{q},\mathbf{p}$. ($\mathbf{u}$ and $\mathbf{v}$ are given vectors.) The correspondence rule is a linear mapping from classical phase space functions into operators and is completely characterized by the way it acts on the generating function $\exp[\mathrm{i}(\mathbf{uq}+\mathbf{vp})]$. This implies for the monomial $M(n,m)$ the mapping

$$M(n,m) \to \frac{1}{\mathrm{i}^{n+m}} \frac{\partial^{n+m} D_\Omega(\mathbf{u},\mathbf{v};\underline{\mathrm{q}},\underline{\mathrm{p}})}{\partial u_{\mu_1}\dots\partial u_{\mu_n}\partial v^{\nu_1}\dots\partial v^{\nu_m}}\bigg|_{\mathbf{u}=\mathbf{v}=0} \quad . \tag{2.8.12}$$

From this mapping general correspondence rules together with their corresponding ordering prescriptions can be generated by choosing a particular function $\Omega(\mathbf{u},\mathbf{v})$. In Table 2.1, cf. [361, 572, 637], some standard examples are displayed including the Weyl ordering which we have discussed before.

Table 2.1. Correspondence Rules

Correspondence Rule	$\Omega(u,v)$	Ordering Prescription
Weyl	1	$\frac{1}{2^n}\sum_{l=0}^{n}\binom{n}{l}\underline{\mathrm{q}}^{n-l}\underline{\mathrm{p}}^m\underline{\mathrm{q}}^l$
Symmetric	$\cos\frac{uv}{2}$	$\frac{1}{2}(\underline{\mathrm{q}}^n\underline{\mathrm{p}}^m+\underline{\mathrm{p}}^m\underline{\mathrm{q}}^n)$
Standard	$\exp\left(-\mathrm{i}\frac{uv}{2}\right)$	$\underline{\mathrm{q}}^n\underline{\mathrm{p}}^m$
Anti-standard	$\exp\left(\mathrm{i}\frac{uv}{2}\right)$	$\underline{\mathrm{p}}^m\underline{\mathrm{q}}^n$
Born–Jordan	$\sin\frac{uv}{2}\Big/\frac{uv}{2}$	$\frac{1}{m+1}\sum_{l=0}^{n}\underline{\mathrm{p}}^{m-l}\underline{\mathrm{q}}^n\underline{\mathrm{p}}^l$

Therefore we can obtain any *ordering prescription* from the following construction: Given a classical Hamiltonian $H(\mathbf{p},\mathbf{q})$, the corresponding quantum Hamiltonian $\underline{\mathrm{H}}$ is calculated as follows

$$\underline{\mathrm{H}}(\underline{\mathrm{p}},\underline{\mathrm{q}}) = \int \exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{u}\cdot\underline{\mathrm{p}} + \frac{\mathrm{i}}{\hbar}\mathbf{v}\cdot\underline{\mathrm{q}}\right)\Omega(\mathbf{u},\mathbf{v})\hat{H}(\mathbf{u},\mathbf{v})\,d\mathbf{u}\,d\mathbf{v} \quad , \tag{2.8.13}$$

where $\hat{H}$ is the Fourier transform of the classical Hamiltonian H

$$\hat{H}(\mathbf{u},\mathbf{v}) = \frac{1}{(2\pi\hbar)^{2D}}\int \exp\left(-\frac{\mathrm{i}}{\hbar}\mathbf{u}\cdot\mathbf{p} - \frac{\mathrm{i}}{\hbar}\mathbf{u}\cdot\mathbf{q}\right)H(\mathbf{p},\mathbf{q})\,d\mathbf{p}\,d\mathbf{q} \quad . \tag{2.8.14}$$

Obviously, choosing a particular *classical Hamiltonian* produces a particular *Hamiltonian function* which is actually given by the following matrix element, e.g. [690]

$$H_{\text{eff}}(\mathbf{p},\mathbf{q}'',\mathbf{q}') = \langle\mathbf{q}''|\underline{\mathrm{H}}(\underline{\mathrm{p}},\underline{\mathrm{q}})|\mathbf{q}'\rangle \quad . \tag{2.8.15}$$

The momentum $\mathbf{p}$ is taken at the endpoint. H_{eff} is called the *effective Hamiltonian*. For the Weyl, the symmetric-rule (SR) and anti-standard (AS) ordering, respectively, we obtain the corresponding effective Hamiltonian functions according to

$$H_{\text{eff}}(\mathbf{p},\mathbf{q}'',\mathbf{q}') = H(\mathbf{p},\tfrac{1}{2}(\mathbf{q}'+\mathbf{q}'')) + \Delta V_{\text{Weyl}}(\tfrac{1}{2}(\mathbf{q}'+\mathbf{q}'')) \ , \tag{2.8.16}$$

$$H_{\text{eff}}(\mathbf{p},\mathbf{q}'',\mathbf{q}') = \tfrac{1}{2}[H(\mathbf{p},\mathbf{q}'') + H(\mathbf{p},\mathbf{q}')] + \Delta V_{\text{SR}}(\mathbf{q}'') \ , \tag{2.8.17}$$

$$H_{\text{eff}}(\mathbf{p},\mathbf{q}'',\mathbf{q}') = H(\mathbf{p},\mathbf{q}') + \Delta V_{\text{AS}}(\mathbf{q}') \ , \tag{2.8.18}$$

with the well defined quantum potentials of order $\hbar^2$

$$\Delta V_{\text{Weyl}} = \frac{\hbar^2}{8m}(g^{ab}\Gamma^d_{ac}\Gamma^c_{bd} - R) = \frac{\hbar^2}{8m}\Big[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + 2g^{ab}{}_{,ab}\Big] \ , \tag{2.8.19}$$

$$\Delta V_{\text{SR}} = \frac{\hbar^2}{8m}\Big[g^{ab}{}_{,ab} - 2g^{-1/4}\Delta_{\text{LB}}g^{1/4}\Big] \ , \tag{2.8.20}$$

$$\Delta V_{\text{AS}} = \frac{\hbar^2}{6m}R \ . \tag{2.8.21}$$

Choosing $\Omega(\mathbf{u},\mathbf{v}) = \exp[\mathrm{i}(1-2\alpha)\mathbf{u}\cdot\mathbf{v}]$ [494] yields ($\bar{\mathbf{q}} = \frac{1}{2}(\mathbf{q}'+\mathbf{q}'')$)

$$\begin{aligned} H_{\text{eff}}(\mathbf{p},\bar{\mathbf{q}};\alpha) &= \frac{1}{2m}g^{ab}(\bar{\mathbf{q}})p_a p_b \\ &+ \frac{\mathrm{i}\hbar}{m}(\tfrac{1}{2}-\alpha)p_a g^{ab}{}_{,b}(\bar{\mathbf{q}}) - \frac{\hbar^2}{2m}(\tfrac{1}{2}-\alpha)^2 g^{ab}{}_{,ab}(\bar{\mathbf{q}}) + V(\bar{\mathbf{q}}) + \Delta V_{\text{Weyl}}(\bar{\mathbf{q}}) \ . \end{aligned} \tag{2.8.22}$$

$\alpha = \frac{1}{2}$ corresponds to the Weyl ordering prescription which is particularly simple and nice.

2.8.2 Weyl Ordering. Let us consider the Weyl ordering prescription in some detail. For a product of powers of position and momentum operators it yields

$$\langle q''|(\underline{q}^n\underline{p}^m)_{\text{Weyl}}|q'\rangle = \int\frac{\mathrm{d}p}{2\pi\hbar}p^m\,\mathrm{e}^{\mathrm{i}\,p(q''-q')/\hbar}\left(\frac{q'+q''}{2}\right)^n \ , \tag{2.8.23}$$

and it turns out that all coordinate-dependent quantities must be evaluated at *mid-point coordinates*. The short-time approximation for the matrix element $\langle\mathbf{q}_j|\,\mathrm{e}^{-\mathrm{i}\epsilon\underline{\mathrm{H}}/\hbar}\,|\mathbf{q}_{j-1}\rangle$ with the Hamiltonian (2.8.1) is consequently given by

$$\Big\langle\mathbf{q}_j\Big|\exp\Big[-\mathrm{i}\,\epsilon\underline{\mathrm{H}}(\underline{\mathbf{p}},\underline{\mathbf{q}})/\hbar\Big]\Big|\mathbf{q}_{j-1}\Big\rangle = \frac{1}{(2\pi\hbar)^D}$$

$$\times\left\langle \mathbf{q}_j \middle| \int \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}\,\mathrm{e}^{-\mathrm{i}\epsilon\underline{H}(\underline{\mathbf{p}},\underline{\mathbf{q}})/\hbar}\int \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}\exp\left[\mathrm{i}(\mathbf{q}-\underline{\mathbf{q}})\cdot\mathbf{u}+\mathrm{i}(\mathbf{p}-\underline{\mathbf{p}})\cdot\mathbf{v}\right]\middle|\mathbf{q}_{j-1}\right\rangle$$

$$=\frac{1}{(2\pi\hbar)^D}\int_{\mathbb{R}^D}\mathrm{d}\mathbf{p}_j\exp\left[\frac{\mathrm{i}}{\hbar}\mathbf{p}_j\cdot\Delta\mathbf{q}_j-\frac{\mathrm{i}\epsilon}{\hbar}H_{\mathrm{eff}}(\mathbf{p}_j,\bar{\mathbf{q}}_j)\right]\ , \tag{2.8.24}$$

where $\bar{\mathbf{q}}_j=\frac{1}{2}(\mathbf{q}_j+\mathbf{q}_{j-1})$ is the jth mid-point coordinate and $\Delta\mathbf{q}_j$ denotes the difference $\Delta\mathbf{q}_j=\mathbf{q}_j-\mathbf{q}_{j-1}$. The effective Hamiltonian is given by

$$H_{\mathrm{eff}}(\mathbf{p}_j,\bar{\mathbf{q}}_j)=\frac{1}{2m}g^{ab}(\bar{\mathbf{q}}_j)p_{a,j}p_{b,j}+V(\bar{\mathbf{q}}_j)+\Delta V_{\mathrm{Weyl}}(\bar{\mathbf{q}}_j)\ . \tag{2.8.25}$$

Inserting (2.8.24) in the composition law, see (2.2.2), we obtain the *Hamiltonian path integral* [135, 221, 331, 367, 563, 698, 786] in the mid-point prescription[4]

$$\begin{aligned}K(\mathbf{q}'',\mathbf{q}';T)&=\lim_{N\to\infty}\prod_{k=1}^{N-1}\int_{\mathbb{R}^D}\mathrm{d}\mathbf{q}_k\cdot\prod_{l=1}^{N}\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{p}_l}{(2\pi\hbar)^D}\\&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\mathbf{p}_j\cdot\Delta\mathbf{q}_j-\epsilon H_{\mathrm{eff}}(\mathbf{p}_j,\bar{\mathbf{q}}_j)\right]\right\}\\&=:\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathcal{D}(\mathbf{q}(t),\mathbf{p}(t))\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}[\mathbf{p}\cdot\dot{\mathbf{q}}-H_{\mathrm{eff}}(\mathbf{p},\mathbf{q})]\mathrm{d}t\right\}\ .\end{aligned} \tag{2.8.26}$$

Note the asymmetry in the above integrations over $\mathbf{q}$ and $\mathbf{p}$: while there are only $N-1$ integrations over $\mathbf{q}$, there are N integrations over $\mathbf{p}$; furthermore, the $\mathbf{q}$-paths are fixed at the end points, as usual; the $\mathbf{p}$-paths, however, are not restricted at the end points.

Integrating out the momenta $\mathbf{p}_j$ by means of the D-dimensional Gaussian integral (2.3.5) yields the *Lagrangian path integral* in the mid-point prescription (MP = *M*id-*P*oint):

$$\begin{aligned}&K(\mathbf{q}'',\mathbf{q}';T)\\&=[g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathcal{D}_{\mathrm{MP}}\mathbf{q}(t)\sqrt{g(\mathbf{q})}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathrm{d}t\,\mathcal{L}_{\mathrm{eff}}(\mathbf{q},\dot{\mathbf{q}})\right]\\&:=[g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\left(\int_{\mathbb{R}^D}\prod_{l=1}^{N-1}\mathrm{d}\mathbf{q}_l\right)\prod_{k=1}^{N}\sqrt{g(\bar{\mathbf{q}}_k)}\end{aligned}$$

[4] Note the slight discrepancy in the j-summation in comparison to (2.2.6). The reordering of the summation affects, however, only terms of $O(\epsilon^2)$ which can be neglected.

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}g_{ab}(\bar{\mathbf{q}}_j)\Delta q_j^a\Delta q_j^b - \epsilon V(\bar{\mathbf{q}}_j) - \epsilon\Delta V_{\mathrm{Weyl}}(\bar{\mathbf{q}}_j)\right)\right] \tag{2.8.27}$$

with the effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}}(\mathbf{q},\dot{\mathbf{q}}) = \frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{Weyl}}(\mathbf{q}) \ . \tag{2.8.28}$$

The mid-point prescription arises here in a very natural way as a consequence of the Weyl ordering prescription. It is a general feature that ordering prescriptions lead to *specific* lattices and that different lattices give *different* quantum corrections $\Delta V \propto \hbar^2$. For a thorough discussion of further subtleties of the path integral, see e.g. Babbit [43], Keller and McLaughlin [571], and Nelson [721].

To prove that the path integral (2.8.27) is indeed the correct one, one has to show that with the corresponding short-time kernel

$$\begin{aligned} K(\mathbf{q}_j,\mathbf{q}_{j-1};\epsilon) &= \left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{D/2}[g(\mathbf{q}_{j-1})g(\mathbf{q}_j)]^{-1/4}\sqrt{g(\bar{\mathbf{q}}_j)} \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2\epsilon}g_{ab}(\bar{\mathbf{q}}_j)\Delta q_j^a\Delta q_j^b - \epsilon V(\bar{\mathbf{q}}_j) - \epsilon\Delta V_{\mathrm{Weyl}}(\bar{\mathbf{q}}_j)\right)\right] \end{aligned} \tag{2.8.29}$$

the Schrödinger equation (2.8.1) follows, e.g. [464, 736], from the time-evolution equation

$$\Psi(\mathbf{q}'',t'') = \int_{\mathbb{R}^D}\mathrm{d}\mathbf{q}'\sqrt{g(\mathbf{q}')}\,K(\mathbf{q}'',\mathbf{q}';t'',t')\Psi(\mathbf{q}',t') \ . \tag{2.8.30}$$

Let us emphasize that this procedure is nothing but a formal proof of the path integral. A rigorous proof *must* include at least two more ingredients

i) One must show that in the limit $N \to \infty$ the path integral representation for K is in fact the matrix element of the time-evolution operator $\underset{\sim}{U}(t'',t')$ for all $\Psi \in \mathcal{H}$ ($\mathcal{H}$: relevant Hilbert space).
ii) One must show that the domain $\mathcal{D}$ of the infinitesimal generator of the kernel K is in fact identical with the domain of the Hamiltonian corresponding to the Schrödinger equation (2.8.1), i.e., the infinitesimal generator *is* the (self-adjoint) Hamiltonian.

Concerning our formulation of the path integral for the quantum motion on curved manifolds, one might ask the following questions:

i) *Is it really necessary to include the quantum potential ΔV?* In our approach the emergence of the quantum potential ΔV is absolutely unavoidable. Once the quantum Hamiltonian is defined to be given by the Laplace–Beltrami operator, a particular quantum potential emerges as an unavoidable consequence of the chosen ordering prescription of position and momentum operators, and enters the evaluation of the short-time

matrix element of the time-evolution operator. However, one is still free to make further manipulations in the lattice formulation of the path integral in order to cast the effective Lagrangian into a convenient form, and there are known examples, indeed, where the quantum potential is cancelled by another contribution.
In fact, a path integral formulation without an $\hbar^2$ quantum potential can be developed. According to Kleinert [611, 613] it is based on the equation for the straightest lines, i.e.,

$$\ddot{q}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = 0 \ , \tag{2.8.31}$$

on some Riemannian manifold $\mathbb{M}$ with coordinates $\mathbf{q}$. The (full) affine connection $\Gamma^\lambda_{\mu\nu}$ is constructed by $\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} + K^\lambda_{\mu\nu}$, where $\bar{\Gamma}^\lambda_{\mu\nu}$ are the usual Christoffel symbols, and $K_{\mu\nu\lambda} := S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$ is the contortion tensor with $S_{\mu\nu\lambda}$ the torsion tensor. The result is finally the lattice formulation of the path integral as in Sect. 6.1.1.5. Therefore this path integral formulation includes torsion in space, respectively space-time.
ii) *Is it always necessary to use the lattice definition of path integrals?* Our discussion is based on the *time-sliced* definition of the path integral, as it has been introduced by Feynman and is usually used in the physics literature. It is true that all basic path integrals have been obtained from a time-sliced evaluation, and eventually taking the proper limit $N \to \infty$. The emergence of potentials $\propto \hbar^2$ in the evaluation of the matrix elements of the short-time propagator is closely related to the stochastic nature of the Feynman paths. Of course, for building a conceptual general calculus, one may ask whether it is possible to *define path integration without a limiting procedure.* According to DeWitt–Morette (cf. the above cited literature and other authors) this is possible. In fact, as long as Gaussian path integrals are concerned, things work out perfectly well, and this theory allows a comprehensive formulation of the space-time transformation technique of Sect. 2.10.3, cf. [152, 343, 344, 943]. Recently, Cartier and DeWitt–Morette have proposed a method [147, 240] to avoid the time-slicing definition for path integrals on curved manifolds. The method has been used by LaChapelle [628] to find a spherical path integral formulation.

It is obvious that the mathematical questions coming along with the path integral have attracted the attention of many authors. Among them are Albeverio [11-16, 18-20], Arthurs [34], Babbit [43], Cohen [198], DeWitt [235], DeWitt–Morette et al. [147, 237–248, 629, 710–712], Dowker and Mayes [265, 266, 690], Elworthy and Truman [299, 300], Fischer et al. [343–345], Garrod [367], Gervais and Jevicki [389], Grosche and Steiner [464], Kac [554], T.D.Lee [645], McLaughlin and Schulman [665], Marinov [676], Mizrahi [698], Omote [736], Papanicolaou [758], Prokhorov [787], Steiner [863, 865], Truman [897], and many others who can be found in the literature.

Furthermore, some instructive proofs can be found in the books of Simon [854], and Reed and Simon [794]. Also due to Albeverio et al. [11-16, 18-20] is a wide range of discussions to formulate the Feynman path integral without the delay of going back to the definition of Wiener integrals. Another approach is due to DeWitt–Morette ("Definition Without Limiting Procedure") [237, 246, 699, 701, 711, 712]. Here this formalism is used to set up a rigorous formulation of the semiclassical expansion [238, 699-705].

2.8.3 Product Ordering. To develop another useful lattice formulation for path integrals, we consider again the generic case [422]. We assume that the metric tensor g_{ab} is real and symmetric and has rank $(g_{ab}) = D$, i.e., we have no constraints on the coordinates. Thus one can always find a linear transformation $\mathbf{C}$: $q_a = C_{ab}y_b$ such that the kinetic energy term in the classical Lagrangian $\mathcal{L}$ is equal to $(m/2)\Lambda_{ab}\dot{y}^a\dot{y}^b$ with $\Lambda_{ab} = \sum_c C^t_{ac}g_{cd}C_{db}$ and where $\boldsymbol{\Lambda}$ is diagonal. $\mathbf{C}$ has the form $C_{ab} = u_a^{(b)}$ with $\mathbf{u}^{(b)}$ $(b \in \{1,\dots,D\})$ being the eigenvectors of g_{ab}, and $\Lambda_{ab} = \sum_c f_c^2\delta_{ac}\delta_{bc}$ where $f_a^2 \neq 0$ $(a \in \{1,\dots,D\})$ are the eigenvalues of g_{ab}. Without loss of generality we assume $f_a^2 > 0$ for all $a \in \{1,\dots,D\}$.[5] Thus one can always find a representation for g_{ab} which reads

$$g_{ab}(\mathbf{q}) = \sum_{c=1}^{D} h_{ac}(\mathbf{q})h_{bc}(\mathbf{q}) \quad . \tag{2.8.32}$$

Here the $h_{ab} = \sum_c C_{ac}f_cC_{cb} = \sum_c u_c^{(a)}f_cu_c^{(b)}$ are real symmetric $D \times D$ matrices and satisfy $h_{ab}h^{bc} = \delta_a^c$. Because there exists the orthogonal transformation $\mathbf{C}$, (2.8.32) yields for the $\mathbf{y}$-coordinate system (denoted by $\mathbb{M}_y$):

$$\Lambda_{ab}(\mathbf{y}) = \sum_c f_c^2(\mathbf{y})\delta_{ac}\delta_{bc} \quad . \tag{2.8.33}$$

(2.8.33) includes the special case $g_{ab} = \Lambda_{ab}$. The Laplace–Beltrami operator expressed in terms of the inverse matrix h^{ab} reads on $\mathbb{M}_q$, i.e., in the original coordinates $\mathbf{q}$ $(h = \det(h_{ab}))$,

$$\Delta_{\mathrm{LB}}^{\mathbb{M}_q} = \sum_{abc}\left\{h^{ac}h^{bc}\frac{\partial^2}{\partial q^a\partial q^b} + \left[\frac{\partial h^{ac}}{\partial q^a}h^{bc} + h^{ac}\frac{\partial h^{bc}}{\partial q^a} + \frac{h_{,a}}{h}h^{ac}h^{bc}\right]\frac{\partial}{\partial q^b}\right\} \quad , \tag{2.8.34}$$

and on $\mathbb{M}_y$

$$\Delta_{\mathrm{LB}}^{\mathbb{M}_y} = \sum_{ab}\frac{1}{f_a^2}\left[\frac{\delta_{bb}}{D}\frac{\partial^2}{\partial y_a^2} + \left(\frac{f_{b,a}}{f_b} - 2\frac{\delta_{bb}}{D}f_{a,a}\right)\frac{\partial}{\partial y_a}\right] \quad . \tag{2.8.35}$$

[5] Actually, the case $f_a^2 < 0$ is possible. This can be seen if one considers pseudo-Euclidean (Minkowski-) spaces with indefinite metric. All the relevant formulæ are also valid in this case, see e.g. [447].

With the help of the momentum operators (2.8.3) we rewrite the Hamiltonian in the "product ordering" form (PF = Product Form)

$$\underline{H} = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}}^{\mathbb{M}_q} + V(\mathbf{q}) = \frac{1}{2m}\sum_{abc} h^{ac}(\mathbf{q})p_a p_b h^{bc}(\mathbf{q}) + V(\mathbf{q}) + \Delta V_{\mathrm{PF}}(\mathbf{q}) \ , \tag{2.8.36}$$

with the well defined quantum potential

$$\Delta V_{\mathrm{PF}}(\mathbf{q}) = \frac{\hbar^2}{8m}\sum_{abc}\Bigg[4h^{ac}{h^{bc}}_{,ab} + 2h^{ac}h^{bc}\frac{h_{,ab}}{h} + 2h^{ac}\left({h^{bc}}_{,b}\frac{h_{,a}}{h} + {h^{bc}}_{,a}\frac{h_{,b}}{h}\right) - h^{ac}h^{bc}\frac{h_{,a}h_{,b}}{h^2}\Bigg] \ . \tag{2.8.37}$$

On $\mathbb{M}_y$ the corresponding $\Delta V_{\mathrm{PF}}^{\mathbb{M}_y}$ is given by

$$\Delta V_{\mathrm{PF}}^{\mathbb{M}_y}(\mathbf{y}) = \frac{\hbar^2}{8m}\sum_{ab}\frac{1}{f_a^2}\Bigg[\left(\frac{f_{b,a}}{f_b}\right)^2 - 4\frac{f_{a,aa}}{f_a}\frac{\delta_{bb}}{D} + 4\frac{f_{a,a}}{f_a}\left(2\frac{f_{a,a}}{f_a}\frac{\delta_{bb}}{D} - \frac{f_{b,a}}{f_b}\right) + 2\left(\frac{f_{b,a}}{f_b}\right)_{,a}\Bigg] \ . \tag{2.8.38}$$

The expressions (2.8.37) and (2.8.38) look somewhat circumstantial, so we display a special case and the connection to the quantum potential ΔV_{Weyl}.

i) Let us assume that Λ_{ab} is proportional to the unit tensor, i.e., $\Lambda_{ab} = f^2\delta_{ab}$. Then $\Delta V_{\mathrm{PF}}^{\mathbb{M}_y}$ simplifies to

$$\Delta V_{\mathrm{PF}}^{\mathbb{M}_y}(\mathbf{y}) = \hbar^2\frac{D-2}{8m}\sum_a\frac{(4-D)f_{,a}^2 + 2f\cdot f_{,aa}}{f^4} \ . \tag{2.8.39}$$

This implies that if the dimension of the space is $D = 2$, then the quantum potential ΔV_{PF} vanishes.

ii) A comparison between (2.8.37) and (2.8.19) gives the connection with the quantum potential corresponding to the Weyl ordering prescription:

$$\Delta V_{\mathrm{PF}}(\mathbf{q}) = \Delta V_{\mathrm{Weyl}}(\mathbf{q}) + \frac{\hbar^2}{8m}\sum_{abc}\left(2h^{ac}{h^{bc}}_{,ab} - {h^{ac}}_{,a}{h^{bc}}_{,b} - {h^{ac}}_{,b}{h^{bc}}_{,a}\right) \ . \tag{2.8.40}$$

In the case of (2.8.33) this yields:

$$\Delta V_{\mathrm{PF}}^{\mathbb{M}_y} = \Delta V_{\mathrm{Weyl}}^{\mathbb{M}_y} + \frac{\hbar^2}{4m}\sum_a\frac{f_{a,a}^2 - f_a f_{a,aa}}{f_a^4} \ . \tag{2.8.41}$$

These equations often simplify practical applications. More special cases have been listed in [447].

Evaluating the short-time matrix elements then gives the *Lagrangian path integral in the "product form"-definition*

$$
\begin{aligned}
&K(\mathbf{q}'',\mathbf{q}';T)\\
&= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{PF}}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}h_{ac}h_{bc}\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{PF}}(\mathbf{q})\right)\mathrm{d}t\right]\\
&:= \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int_{\mathbb{M}}\sqrt{g(\mathbf{q}_k)}\,\mathrm{d}\mathbf{q}_k\\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}h_{ac}(\mathbf{q}_j)h_{bc}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b - \epsilon V(\mathbf{q}_j) - \epsilon\Delta V_{\mathrm{PF}}(\mathbf{q}_j)\right)\right].
\end{aligned}
\tag{2.8.42}
$$

In particular in Chap. 6, we use throughout the path integral formulation of (2.8.42) if not otherwise and explicitly noted.

2.9 Transformation Techniques

2.9.1 General Remarks. Let us consider a one-dimensional path integral

$$
K(x'',x';T) = \int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\sqrt{g(x)}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x) - \Delta V(x)\right)\mathrm{d}t\right] ,
\tag{2.9.1}
$$

where $\Delta V = \hbar^2(\Gamma^2 + 2\Gamma')/8m$ denotes a quantum potential due to a non-trivial metric $\sqrt{g(x)} = \mathrm{e}^{\int_a^x \Gamma(x')\mathrm{d}x'}$. x is a real variable with range $-\infty \le a \le x \le b \le \infty$. It is now assumed that the potential $V + \Delta V$ is so complicated that a direct evaluation of the path integral is not possible. We want to describe a method for transforming a path integral to calculate K or G, respectively. This method is called the *space-time transformation technique* and was originally developed by Duru and Kleinert [279, 280] in order to treat the path integral for the Coulomb potential (based on a time transformation, see e.g. [878, p.201], and the Kustaanheimo–Stiefel transformation, both well known in astronomy [627, 464–470, 507, 630]). However, this was done in a more or less formal manner, and it did not take long before the technique was refined by Inomata [514, 516], Duru and Kleinert [280], and Steiner [863–865]. This was followed by a huge amount of path integral treatments and discussions, see Anderson and Anderson [26], Bernido, Carpio-Bernido and Inomata [85, 142, 143], Cai and Inomata [127], Castrigiano and Stärk [152], Chetouani et al. [177, 179, 187–189], Fischer, Leschke and Müller [343, 344], Grinberg, Marañon and Vucetich [417, 418], Refs. [434, 436, 437, 464–470], Ho and Inomata [495], Inomata [516], Inomata and Kayed [525] (Dirac Coulomb problem), Junker [549], Kleinert [608], Kubo [623], Lawande and Bhagwat [642], Pak and Sökmen [743, 859, 860], and Young and DeWitt–Morette [943].

In order to understand the basic features, let us start by considering a kind of Legendre transformation of the general one-dimensional Hamiltonian:

$$\underline{\mathrm{H}}_E := -\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Gamma(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + V(x) - E \tag{2.9.2}$$

which is a Hermitian operator with respect to the scalar product $(f_1, f_2) = \int \mathrm{d}x\, \sqrt{g(x)}\, f_1^*(x) f_2(x)$. Introducing the momentum operator

$$\underline{\mathrm{p}}_x = \frac{\hbar}{\mathrm{i}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\Gamma(x)\right) , \qquad \Gamma(x) = \frac{\mathrm{d}\ln\sqrt{g(x)}}{\mathrm{d}x} , \tag{2.9.3}$$

$\underline{\mathrm{H}}_E$ can be rewritten as

$$\underline{\mathrm{H}}_E = \frac{\underline{\mathrm{p}}_x^2}{2m} + V(x) + \frac{\hbar^2}{8m}[\Gamma^2(x) + 2\Gamma'(x)] - E \tag{2.9.4}$$

with the corresponding path integral ("promotor" [517, 528]) $K_E(x'', x'; T) = \mathrm{e}^{\mathrm{i}TE/\hbar} K(x'', x'; T)$, where K denotes the path integral (2.9.1). Let us consider the transformation $x = F(q)$, and let $G(q) = \Gamma[F(q)]$. Then we get for the operator $\underline{\mathrm{H}}_E$ expressed in the variable q, which we denote by $\widehat{\underline{\mathrm{H}}}_E$,

$$\widehat{\underline{\mathrm{H}}}_E = -\frac{\hbar^2}{2m}\frac{1}{F'^2(q)}\left[\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \left(G(q)F'(q) - \frac{F''(q)}{F'(q)}\right)\frac{\mathrm{d}}{\mathrm{d}q}\right] + V[F(q)] - E \ . \tag{2.9.5}$$

With the constraint $f[F(q)] = F'^2(q)$ we get for the new Hamiltonian $\tilde{\underline{\mathrm{H}}} := f\widehat{\underline{\mathrm{H}}}_E$:

$$\tilde{\underline{\mathrm{H}}} = \frac{1}{2m}\underline{\mathrm{p}}_q^2 + f[F(q)][V(F(q)) - E] + \Delta V(q) \ , \tag{2.9.6}$$

where $\tilde{\Gamma}(q) = G(q)F'(q) - F''(q)/F'(q)$, $\underline{\mathrm{p}}_q = \frac{\hbar}{\mathrm{i}}(\mathrm{d}/\mathrm{d}q + \tilde{\Gamma}/2)$, and $\Delta V(q)$ denotes the well defined quantum potential

$$\Delta V(q) = \frac{\hbar^2}{8m}\left[3\left(\frac{F''(q)}{F'(q)}\right)^2 - 2\frac{F'''(q)}{F'(q)} + \big(G(q)F'(q)\big)^2 + 2G'(q)F'(q)\right] \ . \tag{2.9.7}$$

The path integral corresponding to the Hamiltonian $\tilde{\underline{\mathrm{H}}}$ is

$$\tilde{K}(q'', q'; s'') = \int\limits_{q(0)=q'}^{q(s'')=q''} \mathcal{D}q(s) \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - f[F(q)][V(F(q)) - E] - \Delta V(q)\right)\mathrm{d}s\right] \ . \tag{2.9.8}$$

Here the "pseudo time" variable $s = s(t)$ is defined by $s(t) = \int_{t'}^{t} d\tau / F'^2(q(\tau))$, with $s(t') = 0$, $s(t'') \equiv s''$ (see below in Sect. 2.9.3). Note that for $G \equiv 0$, ΔV is proportional to the Schwarz derivative of the transformation F.

As is easily checked, we can derive from the short-time kernel of (2.9.8) in the new "time" s via the time-evolution equation

$$\tilde{\Psi}(q'', s'') = \int \widetilde{K}(q'', q'; s'', s') \tilde{\Psi}(q', s') \, \mathrm{d}q' \tag{2.9.9}$$

the time-dependent Schrödinger equation

$$\underline{\widetilde{\mathrm{H}}} \tilde{\Psi}(q, s) = \mathrm{i}\, \hbar \frac{\partial}{\partial s} \tilde{\Psi}(q, s) \ . \tag{2.9.10}$$

2.9.2 Point-Canonical Transformations. The crucial point is now the lattice derivation of $\widetilde{K}(s'')$ and the relation between $\widetilde{K}(s'')$ and K. Let us consider the path integral K in its lattice definition (we assume $J \equiv 1$)

$$\begin{aligned} K(\mathbf{x}'', \mathbf{x}'; T) = &\lim_{N \to \infty} \left(\frac{m}{2\pi \,\mathrm{i}\, \epsilon \hbar} \right)^{ND/2} \\ &\times \prod_{k=1}^{N-1} \int \mathrm{d}\mathbf{x}_k \cdot \exp\left[\frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left(\frac{m}{2\epsilon} (\Delta \mathbf{x}_j)^2 - \epsilon V(\mathbf{x}_j) \right) \right] \ . \end{aligned} \tag{2.9.11}$$

To transform the coordinates $\mathbf{x}$ into the coordinates $\mathbf{q}$ by means of the point canonical transformation [180, 260, 314, 359, 389, 464, 512, 736, 786, 848] $\mathbf{x} = \mathbf{F}(\mathbf{q})$ with $F^a(\mathbf{q})$ denoting the components of $\mathbf{F}$, we use the so-called *mid-point expansion method.* We must expand any dynamical quantity in question which is defined on the points $\mathbf{q}_j$ and $\mathbf{q}_{j-1}$ of the j^{th}-interval in the lattice version about the mid-points $\bar{\mathbf{q}}_j = \frac{1}{2}(\mathbf{q}_j + \mathbf{q}_{j-1})$ keeping terms up to order $(q_j^a - q_{j-1}^a)^3$. Furthermore one must use the path integral equivalence relations [30, 235, 389] (here we incorporate a metric g_{ab} with its inverse g^{ab} in order to state the general formulæ):

$$\Delta q^a \Delta q^b \stackrel{\bullet}{=} \frac{\mathrm{i}\,\epsilon\hbar}{m} g^{ab} \tag{2.9.12}$$

$$\Delta q^a \Delta q^b \Delta q^c \Delta q^d \stackrel{\bullet}{=} \left(\frac{\mathrm{i}\,\epsilon\hbar}{m} \right)^2 \left[g^{ab} g^{cd} + g^{ac} g^{bd} + g^{ad} g^{bc} \right] \tag{2.9.13}$$

$$\begin{aligned} &\Delta q^a \Delta q^b \Delta q^c \Delta q^d \Delta q^e \Delta q^f \\ &\quad \stackrel{\bullet}{=} \left(\frac{\mathrm{i}\,\epsilon\hbar}{m} \right)^3 \Big[g^{ab} g^{cd} g^{cd} + g^{ac} g^{bd} g^{ef} + g^{ad} g^{bc} g^{ef} + g^{ab} g^{ce} g^{df} + g^{ab} g^{cf} g^{de} \\ &\qquad + g^{cd} g^{ae} g^{bf} + g^{cd} g^{af} g^{be} + g^{ac} g^{be} g^{df} + g^{ac} g^{bf} g^{de} + g^{bd} g^{ae} g^{cf} \\ &\qquad + g^{bd} g^{af} g^{ce} + g^{ad} g^{be} g^{cf} + g^{ad} g^{bf} g^{ce} + g^{bc} g^{ae} g^{df} + g^{bc} g^{af} g^{de} \Big] \ . \end{aligned} \tag{2.9.14}$$

These relations are sufficient for all practical purposes. We now have:

$$\begin{aligned}\Delta F^a(\mathbf{q}_j) &\equiv F^a(\mathbf{q}_j) - F^a(\mathbf{q}_{j-1}) = F^a\left(\bar{\mathbf{q}}_j + \frac{\Delta \mathbf{q}_j}{2}\right) - F^a\left(\bar{\mathbf{q}}_j - \frac{\Delta \mathbf{q}_j}{2}\right)\\ &= \Delta q_j^m \frac{\partial F^a(\mathbf{q})}{\partial q^m}\bigg|_{\mathbf{q}=\bar{\mathbf{q}}_j} + \frac{1}{24}\Delta q_j^m \Delta q_j^n \Delta q_j^k \frac{\partial^3 F^a(\mathbf{q})}{\partial q^m \partial q^n \partial q^k}\bigg|_{\mathbf{q}=\bar{\mathbf{q}}_j} + \dots \; .\end{aligned} \tag{2.9.15}$$

Here Δq_j^m denotes $q_j^m - q_{j-1}^m$. This gives the coordinate-transformed path integral

$$\begin{aligned}K\big(F(\mathbf{q}''), F(\mathbf{q}'); T\big) &= [F_{;\mathbf{q}}(\mathbf{q}')F_{;\mathbf{q}}(\mathbf{q}'')]^{-1/2}\\ &\quad\times \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\prod_{l=1}^{N-1}\int \mathrm{d}\mathbf{q}_l \cdot \prod_{k=1}^{N} F_{;\mathbf{q}}(\bar{\mathbf{q}}_k)\\ &\quad\times \exp\Bigg(\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\bigg\{\frac{m}{2\epsilon}\Delta q_j^m \Delta q_j^n F^a_{,m}(\bar{\mathbf{q}}_j)F^a_{,n}(\bar{\mathbf{q}}_j) - \epsilon V(\bar{\mathbf{q}}_j)\\ &\qquad - \frac{\epsilon\hbar^2}{8m}\bigg[(F^a_{,m}(\bar{\mathbf{q}}_j)F^a_{,n}(\bar{\mathbf{q}}_j))^{-1}\left(\frac{F_{;\mathbf{q},m}(\bar{\mathbf{q}}_j)F_{;\mathbf{q},n}(\bar{\mathbf{q}}_j)}{F^2_{;\mathbf{q}}(\bar{\mathbf{q}}_j)} - \frac{F_{;\mathbf{q},mn}(\bar{\mathbf{q}}_j)}{F_{;\mathbf{q}}(\bar{\mathbf{q}}_j)}\right)\\ &\qquad - F^a_{,m}(\bar{\mathbf{q}}_j)F^a_{,nkl}(\bar{\mathbf{q}}_j)F^{-1}_{mnkl}(\bar{\mathbf{q}}_j)\bigg]\bigg\}\Bigg)\\ &\equiv [F_{;\mathbf{q}}(\mathbf{q}')F_{;\mathbf{q}}(\mathbf{q}'')]^{-1/2}\\ &\quad\times \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{MP}}\mathbf{q}(t) F_{;\mathbf{q}}(\mathbf{q},t)\exp\Bigg(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg\{\frac{m}{2}F^a_{,m}F^a_{,n}\dot{q}^n\dot{q}^m - V(\mathbf{q})\\ &\qquad - \frac{\hbar^2}{8m}\bigg[(F^a_{,m}F^a_{,n})^{-1}\left(\frac{F_{;\mathbf{q},m}F_{;\mathbf{q},n}}{F^2_{;\mathbf{q}}} - \frac{F_{;\mathbf{q},mn}}{F_{;\mathbf{q}}}\right) - F^a_{,m}F^a_{,nkl}F^{-1}_{mnkl}\bigg]\bigg\}\Bigg) \; .\end{aligned} \tag{2.9.16}$$

Here $F^a_{,m} = \partial F^a/\partial q^m$, $F_{;\mathbf{q}} = |\mathbf{F}_{;\mathbf{q}}|$ the Jacobian, and F^{-1}_{mnkl} is defined by ($F^a_{j,m} = F^a_{,m}(\bar{\mathbf{q}}_j)$ etc.)

$$\begin{aligned}F^{-1}_{mnkl}(\bar{\mathbf{q}}_j) &= (F^a_{j,m}F^a_{j,n})^{-1}(F^a_{j,k}F^a_{j,l})^{-1}\\ &\quad + (F^a_{j,m}F^a_{j,k})^{-1}(F^a_{j,l}F^a_{j,n})^{-1} + (F^a_{j,m}F^a_{j,l})^{-1}(F^a_{j,k}F^a_{j,n})^{-1} \; .\end{aligned} \tag{2.9.17}$$

The path integral (2.9.16) has the canonical form, i.e., the usual coordinate transformation from flat space to a non-linear coordinate system gives the quantum potential ΔV_{Weyl} without the curvature term. In particular (2.9.16) takes, in the one-dimensional case, the form ($F' \equiv \mathrm{d}F/\mathrm{d}q$) [223, 314, 359, 380, 389, 392, 464, 512, 593, 786, 848]:

$$K\big(F(q''),F(q');T\big)=[F'(q')F'(q'')]^{-1/2}\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{N/2}\prod_{l=1}^{N-1}\int\mathrm{d}q_l$$
$$\times\prod_{k=1}^{N}F'(\bar q_k)\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}\Delta q_j^n\Delta q_j^n F'^2(\bar q_j)-\epsilon V(\bar q_j)-\frac{\epsilon\hbar^2}{8m}\frac{F''^2(\bar q_j)}{F'^4(\bar q_j)}\right)\right]\ .$$
(2.9.18)

It is not difficult to incorporate the explicitly time-dependent one-dimensional coordinate transformation $x=F(q,t)$ [180, 440, 469, 773, 872]:

$$K\big(F(q'',t''),F(q',t');t'',t'\big)$$
$$=\Big[F'(q'',t'')F'(q',t')\Big]^{-1/2}A(q'',q';t'',t')\tilde K(q'',q';t'',t')\ ,\qquad(2.9.19)$$

with the prefactor ($F'(q,t)=\partial F(q,t)/\partial q$, $\dot F(q,t)=\partial F(q,t)/\partial t$, etc.)

$$A(q'',q';t'',t')$$
$$=\exp\left[\frac{\mathrm{i}m}{\hbar}\left(\int^{q''}F'(z,t'')\dot F(z,t'')\mathrm{d}z-\int^{q'}F'(z,t')\dot F(z,t')\mathrm{d}z\right)\right]\ ,\qquad(2.9.20)$$

and the path integral $\tilde K(t'',t')$ is given by

$$\tilde K\big(F(q'',t''),F(q',t');t'',t'\big)=\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}_{\mathrm{MP}}q(t)F'(q,t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(F'^2(q,t)\dot q^2+\dot F^2(q,t)\Big)-V(F(q,t))\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8m}\frac{F''^2(q,t)}{F'^4(q,t)}-m\int^{q}\Big(F'(z,t)\ddot F(z,t)+\dot F(z,t)\dot F'(z,t)\Big)\mathrm{d}z\right]\mathrm{d}t\right\}\ .$$
(2.9.21)

For the case $\dot F'(q,t)\neq0$ this can be simplified to

$$\tilde K\big(F(q'',t''),F(q',t');t'',t'\big)$$
$$=\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}_{\mathrm{MP}}q(t)F'(q,t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}F'^2(q,t)\dot q^2-V(F(q,t))\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8m}\frac{F''^2(q,t)}{F'^4(q,t)}-m\int^{q}F'(z,t)\ddot F(z,t)\,\mathrm{d}z\right)\mathrm{d}t\right]\ .\qquad(2.9.22)$$

In the D-dimensional case $\mathbf{x} = F(\mathbf{q}, t)$ the corresponding transformation formulæ are considerably more complicated and are not stated here, cf. [774].

2.9.3 The Method of Space-Time Transformations. It is obvious that the path integral representation (2.9.21) is not completely satisfactory. Whereas the transformed potential $V(F(q))$ may have a convenient form when expressed in the new coordinate q, the kinetic term $(m/2)F'^2\dot{q}^2$ is in general nasty. Here the so-called "time transformation" comes into play which leads in combination with the "space transformation" already carried out to general "space-time transformations" in path integrals. The time transformation is implemented [279, 280, 464, 495, 514, 516, 613, 773, 863, 865, 870] by introducing a new "pseudo-time" s. To do this, one makes use of the operator identity

$$\frac{1}{\underline{H} - E} = f_r(x,t)\frac{1}{f_l(x,t)(\underline{H} - E)f_r(x,t)}f_l(x,t) \ , \tag{2.9.23}$$

where $\underline{H}$ is the Hamiltonian corresponding to a path integral K, and $f_{l,r}(x,t)$ are functions in x and t, multiplying from the left or from the right, respectively, onto the operator $(\underline{H} - E)$. For the new pseudo-time s one assumes that the constraint

$$\int_0^{s''} \mathrm{d}s f_l\big(F(q(s),s)\big) f_r\big(F(q(s),s)\big) = T = t'' - t' \tag{2.9.24}$$

has for all admissible paths a unique solution $s'' > 0$ given by

$$s'' = \int_{t'}^{t''} \frac{\mathrm{d}\tau}{f_l(x,\tau)f_r(x,\tau)} = \int_{t'}^{t''} \frac{\mathrm{d}\tau}{F'^2(q(\tau),\tau)} \ . \tag{2.9.25}$$

Here one has made the choice $f_l\big(F(q(s),s)\big) = f_r\big(F(q(s),s)\big) = F'\big(q(s),s\big)$ in order that in the final result the metric coefficient in the kinetic energy term is equal to one. A convenient way to derive the corresponding transformation formulæ is to use the energy-dependent Green function G of the kernel K. Let us first consider the time-independent case. For the path integral one obtains by simultaneously implementing the point canonical transformation and the time transformation the following transformation formulæ

$$K(x'',x';T) = \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G(q'',q';E) \ , \tag{2.9.26}$$

$$G(q'',q';E) = \frac{\mathrm{i}}{\hbar}\Big[F'(q'')F'(q')\Big]^{1/2} \int_0^\infty \mathrm{d}s'' \hat{K}(q'',q';s'') \ , \tag{2.9.27}$$

with the transformed path integral $\hat{K}$ given by

$$\hat{K}(q'',q';s'') = \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{1/2}\prod_{k=1}^{N-1}\int \mathrm{d}q_k$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}(\Delta q_j)^2 - \epsilon F'^2(\bar{q}_j)\Big(V(F(\bar{q}_j)) - E\Big) - \epsilon\Delta V(\bar{q}_j)\right)\right]$$
$$\equiv \int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - F'^2(q)\Big(V(F(q) - E\Big) - \Delta V(q)\right)\mathrm{d}s\right] , \tag{2.9.28}$$

with the quantum potential ΔV given by

$$\Delta V(q) = \frac{\hbar^2}{8m}\left(3\frac{F''^2}{F'^2} - 2\frac{F'''}{F'}\right) . \tag{2.9.29}$$

Note that ΔV has the form of a Schwarz derivative of F. For the time-dependent case the formulæ must be modified slightly, and we obtain the space-time transformation formulæ

$$K(x'',x';t'',t') = \Big[F'(q'',t'')F'(q',t')\Big]^{1/2} A(q'',q';t'',t') ,$$
$$\times \int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\,\mathrm{i}\,ET/\hbar}\,G(q'',q';E) \tag{2.9.30}$$
$$G(q'',q';E) = \frac{\mathrm{i}}{\hbar}\int_0^\infty \widehat{K}(q'',q';s'')\mathrm{d}s'' , \tag{2.9.31}$$

with the path integral $\widehat{K}(s'')$ given by

$$\widehat{K}(q'',q';s'') = \int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - F'^2(q,s)\Big(V(F(q,s)) - E\Big) - \Delta V(q,s)\right)\mathrm{d}s\right] . \tag{2.9.32}$$

ΔV denotes the quantum potential

$$\Delta V(q,s) = \frac{\hbar^2}{8m}\left(3\frac{F''^2(q,s)}{F'^2(q,s)} - 2\frac{F'''(q,s)}{F'(q,s)}\right) + mF'^2(q,s)\int^q F'(z,s)\ddot{F}(z,s)\,\mathrm{d}z . \tag{2.9.33}$$

The rigorous lattice derivation is far from being trivial and has been discussed by several authors. Recent attempts to put it on a sound footing can be found in Castrigiano and Stärk [152], Fischer et al. [343, 344] and

Young and DeWitt–Morette [943]. In terms of stochastic processes the time-transformation is formulated as follows:

$$\begin{aligned}
&\int_{\mathcal{C}(\mathbb{R},x')} \mathcal{D}W[x]\delta(x(t)-x')\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} V(x(t))\,\mathrm{d}t\right)\\
&= \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \int_{\mathcal{C}(\mathbb{R},q')} \mathcal{D}W[q]\delta(F(q(s))-x')\\
&\qquad \times \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^{s''}\Big(F'^2(q)\big(V(F(q)-E)+\Delta V(q)\Big)\mathrm{d}s\right] . \qquad (2.9.34)
\end{aligned}$$

Here $\mathcal{C}(\mathbb{R},x')$ denotes the set of paths in $\mathbb{R}$ which start at x' at t', the δ-functions describe the boundary condition, and $\mathcal{D}W[x]$ is the stochastic measure for the Feynman process in real time, or the Wiener process in imaginary time after a Wick rotation.

Finally, let us consider a pure time transformation in a path integral. Set

$$\begin{aligned}
G(\mathbf{q}'',\mathbf{q}';E) &= \sqrt{f(\mathbf{q}')f(\mathbf{q}'')}\\
&\quad\times \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s''\Big\langle \mathbf{q}''\Big|\exp\Big(-\mathrm{i}s''\sqrt{f}\,(\underline{\mathrm{H}}-E)\sqrt{f}/\hbar\Big)\Big|\mathbf{q}'\Big\rangle , \qquad (2.9.35)
\end{aligned}$$

which corresponds to the introduction of the new "pseudo-time" $s''=\int_{t'}^{t''}\mathrm{d}s\,/f(\mathbf{q}(s))$, and assume that the Hamiltonian $\underline{\mathrm{H}}$ has the product ordered form. Then[6]

$$G(\mathbf{q}'',\mathbf{q}';E)=\frac{\mathrm{i}}{\hbar}(f'f'')^{\frac{1}{2}(1-D/2)}\int_0^\infty \tilde{K}(\mathbf{q}'',\mathbf{q}';s'')\mathrm{d}s'' \qquad (2.9.36)$$

with the transformed path integral

$$\begin{aligned}
\tilde{K}(\mathbf{q}'',\mathbf{q}';s'') &= \int\limits_{\mathbf{q}(0)=\mathbf{q}'}^{\mathbf{q}(s'')=\mathbf{q}''} \mathcal{D}\mathbf{q}(s)\sqrt{\tilde{g}}\\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\tilde{h}_{ac}\tilde{h}_{cb}\dot{q}^a\dot{q}^b - f\Big(V(\mathbf{q})+\Delta V_{\mathrm{PF}}(\mathbf{q})-E\Big)\right)\mathrm{d}s\right] .
\end{aligned}$$
$$(2.9.37)$$

Here $\tilde{h}_{ac}=h_{ac}/\sqrt{f}$, $\sqrt{\tilde{g}}=\det(\tilde{h}_{ac})$ and (2.9.37) is of the canonical product form. It is obvious that only in certain particular cases can the metric terms $\tilde{h}$ be transformed to unity. The transformation to unity is always possible in one dimension, though, and this is actually the case in most papers found in the literature.

[6] Note the parametrization invariance in the case $D=2$, which can be interpreted as a gauge transformation, cf. [356].

2.9.4 Space-Time Transformations in Radial Path Integrals. In Sect. 2.7 we discussed the path integral in spherical coordinates for systems that are invariant under rotations, i.e., for which the Lagrangian reads ($\lim_{r\to 0}[r^2V(r)] = 0$)

$$\mathcal{L} = \frac{m}{2}\dot{\mathbf{x}}^2 - V(r) \tag{2.9.38}$$

with $r = |\mathbf{x}|$. Restricting ourselves, without loss of generality, to the three-dimensional case, $D = 3$, we have for the Feynman kernel the following *"partial wave" expansion* (see (2.7.14))

$$K(r'', \Omega'', r', \Omega'; T) = \frac{1}{4\pi r' r''}\sum_{l=0}^{\infty}(2l+1)K_l(r'', r'; T)P_l(\cos\Theta) \ , \tag{2.9.39}$$

where Θ denotes the angle between Ω' and Ω'' and the *radial kernel* K_l is given by the *radial path integral* (see (2.7.15))

$$K_l(r'', r'; T|V) = \int\limits_{r(0)=r'}^{r(T)=r''} \mathcal{D}r(t)\mu_{l+\frac{1}{2}}[r^2]\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right] \ . \tag{2.9.40}$$

Since K_l can be considered as a functional of the potential V, we have stated the dependence on V explicitly on the left hand side. Even this one-dimensional path integral will, for a given potential $V(r)$, in general not be directly tractable. We are thus led [863] to perform in the radial path integral (2.9.40) a non-linear *space-time transformation* $t \to s$, $r \to R$, where the new (stochastic) path-dependent "time" $s = s(t; r(t))$ and the new radial variable $R = R(s)$ are defined by the *stochastic equations* (f, g real and positive, $g(0) = 0$, $s(0; r(0)) = 0$)

$$\mathrm{d}t = f(r)\mathrm{d}s \ , \qquad r = g(R) \ . \tag{2.9.41}$$

We have $0 \leq s \leq s''$, where s'' is path dependent and is defined by the stochastic functional [532, p.273]

$$s'' = \int_0^T \frac{\mathrm{d}t}{f(r(t))} \ . \tag{2.9.42}$$

In order to fix the functions f and g, let us consider the transformation of the Legendre-transformed radial action ($\dot{R} = \mathrm{d}R/\mathrm{d}s$, $g' = \mathrm{d}g/\mathrm{d}R$)

$$\begin{aligned} R_l[r] + ET &:= \int_0^T\left(\frac{m}{2}\dot{r}^2 - \frac{\hbar^2 l(l+1)}{2mr^2} - V(r) + E\right)\mathrm{d}t \\ &\doteq \int_0^{s''}\left[\frac{m}{2}\left(\frac{g'^2}{f(g)}\right)\dot{R}^2 - \frac{\hbar^2 l(l+1)}{2m}\left(\frac{f(g)}{g^2}\right) - f(g)\Big(V(g) - E\Big) - \Delta V\right]\mathrm{d}s \ , \end{aligned} \tag{2.9.43}$$

where ΔV denotes a *quantum potential* of order $\hbar^2$ which has to be added if the action integral is properly treated as a stochastic integral [863]. Obviously, the transformed action takes the canonical form of a radial action in the new variable R if f and g satisfy the constraint equations

$$\frac{[g'(R)]^2}{f(g(R))} = 1 \ , \qquad \frac{f(g(R))}{[g(R)]^2} = \frac{a}{R^2} \tag{2.9.44}$$

for $R \in (0, \infty)$ with a a positive constant. The solution of (2.9.44) is given by $(a = 4/(2-\nu)^2)$

$$f(r) = \frac{4}{(2-\nu)^2} r^\nu \ , \qquad g(R) = R^{2/(2-\nu)} \ , \qquad \nu < 2 \ , \tag{2.9.45}$$

which leads to the non-linear *space-time transformation*

$$\mathrm{d}s = \frac{(2-\nu)^2}{4} \frac{\mathrm{d}t}{r^\nu} \ , \qquad R = r^{1-\nu/2} \ , \qquad \nu < 2 \tag{2.9.46}$$

and the *quantum potential*

$$\Delta V(R) = \frac{\hbar^2 \nu(4-\nu)}{8(2-\nu)^2 m R^2} \ . \tag{2.9.47}$$

Note that $\nu = 0$ corresponds to the identity transformation. The additional term ΔV having the form of a centrifugal potential is a pure *quantum correction* and is a direct consequence of the *stochastic nature* of the Feynman paths which are "continuous but possess no derivative" [326]. (See the general discussion in Sect. 2.9.3.) It arises from the kinetic energy term $\frac{m}{2}\dot{r}^2$ in the path integral (2.9.40) which in the lattice definition is replaced by $\sum_{j=1}^N m(\Delta r_j)^2/2\epsilon^2$ with $\Delta r_j = r_j - r_{j-1}$ (see (2.7.15)). Since Δr_j is under the path integral of order $\sqrt{\epsilon}$, i.e., $\Delta r_j/\epsilon$ diverges like $\epsilon^{-1/2}$ as $\epsilon \to 0$, one has to keep all terms up to order $(R_j - R_{j-1})^4$ when performing the space-time transformation (2.9.46).

We then obtain for the Legendre-transformed radial action

$$\begin{aligned} R_l[r] + ET &= \int_0^{s''} \left[\frac{m}{2}\dot{R}^2 - \frac{\hbar^2 L_\nu(L_\nu+1)}{2mR^2} - W_\nu(R) \right] \mathrm{d}s \\ &=: \bar{R}_{L_\nu}[R] \end{aligned} \tag{2.9.48}$$

with the *new* (energy-dependent) *potential*

$$W_\nu(R) = \frac{4}{(2-\nu)^2} R^{2\nu/(2-\nu)} \Big[V(R^{2/(2-\nu)}) - E \Big] \tag{2.9.49}$$

and the *effective angular momentum*

$$L_\nu = \frac{4l+\nu}{2(2-\nu)} \ . \tag{2.9.50}$$

The last identification follows from (see (2.1.25))

$$\frac{\hbar^2 l(l+1)}{2m}\left(\frac{f(g)}{g^2}\right)+\Delta V=\frac{\hbar^2}{2mR^2}\left[l(l+1)\frac{4}{(2-\nu)^2}+\frac{\nu(4-\nu)}{4(2-\nu)^2}\right]$$
$$=:\frac{\hbar^2 L_\nu(L_\nu+1)}{2mR^2}\ . \tag{2.9.51}$$

To obtain a transformation formula for our radial path integral (2.9.40), we have to consider the energy-dependent Green function $G(r'',\Omega'',r',\Omega';E)$ defined in (2.1.25). Inserting the expansion (2.9.39) we obtain a corresponding "partial wave" expansion for G with the *radial Green function*

$$G_l(r'',r';E|V):=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}K_l(r'',r';T|V)\ . \tag{2.9.52}$$

Performing the space-time transformation (2.9.46) we derive the following *transformation formula* [863] ($\nu<2$)[7]

$$\begin{aligned}
&G_l(r'',r';E|V)\\
&=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\int\limits_{r(0)=r'}^{r(T)=r''}\mathcal{D}r(t)\mu_{l+\frac12}[r^2]\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot r^2-V(r)+E\right)\mathrm{d}t\right]\\
&=\frac{2}{2-\nu}(r'r'')^{\nu/4}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s''\\
&\qquad\times\int\limits_{R(0)=r'^{1-\nu/2}}^{R(s'')=r''^{1-\nu/2}}\mathcal{D}R(s)\mu_{L_\nu+\frac12}[R^2]\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot R^2-W_\nu(R)\right)\mathrm{d}s\right]\ .
\end{aligned} \tag{2.9.53}$$

The last equation can be written in the compact form

$$G_l(r'',r';E|V)=\frac{2}{2-\nu}(r'r'')^{\nu/4}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s''K_{L_\nu}\left(r''^{1-\nu/2},r'^{1-\nu/2};s''\middle|W_\nu\right)\ . \tag{2.9.54}$$

Equation (2.9.54) connects the radial Green function G_l of the original quantum system with potential V to the radial path integral K_{L_ν} of a new quantum system with potential W_ν, equation (2.9.49), and effective angular momentum (2.9.50).

If the Schrödinger wave function in spherical coordinates is written in the form $\Psi_{nlm}(r,\vartheta,\varphi)=r^{-1}u_{nl}(r)Y_l^m(\vartheta,\varphi)$ in terms of the reduced *radial wave function* $u_{nl}(r)$ satisfying $u_{nl}(r)=O(r^{l+1})$ for $r\to 0$, we have the *spectral decompositions*

[7] For the subtleties of translating the centrifugal potential into the functional weight, see [343–345, 865] and Sect. 3.3 on the radial harmonic oscillator.

$$K_l(r'', r'; T|V) = \sum_{n=0}^{\infty} u_{nl}(r'')u_{nl}(r')\,\mathrm{e}^{-\mathrm{i}E_{nl}T/\hbar}\,\Theta(T) \ , \tag{2.9.55}$$

$$G_l(r'', r'; E|V) = \sum_{n=0}^{\infty} \frac{u_{nl}(r'')u_{nl}(r')}{E_{nl} - E - \mathrm{i}\epsilon} \ . \tag{2.9.56}$$

Here n denotes the radial quantum number.

For a given potential $V(r)$ it may be possible to choose ν in the transformation formula (2.9.54) in such a way that the new potential W_ν is at most quadratic in R. Then the space-time transformation may enable us to transform non-Gaussian path integrals into Gaussian ones (or Besselian path integrals, see Sect. 3.3) which are easy to calculate. One important example is the hydrogen atom; see the following section.

2.10 Exact Path Integral Treatment of the Hydrogen Atom

The Lagrangian of the *hydrogen atom* has the form (2.9.38) with $V(r) = -e^2/r$ being the *Coulomb potential* ($-e$ is the charge of the electron).[8] Due to the $1/r$-singularity, the path integral of the hydrogen atom is highly non-Gaussian and cannot directly be solved. The first path integral treatment of the hydrogen atom was carried out by Kleinert and Duru [279, 280]. In this section we shall apply the transformation formula (2.9.54) to the Coulomb potential. To our knowledge, this treatment [864] represents the simplest path integral derivation of the discrete energy spectrum and the (normalized) wave functions of the hydrogen atom.

Since we are interested in the discrete spectrum only, we choose $E < 0$. For $\nu = 1$ we obtain from (2.9.50) $L_1 = 2l + \frac{1}{2}$ and from (2.9.49) the new potential

$$\begin{aligned} W_1(R) &= -4e^2 - 4ER^2 \\ &= -4e^2 + \frac{\Omega^2}{2}R^2 \end{aligned} \tag{2.10.1}$$

which describes a radial harmonic oscillator with "frequency" $\Omega := 2\sqrt{-2E}$. Inserting (2.10.1) into the transformation formula (2.9.54) for $\nu = 1$ gives the radial Green function G_l^H of the hydrogen atom ($E < 0$)

$$\begin{aligned} G_l^H(r'', r'; E) &= 2\,\mathrm{i}(r'r'')^{1/4} \int_0^\infty \mathrm{d}s\, K_{2l+\frac{1}{2}}\left(\sqrt{r''}, \sqrt{r'}; s \,\middle|\, -4e^2 + \frac{\Omega^2}{2}R^2\right) \\ &= 2\,\mathrm{i}(r'r'')^{1/4} \int_0^\infty \mathrm{d}s\, \mathrm{e}^{4\mathrm{i}e^2 s}\, K^{\mathrm{osc}}_{2l+\frac{1}{2}}(\sqrt{r''}, \sqrt{r'}; s) \\ &= 2(r'r'')^{1/4} G^{\mathrm{osc}}_{2l+\frac{1}{2}}(\sqrt{r''}, \sqrt{r'}; 4e^2) \ . \end{aligned} \tag{2.10.2}$$

[8] Units $\hbar = m = 1$ are used in this section.

The remarkable equation (2.10.2) relates the radial Green function of the hydrogen atom (at energy $E < 0$) to the radial Green function $G^{\mathrm{osc}}_{2l+\frac{1}{2}}$ of a *harmonic oscillator* with "frequency" Ω (at "energy" $4e^2$), analytically continued to (unphysical) half-integer angular momentum $2l + \frac{1}{2}$. Since the radial Green function of a harmonic oscillator can be calculated with path integral methods, see Sect. 6.4.1, we have obtained a complete path integral treatment of the hydrogen atom. For a calculation of the wave functions and energy levels using the spectral representation (2.9.56), see the original paper [864]. The result is given in Sects. 6.8.2, and 6.8.6.3.

2.11 The Path Integral in Parabolic Coordinates

For completeness we will now consider parabolic coordinates. It is sufficient to study only the two-dimensional case, because higher dimensional parabolic coordinates are a combination of two-dimensional parabolic coordinates and spherical coordinates. For other coordinate systems, see Table 2.2 and the discussion at the end of this section.

Parabolic coordinates have the property that they lead to a separation of coordinates for special physical systems (we cite only path integral treatments), e.g., in electric fields like the Holt potential [458], distorted Coulomb potentials [189, 436, 437], and the Kaluza–Klein monopole system [432, 461]. Whereas there are systems which are only separable in parabolic coordinates, e.g., the Coulomb potential plus a linear electric field [189, 540], or distorted Coulomb fields [436, 437], parabolic coordinates often offer an alternative coordinate system in comparison to the better known spherical coordinates.

We consider two-dimensional parabolic coordinates ($\mathbf{x} = (x, y)$, $\xi \in \mathbb{R}, \eta \in \mathbb{R}^+$)

$$x = \tfrac{1}{2}(\eta^2 - \xi^2) \ , \qquad y = \xi\eta \ . \tag{2.11.1}$$

The two-dimensional potential $V(\mathbf{x})$ is assumed to have the form[9]

$$V(\mathbf{x}) = \frac{V_\xi(\xi) + V_\eta(\eta)}{\xi^2 + \eta^2} \ . \tag{2.11.2}$$

Therefore we can rewrite the path integral in the following form ($D = 2$)

$$K(\mathbf{x}'', \mathbf{x}'; T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\mathrm{d}t\right]$$

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^N \prod_{k=1}^{N-1}\int_{\mathbb{R}}\mathrm{d}\xi_k\int_0^\infty \mathrm{d}\eta_k\,(\xi_k^2 + \eta_k^2)$$

[9] Note that in this section $V_\xi(\xi)$, etc., does not mean the derivative of V with respect to ξ.

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=0}^{N-1}\left(\frac{m}{2\epsilon}\sqrt{(\xi_j^2+\eta_j^2)(\xi_{j-1}^2+\eta_{j-1}^2)}\,(\Delta^2\xi_j+\Delta^2\eta_j)\right.\right.$$
$$\left.\left. -\,\epsilon\frac{V_\xi(\xi_j)+V_\eta(\eta_j)}{\xi_j^2+\eta_j^2}\right)\right] \tag{2.11.3}$$

$$\equiv \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2+\eta^2)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)-\frac{V_\xi(\xi)+V_\eta(\eta)}{\xi^2+\eta^2}\right)\mathrm{d}t\right] . \tag{2.11.4}$$

This path integral can indeed be separated into the variables η and ξ. However, things do not look so obvious, and more elaborate path integral techniques are needed, i.e., coordinate transformation and "time substitution". The trick here is to get rid of the factor $(\xi^2+\eta^2)$ by a proper redefinition of the time slicing $t_j - t_{j-1}$ according to Sect. 2.9.3, i.e., $t_j - t_{j-1} \to \sqrt{(\xi_j^2+\eta_j^2)(\xi_{j-1}^2+\eta_{j-1}^2)}\,(s_j - s_{j-1})$. This yields the "time-transformed" path integral

$$K(\mathbf{x}'',\mathbf{x}';T) = \int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s''\int\limits_{\xi(0)=\xi'}^{\xi(s'')=\xi''}\mathcal{D}\xi(s)\int\limits_{\eta(0)=\eta'}^{\eta(s'')=\eta''}\mathcal{D}\eta(s)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}(\dot\xi^2+\dot\eta^2)+E(\xi^2+\eta^2)-V_\xi(\xi)-V_\eta(\eta)\right)\mathrm{d}s\right] , \tag{2.11.5}$$

and in this path integral the separation of the coordinates ξ and η is clearly achieved.

The case $V_\xi = V_\eta \equiv 0$ describes the motion of a free particle in two-dimensional Euclidean space in terms of parabolic coordinates, and we can insert the path integral of the one-dimensional harmonic oscillator (see Sect. 3.2). Inserting the solution for each of the coordinates ξ and η we obtain ($\omega^2 = -2E/m$) [280]

$$K_0(x'',x';T) = \int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\frac{m\omega}{\pi\,\mathrm{i}\,\hbar\sin\omega s''}$$
$$\times \exp\left[\frac{\mathrm{i}m\omega}{2\hbar}({\xi'}^2+{\xi''}^2+{\eta'}^2+{\eta''}^2)\cot\omega s''\right]\cosh\left(\frac{m\omega(\xi'\xi''+\eta'\eta'')}{\mathrm{i}\,\hbar\sin\omega s''}\right) \tag{2.11.6}$$

$$= \sum_{\mathrm{e,o}}\int_{\mathbb{R}}\mathrm{d}\zeta\int_{\mathbb{R}}\mathrm{d}k\,\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi'',\eta'')\Psi_{k,\zeta}^{(\mathrm{e,o})\,*}(\xi',\eta')\,\mathrm{e}^{-\mathrm{i}\hbar k^2T/2m} , \tag{2.11.7}$$

where $\sum_{\mathrm{e,o}}$ denotes the summation over even and odd states respectively; the functions $\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta)$ are given by

$$\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta)=\frac{1}{\sqrt{24\pi^2}} \times\begin{pmatrix} \left|\Gamma(\frac{1}{4}-\frac{\mathrm{i}\zeta}{2k})\right|^2 E^{(0)}_{-\frac{1}{2}+\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi)E^{(0)}_{-\frac{1}{2}-\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta) \\ \left|\Gamma(\frac{3}{4}-\frac{\mathrm{i}\zeta}{2k})\right|^2 E^{(1)}_{-\frac{1}{2}+\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi)E^{(1)}_{-\frac{1}{2}-\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta)\end{pmatrix}, \tag{2.11.8}$$

which are normalized according to

$$\int_0^\infty \mathrm{d}\eta\int_{\mathbb{R}}\mathrm{d}\xi\,(\xi^2+\eta^2)\Psi_{k',\zeta'}^{(\mathrm{e,o})\,*}(\xi,\eta)\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta)=\delta(k'-k)\delta(\zeta'-\zeta)\ . \tag{2.11.9}$$

Here the $E_\nu^{(0,1)}(z)$ are even and odd parabolic cylinder functions [413], respectively

$$E_\nu^{(0)}(z)=\sqrt{2}\,\mathrm{e}^{-z^2/4}\,{}_1F_1\bigg(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\bigg) \tag{2.11.10}$$

$$E_\nu^{(1)}(z)=\sqrt{2}\,\mathrm{e}^{-z^2/4}\,{}_1F_1\bigg(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\bigg)\ . \tag{2.11.11}$$

Note that in the evaluation of the path integral one actually uses a double covering of the original (x,y)-plane. Furthermore we have taken into account that the mapping is of the "square-root" type which gives rise to a sign ambiguity [280]. Thus the two contributions arise in (2.11.6). The last line of (2.11.7) is then best obtained by considering a "Coulomb regularization" $-q^2/r$ in $\mathbb{R}^2$, performing a momentum variable transformation $(k_\xi,k_\eta)\to\big(\frac{1}{2k}(\frac{1}{a}+\zeta),\frac{1}{2k}(\frac{1}{a}-\zeta)\big)$ $(a=\hbar^2/mq^2)$ with the new variables (k,ζ), and finally setting $q=0$, i.e. $a=\infty$.

The discussion of three-dimensional parabolic, and two- and three-dimensional spheroidal coordinates is similar, cf. [444, 447]. In the case of elliptic and spheroidal coordinates, particular expansion theorems of plane waves into Mathieu and spheroidal wave functions are needed, cf. [444, 447, 692]. Higher dimensional cases are treated similarly, including generalizations to spaces of positive and negative constant curvature with corresponding coordinate system representations. In Table 2.2 we have listed the coordinate systems in two- and three-dimensional Euclidean space which separate the free particle Schrödinger, respectively Helmholtz equation [119, 447, 462, 714] (d,k', and the a_i are positive parameters which characterize the specific coordinate system, P_i and L_i $(i=1,2,3)$ are the momentum and angular-momentum operators, respectively, and $\{\cdot,\cdot\}$ denotes the anticommutator).

Table 2.2. Coordinate systems in Euclidean space

Observable I in $\mathbb{R}^2$	Coordinate system
$I = P_1^2 - P_2^2$	Cartesian
$I = L_3^2$	Polar
$I = L_3^2 + d^2(P_1^2 - P_2^2)$	Elliptic
$I = \{L_3, P_2\}$	Parabolic
Observables I_1, I_2 in $\mathbb{R}^3$	**Coordinate system**
$I_1 = P_2^2, I_2 = P_3^2$	Cartesian
$I_1 = L_3^2, I_2 = P_3^2$	Circular Polar
$I_1 = L_3^2 + d^2 P_1^2, I_2 = P_3^2$	Circular Elliptic
$I_1 = \{L_3, P_2\}, I_2 = P_3^2$	Circular Parabolic
$I_1 = \boldsymbol{L}^2, I_2 = L_1^2 + k'^2 L_2^2$	Sphero-Elliptic
$I_1 = \boldsymbol{L}^2, I_2 = L_3^2$	Spherical
$I_1 = \boldsymbol{L}^2 - d^2(P_1^2 + P_2^2), I_2 = L_3^2$	Prolate Spheroidal
$I_1 = \boldsymbol{L}^2 + d^2(P_1^2 + P_2^2), I_2 = L_3^2$	Oblate Spheroidal
$I_1 = \{L_1, P_2\} - \{L_2, P_1\}, I_2 = L_3^2$	Parabolic
$I_1 = \boldsymbol{L}^2 + (a_2 + a_3)P_3^2 + (a_1 + a_3)P_2^2 + (a_1 + a_2)P_1^2$, $I_2 = a_1^2 L_1^2 + a_2 L_2^2 + a_3 L_3^2 + a_2 a_3 P_3^2 + a_1 a_3 P_2^2 + a_1 a_2 P_1^2$	Ellipsoidal
$I_1 = L_3^2 - d^2 P_3^2 + d(\{L_2, P_1\} + \{L_1, P_2\})$, $I_2 = d(P_2^2 - P_1^2) + \{L_2, P_1\} - \{L_1, P_2\}$	Paraboloidal

Each coordinate system can be characterized by a set of observables which commute with the Hamiltonian, e.g., in three-dimensional spherical coordinates $L_3, \mathbf{L}^2$. The fact that a physical problem is separable in more than one coordinate system has the consequence that there are additional functionally independent observables. These systems are called superintegrable. Note that separability in spherical and parabolic coordinates implies separability in prolate spheroidal coordinates and the existence of a *Runge–Lenz vector*. This scheme can be generalized to higher dimensional spaces of constant curvature, i.e., including spheres and hyperboloids [462, 735].

3 Basic Path Integrals

In this chapter we present several path integral solutions which we call *basic path integral solutions*. They are:

i) *The free particle.*
ii) *The harmonic oscillator and the general quadratic Lagrangian.* Here we will discuss the general quadratic Lagrangian allowing the various coefficients to have an arbitrary time dependence. The evaluation will be presented in an elementary way. This path integral solution (with the free particle as a special case) is our first basic path integral solution, the *Gaussian path integral.*
iii) *Path integration over group manifolds.* Here we will include:
   a) *The general formalism for path integrals over group manifolds.*
   b) *Path integration in spherical coordinates.* We will discuss the various features of properly defined path integrals including the "Besselian functional weight". The most important result is the path integral identity derived from the *radial harmonic oscillator*, the *Besselian path integral*, the second of the basic path integral solutions.
   c) *Path integration over the group manifolds* SU(2) *and* SU(1, 1). Here the two basic path integral solutions for the Pöschl–Teller and modified Pöschl–Teller potential are derived, the two *Legendrian path integrals.*
   d) We also use an alternative coordinate space representation on SU(1, 1) to derive a path integral representation of the inverted Liouville problem.
   e) We state several expansion formulæ for elliptic and spheroidal coordinates in two- and three-dimensional flat space, two- and three-dimensional Minkowski space, and on the two-dimensional sphere in terms of elliptic and spheroidal wave functions.

Whereas the four most important basic path integral solutions are derived from the path integrals of the harmonic oscillator, the radial harmonic oscillator, the Pöschl–Teller and the modified Pöschl–Teller potential, many other path integral identities can be derived from special cases of the former. Among them the following path integral solutions can be mentioned: the free particle, the linear potential, the quantum motion on the sphere, and the quantum motion on hyperboloids.

3.1 The Free Particle

Let us start with the simplest example, i.e., with the free quantum motion in D-dimensional Euclidean space. From the representation ($\mathbf{x} \in \mathbb{R}^D$, $\Delta\mathbf{x}_j = \mathbf{x}_j - \mathbf{x}_{j-1}$)

$$K_0(\mathbf{x}'', \mathbf{x}'; T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathbf{x}}^2 \mathrm{d}t\right)$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{ND/2} \prod_{k=1}^{N-1} \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}_k \exp\left[\frac{\mathrm{i}\, m}{2\epsilon\hbar}\sum_{j=1}^{N} (\Delta\mathbf{x}_j)^2\right] \qquad (3.1.1)$$

it is obvious that the kernel factorizes into a D-dimensional product of the one-dimensional kernel, see (2.1.35), (2.1.58) and Sect. 2.3:

$$K_0(\mathbf{x}'', \mathbf{x}'; T) = \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{D/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar T}(\mathbf{x}'' - \mathbf{x}')^2\right] \qquad (3.1.2)$$

$$= \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} \exp\left[\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}'' - \mathbf{x}') - \frac{\mathrm{i}}{\hbar} T \frac{\hbar^2\mathbf{k}^2}{2m}\right] \mathrm{d}\mathbf{k} \ . \qquad (3.1.3)$$

From this representation the normalized wave functions and the energy spectrum can be read off[1]

$$\Psi_{\mathbf{k}}(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{D/2}} \ , \qquad E_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m} \qquad (\mathbf{x}, \mathbf{k} \in \mathbb{R}^D) \ . \qquad (3.1.4)$$

Let us note that one can easily incorporate the effect of a *constant* vector-potential $\mathbf{A}$ coupled to $\dot{\mathbf{x}}$ which gives rise to the Lagrangian

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}} \ . \qquad (3.1.5)$$

By a simple shift of variables we obtain

$$K^{(\mathbf{A})}(\mathbf{x}'', \mathbf{x}'; T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}}\right)\mathrm{d}t\right]$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{D/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar T}(\mathbf{x}'' - \mathbf{x}')^2 + \frac{\mathrm{i}\, e}{c\hbar}\mathbf{A}\cdot(\mathbf{x}'' - \mathbf{x}')\right] \qquad (3.1.6)$$

$$= \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} \exp\left[\mathrm{i}\left(\mathbf{k} + \frac{e\mathbf{A}}{c\hbar}\right)\cdot(\mathbf{x}'' - \mathbf{x}') - \frac{\mathrm{i}\,\hbar T}{2m}\mathbf{k}^2\right] \mathrm{d}\mathbf{k} \qquad (3.1.7)$$

with respective energies and wave functions

$$E_{\mathbf{k}}^{(\mathbf{A})} = \frac{\hbar^2}{2m}\mathbf{k}^2 \ , \qquad \Psi_{\mathbf{k}}^{(\mathbf{A})}(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{D/2}}\,\mathrm{e}^{\mathrm{i}(e/c\hbar)\mathbf{A}\cdot\mathbf{x}} \ . \qquad (3.1.8)$$

---

[1] Notice, that here we use the wavenumber $\boldsymbol{k} := \boldsymbol{p}/\hbar$, while in (2.1.29)–(2.1.35) we have used (in the one-dimensional case) the momentum $\boldsymbol{p}$.

3.2 The Quadratic Lagrangian

Next, we consider the time-dependent harmonic oscillator with a driving force and a magnetic field, i.e., we consider the path integral for the one-dimensional Lagrangian ($x \in \mathbb{R}$)

$$\mathcal{L}(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - \frac{c(t)}{2}x^2 + b(t)x\dot{x} - e(t)x \ . \tag{3.2.1}$$

We assume that the various coefficients may be time-dependent as indicated. In contrast to the next case, we restrict ourselves here to the case where the mass m of the particle is time-independent. The present case still allows a rather elementary path integral treatment. In the following section this restriction is abandoned which requires an alternative and more general method for the evaluation of the path integral for the harmonic oscillator. The general case is again discussed in Chap. 4 in the more elaborated context of the semiclassical expansion. Note that the explicit time dependence of the coefficients requires the time ordered formulation of the time-evolution operator. This difficulty does not affect the general feature of the formulation of the path integral, see remark x) in Sect. 2.2. Note that in this case the physical system is no longer conservative, i.e., the conservation of energy is violated.

The path integral has the following form, where the mid-point formulation is assumed

$$K(x'',x';t'',t') = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}x(t)\exp\left(\frac{\mathrm{i}}{\hbar}R[x]\right) \ . \tag{3.2.2}$$

The ordering problem appears in the "b(t)-term", where we have in the corresponding Hamiltonian an xp-term. This has the consequence that we have to use the mid-point formulation according to

$$\begin{aligned}
&K(x'',x';t'',t')\\
&= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{c(t)}{2}x^2 + b(t)x\dot{x} - e(t)x\right)\mathrm{d}t\right]\\
&= \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_{\mathbb{R}}\mathrm{d}x_k\\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}(\Delta x_j)^2 - \epsilon\frac{c_j}{2}\bar{x}_j^2 + b_j\bar{x}_j\Delta x_j - \mathrm{e}_j\,\bar{x}_j\right)\right] \ .
\end{aligned} \tag{3.2.3}$$

($c_j = c(t_j)$, etc., $\bar{x}_j = (x_j + x_{j-1})/2$). We expand the path $x(t)$ about the classical path $x_{\mathrm{Cl}}(t)$, i.e.,

$$x(t) = x_{\mathrm{Cl}}(t) + q(t) \ ,$$

where $q(t)$ denotes the quantum fluctuations about the classical path with $q(t') = q(t'') = 0$. The classical path obeys the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}(x_{\mathrm{Cl}},\dot{x}_{\mathrm{Cl}})}{\partial\dot{x}_{\mathrm{Cl}}} - \frac{\partial\mathcal{L}(x_{\mathrm{Cl}},\dot{x}_{\mathrm{Cl}})}{\partial x_{\mathrm{Cl}}} = 0 \ , \tag{3.2.4}$$

with the boundary conditions $x_{\mathrm{Cl}}(t') = x'$, $x_{\mathrm{Cl}}(t'') = x''$. Expanding about the classical path we obtain for the action

$$\begin{aligned} R[x] = R[x_{\mathrm{Cl}} + q] &= R[x_{\mathrm{Cl}}] + \int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2 - \frac{c}{2}q^2 + bq\dot{q}\right]\mathrm{d}t \\ &= R[x_{\mathrm{Cl}}] + \int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2 - \frac{1}{2}(c+\dot{b})q^2\right]\mathrm{d}t \ . \end{aligned} \tag{3.2.5}$$

The linear term in $q(t)$ vanishes due to the Euler–Lagrange equation (3.2.4). This gives for the path integral the factorization (see also Sect. 2.4)

$$K(x'',x';t'',t') = \exp\left(\frac{\mathrm{i}}{\hbar}R[x_{\mathrm{Cl}}]\right)F(t'',t') \ , \tag{3.2.6}$$

$$F(t'',t') = \int\limits_{q(t')=0}^{q(t'')=0}\mathcal{D}q(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{q}^2 - \omega^2(t)q^2)\mathrm{d}t\right] \ , \tag{3.2.7}$$

where we have used the abbreviation $m\omega^2(t) := c(t) + \dot{b}(t)$. Now

$$\begin{aligned} F(t'',t') = \lim_{N\to\infty} F_N := \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_{\mathbb{R}}\mathrm{d}q_k \\ \times\exp\left[-\frac{m}{2\,\mathrm{i}\,\epsilon\hbar}\sum_{j=1}^{N}\left((\Delta q_j)^2 - \epsilon^2\omega_j^2q_j^2\right)\right] \ , \end{aligned} \tag{3.2.8}$$

i.e., a path integral for a simple harmonic oscillator with time-dependent frequency and the boundary conditions $q' = q'' = 0$ remains. Let us introduce the $(N-1)\times(N-1)$ matrix $\mathbf{B}_N$:

$$\mathbf{B}_N = \begin{pmatrix} 2-\epsilon^2\omega_1^2 & -1 & \dots & 0 & 0 \\ -1 & 2-\epsilon^2\omega_2^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2-\epsilon^2\omega_{N-2}^2 & -1 \\ 0 & 0 & \dots & -1 & 2-\epsilon^2\omega_{N-1}^2 \end{pmatrix} \ . \tag{3.2.9}$$

Thus we get (see (2.3.4))

$$F_N = \left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{N/2}\int_{\mathbb{R}^{N-1}}\mathrm{d}\mathbf{q}\exp\left(-\frac{m}{2\,\mathrm{i}\,\epsilon\hbar}\mathbf{q}^{\mathrm{t}}\mathbf{B}_N\mathbf{q}\right)$$

$$= \sqrt{\frac{m}{2\pi \mathrm{i}\,\epsilon\hbar \det \mathbf{B}_N}} \ . \tag{3.2.10}$$

Therefore[2]

$$F(t'',t') = \sqrt{\frac{m}{2\pi \mathrm{i}\,\hbar f(t'',t')}} \ , \qquad \text{where} \qquad f(t'',t') = \lim_{N\to\infty} \epsilon \det \mathbf{B}_N \ . \tag{3.2.11}$$

Our final task is to determine $f(t'',t')$. Let us consider the $j \times j$ matrix $\mathbf{B}_j$ for which the following recursion relation holds:

$$\det \mathbf{B}_{j+1} = (2 - \epsilon^2\omega_{j+1}^2) \det \mathbf{B}_j - \det \mathbf{B}_{j-1}$$

with $\det \mathbf{B}_1 = 2 - \epsilon^2\omega_1^2$ and $\det \mathbf{B}_0 = 1$. Let us define the quantity $g_j = \epsilon \det \mathbf{B}_j$, then we have

$$g_{j+1} - 2g_j + g_{j-1} = -\epsilon^2\omega_{j+1}^2 g_j \ . \tag{3.2.12}$$

Turning to the continuum limit we find for the function $g(t)$ with $g(t_j) = g_j$ the differential equation:

$$\ddot{g}(t) + \omega^2(t)g(t) = 0 \ , \quad \text{with} \quad g(t') = 0 \ , \quad \dot{g}(t') = 1 \ . \tag{3.2.13}$$

The boundary conditions follow from $g(t') = g_0 = \lim_{\epsilon\to 0} \epsilon \det \mathbf{B}_0 = 0$ and $\dot{g}(t') = \lim_{\epsilon\to 0} \big[g(t'+\epsilon) - g(t')\big]/\epsilon = \lim_{\epsilon\to 0}(\det B_1 - \det B_0) = 1$. Finally we have to insert $g(t'') = f(t'',t')$ into (3.2.11).

At once we get in the free particle case $g(t) = t - t'$, i.e., $g(t'') = t'' - t' = T$ which gives the correct result $f(t'',t') = T$, see (2.3.15).

In the case of the usual harmonic oscillator with $\omega(t) = \omega$ (time-independent) one obtains

$$g_{\text{osc}}(t) = \frac{1}{\omega} \sin \omega(t - t') \ , \tag{3.2.14}$$

and thus $f(t'',t') = \sin \omega T/\omega$, see (2.4.18). To obtain the complete path integral solution for the harmonic oscillator we must calculate its classical action. It is a straightforward calculation to show that it is given by

$$\begin{aligned} R_{\text{osc,Cl}} &= R_{\text{osc,Cl}}(x'',x';T) := R_{\text{osc}}[x_{\text{Cl}}] \\ &= \frac{m\omega}{2\sin\omega T}\Big[(x''^2 + x'^2)\cos\omega T - 2x''x'\Big] \ . \end{aligned} \tag{3.2.15}$$

Thus we obtain the following result for the *Feynman kernel of the harmonic oscillator* which is the first of our basic path integral solutions ($0 < \omega T < \pi$):

[2] The expression for $F(t'',t')$ should be compared with the Gaussian path integral (2.3.16) expressed in terms of the functional determinant, i.e., $2f(t'',t') = \det \underset{\sim}{A}$ with $\underset{\sim}{A} := -\mathrm{d}^2/\mathrm{d}t^2 - \omega^2(t)$. See also Sect. 2.4 for the case $\omega(t) = \omega = \text{const}$.

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2-\omega^2x^2)\,\mathrm{d}t\right]$$
$$= \sqrt{\frac{m\omega}{2\pi\mathrm{i}\,\hbar\sin\omega T}}\exp\left\{\mathrm{i}\frac{m\omega}{2\hbar}\left[(x'^2+x''^2)\cot\omega T - 2\frac{x'x''}{\sin\omega T}\right]\right\} \ . \tag{3.2.16}$$

By use of the *Mehler formula* [303, Vol. III, p. 272]:

$$\mathrm{e}^{-(x^2+y^2)/2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{z}{2}\right)^n H_n(x)H_n(y)$$
$$= \frac{1}{\sqrt{1-z^2}}\exp\left[\frac{4xyz-(x^2+y^2)(1+z^2)}{2(1-z^2)}\right] \ , \tag{3.2.17}$$

where H_n denote the Hermite polynomials, we can expand[3] the Feynman kernel according to (identify $x \equiv \sqrt{m\omega/\hbar}\,x'$, $y \equiv \sqrt{m\omega/\hbar}\,x''$ and $z \equiv \mathrm{e}^{-\mathrm{i}\omega T}$):

$$K(x'',x';T) = \sum_{n=0}^{\infty}\Psi_n(x')\Psi_n(x'')\,\mathrm{e}^{-\mathrm{i}TE_n/\hbar} \ , \tag{3.2.18}$$

and then read off the energy spectrum and the normalized wave functions

$$E_n = \hbar\omega(n+\tfrac{1}{2}) \ , \tag{3.2.19}$$
$$\Psi_n(x) = \left(\frac{m\omega}{2^{2n}\pi\hbar(n!)^2}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ . \tag{3.2.20}$$

Equation (3.2.16) is as it stands well defined only for $0 < \omega T < \pi$. Let us investigate it in other time intervals [273] and set

$$T = \frac{n\pi}{\omega}+\tau, \qquad n\in\mathbb{N}_0, \quad 0<\tau<\pi/\omega \ . \tag{3.2.21}$$

Then we have $\sin\omega T = \mathrm{e}^{\mathrm{i}\pi n}\sin\omega\tau$, $\cos\omega T = \mathrm{e}^{\mathrm{i}\pi n}\cos\omega\tau$ and (3.2.16) becomes for $n\pi < \omega T < (n+1)\pi$

$$K_{\mathrm{osc}}(x'',x';T) = \sqrt{\frac{m\omega}{2\pi\hbar|\sin\omega T|}}\exp\left[-\frac{\mathrm{i}\pi}{2}\left(\frac{1}{2}+n\right)\right]$$
$$\times\exp\left\{\frac{\mathrm{i}\, m\omega}{2\hbar\sin\omega T}\left[(x'^2+x''^2)\cos\omega T - 2x'x''\right]\right\} \ , \tag{3.2.22}$$

[3] The Mehler formula (3.2.17) holds for $x, y \in \mathbb{R}$ and $|z| < 1$, and thus it appears that we have a problem, since in our case we have $z = \mathrm{e}^{-\mathrm{i}\omega T}$ and therefore $|z| = 1$. There is, of course, no real problem since the Feynman kernel and the expansion (3.2.18) are well defined as distributions. Nevertheless, it may sometimes be worthwhile to consider the Euclidean Feynman kernel, see remark vi) in Sect. 2.2, for which $|z| = |\,\mathrm{e}^{-\omega T}\,| < 1$, and no convergence problems arise.

which is the correct formula for the propagator of the harmonic oscillator. Notice that the last formula holds for all $T > 0$, if one replaces the phase factor $-\mathrm{i}(\pi/2)(n+\frac{1}{2})$ by $-\mathrm{i}(\pi/4) - \mathrm{i}(\pi/2)\nu(T)$, where $\nu(T) := [\omega T/\pi]$ is the so-called *Morse index* (see Chap. 5, $[x]$ denotes the integer part of x). One then obtains the correct relation

$$\lim_{T\to(n\pi/\omega)^+} K_{\mathrm{osc}}(x'',x';T) = \mathrm{e}^{-\mathrm{i}n\pi/2}\,\delta(x''-(-1)^n x') \tag{3.2.23}$$

which generalizes the initial condition (2.1.23) for $T \to 0^+$.

Finally we observe that the path integral for the D-dimensional harmonic oscillator can be expressed in terms of the *Morette–Van Hove determinant* (see Sect. 5.2, remark v)) as follows

$$\begin{aligned} K_{\mathrm{osc}}(\mathbf{x}'',\mathbf{x}';T) = \frac{1}{(2\pi\mathrm{i}\hbar)^{D/2}}&\left|\det\left(-\frac{\partial^2 R_{\mathrm{osc,Cl}}}{\partial x''^a\partial x'^b}\right)\right|^{1/2} \\ &\times\exp\left[\frac{\mathrm{i}}{\hbar}R_{\mathrm{osc,Cl}}(\mathbf{x}'',\mathbf{x}';T) - \mathrm{i}\frac{\pi}{2}D\nu(T)\right] , \end{aligned} \tag{3.2.24}$$

where Hamilton's principal function is given by

$$R_{\mathrm{osc,Cl}}(\mathbf{x}'',\mathbf{x}';T) = \frac{m\omega}{2\sin\omega T}\left[(\mathbf{x}'^2+\mathbf{x}''^2)\cos\omega T - 2\mathbf{x}'\cdot\mathbf{x}''\right] . \tag{3.2.25}$$

3.3 The Radial Harmonic Oscillator

Let us now discuss the most important application of the radial path integral (2.7.15), namely the radial harmonic oscillator with $V(r) = \frac{1}{2}m\omega^2 r^2$. The original calculation is due to Peak and Inomata [771], and has been investigated by Duru [274], Goovaerts [402], and Inomata [518]. We present the more general case with time-dependent frequency $\omega(t)$ following Goovaerts [402].

We have to study the kernel

$$\begin{aligned} K_l(r'',r';t'',t') &= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left[\frac{\mathrm{i}}{\hbar}\frac{m}{2}\int_{t'}^{t''}(\dot r^2-\omega^2(t)r^2)\mathrm{d}t\right] \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{N/2}\prod_{n=1}^{N-1}\int_0^\infty \mathrm{d}r_n\mu^N_{l+\frac{D-2}{2}}[r^2] \\ &\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\frac{m}{2}\sum_{j=1}^{N}\left(\frac{(r_j-r_{j-1})^2}{\epsilon}-\epsilon\omega_j^2 r_j^2\right)\right] \\ &= \sqrt{r'r''}\lim_{N\to\infty}\left(\frac{\alpha}{\mathrm{i}}\right)^N \mathrm{e}^{\mathrm{i}\alpha(r'^2+r''^2)/2} \end{aligned}$$

$$\times \prod_{j=1}^{N-1} \int_0^\infty r_j \mathrm{d}r_j \exp\left[\mathrm{i}(\beta_1 r_1^2 + \beta_2 r_2^2 + \ldots + \beta_{N-1} r_{N-1}^2)\right]$$
$$\times \left[I_{l+\frac{D-2}{2}}(-\mathrm{i}\,\alpha r_0 r_1) \times \ldots \times I_{l+\frac{D-2}{2}}(-\mathrm{i}\,\alpha r_{N-1} r_N)\right] \tag{3.3.1}$$

($\alpha = m/\epsilon\hbar$, $\beta_j = \alpha(1-\epsilon^2\omega_j^2/2)$). By means of the integral formula [413, p. 718]

$$\int_0^\infty x\,\mathrm{e}^{-\gamma x^2} J_\nu(\alpha x) J_\nu(\beta x)\mathrm{d}x = \frac{1}{2\gamma}\mathrm{e}^{-(\alpha^2+\beta^2)/4\gamma} I_\nu\left(\frac{\alpha\beta}{2\gamma}\right) , \tag{3.3.2}$$

and its analytic continuation to pure imaginary argument [771], the convoluted integrations are performed through a recursion and it is found that the kernel K_l can be cast into the form

$$K_l(r'',r';t'',t') = \sqrt{r'r''}\,\frac{m}{\mathrm{i}\,\hbar\eta(T)}$$
$$\times \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(\frac{\xi(T)}{\eta(T)}r'^2 + \frac{\dot\eta(T)}{\eta(T)}r''^2\right)\right] I_{l+\frac{D-2}{2}}\left(\frac{mr'r''}{\mathrm{i}\,\hbar\eta(T)}\right) . \tag{3.3.3}$$

The quantities $\eta(T)$ and $\xi(T)$, respectively, are determined by the following differential equations with boundary conditions

$$\ddot\eta + \omega^2(t)\eta = 0 , \qquad \eta(t') = 0 , \qquad \dot\eta(t') = 1 , \tag{3.3.4}$$
$$\ddot\xi + \omega^2(t)\xi = 0 , \qquad \xi(t') = 1 , \qquad \dot\xi(t') = 0 . \tag{3.3.5}$$

In particular for $\omega(t) = \omega$ (time independent) we obtain

$$\eta(t) = \frac{1}{\omega}\sin\omega(t-t') , \qquad \dot\eta(t) = \cos\omega(t-t') , \qquad \xi(t) = \cos\omega(t-t') .$$

This yields the radial path integral solution for the radial harmonic oscillator with time-independent frequency

$$K_l(r'',r';T) = \sqrt{r'r''}\,\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}$$
$$\times \exp\left[\frac{\mathrm{i}\,m\omega}{2\hbar}(r'^2 + r''^2)\cot\omega T\right] I_{l+\frac{D-2}{2}}\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sin\omega T}\right) . \tag{3.3.6}$$

The next step is to calculate with the help of (3.3.6) the energy-levels and wave functions. For this purpose we use the *Hille-Hardy formula* [303, Vol. II, p. 189] ($|t| < 1$)

$$\frac{t^{-\alpha/2}}{1-t}\exp\left[-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right] I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right)$$
$$= \sum_{n=0}^\infty \frac{t^n n!\,\mathrm{e}^{-\frac{1}{2}(x+y)}}{\Gamma(n+\alpha+1)}(xy)^{\alpha/2} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) , \tag{3.3.7}$$

where $L_n^{(\alpha)}$ are Laguerre polynomials. With the substitution $t = \mathrm{e}^{-2\mathrm{i}\omega T}$, $x = m\omega r'^2/\hbar$ and $y = m\omega r''^2/\hbar$ in (3.3.6) we get finally:

$$K_l(r'', r'; T) = \sum_{N=0}^{\infty} \mathrm{e}^{-\mathrm{i}TE_N/\hbar} R_N^l(r') R_N^l(r'') \tag{3.3.8}$$

$$E_n = \hbar\omega\left(N + \frac{D}{2}\right) \tag{3.3.9}$$

$$R_N^l(r) = \sqrt{\frac{2m\omega}{\hbar r^{D-2}} \cdot \frac{\Gamma(\frac{N-l}{2}+1)}{\Gamma(\frac{N+l+D}{2})}} \left(\frac{m\omega}{\hbar}r^2\right)^{l+\frac{D-2}{2}} \times \exp\left(-\frac{m\omega}{\hbar}r^2\right) L_{\frac{N-l}{2}}^{(l+\frac{D-2}{2})}\left(\frac{m\omega}{\hbar}r^2\right) . \tag{3.3.10}$$

The path integral for the harmonic oscillator suggests a generalization in the index l. For this purpose we consider equation (2.7.15) for $D = 3$ with the functional weight $\mu_{l+\frac{1}{2}}[r^2]$

$$K_l(r'', r'; T) = \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{1}{2}}[r^2] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right] . \tag{3.3.11}$$

The functional weight corresponds to the centrifugal potential $\frac{\hbar^2 l(l+1)}{2mr^2}$ in the Schrödinger equation. Replacing $l + \frac{1}{2}$ by λ corresponds to the "centrifugal potential" $V_\lambda(r) := \hbar^2(\lambda^2 - \frac{1}{4})/2mr^2$ in the Schrödinger equation where λ can also be complex, provided $\Re\lambda > \frac{1}{4}$. This situation corresponds to the classical radial Lagrangian $\mathcal{L} = \frac{m}{2}\dot{r}^2 - \hbar^2(\lambda^2 - \frac{1}{4})/2mr^2 - V(r)$. With $V(r) = \frac{m}{2}\omega^2 r^2$ this leads to the path integral expression

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{\lambda^2 - \frac{1}{4}}{2mr^2} - \frac{m}{2}\omega^2 r^2\right)\mathrm{d}t\right] . \tag{3.3.12}$$

Let us remark that in the literature one often uses the asymptotic form of the modified Bessel function, e.g. [771]

$$I_\nu(z) \simeq (2\pi z)^{-\frac{1}{2}} \mathrm{e}^{z - (\nu^2 - \frac{1}{4})/2z} \qquad (|z| \gg 1, \quad \Re(z) > 0) , \tag{3.3.13}$$

and the functional weight becomes (ignoring the condition $\Re(z) > 0$):

$$\mu_\lambda^N[r^2] \overset{(\epsilon\to 0)}{\simeq} \exp\left\{-\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N} \epsilon \frac{\hbar^2(\lambda^2 - \frac{1}{4})}{2mr_j r_{j-1}}\right\} . \tag{3.3.14}$$

According to Fischer, Leschke and Müller [343] the functional weight approach according to (3.3.11) defines expressions such as (3.3.12) in a unique way such that we can use the asymptotic form of the functional weight in the path integral still and thus have the following equivalence in terms of the functional weight formulation (for more details we refer to the literature, e.g. [343, 464, 528, 771, 865])

$$
\begin{aligned}
&\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{\lambda^2-\frac{1}{4}}{2mr^2} - \frac{m}{2}\omega^2 r^2\right)\mathrm{d}t\right] \\
&\quad\equiv \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2 r^2)\,\mathrm{d}t\right] \\
&\quad= \frac{\sqrt{r'r''}m\omega}{\mathrm{i}\,\hbar\sin\omega T}\exp\left[\frac{\mathrm{i}\,m\omega}{2\hbar}\left(r'^2+r''^2\right)\cot\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sin\omega T}\right) \quad . \qquad (3.3.15)
\end{aligned}
$$

This path integral identity is the second of our basic path integrals, the *Besselian path integral.*

3.4 Path Integration Over Group Manifolds

Before we discuss in detail the path integration on specific group manifolds, let us start with some general remarks. We are interested in a general formalism which tells us how to perform path integration on a group manifold. However, we must first ask, what are the relevant objects in the group manifold we want to concentrate on? Is it possible to perform the group path integration in such a way that we can calculate the kernel explicitly? Is it more appropriate to consider an expansion into the characters of the group, respectively into matrix elements in the coordinate space representation? As we will see, it depends on the group which of the alternatives can be applied. Actually, most authors have concentrated on a character expansion of the path integral, see e.g. [104, 262, 263, 679, 726, 776]. However, this is not the only possibility. It has the advantage to be coordinate independent, though. For several applications, an expansion into the matrix elements in the coordinate space representation is nevertheless desirable. Path integral identities can be extracted from such spectral expansions. In some cases it is also possible to state the kernel explicitly, and in other cases the Green function can be explicitly stated. In the following we present some general arguments that tell us which calculational tools can be used to consider and perform path integration on a group manifold.

In our presentation we follow the lines of reasoning of Böhm and Junker [104] (see also [106, 523, 551]). First we will give an introduction to the general theory, followed by the path integration over SU(2) and SU(1, 1). These two group manifolds and the path integral identities derived from them will be of great importance in the applications, where we have to deal with Pöschl–Teller potential and modified Pöschl–Teller potential problems, respectively.

3.4.1 General Formalism. In order to set up our notation we start with a D-dimensional flat space with an indefinite metric according to

$$(g_{ab}) = \mathrm{diag}\,(\underbrace{+1,\ldots,+1}_{p\ \text{times}},\underbrace{-1,\ldots,-1}_{q\ \text{times}}) \ . \tag{3.4.1}$$

Consequently, we write a classical Lagrangian on the lattice for such an indefinite metric as follows

$$\mathcal{L}^N(\mathrm{x}_j,\Delta\,\mathrm{x}_j) = \frac{m}{2\epsilon}\Big[(\Delta x_j^1)^2+\ldots+(\Delta x_j^p)^2-(\Delta x_j^{p+1})^2-\ldots-(\Delta x_j^{p+q})^2\Big]-V(\mathrm{x}_j)\,. \tag{3.4.2}$$

In order to take into account the indefinite metric in this *pseudo-Euclidean space* $E_{p,q}$ endowed with the metric (3.4.1), we must change the measure in the path integral according to

$$\begin{aligned}K(\mathrm{x}'',\mathrm{x}';T) = \lim_{N\to\infty}&\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{Np/2}\left(\frac{\mathrm{i}\,m}{2\pi\epsilon\hbar}\right)^{Nq/2}\prod_{l=1}^{N-1}\int_{\mathbb{R}^{p+q}}\mathrm{d}\mathrm{x}_l\\ &\times\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N\left(\frac{m}{2\epsilon}\left(\sum_{k=1}^p(\Delta x_j^k)^2-\sum_{k=p+1}^{p+q}(\Delta x_j^k)^2\right)-\epsilon V(\mathrm{x}_j)\right)\right] \ .\end{aligned} \tag{3.4.3}$$

To regularize such a path integral, the various integrals must be treated separately [104]: in the integrations over the variables with positive signature (compact variables) in the metric, a small positive imaginary part has to be added to the mass ($m\to m+\mathrm{i}\,\eta$, $\eta>0$), whereas in the variables with negative signature (non-compact variables) a small negative imaginary part ($\eta\to m-\mathrm{i}\,\eta$, $\eta>0$) must be added.

We consider the scalar product

$$(\mathrm{x},\mathrm{x}) = (x^1)^2+\ldots+(x^p)^2-(x^{p+1})^2-\ldots-(x^{p+q})^2 \ , \tag{3.4.4}$$

and introduce the sets

$$\left.\begin{aligned}T_{+1} &= \{\mathrm{x}\,|\,(\mathrm{x},\mathrm{x})>0\} \qquad &&(\text{timelike})\\ T_{-1} &= \{\mathrm{x}\,|\,(\mathrm{x},\mathrm{x})<0\} \qquad &&(\text{spacelike})\ .\end{aligned}\right\} \tag{3.4.5}$$

We want to achieve in the following an expansion of $\exp[z(\mathrm{e}_{j-1},\mathrm{e}_j)]$ in terms of the matrix elements of the corresponding group representations (in particular for rotation groups). Let G be such a group and $g\in G$. Let $\mathcal{L}^2$ be a linear vector space of functions f, actually $\mathcal{L}^2=\mathcal{L}^2(\mathbb{R}^{p+q})$. Then

$$f(\mathrm{x})\in\mathcal{L}^2 \leftrightarrow f(g\,\mathrm{x})\in\mathcal{L}^2 \ , \qquad \mathrm{x}\in\mathbb{R}^{p+q}$$

for any $g\in G$. A regular representation of G is given by

$$D(g)f(\mathrm{x}) = f(g^{-1}\,\mathrm{x}) \ . \tag{3.4.6}$$

Such a representation D is decomposable into a discrete sum of unitary irreducible representations D^l in the Hilbert space $\mathcal{L}^2$. Furthermore, let H be a subgroup of G which leaves the non-zero vector $a \in \mathcal{L}^2$ invariant, i.e.,

$$D^l(h)a = a, \qquad h \in H \subset G \ . \tag{3.4.7}$$

Let (G, H) be a Gelfand pair, then G/H is a symmetric space and

$$\mathcal{L}^2(G/H) = \bigoplus_{l \in \Lambda} D^l, \tag{3.4.8}$$

where $l \in \Lambda$ denotes the class one representations. With each vector $f \in \mathcal{L}^2$ we may associate a scalar function

$$f^l(g) = (D^l(g)f, a) \ . \tag{3.4.9}$$

The $f^l(g)$ are called spherical functions of the representation $D^l(g)$. Choosing a basis $\{b_i\}$ in $\mathcal{L}^2$ so that $b_0 = a$, the matrix elements of $D^l(g)$ are given by

$$D^l_{mn}(g) = (D^l(g)b_m, b_n) \ . \tag{3.4.10}$$

In particular the D^l_{0m} are called associated spherical functions and the D^l_{00} the zonal spherical functions [911]. We have the identities ($h \in H$)

$$D^l_{m0}(gh) = D^l_{m0}(g) \ , \qquad D^l_{00}(h^{-1}gh) = D^l_{00}(g) \ . \tag{3.4.11}$$

An important property of the spherical functions is that they are eigenfunctions of the corresponding Laplace–Beltrami operator Δ_{LB} on the homogeneous space $M = G/H$. Of course, the Hilbert space $\mathcal{L}^2$ is spanned by a complete set of associated spherical functions D^l_{0m}.

We introduce harmonic analysis, i.e., a generalized Fourier transformation on locally compact groups: let $f \in \mathcal{L}^2$ and D^l_{mn} as defined in (3.4.10). The Fourier transform $\hat{f}^l_{mn}$ of $f(g)$ is defined as

$$f(g) = \int \mathrm{d}E_l \, d_l \sum_{m,n} \hat{f}^l_{mn} D^l_{mn}(g) \ , \tag{3.4.12}$$

$$\hat{f}^l_{mn} = \int_G f(g) D^{l\,*}_{mn}(g^{-1}) \, \mathrm{d}g \ , \tag{3.4.13}$$

and $\mathrm{d}g$ is the invariant group (Haar) measure. $\int \mathrm{d}E_l$ stands for a Lebesgue–Stieltjes integral to include discrete ($\int \mathrm{d}E_l \to \sum_l$) as well as continuous representations. $\int \mathrm{d}E_l$ is to be taken over the complete set $\{l\}$ of class one representations. d_l denotes (in the compact case) the dimension of the representation and we take

$$d_l \int_G D^l_{mn}(g) D^{l'\,*}_{m'n'}(g) \, \mathrm{d}g = \delta(l, l') \delta_{m,m'} \delta_{n,n'} \tag{3.4.14}$$

as a definition for d_l. $\delta(l,l')$ can denote a Kronecker delta, respectively, a δ-function, depending on whether the quantity l is a discrete or continuous parameter.

We have furthermore the group composition law

$$D^l_{mn}(g_a^{-1}g_b) = \sum_k D^{l\,*}_{kn}(g_a)D^l_{km}(g_b) \ . \tag{3.4.15}$$

We must analyze under which conditions $(\mathbf{e}_{j-1},\mathbf{e}_j)$ can be expressed in terms of the group elements g_{j-1} and g_j in order to apply the expansion (3.4.15). Let G be a transformation group of H_α, i.e.,

$$\mathbf{e} = g\mathbf{a}, \qquad \mathbf{e},\mathbf{a} \in H_\alpha \tag{3.4.16}$$

where $g \in G$ denotes a $(D \times D)$ matrix representation of G $(D = p+q)$, and $\mathbf{e},\mathbf{a} \in T_\alpha$ with the same $\alpha = \pm 1$. The unit sphere H_α is covered by all possible rotations. Possible choices for G are groups which contain $\mathrm{SO}(p,q)$, respectively $\mathrm{SU}(u,v)$. For example, the four-dimensional sphere $S^{(3)}$ is isomorphic to the group manifold SU(2) and instead of SO(4) we may choose SU(2) as a transformation group of $S^{(3)} = \mathrm{SO}(4)/\mathrm{SO}(3)$, i.e., a rotation group.

Generally, we are most interested in a path integral representation in generalized spherical coordinates in some geometry in the sets $T_\alpha \in E_{p,q}$, i.e.,

$$\mathbf{x} = r\boldsymbol{\Omega} \equiv r\mathbf{e}(\vartheta^1,\dots,\vartheta^{p+q-1}) \ , \qquad \nu = 1,\dots,p+q \ , \tag{3.4.17}$$

where the $\mathbf{e}$ are unit vectors in T_α. The $\mathbf{e}$ span the unit sphere H_α: $H_\alpha = \{\mathbf{e}|(\mathbf{e},\mathbf{e}) = \alpha\}$, $\alpha = +1,-1$. We express the Lagrangian (3.4.2) restricted to T_α in terms of the spherical coordinates (3.4.17) yielding $(\mathrm{x} \in \mathbb{R}^{p+q})$

$$\begin{aligned}
&\mathcal{L}(\mathrm{x}_j,\Delta\mathrm{x}_j)\\
&= \pm\frac{m}{2\epsilon^2}\{(\Delta r_j)^2 + 2r_{j-1}r_j[1 \mp (\mathrm{e}_{j-1},\mathrm{e}_j)]\} - V(\mathrm{x}_j) - \Delta V(\mathrm{x}_j)\\
&= \pm\frac{m}{2\epsilon^2}[r_{j-1}^2 + r_j^2 \mp 2r_{j-1}r_j(\mathrm{e}_{j-1},\mathrm{e}_j)] - V(\mathrm{x}_j) - \Delta V(\mathrm{x}_j) \ .
\end{aligned} \tag{3.4.18}$$

Therefore we obtain the path integral representation

$$\begin{aligned}
K(\mathrm{x}'',\mathrm{x}';T) &= \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{Np/2}\left(\frac{\mathrm{i}\,m}{2\pi\epsilon\hbar}\right)^{qN/2}\prod_{j=1}^{N-1}\int_0^\infty r_j^{p+q-1}\mathrm{d}r_j\int \mathrm{d}\boldsymbol{\Omega}_j\\
&\times\exp\Bigg[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^N\Bigg(\pm\frac{m}{2\epsilon}(r_{j-1}^2 + r_j^2) - \frac{m}{\epsilon}r_{j-1}r_j(\mathrm{e}_{j-1},\mathrm{e}_j)\\
&\qquad - \epsilon\Delta V(r_j,\boldsymbol{\Omega}_j) - \epsilon V(r_j,\boldsymbol{\Omega}_j)\Bigg)\Bigg] \ .
\end{aligned} \tag{3.4.19}$$

In (3.4.18) and (3.4.19) ΔV denotes a quantum potential constructed in the usual way in terms of the metric, but with the modification that the kinetic terms of the Lagrangian are rewritten in terms of $(\mathbf{e}_{j-1}, \mathbf{e}_j)$ which requires a carefully calculated Taylor expansion (compare the discussion of the sphere and the hyperboloid).

Although a path integral representation in spherical coordinates may be in most cases sufficient and convenient, it is not the only possibility. In particular in the case of homogeneous spaces, the corresponding path integral allows as many spectral expansions in coordinate space representations as there are separable coordinate systems in this space. The path integral representations in spherical coordinates have allowed the derivation of several of the basic path integrals, among them the path integral of the radial harmonic oscillator, the Pöschl–Teller and the modified Pöschl–Teller potential (see below in the discussion of the path integration on the SU(2) and SU(1, 1) group manifolds). In Sect. 3.4.2 we discuss interbases expansions which allow us to switch in the path integral from one coordinate space representation to another. These new coordinate space representations of a path integral in a homogeneous space can then give rise to new and more complicated path integral identities. The method has been extensively worked out in [447], and several expansions are listed in the table of path integrals.

In the following we concentrate on two principal possibilities of harmonic analysis of a path integral on a group space. They are

i) *H_α is isomorphic to the group manifold G: $H_\alpha \simeq G$.* $H_\alpha \simeq G$ is quite a strong constraint, so it is not surprising that there is only a very limited number of groups which satisfy it. The harmonic analysis in these cases is performed by the characters of the group $\chi^l(g)$. In the following table the appropriate requirements are listed, i.e., $\dim H_\alpha = \dim G$.

Table 3.1. Dimensions of homogeneous spaces

G	$\dim G$	$\dim H_\alpha$	$\dim G = \dim H_\alpha$
$SO(p,q)$	$(p+q)(p+q-1)/2$	$p+q-1$	$p+q=2$
$SU(u,v)$	$(u+v)^2$	$2(u+v)-1$	$u+v=2$

We see that only the four cases

$$\boxed{SO(2), \quad SO(1,1), \quad SU(2), \quad SU(1,1)}$$

remain. For the one-parameter groups SO(2) and SO(1, 1) the irreducible representations are one-dimensional and in fact the general Fourier transformation (3.4.13) is reduced to the usual Fourier and Laplace transformation, respectively. Therefore we are left as the only non-trivial examples with SU(2) and SU(1, 1), which we discuss in two subsequent sections.

ii) *H_α is given by a group quotient:* $H_\alpha = G/H$. This case describes motion on quotient space group manifolds, i.e., on a homogeneous space. Examples are the motion in (pseudo-) Euclidean spaces, and on spheres and hyperboloids. However, the method is more general and it is not restricted to these two cases. The harmonic analysis in this case is performed by means of the zonal spherical harmonics $D^l_{00}(g)$ [911].

3.4.2 Interbases Expansions. Other important tools in group path integrations can be derived by interbases expansions, i.e., for problems which are separable in more than one coordinate system. In the case of potential problems, these potentials are called superintegrable. This property is very closely related to the fact that such problems have several integrals of motion, and the underlying dynamical symmetry group allows the representation of the problem in several coordinate space representations of the group. Superintegrable systems can be found in Euclidean space [305, 458, 668], in spaces of constant curvature [457–462], and in the theory of monopoles [461]. The basic formula is quite simple, being

$$|\mathbf{k}\rangle = \int \mathrm{d}E_{\mathbf{l}}\, C_{\mathbf{l},\mathbf{k}} |\mathbf{l}\rangle \quad , \tag{3.4.20}$$

where $|\mathbf{k}\rangle$ stands for the momentum space representation of the eigenfunctions with quantum numbers $\mathbf{k}$, and $\int \mathrm{d}E_{\mathbf{l}}$ is the expansion with respect to $\mathbf{l}$ with coefficients $C_{\mathbf{l},\mathbf{k}}$ which can be discrete, continuous or both. The difficulty is, provided that one has two momentum space representations in the quantum numbers $\mathbf{k}$ and $\mathbf{l}$, respectively, to find the expansion coefficients. The expansions which involve Cartesian coordinates and spherical coordinates are well known. For the problem of the free quantum motion in Euclidean space, this means that exponentials representing plane waves are expanded in terms of Bessel functions and circular (polar) waves (a discrete interbases expansion). This expansion actually is the very starting point in the formulation of radial path integrals, and the path integral evaluation of the radial harmonic oscillator, see Sect. 2.7. Because the expansion coefficients are unitary, i.e. $(C_{\mathbf{l},\mathbf{k}})^{-1} = C^*_{\mathbf{l},\mathbf{k}}$, we can insert them twice in the spectral expansion of the Feynman kernel and obtain the identity

$$\int \mathrm{d}E_{\mathbf{k}}\, \Psi_{\mathbf{k}}(\mathbf{x}'')\Psi^*_{\mathbf{k}}(\mathbf{x}')\,\mathrm{e}^{-\mathrm{i}E_{\mathbf{k}}T/\hbar} = \int \mathrm{d}E_{\mathbf{l}}\, \Phi_{\mathbf{l}}(\mathbf{u}'')\Phi^*_{\mathbf{l}}(\mathbf{u}')\,\mathrm{e}^{-\mathrm{i}E_{\mathbf{l}}T/\hbar} \quad , \tag{3.4.21}$$

where we have chosen the coordinate space representation of the wave functions with coordinates $\mathbf{x}$ and quantum numbers $\mathbf{k}$, respectively coordinates $\mathbf{u}$ and quantum numbers $\mathbf{l}$, representing two equivalent solutions of the same problem.

This general method of changing a coordinate basis in quantum mechanics can now be used in the path integral. We assume that we can expand the short-time kernel, respectively the exponential $\mathrm{e}^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}$, in terms of matrix elements of a group in the desired coordinate space representation. We can then change the coordinate basis by means of (3.4.21). Due to the unitarity of the expansion coefficients $C_{\mathbf{l},\mathbf{k}}$ the short-time kernel is expanded in the new coordinate basis, and the orthonormality of the basis allows us to perform explicitly the path integral, exactly in the same way as in the original coordinate basis.

The use of interbases expansions for the calculation of path integrals has been indispensable in almost all cases of the non-trivial basic path integrals. At the basis for the evaluation of the radial path integral are the interbases expansions (2.7.12), which expand a plane wave into Bessel functions. Interbases expansions must also be used to evaluate path integrals in other coordinate systems, like elliptical and spheroidal coordinate systems, see e.g. [447] for many examples.

3.4.3 Path Integrals on Homogeneous Spaces. Let us consider a path integral representation on a homogeneous space $\mathbb{M}$ [519, 523, 911]. Let a group G be a transformation on this space $\mathbb{M}$. If G acts transitively on $\mathbb{M}$, then $\mathbb{M}$ is a homogeneous space with respect to G. In order to convert a path integral on a homogeneous space into one on the corresponding group manifold, we require that the short-time kernel has some invariance properties: We assume that the short-time kernel is symmetric under the interchange of the two end points q' and q'' ($\mathrm{q}', \mathrm{q}'' \in \mathbb{M}$, the space $\mathbb{M}$ is allowed to have an indefinite metric), and is invariant under the action of $g \in G$, i.e.,

$$K(\mathrm{q}_j, \mathrm{q}_{j-1}; \epsilon) = K(\mathrm{q}_{j-1}, \mathrm{q}_j; \epsilon) = K(g\,\mathrm{q}_j, g\,\mathrm{q}_{j-1}; \epsilon) \tag{3.4.22}$$

for all $g \in G$. For an arbitrary fixed point q_a we introduce the function

$$k(g; \epsilon) = K(g\,\mathrm{q}_a, \mathrm{q}_a; \epsilon) \ . \tag{3.4.23}$$

We then derive

$$K(\mathrm{q}_j, \mathrm{q}_{j-1}; \epsilon) = k(g^{-1}g'; \epsilon) = k({g'}^{-1}g; \epsilon) \tag{3.4.24}$$

with $\mathrm{q}_j = g\,\mathrm{q}_a$, $\mathrm{q}_{j-1} = g'\,\mathrm{q}_a$. Consequently, we can define the Feynman path integral as the limit of a multi-convolution (denoted by $*$)

$$K(\mathrm{q}'', \mathrm{q}'; T) = \lim_{N\to\infty} \prod_{j=1}^{N} *k(g_{j-1}^{-1} g_j) \ , \tag{3.4.25}$$

where $\mathrm{q}_j = g_j\,\mathrm{q}_a$, $j = 0, \ldots, N$. Thus, the path integral in a homogeneous space is reduced to a convolution on a group manifold G. Due to the property of a homogeneous space $H_\alpha = G/H$, we know that the harmonic analysis can be performed by the zonal spherical harmonics, i.e., we can expand by using the subgroup composition $g = ab$ with $a \in A$, $b \in B$ such that $G = A \otimes B$ according to ($\lambda_l := \hat{f}^l_{00}$)

$$k(ab;\epsilon) = \int \mathrm{d}E_l\, d_l \lambda_l(b;\epsilon) D^l_{00}(a) \ . \tag{3.4.26}$$

The expansion coefficients in turn are given by

$$\lambda_l(b;\epsilon) = \int_A \mathrm{d}a\, k(ab;\epsilon) D^l_{00}(a^{-1}) \ . \tag{3.4.27}$$

This expansion allows us to perform the convolution on the group manifold G yielding

$$K(\mathrm{q}'',\mathrm{q}';T) = \int \mathrm{d}E_l\, d_l K_l(b'^{-1}b'';T) D^l_{00}(a'^{-1}a'') \ , \tag{3.4.28}$$

where

$$K_l(b'^{-1}b'';T) = \lim_{N\to\infty} \prod_{j=1}^{N} \lambda_l(b_{j-1}^{-1}b_j;\epsilon) \ . \tag{3.4.29}$$

A special case can be stated if the subgroup B consists of the unit element e alone. Then $\lambda_l(e,\epsilon) \equiv \lambda_l(\epsilon)$, and we can explicitly state the corresponding limit yielding the expansion

$$K(\mathrm{q}'',\mathrm{q}';T) = \int \mathrm{d}E_l\, d_l\, \mathrm{e}^{\dot\lambda_l(0)T}\, D^l_{00}(a'^{-1}a'') \tag{3.4.30}$$

$$= \int \mathrm{d}E_l \sum_m \mathrm{e}^{-\mathrm{i}E_lT/\hbar} \langle \mathrm{q}''\,|lm\rangle\langle lm|\,\mathrm{q}'\rangle \ , \tag{3.4.31}$$

$$\langle \mathrm{q}\,|lm\rangle = \sqrt{d_l}\, D^l_{m0}(a) \ , \tag{3.4.32}$$

$$E_l = \mathrm{i}\,\hbar\dot\lambda_l(0) \ . \tag{3.4.33}$$

Let us emphasize again that the index l can be discrete as well as continuous. If the homogeneous space is compact, l is a discrete quantum number. Examples are the path integral representation on the sphere or path integral representations in compact subspaces of non-compact spaces. In non-compact spaces such as the hyperboloid Λ^{D-1} and (pseudo-) Euclidean space, l is a continuous parameter.

3.4.4 The SU(2) Path Integral.

3.4.4.1 Cylindrical Coordinates. The group manifold SU(2) is of particular interest because it serves as a model for spin. Considering a quantum mechanical spherical top one can distinguish an external motion and an internal motion. Its path integral can be separated in terms of these two independent motions and it is not required that an interaction has to be turned on. The internal motion is now interpreted as "spin" taking on integer as well as half integer values. The first discussion of motion on a group manifold was due to Schulman [826] in his discussion of the spherical top in terms of Euler angles.

Schulman made an analysis comparing different approaches, i.e., as seen as motion on a curved manifold and by exact summation of the classical paths using the semiclassical approximation. In this particular case, the dimension of the group manifold is isomorphic to the covering unit sphere $\mathcal{H}_\alpha$, e.g., we have $\mathrm{SU}(2) \simeq S^{(3)}$. Schulman was interested in constructing a path integral for spin, and in fact in the matrix elements of the $(2J+1)$-dimensional unitary irreducible representation of the Hilbert space of SU(2) the "angular" momentum can take on integer as well as half-integer values.

The eigenvalues for the corresponding Casimir operator take on the values $L_J = J(J+1)$ ($J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$). However, since $\mathrm{SU}(2) \simeq S^{(3)}$, we can look at this group manifold from the point of view of motion on the sphere $S^{(3)}$. On the S^{D-1}-sphere in turn the eigenvalues of the Laplace–Beltrami operator are given by $L_l = l(l+D-2)$, thus we have for the $S^{(3)}$-sphere $L_l = l(l+2) = \frac{1}{4}L_J$, where we identified $l = 2J$ and the half-integer representations are "hidden".

In order to derive the correct Feynman kernel for the SU(2) path integral we start by considering a specific coordinate representation. This enables us to determine the correct quantum potential to be included in the effective Lagrangian. Rewriting the coordinate representations in terms of $\mathrm{Tr}(g_a^{-1}g_b)$ gives us finally the appropriate form to deal with the various expansions.

Motion on SU(2) means that the quantum mechanical motion is subject to the constraint

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \ , \tag{3.4.34}$$

i.e., it takes place on the $S^{(3)}$-sphere. A suitable coordinate representation, called cylindrical, has the form $(0 \le \varphi < 2\pi, 0 \le \vartheta \le \pi, 0 \le \psi < 4\pi)$

$$\begin{aligned} x_1 &= \sin\frac{\vartheta}{2}\sin\frac{\varphi-\psi}{2} \ , \quad & x_2 &= \sin\frac{\vartheta}{2}\cos\frac{\varphi-\psi}{2} \ , \\ x_3 &= \cos\frac{\vartheta}{2}\sin\frac{\varphi+\psi}{2} \ , \quad & x_4 &= \cos\frac{\vartheta}{2}\cos\frac{\varphi+\psi}{2} \ . \end{aligned} \tag{3.4.35}$$

The classical Lagrangian reads as

$$\mathcal{L}(x,\dot{x}) = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2) = \frac{m}{8}(\dot{\vartheta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\vartheta) \ . \tag{3.4.36}$$

We read off the metric tensor and its inverse, respectively

$$(g_{ab}) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\vartheta \\ 0 & \cos\vartheta & 1 \end{pmatrix} , \qquad (g^{ab}) = 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\sin^2\vartheta} & -\dfrac{\cos\vartheta}{\sin^2\vartheta} \\ 0 & -\dfrac{\cos\vartheta}{\sin^2\vartheta} & \dfrac{1}{\sin^2\vartheta} \end{pmatrix} \tag{3.4.37}$$

and $g = \det(g_{ab}) = (\sin\vartheta/8)^2$. We have $\Gamma_\varphi = \Gamma_\psi = 0$, $\Gamma_\vartheta = \cot\vartheta$, and for the momentum operators we obtain

$$p_\varphi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\varphi} \ , \qquad p_\psi = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial\psi} \ , \qquad p_\vartheta = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{2}\right) \ . \tag{3.4.38}$$

The quantum potential ΔV is found to be

$$\Delta V_{\mathrm{Weyl}}(\vartheta)=\Delta V_{\mathrm{PF}}(\vartheta)\equiv\Delta V(\vartheta)=-\frac{\hbar^2}{2m}\left(1+\frac{1}{\sin^2\vartheta}\right)\ . \tag{3.4.39}$$

Thus we have the necessary ingredients to write down the path integral for quantum motion on SU(2) in terms of Euler angles

$$K^{\mathrm{SU}(2)}(\mathbf{x}'',\mathbf{x}';T)=\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\frac{\sin\vartheta}{8}\mathcal{D}\varphi(t)\mathcal{D}\psi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{8}(\dot\vartheta^2+\dot\varphi^2+\dot\psi^2+2\dot\varphi\dot\psi\cos\vartheta)+\frac{\hbar^2}{2m}\left(1+\frac{1}{\sin^2\vartheta}\right)\right]dt\right\}\ . \tag{3.4.40}$$

On the other hand we know that the SU(2) path integration can be performed by using the character expansion in the harmonic analysis. We thus obtain for the SU(2) path integral in the character expansion $\chi^l=C^l_{2J}(\cos\frac{\Omega}{2})$ together with the corresponding spectral expansion

$$K^{\mathrm{SU}(2)}(\vartheta'',\psi'',\varphi'',\vartheta',\varphi',\psi';T)$$
$$=\sum_{J=0,\frac12}^{\infty}\frac{2J+1}{2\pi^2}C^1_{2J}\left(\cos\frac{\Omega}{2}\right)\exp\left[-\frac{2\,\mathrm{i}\,\hbar T}{m}J(J+1)\right] \tag{3.4.41}$$
$$=\sum_{J,m,n}\Psi^J_{mn}(\vartheta'',\varphi'',\psi'')\Psi^{J\,*}_{mn}(\vartheta',\varphi',\psi')\exp\left[-\frac{\mathrm{i}\,\hbar T}{2m}2J(2J+2)\right]\ , \tag{3.4.42}$$

where

$$\cos\frac{\Omega}{2}=\sin\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\cos\frac{\varphi''-\varphi'-(\psi''-\psi')}{2}$$
$$+\cos\frac{\vartheta'}{2}\cos\frac{\vartheta''}{2}\cos\frac{\varphi''-\varphi'+(\psi''-\psi')}{2}\ , \tag{3.4.43}$$

and with the wave functions

$$\Psi^J_{mn}(\vartheta,\varphi,\psi)=\sqrt{\frac{2J+1}{4\pi^2}}\,\mathrm{e}^{\mathrm{i}\,m\varphi+\mathrm{i}\,n\psi}\,D^J_{mn}(\cos\vartheta)$$
$$=2^{-m}\sqrt{\frac{2J+1}{8\pi^2}\frac{\Gamma(J-m+1)\Gamma(J+m+1)}{\Gamma(J-n+1)\Gamma(J+n+1)}}\,\mathrm{e}^{\mathrm{i}\,m\varphi+\mathrm{i}\,n\psi}$$
$$\times(1-\cos\vartheta)^{\frac{m-n}{2}}(1+\cos\vartheta)^{\frac{m+n}{2}}P^{(m-n,m+n)}_{J-m}(\cos\vartheta)\ , \tag{3.4.44}$$

where the $P_n^{(a,b)}(x)$ are Jacobi polynomials [413], and the energy spectrum is

$$E_J = \frac{\hbar^2}{2m} 2J(2J+2) \ . \tag{3.4.45}$$

The $P_n^{(a,b)}$ and the D_{mn}^J are related by ($s = J - \frac{1}{2}(\mu+\nu), \mu = |m-m'|, \nu = |m+m'|$, Wigner polynomials) [908]

$$D_{mn}^J(z) = 2^{-(\mu+\nu)/2}\sqrt{\frac{s!(s+\mu+\nu)!}{(s+\mu)!(s+\nu)!}}(1-z)^{\mu/2}(1+z)^{\nu/2}P_s^{(\mu,\nu)}(z) \ . \tag{3.4.46}$$

Note that $2J \in \mathbb{N}$ and thus the energy spectrum is indistinguishable from the motion on the covering $S^{(3)}$-sphere, as it should be.

Note that due to the fact that the group manifold SU(2) is isomorphic to the $S^{(3)}$-sphere, there are six coordinate space representations of the SU(2) path integral, i.e., as many as there are separable coordinate systems for the Laplace–Beltrami operator on the sphere $S^{(3)}$. Therefore the character χ^l may be expanded into six different coordinate space representations of the wave functions. The switching between the various spectral expansions is performed by the relevant interbases expansions. The combinations of corresponding path integral representations and spectral expansions yield path integral identities. In particular, the spherical coordinate system on the $S^{(3)}$-sphere (the three-dimensional rotator) yields a path integral identity for the $1/\sin^2\vartheta$ potential, a special case of the path integral identity of the Pöschl–Teller potential.

3.4.4.2 The Pöschl–Teller Path Integral. Let us consider the (Pöschl–Teller) potential

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x} + \frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right) \ , \qquad 0 < x < \frac{\pi}{2} \ . \tag{3.4.47}$$

Setting $\alpha = m-n$, $\beta = m+n$, $l = J-m$ and substituting $\vartheta \to 2x$ yields by comparison of the path integral representations (3.4.40) and (3.4.42) and by separating off the φ- and ψ-path integrations the basic path integral solution of the Pöschl–Teller potential according to [104, 272, 531]

$$\begin{aligned}&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x}+\frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]\mathrm{d}t\right\}\\ &= \sum_{l=0}^{\infty}\Phi_l^{(\alpha,\beta)}(x'')\Phi_l^{(\alpha,\beta)\,*}(x')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}(\alpha+\beta+2l+1)^2\right] \ ,\end{aligned} \tag{3.4.48}$$

with the wave functions given by

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2l+1)\frac{l!\Gamma(\alpha+\beta+l+1)}{\Gamma(\alpha+l+1)\Gamma(\beta+l+1)}\right]^{1/2} \times (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos 2x) \ . \tag{3.4.49}$$

Here, of course, we can analytically continue from integer values of m and n to, say, real numbers α and β, respectively. For a carefully performed lattice formulation of the Pöschl–Teller path integral a similar functional weight approach as for the radial path integral must be made [424]. However, the path integral representation (3.4.48) can be justified; see Fischer, Leschke and Müller [344].

It is possible to state closed expressions for the (energy dependent) Green function for the Pöschl–Teller potential. It has the form ([344, 617], $0 < x < \pi/2$)

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \\
&\quad\times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\alpha^2-\frac14}{\sin^2 x} + \frac{\beta^2-\frac14}{\cos^2 x}\right)\right]\mathrm{d}t\right\} \\
&= \frac{m}{2\hbar^2}\sqrt{\sin 2x' \sin 2x''}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\
&\quad\times \big(\sin x' \sin x''\big)^{m_1-m_2}\big(\cos x' \cos x''\big)^{m_1+m_2} \\
&\quad\times {}_2F_1\big(m_1-L_E, L_E+m_1+1; m_1-m_2+1; \sin^2 x_<\big) \\
&\quad\times {}_2F_1\big(m_1-L_E, L_E+m_1+1; m_1+m_2+1; \cos^2 x_>\big)
\end{aligned} \tag{3.4.50}$$

with $m_{1/2} = \frac12(\beta \pm \alpha)$, $L_E = -\frac12 + \frac12\sqrt{2mE}/\hbar$. This path integral representation is the third of our basic path integrals.

3.4.5 The SU(1,1) Path Integral. In this section we present the SU(1,1) path integral, first in a particular coordinate space representation in order to derive the relevant formulæ. In the second step, this representation is used to derive the fourth basic path integral, the path integral identity for the modified Pöschl–Teller path integral. Third, in an alternative coordinate space representation, we can present another path integral identity based on the SU(1,1) path integral.

3.4.5.1 Cylindrical Coordinates. As a further path integral on a group manifold we discuss the SU(1,1) path integral. Here again the method of path integration over group manifolds must be applied. It was first calculated by Böhm and Junker [551], and with more detail and in the context of the general framework in Ref. [104].

Motion on the SU(1,1) manifold means that the quantum mechanical motion is subject to the constraint

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \ , \tag{3.4.51}$$

thus the signature of the metric is $(+1, +1, -1, -1)$ and therefore indefinite. A convenient parametrization reads $(0 \leq \varphi, \psi < 2\pi, \tau \geq 0)$

$$\begin{aligned} x_1 &= \sinh\tau\sin\psi \ , \quad x_2 = \sinh\tau\cos\psi \ , \\ x_3 &= \cosh\tau\sin\varphi \ , \quad x_4 = \cosh\tau\cos\varphi \ . \end{aligned} \tag{3.4.52}$$

We obtain the path integral representation

$$\begin{aligned} &K^{\mathrm{SU}(1,1)}(\tau'', \tau', \psi'', \psi', \varphi'', \varphi'; T) \\ &= \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \sinh\tau\cosh\tau \int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''} \mathcal{D}\psi(t) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \\ &\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2}\Big(-\dot\tau^2 - \sinh^2\tau\dot\varphi^2 + \cosh^2\tau\dot\psi^2 \Big) \right.\right. \\ &\qquad\qquad \left.\left. + \frac{\hbar^2}{8m}\left(4 - \frac{1}{\sinh^2\tau} + \frac{1}{\cosh^2\tau} \right) \right] \mathrm{d}t \right\} \ . \end{aligned} \tag{3.4.53}$$

The path integration over the SU(1,1) group manifold is similar to the path integration over the SU(2) manifold performed by a character expansion in the harmonic analysis. The main difference is that we have to take into account a discrete and a continuous spectrum. The spectral expansion in the cylindrical coordinates gives the result [104]

$$\begin{aligned} &K^{\mathrm{SU}(1,1)}(\tau'', \tau', \psi'', \psi', \varphi'', \varphi'; T) = \frac{1}{4\pi^2} \sum_\sigma \\ &\times \left[\sum_{l=-\frac{1}{2},0}^{\infty} \sum_{m,n} \mathrm{e}^{\mathrm{i}\,m(\varphi''-\varphi')+\mathrm{i}\,n(\psi''-\psi')} \Psi_{l,mn,\sigma}^{\mathrm{SU}(1,1)}(\tau'') \Psi_{l,mn,\sigma}^{\mathrm{SU}(1,1)\,*}(\tau') K_l^{\mathrm{SU}(1,1)}(T) \right. \\ &\qquad + \int_0^\infty \mathrm{d}k \sum_{m,n} \mathrm{e}^{\mathrm{i}\,m(\varphi''-\varphi')+\mathrm{i}\,n(\psi''-\psi')} \\ &\qquad\qquad \left. \times \Psi_{-\frac{1}{2}+\mathrm{i}\,k,mn,\sigma}^{\mathrm{SU}(1,1)}(\tau'') \Psi_{-\frac{1}{2}+\mathrm{i}\,k,mn,\sigma}^{\mathrm{SU}(1,1)\,*}(\tau') K_{-\frac{1}{2}+\mathrm{i}\,k}^{\mathrm{SU}(1,1)}(T) \right] \ . \end{aligned} \tag{3.4.54}$$

The summation over the index σ is defined as $\sum_\sigma := \sum_{+,-} + \sum_{0,\frac{1}{2}}$ such that one has

$$\begin{array}{ll} \sigma = + & m, n = l+1, l+2, \dots \\ \sigma = - & m, n = -l-1, -l-2, \dots \\ \sigma = 0 & m, n = 0, \pm 1, \pm 2, \dots \\ \sigma = \dfrac{1}{2} & m, n = \pm\dfrac{1}{2}, \pm\dfrac{3}{2} \ . \end{array}$$

For the quantities $K_l^{\mathrm{SU}(1,1)}(T)$, respectively $K_{-\frac{1}{2}+\mathrm{i}\,k}^{\mathrm{SU}(1,1)}(T)$, one obtains

$$K_l^{\mathrm{SU}(1,1)}(T) = \exp\left(-\frac{\mathrm{i}\,\hbar T}{2m}2l(2l+2)\right) \ , \tag{3.4.55}$$

$$K_{-\frac{1}{2}+\mathrm{i}\,k}^{\mathrm{SU}(1,1)}(T) = \exp\left(\frac{\mathrm{i}\,\hbar T}{2m}(k^2+1)\right) \ . \tag{3.4.56}$$

Note that the relevant spectrum emerging from the spectrum of the group manifold $\mathrm{SU}(1,1)$ is of the form [559]

$$E_{\sigma,j_0} = -\frac{\hbar^2}{2m}[j_0^2 + \sigma(\sigma+2)] \ , \quad \begin{array}{l} \text{continuous spectrum: } j_0 = 0, \sigma = -1+\mathrm{i}\,k \ , \\ \text{discrete spectrum: } j_0 = 2n \ (n \in \mathbb{N}), \sigma = -1 \ . \end{array} \tag{3.4.57}$$

In the notation of [104] we have $E_n \to E_l = -(\hbar^2/2m)2l(2l+2)$ with $l = -\frac{1}{2}, 0, \frac{1}{2} \dots$.

Let us display these wave functions in alternative ways. We introduce $\eta = m-n$, $\nu = m+n$ (for $\sigma = +$, respectively $\nu = -m-n$ for $\sigma = -$), i.e., $m = \frac{\nu+\eta}{2}$, $n = \frac{\nu-\eta}{2}$ (for $\sigma = +$, respectively $m = \frac{\eta-\nu}{2}$, $n = -\frac{\nu+\eta}{2}$ for $\sigma = -$). Further we define $k_1 = \frac{1}{2}(1+\nu)$, $k_2 = \frac{1}{2}(1+\eta)$ and introduce $N \in \mathbb{N}$ by $l = N-n$. This yields ($\kappa = k_1 - k_2 - N$):

$$\begin{aligned}
\Psi_{l,mn}^{\mathrm{SU}(1,1)}(\tau) &= \sqrt{\frac{2(2N+\eta-\nu+1)}{\Gamma^2(1+\eta)}\,\frac{\Gamma(1+N+\eta)\Gamma(\nu-N)}{N!\,\Gamma(\nu-\eta-N)}} \\
&\quad \times (\cosh\tau)^{-\nu}(\sinh\tau)^{\eta}\,{}_2F_1\big(1+N+\eta-\nu, -N; 1+\eta; -\sinh^2\tau\big) \ ,
\end{aligned} \tag{3.4.58}$$

$$\begin{aligned}
&= \left[\frac{2(1-2\kappa)}{\Gamma^2(2k_2)} \cdot \frac{\Gamma(k_1+k_1-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2} \\
&\quad \times (\sinh\tau)^{2k_2-1}(\cosh\tau)^{-2k_1+1} \\
&\quad \times {}_2F_1\big(-k_1+k_2+\kappa, -k_1+k_2-\kappa+1; 2k_2; -\sinh^2\tau\big) \ .
\end{aligned} \tag{3.4.59}$$

${}_2F_1(a,b;c;z)$ is the hypergeometric function. Note that the discrete spectrum is infinite. For continuous wave functions we write them in terms of η, ν and k_1, k_2, respectively. We obtain by explicit insertion of $l \to -\frac{1}{2}+\mathrm{i}\,k$ ($\sigma = 0$) therefore $[\kappa = \frac{1}{2}(1+\mathrm{i}\,k)]$:

$$\begin{aligned}
&\Psi_{-\frac{1}{2}+\mathrm{i}\,k,mn}^{\mathrm{SU}(1,1)}(\tau) \\
&= \frac{\sqrt{2k\sinh\pi k}}{\Gamma(1+\eta)}\Gamma\big(m+\tfrac{1}{2}+\tfrac{\mathrm{i}k}{2}\big)\Gamma\big(\tfrac{1}{2}-n-\tfrac{\mathrm{i}k}{2}\big)(\cosh\tau)^{\nu}(\sinh\tau)^{\eta}
\end{aligned}$$

$$\times\,{}_2F_1\left(\frac{\nu+\eta-\mathrm{i}\,k+1}{2},\frac{\nu+\eta+1+\mathrm{i}\,k}{2};1+\eta;-\sinh^2\tau\right)\ , \tag{3.4.60}$$

$$\begin{aligned}
&= \frac{\sqrt{2k\sinh\pi k}}{\Gamma(2k_2)}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\\
&\qquad\qquad\times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2}\\
&\quad\times(\cosh\tau)^{2k_1-1}(\sinh\tau)^{2k_2-1}\\
&\quad\times\,{}_2F_1\left(k_1+k_2-\kappa,k_1+k_2+\kappa-1;2k_2;-\sinh^2\tau\right)\ .
\end{aligned} \tag{3.4.61}$$

3.4.5.2 The Modified Pöschl–Teller Path Integral. With the results of the previous Sect. 3.4.5.1 we can derive a path integral identity for the modified Pöschl–Teller potential which is defined as ($r>0$)

$$V^{(\eta,\nu)}(r)=\frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac14}{\sinh^2 r}-\frac{\nu^2-\frac14}{\cosh^2 r}\right)\ ,\qquad r>0\ . \tag{3.4.62}$$

We obtain by comparison of the representation (3.4.53) with the spectral representation (3.4.54) together with (3.4.56) and separating off the φ and ψ path integration the basic path integral:

$$\begin{aligned}
&\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2-\frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac14}{\sinh^2 r}-\frac{\nu^2-\frac14}{\cosh^2 r}\right)\right]\mathrm{d}t\right\}\\
&=\sum_{n=0}^{N_M}\Psi_n^{(\eta,\nu)}(r')\Psi_n^{(\eta,\nu)}(r'')\exp\left\{+\frac{\mathrm{i}\hbar T}{2m}\Big[2(k_1-k_2-n)-1\Big]^2\right\}\\
&\quad+\int_0^\infty\mathrm{d}k\,\Psi_k^{(\eta,\nu)}(r'')\Psi_k^{(\eta,\nu)*}(r')\exp\left(-\frac{\mathrm{i}\hbar T}{2m}k^2\right)\ .
\end{aligned} \tag{3.4.63}$$

Here we have introduced the numbers k_1,k_2 defined by [350]: $k_1=\frac12(1\pm\nu)$, $k_2=\frac12(1\pm\eta)$, where the correct sign depends on the boundary conditions for $r\to0$ and $r\to\infty$, respectively. In particular for $\eta^2=\frac14$, i.e., $k_2=\frac14,\frac34$, we obtain wave functions with even and odd parity, respectively. The number N_M denotes the maximal number of states with $0,1,\dots,N_M<k_1-k_2-\frac12$. The bound state wave functions read in two alternative formulations

$$\begin{aligned}
\Psi_n^{(k_1,k_2)}(r)&=N_n^{(k_1,k_2)}(\sinh r)^{2k_2-\frac12}(\cosh r)^{-2k_1+\frac32}\\
&\quad\times\,{}_2F_1(-k_1+k_2+\kappa,-k_1+k_2-\kappa+1;2k_2;-\sinh^2 r)\ ,
\end{aligned} \tag{3.4.64a}$$

$$N_n^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}, \tag{3.4.64b}$$

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)}(\sinh r)^{\eta+1/2}(\cosh r)^{n-\nu+1/2} \times {}_2F_1(-n, \nu - n; 1+\eta; \tanh^2 r) \ , \tag{3.4.65a}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(1+\eta)}\left[\frac{2(\nu-\eta-2n-1)\Gamma(n+1+\eta)\Gamma(\nu-n)}{\Gamma(\nu-\eta-n)n!}\right]^{1/2} \tag{3.4.65b}$$

($\kappa = k_1 - k_2 - n$). Note the equivalent formulation

$$\tilde{\Psi}_n^{(\alpha,\beta)}(r) = \left[\frac{\beta n!\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right]^{1/2}(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta-1}{2}}P_n^{(\alpha,\beta)}(x)\,, \tag{3.4.66}$$

with the substitutions $\alpha = 2k_2 - 1$, $\beta = 2(k_1 - k_2 - n) - 1 = 2\kappa - 1$, $x = 2/\cosh^2 r - 1$ and the incorporation of the appropriate measure term, i.e., $\mathrm{d}r = \left[(1+x)\sqrt{2(1-x)}\right]^{-1}\mathrm{d}x$. The scattering states are given by (we are using two alternative formulations)

$$\Psi_k^{(k_1,k_2)}(r) = N_k^{(k_1,k_2)}(\cosh r)^{2k_1-\frac{1}{2}}(\sinh r)^{2k_2-\frac{1}{2}} \times {}_2F_1(k_1+k_2-\kappa, k_1+k_2+\kappa-1; 2k_2; -\sinh^2 r) \ , \tag{3.4.67a}$$

$$N_k^{(k_1,k_2)} = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa) \times \Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2}, \tag{3.4.67b}$$

$$\Psi_k^{(\eta,\nu)}(r) = N_k^{(\eta,\nu)}(\cosh r)^{\mathrm{i}k}(\tanh r)^{\eta+1/2} \times {}_2F_1\left(\frac{\nu+\eta+1-\mathrm{i}k}{2}, \frac{\eta-\nu+1-\mathrm{i}k}{2}; 1+\eta; \tanh^2 r\right) \ , \tag{3.4.68a}$$

$$N_k^{(\eta,\nu)} = \frac{1}{\Gamma(1+\eta)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\Gamma\left(\frac{\nu+\eta+1-\mathrm{i}k}{2}\right)\Gamma\left(\frac{\eta-\nu+1-\mathrm{i}k}{2}\right) \ . \tag{3.4.68b}$$

$[\kappa = \frac{1}{2}(1+\mathrm{i}k)]$. Of course, in the path integral formulation of the modified Pöschl–Teller potential a functional weight interpretation must be used for the Pöschl–Teller potential in order to have a proper short-time behaviour, respectively a lattice regularization.

Similarly as for the Pöschl–Teller potential one can derive a path integral identity for the modified Pöschl–Teller potential. One gets ($m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2mE}/\hbar)$, $L_\nu = \frac{1}{2}(-1+\nu)$, $r > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2-\frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r}-\frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\}$$
$$=\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\left(\cosh r_<\right)^{-(m_1-m_2)}\left(\tanh r_<\right)^{m_1+m_2+\frac{1}{2}}$$
$$\times\left(\cosh r_>\right)^{-(m_1-m_2)}\left(\tanh r_>\right)^{m_1+m_2+\frac{1}{2}}$$
$$\times\,{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;\cosh^{-2}r_<\right)$$
$$\times\,{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh^2 r_>\right)\ . \tag{3.4.69}$$

3.4.5.3 Horocyclic Coordinates and the Inverted Liouville Problem. Let us turn to the horocyclic system. It parametrizes the coordinates on the hyperboloid (3.4.52) as follows ($\mathrm{v}=(v_0,v_1)\in\mathbb{R}^{(1,1)}$, $u\in\mathbb{R}$)

$$\begin{aligned} x_1&=\tfrac{1}{2}[\mathrm{e}^{-u}+e^u(1-\mathrm{v}^2)]\ , &\quad x_3&=v_1\,\mathrm{e}^u\ ,\\ x_2&=v_0\,\mathrm{e}^u\ , & x_4&=\tfrac{1}{2}[-\mathrm{e}^{-u}+e^u(1+\mathrm{v}^2)]\ , \end{aligned} \tag{3.4.70}$$

with the domain of the coordinates as indicated above. As known from the theory of harmonic analysis on this manifold one has a discrete and a continuous series in the variable u with wave functions corresponding to ($\mathrm{k}=(k_0,k_1)\in\mathbb{R}^{(1,1)},\mathrm{k}^2=k_0^2-k_1^2>0$ is taken in the physical domain, $\tilde{J}_{\mathrm{i}k}(z)=[J_{\mathrm{i}k}(z)+J_{-\mathrm{i}k}(z)]$, α is the parameter of the non-unique self-adjoint extension) [559, 696, 893]

Discrete series ($n\in\mathbb{N}$, $0<\alpha\le 2$):

$$\Psi_{k_0,k_1,n}(v_0,v_1,u)=\frac{\mathrm{e}^{\mathrm{i}k_0v_0-\mathrm{i}k_1v_1}}{2\pi}\sqrt{2(2n+\alpha)}\,J_{2n+\alpha}(|\mathrm{k}|\,\mathrm{e}^{-u})\ , \tag{3.4.71}$$

Continuous series ($|\mathrm{k}|>0$):

$$\Psi_{k_0,k_1,k}(v_0,v_1,u)=\frac{\mathrm{e}^{\mathrm{i}k_0v_0-\mathrm{i}k_1v_1}}{2\pi}\sqrt{\frac{k}{2\sinh\pi k}}\,\tilde{J}_{\mathrm{i}k}(|\mathrm{k}|\,\mathrm{e}^{-u})\ . \tag{3.4.72}$$

These wave functions form the matrix element expansion of the Titchmarch transformation. According to the general theory of the path integration on group manifolds we have to calculate the quantity $\hat{f}^{\lambda}_{mn}$. Since this expression is actually independent of the representation one chooses we can take the result of Böhm and Junker [104] and we have in the limit $\epsilon \to 0$ the result ($|\mathrm{k}|^2 = k_0^2 - k_1^2 > 0$, i.e., k in the physical domain)

$$\text{Discrete series } (n \in \mathbb{N})\text{: } K_n^{\mathrm{SU}(1,1)} = \exp\left(\frac{\mathrm{i}\hbar T}{2m}(4n^2 - 1)\right) , \tag{3.4.73}$$

$$\text{Continuous series } (p > 0)\text{: } K_{-\frac{1}{2}+\mathrm{i}k}^{\mathrm{SU}(1,1)} = \exp\left(- \frac{\mathrm{i}\hbar T}{2m}(k^2 + 1)\right) . \tag{3.4.74}$$

Putting everything together we arrive at the following path integral representation

$$\begin{aligned}
&\int\limits_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t)\,\mathrm{e}^{2u} \int\limits_{\mathrm{v}(t')=\mathrm{v}'}^{\mathrm{v}(t'')=\mathrm{v}''} \mathcal{D}\,\mathrm{v}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{u}^2 - \mathrm{e}^{2u}\,\dot{\mathrm{v}}^2) - \frac{\hbar^2}{2m}\right)\mathrm{d}t\right] \\
&= \mathrm{e}^{u'+u''} \int_{\mathbb{R}^{(1,1)}} \mathrm{dk}\, \frac{\mathrm{e}^{\mathrm{i}\,\mathrm{k}\cdot(\mathrm{v}''-\mathrm{v}')}}{4\pi^2} \\
&\quad \times \left[\sum_{n\in\mathbb{N}} 2(2n+\alpha) J_{2n+\alpha}(|\mathrm{k}|\,\mathrm{e}^{-u''}) J_{2n+\alpha}(|\mathrm{k}|\,\mathrm{e}^{-u'})\,\mathrm{e}^{\mathrm{i}\hbar(4n^2-1)T/2m} \right. \\
&\quad \left. + \int_0^\infty \frac{k\mathrm{d}k}{2\sinh\pi k} \tilde{J}_{\mathrm{i}k}(|\mathrm{k}|\,\mathrm{e}^{-u'}) \tilde{J}_{\mathrm{i}k}(|\mathrm{k}|\,\mathrm{e}^{-u''})\,\mathrm{e}^{-\mathrm{i}\hbar(k^2+1)T/2m} \right] .
\end{aligned} \tag{3.4.75}$$

This result enables us to derive the path integral identity for the inverted Liouville problem. Separating off the (v_0, v_1)-path integrations we obtain

$$\begin{aligned}
&\int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{y}^2 + \frac{\hbar^2\kappa^2}{2m}\,\mathrm{e}^{2y}\right)\mathrm{d}t\right] \\
&= \sum_{n\in\mathbb{N}} 2(2n+\alpha)\,\mathrm{e}^{2\,\mathrm{i}\,\hbar T n^2/m}\, J_{2n+\alpha}(\kappa\,\mathrm{e}^{y''}) J_{2n+\alpha}(\kappa\,\mathrm{e}^{y'}) \\
&\quad + \int_0^\infty \frac{k\mathrm{d}k}{2\sinh\pi k} \tilde{J}_{\mathrm{i}k}(\kappa\,\mathrm{e}^{y'}) \tilde{J}_{\mathrm{i}k}(\kappa\,\mathrm{e}^{y''})\,\mathrm{e}^{-\mathrm{i}\hbar k^2 T/2m} .
\end{aligned} \tag{3.4.76}$$

3.4.6 Expansion Formulæ. In this section we give a short list of expansion formulæ which emerge from a particular coordinate representation of the Euclidean group, say in $\mathbb{R}^2$ and $\mathbb{R}^3$, and in pseudo-Euclidean spaces or hyperboloids, respectively. The expansion of plane waves in spherical waves (2.7.12) has already been exploited in Sect. 2.7.

3.4.6.1 Elliptic and Spheroidal Coordinates in Flat Space. As has been shown in [447] expansion theorems for elliptic and spheroidal coordinates can be used to derive explicit path integral representations in these coordinates; the expansion formulæ are in fact interbases expansions. Let us start with the two-dimensional case. We consider for an arbitrary α the periodic and non-periodic Mathieu functions $\mathrm{me}_n, \mathrm{Me}_n^{(1)}$, and the corresponding even and odd Mathieu functions $\mathrm{ce}_n, \mathrm{se}_n, \mathrm{Mc}_n^{(1)}, \mathrm{Ms}_n^{(1)}$. We have the relations $\mathrm{ce}_n(z;h^2) = \mathrm{me}_n(z;h^2)/\sqrt{2}$, $M_n^{(1)}(z) = \mathrm{Mc}_n^{(1)}(z;h)$ $(n = 0,1,\dots)$ and $\mathrm{se}_n(z;h^2) = \mathrm{i}\cdot\mathrm{me}_n(z;h^2)/\sqrt{2}$, $M_{-n}^{(1)}(z) = (-1)^{-n}\,\mathrm{Ms}_n^{(1)}(z;h)$ $(n = 1,2,\dots,$ $h = kd/2$, k is the wave number, d is the distance between the foci of the ellipse). The orthonormality relations are given by [692]

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{ce}_n(\vartheta)\,\mathrm{ce}_l^*(\vartheta)\,\mathrm{d}\vartheta &= \frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{se}_n(\vartheta)\,\mathrm{se}_l^*(\vartheta)\,\mathrm{d}\vartheta\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{me}_n(\vartheta)\,\mathrm{me}_l^*(\vartheta)\,\mathrm{d}\vartheta = \delta_{nl}\ ,\end{aligned}\tag{3.4.77}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\mathrm{ce}_n(\vartheta)\,\mathrm{se}_l^*(\vartheta)\,\mathrm{d}\vartheta = 0\ .\tag{3.4.78}$$

$$\begin{aligned}&d^2\int_0^\infty\mathrm{d}\mu\int_{-\pi}^{\pi}\mathrm{d}\nu(\sinh^2\mu+\sin^2\nu)M_n^{(1)}(\mu;h)M_l^{(1)*}(\mu;h')\,\mathrm{me}_n(\nu;h^2)\,\mathrm{me}_l^*(\nu;h'^2)\\ &= \frac{2\pi}{k}\delta_{nl}\delta(k-k')\ .\end{aligned}\tag{3.4.79}$$

We have the following expansion of plane waves into elliptical waves [692, p. 185]

$$\begin{aligned}\exp\Big[\mathrm{i}\,k(x\cos\alpha+y\sin\alpha)\Big] &= 2\sum_{n=0}^{\infty}\mathrm{i}^n\,\mathrm{ce}_n(\alpha;h^2)M_n^{(1)}(\mu;h)\,\mathrm{ce}_n(\nu;h^2)\\ &\quad+2\sum_{n=1}^{\infty}\mathrm{i}^{-n}\,\mathrm{se}_n(\alpha;h^2)M_{-n}^{(1)}(\mu;h)\,\mathrm{se}_n(\nu;h^2)\ .\end{aligned}\tag{3.4.80}$$

In the limit $d \to 0$ the functions $M_n^{(1)}$ yield J_n-Bessel functions, and me_n exponentials.

In three dimensions one has for prolate-spheroidal coordinates the expansion [692, p. 315] in spheroidal wave functions $S_l^{n\,(1)}(z;kd), \mathrm{ps}_l^n(z;k^2d^2)$

$$\begin{aligned}&\exp\Big[\mathrm{i}\,kd(\sinh\mu\sin\nu\sin\vartheta\cos\varphi+\cosh\mu\cos\nu\cos\vartheta)\Big]\\ &=\sum_{l=0}^{\infty}\sum_{n=-l}^{l}(2l+1)\,\mathrm{i}^{l+2n}\\ &\qquad\times S_l^{n\,(1)}(\cosh\mu;kd)\,\mathrm{ps}_l^n(\cos\nu;k^2d^2)\,\mathrm{ps}_l^{-n}(\cos\vartheta;k^2d^2)\,\mathrm{e}^{\mathrm{i}\,n\varphi}\ .\end{aligned}\tag{3.4.81}$$

The orthonormality relations are given by

$$
\begin{aligned}
& d^3 \int_0^\infty \mathrm{d}\mu \int_0^\pi \mathrm{d}\nu (\sinh^2\mu + \sin^2\nu) \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{\mathrm{i}\varphi(n-n')} \\
& \qquad \times S_l^{n\,(1)}(\cosh\mu; kd) S_{l'}^{n'\,(1)}(\cosh\mu; k'd)\, \mathrm{ps}_l^n(\cos\nu; k^2d^2)\, \mathrm{ps}_{l'}^{n'}(\cos\nu; k'^2d^2) \\
& = \frac{2\pi^2}{2l+1} \frac{(l+n)!}{(l-n)!} \frac{1}{k^2} \delta_{nn'}\delta_{ll'}\delta(k-k') \ , \qquad (3.4.82)
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^\pi \sin\vartheta\, \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{ps}_l^{-n}(\cos\vartheta; k^2d^2)\, \mathrm{ps}_{l'}^{-n'\,*}(\cos\vartheta; k^2d^2)\, \mathrm{e}^{\mathrm{i}\varphi(n-n')} \\
& = \frac{4\pi}{2l+1} \frac{(l-n)!}{(l+n)!} \delta_{nn'}\delta_{ll'} \ . \qquad (3.4.83)
\end{aligned}
$$

In the limit $d \to 0$ the functions $S_l^{n\,(1)}$ yield $J_{l+1/2}$-Bessel functions and the ps_l^{-n} Legendre polynomials, respectively. The case of oblate spheroidal coordinates is similar and follows from analytic continuation, cf. [447].

3.4.6.2 Spherical Coordinates in Pseudo-Euclidean Space. The expansion formula for (pseudo)-spherical coordinates on the hyperboloid has been important in the evaluation of the path integral on the hyperboloid, cf. [104, 466]. Actually, it describes the interbases expansion of plane waves in a space with a Minkowski space-time metric into spherical waves. It has for some α the form [413, p. 804]

$$
\mathrm{e}^{-z\cosh\alpha} = \sqrt{\frac{2}{\pi z}} (z\sinh\alpha)^{\frac{3-D}{2}} \int_0^\infty \left| \frac{\Gamma(\mathrm{i}k + \frac{D-2}{2})}{\Gamma(\mathrm{i}k)} \right|^2 \mathcal{P}_{\mathrm{i}k-\frac{1}{2}}^{\frac{3-D}{2}}(\cosh\alpha) K_{\mathrm{i}k}(z)\mathrm{d}k \,, \qquad (3.4.84)
$$

which for $D = 2$ takes on the form

$$
\mathrm{e}^{-z\cosh\alpha} = \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k\alpha} K_{\mathrm{i}k}(z) \ . \qquad (3.4.85)
$$

$K_\nu(z)$ is the MacDonald Bessel function, and $\mathcal{P}_\mu^\nu(z)$ is the associated Legendre function. This formula is valid in D dimensions. For more details, cf. [447].

3.4.6.3 Elliptic and Spheroidal Coordinates in Pseudo-Euclidean Space. Similarly to Euclidean space, expansion formulæ in elliptic and spheroidal coordinates can be found which have the correct limit for $d \to 0$. $\mathrm{Me}_\nu(z; d^2) \propto \mathrm{e}^{\nu z}$ and $M_\nu^{(3)}(z/d; d) \propto H_\nu^{(1)}(z)$ $(d \to 0)$, and they yield the wave functions of the polar system. They obey the orthonormality relations $(h = kd/2)$

$$
\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\tau\, \mathrm{Me}_{\mathrm{i}k}(\tau; h^2)\, \mathrm{Me}^*_{\mathrm{i}k'}(\tau; h^2) = \delta(k'-k) \ , \qquad (3.4.86)
$$

$$
\begin{aligned}
& \frac{p\,\mathrm{e}^{-\pi k}}{8\pi} d^2 \int_{\mathbb{R}} \mathrm{d}a \int_0^\infty \mathrm{d}b\, (\sinh^2 a - \sinh^2 b) \\
& \qquad \times \mathrm{Me}_{\mathrm{i}k}(b; \tfrac{p^2d^2}{4})\, \mathrm{Me}^*_{\mathrm{i}k'}(b; \tfrac{p'^2d^2}{4}) M_{\mathrm{i}k}^{(3)}(a; \tfrac{pd}{2}) M_{\mathrm{i}k'}^{(3)\,*}(a; \tfrac{p'd}{2}) \\
& = \delta(k'-k)\delta(p'-p) \ . \qquad (3.4.87)
\end{aligned}
$$

Therefore we have the expansions for plane waves in the coordinates (v_0, v_1) with metric $(+,-)$

$$\begin{aligned}&\exp\left[\mathrm{i}\,p(v_0\cosh\tau - v_1\sinh\tau)\right]\\&\quad=\frac{1}{2}\int_{\mathbb{R}}\mathrm{d}k\;\mathrm{e}^{-\pi k/2}\,\mathrm{Me}_{\mathrm{i}\,k}(b;\tfrac{p^2d^2}{4})\,\mathrm{Me}_{\mathrm{i}\,k}(\tau;\tfrac{p^2d^2}{4})M^{(3)}_{\mathrm{i}\,k}(a;\tfrac{pd}{2})\;.\end{aligned}\tag{3.4.88}$$

In three dimensions matters are a little more complicated. We must extend the expansion of the two-dimensional case to three dimensions with the restriction that for $d \to 0$ we get back the spherical system. The proper spheroidal functions consequently are [692] $\mathrm{Ps}^{\mu}_{\nu}(z;\gamma^2)$ and $S^{\mu\,(3)}_{\mathrm{i}\,k-1/2}(z/\gamma;\gamma)$. They have the asymptotic behaviour $\mathrm{Ps}^{\mu}_{\nu}(z;\gamma^2)\propto\mathcal{P}^{\mu}_{\nu}(z)$ and $S^{\mu\,(3)}_{\mathrm{i}\,k-1/2}(z/\gamma;\gamma)$ $\propto\sqrt{\pi/2z}\,H^{(1)}_{\mathrm{i}\,k}(z)$ $(\gamma = pd,\ d\to 0)$, giving the spherical wave functions. From these considerations we derive by using the theory of spheroidal functions [692] the following interbases expansion [447]

$$\begin{aligned}&\exp\Big[\mathrm{i}\,pd(\cosh\xi\cosh\eta\cosh\alpha-\sinh\xi\sinh\eta\sinh\alpha\cos\varphi)\Big]\\&=\frac{1}{\pi^{3/2}}\sum_{n\in\mathbb{Z}}\int_0^\infty\mathrm{d}k\,k\sinh\pi k\big|\Gamma(\tfrac12+\mathrm{i}\,k+n)\big|^2\,\mathrm{e}^{-\pi k/2}\,\mathrm{e}^{\mathrm{i}\,n\varphi}\\&\qquad\times\mathrm{Ps}^{-n}_{\mathrm{i}\,k-1/2}(\cosh\eta;p^2d^2)\,\mathrm{Ps}^{-n}_{\mathrm{i}\,k-1/2}(\cosh\alpha;p^2d^2)S^{n\,(3)}_{\mathrm{i}\,k-1/2}(\cosh\xi;pd)\;.\end{aligned}\tag{3.4.89}$$

The spheroidal wave functions obey the orthonormality relations

$$\begin{aligned}&\int_0^{2\pi}d\psi\int_0^\infty\sinh\alpha\,\mathrm{d}\alpha\,\mathrm{Ps}^{-n}_{\mathrm{i}\,k-1/2}(\cosh\alpha)\,\mathrm{Ps}^{-n'}_{\mathrm{i}\,k'-1/2}(\cosh\alpha)\;\mathrm{e}^{\mathrm{i}\,\psi(n-n')}\\&=\frac{2\pi^2}{k\sinh\pi k}\big|\Gamma(\tfrac12+\mathrm{i}\,k+n)\big|^{-2}\delta_{nn'}\delta(k-k')\;,\end{aligned}\tag{3.4.90}$$

$$\begin{aligned}&d^3\int_0^\infty\mathrm{d}\xi\int_0^\infty\mathrm{d}\eta\,(\sinh^2\xi-\sinh^2\eta)\sinh\xi\sinh\eta\int_0^{2\pi}\mathrm{d}\varphi\;\mathrm{e}^{\mathrm{i}\,\varphi(n-n')}\\&\qquad\times\mathrm{Ps}^{-n}_{\mathrm{i}\,k-1/2}(\cosh\eta;p^2d^2)\,\mathrm{Ps}^{-n'\,*}_{\mathrm{i}\,k'-1/2}(\cosh\eta;p'^2d^2)\\&\qquad\times S^{n\,(3)}_{\mathrm{i}\,k-1/2}(\cosh\xi;pd)S^{n'\,(3)\,*}_{\mathrm{i}\,k'-1/2}(\cosh\xi;p'd)\\&=\frac{\pi}{k\sinh\pi k}\big|\Gamma(\tfrac12+\mathrm{i}\,k+n)\big|^{-2}\frac{2\pi}{p^2}\,\mathrm{e}^{\pi k}\,\delta_{nn'}\delta(k-k')\delta(p-p')\;.\end{aligned}\tag{3.4.91}$$

The case of oblate spheroidal coordinates is, of course, similar.

4 Perturbation Theory

In this chapter we study perturbation theory within the path integrals. We summarize four approaches for the evaluation of perturbation expansions which are:

i) *Path integration of a perturbation expansion after the method of Devreese and Goovaerts.* In this method one expands the potential term in the action in a perturbation expansion, performs a Laplace transformation of the potential (if possible), and reduces therefore the original path integral problem to that of a specific quadratic Lagrangian. The latter path integral problem can be solved exactly and one is left with a sum in powers of the coupling constant for the (energy dependent) Green function of the original potential problem.

ii) *Time-ordered perturbation expansion of the path integral and boundary problems in path integrals.* This expansion method is the original perturbative approach: if it is not possible to obtain for a quantum mechanical problem an analytical expression of the kernel, respectively its Green function, one expands the Feynman path integral about a known solution in powers of the coupling. This requires that the coupling constants are "small" in order that the perturbation expansion makes sense. Usually, it is not possible to perform all the integrations involved except for specific problems. The most important among these specific problems is point interactions which are treated in some detail. They belong to the class of exactly solvable quantum mechanical problems, and lead also to the incorporation of explicit boundary conditions in the path integral.

iii) *Effective potentials in partition functions.* In many applications in statistical physics approximate solutions of the generating functional are sufficient for a numerical investigation. Instead of exactly solving a path integral problem, one is interested in a "good and fast" algorithm. The theory of the effective potential in partition functions provides such an algorithm. It exploits the solution of the harmonic oscillator in order to derive a prescription for the numerical determination of an effective potential.

iv) *The semiclassical expansion about the harmonic expansion.* The semiclassical expansion makes full use of the information contained in a general quadratic Lagrangian, respectively Hamiltonian. It allows one to expand an arbitrary dynamical quantum mechanical problem about its semiclassical solution in powers of $\hbar$ up to any order. The expansion in powers of $\hbar$ is based on moment formulæ which are the quantum mechanical analogue of Wick's theorem.

4.1 Path Integration and Perturbation Theory

In this section we present the perturbative method for path integrals for quite arbitrary potentials. The path integral is rewritten in terms of a specific quadratic Lagrangian for which the path integral can be solved exactly. Finally, one is left with a perturbation expansion in powers of the coupling constant of the potential.

The method was originally developed by Feynman [340] and Devreese et al. [405, 408, 409] with the explicit treatment of the Coulomb potential. Further examples are the usual harmonic oscillator treated by Grosjean and Goovaerts [472] (slightly modified) and the δ-function potential by Goovaerts, Babceno and Devreese [404]. The drawback of the method is that it is quite involved for all the standard examples such as the harmonic oscillator, the Coulomb potential, or the δ-function. The case of the harmonic oscillator leads to a very cumbersome perturbation expansion, and in the case of the Coulomb potential, only the wave functions of a definite parity could be extracted. Also, the principal problem that the kernel of the Coulomb potential is not known in closed form cannot be resolved. However, this method showed for the first time that it was possible to extract all the relevant information of the discrete and continuous spectrum of the Coulomb potential by a genuine path integral approach.[1]

The first object which was studied by this perturbative method was not the Feynman kernel $K(\mathbf{x}'',\mathbf{x}';T)$ itself, but instead the expression

$$W_0(\mathbf{x}') \equiv \int \mathrm{d}\mathbf{x}'' K(\mathbf{x}'',\mathbf{x}';T) \ . \tag{4.1.1}$$

This is not a serious drawback as long as the calculation of wave functions and energy levels is concerned, but it is, of course, a more conceptual one because one is interested in the entire Feynman kernel and not in an "averaged" one. As already said, the calculation of $K(T)$ for the Coulomb potential is not possible in the rigorous sense that the Feynman kernel can be expressed, say, in terms of elementary functions such as the harmonic oscillator. An exact expression for the Coulomb potential can only be achieved for the resolvent kernel $G(E)$ (energy dependent Green function), the Fourier transform of $K(T)$. Another drawback of W_0 in (4.1.1) is that W_0 only contains the wave functions and energy levels for even parity. Due to the general properties

[1] It must be noted that Gutzwiller studied the hydrogen atom in 1967 [479] by means of path integrals. He found an exact Green function for the bound state energy levels in polar coordinates in momentum space by using a semiclassical approximation. Actually, Gutzwiller developed in these papers his periodic orbit theory, see Chap. 5.

of the Feynman kernel, wave functions with $\Psi(\mathbf{x}'') = -\Psi(-\mathbf{x}'')$ do not contribute to W_0. This, however, can be circumvented by considering instead the quantity

$$W_1(\mathbf{x}') = \int d\mathbf{x}''\mathbf{x}'' \cdot \nabla_{\mathbf{x}'} K(\mathbf{x}'', \mathbf{x}'; T) \quad , \tag{4.1.2}$$

so that only states with odd parity contribute to W_1. The whole picture then emerges from a proper combination of W_0 and W_1.

We present the improved method for a perturbative calculation of the entire kernel $K(T)$ following Goovaerts and Broeckx [405]. From the result it is then obvious how to derive the two quantities W_0 and W_1, respectively.

The general method for the time-ordered perturbation expansion is quite simple. Let us assume that we have a potential $W(\mathbf{x}) = V(\mathbf{x}) + \tilde{V}(\mathbf{x})$ in the path integral and suppose that $W(\mathbf{x})$ is so complicated that a direct path integration is not possible. However, the path integral $K^{(V)}$ corresponding to $V(\mathbf{x})$ is assumed to be known. We expand the integrand of the path integral containing $\tilde{V}(\mathbf{x})$ in a perturbation expansion about $V(\mathbf{x})$. The result has a simple interpretation on the lattice: the initial kernel corresponding to $V(\mathbf{x})$ propagates during the short-time interval ϵ unperturbed, then it interacts with $\tilde{V}(\mathbf{x})$ in order to propagate again in another short-time interval ϵ unperturbed, and so on, up to the final state. One then obtains the following series expansion (see also, e.g., [65, 68, 340, 404–408, 430, 621, 642, 830, 876], $\mathbf{x} \in \mathbb{R}^D$)

$$\begin{aligned}
K(\mathbf{x}'', \mathbf{x}'; T) &= \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}) - \tilde{V}(\mathbf{x})\right)dt\right] \\
&= K^{(V)}(\mathbf{x}'', \mathbf{x}'; T) + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\prod_{j=1}^{n}\int_{\mathbb{R}^D} d\mathbf{x}_j \int_{t'}^{t''} dt_j \\
&\quad \times K^{(V)}(\mathbf{x}_1, \mathbf{x}'; t_1)\tilde{V}(\mathbf{x}_1)K^{(V)}(\mathbf{x}_2, \mathbf{x}_1; t_2 - t_1) \times \ldots \\
&\quad \ldots \times \tilde{V}(\mathbf{x}_{n-1})K^{(V)}(\mathbf{x}_n, \mathbf{x}_{n-1}; t_n - t_{n-1})\tilde{V}(\mathbf{x}_n)K^{(V)}(\mathbf{x}'', \mathbf{x}_n; T - t_n) \\
&= K^{(V)}(\mathbf{x}'', \mathbf{x}'; T) + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \prod_{j=1}^{n}\int_{t'}^{t_{j+1}} dt_j \int_{\mathbb{R}^D} d\mathbf{x}_j \\
&\quad \times K^{(V)}(\mathbf{x}_1, \mathbf{x}'; t_1 - t')\tilde{V}(\mathbf{x}_1)K^{(V)}(\mathbf{x}_2, \mathbf{x}_1; t_2 - t_1) \times \ldots \\
&\quad \ldots \times \tilde{V}(\mathbf{x}_{n-1})K^{(V)}(\mathbf{x}_n, \mathbf{x}_{n-1}; t_n - t_{n-1})\tilde{V}(\mathbf{x}_n)K^{(V)}(\mathbf{x}'', \mathbf{x}_n; t'' - t_n) \ .
\end{aligned} \tag{4.1.3}$$

In the second step we have ordered the time as $t' = t_0 < t_1 < t_2 < \cdots < t_{n+1} = t''$ $(T = t'' - t')$ and paid attention to the fact that $K(t_j - t_{j-1})$ is different from zero only if $t_j > t_{j-1}$.

We consider a D-dimensional path integral with a potential $V(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^D$, apply the above perturbation expansion and get (we use the Euclidean path integral, see p. 34)

$$K_E(\mathbf{x}'',\mathbf{x}';T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(t)\exp\left[-\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2+V(\mathbf{x})\right)\mathrm{d}t\right] \ . \tag{4.1.4}$$

Expanding the exponential yields ($n = 0$ term $\equiv 1$)

$$\begin{aligned} K_E(\mathbf{x}'',\mathbf{x}';T) &= \sum_{n=0}^{\infty}\frac{(-\hbar)^{-n}}{n!} \\ &\times \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(t)\exp\left(-\frac{m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2\mathrm{d}t\right)\left(\int_{t'}^{t''}V(\mathbf{x}(t))\mathrm{d}t\right)^n \ . \end{aligned} \tag{4.1.5}$$

Introducing the Fourier transform $\widehat{V}(\mathbf{k})$ of the potential $V(\mathbf{x})$

$$V(\mathbf{x}) = \frac{1}{(2\pi\hbar)^D}\int_{\mathbb{R}^D}\mathrm{d}\mathbf{k}\ \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}/\hbar}\,\widehat{V}(\mathbf{k}) \ , \tag{4.1.6}$$

we obtain for the path integral

$$K_E(\mathbf{x}'',\mathbf{x}';T) = \sum_{n=0}^{\infty}\frac{(-\hbar)^{-n}}{n!}\prod_{j=1}^{n}\int_{t'}^{t''}\mathrm{d}t_j\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_j}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_j)P_n(\mathbf{x}'',\mathbf{x}';T) \ , \tag{4.1.7}$$

$$P_n(\mathbf{x}'',\mathbf{x}';T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(t)\exp\left[-\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2-\mathbf{k}_n(t)\cdot\mathbf{x}\right)\mathrm{d}t\right] \tag{4.1.8}$$

with the D-dimensional vector

$$\mathbf{k}_n(t) = \mathrm{i}\sum_{j=1}^{n}\delta(t-t_j)\mathbf{k}_j \ . \tag{4.1.9}$$

The path integrals $P_n(T)$ are path integrals for a linear potential, and therefore can be exactly determined by solving the classical Euler–Lagrange equations and inserting this solution into the path integral solution for the general quadratic Lagrangian. In order to perform the n-fold time integrations in the perturbation expansion, one exploits the convolution theorem of the Laplace transformation. This finally leads to the following two alternative expressions for the perturbation expansion ($c > 1$)

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}T\,\mathrm{e}^{sT/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}_E\mathbf{x}(t)\exp\left[-\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2+V(\mathbf{x})\right)\mathrm{d}t\right]\\
&=\sum_{n=0}^{\infty}(-1)^n\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_0}{(2\pi\hbar)^D}\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_1)\cdots\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_n}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_n)\\
&\qquad\times\frac{\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{x}'\cdot\sum_{j=1}^n\mathbf{k}_j-\frac{\mathrm{i}}{\hbar}\mathbf{x}''\cdot\mathbf{k}_0\right)}{[s+(\mathbf{k}_0^2/2m)]\dots[s+(\mathbf{k}_0+\dots+\mathbf{k}_n)^2/2m]}
\end{aligned}\tag{4.1.10}$$

$$\begin{aligned}
&=\sum_{n=0}^{\infty}(-1)^n\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_0}{(2\pi\hbar)^D}\prod_{j=1}^{n}\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_j}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_j-\mathbf{k}_{j-1})\\
&\qquad\times\frac{\exp\left[\frac{\mathrm{i}}{\hbar}(\mathbf{x}'\cdot\mathbf{k}_n-\mathbf{x}''\cdot\mathbf{k}_0)\right]}{(s+\mathbf{k}_0^2/2m)\dots(s+\mathbf{k}_n^2/2m)}\quad .
\end{aligned}\tag{4.1.11}$$

This method therefore allows one to perform the path integration as explicitly as possible. In the final expression path integrations no longer occur. However, one is left with convolutions in Fourier space together with an infinite summation. It is obvious that such a complicated formula allows the necessary final manipulations only for particular problems. Examples can be found in [401, 404–409].

4.2 Summation of the Perturbation Series for δ- and δ'-Potentials

In Sect. 3.3 we discussed some basic path integral solutions. However, there are several potential and boundary problems which do not fall into these classes and are exactly solvable quantum mechanical problems nevertheless. The δ-function potential and potential problems with a δ-function perturbation belong to these kinds of problems. Based on a functional analytic approach, point interactions as solvable quantum models have been thoroughly discussed by Albeverio et al. [17].

In this section we discuss all these non-Gaussian path integral problems which can be interpreted as path integrals with point interactions, respectively boundary conditions. As it turns out, the corresponding Green functions can be written as a quotient of two determinants with the unperturbed Green functions taken at the perturbation points as entries [17, 439], cf., the section about point interactions in the table of path integrals. However, this simple feature does not hold for multiple point interactions in the path integral representation for the one-dimensional Dirac particle. The former simplification does not work in general, and the corresponding Green functions are matrices with determinants as entries which are in turn also determinants, and so on, cf. [448].

4.2.1 One-Dimensional Point Interaction. In order to incorporate a one-dimensional point interaction, i.e., a one-dimensional δ-function perturbation we consider in (4.1.3) the following potential in the path integral [430]

$$W(x) = V(x) - \gamma\delta(x-a) \ . \tag{4.2.1}$$

The path integral for this potential problem has the form

$$K(x'',x';T) = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - W(x)\right)\mathrm{d}t\right] \ . \tag{4.2.2}$$

Let us assume that the path integral for the potential $V(\mathbf{x})$ is known, i.e., the path integral

$$K^{(V)}(x'',x';T) := \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right] \ , \tag{4.2.3}$$

and also the (energy dependent) Green function

$$\begin{aligned} G^{(V)}(x'',x';E) &= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} K^{(V)}(x'',x';T) \\ K^{(V)}(x'',x';T) &= \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} G^{(V)}(x'',x';E) \ . \end{aligned} \tag{4.2.4}$$

Introducing the Green function $G(E)$ of the perturbed system similarly to (4.2.4) we are able to sum exactly the perturbation expansion, due to the convolution theorem of Fourier transformation, yielding

$$G^{(\delta)}(x'',x';E) = G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E) - 1/\gamma} \ . \tag{4.2.5}$$

Here it is assumed that $G^{(V)}(a,a;E)$ actually exists and the energy levels $E_n^{(\delta)}$ of the perturbed problem $W(x)$ are therefore determined in a unique way by the equation

$$G^{(V)}(a,a;E_n^{(\delta)}) = \frac{1}{\gamma} \ . \tag{4.2.6}$$

This is in general an implicit (transcendental) equation. The corresponding wave functions are given by

$$\Psi_i^{(\delta)}(x) = \left[\lim_{E\to E_i^{(\delta)}} \frac{E_i^{(\delta)} - E}{1/\gamma - G^{(V)}(a,a;E)}\right]^{1/2} G^{(V)}(x,a;E_i^{(\delta)}) \ . \tag{4.2.7}$$

4.2.2 Point Interaction for the Dirac Particle. We consider a point interaction in the path integral for the one-dimensional Dirac particle. From the theory of [508] we have for the one-dimensional Dirac particle

$$\mathbf{K}^{(V)}(x'',x';T)=\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}\nu(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathbf{W}(x)\mathrm{d}t\right)\ ,\tag{4.2.8}$$

with $\mathbf{W}$ a matrix-valued potential, and the Green function $\mathbf{G}^{(V)}(E)$ in its matrix representation is defined as

$$\mathbf{G}^{(V)}(x'',x';E)=\begin{pmatrix}G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'',x';E)\\ G_{21}^{(V)}(x'',x';E) & G_{22}^{(V)}(x'',x';E)\end{pmatrix}\ .\tag{4.2.9}$$

We first consider a δ-function perturbation in the electron (= "+") component, i.e., $\tilde{\mathbf{V}}=-\alpha\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\delta(x-a)$. We obtain by inserting it into the path integral and summing the perturbation expansion

$$\begin{aligned}\mathbf{G}^{(\delta+)}(x'',x';E)&=\mathbf{G}^{(V)}(x'',x';E)+\frac{1}{1/\alpha-G_{11}^{(V)}(a,a;E)}\\ &\times\begin{pmatrix}G_{11}^{(V)}(a,x';E)G_{11}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,x';E)G_{12}^{(V)}(x'',a;E)\\ G_{21}^{(V)}(a,x';E)G_{11}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,x';E)G_{12}^{(V)}(x'',a;E)\end{pmatrix}\ .\end{aligned}\tag{4.2.10}$$

Similarly for the positron (= "−") component, i.e., $\tilde{\mathbf{V}}=(4m^2\beta c^2/\hbar^2)\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\delta(x-a)$ (the constants have been chosen for convenience)

$$\begin{aligned}\mathbf{G}^{(\delta-)}(x'',x';E)&=\mathbf{G}^{(V)}(x'',x';E)-\frac{1}{\hbar^2/4m^2c^2\beta+G_{22}^{(V)}(a,a;E)}\\ &\times\begin{pmatrix}G_{12}^{(V)}(a,x';E)G_{21}^{(V)}(x'',a;E) & G_{12}^{(V)}(a,x';E)G_{22}^{(V)}(x'',a;E)\\ G_{22}^{(V)}(a,x';E)G_{21}^{(V)}(x'',a;E) & G_{22}^{(V)}(a,x';E)G_{22}^{(V)}(x'',a;E)\end{pmatrix}\ .\end{aligned}\tag{4.2.11}$$

The investigation of the two problems shows that in the non-relativistic limit the path integral incorporation of the δ- and δ'-function perturbations, respectively, emerge. The former case has already been discussed in Sect. 4.2.1, the latter we state in the next section.

4.2.3 One-Dimensional δ'-Function. The case of the incorporation of a δ'-function perturbation is as it stands not well defined due to the ultraviolet divergence. In order to make the problem well defined one regularizes it by considering a δ-function perturbation for the one-dimensional Dirac particle (see above) and its corresponding path integral formulation. The non-relativistic limit of the point interaction in the positron component is due to the particular feature of the emerging boundary condition called a δ'-function perturbation. Performing in (4.2.11) this limit yields the following result for the corresponding Green function [446, 448]

$$G^{(\delta')}(x'',x';E) = \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}$$

$$\times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x) + \beta\delta'(x-a)\right)\mathrm{d}t\right]$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}_{,x'}(x'',a;E)G^{(V)}_{,x''}(a,x';E)}{1/\beta + \widehat{G}^{(V)}_{,x'x''}(a,a;E)}\ , \tag{4.2.12}$$

$$\widehat{G}^{(V)}_{,xy}(a,a;E) = \left(\frac{\partial^2}{\partial x\partial y}G^{(V)}(x,y;E) - \frac{2m}{\hbar^2}\delta(x-y)\right)\Bigg|_{x=y=a}\ . \tag{4.2.13}$$

Actually, a point interaction in one dimension has a four-parameter family of self-adjoint extensions [17, 73, 144, 145]. These four parameters can be successively incorporated into the path integral, cf. [448].

4.2.4 Boundary Value Problems. Actually, every point interaction in the path integral corresponds to a particular boundary condition. However, in the following we want to discuss two specific kinds of boundary conditions, i.e., Dirichlet and Neumann boundary conditions, respectively.

We present the incorporation of (Dirichlet or Neumann) boundary condition at one point $a \in \mathbb{R}$, where the motion can take place on the right or left hand side of a, i.e., $x > a$ or $x < a$, respectively. To take into account a second boundary condition $b \in \mathbb{R}$ we just repeat the procedure, and the motion is restricted to $a < x < b$. This approach turns out to be extremely useful for potentials which are restricted to $x \equiv r > 0$, i.e., potential problems which are defined and solvable in $\mathbb{R}$ are interpreted as radial potentials. This restriction can alter the properties of a potential completely, for instance, from a pure scattering potential to a confinement potential [439]. If the correct boundary condition is not taken into account, the results can be very misleading [275].

4.2.4.1 Dirichlet Boundary Conditions. In (4.2.5) we consider the limit $\gamma \to -\infty$ which has the effect that an impenetrable wall appears at $x = a$ [439]. We set $\lim_{\gamma\to-\infty} G^{(\delta)}(E) \equiv G^{(x=a)}(E)$, i.e., we obtain

$$G^{(x=a)}(x'',x';E)$$

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(x=a)}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right]$$

$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a; E) G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E)} \ . \tag{4.2.14}$$

Bound states are determined by the equation

$$G^{(V)}(a, a; E_n) = 0 \ . \tag{4.2.15}$$

Again, the wave functions are given by the residua of $G(E)$ at $E = E_n$, i.e.,

$$\Psi_n(x) = \lim_{E \to E_n} \left[-(E_n - E) \frac{G^{(V)}(x, a; E)}{G^{(V)}(a, a; E)} \right] \ . \tag{4.2.16}$$

4.2.4.2 Neumann Boundary Conditions. In the case of Neumann boundary conditions one can start by considering a δ'-function perturbation in the path integral and by making the strength infinitely repulsive [17] after summing up the corresponding perturbation expansion. We therefore obtain a path integral formulation for Neumann boundary conditions at $x = a$ (superscripts (D, N) denote in the following Dirichlet, respectively Neumann boundary conditions, for the Green function) [446]

$$G^{(N)}(x'', x'; E)$$

$$= \frac{\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}T \, \mathrm{e}^{\mathrm{i} ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(N)}_{(x=a)} x(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{x}^2 - V(x) \right) \mathrm{d}t \right]$$

$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}_{,x'}(x'', a; E) G^{(V)}_{,x''}(a, x'; E)}{\widehat{G}^{(V)}_{,x'x''}(a, a; E)} \ , \tag{4.2.17}$$

$$\widehat{G}^{(V)}_{,xy}(a, a; E) = \left(\frac{\partial^2}{\partial x \partial y} G^{(V)}(x, y; E) - \frac{2m}{\hbar^2} \delta(x - y) \right) \bigg|_{x=y=a} \ . \tag{4.2.18}$$

Bound states are determined by the equation

$$\widehat{G}^{(V)}_{,x'x''}(a, a; E_n) = 0 \ . \tag{4.2.19}$$

The wave functions are given by the residua of $G(E)$ at $E = E_n$, i.e.,

$$\Psi_n(x) = \lim_{E \to E_n} \left[-(E_n - E) \frac{G^{(V)}_{,y}(x, a; E)}{\widehat{G}^{(V)}_{,xy}(a, a; E)} \right] \ . \tag{4.2.20}$$

4.3 Partition Functions and Effective Potentials

Let us consider[2] the path integral for the one-dimensional density matrix (see (2.2.11))

$$\rho(x'', x'; \beta) = \int\limits_{x(0)=x'}^{x(\beta\hbar)=x''} \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} \left(\frac{m}{2}\dot{x}^2 + V(x)\right)\mathrm{d}t \right] . \tag{4.3.1}$$

An interesting object in statistical mechanics is the partition function

$$Z := \int_{\mathbb{R}} \rho(x, x; \beta)\, \mathrm{d}x = \operatorname{Tr} \mathrm{e}^{-\beta \underline{\mathrm{H}}} . \tag{4.3.2}$$

With (4.3.1) we obtain for the partition function

$$\begin{aligned} Z(\beta) &= \oint \mathcal{D}_E x(t) \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} \left(\frac{m}{2}\dot{x}^2 + V(x)\right)\mathrm{d}t \right] \\ &:= \int_{\mathbb{R}} \mathrm{d}x \int\limits_{x(0)=x}^{x(\beta\hbar)=x} \mathcal{D}_E x(t) \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} \left(\frac{m}{2}\dot{x}^2 + V(x)\right)\mathrm{d}t \right] . \end{aligned} \tag{4.3.3}$$

We can easily obtain a first approximation. Since we are dealing with closed paths, we can assume that for small "times" $\tau = \beta\hbar = \hbar/k_B T$, i.e., large temperature T, the contributions from the potential become important only if the particle remains near the initial (and final) point x. Going farther away increases $\dot{x}$ which gets exponentially damped. Therefore

$$\begin{aligned} Z \simeq Z_{\mathrm{Cl}} &:= \int_{\mathbb{R}} \mathrm{d}x\ \mathrm{e}^{-\tau V(x)/\hbar} \int\limits_{x(0)=x}^{x(\beta\hbar)=x} \mathcal{D}_E x(t) \exp\left(-\frac{m}{2\hbar}\int_0^{\beta\hbar} \dot{x}^2\, \mathrm{d}t \right) \\ &= \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int_{\mathbb{R}} \mathrm{d}x\ \mathrm{e}^{-\beta V(x)} . \end{aligned} \tag{4.3.4}$$

Z_{Cl} is known as the classical limit of the partition function. Of course, this first rough approximation deserves further refinement in order to obtain better results. And indeed, many methods are known to improve the rough estimate (4.3.4), cf., the approach of Wigner [931] and Kirkwood [594], and later, e.g., Friesner and Levy [353], [638], Miller [694], Schweizer, Stratt, Chandler and Wolynes [837], Feynman [334, 340], Feynman and Kleinert [341, 614], Giachetti and Tognetti [390]. See also the review article of Leschke [646], and Chap. 4 in [801].

In order to improve (4.3.4) and to preserve the general feature of the partition function, we consider

[2] Here we follow closely Feynman [334, 340]. See also [341, 390, 614].

$$Z_{\mathrm{Cl},W} := \sqrt{\frac{m}{2\pi\beta\hbar^2}} \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\beta W(x)} \quad . \tag{4.3.5}$$

Here we have introduced an effective potential $W(x)$ which will be defined following Feynman and Hibbs [340]. For this purpose let us consider two (Helmholtz) free energies F and F', corresponding to the actions S and S' (in this section we use $S := R_E$ for the Euclidean action)

$$Z = \mathrm{e}^{-\beta F} = \oint \mathcal{D}_E x(t)\, \mathrm{e}^{-S/\hbar} \ , \qquad Z' = \mathrm{e}^{-\beta F'} = \oint \mathcal{D}_E x(t)\, \mathrm{e}^{-S'/\hbar} \quad . \tag{4.3.6}$$

We define for some observable $\mathcal{A}$ the expectation value by

$$\langle \mathcal{A} \rangle_S = \oint \mathcal{D}_E x(t) \mathcal{A}\, \mathrm{e}^{-S/\hbar} \Big/ \oint \mathcal{D}_E x(t)\, \mathrm{e}^{-S/\hbar} \quad . \tag{4.3.7}$$

Consequently we have $\langle \mathrm{e}^{-(S-S')/\hbar} \rangle_{S'} = \mathrm{e}^{-\beta(F-F')}$. We want to determine a best lower bound using a trial function F in the following way. Because of the convexity of the exponential function $\langle \mathrm{e}^x \rangle \geq \mathrm{e}^{\langle x \rangle}$ (Jensen's inequality [398, 885, 890]) we then have $\mathrm{e}^{-\langle S-S' \rangle_{S'}/\hbar} \leq \mathrm{e}^{-\beta(F-F')}$ and therefore

$$F \leq F' + \frac{1}{\beta\hbar} \langle S - S' \rangle_{S'} \ , \qquad Z \geq Z'\, \mathrm{e}^{-\langle S-S' \rangle_{S'}/\hbar} \quad . \tag{4.3.8}$$

We define $\Delta = \langle S - S' \rangle_{S'}/\beta\hbar$; then our minimization problem is

$$F \leq F' + \Delta \ , \tag{4.3.9}$$

and the expression $F' + \Delta$ must be minimized with respect to some parameters suitably chosen.

Let us apply this method to a simple ansatz for an effective potential, i.e., we consider

$$S = \int_0^{\beta\hbar} \left(\frac{m}{2} \dot{x}^2 + V(x(s)) \right) \mathrm{d}s \ , \qquad S' = \frac{m}{2} \int_0^{\beta\hbar} \dot{x}^2\, \mathrm{d}s + \beta\hbar W(\bar{x}) \tag{4.3.10}$$

where we have introduced for a given path $x(s)$ an average position $\bar{x}$

$$\bar{x} = \frac{1}{\beta\hbar} \int_0^{\beta\hbar} x(s)\, \mathrm{d}s \quad . \tag{4.3.11}$$

We obtain

$$\Delta = \frac{1}{\beta\hbar} \left\langle \int_0^{\beta\hbar} [V(x) - W(\bar{x})] \mathrm{d}s \right\rangle_{S'}$$

$$= \frac{\oint \mathcal{D}_E x(s) \left(\int_0^{\beta\hbar} \frac{V(x)}{\beta\hbar} \mathrm{d}s - W(\bar{x}) \right) \exp\left(-\int_0^{\beta\hbar} \frac{m\dot{x}^2}{2\hbar} \mathrm{d}s \right) \mathrm{e}^{-\beta W(\bar{x})}}{\oint \mathcal{D}_E x(s) \exp\left(-\frac{m}{2\hbar} \int_0^{\beta\hbar} \dot{x}^2 \mathrm{d}s \right) \mathrm{e}^{-\beta W(\bar{x})}}$$

$$= \frac{\int_{\mathbb{R}} \mathrm{d}\,\bar{x} \left[\overline{V(\bar{x})} - W(\bar{x})\right] \mathrm{e}^{-\beta W(\bar{x})}}{\int_{\mathbb{R}} \mathrm{d}\,\bar{x}\ \mathrm{e}^{-\beta W(\bar{x})}} , \tag{4.3.12}$$

where the path integrals have been explicitly evaluated, see [340,341]. Here we have introduced the average potential $\overline{V(\bar{x})}$, i.e., averaged over a Gaussian

$$\overline{V(\bar{x})} = \sqrt{\frac{6m}{\pi\beta\hbar^2}} \int_{\mathbb{R}} V(x) \exp\left[-\frac{6m}{\beta\hbar^2}(x-\bar{x})^2\right] \mathrm{d}x \ . \tag{4.3.13}$$

Then one finds that $F' + \Delta$ takes its minimum for

$$W(\bar{x}) = \overline{V(\bar{x})} \ , \tag{4.3.14}$$

which is the classical result of Feynman and Hibbs [340].

4.4 Semiclassical Expansion About the Harmonic Approximation

For the discussion of the semiclassical expansion about the harmonic approximation let us consider the Lagrangian path integral

$$K(x'', x'; t'', t') = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{x}^2(t) - V(x(t), t)\right)\mathrm{d}t\right] \tag{4.4.1}$$

for a given potential $V(x,t)$. In general x can be a D-dimensional coordinate, but without loss of generality we only consider the case $D = 1$. Taylor expansion of the potential $V(x,t)$ around its assumed minimum at $x = 0$ $(V(0,t) = \partial V(0,t)/\partial x = 0)$ gives

$$K(x'', x'; t'', t') = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}\dot{x}^2 - \frac{V''(t)}{2}x^2(t) - \sum_{n=3}^{\infty} \frac{V^{(n)}(t)}{n!} x^n(t)\right]\mathrm{d}t\right\} , \tag{4.4.2}$$

where $V^{(n)}(t) = \partial^n V(0,t)/\partial x^n$ (including an explicit time dependence). Retaining only the quadratic term gives the leading semiclassical approximation about the harmonic oscillator. However, let us consider the higher orders. We call the kernel with the quadratic terms K_{GHO}. Developing the higher order term in the exponential yields

$$K(x'', x'; t'', t') = K_{\mathrm{GHO}}(x'', x'; t'', t') \left\{ 1 + \sum_{j=1}^{\infty} \frac{1}{j!}\left(-\frac{\mathrm{i}}{\hbar}\right)^j \sum_{n_1=3}^{\infty} \cdots \sum_{n_j=3}^{\infty} \right.$$

$$\times \int_{t'}^{t''} \frac{dt_1 \ldots dt_j}{n_1! \ldots n_j!} V^{(n_1)}(t_1) \ldots V^{(n_j)}(t_j) \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) x^{n_1}(t_1) \ldots x^{n_j}(t_j)$$

$$\left. \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{1}{2}V''(t)x^2(t)\right)dt\right]\right\} \ . \tag{4.4.3}$$

The task is therefore to calculate the "moments" in (4.4.3). It turns out that by an appropriate summation the semiclassical expansion around the harmonic approximation (4.4.3) can be cast into

$$K(x'',x';t'',t') = K_{\mathrm{GHO}}(x'',x';t'',t')\left[1 + \sum_{j=1}^{\infty} \hbar^j K_j(x'',x';t'',t')\right] \ , \tag{4.4.4}$$

where the K_j are in fact of $O(1)$ with respect to $\hbar$. This can be seen from (4.4.3) in the following way. Let us consider $j = 1$. The first non-vanishing term comes from the x^4 contribution which yields in the path integral a term $\propto \hbar^{4/2}/\hbar = \hbar$; the next term of the same order comes from $j = 2$ and the $(x^3)^2$ contribution which yields a term $\propto \hbar^{6/2}/\hbar^2 = \hbar$, and all terms of odd powers in x vanish, etc., cf. [238].

Whereas in many cases a semiclassical expansion around the harmonic approximation is sufficient,[3] we want to exploit the full power of the theory. In order to do this, we switch from the Lagrangian formulation of the path integral to its corresponding Hamiltonian version. This formulation allows us to calculate moments not only with respect to x but also with respect to its conjugate momentum p. Our presentation can only sketch the most important features of the theory; for further details the reader is referred to the relevant literature, cf. [238, 699–705].

K_{GHO} is the kernel corresponding to the general quadratic time-dependent Hamiltonian

$$H_{\mathrm{GHO}} = \frac{1}{2m}g(t)p^2(t) + \frac{1}{2}f(t)q^2(t) + k(t)p(t)q(t) \ , \tag{4.4.5}$$

i.e., we have in this formulation the Hamiltonian path integral

[3] Note that in the semiclassical approximation for radial problems the so-called Langer modification must be taken into account, otherwise the angular dependence comes out wrong. This is particularly important in the Coulomb problem whose WKB formula gives *with* the Langer modification the exact energy levels, cf. [384, 632].

$$K_{\mathrm{GHO}}(q'',q';t'',t') = \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}(q(t),p(t))$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(p(t)\dot{q}(t) - \frac{1}{2m}g(t)p^2(t) - \frac{1}{2}f(t)q^2(t) - k(t)p(t)q(t)\right)\mathrm{d}t\right] . \tag{4.4.6}$$

Here we assume an appropriate operator ordering in the Hamiltonian, e.g., Weyl ordering, with the corresponding lattice definition of the path integral, e.g., the mid-point lattice. Because this feature is not relevant for the theory of the semiclassical expansion around the harmonic approximation, we skip the corresponding notation.

We call $\mathcal{D}w(p,q)$ the Gaussian measure defined by

$$\mathcal{D}w(p,q) = \mathcal{D}(q(t),p(t))$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(p(t)\dot{q}(t) - \frac{1}{2m}g(t)p^2(t) - \frac{1}{2}f(t)q^2(t) - k(t)p(t)q(t)\right)\mathrm{d}t\right] . \tag{4.4.7}$$

In this formulation we have the identity

$$K_{\mathrm{GHO}}(q'',q';t'',t') = \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) . \tag{4.4.8}$$

We introduce the Fourier transform in the functional integral defined by

$$\mathcal{F}\mathcal{D}w(\mu,\nu) = \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)\, \mathrm{e}^{-\mathrm{i}\langle\mu,q\rangle - \mathrm{i}\langle\nu,p\rangle} . \tag{4.4.9}$$

Here $\langle\mu,q\rangle = \int_{t'}^{t''} q(s)\,\mathrm{d}\mu(s)$, so that if $\mathrm{d}\mu(t) = \delta(t-s)\mathrm{d}s$, then $\langle\mu,q\rangle = q(t)$, i.e, μ,ν are integration measures, and $\langle\cdot,\cdot\rangle$ is a scalar product with respect to these measures. Of crucial importance is that $\mathcal{F}\mathcal{D}w(\mu,\nu)$ can be evaluated. The result is

$$\mathcal{F}\mathcal{D}w(\mu,\nu) = \exp\left[-\mathrm{i}\langle\mu,\bar{q}\rangle - \mathrm{i}\langle\nu,\bar{p}\rangle - \frac{\mathrm{i}\hbar}{2}\int_{t'}^{t''}\int_{t'}^{t''} G_{ab}(t,s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s)\right.$$
$$\left. -\mathrm{i}\hbar\int_{t'}^{t''}\int_{t'}^{t''} \bar{G}(t,s)\,\mathrm{d}\mu(t)\,\mathrm{d}\nu(s) - \frac{\mathrm{i}\hbar}{2}\int_{t'}^{t''}\int_{t'}^{t''} G_p(t,s)\,\mathrm{d}\nu(t)\,\mathrm{d}\nu(s)\right] \tag{4.4.10}$$

$$= \exp\left[-\mathrm{i}\int_{t'}^{t''} \bar{r}(t)\,\mathrm{d}\tilde{\alpha}(t) - \frac{\mathrm{i}\hbar}{2}\int_{t'}^{t''}\int_{t'}^{t''} \mathrm{d}\;\alpha(t)\mathcal{G}(t,s)\,\mathrm{d}\tilde{\alpha}(t)\right] \quad . \tag{4.4.11}$$

Here we have introduced the notation

$$\mathrm{d}\,\alpha(t) = (\mathrm{d}\mu(t), \mathrm{d}\nu(t)) \;, \qquad \mathrm{d}\,\tilde{\alpha}(t) = \begin{pmatrix}\mathrm{d}\mu(t)\\ \mathrm{d}\nu(t)\end{pmatrix}, \tag{4.4.12}$$

$$\bar{r}(t) = (\bar{q}(t), \bar{p}(t)) = (q_{\mathrm{Cl},0}(t), p_{\mathrm{Cl},0}(t)) \tag{4.4.13}$$

which are the classical position and momentum, respectively, corresponding to the Hamiltonian H_{GHO} related by

$$p_{\mathrm{Cl},0}(t) = \frac{m}{g(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t} - k(t)\right)q_{\mathrm{Cl},0}(t) \quad . \tag{4.4.14}$$

Finally we have

$$\mathcal{G}(t,s) = \begin{pmatrix} G_{ab}(t,s) & \bar{G}(t,s) \\ \bar{G}(t,s) & G_p(t,s)\end{pmatrix} \tag{4.4.15}$$

($(t,s) \in [t',t'']$), which is called the Feynman–Green function of the small disturbance operator (the Jacobi operator, i.e., the differential operator emerging from the second variation of the classical Lagrangian, respectively Hamiltonian) in phase space, where

$$\mathcal{O} = \begin{pmatrix} -f(t) & -k(t) - \dfrac{\mathrm{d}}{\mathrm{d}t} \\ -k(t) + \dfrac{\mathrm{d}}{\mathrm{d}t} & -\dfrac{g(t)}{m}\end{pmatrix} \tag{4.4.16}$$

so that

$$\mathcal{O}\mathcal{G}(t,s) = \delta(t-s)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{4.4.17}$$

($(t,s) \in [t',t'']$). We can further specify the quantities G_{ab}, G_p and $\bar{G}$, respectively:

$G_{ab}(t,s)$ is the Feynman–Green function of the small disturbance operator $\mathcal{S}$ in configuration space defined by

$$\mathcal{S} = -\frac{m}{g(t)}\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\dot{g}(t)}{g(t)}\frac{\mathrm{d}}{\mathrm{d}t} - \dot{k}(t) + \frac{1}{m}f(t)g(t) - k^2(t) + \frac{\dot{g}(t)k(t)}{g(t)}\right] \tag{4.4.18}$$

so that

$$\mathcal{S}G_{ab}(t,s) = \delta(t-s) \;, \quad G_{ab}(t,s) = G_{ab}(s,t) \;, \quad G(0,t) = G(t'',t) = 0 \tag{4.4.19}$$

($(t,s) \in [t',t'']$, note that $\mathcal{S}q_{\mathrm{Cl},0}(t) = 0$). Furthermore we have

$$\bar{G}(t,s) = \frac{m}{g(t)}\left(\frac{\partial}{\partial s} - k(s)\right)G_{ab}(t,s) \tag{4.4.20}$$

$$G_p(t,s) = \frac{m^2}{g(t)g(s)}\left(\frac{\partial}{\partial t} - k(t)\right)\left(\frac{\partial}{\partial s} - k(s)\right)G_{ab}(t,s) - \frac{m}{g(t)}\delta(t-s) \ . \tag{4.4.21}$$

The functions G_{ab} and G_p are continuous, whereas $\bar{G}$ has a discontinuity at $t = s$:

$$\left(\lim_{t\to s} - \lim_{s\to t}\right)\bar{G}(t,s) = 1 \ . \tag{4.4.22}$$

Let us consider now "cylindrical functionals" on the space of paths. By this notion we denote functions of the coordinates and momenta in phase space, subject to path integration, appearing polynomially in the path integral. Thus we want to calculate

$$I(\{\mu,\nu\}) = \int F(\langle\boldsymbol{\mu},q\rangle,\langle\boldsymbol{\nu},p\rangle)\mathcal{D}w(p,q) \ , \tag{4.4.23}$$

where $\langle\boldsymbol{\mu},q\rangle$ and $\langle\boldsymbol{\nu},p\rangle$ denote sets of n- and m-dimensional measures, respectively. They represent the corresponding generalization of the one-dimensional measure $d\mu(s)$: for instance we can set $d\boldsymbol{\mu} := d\mu(s)(\delta(s-t_1),\ldots,\delta(s-t_n))$ and $d\boldsymbol{\nu} := d\nu(s)(\delta(s-t_1),\ldots,\delta(s-t_m))$. Then it follows that $\langle\boldsymbol{\mu},q\rangle = (q(t_1),\ldots,q(t_n))$ and $\langle\boldsymbol{\nu},p\rangle = (p(t_1),\ldots,p(t_m))$. Now consider the mapping from the space of paths $\mathcal{P}$ in phase space $\mathbb{R}^{n+m}$: $\mathcal{P} \to \mathbb{R}^{n+m}$ so that $(p,q) \to (\mathbf{u},\mathbf{v}) \in \mathbb{R}^{n+m}$. Then

$$I(\{q,p\}) = \int F(\mathbf{u},\mathbf{v})\mathcal{D}w(\mathbf{u},\mathbf{v}) \ . \tag{4.4.24}$$

Considering the Fourier transformation

$$\begin{aligned}\mathcal{D}w(\mathbf{u},\mathbf{v}) &= \mathcal{F}^{-1}_{\xi,\eta}\left[\mathcal{F}\mathcal{D}w(\boldsymbol{\xi}\cdot\mathbf{u},\boldsymbol{\eta}\cdot\mathbf{v})\right] \\ &= \frac{d\mathbf{u}\,d\mathbf{v}}{(2\pi)^{n+m}}\int_{\mathbb{R}^{n+m}} e^{i\boldsymbol{\xi}\cdot\mathbf{u}+i\boldsymbol{\eta}\cdot\mathbf{v}}\,\mathcal{F}w(\boldsymbol{\xi}\cdot\mathbf{u},\boldsymbol{\eta}\cdot\mathbf{v})\,d\boldsymbol{\xi}\,d\boldsymbol{\eta} \ ,\end{aligned} \tag{4.4.25}$$

we obtain

$$I(\{\mu,\nu\}) = \int_{\mathbb{R}^{n+m}} d\mathbf{u}\,d\mathbf{v}F(\mathbf{u},\mathbf{v})\int_{\mathbb{R}^{n+m}} e^{i\boldsymbol{\xi}\cdot\mathbf{u}+i\boldsymbol{\eta}\cdot\mathbf{v}}\,\mathcal{F}w(\boldsymbol{\xi}\cdot\mathbf{u},\boldsymbol{\eta}\cdot\mathbf{v})\,d\boldsymbol{\xi}\,d\boldsymbol{\eta} \ . \tag{4.4.26}$$

In particular, if the functional F depends either only on q, or on p, we obtain

$$\int F(\langle\boldsymbol{\nu},q\rangle)\mathcal{D}w(p,q) = \int_{\mathbb{R}^n}\frac{d\mathbf{u}}{(2\pi)^n}F(\mathbf{u})\int_{\mathbb{R}^n}\mathcal{F}w(\boldsymbol{\xi}\cdot\mathbf{u},0)\,e^{i\boldsymbol{\xi}\cdot\mathbf{u}}\,d\boldsymbol{\xi} \tag{4.4.27}$$

$$\int F(\langle\boldsymbol{\nu},p\rangle)\mathcal{D}w(p,q) = \int_{\mathbb{R}^m}\frac{d\mathbf{v}}{(2\pi)^m}F(\mathbf{v})\int_{\mathbb{R}^m}\mathcal{F}w(0,\boldsymbol{\eta}\cdot\mathbf{v})\,e^{i\boldsymbol{\eta}\cdot\mathbf{v}}\,d\boldsymbol{\eta} \ . \tag{4.4.28}$$

For the case that all q's and p's appear only linearly, (4.4.26) yields

$$\begin{aligned}
&\int \langle \boldsymbol{\mu}, q\rangle \langle \boldsymbol{\nu}, p\rangle \mathcal{D}w(p,q) \\
&= \mathrm{i}^{m+n} \frac{\partial^{n+m}\mathcal{F}w(\boldsymbol{\xi}\cdot\mathbf{u}, \boldsymbol{\eta}\cdot\mathbf{v})}{\partial\xi^1 \ldots \partial\xi^n \partial\eta^1 \ldots \partial\eta^m} \mathcal{F}w(\boldsymbol{\xi}\cdot\mathbf{u}, \boldsymbol{\eta}\cdot\mathbf{v})\Big|_{\boldsymbol{\xi}=\boldsymbol{\eta}=0} \\
&= \mathrm{i}^{m+n}\, \mathcal{H}\Big(-\frac{\mathrm{i}\, c_1}{2}, \ldots, -\frac{\mathrm{i}\, c_{m+n}}{2} \Big) , \qquad (4.4.29)
\end{aligned}$$

with the quantities c_i defined by $c_i = \langle \mu_i, \bar{q}\rangle$, $i = 1, \ldots, n$, $c_i = \langle \nu_i, \bar{p}\rangle$, $i = n+1, \ldots, n+m$. Here $\mathcal{H}$ is a generalized Hermite polynomial defined by

$$\mathcal{H}(A_{ij}x^i, \ldots, A_{mn}x^m) = (-1)^{n+m} \exp(A_{ij}x^i x^j) \frac{\partial^{n+m} \exp(-A_{ij}x^i x^j)}{\partial x^1 \ldots \partial x^{n+m}} \qquad (4.4.30)$$

with matrix $A = \frac{\mathrm{i}}{2}\int_{t'}^{t''}\int_{t'}^{t''} \mathrm{d}t\mathrm{d}s\, \mathcal{G}$. Explicitly $(y_i = A_{ij}x^j)$

$$\mathcal{H}(y_1, \ldots y_n) = \sum_{k=0}^{n} (-1)^{\frac{k}{2}} 2^{n-\frac{k}{2}} \sum_{i_1,\ldots,i_n} x_{i_1} \ldots x_{i_{n-k}} A_{i_{n-k+1}, I_{n-k+2}} \ldots A_{i_{n-1}, i_n} , \qquad (4.4.31)$$

where the summation runs over all *even* k up to the largest even number smaller than or equal to n, and the second summation runs over all different combinations of the indices i_j. There is a total of $(2m-1)!!$ different terms for $n = 2m$. Using the integral representation

$$\int_{\mathbb{R}^n} \Phi(\mathbf{b}\cdot\mathbf{u}) \exp\Big(-\frac{1}{2}\mathbf{u}^\dagger \mathbf{A}\mathbf{u}\Big)\, \mathrm{d}\mathbf{u} = \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{u})\, \mathrm{e}^{-\mathbf{u}^2/2c^2}}{(2\pi)^{n-1} c\sqrt{\det \mathbf{A}}}\, \mathrm{d}\mathbf{u} \qquad (4.4.32)$$

with $c^2 = b^i b^j (A_{ij})^{-1}$, the elementary Gaussian integral, we get for the Gaussian measure of the general time-dependent quadratic Hamiltonian

$$\begin{aligned}
&\int F(\langle \boldsymbol{\mu}, q\rangle, \langle \boldsymbol{\nu}, p\rangle) \mathcal{D}w(p,q) \\
&= \int_{\mathbb{R}^{n+m}} \frac{\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}}{(2\pi\mathrm{i}\,\hbar)^{\frac{n+m}{2}}\sqrt{\det \mathbf{W} \det \mathbf{S}}} F(\mathbf{u}, \mathbf{v}) \\
&\quad \times \exp\Big\{ \frac{\mathrm{i}}{2\hbar}\Big[(\mathbf{v}-\mathbf{b})^\dagger \mathbf{S}^{-1}(\mathbf{v}-\mathbf{b}) - 2(\mathbf{u}-\mathbf{a})^\dagger (\mathbf{W}^{-1}\mathbf{C}\mathbf{S}^{-1})(\mathbf{v}-\mathbf{b}) \\
&\qquad + (\mathbf{u}-\mathbf{a})^\dagger (\mathbf{W}^{-1} + \mathbf{W}^{-1}\mathbf{S}\mathbf{C}^{-1}\tilde{\mathbf{C}}\mathbf{W}^{-1})(\mathbf{u}-\mathbf{a})\Big]\Big\} , \qquad (4.4.33)
\end{aligned}$$

with the quantities $(\mathbf{a} = \langle \boldsymbol{\mu}, \bar{q}\rangle, \mathbf{b} = \langle \boldsymbol{\nu}, \bar{p}\rangle)$

$$\left.\begin{aligned}\mathbf{W} &= \int_{t'}^{t''}\int_{t'}^{t''} G_{ab}(t,s)\,\mathrm{d}\boldsymbol{\mu}(t)\otimes\mathrm{d}\boldsymbol{\mu}(s)\ ,\\ \mathbf{C} &= \int_{t'}^{t''}\int_{t'}^{t''} \bar{G}(t,s)\,\mathrm{d}\boldsymbol{\mu}(t)\otimes\mathrm{d}\boldsymbol{\nu}(s)\ ,\\ \mathbf{V} &= \int_{t'}^{t''}\int_{t'}^{t''} G_p(t,s)\,\mathrm{d}\boldsymbol{\nu}(t)\otimes\mathrm{d}\boldsymbol{\nu}(s)\end{aligned}\right\} \tag{4.4.34}$$

and $\mathbf{S} = \mathbf{V} - \tilde{\mathbf{C}}\mathbf{W}^{-1}\mathbf{C}, \tilde{\mathbf{C}} = \mathbf{C}^t$. Furthermore we have set $\mathrm{d}\boldsymbol{\mu}(t)\otimes\mathrm{d}\boldsymbol{\mu}(s) := (\mathrm{d}\mu_i(t)\,\mathrm{d}\mu_j(s))$, $i = 1,\dots,n, j = 1,\dots,m$, etc.

Summarizing, we obtain for the semiclassical expansion formula (4.4.3) about the harmonic approximation the following result (assuming that the potential is not time dependent)

$$\begin{aligned}K(x'',x';t'',t') &= K_{\mathrm{GHO}}(x'',x';t'',t')\\ &\times\Bigg(1+\sum_{j=1}^{\infty}\frac{1}{j!}\left(-\frac{\mathrm{i}}{\hbar}\right)^j\sum_{n_1=3}^{\infty}\cdots\sum_{n_j=3}^{\infty}\int_{t'}^{t''}\frac{\mathrm{d}t_1\dots\mathrm{d}t_j}{n_1!\dots n_j!}\\ &\times V^{(n_1)}(t_1)\dots V^{(n_j)}(t_j)(\mathrm{i}\hbar)^{N/2}\sum_{\{i_1,i_2,\dots,i_{2m}\}}G_{ab}(t_{i_1},t_{i_2})\dots G_{ab}(t_{i_{N-1}},t_{i_N})\Bigg)\ ,\end{aligned} \tag{4.4.35}$$

for $N = \sum_i n_i$ even, and (4.4.35) is zero for N odd. Here $\{i_1, i_2, \dots, i_{2m}\}$ denote all partitions which can be constructed from the $n_1\cdot t_1 + \dots + n_k\cdot t_k$ different t_i (equal times $t_i = t_j$ included), and G_{ab} as in (4.4.18).

For the general time-dependent case, the formula (4.4.33) must be used, where also explicitly time-dependent Hamiltonians according to (4.4.5) can be taken into account. For details we refer to [238, 699–705].

5 Semiclassical Theory

5.1 Semiclassical Theory and Quantum Chaos

In the last few decades, quantum properties of classically non-integrable Hamiltonian systems have been extensively investigated by means of the semiclassical approximation to the Feynman path integral. In particular, in the field of *quantum chaos* (see e.g. [89, 483, 869]), the semiclassical time-evolution kernel is the starting point for the derivation of the *Gutzwiller trace formula* [479, 480, 483] which nowadays is the basic tool for understanding quantum spectra of complex systems whose classical limit is strongly chaotic.

During the age of the *old* quantum theory which started with Planck in 1900, there already existed a powerful *semiclassical quantization rule* for simple quantum systems, the famous *Bohr–Sommerfeld quantum conditions* $\int p_i \, \mathrm{d}q_i = 2\pi n_i \hbar$, $i = 1, \ldots, D$, for systems with D degrees of freedom, where the n_i are integer quantum numbers. (Notice that there is *no* summation involved in $p_i \, \mathrm{d}q_i$.)

It was emphasized, however, in 1917 by Einstein [296] that the products $p_i \, \mathrm{d}q_i$ are in general not invariant and thus the quantum conditions have no invariant meaning, but rather depend on the choice of the coordinate system in which the classical motion is separable (if at all). By analysing a simple example, the two-dimensional motion of a particle under an attractive central force, Einstein found a general coordinate-invariant formulation of the quantum conditions in terms of the classical action integrals I_k $(k = 1, \ldots, D)$

$$I_k := \frac{1}{2\pi} \oint_{\gamma_k} \sum_{i=1}^{D} p_i \, \mathrm{d}q_i = n_k \hbar \ , \tag{5.1.1}$$

noticing that the line integrals of the canonical one-form $\sum_{i=1}^{D} p_i \, \mathrm{d}q_i$ taken over a complete set of topologically inequivalent ("irreducible") closed loops γ_k are invariant. In contrast to the original version of the quantization conditions, it is not necessary to perform explicitly the separation of variables; indeed, one need not require the motion to be separable, but only to be multiply periodic.[1] However, Einstein pointed out that the conditions (5.1.1) can only be written down in the case of very special systems for which there exist D independent integrals of the $2D$ canonical equations of motion of the form

[1] Indeed, an integrable system may not be necessarily separable, e.g., the Toda lattice, cf. [483, p.41].

$R_k(p,q) = \text{const.}$, where the R_k have the property that the relevant manifolds in $2D$-dimensional phase space have the shape of D-dimensional tori. In modern terminology, these systems are called *integrable systems.*[2] As a result, the trajectories of integrable systems wind round these D-dimensional tori which in turn cause the motion of integrable systems to be very *regular* in the sense that even the long-time behaviour is well under control. Indeed, in integrable systems, trajectories with neighbouring initial conditions separate only as some power of time.

Einstein was the first physicist to realize the important rôle played by invariant tori in phase space which he called "Trakte" [296]. However, the integrable systems forming the standard "textbook systems" with their clockwork predictability are not typical; that is almost all dynamical systems are *non-integrable* in the sense that there exist no constants of motion besides the energy and therefore no invariant tori in phase space. *Ergodicity* [32] implies that almost all trajectories fill – in the absence of invariant tori – the whole $(2D-1)$-dimensional energy surface densely. Today, our knowledge of classical dynamics is very rich [31, 483, 652], and most natural scientists begin to appreciate the importance of *chaos* in complex systems. It is now commonly recognized that generic systems execute a very irregular, *chaotic* motion which is unpredictable, that is the trajectories depend sensitively on the initial conditions such that neighbouring trajectories in phase space separate at an exponential rate. Einstein [296] made the crucial remark that the absence of tori excludes the formulation of the quantum conditions (5.1.1).

Shortly after the discovery of the Schrödinger equation, a *semiclassical* approach was devised, known as the *WKB-method*, named after Wentzel [918], Kramers [622] and Brillouin [120]. In the *semiclassical limit* one studies the behaviour of quantum mechanical quantities like energy levels, wave functions, barrier penetration probabilities, decay rates, or the S-matrix as Planck's constant tends to zero. This limit is different from the classical limit, for which $\hbar$ is precisely equal to zero, because, in general, quantal functions are non-analytic in $\hbar$ as $\hbar$ goes to zero.

Einstein's *torus quantization* for integrable systems was rediscovered and slightly generalized by Keller [570], who made clear that a sound mathematical derivation of the semiclassical behaviour of quantum mechanics requires a detailed knowledge of the underlying classical phase space structure. In the case of *integrable* systems, he was able to give the most general semiclassical quantization rule ($k = 1, \ldots, D$)

$$I_k = \left(n_k + \frac{\beta_k}{4}\right)\hbar \tag{5.1.2}$$

[2] Here and in the following, we only consider Hamiltonian systems, that is motion governed by Newton's equations without dissipation. The D constants R_k are assumed to be "smooth enough" and to be *in involution*, i.e., their Poisson brackets with each other vanish, see Arnold [31], and Lichtenberg and Lieberman [652] for further details.

which turns out to be exactly Einstein's torus quantization (5.1.1) apart from corrections arising from the *Maslov indices* β_k (see the discussion in Sect. 5.2). Today this quantization condition for integrable systems goes under the name of *EBK-quantization*, for Einstein, Brillouin, and Keller.

As already mentioned, Einstein [296] made the important observation that ergodic systems possess no invariant tori and that his quantization method cannot therefore be applied. In fact, it is known [483] that the phase space of strongly chaotic systems carries two mutually transverse foliations, each with leaves of D dimensions. Every trajectory is the intersection of two manifolds, one from each foliation. The distance between two neighbouring trajectories increases exponentially along the unstable manifold, and decreases exponentially along the stable manifold. Obviously the EBK-construction based on invariant tori is no longer possible and there remained the difficult task of finding a semiclassical quantization method for chaotic systems. It took another decade until Gutzwiller [479, 480] opened up the royal road towards an answer to "Einstein's question: how can classical mechanics give us any hints about the quantum-mechanical energy levels when the classical system is ergodic?" [483, p.282].

Gutzwiller's semiclassical theory for strongly chaotic systems starts from the Feynman path integral and the important fact that in the semiclassical limit when $\hbar$ tends to zero the leading contribution to the path integral comes from the *classical orbits*. This will be discussed in Sect. 5.2. Taking the trace of the Feynman kernel, the contributions come from the classical paths which are *closed* in *coordinate space*. Finally, if one performs a Fourier transform, one obtains the (energy dependent) Green function which is given by a formal sum over all classical paths which are *closed* in *phase space*, i.e., over all *periodic orbits*. The resulting *Gutzwiller trace formula* or *periodic-orbit theory* will be discussed in Sect. 5.4. At present, it provides the only substitute, appropriate for quantum systems whose classical limit is strongly chaotic, for the EBK-quantization rules applicable to integrable systems.

In Sects. 5.2–5.4 we can only give a short summary of the semiclassical theory and its applications in the field of quantum chaos. For more details, the reader is referred to the literature. Our presentation follows in several parts closely the recent review by Bolte [109], and also Bolte and Steiner [111]. For the early history of the semiclassical expansion, see Chap. 1.

5.2 Semiclassical Expansion of the Feynman Path Integral

Let us consider a particle of mass m moving in $\mathbb{R}^D$ in the potential $V(\mathbf{x})$ with classical Lagrangian ($\mathbf{x} \in \mathbb{R}^D$)

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}) \qquad (5.2.1)$$

and the associated path integral representation

$$K(\mathbf{x}'',\mathbf{x}';T) = \int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(T)=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\, \mathrm{e}^{\mathrm{i}\, R[\mathbf{x}(t)]/\hbar} \tag{5.2.2}$$

determining its quantum-mechanical time evolution. The path integral (5.2.2) can be considered as an infinite-dimensional generalization of the following N-dimensional integral

$$I(\hbar) = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{x}\, a(\mathbf{x})\, \mathrm{e}^{\mathrm{i}\, \phi(\mathbf{x})/\hbar} \ , \tag{5.2.3}$$

where $a \in C_0^\infty(\mathbb{R}^N)$, and $\phi \in C^\infty(\mathbb{R}^N)$ is a real valued function. The leading asymptotics of integrals of this type can be easily calculated in the semiclassical limit $\hbar \to 0$ by the *method of stationary phase*. The idea is that the phase factor $\mathrm{e}^{\mathrm{i}\, \phi/\hbar}$ oscillates for $\hbar \to 0$ so fast that the main contribution to the integral (5.2.3) comes from the stationary points $\mathbf{x}_\gamma$ of the phase where $\nabla\phi(\mathbf{x}) = 0$. Indeed, one obtains [112, 419] $I(\hbar) = O(\hbar^n)$ as $\hbar \to 0$ for all $n \in \mathbb{N}$ if $\nabla\phi(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in the support of a. Thus $I(\hbar)$ decreases in this case faster than any power of $\hbar$, which we denote as $I(\hbar) = O(\hbar^\infty)$, and one concludes that all contributions to $I(\hbar)$ that behave as some power of $\hbar$ must be due to the stationary points of the phase. Let us therefore assume that $\phi(\mathbf{x})$ has a finite number of isolated non-degenerate stationary points $\mathbf{x}_\gamma \in \operatorname{supp} a(\mathbf{x})$, i.e., $\nabla\phi(\mathbf{x}_\gamma) = 0$ with

$$\det\left(\frac{\partial^2\phi(\mathbf{x}_\gamma)}{\partial x^a \partial x^b}\right) \neq 0 \ . \tag{5.2.4}$$

Then [419]

$$\begin{aligned} I(\hbar) = (2\pi\hbar)^{N/2} \sum_\gamma a(\mathbf{x}_\gamma) &\left|\det\left(\frac{\partial^2\phi(\mathbf{x}_\gamma)}{\partial x^a \partial x^b}\right)\right|^{-1/2} \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar}\phi(\mathbf{x}_\gamma) + \mathrm{i}\,\frac{\pi}{4}\operatorname{sgn}\left(\frac{\partial^2\phi(\mathbf{x}_\gamma)}{\partial x^a \partial x^b}\right)\right] \cdot \Big(1 + O(\hbar)\Big) \ . \end{aligned} \tag{5.2.5}$$

Here $\operatorname{sgn}\left(\partial^2\phi(\mathbf{x}_\gamma)/\partial x^a \partial x^b\right) := 2r - N$ denotes the difference of the number of positive and negative eigenvalues of the matrix $\left(\partial^2\phi(\mathbf{x}_\gamma)/\partial x^a \partial x^b\right)$, where $r \leq N$ counts the number of positive and $N - r$ the number of negative eigenvalues.

In order to see even better the analogy with the semiclassical asymptotics of the path integral (5.2.2), we have to consider the more general N-dimensional integral

$$I(\mathbf{y};\hbar) = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{x}\, a(\mathbf{x},\mathbf{y})\, \mathrm{e}^{\mathrm{i}\, \phi(\mathbf{x},\mathbf{y})/\hbar} \ , \tag{5.2.6}$$

where the amplitude a and the real phase ϕ are smooth functions of a parameter $\mathbf{y} \in \mathbb{R}^M$. Let $\mathbf{x}_\gamma \in \mathbb{R}^N$ be the isolated non-degenerate stationary points of the phase, $\nabla_\mathbf{x}\phi(\mathbf{x}_\gamma, \mathbf{y}) = 0$, $\det\left(\partial^2\phi(\mathbf{x}_\gamma, \mathbf{y})/\partial x^a \partial x^b\right) \neq 0$. Then there exists a local diffeomorphism $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ in a neighbourhood of every stationary point $\mathbf{x}_\gamma$. We denote the inverse function as $\mathbf{x}_\gamma(\mathbf{y})$ which solves the condition $\nabla_\mathbf{x}\phi(\mathbf{x}_\gamma(\mathbf{y}), \mathbf{y}) = 0$. We then obtain the following result. Let $a(\mathbf{x}, \mathbf{y}) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^M)$ be an amplitude such that $\{\mathbf{x} \in \mathbb{R}^N; a(\mathbf{x}, \mathbf{y}) \neq 0 \text{ for some } \mathbf{y} \in \mathbb{R}^M\}$ is contained in a compact set in $\mathbb{R}^N$, and let $\phi(\mathbf{x}, \mathbf{y}) \in C^\infty(\mathbb{R}^N \times \mathbb{R}^M)$ be a real valued phase with isolated non-singular stationary points $\mathbf{x}_\gamma(\mathbf{y})$. Then the following asymptotics hold [419]

$$I(\mathbf{y}; \hbar) = (2\pi\hbar)^{N/2} \sum_\gamma a(\mathbf{x}_\gamma(\mathbf{y}), \mathbf{y}) \left| \det\left(\frac{\partial^2\phi(\mathbf{x}_\gamma(\mathbf{y}), \mathbf{y})}{\partial x^a \partial x^b}\right)\right|^{-1/2}$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\phi(\mathbf{x}_\gamma(\mathbf{y}), \mathbf{y}) + \mathrm{i}\frac{\pi}{4}\operatorname{sgn}\left(\frac{\partial^2\phi(\mathbf{x}_\gamma(\mathbf{y}), \mathbf{y})}{\partial x^a \partial x^b}\right)\right] \cdot \Big(1 + O(\hbar)\Big) \ . \tag{5.2.7}$$

To apply the method of stationary phase to the path integral (5.2.2), we have to find the stationary points of the action

$$R[\mathbf{x}(t)] = \int_0^T \left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right) \mathrm{d}t \ , \tag{5.2.8}$$

i.e., the solutions of the variational problem $\delta R = 0$ with the end-point conditions $\mathbf{x}(0) = \mathbf{x}', \mathbf{x}(T) = \mathbf{x}''$. But according to Hamilton's principle these are just the classical paths $\mathbf{x}_{\mathrm{Cl}}(t)$ which connect $\mathbf{x}'$ and $\mathbf{x}''$ in time T. Since the time T is fixed, but not the energy E of the classical paths, there will usually exist several solutions to the variational problem, i.e., several classical trajectories $\mathbf{x}_\gamma(t)$ which fulfil the required boundary conditions $\mathbf{x}_\gamma(0) = \mathbf{x}'$ and $\mathbf{x}_\gamma(T) = \mathbf{x}''$. Denoting the classical action evaluated along an actual path $\mathbf{x}_\gamma(t)$ of the system as

$$R_\gamma = R_\gamma(\mathbf{x}'', \mathbf{x}'; T) := R[\mathbf{x}_\gamma(t)] \tag{5.2.9}$$

(*Hamilton's principal function* for the trajectory $\mathbf{x}_\gamma(t)$), one obtains for $T > 0$ the following *semiclassical expansion of the D-dimensional Feynman kernel*

$$K(\mathbf{x}'', \mathbf{x}'; T) = \frac{1}{(2\pi\mathrm{i}\,\hbar)^{D/2}} \sum_\gamma \left|\det\left(-\frac{\partial^2 R_\gamma}{\partial x''^a \partial x'^b}\right)\right|^{1/2}$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}R_\gamma(\mathbf{x}'', \mathbf{x}'; T) - \mathrm{i}\frac{\pi}{2}\nu_\gamma\right] \cdot \Big(1 + O(\hbar)\Big) \ . \tag{5.2.10}$$

Here several remarks are in order:

i) The *orbit formula* (5.2.10) gives the leading term of the Feynman kernel in the semiclassical limit $\hbar \to 0$ as a sum over all classical trajectories, also called orbits, which connect $\mathbf{x}'$ and $\mathbf{x}''$ in time T.

ii) In deriving this formula from the path integral (5.2.2), the Feynman paths $\mathbf{x}(t)$ have been expanded about the stationary points, i.e., $\mathbf{x}(t)$ has been decomposed into $\mathbf{x}(t) = \mathbf{x}_\gamma(t) + \mathbf{q}(t)$ with $\mathbf{q}(0) = \mathbf{q}(T) = 0$:

$$K(\mathbf{x}'', \mathbf{x}'; T) = \sum_\gamma \int\limits_{\mathbf{q}(0)=0}^{\mathbf{q}(T)=0} \mathcal{D}\mathbf{q}(t)\, \mathrm{e}^{\mathrm{i}\, R[\mathbf{x}_\gamma(t)+\mathbf{q}(t)]/\hbar} \quad . \tag{5.2.11}$$

Then the action $R[\mathbf{x}_\gamma + \mathbf{q}]$ has been expanded about the given classical path to second order in the quantum fluctuation $\mathbf{q}(t)$ leading to the path integral for a D-dimensional harmonic oscillator with time dependent frequencies whose path integral solution is known, see Sect. 3.2. The result is

$$K(\mathbf{x}'', \mathbf{x}'; T) = \frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{D/2}} \sum_\gamma \frac{\mathrm{e}^{\mathrm{i}\, R_\gamma(\mathbf{x}'',\mathbf{x}';T)/\hbar - \mathrm{i}\,\pi\nu_\gamma/2}}{|\det \mathbf{J}_\gamma(T)|^{1/2}} \cdot \Big(1 + O(\hbar)\Big) \ , \tag{5.2.12}$$

where the matrix valued function $\mathbf{J}_\gamma(T)$ turns out to have a geometric interpretation as the so-called *Jacobi field*, see e.g. [828], of the classical trajectory emerging from the initial point $\mathbf{x}'$. Furthermore, the number ν_γ, which enters the above derivation as the number of negative eigenvalues of the matrix $\mathbf{J}_\gamma(T)$, is known as the *Morse index* of the trajectory and has the geometric meaning of the number of *conjugate points* on the trajectory $\mathbf{x}_\gamma(t)$, $0 \le t \le T$, conjugate to $\mathbf{x}'$, counted with multiplicities. In the following we shall present only a short discussion of this aspect of classical mechanics which deals with the *calculus of variations in the large* of classical trajectories.

iii) Consider the family $\mathbf{x}(\mathbf{p}'; t)$ of classical trajectories which start at $\mathbf{x}'$ at time $t = 0$, $\mathbf{x}(\mathbf{p}'; 0) = \mathbf{x}'$, parametrized by their initial momenta $\mathbf{p}' = m\dot{\mathbf{x}}(\mathbf{p}'; 0)$. Thus $\mathbf{x}(\mathbf{p}'; t)$ are trajectories spreading out from $\mathbf{x}'$ in different directions with different velocities. One defines the $D \times D$ matrix $\mathbf{J}(\mathbf{p}'; t)$ by

$$J_{kl}(\mathbf{p}'; t) := \frac{\partial x_l(\mathbf{p}'; t)}{\partial p'_k} \quad . \tag{5.2.13}$$

Since $\mathbf{x}(\mathbf{p}'; 0) = \mathbf{x}'$ for all momenta $\mathbf{p}'$, one obtains that $\mathbf{J}(\mathbf{p}'; 0) = 0$, and furthermore

$$\dot{J}_{kl}(\mathbf{p}'; 0) = \frac{1}{m} \frac{\partial p'_l}{\partial p'_k} = \frac{1}{m} \delta_{kl} \quad . \tag{5.2.14}$$

The matrix $\mathbf{J}(\mathbf{p}'; t)$ now describes the deviation of a trajectory $\mathbf{x}(\mathbf{p}' + \boldsymbol{\epsilon}; t)$ from $\mathbf{x}(\mathbf{p}'; t)$ at $t > 0$ to first order in $\boldsymbol{\epsilon}$,

$$x_k(\mathbf{p}' + \boldsymbol{\epsilon}; t) = x_k(\mathbf{p}'; t) + J_{kl}(\mathbf{p}'; t)\epsilon_l + O(\boldsymbol{\epsilon}^2) \quad . \tag{5.2.15}$$

All trajectories $\mathbf{x}(\mathbf{p}';t)$ are solutions of the equation of motion

$$m\ddot{x}_l + \frac{\partial V}{\partial x_l} = 0 \ , \tag{5.2.16}$$

which yields after taking the derivative with respect to p'_k the *Jacobi differential equation*

$$m\ddot{J}_{kl}(\mathbf{p}';t) + \sum_{n=1}^{D} \frac{\partial^2 V(\mathbf{x}(\mathbf{p}';t))}{\partial x_l \partial x_n} J_{kn}(\mathbf{p}';t) = 0 \ . \tag{5.2.17}$$

Thus the Jacobi field $\mathbf{J}(\mathbf{p}';t)$ satisfies the differential equation

$$m\ddot{\mathbf{J}}(\mathbf{p}';t) + \nabla \otimes \nabla\, V(\mathbf{x}(\mathbf{p}';t))\mathbf{J}(\mathbf{p}';t) = 0 \tag{5.2.18}$$

with initial conditions $\mathbf{J}(\mathbf{p}';0) = 0$ and $\dot{\mathbf{J}}(\mathbf{p}';0) = \mathbb{1}/m$. The uniqueness of the solutions of the Jacobi equation (5.2.18) implies that the matrix valued function $\mathbf{J}_\gamma(T)$, whose determinant enters the semiclassical formula (5.2.12), can be identified as the Jacobi field along the classical trajectory $\mathbf{x}_\gamma$, $\mathbf{J}_\gamma(T) = \mathbf{J}(\mathbf{p}';T)$. The geometric meaning of the Jacobi field can be visualized by a change of the point of view: presently we consider the problem of investigating all solutions $\mathbf{x}_\gamma(t)$ of the classical equations of motion with fixed boundary conditions at $t=0$ and $t=T$. Now we view a given trajectory $\mathbf{x}_\gamma(t)$ with initial conditions $\mathbf{x}_\gamma(0) = \mathbf{x}'$ and $m\dot{\mathbf{x}}_\gamma(0) = \mathbf{p}'$ as a function of the end-point of the interval $[0,T]$, i.e., $\mathbf{x}_\gamma(T)$. The determinant of the Jacobi field $\mathbf{J}(\mathbf{p}';T)$ of this trajectory vanishes at t_c if there exists another trajectory $\mathbf{x}(\mathbf{p}'+\boldsymbol{\epsilon};t)$ with initial conditions $\mathbf{x}(\mathbf{p}'+\boldsymbol{\epsilon};0) = \mathbf{x}'$ and $m\dot{\mathbf{x}}(\mathbf{p}'+\boldsymbol{\epsilon};0) = \mathbf{p}'+\boldsymbol{\epsilon}$ that intersects $\mathbf{x}_\gamma(T)$ at $T = t_c$, i.e., $\mathbf{x}(\mathbf{p}'+\boldsymbol{\epsilon};t_c) = \mathbf{x}_c = \mathbf{x}_\gamma(t_c)$. This intersection takes place at the *focal point* $\mathbf{x}_c$ which is also said to be *conjugate* to $\mathbf{x}' = \mathbf{x}_\gamma(0)$. For small times T such that on $\mathbf{x}_\gamma(t)$ there occur no conjugate points in the interval $0 \le t \le T$, all eigenvalues of $\mathbf{J}(\mathbf{p}';T)$ are positive. This result derives from the fact that then the trajectory γ minimizes the action functional (5.2.8) in Hamilton's principle. At a conjugate point $\mathbf{x}_c$ at least one eigenvalue vanishes, and the multiplicity of this zero mode is given by the dimension of that subspace of momentum space for which $\mathbf{x}(\mathbf{p}'+\boldsymbol{\epsilon};t_c) = \mathbf{x}_c$. Increasing T then turns these eigenvalues of $\mathbf{J}(\mathbf{p}';T)$ negative. Thus, for arbitrary T the number ν_γ of negative eigenvalues of $\mathbf{J}_\gamma(T) = \mathbf{J}(\mathbf{p}';T)$, the Jacobi field along the classical trajectory $\mathbf{x}_\gamma(t)$, is the number of conjugate points on $\mathbf{x}_\gamma(t)$ for $0 \le t \le T$ counted with multiplicities. The number ν_γ of conjugate points can be identified with the so-called *Morse index* of the trajectory $\mathbf{x}_\gamma(t)$ as a result of the *Morse index theorem* [697, 713, 828].

iv) The final step in the derivation of (5.2.10) consists in expressing the determinant of the Jacobi field $\mathbf{J}_\gamma(T)$ evaluated at the final time T by Hamilton's principal function R_γ, see (5.2.9). It is well known that $R_\gamma(\mathbf{x}'',\mathbf{x}';T)$

is the generating function of the *canonical transformation*, commonly labelled as type 1, which corresponds to the classical time evolution in phase space *backwards* in time, i.e., $(\mathbf{p}'', \mathbf{x}'') \overset{R_\gamma}{\mapsto} (\mathbf{p}', \mathbf{x}')$ with $\mathbf{x}''$ and $\mathbf{x}'$ as independent variables, and the final and initial momenta $\mathbf{p}''$ and $\mathbf{p}'$, respectively, as dependent variables. The latter are given by the relations

$$\mathbf{p}'' = \nabla_{\mathbf{x}''} R_\gamma \ , \qquad \mathbf{p}' = -\nabla_{\mathbf{x}'} R_\gamma \ . \tag{5.2.19}$$

The corresponding classical trajectory $\mathbf{x}_\gamma(t)$ connecting $\mathbf{x}'$ and $\mathbf{x}''$ in time T has energy E_γ which can be calculated from R_γ as follows

$$E_\gamma = -\frac{\partial R_\gamma}{\partial T} \ . \tag{5.2.20}$$

With

$$(\mathbf{J}_\gamma(T))_{kl} = J_{kl}(\mathbf{p}'; T) = \frac{\partial x^l(\mathbf{p}'; T)}{\partial p'_k} = \left(\frac{\partial p'_k}{\partial x''^l}\right)^{-1} = -\left[\frac{\partial}{\partial x''^l}\left(\frac{\partial R_\gamma}{\partial x'^k}\right)\right]^{-1}$$

we derive

$$\det \mathbf{J}_\gamma(T) = \left[\det\left(-\frac{\partial^2 R_\gamma}{\partial x''^a \partial x'^b}\right)\right]^{-1} \ , \tag{5.2.21}$$

and thus (5.2.12) is identical to the semiclassical expansion (5.2.10).

v) It has become a widespread though inaccurate custom to call the determinant which occurs through its square root in the semiclassical formula (5.2.10) for the Feynman kernel the Van Vleck determinant on the basis of Van Vleck's work of 1928 [906]. An improved designation of this determinant is sometimes given as the Van Vleck–Pauli–Morette determinant [613, Sect. 4.3]. It even happens that the semiclassical formula itself is unduly called Van Vleck's formula. Recently, convincing evidence has been provided [193] that the semiclassical formula (however, for *small*[3] times T only and *without* the summation over the classical paths γ and the corrections coming from the Morse index) is due to Pauli [765], and that the determinant which occurs in Pauli's formula is, up to a sign factor, the same as that due to Morette [710], and Van Hove [905]. Thus in the following we call the semiclassical expansion (5.2.10) *Pauli's formula* and the determinant the *Morette–Van Hove determinant*. See also the historical discussion in Chap. 1, p. 11.

vi) Notice that Pauli's semiclassical formula (5.2.10) is exact for systems whose Lagrangian is at most quadratic in $\mathbf{x}$ like the free particle (2.1.59) and the harmonic oscillator (3.2.24).

[3] Here small T means $T < t_m$ with $t_m := \min_\gamma t_c^\gamma$, where t_c^γ denotes the time at which there exists on a given trajectory $x_\gamma(t)$ the first conjugate point conjugate to $x_\gamma(0) = x'$.

vii) The singularities of the Morette–Van Hove determinant which arise from the zero modes of the Jacobi field at conjugate points were investigated for the first time by Choquard [192] for conservative systems with non-singular, confining potentials. This is a class of systems which allows an infinity of trajectories passing through $\mathbf{x}'$ at time $t' = 0$ and $\mathbf{x}''$ at time T. It was shown in [192] that the manifold of conjugate points is given by $\partial x'^k / \partial E_\gamma = -[\partial^2 R_\gamma / \partial x'^k \partial T]^{-1} = 0$.

The first person to go beyond the conjugate points was Gutzwiller in 1967 [479] who made the formula (5.2.10) as the starting point for the derivation of his trace formula, see Sect. 5.4. For a rigorous mathematical derivation of the semiclassical formula (5.2.10), see [477].

5.3 Semiclassical Expansion of the Green Function

The Green function G is defined as the Fourier transform of the Feynman kernel, see (2.1.25)

$$G(\mathbf{x}'', \mathbf{x}'; E) := \frac{\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}T \, \mathrm{e}^{\mathrm{i}(E + \mathrm{i}\,\epsilon)T/\hbar} \, K(\mathbf{x}'', \mathbf{x}'; T) \ . \tag{5.3.1}$$

A formal way to derive the semiclassical asymptotics of the Green function is to insert into (5.3.1) the asymptotic expansion (5.2.10) for K, then to interchange the summation over classical paths with the time integration, and finally to evaluate the remaining Fourier transform by the method of stationary phase. However, there one encounters a serious problem. It is well known that the two limits $\hbar \to 0$ and $T \to \infty$ do not commute. On the other hand, one requires in (5.3.1) the kernel for $T \to \infty$, and thus the semiclassical formula for K, which holds for $\hbar \to 0$, cannot be used directly. Since both G and K are distributions, one should regularize them with suitable test functions. Here we shall ignore this problem, but shall come back to it in Sect. 5.4.

Inserting the semiclassical expansion (5.2.10) for K into (5.3.1) leads to

$$\begin{aligned} G(\mathbf{x}'', \mathbf{x}'; E) &= \frac{\mathrm{i}}{\hbar} \frac{1}{(2\pi \mathrm{i}\, \hbar)^{D/2}} \sum_\gamma \int_0^\infty \mathrm{d}T \left| \det \left(- \frac{\partial^2 R_\gamma}{\partial x''^a \partial x'^b} \right) \right|^{1/2} \\ &\times \exp \left\{ \frac{\mathrm{i}}{\hbar} \Big[R_\gamma(\mathbf{x}'', \mathbf{x}'; T) + ET \Big] - \mathrm{i}\, \frac{\pi}{2} \nu_\gamma \right\} \cdot \Big(1 + O(\hbar) \Big) \ . \end{aligned} \tag{5.3.2}$$

Evaluating the time integration by the method of stationary phase, we obtain

$$\frac{\partial}{\partial T} \Big[R_\gamma(\mathbf{x}'', \mathbf{x}'; T) + ET \Big] = 0 \tag{5.3.3}$$

and with $\partial R_\gamma / \partial T = -E_\gamma$ the following condition for the stationary points in T for a given energy E

$$E_\gamma = E_\gamma(\mathbf{x}'', \mathbf{x}'; T) = E \ . \tag{5.3.4}$$

The solutions to this condition are the travelling times $T_\gamma = T_\gamma(E)$ of classical trajectories $\mathbf{x}_\gamma(t)$ which go from $\mathbf{x}'$ to $\mathbf{x}''$ in time T_γ with energy E. While in the semiclassical formula for K the energy of the classical trajectories was not fixed, the leading semiclassical contribution to the Green function G comes just from those classical trajectories which possess fixed energy E.

Evaluating the integral in (5.3.2) at a given stationary point $T = T_\gamma > 0$, we obtain the following phase

$$R_\gamma(\mathbf{x}'', \mathbf{x}'; T_\gamma) + ET_\gamma = S_\gamma(\mathbf{x}'', \mathbf{x}'; E) := \int_\gamma \mathbf{p} \cdot \mathrm{d}\mathbf{x} \ , \tag{5.3.5}$$

which is just the *classical action* of the given trajectory γ with energy E.

The final result for the *semiclassical expansion of the Green function* reads

$$\begin{aligned} G(\mathbf{x}'', \mathbf{x}'; E) = \bar{G}(\mathbf{x}'', \mathbf{x}'; E) &+ \frac{\mathrm{i}}{\hbar} \frac{1}{(2\pi \mathrm{i}\, \hbar)^{(D-1)/2}} \\ \times \sum_\gamma \sqrt{D_\gamma(\mathbf{x}'', \mathbf{x}'; E)} &\exp\left[\frac{\mathrm{i}}{\hbar} S_\gamma(\mathbf{x}'', \mathbf{x}'; E) - \mathrm{i}\frac{\pi}{2}\mu_\gamma\right] \cdot \Big(1 + O(\hbar)\Big) \ , \end{aligned} \tag{5.3.6}$$

where

$$D_\gamma(\mathbf{x}'', \mathbf{x}'; E) := \left| \det \begin{pmatrix} \dfrac{\partial^2 S_\gamma}{\partial x''^a \partial x'^b} & \dfrac{\partial^2 S_\gamma}{\partial x''^a \partial E} \\ \dfrac{\partial^2 S_\gamma}{\partial x'^a \partial E} & \dfrac{\partial^2 S_\gamma}{\partial E^2} \end{pmatrix} \right| \ , \tag{5.3.7}$$

and $\bar{G}$ denotes the contribution to the Green function from the stationary point at $T = 0$. Here the index μ_γ counts the number of points on $\mathbf{x}_\gamma$ conjugate to $\mathbf{x}'$ in energy E rather than in time and is defined as

$$\mu_\gamma := \begin{cases} \nu_\gamma \ , & D_T^2 R_\gamma > 0 \ , \\ \nu_\gamma + 1 \ , & D_T^2 R_\gamma < 0 \ , \end{cases} \tag{5.3.8}$$

with $D_T^2 R_\gamma$ denoting the second derivative of R_γ:

$$D_T^2 R_\gamma := \frac{\partial^2}{\partial T^2} R_\gamma(\mathbf{x}'', \mathbf{x}'; T) \bigg|_{T = T_\gamma} \ . \tag{5.3.9}$$

In the above derivation we have assumed that the stationary points are non-degenerate, i.e., $D_T^2 R_\gamma \neq 0$ which states that $\mathbf{x}''$ must not be conjugate to $\mathbf{x}'$ in energy on any of the trajectories $\mathbf{x}_\gamma$.

5.4 The Gutzwiller Trace Formula

In seeking a substitute for the EBK-quantization condition (5.1.2) in the case of strongly chaotic systems, Gutzwiller [479] concentrated on the *spectral density*[4]

$$d(E) := \sum_{n=0}^{\infty} \delta(E - E_n) \ . \tag{5.4.1}$$

Using the spectral representation (2.1.50)[5]

$$\begin{aligned} G(\mathbf{x}'', \mathbf{x}'; E) &= \sum_{n=0}^{\infty} \frac{\Psi_n(\mathbf{x}'')\Psi_n^*(\mathbf{x}')}{E_n - E - \mathrm{i}\,\epsilon} \\ &= \sum_{n=0}^{\infty} P\frac{\Psi_n(\mathbf{x}'')\Psi_n^*(\mathbf{x}')}{E_n - E} + \mathrm{i}\,\pi \sum_{n=0}^{\infty} \Psi_n(\mathbf{x}'')\Psi_n^*(\mathbf{x}')\delta(E_n - E) \end{aligned} \tag{5.4.2}$$

we observe that the spectral density $d(E)$ can be obtained from the trace of the Green function

$$\mathrm{Tr}(\underset{\sim}{\mathrm{H}} - E - \mathrm{i}\,\epsilon)^{-1} = \sum_{n=0}^{\infty} \frac{1}{E_n - E - \mathrm{i}\,\epsilon} = \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, G(\mathbf{x}, \mathbf{x}; E) \tag{5.4.3}$$

as follows

$$d(E) = \frac{1}{\pi}\Im \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, G(\mathbf{x}, \mathbf{x}; E) \ . \tag{5.4.4}$$

In general, the resolvent of $\underset{\sim}{\mathrm{H}}$ is not of trace class which manifests itself as a divergence of the infinite sum in (5.4.3).[6] This problem can be overcome, however, if one considers a smeared level density with a suitable test function; see remark vi) below.

To obtain the Gutzwiller trace formula, we insert the semiclassical expansion (5.3.6) for G into (5.4.4)[7]

$$\begin{aligned} d(E) = \bar{d}(E) - 2\Im \Bigg\{ &\frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{(D+1)/2}} \sum_{\gamma} \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x} \sqrt{D_\gamma(\mathbf{x}, \mathbf{x}; E)} \\ &\times \exp\left[\frac{\mathrm{i}}{\hbar} S_\gamma(\mathbf{x}, \mathbf{x}; E) - \mathrm{i}\,\frac{\pi}{2}\mu_\gamma\right] \cdot \Big(1 + O(\hbar)\Big) \Bigg\} \ , \end{aligned} \tag{5.4.5}$$

where the sum over γ runs over all *closed* classical trajectories $\mathbf{x}_\gamma$ starting out at $\mathbf{x}$ and returning to this point after a time $T_\gamma = T_\gamma(E) > 0$.

[4] Here we assume that the quantum system whose classical limit is described by the classical Lagrangian (5.2.1) has only a discrete energy spectrum $E_0 \le E_1 \le \dots$.

[5] Here P denotes the principal value.

[6] E.g., for two-dimensional Euclidean billiards it follows from Weyl's law [49, 919] that $E_n = O(n)$ for $n \to \infty$.

[7] $\bar{d}(E)$ is the contribution derived from the term $\bar{G}$ in (5.3.6); see remark v) below.

The method of stationary phase applied to the integral in (5.4.5) leads to the condition

$$\left[\nabla_{\mathbf{x}''} S_\gamma(\mathbf{x}'',\mathbf{x}';E) + \nabla_{\mathbf{x}'} S_\gamma(\mathbf{x}'',\mathbf{x}';E)\right]_{\mathbf{x}''=\mathbf{x}'=\mathbf{x}} = 0 \tag{5.4.6}$$

for $\mathbf{x} \in \mathbb{R}^D$. From the definition of the classical action S_γ as a Legendre transformation of R_γ, see (5.3.5), one derives

$$\nabla_{\mathbf{x}''} S_\gamma(\mathbf{x}'',\mathbf{x}';E) = \mathbf{p}'' , \quad \nabla_{\mathbf{x}'} S_\gamma(\mathbf{x}'',\mathbf{x}';E) = -\mathbf{p}' , \quad \frac{\partial}{\partial E} S_\gamma(\mathbf{x}'',\mathbf{x}';E) = T_\gamma , \tag{5.4.7}$$

and the condition (5.4.6) yields $\mathbf{p}'' - \mathbf{p}' = 0$, i.e., the closed trajectories $\mathbf{x}_\gamma$ must have identical initial momentum $\mathbf{p}'$ and final momentum $\mathbf{p}''$. Thus the condition of stationary phase applied to (5.4.5) picks out all points $\mathbf{x}$ in configuration space that lie on some closed orbit $\mathbf{x}_\gamma$ with $T_\gamma > 0$ and with the additional property that initial and final momenta are equal. That is the stationary points are those orbits which are *closed in phase space*, i.e., the *periodic orbits* γ *of period* $T_\gamma > 0$. These stationary points can never be isolated since a periodic orbit γ itself is a one-dimensional manifold of stationary points, along which the classical action $S_\gamma(\mathbf{x},\mathbf{x};E)$ is constant, i.e, for $\mathbf{x}$ on the periodic orbit γ

$$S_\gamma(\mathbf{x},\mathbf{x};E) = \oint_\gamma \mathbf{p}\cdot d\mathbf{x} =: S_\gamma(E) \ . \tag{5.4.8}$$

Here we shall not give the somewhat subtle calculation of the integral (5.4.5), but rather refer the reader to the literature [109, 111, 479, 480, 483].

We then obtain the *Gutzwiller trace formula*

$$\begin{aligned} d(E) &= \bar{d}(E) \\ &+ \frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{T_{\gamma_p}}{|\det(\mathbf{M}_{\gamma_p}^k - \mathbb{1})|^{1/2}} \cos\left(\frac{1}{\hbar} k S_{\gamma_p} - \frac{\pi}{2} k \tilde{\mu}_{\gamma_p}\right) \cdot \left(1 + O(\hbar)\right) \\ &+ O(\hbar^\infty) \ . \end{aligned} \tag{5.4.9}$$

Here several remarks are in order

i) The sums in (5.4.9) run over all *primitive periodic orbits* γ_p with period T_{γ_p}, and their k-fold repetitions, $k \in \mathbb{N}$.[8]
ii) The matrix $\mathbf{M}_{\gamma_p}$ is the so-called *monodromy matrix* and is given by a linearization of the *Poincaré recurrence map* P_{γ_p}. To simplify the discussion, we mention only the case of systems with two degrees of freedom. Then $\mathbf{M}_\gamma$ is a 2×2 matrix with two eigenvalues Λ_γ and Λ_γ^{-1}. Furthermore, since $\det \mathbf{M}_\gamma = 1$,

[8] A single traversal of the set of points on a periodic orbit is called the *primitive periodic orbit* γ_p corresponding to γ.

$$\det(\mathbf{M}_\gamma - \mathbb{1}) = 2 - \mathrm{Tr}\,\mathbf{M}_\gamma = 2 - \Lambda_\gamma - \Lambda_\gamma^{-1} \; , \tag{5.4.10}$$

which implies that

$$\Lambda_\gamma^{\pm 1} = \frac{1}{2}\left(\mathrm{Tr}\,\mathbf{M}_\gamma \pm \sqrt{(\mathrm{Tr}\,\mathbf{M}_\gamma)^2 - 4}\right) . \tag{5.4.11}$$

We distinguish four cases:

a) $\underline{\mathrm{Tr}\,\mathbf{M}_\gamma > 2}$: In this case one obtains two real eigenvalues $\Lambda_\gamma^{\pm 1} = \mathrm{e}^{\pm u_\gamma}$, $u_\gamma > 0$, and γ is *unstable.* (A nearby trajectory locally separates away from γ at a rate $\mathrm{e}^{\lambda_\gamma T_\gamma}$, where $\lambda_\gamma = u_\gamma / T_\gamma$ is the *Lyapunov exponent* of γ.) Such a periodic orbit is called *hyperbolic*, and

$$|\det(\mathbf{M}_\gamma - \mathbb{1})|^{1/2} = 2 \sinh \tfrac{u_\gamma}{2} \; . \tag{5.4.12}$$

b) $\underline{\mathrm{Tr}\,\mathbf{M}_\gamma < -2}$: Again one obtains two real eigenvalues, but now negative ones, $\Lambda_\gamma^{\pm 1} = -\mathrm{e}^{\pm u_\gamma}$, which again means that γ is *unstable.* In this case the periodic orbit is called *inverse hyperbolic*, and

$$|\det(\mathbf{M}_\gamma - \mathbb{1})|^{1/2} = 2 \cosh \tfrac{u_\gamma}{2} \; . \tag{5.4.13}$$

c) $\underline{|\mathrm{Tr}\,\mathbf{M}_\gamma| < 2}$: Now both eigenvalues of $\mathbf{M}_\gamma$ are complex, $\Lambda_\gamma^{\pm 1} = \mathrm{e}^{\pm \mathrm{i} v_\gamma}$, $v_\gamma \in (0, \pi)$. γ is a *locally stable orbit* (in the linear approximation of the dynamics) and neighbouring trajectories wind around it. In this case γ is called *elliptic*, and

$$|\det(\mathbf{M}_\gamma - \mathbb{1})|^{1/2} = 2 \sin \tfrac{v_\gamma}{2} \; . \tag{5.4.14}$$

d) $\underline{|\mathrm{Tr}\,\mathbf{M}_\gamma| = 2}$: This case implies that both eigenvalues are either $\Lambda_\gamma^{\pm 1} = +1$ or $\Lambda_\gamma^{\pm 1} = -1$. Such a periodic orbit is called *parabolic.* Once $\Lambda_\gamma = 1$, one observes that the trace formula becomes inapplicable since then $\det(\mathbf{M}_\gamma - \mathbb{1}) = 0$ and the respective term in the trace formula diverges. Otherwise $|\det(\mathbf{M}_\gamma - \mathbb{1})|^{1/2} = 2$.

iii) In the derivation of the Gutzwiller trace formula (5.4.9) we have assumed that all periodic points $\mathbf{x} \in \mathbb{R}^D$ are contained in smooth connected one-dimensional manifolds, i.e., all periodic orbits are single isolated periodic orbits.

iv) The index $\tilde{\mu}_\gamma$ is called the *Maslov index* and denotes μ_γ plus the number of negative eigenvalues of the second variation of the classical action S_γ evaluated in local coordinates along the periodic orbit.

v) The first term in the trace formula $\bar{d}(E)$, which is due to the singularity of the Feynman kernel at $T = 0$ and is proportional to $\mathrm{vol}(\Omega_E)$, corresponds to the *Thomas–Fermi approximation* in the general case, and to the *Weyl term* [49, 919] in the integrated level density for billiards. Here $\mathrm{vol}(\Omega_E)$ denotes the volume of the energy hypersurface Ω_E which is defined by $\Omega_E = \{(\mathbf{p}, \mathbf{x}) | H(\mathbf{p}, \mathbf{x}) = E\}$ of energy E in phase space

$$\begin{aligned}\mathrm{vol}(\Omega_E) &= \int_{\mathbb{R}^D} \mathrm{d}\mathbf{p} \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, \delta(E - H(\mathbf{p},\mathbf{x})) \\ &= m\, \mathrm{vol}(S^{(D-1)}) \int\limits_{V(\mathbf{x})\leq E} \mathrm{d}\mathbf{x} \left[2m(E - V(\mathbf{x}))\right]^{(D-2)/2} .\end{aligned} \tag{5.4.15}$$

The leading asymptotic term is given by

$$\bar{d}(E) = \frac{\mathrm{vol}(\Omega_E)}{(2\pi\hbar)^D}\left(1 + O(\hbar)\right) . \tag{5.4.16}$$

vi) The original Gutzwiller trace formula (5.4.9) has serious convergence problems which arise from the fact that for strongly chaotic systems the number of primitive periodic orbits whose periods T_{γ_p} are smaller than T increases exponentially for $T \to \infty$. If one considers, however, a smeared level density with a suitable test function, it is possible to derive a generalized version of the Gutzwiller trace formula which is absolutely convergent [852, 853].

The Gutzwiller trace formula establishes a striking *duality relation* between the quantal energy spectrum $\{E_n\}$ and the classical periodic orbits $\{\gamma\}$. It is a beautiful manifestation of Feynman's space-time view of quantum mechanics, where the classical trajectories, which had been abandoned by Heisenberg and Bohr, play an important rôle in understanding quantum mechanical systems whose classical limit is strongly chaotic.

The literature on the application of the Gutzwiller trace formula is huge. Here we can only give a personal selection [36–39, 45, 88–90, 109, 111, 352, 479–483, 850–853, 869, 933].

Whereas the Gutzwiller trace formula is a semiclassical expression, there are Hamiltonian systems for which the trace formula turns out to be *exact* and is given by the famous *Selberg trace formula* [483, 489, 844, 909, 910]. These systems are prototype examples of strongly chaotic systems and are given by the geodesic flow on Riemannian manifolds with constant negative Gaussian curvature. See [36, 37, 39–42, 48, 108, 110, 480, 483, 689, 866, 868, 869] for applications of the Selberg trace formula and the Selberg zeta function in the field of quantum chaos.

6 Table of Path Integrals

We present in the following *table of path integrals* exactly solvable path integrals according to the following classification scheme:

1) *General Formulæ.* This includes the different lattice definitions of path integrals on curved manifolds, transformation formulæ for canonical and time transformations, separable coordinate systems, and some perturbation methods.
2) *The General Quadratic Lagrangian.* Here we list the general formulæ for quadratic Lagrangians, including many explicit examples with electric and magnetic fields, couplings between oscillators in higher dimensions, two-time actions, some formulæ concerning the semi-classical approximation, trace formulæ, and, of course, the harmonic oscillator in its many appearances and modifications
3) *Discontinuous Potentials.* Here we state path integrals in half-spaces and boxes.
4) *The Radial Harmonic Oscillator.* This section includes Besselian type path integrals such as the Morse oscillator, motion in radial sectors, the Calogero model, and the general Besselian path integral, which is of the Natanzon type, cf. Table 6.3.
5) *The Pöschl–Teller Potential.* Path integrals related to the trigonometric version of the Pöschl–Teller potential are listed.
6) *The Modified Pöschl–Teller Potential.* This section contains path integrals related to the Rosen–Morse and Manning–Rosen potential, hyperbolic barriers, and the general Legendrian path integral, which is of the Natanzon type, cf. Table 6.3.
7) *Motion on Group Spaces and Homogeneous Spaces.* Path integrals for the quantum motion on homogeneous manifolds are listed, including some particular coordinate space representations, general expressions for path integrals on group spaces, and on spheres and hyperboloids.
8) *Coulomb Potentials.* Here we list all path integrals which are of the Coulomb type. They are related by means of a space-time transformation to Besselian path integrals.
9) *Magnetic Monopole and Anyon Systems.* In this section path integrals for monopoles, dyons, anyons, and applications to cosmology are cited.

10) *Motion in Hyperbolic Space.* Here we list path integrals for the quantum motion on hyperboloids. Some emphasis is on the hyperbolic plane, i.e., the Poincaré upper half-plane (Lobachevski space). Also the cases of magnetic fields are included, as well as the Higgs oscillator and the Kepler–Coulomb problem in spaces of constant curvature.
11) *Explicitly Time-Dependent Problems.* Here we list some general formulæ and specific examples of how to incorporate an explicit time dependence in the path integral. The general feature of this dependence is a "Galilean"-type modification of the usual potential problems.
12) *Point Interactions.* This section lists path integrals with point interactions, i.e., which are usually described by δ functions. General formulæ and some examples are presented. More general examples can be constructed by the interested reader by simply inserting some other path integral solution.
13) *Boundary Value Problems.* This section contains path integrals for the motion constrained by impenetrable walls and boxes with general boundary conditions. It generalizes Sect. 6.3, and includes the method of how to incorporate boundary conditions and absolute value problems from known unconstrained path integrals. Similarly to Sect. 6.12 general formulæ and some examples are presented.
14) *Coherent States.* Here the important coherent state path integral is given, together with several applications and generalizations to higher dimensions. Here we use $\hbar = 1$ throughout.
15) *Fermions.* Here the most important applications of the coherent state path integral are listed, i.e., the path integral formulation for fermions.
16) *Supersymmetric Quantum Mechanics.* Some supersymmetric path integral formulations and solutions are given.

In particular, the path integrals corresponding to solutions of the harmonic oscillator, respectively the general quadratic Lagrangian, are called *Gaussian path integrals* (section 6.2), those corresponding to the solutions of the radial harmonic oscillator are called *Besselian path integrals* (section 6.4), and those corresponding to the path integral solutions of the Pöschl–Teller and modified Pöschl–Teller potential are called *Legendrian path integrals* (Sect. 6.5 and 6.6), respectively. We call the *General Besselian* and *Legendrian path integral* solutions the *Basic path integrals.*

In the case of general quantum mechanical problems, more than just one of the basic path integral solutions is required. However, such problems can be conveniently put into a hierarchy according to which of the basic path integrals is the most important one for its solution. This classification scheme is listed in Table 6.1.

Table 6.1. Applications of potential problems (examples)

Quadratic Lagrangian	Radial harmonic oscillator	Pöschl–Teller potential	Modified Pöschl–Teller pot.
Infinite square well	Coulomb potential	Scarf pots.	Reflectionless pot.
Linear potential	Morse potential	Symmetric top	Rosen–Morse pot.
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood–Saxon pot.
Forced oscillator	Motion in a section	Higgs oscillator on spheres	Hultén pot.
Saddle point potential	Calogero model	Smorodinsky–Winternitz pot.	Manning–Rosen potential
Uniform magnetic field	Aharonov–Bohm potential		Hyperbolic Scarf potential
Driven coupled oscillators	Natanzon potential		Hyperbolic barrier potential
Two-time action (polaron)	Smorodinsky–Winternitz potentials		Hyperbolic spaces of rank one
Second derivative Lagrangians	Coulomb-like pots. in polar and parabolic coordinates		Kepler problem on (pseudo-) spheres
Semiclassical expansion	Non-relativistic monopoles		Natanzon potentials
Generating functional	Kaluza–Klein monopole		Hyperbolic strip
Moments formula	Poincaré plane + magnetic field + potentials		Higgs oscillator on pseudospheres
Effective potential	Dirac Coulomb problem		Hermitian spaces
Anharmonic oscillator	Anyons		Smorodinsky–Winternitz pots.

Usually, the Pöschl–Teller wave functions (Sect. 6.5) are denoted by $\Phi_n^{(\alpha,\beta)}(x)$ ($|x| < 1, n \in \mathbb{N}$), and the modified Pöschl–Teller wave functions (Sect. 6.6) are denoted by $\Psi_n^{(\eta,\nu)}(r)$ ($r > 1$) for the (finite) discrete spectrum and $\Psi_k^{(\eta,\nu)}(r)$ ($r > 1, k > 0$) for the continuous spectrum.

It is obvious that all potential problems can be generalized to more complicated problems, i.e., one can add an additional explicit time-dependence, implement a δ function perturbation, and consider problems in half-spaces and infinite boxes, respectively. These cases are generally omitted, because they just represent a combination of known results. Here we display the relevant basic formulæ, and some simple examples for illustration.

Table 6.2. Non-standard problems (examples)

Group path integration	Perturbation expansions
Euclidean space	δ-functions
Pseudo-Euclidean space	δ'-functions
Spheres	Point interaction for Dirac particle
Single-sheeted pseudospheres	Dirichlet boundary conditions
Double-sheeted pseudospheres	Neumann boundary conditions
Bispherical coordinates	Boxes and radial rings
Pseudo-bispherical coordinates	Absolute value potentials
Klein–Gordon propagator	Point interactions in $\mathbb{R}^{2,3}$
	Discontinuous potentials

However, there are non-standard cases where at least an implicit quantum mechanical solution exists. These are potential problems together with a δ-function perturbation, step potentials, explicit time dependent problems and boundary value problems, respectively. These we list in Table 6.2. We also include path integrals over group spaces in this table.

Note that we use the notation $q_{\gtrless}$ for the greater, respectively smaller, of two coordinate values q', q'', where q may be any coordinate. $\mathbf{x}$ denotes a D-dimensional Cartesian coordinate vector, and $\mathbf{q}$ (q) a D-dimensional (Minkowski) coordinate vector in some curved manifold $\mathbb{M}$. $\mathbf{q}_1 \cdot \mathbf{q}_2$ denotes a scalar product in a Riemannian space with positive definite metric, and $\mathrm{x} \cdot \mathrm{y}$ a scalar product in a pseudo-Riemannian space with indefinite (Minkowski-like) metric. If not noted otherwise, the variables x, y, z will stand for real variables, i.e., $x, y, z \in \mathbb{R}$, and the variable r will have the range $r > 0$.

We remark that a list of special path integrals based on the solution of the general quadratic Lagrangian and of the moment formulæ can be found in Appendix D of [801]. Most of the path integrals in this list are explicitly contained also in our table; those which are not listed here can be derived from the corresponding general cases in combination with general formulæ, respectively. In particular, we list two path integral solutions of [801] based on reflection symmetry.

By a time-Fourier transformation we indicate that a (space-) time transformation is necessary in order to evaluate the given path integral.

In case that one model is a generalization of a former one, not all details, e.g., wave function expansion, are repeated.

If not explicitly stated otherwise, we always use the product lattice definition of path integrals, cf. Sect. 2.5.5.3.

Table 6.3. Natanzon potentials

$R(z)$	$V(z),\quad \Delta V(z)=\frac{\hbar^2}{2m}\left(\left(\frac{z''}{z'}\right)^2-\frac{z'''}{z'}\right)$
$R=\sigma_2 z^2+\sigma_1 z+c_0$	$\frac{\hbar^2}{2m}\frac{g_2 z^2+g_1 z+\eta}{R(z)}+\Delta V(z),\ z>0$
$R=c_0,\ x\in\mathbb{R}$ $h=e^{2x/\sqrt{c_0}}$	$\frac{\hbar^2}{2mc_0}\left(g_2\,e^{4x/\sqrt{c_0}}+g_1\,e^{2x/\sqrt{c_0}}+\eta+1\right)$
$R=\sigma_1 z,\ r>0$ $h=r^2/\sigma_1$	$\frac{\hbar^2}{2m}\left(\frac{g_2}{\sigma_1^2}r^2+\frac{g_1}{\sigma_1}+\frac{\eta+3/4}{r^2}\right)$
$R=\sigma_2 z^2,\ r>0$ $h=2r/\sqrt{\sigma_2}$	$\frac{\hbar^2}{2m}\left(\frac{g_2}{\sigma_2}+\frac{g_1/\sqrt{\sigma_2}}{r}+\frac{\eta}{4r^2}\right)$
$R=a_0 z^2+b_0 z+c_0$	$\frac{\hbar^2}{2m}\frac{fz(z-1)+h_0(1-z)+h_1}{R(z)}+\Delta V(z),\ z\in(0,1)$
$R=1,\ x\in\mathbb{R}$ $z=\frac{1}{2}(1+\tanh x)$	$\frac{\hbar^2}{2m}\left(h_1+\frac{h_0}{2}+1-\frac{h_0}{2}\tanh x-\frac{f/4}{\cosh^2 x}\right)$
$R=z,\ r>0$ $z=\tanh^2 r$	$\frac{\hbar^2}{2m}\left(h_1+1+\frac{f-3/4}{\cosh^2 r}+\frac{h_0+h_1+3/4}{\sinh^2 r}\right)$
$R=1-z,\ r>0$ $z=1/\cosh^2 r$	$\frac{\hbar^2}{2m}\left(h_0+h_1+1-\frac{f+3/4}{\cosh^2 r}+\frac{h_1+3/4}{\sinh^2 r}\right)$
$R=z^2,\ r>0$ $z=1-e^{-2r}$	$\frac{\hbar^2}{2m}\left(\frac{h_1}{4}+f+1+\frac{h_0+h_1}{4\sinh^2 r}+\left(\frac{h_1}{2}-f\right)\coth r\right)$
$R=4z(1-z),\ \varphi=(0,\pi)$ $z=\frac{1}{2}(1-\cos\varphi)$	$\frac{\hbar^2}{8m}\left(\frac{h_0+h_1+3/4}{\sin^2(\varphi/2)}+\frac{h_1+3/4}{\cos^2(\varphi/2)}-(f+1)\right)$

Table 6.4. Gaussian and Besselian path integrals

The free particle ($\mathbf{x} \in \mathbb{R}^D$, Sect.6.2):
$\displaystyle\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathbf{x}}^2\, \mathrm{d}t\right) = \left(\frac{m}{2\pi\, \mathrm{i}\, \hbar T}\right)^{D/2} \exp\left(\frac{\mathrm{i}\, m}{2\hbar T}\lvert\mathbf{x}'' - \mathbf{x}'\rvert^2\right) .$
The harmonic oscillator ($x \in \mathbb{R}$, Sect.6.2):
$\displaystyle\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} (\dot{x}^2 - \omega^2 x^2)\, \mathrm{d}t\right] = \left(\frac{m\omega}{2\pi\, \mathrm{i}\, \hbar \sin\omega T}\right)^{1/2} \exp\left\{-\frac{m\omega}{2\,\mathrm{i}\,\hbar}\left[(x'^2 + x''^2)\cot\omega T - 2\frac{x'x''}{\sin\omega T}\right]\right\} .$
The radial harmonic oscillator ($r > 0$, Sect. 6.4):
$\displaystyle\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{\lambda^2 - \frac{1}{4}}{2mr^2} - \frac{m}{2}\omega^2 r^2\right)\mathrm{d}t\right] = \frac{m\omega\sqrt{r'r''}}{\mathrm{i}\,\hbar \sin\omega T} \exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(r'^2 + r''^2)\cot\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sin\omega T}\right) .$
The Coulomb potential ($r > 0, \kappa = q_1 q_2\sqrt{-m/2E}/\hbar$, Sect. 6.8):
$\displaystyle\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\, ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 + \frac{q_1 q_2}{r} - \frac{\hbar^2}{2m}\frac{\lambda^2 - \frac{1}{4}}{r^2}\right)\mathrm{d}t\right]$ $\displaystyle = \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(\frac{1}{2} + \lambda - \kappa)}{\Gamma(2\lambda + 1)} W_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) .$

Table 6.5. Legendrian path integrals

The Pöschl–Teller potential ($|x| < \pi/2, L_E = -\frac{1}{2} + \frac{1}{2}\sqrt{2mE}/\hbar, m_{1,2} = \frac{1}{2}(\beta \pm \alpha)$, $n \in \mathbb{N}_0$, Sect. 6.5):

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x} + \frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]\mathrm{d}t\right\}$$

$$= \frac{m}{2\hbar^2}\sqrt{\sin 2x' \sin 2x''}\,\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\big(\sin x' \sin x''\big)^{m_1-m_2}\big(\cos x' \cos x''\big)^{m_1+m_2}$$

$$\times {}_2F_1\big(m_1-L_E, L_E+m_1+1; m_1-m_2+1; \sin^2 x_<\big)\,{}_2F_1\big(m_1-L_E, L_E+m_1+1; m_1+m_2+1; \cos^2 x_>\big) ,$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[\frac{2(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2}(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x) , \quad E_n = \frac{\hbar^2}{2m}(\alpha+\beta+2n+1)^2 .$$

The modified Pöschl–Teller potential ($r > 0, L_\nu = \frac{1}{2}(\nu-1), m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2mE}/\hbar)$, $k_{1,2} = \frac{1}{2}(\eta \pm \nu + 1 - \mathrm{i}\,k)$, Sect. 6.6):

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r} - \frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]\mathrm{d}t\right\}$$

$$= \frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}(\cosh r' \cosh r'')^{-(m_1-m_2)}(\tanh r' \tanh r'')^{m_1+m_2+1/2}$$

$$\times {}_2F_1\big(m_1-L_\nu, L_\nu+m_1+1; m_1-m_2+1; \cosh^{-2} r_<\big)\,{}_2F_1\big(m_1-L_\nu, L_\nu+m_1+1; m_1+m_2+1; \tanh^2 r_>\big) ,$$

$$\Psi_n^{(\eta,\nu)}(r) = \left[\frac{2(\nu-\eta-2n-1)\Gamma(n+1+\eta)\Gamma(\nu-n)}{\Gamma^2(1+\eta)\Gamma(\nu-\eta-n)n!}\right]^{1/2}(\sinh r)^{\eta+\frac{1}{2}}(\cosh r)^{n-\nu+1/2}{}_2F_1\big(-n, \nu-n; 1+\eta; \tanh^2 r\big) ,$$

$$E_n = -\frac{\hbar^2}{2m}(2n+\eta-\nu-1)^2 , \qquad n = 0, 1, \dots \left[\tfrac{1}{2}(\nu-\eta-1)\right] ,$$

$$\Psi_k^{(\eta,\nu)}(r) = \frac{1}{\Gamma(1+\eta)}\sqrt{\frac{k\sinh \pi k}{2\pi^2}}\,\Gamma(k_1)\Gamma(k_2)(\cosh r)^{\mathrm{i}\,k}(\tanh r)^{\eta+1/2}{}_2F_1(k_1, k_2; 1+\eta; \tanh^2 r) .$$

6.1 General Formulæ

6.1.1 Path Integral Formulation on Curved Manifolds.

$$\langle \mathbf{q}''|\mathrm{U}(T)|\mathbf{q}'\rangle = \langle \mathbf{q}''|\,\mathrm{e}^{-\mathrm{i}\,HT(\underline{\mathrm{p}},\underline{\mathrm{q}})/\hbar}\,|\mathbf{q}'\rangle\Theta(T)$$

$$= \left\langle \mathbf{q}''\left|\exp\left(-\frac{\mathrm{i}T}{2m\hbar}g^{-1/4}(\underline{\mathrm{q}})\underline{\mathrm{p}}_a g^{1/2}(\underline{\mathrm{q}})g^{ab}(\underline{\mathrm{q}})\underline{\mathrm{p}}_b g^{-1/4}(\underline{\mathrm{q}}) + V(\underline{\mathrm{q}})\right)\right|\mathbf{q}'\right\rangle\Theta(T)$$

6.1.1.1 Pre-Point Formulation [69,76,173,228–235,264,265,314,368,416,494, 565,568,636,637,690,761,797,899,914,915] after DeWitt (DW):

$$= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{DW}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) + \frac{R\hbar^2}{6m}\right)\mathrm{d}t\right]. \tag{6.1.1}$$

6.1.1.2 Symmetric Rule (SR) [26,76,264,266,690,888,894,939]:

$$= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{SR}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{SR}(\mathbf{q})\right)\mathrm{d}t\right]. \tag{6.1.2}$$

6.1.1.3 Mid-Point Formulation (MP) [198,217,260,346,358,359,380,389,392, 464,494,565,568,572,593,647,648,645,665,676,698,736,811,815,888,914,915]:

$$= [g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4} \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{MP}}\mathbf{q}(t)\sqrt{g}$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{MP}}(\mathbf{q})\right)\mathrm{d}t\right]. \tag{6.1.3}$$

6.1.1.4 Product-Form Formulation [422,447,470] ($h_{ac}h_{cb} = g_{ab}$):

$$= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{PF}}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}h_{ac}h_{bc}\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{PF}}(\mathbf{q})\right)\mathrm{d}t\right]. \tag{6.1.4}$$

6.1.1.5 Vielbein Formulation [611–613,616] ($\mathrm{d}x^a = e^a_\mu(q)\,\mathrm{d}q^\mu$, $\Delta x^a_j = e^a_{j;\mu}\Delta q^\mu_j - e^a_{j;\mu,\nu}\Delta q^\mu_j \Delta_j q^\nu/2 + e^a_{j;\mu,\nu\lambda}\Delta q^\mu_j \Delta_j q^\nu \Delta q^\lambda_j/6$):

$$= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\,\epsilon\hbar}\right)^{D/2}\int\prod_{j=1}^{N-1}\mathrm{d}\Delta x^a_j\exp\left[\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left(\frac{m}{2\epsilon}\Delta^2 x^a_j - V(\mathbf{q}_j)\right)\right]. \tag{6.1.5}$$

$\Delta V_{\mathrm{MP}} = \Delta V_{\mathrm{Weyl}}, \Delta V_{\mathrm{SR}}, \Delta V_{\mathrm{PF}}$ as in (2.8.19), (2.8.20) and (2.8.40), respectively, and R is the scalar curvature.

6.1.2 Phase-Space Formulation, Hamiltonian Path Integral.
[26,135, 245,248,340,375,380,394,564,621,634,635,677,685,701,703,704,732,772,788,828, 862,894]

$$\left\langle \mathbf{q}'' \middle| \exp\left(-\frac{\mathrm{i}T}{2m\hbar} g^{-1/4}(\underline{\mathrm{q}})\underline{\mathrm{p}}_a g^{1/2}(\underline{\mathrm{q}}) g^{ab}(\underline{\mathrm{q}})\underline{\mathrm{p}}_b g^{-1/4}(\underline{\mathrm{q}}) + V(\underline{\mathrm{q}})\right) \middle| \mathbf{q}' \right\rangle \Theta(T)$$
$$= \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{MP}}(q(t), p(t))$$
$$\times \exp\left\{ \frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\mathbf{p}\cdot\dot{\mathbf{q}} - \left(\frac{m}{2} g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{MP}}(\mathbf{q})\right)\right] \mathrm{d}t \right\} . \tag{6.1.6}$$

6.1.3 Path Integral with Magnetic Field.
[340,509,698,828]

$$\left\langle \mathbf{q}'' \middle| \exp\left[-\frac{\mathrm{i}T}{2m\hbar} \underline{\mathrm{g}}^{-\frac{1}{4}} \left(\underline{\mathrm{p}}_b - \frac{e}{c}\underline{\mathrm{A}}_b \right) \underline{\mathrm{g}}^{\frac{1}{2}} \mathrm{g}^{ab} \left(\underline{\mathrm{p}}_b - \frac{e}{c}\underline{\mathrm{A}}_b \right) \underline{\mathrm{g}}^{-\frac{1}{4}} + V \right) \middle| \mathbf{q}' \right\rangle \Theta(T)$$
$$= [g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4} \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{MP}}\mathbf{q}(t)\sqrt{g(\mathbf{q})}$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2} g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b + \frac{e}{c}\dot{\mathbf{q}}\cdot\mathbf{A} - V(\mathbf{q}) - \Delta V_{\mathrm{MP}}(\mathbf{q})\right) \mathrm{d}t \right] . \tag{6.1.7}$$

6.1.4 Phase-Space Path Integral by Localization – Duistermaat–Heckman Formula.
[99,268,269,284,492,573,726,744] (ϕ^a are the coordinates in phase-space, θ_a are the conjugates, H is the Hamiltonian, and ω_{ab} is an antisymmetric tensor given by the symplectic 2-form: $\omega_{ab} = \partial_a\theta_b - \partial_b\theta_a$, c^a are Grassmann variables for the BRST mechanism, and the tensors Ω_{ab} and R_{ab} can be expressed as geometric quantities in terms of the metric g_{ab} and the Hamiltonian vector χ_H^a which is a Killing vector of g_{ab}, i.e., $\Omega_{ab} = 2\chi^H_{a;b}, R_{ab} = \frac{1}{2}R_{abcd}c^c_{\mathrm{Cl}}c^d_{\mathrm{Cl}}$, and the localization is defined via $\mathcal{X}_H^a = -\omega^{ab}\partial_b H$)

$$\int \mathcal{D}\boldsymbol{\phi}\mathcal{D}\mathrm{c}\exp\left[\mathrm{i}\int_0^T \mathrm{d}t(\Theta_a\dot{\phi}^a - H(\boldsymbol{\phi}) + \tfrac{1}{2}c^a\omega_{ab}c^b)\right]$$
$$= \int_{\chi_H^a} \mathrm{d}\boldsymbol{\phi}_{\mathrm{Cl}}\,\mathrm{d}\mathbf{c}_{\mathrm{Cl}}\exp\left[-\mathrm{i}TH_{\mathrm{Cl}}(\boldsymbol{\phi}_{\mathrm{Cl}}) + \tfrac{\mathrm{i}}{2}Tc^a_{\mathrm{Cl}}\omega_{ab}c^b_{\mathrm{Cl}}\right] \sqrt{\det\frac{T(\Omega^a_b + R^a_b)/2}{\sinh(\frac{T}{2}(\Omega^a_b + R^a_b))}} . \tag{6.1.8}$$

6.1.5 Phase-Space Path Integration via Hamilton–Jacobi Coordinates. [26,56,276,278,726,848]

6.1.5.1 Evaluation with Generating Function F_1.

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}(q(t),p(t)) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[p\dot{q}-\left(\frac{p^2}{2m}+V(q)\right)\right]\mathrm{d}t\right\}$$
$$= \int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left\{\frac{\mathrm{i}}{\hbar}\Big[P''(Q''-Q')+F_1(q'',Q'',p,t'')-F_1(q',Q',p,t')\Big]\right\} , \tag{6.1.9}$$

with $f(q)=p^2/2m+V(q), P=-\partial F_1/\partial Q$, where:

$$F_1(q,Q,p,t)=\sqrt{2m}\int^q \sqrt{f(Q)-V(q')}\ \mathrm{d}q' - f(Q)t \ . \tag{6.1.10}$$

6.1.5.2 Evaluation with Generating Function F_2. [56]

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}(q(t),p(t)) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[p\dot{q}-\left(\frac{p^2}{2m}+V(q)\right)\right]\mathrm{d}t\right\}$$
$$= \int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}F_2(q'',P'',t'')-\frac{\mathrm{i}}{\hbar}F_2(q',P',t')\right] , \tag{6.1.11}$$

with $p=\sqrt{2m(P^2-V(q))}, Q=\partial F_2/\partial P$, where:

$$F_2(q,P,t)=\sqrt{2m}\int^q \sqrt{P^2-V(q')}\ \mathrm{d}q' - P^2 t \ . \tag{6.1.12}$$

6.1.6 Path Integral for Classical Mechanics. [412,889]

$$K(\boldsymbol{\phi}'',\boldsymbol{\phi}';T) = \langle\boldsymbol{\phi}(T)|\exp(-\underset{\sim}{L}T)|\boldsymbol{\phi}(0)\rangle$$
$$= \int\limits_{\boldsymbol{\phi}(0)=\boldsymbol{\phi}'}^{\boldsymbol{\phi}(T)=\boldsymbol{\phi}''} \mathcal{D}\boldsymbol{\phi}(t)\mathcal{D}A(t)\mathcal{D}\mathrm{C}(t)\mathcal{D}\bar{\mathrm{C}}(t)\delta^{2n}(\mathrm{C}_0)\exp\left(\mathrm{i}\int_0^T \mathcal{L}(\boldsymbol{\phi},\mathrm{C})\mathrm{d}t\right) , \tag{6.1.13}$$

$$\mathcal{L}(\boldsymbol{\phi},\mathrm{C}) = A_a\big[\dot{\phi}^a-\omega^{ab}\partial_b H(\boldsymbol{\phi})\big] + \mathrm{i}\,\bar{C}_a\left(\partial - t\delta^a_b - \frac{\partial h^a}{\partial\phi^b}\right)C^b \ . \tag{6.1.14}$$

Here $\boldsymbol{\phi}(t)=(q^1,\dots,q^n,p_1,\dots,p_n)$ is a coordinate on a $2n$-dimensional phase-space, $H(\boldsymbol{\phi})$ is the Hamiltonian, $\omega^{ab}=-\omega^{ba}$ is the symplectic matrix and $h(\boldsymbol{\phi})$ is the Hamiltonian vector field. The Hamiltonian equations of motion are $\dot{\phi}(t)=\omega^{ab}\partial_b H(\boldsymbol{\phi}(t))\equiv h^a(\boldsymbol{\phi}(t))$. Any density function $\rho(\boldsymbol{\phi},t)$ on phase space evolves in time according to the Liouville equation $\partial_t \rho(\boldsymbol{\phi}, t) = -\{\rho, H\} \equiv -\underline{\mathrm{L}}\rho(\boldsymbol{\phi}, t)$ with formal solution $\rho(\boldsymbol{\phi}, t) = \mathrm{e}^{-\underline{\mathrm{L}}t}\rho(\boldsymbol{\phi}, 0)$. Λ_a is an auxiliary field, $C_a, \bar{C}_a$ are Grassmann variables.

6.1.7 Fourier-Mode Expansion. [217,307,340,341,365,478,613,809,915] (cf. also the special case on page 40, $\hbar = m = 1$)

$$\int\limits_{x(0)=x'}^{x(\beta)=x''} \mathcal{D}_E x(t) \exp\left[-\int_0^\beta \left(\frac{1}{2}\dot{x}^2 + V(x)\right)\mathrm{d}t\right]$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}x_0}{\sqrt{2\pi\beta}} \int \prod_{n=1}^{\infty} \frac{\beta\omega_n^2}{\pi} \mathrm{d}\Re(x_n)\,\mathrm{d}\Im(x_n)$$

$$\times \exp\left[-\beta\sum_{n=1}^{\infty}\omega_n^2|x_n|^2 - \int_0^\beta V(x(t))\mathrm{d}t\right] , \tag{6.1.15}$$

with the Fourier expansion of the paths ($x_0 = \int_0^\beta x(t)\,\mathrm{d}t/\beta, \omega_n = 2\pi n/\beta$)

$$x(t) = x_0 + \sum_{n=1}^{\infty}\left(x_n\,\mathrm{e}^{\mathrm{i}\,\omega_n t} + x_n^*\,\mathrm{e}^{-\mathrm{i}\,\omega_n t}\right) , \qquad x_n = \frac{1}{\beta}\int_0^\beta x(t)\,\mathrm{e}^{-\mathrm{i}\,\omega_n t}\,\mathrm{d}t . \tag{6.1.16}$$

6.1.8 Cameron–Martin Formula. [13,132,133,219,237,712,242,300,376, 554,897] ($y(t) = (Ax)(t), J(s) = K_2(s,s) - K_1(s,s)$)

$$\int\limits_{y(0)=y'}^{y(T)=y''} \mathcal{D}y(t)|D|\,\mathrm{e}^{-\mathrm{i}\,\pi\,\mathrm{ind}(A)/2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \left(\frac{m}{2}\dot{y}^2 - V(y)\right)\mathrm{d}t\right]$$

$$= \int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[\frac{m}{2}\dot{x}^2 - V(x + Ax)\right]\mathrm{d}t + \frac{\mathrm{i}\,m}{2\hbar}\Phi[x]\right\} , \tag{6.1.17}$$

$$K(t,s) := \begin{cases} K_1(t,s) & 0 \le t < s, \quad 0 < s \le T, \\ K_2(t,s) & s < t \le T, \quad 0 \le s < T, \\ \frac{1}{2}[K_1(s,s) + K_2(t,t)] & t = s, \qquad\quad 0 \le s \le T, \end{cases} \tag{6.1.18}$$

$$D = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{k=1}^{n}\int_0^T \mathrm{d}s_k \det\begin{pmatrix} K(s_1,s_1) & \dots & K(s_1,s_n) \\ \vdots & \ddots & \vdots \\ K(s_n,s_1) & \dots & K(s_n,s_n) \end{pmatrix} , \tag{6.1.19}$$

$$\Phi[x] = \int_0^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^T K(t,s)x(s)\,\mathrm{d}s\right]^2 \mathrm{d}t$$

$$+2\int_0^T\left[\int_0^T \frac{\partial}{\partial t}K(t,s)x(s)\,\mathrm{d}s\right]\mathrm{d}x(t)+\int_0^T J(t)\,\mathrm{d}x^2(t)\ . \tag{6.1.20}$$

6.1.9 Coordinate Transformation. [30,180,223,314,359,380,389,392,440, 464,469,470,512,593,773,775,786,848,872] ($x = F(q,t)$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right]$$

$$=\exp\left(\frac{\mathrm{i}\,m}{\hbar}\left(\int_{q'}^{q''}F'(z,t'')\dot{F}(z,t'')\mathrm{d}z\right)\int\limits_{q(t')=q'}^{q(t'')=q''}\mathcal{D}_{\mathrm{MP}}q(t)F'(q,t)\right.$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(F'^2(q,t)\dot{q}^2+\dot{F}^2(q,t)\right)-V(F(q,t))-\frac{\hbar^2}{8m}\frac{F''^2(q,t)}{F'^4(q,t)}\right.\right.$$

$$\left.\left.-m\int^q\left(F'(z,t)\ddot{F}(z,t)+\dot{F}(z,t)\dot{F}'(z,t)\right)\mathrm{d}z\right]\mathrm{d}t\right\}\ . \tag{6.1.21}$$

6.1.10 Space-Time (Duru–Kleinert) Transformation.

6.1.10.1 One-Dimensional Path Integrals. [26,152,180,279,280,343,344,440, 464,469,470,514,517,549,552,605,608,613,623,743,773,775,863,872,943]
One considers the transformations $x = F(q,t)$ and $\mathrm{d}t/F'^2(q,t) = \mathrm{d}s$:

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right]$$

$$=\Big[F'(q'',t'')F'(q',t')\Big]^{1/2}A(q'',q';t'',t')\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\,G(q'',q';E)\ , \tag{6.1.22}$$

$$A(q'',q';t'',t')$$
$$=\exp\left[\frac{\mathrm{i}\,m}{\hbar}\left(\int^{q''}F'(z,t'')\dot{F}(z,t'')\,\mathrm{d}z-\int^{q'}F'(z,t')\dot{F}(z,t')\,\mathrm{d}z\right)\right]\ , \tag{6.1.23}$$

$$G(q'',q';E)=\frac{\mathrm{i}}{\hbar}\int_0^\infty \widehat{K}(q'',q';s'')\mathrm{d}s''\ , \tag{6.1.24}$$

$$\widehat{K}(q'',q';s'')=\int\limits_{q(0)=q'}^{q(s'')=q''}\mathcal{D}q(s)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{q}^2 - F'^2(q,s)\Big(V(F(q,s)) - E\Big) - \Delta V(q,s)\right)\mathrm{d}s\right] , \tag{6.1.25}$$

$$\Delta V(q,s) = \frac{\hbar^2}{8m}\left(3\frac{F''^2(q,s)}{F'^2(q,s)} - 2\frac{F'''(q,s)}{F'(q,s)}\right) + mF'^2(q,s)\int^q F'(z,s)\ddot{F}(z,s)\mathrm{d}z . \tag{6.1.26}$$

For $F(q,t) \equiv F(q)$, i.e., if the transformation is time independent, we have

$$\Delta V(q) = \frac{\hbar^2}{8m}\left(3\frac{F''^2(q)}{F'^2(q)} - 2\frac{F'''(q)}{F'(q)}\right) . \tag{6.1.27}$$

6.1.10.2 Transformation in D Dimensions. [774]

$$\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{MP}}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{MP}}(\mathbf{q})\right)\mathrm{d}t\right]$$

$$= [f(\mathbf{Q}')f(\mathbf{Q}'')]^{(2-D)/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \int\limits_{\mathbf{Q}(0)=\mathbf{Q}'}^{\mathbf{Q}(s)=\mathbf{Q}''} \mathcal{D}_{\mathrm{MP}}\mathbf{Q}(s)\sqrt{G}$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}g_{ab}(\mathbf{Q})\dot{Q}^a\dot{Q}^b - \tilde{V}(\mathbf{Q})\right)\mathrm{d}s\right] \tag{6.1.28}$$

$$\tilde{V}(\mathbf{Q}) = f(\mathbf{Q})[V(\mathbf{q}(\mathbf{Q})) - E] - \frac{\hbar^2(D-2)}{8m}\left[\Gamma^{\lambda\mu}_{\lambda}(\mathbf{Q})\frac{f_{,\mu}(\mathbf{Q})}{f(\mathbf{Q})}\right.$$
$$\left. - G^{\lambda\mu}(\mathbf{Q})\left(\frac{(D-6)}{4}\frac{f_{,\lambda}(\mathbf{Q})f_{,\mu}(\mathbf{Q})}{f^2(\mathbf{Q})} + \frac{f_{,\lambda\mu}(\mathbf{Q})}{f(\mathbf{Q})}\right)\right] . \tag{6.1.29}$$

Here a one-to-one coordinate transformation according to $\mathbf{q} \mapsto \mathbf{Q}$ has been performed via $q^a = q^a(\mathbf{Q})$. The metric tensor transforms according to $g_{ab}(\mathbf{q}) \mapsto G_{\lambda\mu}(\mathbf{Q}) = f(\mathbf{Q})e^a{}_\lambda(\mathbf{Q})e^b{}_\mu(\mathbf{Q})g_{ab}(\mathbf{q}(\mathbf{Q}))$, together with the time transformation $\mathrm{d}t = f(\mathbf{Q}(s))\mathrm{d}s$. The Γ^{ab}_c are the usual Christoffel symbols, and the $e^a{}_\lambda$ are the coefficients of the transformation $\mathrm{d}q^a = e^a{}_\lambda\,\mathrm{d}Q^\lambda$. A possible vector potential $A_a(\mathbf{q})$ can be incorporated into the transformation according to $A_a(\mathbf{q}) \mapsto e^a{}_\lambda(\mathbf{Q})A_a(\mathbf{q}(\mathbf{Q}))$.

6.1.10.3 Transformation for Radial Path Integrals. [863,865]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\mu_{l+1/2}[r^2]\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right]$$

$$= \frac{2(r'r'')^{\nu/4}}{2-\nu}\int_0^\infty \mathrm{d}s'' \int\limits_{R(0)=R'}^{R(s'')=R''} \mathcal{D}R(s)\mu_{L_\nu+1/2}[R^2]$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{R}^2 - W_\nu(R)\right)\mathrm{d}s\right] , \tag{6.1.30}$$

$$W_\nu(R) = \left(\frac{2}{2-\nu}\right)^2 R^{\frac{2\nu}{2-\nu}}\left[V(R^{2/(2-\nu)}) - E\right] + \frac{\hbar^2}{8mR^2}\frac{\nu(4-\nu)}{(2-\nu)^2} , \tag{6.1.31}$$

with the new angular momentum $L_\nu = (4l+\nu)/2(2-\nu)$ with respect to the space-time transformation $\mathrm{d}t = (\frac{2}{2-\nu})^2 r^\nu \mathrm{d}s, r = R^{2/(2-\nu)}, \nu < 2$.

6.1.11 Time Transformation. [608] One considers the time transformation $\mathrm{d}t/f(x) = \mathrm{d}s$ ($\tilde{h}_{ac} = h_{ac}/\sqrt{f}$ and $\sqrt{\tilde{g}} = \det(\tilde{h}_{ac})$):

$$\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}\mathbf{q}(t)\sqrt{g}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b - V(\mathbf{q})\right)\mathrm{d}t\right]$$

$$= (f'f'')^{\frac{1}{2}(1-D/2)}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{\mathrm{PF}}\mathbf{q}(t)\sqrt{\tilde{g}}$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\tilde{h}_{ac}\tilde{h}_{cb}\dot{q}^a\dot{q}^b - f\big(V(\mathbf{q}) + \Delta V_{\mathrm{PF}}(\mathbf{q}) - E\big)\right)\mathrm{d}s\right] . \tag{6.1.32}$$

6.1.12 Separation of Variables. [429,875]

$$\int\limits_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''} \mathcal{D}\mathbf{z}(t)f^d(\mathbf{z})\sqrt{g(\mathbf{z})}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(g_{ab}(\mathbf{z})\dot{z}^a\dot{z}^b + f^2(\mathbf{z})\dot{\mathbf{x}}^2\right) - \left(\frac{V(\mathbf{x})}{f^2(\mathbf{z})} + W(\mathbf{z})\right)\right]\mathrm{d}t\right\}$$

$$= [f(\mathbf{z}')f(\mathbf{z}'')]^{-d/2}\int \mathrm{d}E_l\,\Psi_l(\mathbf{x}'')\Psi_l^*(\mathbf{x}')\int\limits_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''} \mathcal{D}\mathbf{z}(t)\sqrt{g(\mathbf{z})}$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}(\mathbf{z})\dot{z}^a\dot{z}^b - W(\mathbf{z}) - \frac{E_l}{f^2(\mathbf{z})}\right)\mathrm{d}t\right] . \tag{6.1.33}$$

6.1.13 Transformation Formula for Separable Coordinate Systems.
[216,297,444,447,513]

$$\int\limits_{\boldsymbol{\varrho}(t')=\boldsymbol{\varrho}'}^{\boldsymbol{\varrho}(t'')=\boldsymbol{\varrho}''} \mathcal{D}\boldsymbol{\varrho}(t)|\mathbf{h}|^D \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\mathbf{h}^2\cdot\dot{\boldsymbol{\varrho}}^2 - \Delta V_{\mathrm{PF}}(\boldsymbol{\varrho})\right)\mathrm{d}t\right]$$

$$= \int\limits_{\boldsymbol{\varrho}(t')=\boldsymbol{\varrho}'}^{\boldsymbol{\varrho}(t'')=\boldsymbol{\varrho}''} \mathcal{D}\boldsymbol{\varrho}(t)\prod_{i=1}^{D}\sqrt{\frac{S}{M_i}}\cdot\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}S\frac{\dot{\varrho}_i^2}{M_i} - \Delta V_{\mathrm{PF},i}(\boldsymbol{\varrho})\right)\mathrm{d}t\right]$$

$$= (S'S'')^{\frac{1}{2}(1-D/2)}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\prod_{i=1}^{D}(M_i'M_i'')^{1/4}\int\limits_{\varrho_i(0)=\varrho_i'}^{\varrho_i(s'')=\varrho_i''}\mathcal{D}\varrho_i(s)$$

$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{\varrho}_i^2 + \frac{\hbar^2}{2m}\sum_{j=1}^{D}k_j^2\Phi_{ij}(\varrho_i) - \frac{\hbar^2}{8m}(\Gamma_i^2 + 2\Gamma_i')\right)\mathrm{d}s\right] . \tag{6.1.34}$$

Here it is assumed that the Laplace–Beltrami operator can be written as

$$\Delta_{\mathrm{LB}} = \sum_{i=1}^{D}\frac{1}{\prod_{j=1}^{D}h_j(\boldsymbol{\varrho})}\frac{\partial}{\partial\varrho_i}\left(\frac{\prod_{k=1}^{D}h_k(\boldsymbol{\varrho})}{h_i^2(\boldsymbol{\varrho})}\frac{\partial}{\partial\varrho_i}\right) , \tag{6.1.35}$$

where the h_i are the Lamé coefficients in the line element$ds^2 = \sum_{i=1}^{D} h_i^2(d\varrho_i)^2$. Furthermore, a factorization of the Lamé coefficients h_i is assumed according to

$$\frac{\prod_{j=1}^{D}h_j(\boldsymbol{\varrho})}{h_i^2(\boldsymbol{\varrho})} = M_{i1}\prod_{j=1}^{D}f_j(\varrho_j) \tag{6.1.36}$$

such that

$$M_{i1}(\varrho_1,\ldots,\varrho_{i-1},\varrho_{i+1},\ldots,\varrho_D) = \frac{\partial S}{\partial\Phi_{i1}} = \frac{S(\boldsymbol{\varrho})}{h_i^2(\boldsymbol{\varrho})} , \tag{6.1.37}$$

where $h^{1/2}/S(\boldsymbol{\varrho}) = \prod_{i=1}^{D} f_i(\varrho_i)$, $h = \prod_{i=1}^{D} h_i(\boldsymbol{\varrho})$, and S is the Stäckel determinant [714]

$$S(\boldsymbol{\varrho}) = \begin{vmatrix} \Phi_{11}(\varrho_1) & \Phi_{12}(\varrho_1) & \ldots & \Phi_{1D}(\varrho_1) \\ \Phi_{21}(\varrho_2) & \Phi_{22}(\varrho_2) & \ldots & \Phi_{2D}(\varrho_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{D1}(\varrho_D) & \Phi_{D2}(\varrho_D) & \ldots & \Phi_{DD}(\varrho_D) \end{vmatrix} . \tag{6.1.38}$$

M_{i1} is called the cofactor of Φ_{i1}. The separation constants k_i^2 are the eigenvalues of the operators $(k = 1,\ldots,D)$

$$I_k = \sum_{i=1}^{D}\left(\Phi^{-1}\right)_{ik}\left[-\frac{\hbar^2}{2m}\frac{1}{f_i}\frac{\mathrm{d}}{\mathrm{d}\varrho_i}\left(f_i\frac{\mathrm{d}}{\mathrm{d}\varrho_i}\right) + v_i(\varrho_i)\right] , \tag{6.1.39}$$

with $V(\boldsymbol{\varrho}) = \sum_i v_i(\varrho_i)/h_i^2$, with $E = \hbar^2 k_1^2/2m$ the energy, and $\Gamma_i = f_i$.

6.1.14 Semiclassical Separation Formula for Two-Dimensional Systems. [356,447,462]

$$\int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}} \mathrm{e}^{-\mathrm{i}ET/\hbar} \int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}(q(t),p(t))$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\dot{\mathbf{q}}\cdot\mathbf{p} - \frac{1}{h_1(q_1)+h_2(q_2)}\left(\frac{\mathbf{p}^2}{2m} + V_1(q_1) + V_2(q_2)\right)\right]\mathrm{d}t\right\}$$
$$\overset{(\hbar\to 0)}{\simeq} \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s'' \frac{\exp\left[\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(q_1'',q_1';s'') + \frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(q_2'',q_2';s'')\right]}{2\pi\,\mathrm{i}\,\hbar\sqrt{\partial q_1(s'')/\partial p_1(0)\partial q_2(s'')/\partial p_2(0)}} \ , \tag{6.1.40}$$
$$R_{\mathrm{Cl}}(q_i'',q_i';s'') = \int_0^{s''}\left[p_{i,\mathrm{Cl}}\dot{q}_{i,\mathrm{Cl}} - \left(\frac{p_{i,\mathrm{Cl}}^2}{2m} + V_i(q_{i,\mathrm{Cl}}) - Eh_i(q_{i,\mathrm{Cl}})\right)\right]\mathrm{d}s \ . \tag{6.1.41}$$

6.1.15 Integral Equation for Perturbation Expansion. [340,828] ($K^{(V)}$ and $K^{(0)}$ denote the perturbed and the unperturbed kernel, respectively, with perturbation potential $V(x)$, $x \in \mathbb{R}$)

$$K^{(V)}(x'',x';t'',t') = K^{(0)}(x'',x';t'',t')$$
$$- \int_{t'}^{t''}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}x\,K^{(0)}(x'',x';t'',t')V(x)K^{(V)}(x'',x';t'',t') \ . \tag{6.1.42}$$

6.1.16 Perturbation Expansion for Path Integrals. [65,340,404-408,430, 621,642,830,876]

$$K(\mathbf{x}'',\mathbf{x}';T) = \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}) - \tilde{V}(\mathbf{x})\right)\mathrm{d}t\right]$$
$$= K^{(V)}(\mathbf{x}'',\mathbf{x}';T) + \sum_{n=1}^{\infty}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \prod_{j=1}^{n}\int_{t'}^{t_{j+1}}\mathrm{d}t_j\int_{\mathbb{R}^D}\mathrm{d}\mathbf{x}_j$$
$$\times K^{(V)}(\mathbf{x}_1,\mathbf{x}';t_1-t')\widetilde{V}(\mathbf{x}_1)K^{(V)}(\mathbf{x}_2,\mathbf{x}_1;t_2-t_1)\times\dots$$
$$\dots\times\widetilde{V}(\mathbf{x}_{n-1})K^{(V)}(\mathbf{x}_n,\mathbf{x}_{n-1};t_n-t_{n-1})\widetilde{V}(\mathbf{x}_n)K^{(V)}(\mathbf{x}'',\mathbf{x}_n;t''-t_n) \ . \tag{6.1.43}$$

6.1.17 Exact Path Integration of Perturbation Expansion. [404–409, 472] $\left(\widehat{V}(\mathbf{k})=\int_{\mathbb{R}^D}\mathrm{d}\mathbf{x}\,\mathrm{e}^{-\mathrm{i}\,\mathbf{kx}}\,V(\mathbf{x}), c>1\right)$

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\mathrm{d}T\,\mathrm{e}^{sT/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}_E\mathbf{x}(t)\exp\left[-\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2+V(\mathbf{x})\right)\mathrm{d}t\right]\\
&=\sum_{n\in\mathbb{N}_0}(-1)^n\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_0}{(2\pi\hbar)^D}\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_1}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_1)\cdots\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_n}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_n)\\
&\quad\times\frac{\exp\left(\frac{\mathrm{i}}{\hbar}\mathbf{x}'\cdot\sum_{j=1}^n\mathbf{k}_j-\frac{\mathrm{i}}{\hbar}\mathbf{x}''\mathbf{k}_0\right)}{[s+(\mathbf{k}_0^2/2m)]\dots[s+(\mathbf{k}_0+\dots+\mathbf{k}_n)^2/2m]}\ ,
\end{aligned}\tag{6.1.44}$$

$$\begin{aligned}
&=\sum_{n\in\mathbb{N}_0}(-1)^n\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_0}{(2\pi\hbar)^D}\prod_{j=1}^n\int_{\mathbb{R}^D}\frac{\mathrm{d}\mathbf{k}_j}{(2\pi\hbar)^D}\widehat{V}(\mathbf{k}_j-\mathbf{k}_{j-1})\\
&\quad\times\frac{\exp\left[\frac{\mathrm{i}}{\hbar}(\mathbf{x}'\cdot\mathbf{k}_n-\mathbf{x}''\cdot\mathbf{k}_0)\right]}{(s+\mathbf{k}_0^2/2m)\dots(s+\mathbf{k}_n^2/2m)}\ .
\end{aligned}\tag{6.1.45}$$

6.1.18 Effective Potential. [72,334,340,341,390,539,613,614,915]

$$\begin{aligned}
&\oint\mathcal{D}_E x(t)\exp\left[-\frac{1}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2+V(x)\right)\mathrm{d}t\right]\\
&:=\int_{\mathbb{R}}\mathrm{d}x\int\limits_{x(0)=x}^{x(T)=x}\mathcal{D}_E x(t)\exp\left[-\frac{1}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2+V(x)\right)\mathrm{d}t\right]\\
&\overset{(T\to0)}{\simeq}\sqrt{\frac{m}{2\pi\hbar T}}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{-TW_{\mathrm{eff}}(x)/\hbar}\ ,
\end{aligned}\tag{6.1.46}$$

with an effective potential according to, e.g.,

$$W_{\mathrm{eff}}(x)=V(x)\ ,\tag{6.1.47}$$

$$W_{[340]}(x)=\sqrt{\frac{6m}{\pi\hbar T}}\int_{\mathbb{R}}V(y)\exp\left[-\frac{6m}{T\hbar}(x-y)^2\right]\mathrm{d}y\ ,\tag{6.1.48}$$

$$W_{[341,390,613]}(x)=W_1(x_0)\ .\tag{6.1.49}$$

The quantity $W_1(x_0)$ is evaluated in the following way. One considers the smeared version of the potential $V(x)$ according to

$$V_{a^2(x_0)}(x_0)=\int_{\mathbb{R}}\frac{\mathrm{d}x}{2\pi a^2}V(x)\exp\left(-\frac{(x-x_0)^2}{2a^2}\right)\ ,\tag{6.1.50}$$

$$a^2(x_0) = \frac{1}{T\Omega^2(x_0)}\left(\frac{T\Omega(x_0)}{2}\coth\frac{\Omega(x_0)T}{2} - 1\right) , \tag{6.1.51}$$

where $\Omega(x_0)$ is the frequency of a harmonic oscillator in the trial Lagrangian which emerges in a Fourier mode expansion of the partition function. Then one considers the quantity

$$\tilde{W}_1(x_0, a^2, \Omega) = V_{a^2(x_0)}(x_0) - \frac{1}{2}\Omega^2(x_0)a^2(x_0) + \frac{2}{\hbar T}\frac{\sinh\frac{\Omega T}{2}}{\Omega T} , \tag{6.1.52}$$

and minimizes it such that the equations

$$a^2(x_0) = \frac{\left(\frac{\Omega T}{2}\coth\frac{\Omega T}{2} - 1\right)}{T\Omega^2(x_0)} , \qquad \Omega^2(x_0) = \frac{\partial^2}{\partial x_0^2}V_{a^2}(x_0) \tag{6.1.53}$$

are fulfilled. The emerging effective potential is denoted by $W_1(x_0)$ and inserted into the expression for the partition function.

6.1.19 Constraint Path Integral. [100,310,348,357,358,366,567,716,845, 849] ($\varphi_1(\mathbf{x}) \equiv f(\mathbf{x}), \varphi_2(\mathbf{x},\mathbf{p}) \equiv \mathbf{p}\cdot\nabla_{\mathbf{x}} f(\mathbf{x})$ are constraints on $H(\mathbf{p},\mathbf{x})$).

$$\begin{aligned} K(\mathbf{x}'',\mathbf{x}';T) = &\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\mathrm{MP}}\mathbf{x}(t)\mathcal{D}\mathbf{p}(t)\sqrt{|\det\{\varphi_1,\varphi_2\}|} \\ &\times \delta(\varphi_1)\delta(\varphi_2)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Big[\mathbf{p}\cdot\dot{\mathbf{x}} - H(\mathbf{p},\mathbf{x})\Big]\mathrm{d}t\right\} . \end{aligned} \tag{6.1.54}$$

6.1.20 Time-Ordered Correlation Functions. [3,217,330,340,376,397,511, 534,688,790,799,801] (D_F is the Feynman propagator, see also p.54)

$$\begin{aligned} \Big\langle T\big(\underline{x}(t_1)\dots\underline{x}(t_n)\big)\Big\rangle_R &:= \int \mathcal{D}x(t)x(t_1)\dots x(t_n)\,\mathrm{e}^{\mathrm{i}R[x(t)]/\hbar} \\ &= \frac{(-\mathrm{i}\hbar)^n}{Z[J]}\frac{\delta^n Z[J]}{\delta J(t_1)\dots\delta J(t_n)}\bigg|_{J=0} , \end{aligned} \tag{6.1.55}$$

$$Z[J] = \int \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}R[x(t)] + \frac{\mathrm{i}}{\hbar}\int_0^\infty J(t)x(t)\,\mathrm{d}t\right] \tag{6.1.56}$$

$$= \exp\left[-\frac{\mathrm{i}}{2\hbar m}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathrm{d}t\,\mathrm{d}t'\,J(t)D_F(t-t')J(t')\right] , \tag{6.1.57}$$

$$\Big\langle T\big(\underline{x}(t_1)\underline{x}(t_2)\big)\Big\rangle_R = \frac{(-\mathrm{i}\hbar)^2}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(t_1)\delta J(t_2)}\bigg|_{J=0} = \frac{\mathrm{i}\hbar}{m}D_F(t_1-t_2) . \tag{6.1.58}$$

6.2 The General Quadratic Lagrangian

6.2.1 General D-Dimensional Quadratic Lagrangian. [5,43,221,229,235, 248,258,308,321,325,326,340,367,376,415,471,582,631,649,676,680,681,749,828, 892,894,914,915]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\mathrm{MP}}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[(\dot{\mathbf{x}},\mathbf{x})\mathbf{A}\begin{pmatrix}\dot{\mathbf{x}}\\ \mathbf{x}\end{pmatrix}+2\mathbf{d}(t)\cdot\dot{\mathbf{x}}+2\mathbf{e}(t)\cdot\mathbf{x}\right]dt\right\}$$

$$=\frac{\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')\right)}{\sqrt{(\mathrm{i}\,\pi\hbar)^D\Delta(t',t'')}}\ , \tag{6.2.1}$$

$$=\left(\frac{1}{2\pi\mathrm{i}\,\hbar}\right)^{D/2}\sqrt{\det\left(-\frac{\partial^2 R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')}{\partial x''^a\partial x'^b}\right)}\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')\right)\ . \tag{6.2.2}$$

The determinant $\Delta(t'',t')$ is given by

$$\Delta(t'',t')=\begin{vmatrix} F_{11}(t') & \dots & F_{12D}(t')\\ F_{11}(t'') & \dots & F_{12D}(t'')\\ \vdots & \ddots & \vdots\\ F_{D1}(t') & \dots & F_{D2D}(t')\\ F_{D1}(t'') & \dots & F_{D2D}(t'')\end{vmatrix}\ , \tag{6.2.3}$$

with the matrix $\mathbf{F}=(F_{jk})$, giving the solutions of the corresponding Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}(\mathbf{x},\dot{\mathbf{x}},t)}{\partial\dot{x}_j}=\frac{\partial\mathcal{L}(\mathbf{x},\dot{\mathbf{x}},t)}{\partial x_j} \tag{6.2.4}$$

of the Lagrangian in the path integral (6.2.1) with solutions

$$\begin{pmatrix}\dot{\mathbf{x}}_{\mathrm{Cl}}(t)\\ \mathbf{x}_{\mathrm{Cl}}(t)\end{pmatrix}=\mathbf{F}(t)\mathbf{C}(t'',t')+\mathbf{X}(t) \tag{6.2.5}$$

with the vector $\mathbf{C}$ appropriate for the boundary conditions $\mathbf{x}_{\mathrm{Cl}}(t')=\mathbf{x}'$ and $\mathbf{x}_{\mathrm{Cl}}(t'')=\mathbf{x}''$. The vector $\mathbf{X}$ is a particular solution of the generalized Euler–Lagrange equations, with the Wronskian

$$\begin{vmatrix} F_{11}(t) & \dots & F_{12D}(t)\\ \dot{F}_{11}(t) & \dots & \dot{F}_{12D}(t)\\ \vdots & \ddots & \vdots\\ F_{D1}(t) & \dots & F_{D2D}(t)\\ \dot{F}_{D1}(t) & \dots & \dot{F}_{D2D}(t)\end{vmatrix}=\begin{vmatrix} a_{11}(t) & \dots & a_{1D}\\ \vdots & \ddots & \vdots\\ a_{D1}(t) & \dots & a_{DD}\end{vmatrix}^{-1}\ . \tag{6.2.6}$$

$R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')$ is the action evaluated along the classical path, and the Morette–Van Hove determinant is $\det\left(-\partial^2 R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')/\partial x^{a''}\partial x^{b'}\right)$.

6.2.1.1 One-Dimensional General Quadratic Lagrangian. [219,257,325,326, 340,401,473,552,828,937]

$$K(x'',x';t'',t') = \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Big(a(t)\dot{x}^2 + 2b(t)\dot{x}x + c(t)x^2 + 2d(t)\dot{x} + 2e(t)x\Big)\mathrm{d}t\right]$$

$$= \sqrt{\frac{1}{2\pi\mathrm{i}\hbar}\det\left(-\frac{\partial^2 R_{\mathrm{Cl}}(x'',x')}{\partial x'\partial x''}\right)}\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(x'',x')\right) , \tag{6.2.7}$$

$$= \frac{\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(x'',x')\right)}{\sqrt{\mathrm{i}\,\pi\hbar(y_2''y_1' - y_1''y_2')}} . \tag{6.2.8}$$

$y_1(t)$ is a non-trivial solution of $a(t)\ddot{x} + \dot{a}(t)\dot{x} + (\dot{b}(t) - c(t))x = 0$, $y_2(t) = y_1(t)\int_\tau^t \mathrm{d}s/(a(s)y_1^2(s))$, τ suitably chosen, with the Wronskian

$$\begin{vmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{vmatrix} = \frac{1}{a(t)} . \tag{6.2.9}$$

6.2.1.2 D-Dimensional Free Particle. [106,325,326,340,613,826,876,927]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2\mathrm{d}t\right)$$

$$= \left(\frac{m}{2\pi\mathrm{i}\,\hbar T}\right)^{D/2}\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}|\mathbf{x}''-\mathbf{x}'|^2\right) \tag{6.2.10}$$

$$= \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D}\mathrm{d}\mathbf{k}\exp\left[\mathrm{i}\,\mathbf{k}\cdot(\mathbf{x}''-\mathbf{x}') - \mathrm{i}\,T\frac{\hbar|\mathbf{k}|^2}{2m}\right] . \tag{6.2.11}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2\mathrm{d}t\right)$$

$$= 2\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2\pi\mathrm{i}\,\hbar}\right)^{D/2}\left(\frac{m}{2E}|\mathbf{x}''-\mathbf{x}'|^2\right)^{\frac{1}{2}(1-D/2)} K_{1-\frac{D}{2}}\left(\frac{|\mathbf{x}''-\mathbf{x}'|}{\hbar}\sqrt{-2mE}\right) . \tag{6.2.12}$$

For $D = 1, 2, 3$ one has

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathbf{x}}^2 \mathrm{d}t\right)$$

$$\stackrel{(D=1)}{=} \frac{1}{\hbar}\sqrt{-\frac{m}{2E}} \exp\left(-\frac{|x''-x'|}{\hbar}\sqrt{-2mE}\right) , \tag{6.2.13}$$

$$\stackrel{(D=2)}{=} \frac{m}{\pi\hbar^2} K_0\left(\frac{|\mathbf{x}''-\mathbf{x}'|}{\hbar}\sqrt{-2mE}\right) , \tag{6.2.14}$$

$$\stackrel{(D=3)}{=} \frac{m}{2\pi\hbar^2|\mathbf{x}''-\mathbf{x}'|} \exp\left(-\frac{|\mathbf{x}''-\mathbf{x}'|}{\hbar}\sqrt{-2mE}\right) . \tag{6.2.15}$$

The D-dimensional free particle can also be expressed in terms of the D-dimensional Euclidean group path integral [106]. Cf. [444] and 6.4.1.4, 6.7.1.5–8 for the expansion of the propagator in various coordinate systems which separate the path integrations in $D = 2$ and $D = 3$.

6.2.1.3 Free Particle with Vector Potential. [325,326,340,395,590]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}}\right)\mathrm{d}t\right]$$

$$= \frac{1}{(2\pi)^D}\int_{\mathbb{R}^D} \mathrm{d}\mathbf{k} \exp\left[\mathrm{i}\left(\mathbf{k} + \frac{e}{c\hbar}\mathbf{A}\right)\cdot(\mathbf{x}''-\mathbf{x}') - \frac{\mathrm{i}}{\hbar}T\frac{\hbar^2\mathbf{k}^2}{2m}\right] , \tag{6.2.16}$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{D/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar T}(\mathbf{x}''-\mathbf{x}')^2 + \frac{\mathrm{i}\, e\mathbf{A}}{c\hbar}\cdot(\mathbf{x}''-\mathbf{x}')\right] . \tag{6.2.17}$$

6.2.1.4 Linear Potential. [10,146,231,257,278,318,340,374,661,664,765,801]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - kx\right)\mathrm{d}t\right]$$

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2}\frac{(x''-x')^2}{T} - \frac{kT}{2}(x'+x'') - \frac{k^2T^3}{24m}\right)\right] , \tag{6.2.18}$$

$$= \int_{\mathbb{R}} \mathrm{d}E\, \mathrm{e}^{-\mathrm{i}ET/\hbar}\left(\frac{2m}{\hbar^2\sqrt{k}}\right)^{2/3}$$

$$\times \mathrm{Ai}\left[\left(x''-\frac{E}{k}\right)\left(\frac{2mk}{\hbar^2}\right)^{1/3}\right] \mathrm{Ai}\left[\left(x'-\frac{E}{k}\right)\left(\frac{2mk}{\hbar^2}\right)^{1/3}\right] . \tag{6.2.19}$$

The Green function is given by [439,715]

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-kx\right)dt\right]$$
$$=\frac{4}{3}\frac{m}{\hbar^2}\left[\left(x'-\frac{E}{k}\right)\left(x''-\frac{E}{k}\right)\right]^{1/2}$$
$$\times K_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_>-\frac{E}{k}\right)^{3/2}\right]I_{1/3}\left[\frac{\sqrt{8mk}}{3\hbar}\left(x_<-\frac{E}{k}\right)^{3/2}\right] \ . \tag{6.2.20}$$

6.2.1.5 A Free Particle in a Uniform Magnetic Field. [164,257,340,395,649,680, 681,745–749,756,765,771] ($\omega=eB/mc$, $\mathbf{x}=(x,y)=\varrho(\cos\varphi,\sin\varphi)\in\mathbb{R}^2$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{i\,m}{2\hbar}\int_{t'}^{t''}[\dot{\mathbf{x}}^2+\omega(x\dot{y}-y\dot{x})]dt\right\}$$
$$=\frac{m\omega}{4\pi i\hbar\sin(\omega T/2)}$$
$$\times\exp\left[\!\left[\frac{i\,m\omega}{4\hbar}\left\{\left[(x''-x')^2+(y''-y')^2\right]\cot\frac{\omega T}{2}+2(x'y''-x''y')\right\}\right]\!\right] \ , \tag{6.2.21}$$
$$=\frac{m\omega}{4\pi i\hbar\sin(\omega T/2)}$$
$$\times\exp\left\{\frac{i\,m\omega}{4\hbar\sin(\omega T/2)}\left[(\varrho'^2+\varrho''^2)\cos\frac{\omega T}{2}-2\varrho'\varrho''\cos\left(\varphi''-\varphi'-\frac{\omega T}{2}\right)\right]\right\} \ , \tag{6.2.22}$$
$$=\sum_{n\in\mathbb{N}_0}\sum_{\nu\in\mathbb{Z}}\Psi_{n,\nu}(\varrho'',\varphi'')\Psi^*_{n,\nu}(\varrho',\varphi')\,e^{-iE_nT/\hbar} \ , \tag{6.2.23}$$
$$\Psi_{n,\nu}(\varrho,\varphi)=\left[\frac{m\omega}{2\pi\hbar}\frac{n!}{(n+|\nu|)!}\right]^{1/2}\left(\frac{m\omega}{2\hbar}\varrho^2\right)^{|\nu|/2}$$
$$\times\exp\left(i\nu\varphi-\frac{m\omega}{4\hbar}\varrho^2\right)L_n^{(|\nu|)}\left(\frac{m\omega}{2\hbar}\varrho^2\right) \ , \tag{6.2.24}$$
$$E_n=\omega\hbar(n+\tfrac{1}{2}) \ . \tag{6.2.25}$$

The Green function is given by

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{i\,m}{2\hbar}\int_{t'}^{t''}\left[\dot{\mathbf{x}}^2+\omega(x\dot{y}-y\dot{x})\right]dt\right\}$$

$$= \sum_{\nu \in \mathbb{Z}} \frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi} \frac{2\Gamma(\frac{1}{2} - \frac{E}{\hbar\omega})}{\hbar\omega\varrho'\varrho''|\nu|!}$$
$$\times W_{E/\hbar\omega+|\nu|/2,|\nu|/2}\left(\frac{m\omega}{2\hbar}\varrho_>^2\right) M_{E/\hbar\omega+|\nu|/2,|\nu|/2}\left(\frac{m\omega}{2\hbar}\varrho_<^2\right) . \tag{6.2.26}$$

6.2.1.6 Free Particle in a Crossed Time-Dependent Electric and Magnetic Field. [159,319,322,500,545,728] ($\omega = eB/mc$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m\omega}{2}(x\dot{y} - y\dot{x}) + qE_y(t)y + qE_z(t)z\right)\mathrm{d}t\right]$$
$$= \left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{3/2} \frac{\omega T}{2\sin\frac{\omega T}{2}} F(z'', z'; T)$$
$$\times \exp\left\{\frac{\mathrm{i}m\omega}{4\hbar}\cot\frac{\omega T}{2}\left[(x''-x')^2 + (y''-y')^2\right] + \frac{\mathrm{i}m\omega}{2\hbar}(x'y'' - x''y')\right.$$
$$+ \frac{\mathrm{i}q}{\hbar\sin\omega T}\left[y''E''(t'',t') + y'E'(t'',t') - \frac{q}{m\omega}E(t'',t')\right]$$
$$\left. + \frac{\mathrm{i}q\omega}{4\hbar\tan\frac{\omega T}{2}}E(\omega T)\left[qE(\omega T) - 2\left(H(\omega T)(x'-x'') + (y'+y'')\tan\frac{\omega T}{2}\right)\right]\right\}, \tag{6.2.27}$$

with

$$F(z'', z'; T) = \exp\left[\frac{\mathrm{i}m}{2\hbar T}\left((z''-z')^2\right.\right.$$
$$+ \frac{2qz'}{m}\int_{t'}^{t''}\mathrm{d}t\, E_z(t)(t''-t) + \frac{2qz''}{m}\int_{t'}^{t''}\mathrm{d}t\, E_z(t)(t-t')$$
$$\left.\left. - \frac{2q^2}{m^2}\int_{t'}^{t''}\mathrm{d}t\, E_z(t)(t''-t)\int_{t'}^{t''}\mathrm{d}s\, E_z(s)(s-t')\right)\right] , \tag{6.2.28}$$

and the quantities

$$E(\omega T) = \frac{E_1(t'',t') + E_2(t'',t')}{m\sin\omega T} , \quad H(\omega T) = 1 - 2\frac{\tan\frac{\omega T}{2}}{\omega T} , \tag{6.2.29}$$

$$E'(t'',t') = \int_{t'}^{t''}\mathrm{d}t\, E_y(t)\sin\omega(t''-t) , \tag{6.2.30a}$$

$$E''(t'',t') = \int_{t'}^{t''}\mathrm{d}t\, E_y(t)\sin\omega(t-t')\mathrm{d}t , \tag{6.2.30b}$$

$$E(t'',t') = \int_{t'}^{t''} \mathrm{d}t\, E_y(t) \sin\omega(t''-t) \int_{t'}^{t} \mathrm{d}s\, E_y(s) \sin\omega(s-t') \ , \tag{6.2.30c}$$

$$E_1(t'',t') = \int_{t'}^{t''} \mathrm{d}t\, E_y(t) \sin\omega(t''-t) \int_{t'}^{t} \mathrm{d}s\, \sin\omega(s-t') \ , \tag{6.2.30d}$$

$$E_2(t'',t') = \int_{t'}^{t''} \mathrm{d}t\, \sin\omega(t''-t) \int_{t'}^{t} \mathrm{d}s\, E_y(s) \sin\omega(s-t') \ . \tag{6.2.30e}$$

In the case that the electric fields are time-independent we get ($v_d = qE_z/m$, $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$) [500,728]

$$\begin{aligned}
&\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m\omega}{2}(x\dot{y} - y\dot{x}) + qE_y y + qE_z z\right)\mathrm{d}t\right] \\
&= \left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{3/2} \frac{\omega T}{2\sin\frac{\omega T}{2}} \\
&\times \exp\left[\frac{\mathrm{i}m}{2\hbar T}(z''-z')^2 + \frac{\mathrm{i}m\omega v_d T}{2\hbar}(y'+y'') + \frac{\mathrm{i}qE_y T}{2\hbar}\left((z'+z'') + \frac{qE_y T^2}{12m}\right)\right] \\
&\times \exp\left[\frac{\mathrm{i}m\omega}{4\hbar}\cot\frac{\omega T}{2}\left[(x''-x')^2 + (y''-y')^2\right] + \frac{\mathrm{i}m\omega}{2\hbar}(x'y'' - x''y')\right] \\
&\times \exp\left[\frac{\mathrm{i}mv_d}{4\hbar\tan(\omega T/2)}\left(\omega T - 2\tan\frac{\omega T}{2}\right)\left[2(x'-x'') + v_d T\right]\right] \ .
\end{aligned} \tag{6.2.31}$$

On caustics the Feynman kernel has the form [160]

$$\begin{aligned}
K\left(\mathbf{x}'',\mathbf{x}';\frac{2k\pi}{\omega}\right) &= 2\,\mathrm{e}^{-\mathrm{i}k\pi/2}\delta(x'-x''-v_d T)\delta(y'-y'') \\
&\quad\times \exp\left[-\frac{\mathrm{i}m\omega v_d^2 T^2}{4\hbar} + \frac{\mathrm{i}m\omega v_d T}{2\hbar}(y''-y')\right] \ .
\end{aligned} \tag{6.2.32}$$

6.2.1.7 Harmonic Oscillator. [26,122,202,257,273,278,301,325,326,340,472,501, 528,552,571,639,644,684,750,801,805,887]

$$\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2 - \omega^2 x^2)\,\mathrm{d}t\right] \\
&= \left(\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega T}\right)^{1/2} \exp\left\{\frac{\mathrm{i}m\omega}{2\mathrm{i}\hbar}\left[(x'^2 + x''^2)\cot\omega T - 2\frac{x'x''}{\sin\omega T}\right]\right\},
\end{aligned} \tag{6.2.33}$$

$$= \left(\frac{m\omega\, \mathrm{e}^{-\,\mathrm{i}\,\pi(\frac{1}{2}+n)}}{2\pi\hbar\sin\omega\tau}\right)^{1/2} \exp\left\{\frac{\mathrm{i}\,m\omega}{2\hbar\sin\omega T}\Big[(x'^2+x''^2)\cos\omega T - 2x'x''\Big]\right\} , \tag{6.2.34}$$

$$= \sum_{n\in\mathbb{N}_0}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^n n!}\exp\left(-\frac{m\omega}{2\hbar}(x'^2+x''^2)\right)$$
$$\times H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x''\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x'\right)\mathrm{e}^{-\,\mathrm{i}\,\omega T(n+1/2)} . \tag{6.2.35}$$

($T = n\pi/\omega+\tau$, $n\in\mathbb{N}_0$; $0<\tau<\pi/\omega$): [273,501,651,684,795]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2-\omega^2x^2)\,\mathrm{d}t\right]\Bigg|_{T=\frac{n\pi}{\omega}+} = \mathrm{e}^{-\,\mathrm{i}\,\pi n/2}\,\delta(x'-(-1)^n x'') . \tag{6.2.36}$$

The Green function is given by [46]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2-\omega^2x^2)\,\mathrm{d}t\right]$$
$$= \sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right)D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}\,x_>\right)D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,x_<\right) . \tag{6.2.37}$$

The following recurrence relation holds for the D-dimensional harmonic oscillator kernel$K^{(D)}(T)$ ($\nu = \mathbf{x}'\cdot\mathbf{x}''$)

$$K^{(D)}(\mathbf{x}'',\mathbf{x}';T) = \frac{1}{2\pi}\frac{\partial}{\partial\nu}K^{(D-2)}(\mathbf{x}'',\mathbf{x}';T) . \tag{6.2.38}$$

The Green function is given by ($\xi = \frac{1}{2}(|\mathbf{x}'+\mathbf{x}''|+|\mathbf{x}''-\mathbf{x}'|)$, $\eta = \frac{1}{2}(|\mathbf{x}'+\mathbf{x}''|-|\mathbf{x}''-\mathbf{x}'|)$, $D = 1,3,5\ldots$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{\mathbf{x}}^2-\omega^2\mathbf{x}^2)\,\mathrm{d}t\right]$$
$$= \sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right)\left(\frac{1}{2\pi}\right)^{\frac{D-1}{2}}\left[\frac{1}{\eta^2-\xi^2}\left(\eta\frac{\partial}{\partial\xi}-\xi\frac{\partial}{\partial\eta}\right)\right]^{\frac{D-1}{2}}$$
$$\times D_{-\frac{1}{2}+E/\hbar\omega}\left(\sqrt{\frac{2m\omega}{\hbar}}\,\xi\right)D_{-\frac{1}{2}+E/\hbar\omega}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,\eta\right) . \tag{6.2.39}$$

6.2.1.8 Repelling Harmonic Oscillator. [146,281,359,485,664,753]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2+\omega^2x^2)\,\mathrm{d}t\right]$$

$$= \left(\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sinh\omega T}\right)^{1/2}\exp\left\{-\frac{m\omega}{2\,\mathrm{i}\,\hbar}\left[(x'^2+x''^2)\coth\omega T-2\frac{x'x''}{\sinh\omega T}\right]\right\}, \tag{6.2.40}$$

$$= \frac{1}{8\pi^2\hbar}\sqrt{\frac{m}{\hbar\omega}}\int_{\mathbb{R}}\mathrm{d}E\,\exp\left(-\frac{\mathrm{i}}{\hbar}TE+\frac{\pi E}{2\omega\hbar}\right)$$
$$\times\left[\left|\Gamma\left(\frac{1}{4}+\frac{\mathrm{i}\,E}{2\hbar\omega}\right)\right|^2 E^{(0)}_{-\frac{1}{2}+\mathrm{i}\,E/\hbar\omega}\left(\sqrt{\frac{2m\omega}{\mathrm{i}\,\hbar}}\,x_<\right)E^{(0)}_{-\frac{1}{2}-\mathrm{i}\,E/\hbar\omega}\left(\sqrt{\frac{2m\,\mathrm{i}\,\omega}{\hbar}}\,x_>\right)\right.$$
$$\left.+\left|\Gamma\left(\frac{3}{4}+\frac{\mathrm{i}\,E}{2\hbar\omega}\right)\right|^2 E^{(1)}_{-\frac{1}{2}+\mathrm{i}\,E/\hbar\omega}\left(\sqrt{\frac{2m\omega}{\mathrm{i}\,\hbar}}\,x_<\right)E^{(1)}_{-\frac{1}{2}-\mathrm{i}\,E/\hbar\omega}\left(\sqrt{\frac{2m\,\mathrm{i}\,\omega}{\hbar}}\,x_>\right)\right]. \tag{6.2.41}$$

6.2.1.9 Forced Harmonic Oscillator. [158,160,211,257,325,326,340,395,401, 498,562,578–581,801,915].

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{x}^2-\omega^2x^2)+J(t)x\right)\mathrm{d}t\right]$$

$$= \sqrt{\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sin\omega T}}\exp\left[\frac{\mathrm{i}\,m\omega}{2\hbar\sin\omega T}\left((x'^2+x''^2)\cos\omega T-2x'x''\right.\right.$$
$$+\frac{2x''}{m\omega}\int_{t'}^{t''}\mathrm{d}t\,J(t)\sin\omega(t-t')+\frac{2x'}{m\omega}\int_{t'}^{t''}\mathrm{d}t\,J(t)\sin\omega(t''-t)$$
$$\left.\left.-\frac{2}{m^2\omega^2}\int_{t'}^{t''}\mathrm{d}t\,J(t)\int_{t'}^{t}\mathrm{d}s\,J(s)\sin\omega(t''-t)\sin\omega(s-t')\right)\right], \tag{6.2.42}$$

$$= \sum_{n\in\mathbb{N}_0}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^n n!}H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x''\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x'\right)$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}(x'^2+x''^2)-\mathrm{i}\,\omega T\left(n+\frac{1}{2}\right)\right]$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar\sin\omega T}\left(x''\int_{t'}^{t''}\mathrm{d}t\,J(t)\sin\omega(t-t')+x'\int_{t'}^{t''}\mathrm{d}t\,J(t)\sin\omega(t''-t)\right.\right.$$
$$\left.\left.-\frac{1}{m\omega}\int_{t'}^{t''}\int_{t'}^{t}\mathrm{d}s\,\mathrm{d}t\,J(t)J(s)\sin\omega(t''-t)\sin\omega(s-t')\right)\right]. \tag{6.2.43}$$

For the case of a time-independent force $J(t) = k$ we get the kernel $K^{(k)}(T)$

$$K^{(k)}(x'',x';T) = \sqrt{\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sin\omega T}}$$
$$\times\exp\Bigg[-\frac{\mathrm{i}\,k^2T}{2m\hbar\omega^2}+\frac{\mathrm{i}\,m\omega}{2\hbar\sin\omega T}\bigg(({x'}^2+{x''}^2)\cos\omega T-2x'x''$$
$$-\frac{2k}{m\omega^2}(x'+x'')(\cos\omega T-1)-\frac{2k^2}{m^2\omega^4}(\cos\omega T-1)\bigg)\Bigg]\ , \qquad (6.2.44)$$

$$= \sum_{n\in\mathbb{N}_0}\Psi_n^{(k)}(x'')\Psi_n^{(k)}(x')\,\mathrm{e}^{-\mathrm{i}\,E_n^{(k)}T/\hbar}\ , \qquad (6.2.45)$$

$$\Psi_n^{(k)}(x) = \bigg(\frac{m\omega/\pi\hbar}{(2^n n!)^2}\bigg)^{1/4}H_n\Bigg[\sqrt{\frac{m\omega}{\hbar}}\bigg(x-\frac{k}{m\omega^2}\bigg)\Bigg]\exp\Bigg[-\frac{m\omega}{2\hbar}\bigg(x-\frac{k}{m\omega^2}\bigg)^2\Bigg]\ , \qquad (6.2.46)$$

$$E_n = \hbar\omega\bigg(n+\frac{1}{2}\bigg)-\frac{k^2}{2m\omega^2}\ . \qquad (6.2.47)$$

On caustics the kernel has the form [160] ($t_0 = t'' - \pi/2\omega, n \in \mathbb{N}_0$)

$$K\bigg(x'',x';\frac{n\pi}{\omega}\bigg) = \mathrm{e}^{-\mathrm{i}\,n\pi/2}$$
$$\times\exp\Bigg[-\frac{\mathrm{i}}{2\hbar}\bigg(x''\int_{t_0}^{t''}\mathrm{d}tJ(t)\sin\omega(t-t_0)-(-1)^n x'\int_{t'}^{t_0}\mathrm{d}tJ(t)\sin\omega(t_0-t)\bigg)$$
$$-\frac{\mathrm{i}}{2\hbar m\omega}\bigg(\int_{t_0}^{t''}\mathrm{d}tJ(t)\sin\omega(t''-t)\int_{t_0}^{t}\mathrm{d}sJ(s)\sin\omega(s-t_0)$$
$$-(-1)^n\int_{t'}^{t_0}\mathrm{d}tJ(t)\sin\omega(t_0-t)\int_{t_0}^{t}\mathrm{d}sJ(s)\sin\omega(s-t')\bigg)\Bigg]$$
$$\times\,\delta\Bigg[x''+\frac{1}{m\omega}\int_{t_0}^{t''}\mathrm{d}tJ(t)\sin\omega(t''-t)$$
$$-(-1)^n\bigg(\frac{1}{m\omega}\int_{t'}^{t_0}\mathrm{d}tJ(t)\sin\omega(t-t')-x'\bigg)\Bigg]\ . \qquad (6.2.48)$$

6.2.1.10 Time-Dependent Forced Oscillator. [126,158,251,257,319,401,578]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\Bigg[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg(\frac{m}{2}(\dot{x}^2-\omega(t)^2x^2)+J(t)x\bigg)\mathrm{d}t\Bigg]$$

$$
= \sqrt{\frac{m}{2\pi \mathrm{i}\hbar f(t')}} \exp\left[\frac{\mathrm{i}m}{2\hbar f(t')}\left(x''^2\dot{g}(t'') - 2x'x'' - x'^2\dot{f}(t')\right)\right.
$$
$$
+ \frac{\mathrm{i}}{\hbar f(t')}\left(x'\int_{t'}^{t''}\mathrm{d}t\,J(t)f(t) + x''\int_{t'}^{t''}\mathrm{d}t\,J(t)g(t)\right)
$$
$$
\left.-\frac{1}{m}\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t}\mathrm{d}s\,J(t)J(s)f(t)g(s)\right)\Bigg] \ . \tag{6.2.49}
$$

$f(t)$ and $g(t)$, respectively, denote functions defined by solutions of the differential equations

$$
\begin{aligned}
&\ddot{f}(t) + \omega^2(t)f(t) = 0 \ , \qquad f(t'') = 0 \ , \qquad \dot{f}(t'') = -1 \ , \\
&\ddot{g}(t) + \omega^2(t)g(t) = 0 \ , \qquad g(t') = 0 \ , \qquad \dot{g}(t') = 1 \ .
\end{aligned} \tag{6.2.50}
$$

6.2.1.11 Electron in a Saddle-Point Potential and a Magnetic Field. [904] $(\Omega = eB/2mc,\ \mathbf{x} = (x, y) \in \mathbb{R}^2)$

$$
\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\dot{\mathbf{x}}^2 + \gamma^2(x^2 - y^2) + \Omega(\dot{x}y - x\dot{y})\right]\mathrm{d}t\right\}
$$
$$
= \frac{m(\lambda^2+\omega^2)\sqrt{\tau_1(T)\tau_2(T)}}{4\pi\mathrm{i}\hbar}\exp\left\{\frac{\mathrm{i}m(\lambda^2+\omega^2)\tau_1(T)\tau_2(T)}{4\hbar}\right.
$$
$$
\times\left[\sinh\frac{\lambda T}{2}\sin\frac{\omega T}{2}\left(\frac{(x''+x')^2}{\tau_2(T)} - \frac{(y''+y')^2}{\tau_1(T)}\right)\right.
$$
$$
\left.+\cosh\frac{\lambda T}{2}\cos\frac{\omega T}{2}\left(\frac{(x''-x')^2}{\tau_1(T)} + \frac{(y''-y')^2}{\tau_2(T)}\right)\right]
$$
$$
\left.-\frac{\mathrm{i}m\tau_1(T)\tau_2(T)\Omega}{4\hbar}\left(\frac{(x''+x')(y''-y')}{\tau_2^2(T)} - \frac{(x''-x')(y''+y')}{\tau_1^2(T)}\right)\right\} \ . \tag{6.2.51}
$$

Here denote

$$
\omega = +\sqrt{\frac{1}{2}\Omega^2 + \sqrt{\gamma^4 + \frac{1}{4}\Omega^4}} \ , \qquad \lambda = +\sqrt{-\frac{1}{2}\Omega^2 + \sqrt{\gamma^4 + \frac{1}{4}\Omega^4}} \ , \tag{6.2.52}
$$

$$
\frac{1}{\tau_1(T)} = \lambda\left(\cosh\frac{\lambda T}{2}\sin\frac{\omega T}{2} + \sinh\frac{\lambda T}{2}\cos\frac{\omega T}{2}\right)
$$
$$
+\,\omega\left(\cosh\frac{\lambda T}{2}\sin\frac{\omega T}{2} - \sinh\frac{\lambda T}{2}\cos\frac{\omega T}{2}\right) \ , \tag{6.2.53}
$$
$$
\frac{1}{\tau_2(T)} = \lambda\left(-\cosh\frac{\lambda T}{2}\sin\frac{\omega T}{2} + \sinh\frac{\lambda T}{2}\cos\frac{\omega T}{2}\right)
$$

$$+\,\omega\left(\cosh\frac{\lambda T}{2}\sin\frac{\omega T}{2}+\sinh\frac{\lambda T}{2}\cos\frac{\omega T}{2}\right)\ . \tag{6.2.54}$$

6.2.1.12 Three-Dimensional Anisotropic Oscillator with Magnetic Field. [64,162,222,620,673,750] ($\omega = eB/mc$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\Big[\dot{\mathbf{x}}^2-(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)+\omega(x\dot y-y\dot x)\Big]\mathrm{d}t\right\}$$
$$=\sqrt{\frac{\omega_z}{\sin\omega_z T}}\left(\frac{m}{2\pi\,\mathrm{i}\,\hbar}\right)^{3/2}\frac{\lambda_+^2-\lambda_-^2}{\omega\sqrt{D}}$$
$$\times\exp\left\{-\frac{m\omega_z}{2\,\mathrm{i}\,\hbar}\left[(z'^2+z''^2)\cot\omega_z T-2\frac{z'z''}{\sin\omega_z T}\right]+\frac{\mathrm{i}}{\hbar}\frac{m}{2D}\Big[(x'^2+x''^2)a_1\right.$$
$$\left.+\,(y'^2+y''^2)a_2+x'x''a_3+y'y''a_4+(x'y''-x''y')a_5+(x''y''-x'y')a_6\Big]\right\}, \tag{6.2.55}$$

with the quantities

$$\lambda_\pm=\frac{1}{2}(\omega^2+\omega_x^2+\omega_y^2)\pm\frac{1}{2}\sqrt{(\omega^2+\omega_x^2+\omega_y^2)^2-4\omega_x^2\omega_y^2}\ , \tag{6.2.56}$$

$$D=2-2\cos\lambda_+T\cos\lambda_-T+\frac{(\omega_x^2-\omega_y^2)^2+\omega^2(\omega_x^2+\omega_y^2)}{\omega^2\omega_x\omega_y}\sin\lambda_+T\sin\lambda_-T\ , \tag{6.2.57}$$

$$a_1=\frac{\lambda_+^2-\lambda_-^2}{\omega^2\omega_y}\big[(\lambda_+\omega_x-\lambda_-\omega_y)\cos\lambda_+T\sin\lambda_-T\ ,$$
$$+\,(\lambda_+\omega_y-\lambda_-\omega_x)\cos\lambda_-T\sin\lambda_+T\big]\ , \tag{6.2.58a}$$

$$a_2=\frac{\lambda_+^2-\lambda_-^2}{\omega^2\omega_x}\big[(\lambda_+\omega_y-\lambda_-\omega_x)\cos\lambda_+T\sin\lambda_-T\ ,$$
$$+\,(\lambda_+\omega_x-\lambda_-\omega_y)\cos\lambda_-T\sin\lambda_+T\big] \tag{6.2.58b}$$

$$a_3=2\frac{\lambda_+^2-\lambda_-^2}{\omega^2\omega_y}\big[(\lambda_-\omega_x-\lambda_+\omega_y)\sin\lambda_+T-(\lambda_+\omega_x-\lambda_-\omega_y)\sin\lambda_-T\big]\ , \tag{6.2.58c}$$

$$a_4=2\frac{\lambda_+^2-\lambda_-^2}{\omega^2\omega_y}\big[(\lambda_-\omega_y-\lambda_+\omega_x)\sin\lambda_+T-(\lambda_+\omega_y-\lambda_-\omega_x)\sin\lambda_-T\big]\ , \tag{6.2.58d}$$

$$a_5=\frac{2}{\omega}(\lambda_+^2-\lambda_-^2)(\cos\lambda_-T-\cos\lambda_+T)\ , \tag{6.2.58e}$$

$$a_6=\frac{\omega_x^2-\omega_y^2}{\omega}\left[2-\frac{\omega^2+\omega_x^2+\omega_y^2}{\omega_x\omega_y}\sin\lambda_-T\sin\lambda_+T-2\cos\lambda_-T\cos\lambda_+T\right]\ . \tag{6.2.58f}$$

6.2.1.13 Two-Dimensional Anisotropic Oscillator with Magnetic Field. [162] ($\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\Omega^2 = \omega^2 + \omega_y^2, \omega = eB/mc$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\Big[\dot{\mathbf{x}}^2 - (\omega_y^2 y^2 + \omega_z^2 z^2) + \omega(x\dot{y} - y\dot{x})\Big]\mathrm{d}t\right\}$$

$$= \frac{1}{\sqrt{1 - \omega^2 g(\Omega T)/\Omega^3 T}}\left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{3/2}\left(\frac{\omega_z \Omega T^2}{\sin\omega_z T \sin\Omega T}\right)^{1/2}$$

$$\times \exp\left\{\frac{\mathrm{i}\, m}{2\hbar T}(x'' - x')^2 - \frac{m\omega_z}{2\,\mathrm{i}\,\hbar}\left[(z'^2 + z''^2)\cot\omega_z T - 2\frac{z'z''}{\sin\omega_x T}\right]\right\}$$

$$\times \exp\left\{-\frac{m\Omega}{2\,\mathrm{i}\,\hbar}\left[(y'^2 + y''^2)\cot\Omega T - 2\frac{y'y''}{\sin\Omega T}\right]\right\}$$

$$\times \exp\left[\frac{\mathrm{i}\, m\omega}{2\hbar}(x''y'' - x'y') + A_x(T)(x'' - x')^2 \right.$$

$$\left. + A_{xy}(T)(x' - x'')(y' + y'') + A_y(T)(y' + y'')^2\right] , \tag{6.2.59}$$

where $(g(x) = x - 2\tan(x/2), H(x) = 1 - \omega^2 g(x)/\Omega^2 x)$

$$A_x(T) = \frac{\mathrm{i}\, m\omega^2 g(\Omega T)}{2\hbar\Omega^3 T^2 H(\Omega T)},\ A_{xy}(T) = \frac{\mathrm{i}\, m\omega\tan\frac{\Omega T}{2}}{\hbar\Omega T H(\Omega T)},\ A_y(T) = \frac{\mathrm{i}\, m\omega^2\tan^2\frac{\Omega T}{2}}{2\hbar\Omega^2 T H(\Omega T)} . \tag{6.2.60}$$

6.2.1.14 Three-Dimensional Anisotropic Oscillator with Magnetic Field. [771] ($\omega = eB/mc, \Omega^2 = \omega_0^2 + \omega^2$, $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\Big[\dot{\mathbf{x}}^2 - \omega_0^2(x^2 + y^2) - \omega_z^2 z^2 - 2\omega(x\dot{y} - y\dot{x})\Big]\mathrm{d}t\right\}$$

$$= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 - \omega_0^2 r^2) + m\omega r^2\dot{\varphi} + \frac{\hbar^2}{8m}\right)\mathrm{d}t\right]$$

$$= \sqrt{\frac{m\omega_z}{2\pi\,\mathrm{i}\,\hbar\sin\omega_z T}}\,\frac{m\Omega}{2\pi\,\mathrm{i}\,\hbar\sin\Omega T}\exp\left[-\frac{m\Omega}{2\,\mathrm{i}\,\hbar}(r'^2 + r''^2)\cot\Omega T\right]$$

$$\times \exp\left\{-\frac{m\omega_z}{2\,\mathrm{i}\,\hbar\sin\omega_z T}\left[(z'^2 + z''^2)\cos\omega_z T - 2z'z''\right]\right\}$$

$$\times \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\,\nu(\varphi'' - \varphi' + \omega T)} I_\nu\left(\frac{m\Omega r'r''}{\mathrm{i}\,\hbar\sin\Omega T}\right) , \tag{6.2.61}$$

$$= \sum_{n_r, n_z \in \mathbb{N}_0} \sum_{\nu \in \mathbb{Z}} \Psi_{n_r,\nu,n_z}(r'', \varphi'', z'') \Psi^*_{n_r,\nu,n_z}(r', \varphi', z')$$

$$\times \exp\left\{ -\mathrm{i}\, T\Big[\Omega(2n_r + |\nu| + 1) + \omega_z(n_z + \tfrac{1}{2}) - \nu\omega \Big] \right\} , \tag{6.2.62}$$

$$\Psi_{n_r,\nu,n_z}(r, \varphi, z) = \left(\frac{m\Omega}{\pi\hbar} \frac{n_r!}{(n_r + |\nu|)!} \right)^{1/2} \left(\frac{m\Omega}{\hbar} r^2 \right)^{|\nu|/2}$$

$$\times \exp\left(\mathrm{i}\,\nu\varphi - \frac{m\Omega}{2\hbar} r^2 \right) L_{n_r}^{(|\nu|)}\left(\frac{m\Omega}{\hbar} r^2 \right)$$

$$\times \left(\frac{m\omega_z}{(2^{n_z} n_z!)^2 \pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega_z}{2\hbar} z^2 \right) H_{n_z}\left(\sqrt{\frac{m\omega_z}{\hbar}}\, z \right) . \tag{6.2.63}$$

6.2.1.15 Motion in a Penning Trap Potential.
$(\mathbf{x} = (x, y, z),\ \omega = eB/mc,\ \Omega^2 = \omega^2/4 - \omega_0^2/2,\ x = \varrho\cos\varphi,\ y = \varrho\sin\varphi)$

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{ \frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \left[\dot{\mathbf{x}}^2 - \omega_0^2\left(z^2 - \frac{x^2 + y^2}{2} \right) - \omega(\dot{x}y - \dot{y}x) \right] \mathrm{d}t \right\}$$

$$= \sqrt{\frac{m\omega_0}{2\pi\mathrm{i}\,\hbar\sin\omega_0 T}} \frac{m\Omega}{2\pi\mathrm{i}\,\hbar\sin\Omega T}$$

$$\times \exp\left\{ \frac{\mathrm{i}\,m\omega_0}{2\hbar\sin\omega_0 T} \Big[(z'^2 + z''^2)\cos\omega_0 T - 2z'z'' \Big] \right\}$$

$$\times \exp\left\{ \frac{\mathrm{i}\,m\Omega}{2\hbar\sin\Omega T} \left[(\varrho'^2 + \varrho''^2)\cos\Omega T - 2\varrho'\varrho''\cos\left(\varphi'' - \varphi + \frac{\omega T}{2} \right) \right] \right\} . \tag{6.2.64}$$

The trapping condition reads $\omega^2 > 2\omega_0^2$; otherwise, the radial path integral gives a radial repelling oscillator.

6.2.1.16 Harmonic Oscillator with Magnetic Field and Driving Force. [783]
$\Big[\mathbf{x} \in \mathbb{R}^2, \Omega^2 = \omega^2 + 4\omega^2, \omega = eB/mc, \mathbf{J} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\Big]$

$$\int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(T)=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_0^T \left(\frac{m}{2}\left(\dot{\mathbf{x}}^2 - \omega^2\mathbf{x}^2 + \omega\dot{\mathbf{x}}\mathbf{J}\mathbf{x} \right) + \mathbf{F}(t)\mathbf{x} \right) \mathrm{d}t \right]$$

$$= \frac{m\Omega}{4\pi\mathrm{i}\,\hbar\sin(\Omega T/2)} \exp\left(\frac{\mathrm{i}}{\hbar} R_{\mathrm{Cl}}(\mathbf{x}'', \mathbf{x}') \right) , \tag{6.2.65}$$

$$R_{\mathrm{Cl}}(\mathbf{x}'', \mathbf{x}') = \frac{m\Omega}{4} \cot\tfrac{\Omega T}{2} (\mathbf{x}''^2 + \mathbf{x}'^2) - \frac{m}{2}\Omega\mathbf{x}' \frac{\mathrm{e}^{-\mathbf{J}\omega T/2}}{\sin(\Omega T/2)} \mathbf{x}''$$

$$+\frac{1}{\sin(\Omega T/2)}\int_0^T \mathrm{d}t\left[\mathbf{x}'\sin\tfrac{\Omega T}{2}\,\mathrm{e}^{-\mathrm{J}\omega(T-t)/2}+\mathbf{x}''\sin\tfrac{\Omega}{2}(T-t)\,\mathrm{e}^{\mathrm{J}\omega t/2}\right]\mathbf{F}(t)$$
$$-\frac{2}{m\Omega\sin(\Omega T/2)}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\sin\tfrac{\Omega}{2}(T-t)\sin\tfrac{\Omega}{2}s\,\mathbf{F}(t)\,\mathrm{e}^{-\mathrm{J}\Omega(t-s)/2}\,\mathbf{F}(s)\,. \tag{6.2.66}$$

6.2.1.17 Linear Fokker-Planck Process. [349,777,922–924,927]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[-\frac{1}{4D}\int_{t'}^{t''}(\dot{x}+\gamma x)^2\mathrm{d}t\right]$$
$$=\left(2\pi\frac{D}{\gamma}(1-\mathrm{e}^{-2\gamma T})\right)^{-1/2}\exp\left(-\frac{(x''-x'\,\mathrm{e}^{-\gamma T})^2}{2D(1-\mathrm{e}^{-2\gamma T})/\gamma}\right)\,. \tag{6.2.67}$$

6.2.1.18 Motion in a Constant Force Field with Constant Friction. [473,900]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{\mathrm{MP}}x(t)\,\mathrm{e}^{\gamma t/2}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{x}-\frac{g}{\gamma}\right)^2\mathrm{e}^{\gamma t}\right]\mathrm{d}t\right\}$$
$$=\sqrt{\frac{m\gamma}{2\pi\mathrm{i}\hbar(\mathrm{e}^{-\gamma t'}-\mathrm{e}^{-\gamma t''})}}\exp\left\{\frac{\mathrm{i}}{\hbar}\frac{m[\gamma(x''-x')-g(t''-t')]^2}{2\gamma(\mathrm{e}^{-\gamma t'}-\mathrm{e}^{-\gamma t''})}\right\}\,, \tag{6.2.68}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}k\,\exp\left[-\frac{\mathrm{i}\hbar k^2}{2m\gamma}\left(\mathrm{e}^{-\gamma t'}-\mathrm{e}^{-\gamma t''}\right)-\mathrm{i}\frac{pgT}{\gamma}+\mathrm{i}\,p(x''-x')\right]\,. \tag{6.2.69}$$

6.2.1.19 Harmonic Oscillator with Damping. [22,63,163,199,306,381,411,415, 542,552,708,748,749,915] ($\omega_\gamma^2=\omega^2-\gamma^2/4$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{\mathrm{MP}}x(t)\,\mathrm{e}^{\gamma t/2}\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2-\omega^2x^2)\,\mathrm{e}^{\gamma t}\,\mathrm{d}t\right]$$
$$=\sqrt{\frac{m\omega_\gamma}{2\pi\mathrm{i}\hbar\sin\omega_\gamma T}}$$
$$\times\exp\left\{\frac{\mathrm{i}m\omega_\gamma}{2\hbar\sin\omega_\gamma T}\left[\left(x'^2\,\mathrm{e}^{\gamma t'}+x''^2\,\mathrm{e}^{\gamma t''}\right)\cos\omega_\gamma T-2x'x''\,\mathrm{e}^{\gamma(t'+t'')/2}\right]\right.$$
$$\left.+\frac{\mathrm{i}m\gamma}{4\hbar}\left(x''^2\,\mathrm{e}^{\gamma t''}-x'^2\,\mathrm{e}^{\gamma t'}\right)+\gamma\frac{t'+t''}{4}\right\}\,. \tag{6.2.70}$$

6.2.1.20 Inverted Oscillator with Constant Friction. [63] ($\omega_\gamma^2 = \omega^2 + \gamma^2/4$, $\varphi = \operatorname{artanh}\gamma/2\omega_\gamma$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}x(t)\, \mathrm{e}^{\gamma t/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_0^T (\dot{x}^2 + \omega^2 x^2)\, \mathrm{e}^{\gamma t}\, \mathrm{d}t\right]$$
$$= \sqrt{\frac{m\omega_\gamma}{2\pi\,\mathrm{i}\,\hbar \sinh \omega_\gamma T}}$$
$$\times \exp\left[\frac{\mathrm{i}\, m\omega}{2\hbar \sinh\omega_\gamma T \cosh(\omega_\gamma T + \varphi)}\left(\mathrm{e}^{\gamma T}\left(\cosh^2\varphi + \sinh^2\omega_\gamma T\right) x'^2\right.\right.$$
$$\left.\left. + \cosh^2(\omega_\gamma T + \varphi) x''^2 - 2x'x''\, \mathrm{e}^{\gamma T/2} \cosh\varphi \cosh(\omega_\gamma T + \varphi)\right) + \frac{\gamma T}{4}\right]\ . \tag{6.2.71}$$

6.2.1.21 Harmonic Oscillator with Complex Friction. [63] ($\omega_\gamma^2 = \omega^2 - \gamma^2/4$, $\gamma = \gamma_1 + \mathrm{i}\,\gamma_2$, $\varphi = \arctan\gamma/2\omega_\gamma$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}x(t)\, \mathrm{e}^{\gamma t/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_0^T (\dot{x}^2 - \omega^2 x^2)\, \mathrm{e}^{\gamma t}\, \mathrm{d}t\right]$$
$$= \sqrt{\frac{m\omega_\gamma}{2\pi\,\mathrm{i}\,\hbar \sin \omega_\gamma T}}$$
$$\times \exp\left[\frac{\mathrm{i}\, m\omega}{2\hbar \sin\omega_\gamma T \cos(\omega_\gamma T - \varphi)}\left(\mathrm{e}^{\gamma T}\left(\cos^2\varphi - \sin^2\omega_\gamma T\right) x'^2\right.\right.$$
$$\left.\left. + \cos^2(\omega_\gamma T - \varphi) x''^2 - 2x'x''\, \mathrm{e}^{\gamma T/2} \cos\varphi \cos(\omega_\gamma T - \varphi)\right) + \frac{\gamma T}{4}\right]\ . \tag{6.2.72}$$

6.2.1.22 Damped Harmonic Oscillator in a Uniform Magnetic Field. [542] ($\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\omega_l = eB/2mc$, $\Omega^2 = \omega_l^2 + \omega^2 - \gamma^2/4$, $\Omega_3 = \omega^2 - \gamma^2/4$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\mathrm{MP}}\mathbf{x}(t)\, \mathrm{e}^{\gamma t/2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}(\dot{\mathbf{x}}^2 - \omega^2\mathbf{x}^2) + \frac{e}{c}\mathbf{B}\times\mathbf{x}\right) \mathrm{e}^{\gamma t}\, \mathrm{d}t\right]$$
$$= \left(\frac{m\Omega_3}{2\pi\,\mathrm{i}\,\hbar \sin\Omega_3 T}\right)^{1/2} \frac{m\Omega\, \mathrm{e}^{3\gamma(t'+t'')/4}}{2\pi\,\mathrm{i}\,\hbar\sin\Omega T} \exp\left[-\frac{\mathrm{i}\, m\gamma}{4\hbar}\left(\mathrm{e}^{\gamma t'} x'^2 - \mathrm{e}^{\gamma t''} x''^2\right)\right]$$

$$\times \exp\left\{-\frac{m\Omega_3}{2\,\mathrm{i}\,\hbar\sin\Omega_3 T}\left[\left(x_3''^2\,\mathrm{e}^{\gamma t''}+x_3'^2\,\mathrm{e}^{\gamma t'}\right)\cos\Omega_3 T-2x_3'x_3''\,\mathrm{e}^{\frac{\gamma}{2}(t'+t'')}\right]\right\}$$

$$\times \exp\left\{-\frac{m\Omega}{2\,\mathrm{i}\,\hbar\sin\Omega T}\left[\left((x_1'^2+x_2''^2)\,\mathrm{e}^{\gamma t''}+(x_1'^2+x_2'^2)\,\mathrm{e}^{\gamma t'}\right)\cos\Omega T\right.\right.$$

$$\left.\left.-2\,\mathrm{e}^{\frac{\gamma}{2}(t'+t'')}\left[\cos\omega_l T(x_1'x_1''+x_2'x_2'')+\sin\omega_l T(x_1'x_2''-x_2'x_1'')\right]\right\},\tag{6.2.73}$$

$$=\sum_{l\in\mathbb{Z}}\sum_{m,n\in\mathbb{N}_0}\Psi_{l,m,n}(\mathbf{Q}'')\Psi_{l,m,n}^*(\mathbf{Q}')\,\mathrm{e}^{-\,\mathrm{i}\,T(E_{nl}+E_m)/\hbar}\quad,\tag{6.2.74}$$

with the wave functions ($Q_1=r\cos\varphi, Q_2=r\sin\varphi, Q_3=Q_3$, $\mathbf{Q}=\mathrm{e}^{\gamma t/2}\,\mathbf{x}$)

$$\Psi_{l,m,n}(\mathbf{Q})=\mathrm{e}^{3\gamma t/4-(\mathrm{i}\,m\gamma/4\hbar)Q^2}\,F_1(Q_1,Q_2)F_3(Q_3)\quad,\tag{6.2.75}$$

$$F_1(Q_1,Q_2)=\sqrt{\frac{m\Omega}{\pi\hbar}\cdot\frac{n!}{(n+|l|)!}}\left(\frac{m\Omega}{\hbar}r^2\right)^{|l|/2}$$

$$\times\exp\left(-\frac{m\Omega}{2\hbar}r^2\right)L_n^{(|l|)}\left(\frac{m\Omega}{\hbar}r^2\right)\mathrm{e}^{\mathrm{i}\,l\varphi}\quad,\tag{6.2.76}$$

$$F_3(Q_3)=\sqrt{\sqrt{\frac{m\Omega_3}{\pi\hbar}}\cdot\frac{1}{2^m m!}}\exp\left(-\frac{m\Omega_3}{2\hbar}Q_3^2\right)H_m\left(\sqrt{\frac{m\Omega_3}{\hbar}}Q_3\right)\quad,\tag{6.2.77}$$

$$E_{n,l}=\hbar\Omega(2n+|l|+1)+\hbar\omega_l|l|\quad,\qquad E_m=\hbar\Omega_3(m+\tfrac{1}{2})\quad.\tag{6.2.78}$$

6.2.1.23 Damped Harmonic Oscillator with Time-Dependent Driving Force. [1,6,165,542]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{\mathrm{MP}}x(t)e^{\gamma t/2}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{x}^2-\omega^2x^2)+q(t)x\right)\mathrm{e}^{\gamma t}\,\mathrm{d}t\right]$$

$$=\left(\frac{m\Omega}{2\pi\,\mathrm{i}\,\hbar\sin\Omega T}\right)^{1/2}\exp\left\{\frac{m\Omega}{2\,\mathrm{i}\,\hbar\sin\Omega T}\left[2(x'-x_0')(x''-x_0')\,\mathrm{e}^{\gamma(t'+t'')/2}\right.\right.$$

$$\left.\left.-\left((x''-x_0'')^2\,\mathrm{e}^{\gamma t''}+(x'-x_0')^2\,\mathrm{e}^{\gamma t'}\right)\cos\Omega T\right]\right\}$$

$$\times\exp\left\{-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\Delta(t)-\Delta^*(t)\right]\mathrm{d}t-\frac{\mathrm{i}\,m\gamma}{4\hbar}\left(x''^2\,\mathrm{e}^{\gamma t''}-x'^2\,\mathrm{e}^{\gamma t'}\right)\right.$$

$$\left.-\frac{\mathrm{i}}{\hbar}\left[x''\left(p_0(t'')+\frac{m}{2}\gamma x_0''\,\mathrm{e}^{\gamma t''}\right)-x'\left(p_0(t')+\frac{m}{2}\gamma x_0'\,\mathrm{e}^{\gamma t'}\right)\right]\right\}.\tag{6.2.79}$$

Here we denote $\Omega^2 = \omega^2 - \gamma^2/4$, $p_0(t) = m\dot{x}_0(t)\,\mathrm{e}^{\gamma t}$, with $x_0(t)$ a solution of the equation $m\ddot{x}_0(t) + \gamma x_0(t) + m\omega^2 x_0(t) = q(t)$, and $\Delta(t)$ is given by

$$\Delta(t) = \frac{m}{2}(\dot{x}^2 - \omega^2)\,\mathrm{e}^{\gamma t} + \frac{m}{2}\gamma\dot{x}_0(t)x_0(t)\,\mathrm{e}^{\gamma t} + \frac{\mathrm{i}}{4}\gamma \ . \tag{6.2.80}$$

6.2.1.24 Forced Harmonic Oscillator with Time-Dependent Frequency, Damping and Driving Force. [6,44,130,165,321,401,473,542,552,578–581,901,937]

$$\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}x(t)\,\mathrm{e}^{\gamma t/2}\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{x}^2 - \omega^2(t)x^2) + q(t)x\right)\mathrm{e}^{\gamma(t)}\mathrm{d}t\right] \\
&= \frac{\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(x'',x')\right)}{\sqrt{\mathrm{i}\,\pi\hbar[x_2(t'')x_1(t') - x_1(t'')x_2(t')]}} \ ,
\end{aligned} \tag{6.2.81}$$

with $x_1(t), x_2(t)$ determined by

$$\ddot{x}_1 + \gamma(t)\dot{x}_1 + \omega^2(t)x_1 = 0 \ , \qquad x_2(t) = \frac{2x_1(t)}{m}\int^t \frac{\mathrm{e}^{-\gamma s}}{x_1^2(s)}\mathrm{d}s \ . \tag{6.2.82}$$

The classical action has the form

$$\begin{aligned}
R_{\mathrm{Cl}}(x'',x') = {}& \frac{1}{x_2(t'')x_1(t') - x_1(t'')x_2(t')} \\
&\times \left\{ \frac{m}{2}\Big[\dot{x}_2(t'')x_1(t') - \dot{x}_1(t'')x_2(t')\Big]x''^2\,\mathrm{e}^{\gamma(t'')} \right. \\
&- 2x'x'' + \frac{m}{2}[\dot{x}_2(t'')x_1(t') - \dot{x}_1(t'')x_2(t')]x'^2\,\mathrm{e}^{\gamma(t')} \\
&+ [x''x_1(t') - x'x_1(t'')]\int_{t'}^{t''} q(t)x_2(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t \\
&\qquad - [x''x_2(t') - x'x_2(t'')]\int_{t'}^{t''} q(t)x_1(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t \\
&+ \frac{1}{4}x_1(t'')x_1(t')\left(\int_{t'}^{t''} q(t)x_2(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t\right)^2 \\
&+ \frac{1}{4}x_2(t'')x_2(t')\left(\int_{t'}^{t''} q(t)x_1(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t\right)^2 \\
&- \frac{1}{2}x_1(t'')x_2(t')\int_{t'}^{t''} q(t)x_2(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t\int_{t'}^{t} q(s)x_1(s)\,\mathrm{e}^{\gamma(s)}\mathrm{d}s \\
&\left. -\frac{1}{2}x_2(t'')x_1(t')\int_{t'}^{t''} q(t)x_1(t)\,\mathrm{e}^{\gamma(t)}\mathrm{d}t\int_{t'}^{t} q(s)x_2(s)\,\mathrm{e}^{\gamma(s)}\mathrm{d}s \right\} \ .
\end{aligned} \tag{6.2.83}$$

6.2.1.25 Generalized Quadratic Parametric Oscillator. [61,321,681,941]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}_{\mathrm{MP}}(x(t),p(t)) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \left[p\dot{x} - \frac{1}{2}\left(\frac{Z(t)}{m}p^2 + \omega Y(t)(px+xp)\right.\right.\right.$$

$$\left.\left.\left. + X(t)m\omega^2 x^2 - \mu(t)x - \nu(t)p\right)\right]\mathrm{d}t\right\}$$

$$= \frac{1}{2\sqrt{\pi c(T)}} \exp\left\{\Lambda(T) + a(T)x''^2 + h(t)\,\mathrm{e}^{b(T)}\,x'' + c(T)h^2(T)\right.$$

$$\left. - \frac{1}{4c(T)}\left[\mathrm{e}^{b(T)}\,x'' - x' + f(T) + 2h(T)c(T)\right]^2\right\} , \tag{6.2.84}$$

where denote ($X(t)$, $Y(t)$, $Z(t)$ are complex valued)

$$a(T) = \frac{\mathrm{i}}{Z(T)} \times \left[\frac{\dot{\nu}_1(0)\dot{\nu}_2(T) - \dot{\nu}_1(T)\dot{\nu}_2(0) + Y(0)\big[\dot{\nu}_1(T)\nu_2(0) - \nu_1(0)\dot{\nu}_2(T)\big]}{\dot{\nu}_1(0)\nu_2(T) - \nu_1(T)\dot{\nu}_2(0) + Y(0)\big[\nu_1(T)\nu_2(0) - \nu_1(0)\nu_2(T)\big]} - Y(T)\right], \tag{6.2.85a}$$

$$b(T) = -\ln\frac{\dot{\nu}_1(0)\nu_2(T) - \nu_1(T)\dot{\nu}_2(0) + Y(0)\big[\nu_1(T)\nu_2(0) - \nu_1(0)\nu_2(T)\big]}{\dot{\nu}_1(0)\nu_2(0) - \nu_1(0)\dot{\nu}_2(0)} , \tag{6.2.85b}$$

$$c(T) = \frac{\mathrm{i}}{2}\int_0^T \frac{\mathrm{d}t\, Z(t)(\dot{\nu}_1(0)\dot{\nu}_2(0) - \nu_1(0)\dot{\nu}_2(0))}{\dot{\nu}_1(0)\nu_2(t) - \nu_1(t)\dot{\nu}_2(0) + Y(0)\big[\nu_1(t)\nu_2(0) - \nu_1(0)\nu_2(t)\big]}, \tag{6.2.85c}$$

$$\Lambda(T) = \frac{1}{2}b(T) + g(T) , \tag{6.2.85d}$$

$$h(T) = -\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{e}^{-b(t)}\,\big[\mu(t) - 2\,\mathrm{i}\,\hbar\nu(t)a(t)\big]\mathrm{d}t , \tag{6.2.85e}$$

$$f(T) = -\frac{\mathrm{i}}{\hbar}\int_0^T \Big\{\hbar\nu(t)\,\mathrm{e}^{b(t)} - 2\,\mathrm{i}\,c(t)\,\mathrm{e}^{-b(t)}\,\big[\mu(t) - 2\,\mathrm{i}\,\hbar\nu(t)a(t)\big]\Big\}\mathrm{d}t , \tag{6.2.85f}$$

$$g(T) = \int_0^T h(t)\dot{f}(t)\,\mathrm{d}t . \tag{6.2.85g}$$

$\nu_{1,2}(t)$ are two independent solutions of the differential equation

$$\ddot{\nu}(t) - \frac{\dot{Z}(t)}{Z(t)}\dot{\nu}(t) - \left[\omega^2 Y^2(t) - m^2\omega^2 X(t)Z(t) + \frac{Z(t)}{\omega}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{Y(t)}{Z(t)}\right)\right]\nu(t) = 0 . \tag{6.2.86}$$

6.2.1.26 Damped, Forced and Inverted Oscillator. [61,62] For the special case $Z(t) = \mathrm{e}^{-\gamma t}$, $Y(t) = 0$, $X(t) = -\mathrm{e}^{-\gamma t}$, $\mu(t) = \lambda \cos \tilde{\Omega} t$, $\nu(t) = 0$ one finds in (6.2.84)

$$a(T) = \frac{\mathrm{i}\, m\omega}{2\hbar} \frac{\sinh \Omega T}{\cosh(\Omega T + \varphi)} \mathrm{e}^{\gamma T} , \tag{6.2.87a}$$

$$b(T) = \frac{\gamma T}{2} - \ln \frac{\cosh(\Omega T + \varphi)}{\cosh \varphi} , \quad c(T) = \frac{\mathrm{i}\,\hbar}{2m\omega} \frac{\sinh \Omega T}{\cosh(\Omega T + \varphi)} , \tag{6.2.87b}$$

$$h(T) = \frac{\mathrm{i}\,\lambda}{4\hbar \cosh \varphi} W(T) , \qquad f(T) = \frac{1}{4m\omega \cosh \varphi} R(T) , \tag{6.2.87c}$$

$$\Omega^2 = \omega^2 + \frac{\gamma^2}{4} , \qquad \varphi = \operatorname{artanh} \frac{\gamma}{2\Omega} , \qquad \cosh \varphi = \frac{\Omega}{\omega} , \tag{6.2.87d}$$

$$\begin{aligned} W(T) = {} & \frac{\mathrm{e}^{\varphi}\left[1 - \mathrm{e}^{(\Omega + \mathrm{i}\tilde{\Omega} - \gamma/2)T}\right]}{\Omega + \mathrm{i}\tilde{\Omega} - \gamma/2} - \frac{\mathrm{e}^{-\varphi}\left[1 - \mathrm{e}^{-(\Omega - \mathrm{i}\tilde{\Omega} + \gamma/2)T}\right]}{\Omega - \mathrm{i}\tilde{\Omega} + \gamma/2} \\ & + \frac{\mathrm{e}^{\varphi}\left[1 - \mathrm{e}^{(\Omega - \mathrm{i}\tilde{\Omega} - \gamma/2)T}\right]}{\Omega - \mathrm{i}\tilde{\Omega} - \gamma/2} - \frac{\mathrm{e}^{-\varphi}\left[1 - \mathrm{e}^{-(\Omega + \mathrm{i}\tilde{\Omega} + \gamma/2)T}\right]}{\Omega + \mathrm{i}\tilde{\Omega} + \gamma/2} , \end{aligned} \tag{6.2.87e}$$

$$\begin{aligned} R(T) = {} & \frac{1 - \mathrm{e}^{(\Omega + \mathrm{i}\tilde{\Omega} - \gamma/2)T}}{\Omega + \mathrm{i}\tilde{\Omega} - \gamma/2} + \frac{1 - \mathrm{e}^{-(\Omega - \mathrm{i}\tilde{\Omega} + \gamma/2)T}}{\Omega - \mathrm{i}\tilde{\Omega} + \gamma/2} \\ & + \frac{1 - \mathrm{e}^{(\Omega - \mathrm{i}\tilde{\Omega} - \gamma/2)T}}{\Omega - \mathrm{i}\tilde{\Omega} - \gamma/2} + \frac{1 - \mathrm{e}^{-(\Omega + \mathrm{i}\tilde{\Omega} + \gamma/2)T}}{\Omega + \mathrm{i}\tilde{\Omega} + \gamma/2} . \end{aligned} \tag{6.2.87f}$$

6.2.1.27 Driven Coupled Harmonic Oscillators. [230,708] ($\mathbf{x} = (x_1, x_2) \in \mathrm{I\!R}^2$)

$$\begin{aligned} & \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\sum_{j=1,2} \frac{m_j}{2} \left(\dot{x}_j^2 - \omega_j^2 x_j^2 + 2\frac{f_j(t)}{m_j} x_j \right) - \lambda x_1 x_2 \right] \mathrm{d}t \right\} \\ & = \frac{1}{2\pi \mathrm{i}\, \hbar} \left[\frac{m_1 m_2 \Omega_1 \Omega_2}{\sin \Omega_1 T \sin \Omega_2 T} \right]^{1/2} \exp\left[\frac{\dot{\eta}_1}{\mathrm{i}\hbar} \left(\frac{\eta_1}{2} + \sqrt{m_1}\, C x_1 - \sqrt{m_2}\, S x_2 \right) \Bigg|_{t'}^{t''} \right] \\ & \quad \times \exp\left\{ \frac{\mathrm{i}\,\Omega_1}{2\hbar \sin \Omega_1 T} \left[\cos \Omega_1 T \Big(m_1 C^2 x_1''^2 + m_2 S^2 x_2''^2 - 2\sqrt{m_1 m_2}\, S C x_1'' x_2'' \right.\right. \\ & \qquad + \eta_1''^2 + 2\sqrt{m_1}\, C \eta_1'' x_1'' - 2\sqrt{m_2}\, S \eta_1'' x_2'' + (x_1'', x_2'', \eta_1'') \to (x_1', x_2', \eta_1') \Big) \\ & \qquad - 2m_1 C^2 x_1' x_1'' + 2\sqrt{m_1 m_2}\, S C x_1' x_2'' - 2\sqrt{m_1} C \eta_1'' x_1' + 2\sqrt{m_1 m_2}\, S C x_2' x_1'' \\ & \qquad \left.\left. - 2m_2 S^2 x_2' x_2'' + 2\sqrt{m_2}\, S \eta_1' x_2'' - 2\sqrt{m_1}\, C \eta_1' x_1'' + 2\sqrt{m_2}\, S \eta_1'' x_2' - 2\eta_1' \eta_1'' \right] \right\} \end{aligned}$$

$$\times \exp\left\{\text{The above expression with } 1 \leftrightarrow 2 \text{ and } S \to -S\right\}$$

$$\times \exp\left\{-\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t''}\left[\frac{f_1(t)}{\sqrt{m_1}}(C+S)+\frac{f_2(t)}{\sqrt{m_2}}(C-S)\right]\mathrm{d}t\right\} \ , \tag{6.2.88}$$

$$= \sum_{n_1,n_2\in\mathbb{N}_0} \Psi_{n_1,n_2}(x_1'',x_2'')\Psi^*_{n_1,n_2}(x_1',x_2')\,\mathrm{e}^{-\mathrm{i}TE_{n_1,n_2}/\hbar} \ . \tag{6.2.89}$$

Here $S=\sin\varphi$, $C=\cos\varphi=\sqrt{(1+R)/2}$, and

$$R=\frac{\sqrt{m_1m_2(\omega_2^2-\omega_1^2)^2}}{\sqrt{4\lambda^2+m_2m_2(\omega_2^2-\omega_1^2)^2}} \ , \tag{6.2.90a}$$

$$\Omega_{1,2}^2=\frac{1}{2}\left(\omega_1^2+\omega_2^2\mp\sqrt{(\omega_2^2-\omega_1^2)^2+\frac{4\lambda^2}{m_1m_2}}\right) \ , \tag{6.2.90b}$$

$$\eta_i(t)=\frac{1}{\sin\Omega_i t}\left\{\int_{t'}^{t}F_i(s)\sin[\Omega_i(s-t')]\sin[\Omega_i(t''-s)]\,\mathrm{d}s\right.$$
$$\left.+\int_{t}^{t''}F_i(s)\sin[\Omega_i(s-t')]\sin[\Omega_i(t''-s)]\,\mathrm{d}s\right\} \ , \tag{6.2.90c}$$

$$F_1=\sqrt{\frac{1}{m_1}}f_1\cos\varphi-\sqrt{\frac{1}{m_2}}f_2\sin\phi\,,\quad F_2=\sqrt{\frac{1}{m_1}}f_1\sin\varphi-\sqrt{\frac{1}{m_2}}f_2\cos\varphi\,. \tag{6.2.90d}$$

The energy spectrum is given by

$$E_{n_1,n_2}=\hbar\Omega_1(n_1+\tfrac{1}{2})+\hbar\Omega_2(n_2+\tfrac{1}{2}) \ . \tag{6.2.91}$$

The wave functions are

$$\Psi_{n_1,n_2}=\frac{\left[m_1m_2\Omega_1\Omega_2/\pi^2\hbar^2\right]^{1/4}}{\sqrt{2^{n_1+n_2}n_1!n_2!}}\exp\left\{\frac{-1}{2\hbar}\left[\Omega_1(\sqrt{m_1}\,Cx_1-\sqrt{m_2}\,Sx_2+\eta_1)^2\right.\right.$$
$$+\Omega_2(\sqrt{m_1}\,Sx_1+\sqrt{m_2}\,Cx_2+\eta_2)^2+\mathrm{i}\,\dot\eta_1(\eta_1-2\sqrt{m_1}\,Cx_1+2\sqrt{m_2}\,Sx_2)$$
$$\left.\left.+\mathrm{i}\,\dot\eta_2(\eta_2-2\sqrt{m_1}\,Sx_1-2\sqrt{m_2}\,Cx_2)\right]\right\}$$
$$\times\exp\left\{-\frac{\mathrm{i}}{2\hbar}\int^{t}\left[\frac{\eta_1}{\sqrt{m_1}}(f_1C-f_2S)+\frac{\eta_2}{\sqrt{m_2}}(f_1S+f_2C)\right]\mathrm{d}t\right\}$$
$$\times H_{n_1}\left[\sqrt{\frac{\Omega_1}{\hbar}}\,(\sqrt{m_1}\,Cx_1-\sqrt{m_2}\,Sx_2+\eta_1)\right]$$
$$\times H_{n_2}\left[\sqrt{\frac{\Omega_2}{\hbar}}\,(\sqrt{m_1}\,Sx_1+\sqrt{m_2}\,Cx_2+\eta_1)\right] \ . \tag{6.2.92}$$

6.2.1.28 Coupled Harmonic Oscillators. [2,150,230,942] ($\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\sum_{j=1,2}\frac{m}{2}(\dot{x}_j^2-\omega_j^2x_j^2)-\lambda x_1x_2\right)\mathrm{d}t\right]$$

$$\begin{aligned}
&=\frac{m}{2\pi\mathrm{i}\hbar}\left[\frac{\Omega_1\Omega_2}{\sin\Omega_1T\sin\Omega_2T}\right]^{1/2}\\
&\quad\times\exp\left\{\frac{\mathrm{i}m\Omega_1}{4\hbar\sin\Omega_1T}\left[\cos\Omega_1T\Big(x_1''^2+x_2''^2+x_1'^2+x_2'^2\right.\right.\\
&\qquad\left.\left.-2(x_1''x_2''+x_1'x_2')\Big)-2(x_1'x_1''+x_2'x_2''-x_1'x_2''-x_2'x_1'')\right]\right\}\\
&\quad\times\exp\left\{\frac{\mathrm{i}m\Omega_2}{4\hbar\sin\Omega_2T}\left[\cos\Omega_2T\Big(x_1''^2+x_2''^2+x_1'^2+x_2'^2\right.\right.\\
&\qquad\left.\left.+2(x_1''x_2''+x_1'x_2')\Big)-2(x_1'x_1''+x_2'x_2''+x_1'x_2''+x_2'x_1'')\right]\right\}\ ,
\end{aligned}\tag{6.2.93}$$

$$=\sum_{n_1,n_2\in\mathbb{N}_0}\Psi_{n_1,n_2}(x_1'',x_2'')\Psi_{n_1,n_2}(x_1',x_2')\,\mathrm{e}^{-\mathrm{i}TE_{n_1,n_2}/\hbar}\ .\tag{6.2.94}$$

Here $\Omega_{1,2}^2=\omega_{1,2}^2-\lambda/m$, with the same energy spectrum as in (6.2.91), and the wave functions are

$$\begin{aligned}
\Psi_{n_1,n_2}&=\sqrt{\frac{m\sqrt{\Omega_1\Omega_2}}{\pi\hbar 2^{n_1+n_2}n_1!n_2!}}\exp\left[-\frac{m}{4\hbar}\Big(\Omega_1(x_1-x_2)^2+\Omega_2(x_1+x_2)^2\Big)\right]\\
&\quad\times H_{n_1}\left(\sqrt{\frac{m\Omega_1}{2\hbar}}\,(x_1-x_2)\right)H_{n_2}\left(\sqrt{\frac{m\Omega_2}{2\hbar}}\,(x_1+x_2)\right)\ .
\end{aligned}\tag{6.2.95}$$

6.2.1.29 Damped Harmonic Oscillator Coupled to its Dual. [181] ($r=\bar{x}-x$, $R=x\,\mathrm{e}^{\lambda t}+\bar{x}\,\mathrm{e}^{-\lambda t}$, $\Omega=\sqrt{\omega^2-\lambda^2}$, $N=n_1-n_2\in\mathbb{Z}$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{\bar{x}(t')=\bar{x}'}^{\bar{x}(t'')=\bar{x}''}\mathcal{D}\bar{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Big(\dot{x}\dot{\bar{x}}+\lambda(x\dot{\bar{x}}-\dot{x}\bar{x})-\omega^2x\bar{x}\Big)\mathrm{d}t\right]$$

$$\begin{aligned}
&=\sqrt{\frac{\Omega}{\pi\mathrm{i}\hbar\sin\Omega T}}\exp\left[\frac{\mathrm{i}\Omega}{\hbar\sin\Omega T}\Big((R''^2+R'^2)\cos\Omega T-2R'R''\Big)\right]\\
&\quad\times\sqrt{\frac{\mathrm{i}\Omega}{4\pi\hbar\sin\Omega T}}\exp\left[-\frac{\mathrm{i}\Omega}{4\hbar\sin\Omega T}\Big((r''^2+r'^2)\cos\Omega T-2r'r''\Big)\right]\ ,
\end{aligned}\tag{6.2.96}$$

$$= \sum_{n_1,n_2\in\mathbb{N}_0} \Psi_{n_1,n_2}(r'',R'')\Psi_{n_1,n_2}(r',R')\,\mathrm{e}^{-\mathrm{i}T\omega N} \ , \tag{6.2.97}$$

$$\Psi_{n_1,n_2}(r,R) = \sqrt{\frac{\hbar}{\pi\Omega n_1!n_2!2^{n_1+n_2}}} \\ \times \exp\left[-\frac{\Omega}{\hbar}\left(R^2+\frac{r^2}{4}\right)\right] H_{n_1}\left(\sqrt{\frac{2\Omega}{\hbar}}R\right) H_{n_2}\left(\sqrt{\frac{\Omega}{2\hbar}}r\right) \ . \tag{6.2.98}$$

6.2.1.30 Temporal Non-Local Oscillator. [834]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_0^T \left[\dot{x}(t)^2-\omega^2x(t)^2-\alpha x(t+\tfrac{\tau}{2})x(t-\tfrac{\tau}{2})\right]\mathrm{d}t\right\}$$

$$= \left(\frac{m}{2\pi\mathrm{i}\hbar}\frac{\tilde{\omega}^2+\omega^2+\alpha}{2\tilde{\omega}\sin\tilde{\omega}T}\right)^{1/2} \\ \times \exp\left\{-\frac{m}{2\mathrm{i}\hbar}\frac{\tilde{\omega}^2+\omega^2+\alpha}{2\tilde{\omega}\sin\tilde{\omega}T}\left[({x'}^2+{x''}^2)\cos\tilde{\omega}T-2x'x''\right]\right\} \ . \tag{6.2.99}$$

Here $\tilde{\omega}$ is implicitly defined by the roots of the equation $y^2=\omega^2+\alpha\cos(y\tau)$ with solution $y=\tilde{\omega}$.

6.2.1.31 Harmonic Oscillator with Two-Time Quadratic Action, Magnetic Field and Driving Force. [5,21,93,113,121,148,149,161,185,195,218,250,332–340,342,354,474,576,584–588,591,667,680,751,784,813,843,891,902], in particular we cite [149,581,584] ($\mathbf{\Lambda}=e\mathbf{B}/mc$, $\mathbf{x}\in\mathbb{R}^3$)

$$\int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(T)=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(t)\exp\left\{-\frac{1}{\hbar}\int_0^T\left[\frac{m}{2}\left(\dot{\mathbf{x}}^2+\eta^2\mathbf{x}^2-\mathrm{i}(\dot{\mathbf{x}}\times\mathbf{\Lambda})\cdot\mathbf{x}\right)+\mathbf{f}(t)\cdot\mathbf{x}(t)\right]\mathrm{d}t \\ -\frac{ma}{4\hbar}\int_0^T\int_0^T\frac{\omega\cosh\omega(|t-s|-\frac{T}{2})}{\sinh(\omega T/2)}\mathbf{x}(t)\cdot\mathbf{x}(s)\,\mathrm{d}t\,\mathrm{d}s\right\}$$

$$= \left(\frac{m}{4\pi\hbar\omega}\right)^{3/2}\exp\left(-\frac{1}{\hbar}R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')\right)\frac{\sinh^3\frac{\omega T}{2}}{\sinh\frac{\Omega T}{2}\sinh\frac{\psi T}{2}\sinh\frac{\Phi_1 T}{2}\dots\sinh\frac{\Phi_4 T}{2}} \\ \times\left(\frac{\Omega^2-\omega^2}{\Omega^2-\psi^2}\frac{\coth\frac{\Omega T}{2}}{\Omega}+\frac{\psi^2-\omega^2}{\psi^2-\Omega^2}\frac{\coth\frac{\psi T}{2}}{\psi}\right)^{-1/2} \\ \times\left(\sum_{k=1}^4\frac{(\Phi_k^2-\omega^2)(\Phi_k^2-\Omega^2)(\Phi_k^2-\psi^2)}{\prod_{l\neq k}(\Phi_k^2-\Phi_l^2)}\frac{\coth\frac{\Phi_k T}{2}}{\Phi_k}\right)^{-1} \tag{6.2.100}$$

($\mathbf{x}, \mathbf{\Lambda}$ are three dimensional vectors). The classical action is given by (matrix multiplication understood)

$$\begin{aligned}
R_{\mathrm{Cl}}(\mathbf{x}'', \mathbf{x}') &= \frac{m}{2}\omega(\mathbf{x}'' - \mathbf{x}')\left(\frac{\ddot{\mathbf{\Delta}}(0)}{\omega^2} - \mathbf{\Gamma}^2\mathbf{\Delta}(0)\right)(\mathbf{x}'' - \mathbf{x}') \\
&+ \frac{m}{8}\omega(\mathbf{x}' + \mathbf{x}'')\mathbf{\Delta}^{-1}(0)(\mathbf{x}' + \mathbf{x}'') \\
&+ \frac{m}{4}\omega(\mathbf{x}'' - \mathbf{x}')\left(\mathbf{\Gamma} - \frac{\mathbf{L}}{\omega}\right)(\mathbf{x}' + \mathbf{x}'') - \frac{m}{4}\omega(\mathbf{x}' + \mathbf{x}'')\mathbf{\Gamma}(\mathbf{x}'' - \mathbf{x}') \\
&+ \frac{1}{2}\int_0^T (\mathbf{x}'' - \mathbf{x}')\left(\mathbf{\Gamma}\mathbf{\Delta}(T-t) - \frac{\dot{\mathbf{\Delta}}(T-t)}{\omega}\right)\mathbf{f}(t)\,\mathrm{d}t \\
&- \frac{1}{2}\int_0^T \mathbf{f}(t)\left(\mathbf{\Gamma}\mathbf{\Delta}(t) - \frac{\dot{\mathbf{\Delta}}(t)}{\omega}\right)(\mathbf{x}'' - \mathbf{x}')\mathrm{d}t \\
&+ \frac{1}{4}\int_0^T \Big((\mathbf{x}' + \mathbf{x}'')\mathbf{\Delta}^{-1}(0)\mathbf{\Delta}(T-t)\mathbf{f}(t) + \mathbf{f}(t)\mathbf{\Delta}^{-1}(0)\mathbf{\Delta}(t)(\mathbf{x}' + \mathbf{x}'')\Big)\mathrm{d}t \\
&+ \frac{1}{2m\omega}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathbf{f}(t)\Big[\mathbf{\Delta}^{-1}(0)\mathbf{\Delta}(t)\mathbf{\Delta}(T-s) - \mathbf{\Delta}(t,s)\Big]\mathbf{f}(s) \ .
\end{aligned} \tag{6.2.101}$$

Here $\pm\Omega$ and $\pm\Psi$ are solutions of the quartic equation $z^4 - (\eta^2 + \omega^2)z^2 + \omega^2(a+\eta^2) = 0$, and the Φ_k ($k = 1,2,3,4$) are solutions of the quartic equation $z^4 - \Lambda z^3 - (\eta^2 + \omega^2)z^2 + \omega^2\Lambda z + \omega^2(a+\eta^2) = 0$, where $\Lambda = |\mathbf{\Lambda}|$. Furthermore

$$\begin{aligned}
\Delta_\pm &= \pm\omega\sum_{k=1}^4 \frac{(\Phi_k^2 - \omega^2)\,\mathrm{e}^{\pm t\Phi_k}}{(\mathrm{e}^{\pm T\Phi_k} - 1)\prod_{l\neq k}(\Phi_k - \Phi_l)} \ , \\
\Delta_3 &= \frac{\omega}{2}\frac{\Omega^2 - \omega^2}{\Omega^2 - \Psi^2}\frac{\cosh[\Omega(t - T/2)]}{\Omega\sinh\Omega T/2} + \frac{\omega}{2}\frac{\Psi^2 - \omega^2}{\Psi^2 - \Omega^2}\frac{\cosh[\Omega(t - T/2)]}{\Psi\sinh\Psi T/2} \ ,
\end{aligned} \tag{6.2.102}$$

and the matrix $\boldsymbol{\Delta}$ is defined by $\boldsymbol{\Delta} = \mathbf{U}\,\mathrm{diag}(\Delta_+, \Delta_-, \Delta_3)\mathbf{U}^t$ with, e.g.,

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} & 0 \\ \mathrm{i} & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \ , \qquad \mathbf{L} = \mathrm{i}\begin{pmatrix} 0 & -\Lambda_z & \Lambda_y \\ \Lambda_z & 0 & -\Lambda_x \\ -\Lambda_y & \Lambda_x & 0 \end{pmatrix} \tag{6.2.103}$$

in the case that the magnetic field is parallel to the z-axis. In general $\mathbf{U}$ is some matrix diagonalizing $\mathbf{L}$. Finally ($\mathbf{\Gamma} = [\frac{1}{2}\,\mathbb{1} + \dot{\boldsymbol{\Delta}}(0)/\omega]\boldsymbol{\Delta}^{-1}(0)$)

$$\boldsymbol{\Delta}(t,s) = \begin{cases} \boldsymbol{\Delta}(t-s) & t \geq s \ , \\ \boldsymbol{\Delta}(T+t-s) & t \leq s \ . \end{cases} \tag{6.2.104}$$

6.2.1.32 Harmonic Oscillator with Two-Time Quadratic Action and Driving Force. [21,121,658] ($W=\sqrt{(\Omega^2-\omega^2)^2+16\hbar\omega C/m}, X'=\sqrt{m\omega/2\hbar}\,x', X''=\sqrt{m\omega/2\hbar}\,x''$)

$$\int\limits_{x(0)=x'}^{x(T)=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}(\dot{x}^2-\Omega^2x^2)+f(t)x\right)\mathrm{d}t\right.$$
$$\left.-2C\int_0^T\mathrm{d}t\int_0^T\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\omega(t-s)}\,x(t)x(s)\right]$$
$$=2\sqrt{\frac{m\hbar}{\omega}}\sqrt{\frac{\Omega_1}{2\pi\mathrm{i}\hbar\sin\Omega_1T}}\sqrt{\frac{\Omega_2}{2\pi\mathrm{i}\hbar\sin\Omega_2T}}\frac{\mathrm{e}^{\mathrm{i}\omega T/2-(X'^2+X''^2)\tan^2\theta}}{\sqrt{A^2(T)-B^2(T)}}$$
$$\times\exp\left[\frac{b_1(T)}{\cos^2\vartheta}\big((X'^2+X''^2)\cos\Omega_1-2X'X''\big)\right.$$
$$+\frac{2b_1(T)}{\Omega_1}\sqrt{\frac{\omega}{2m\hbar}}\int_0^T\mathrm{d}t\,f(t)\big[X'\sin\Omega_1t+X''\sin\Omega_1(T-t)\big]$$
$$-\frac{\omega b_1(T)\cos^2\vartheta}{m\hbar\Omega_1^2}\int_0^T\mathrm{d}t\int_0^T\mathrm{d}s\,f(t)f(s)\sin\Omega_1s\sin\Omega_1(T-t)$$
$$-\frac{\omega b_2(T)\sin^2\vartheta}{m\hbar\Omega_2^2}\int_0^T\mathrm{d}t\int_0^T\mathrm{d}s\,f(t)f(s)\sin\Omega_2s\sin\Omega_2(T-t)$$
$$\left.+\frac{A(T)\big[D^2(T)+D'^2(T)\big]-2B(T)D(T)D'(T)}{4\big[A^2(T)-B^2(T)\big]}\right]\,,\tag{6.2.105}$$

$$\sin\vartheta=-\sqrt{\frac{\omega^2-\Omega^2+W}{2W}}\,,\qquad\cos\vartheta=\sqrt{\frac{\Omega^2-\omega^2+W}{2W}}\,,\tag{6.2.106a}$$

$$\Omega_{1,2}^2=\frac{\omega^2+\Omega^2\pm W}{2}\,,\qquad b_{1/2}(T)=\frac{\mathrm{i}\,\Omega_{1/2}}{\omega\sin\Omega_{1/2}T}\,,\tag{6.2.106b}$$

$$A(T)=1-b_1(T)\sin^2\vartheta\cos\Omega_1T-b_2(T)\cos^2\vartheta\cos\Omega_2T\,,\tag{6.2.106c}$$

$$B(T)=b_1(T)\sin^2\vartheta+b_2(T)\cos^2\vartheta\,,\tag{6.2.106d}$$

$$D'(T)=2X'\tan\vartheta-2b_1(T)(X'\cos\Omega_1T-X'')\tan\vartheta$$
$$+\sqrt{\frac{2\omega}{m\hbar}}\sin\vartheta\cos\vartheta\int_0^T\mathrm{d}t\,f(t)\left(b_2(T)\frac{\sin\Omega_2t}{\Omega_2}-b_1(T)\frac{\sin\Omega_1t}{\Omega_1}\right)\,,\tag{6.2.106e}$$

$$D''(T)=2X''\tan\vartheta-2b_1(T)(X''\cos\Omega_1T-X')\tan\vartheta$$
$$+\sqrt{\frac{2\omega}{m\hbar}}\sin\vartheta\cos\vartheta\int_0^T\mathrm{d}t\,f(t)\left(b_2(T)\frac{\sin\Omega_2(T-t)}{\Omega_2}-b_1(T)\frac{\sin\Omega_1(T-t)}{\Omega_1}\right)\,.\tag{6.2.106f}$$

6.2.1.33 Harmonic Oscillator with Two-Time Quadratic Action. [161,185,801]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)$$
$$\times \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_0^T \left(\dot{x}^2(t) - \omega_0^2 x^2(t)\right)\mathrm{d}t - \frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, G(t,s)[x(t)-x(s)]^2\right]$$
$$= \sqrt{\frac{m(\Omega_1^2-\Omega_2^2)\sin^2\frac{\omega T}{2}}{4\pi\,\mathrm{i}\,\hbar D \sin\frac{\Omega_1 T}{2}\sin\frac{\Omega_2 T}{2}}}$$
$$\times \exp\left[\!\!\left[\frac{\mathrm{i}\,m}{2\hbar d}\left\{(x''-x')^2\left[2\Omega_1\Omega_2 \sin\frac{\Omega_1 T}{2}\sin\frac{\Omega_2 T}{2}\right.\right.\right.\right.$$
$$+ (\Omega_1^2+\Omega_2^2)\cos\frac{\Omega_1 T}{2}\cos\frac{\Omega_2 T}{2}$$
$$\left. -\Omega_1\Omega_2\left(\frac{(\Omega_1^2-\omega^2)\sin\frac{\Omega_2 T}{2}}{(\Omega_2^2-\omega^2)\sin\frac{\Omega_1 T}{2}} + \frac{(\Omega_2^2-\omega^2)\sin\frac{\Omega_1 T}{2}}{(\Omega_1^2-\omega^2)\sin\frac{\Omega_2 T}{2}}\right)\right]$$
$$\left.\left.\left. + \frac{2\Omega_1\Omega_2(\Omega_1^2-\Omega_2^2)^2\sin\frac{\Omega_1 T}{2}\sin\frac{\Omega_2 T}{2}}{(\Omega_1^2-\omega^2)(\Omega_2^2-\omega^2)}(x'^2+x''^2)\right\}\right]\!\!\right] \quad , \tag{6.2.107}$$

$$G(t,s) = \frac{m\Omega^2\omega}{8}\frac{\cos[\omega(T/2-|t-s|)]}{\sin(\omega T/2)} \quad , \qquad \Omega_0^2 = \omega_0^2 + \Omega^2 \quad , \tag{6.2.108}$$

$$\Omega_{1,2}^2 = \frac{\omega^2+\Omega_0^2 \pm W}{2} \quad , \qquad W = \sqrt{(\Omega_0^2-\omega^2)^2 + 4\Omega^2\omega^2} \quad , \tag{6.2.109a}$$

$$D = \frac{\Omega_1^2-\omega^2}{\Omega_1}\sin\frac{\Omega_2 T}{2}\cos\frac{\Omega_1 T}{2} - \frac{\Omega_2^2-\omega^2}{\Omega_2}\sin\frac{\Omega_1 T}{2}\cos\frac{\Omega_2 T}{2} \quad , \tag{6.2.109b}$$

$$d = 2(\Omega_1^2-\Omega_2^2)\left[\frac{\Omega_1\sin\frac{\Omega_1 T}{2}\cos\frac{\Omega_2 T}{2}}{\Omega_1^2-\omega^2} - \frac{\Omega_2\sin\frac{\Omega_2 T}{2}\cos\frac{\Omega_1 T}{2}}{\Omega_2^2-\omega^2}\right] \quad . \tag{6.2.109c}$$

6.2.1.34 Harmonic Oscillator with Two-Time Quadratic Action and Driving Force. [5,148,576,584,945]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}_E x(t)\exp\left[-\frac{1}{\hbar}\int_0^T\left(\frac{m}{2}(\dot{x}^2 - Ax^2) + f(t)x\right)\mathrm{d}t\right.$$
$$\left. + \frac{m a\omega}{4\hbar}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\frac{\cosh[\omega(T/2-|t-s|)]}{\sin(\omega T/2)}[x(t)-x(s)]^2\right]$$

$$= \left(\frac{m\omega}{8\pi\hbar\Delta(0)}\right)^{1/2} \frac{\sinh(\omega T/2)}{\sinh(\Omega_1 T/2)\sinh(\Omega_2 T/2)}$$
$$\times \exp\left[-\frac{m}{2\hbar\omega}\ddot{\Delta}(0)(x''-x')^2 - \frac{m\omega}{8\hbar\Delta(0)}(x''+x')^2\right.$$
$$-\frac{1}{\hbar}\int_0^T \mathrm{d}t\, f(t)\left(\frac{\dot{\Delta}(t)}{\omega}(x''-x') + \frac{\Delta(t)}{2\Delta(0)}(x'+x'')\right)$$
$$\left.+\frac{1}{2m\omega\hbar}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, f(t)f(s)\left(\frac{\Delta(t)\Delta(s)}{\Delta(0)} - \Delta(|t-s|)\right)\right] . \tag{6.2.110}$$

Here

$$\Omega_{1,2}^2 = \frac{1}{2}(\omega^2 - A) \pm \frac{1}{2}\sqrt{(\omega^2-A^2)^2 - 4\omega^2(a-A)}\ , \tag{6.2.111}$$
$$\Delta(t) = \frac{\omega}{2}\frac{\Omega_1^2-\omega^2}{\Omega_1^2-\Omega_2^2}\frac{\cosh[\Omega_1(t-T/2)]}{\Omega_1\sinh(\Omega_1 T/2)} + \frac{\omega}{2}\frac{\Omega_2^2-\omega^2}{\Omega_2^2-\Omega_1^2}\frac{\cosh[\Omega_2(t-T/2)]}{\Omega_2\sinh(\Omega_2 T/2)}\ . \tag{6.2.112}$$

6.2.1.35 Free Particle with Two-Time Quadratic Action. [185,585,801,818] ($\omega_0 = 0$, $\Omega_2 = 0$, $\Omega_1^2 = \omega^2 + \Omega^2$ and $G(t,s)$ in the notation of the previous case)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_0^T \dot{x}^2\mathrm{d}t - \frac{\mathrm{i}}{\hbar}\int_0^T\mathrm{d}t\int_0^T \mathrm{d}s\, G(t,s)(x(t)-x(s))^2\right]$$
$$= \sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}\frac{\sin\frac{\omega T}{2}}{\omega}\frac{\Omega_1}{\sin\frac{\Omega_1 T}{2}}}$$
$$\times\exp\left[\frac{\mathrm{i}\,m\Omega^2}{2\Omega_1^2\hbar T}(x''-x')^2\left(\frac{\omega^2}{\Omega^2} + \frac{T}{2}\Omega_1\cot\frac{\Omega_1 T}{2}\right)\right]\ , \tag{6.2.113}$$
$$= \frac{1}{2\pi}\frac{\sin(\omega T/2)}{\omega}\frac{\Omega_1}{\sin\frac{\Omega_1 T}{2}}\left(\frac{\omega^2}{\Omega_1^2} + \frac{T}{2}\frac{\Omega^2}{\Omega_1}\cot\frac{\Omega_1 T}{2}\right)^{-1/2}$$
$$\times\int_{\mathbb{R}}\mathrm{d}k\,\exp\left[-\frac{\mathrm{i}\,\hbar k^2}{2m}T\left(\frac{\omega^2}{\Omega_1^2} + \frac{\Omega^2 T}{2\Omega_1}\cot\frac{\Omega_1 T}{2}\right)^{-1} + \mathrm{i}\,k(x''-x')\right]\ . \tag{6.2.114}$$

6.2.1.36 Free Particle with Two-Time Quadratic Action. [93,113,185,249,292, 585,667,748,843,902]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}m}{2\hbar}\int_0^T \dot{x}^2 \mathrm{d}t - \frac{\mathrm{i}}{\hbar}\frac{m\Omega^2}{4T}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\,\big(x(t)-x(s)\big)^2\right]$$

$$= \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\frac{\Omega T}{2\sin\frac{\Omega T}{2}}}\exp\left[\frac{\mathrm{i}\,m\Omega}{4\hbar}\cot\frac{\Omega T}{2}(x''-x')^2\right] , \tag{6.2.115}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\Omega T}{\sin\Omega T}}\int_{\mathrm{IR}} dp\,\exp\left[-\frac{\mathrm{i}\,\hbar k^2}{m\Omega}\tan\frac{\Omega T}{2} + \mathrm{i}\,p(x''-x')\right] . \tag{6.2.116}$$

6.2.1.37 Random Gas Potential. [747] ($\mathbf{x}\in\mathrm{IR}^3$, $\beta = 1/kT$, k is the Boltzmann constant)

$$\int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(\beta\hbar)=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(s)\exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m\Omega^2}{4\beta\hbar}\int_0^{\beta\hbar}\mathrm{d}s(\mathbf{x}(s)-\mathbf{x}(t))^2\right)\mathrm{d}t\right]$$

$$= \left(\frac{m\Omega}{2\pi\hbar\sinh(\beta\hbar\Omega)}\right)^{3/2}\left(\frac{\beta\hbar\Omega}{2k\tanh(\beta\hbar\Omega/2)}\right)^{3/2} \times \exp\left[-\frac{m\Omega}{4\hbar}\coth(\tfrac{1}{2}\beta\hbar\Omega)(\mathbf{x}''-\mathbf{x}')^2\right] . \tag{6.2.117}$$

6.2.1.38 Multiple Harmonic Oscillator with Two-Time Quadratic Action and Driving Force. [151,784] ($\mathbf{x}\in\mathrm{IR}^D$)

$$\int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(T)=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \mathbf{F}(t)\cdot\mathbf{x}(t) - \frac{1}{8}\sum_{i=1}^n \kappa_i\Omega_i\int_0^T \mathrm{d}s\frac{\cos[\Omega_i(T/2-|t-s|)]}{\sin(\Omega_i T/2)}(\mathbf{x}(t)-\mathbf{x}(s))^2\right)\mathrm{d}t\right]$$

$$= G(T)\exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}')\right) . \tag{6.2.118}$$

The pre-exponential factor and the classical action are, respectively,

$$G(T) = \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{D/2}\prod_{j=1}^n\left(\frac{\omega_i}{\Omega_i}\frac{\sin(\Omega_i T/2)}{\sin(\omega_i T/2)}\right)^D , \tag{6.2.119}$$

$$R_{\mathrm{Cl}}(\mathbf{x}'',\mathbf{x}') = \left[\frac{m^2}{4}\sum_{i=1}^n\frac{h_i}{\omega_i}\cot\frac{\omega_i T}{2} + \frac{m}{2T}\left(1 - m\sum_{i=1}^n\frac{h_i}{\omega_i^2}\right)\right]\left(\mathbf{x}''-\mathbf{x}'\right)^2$$

$$
\begin{aligned}
&+\frac{1}{2}\Big(\mathbf{x}''+\mathbf{x}'\Big)\cdot\int_0^T \mathrm{d}t\,\mathbf{F}(t)+(\mathbf{x}''-\mathbf{x}')\cdot\int_0^T \mathrm{d}t\,\mathbf{F}(t)\\
&\times\left[\left(1-m\sum_{i=1}^n\frac{h_i}{\omega_i^2}\right)\left(\frac{t}{T}-\frac{1}{2}\right)+m\sum_{i=1}^n\frac{h_i}{\omega_i^2}\left(\frac{\sin\omega_i t-\sin[\omega_i(T-t)]}{2\sin\omega_i T}\right)\right]\\
&-\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\,\mathbf{F}(t)\cdot\mathbf{F}(s)\Bigg\{\frac{1}{mT}\left(1-m\sum_{i=1}^n\frac{h_i}{\omega_i^2}\right)(T-t)s\\
&+\sum_{i=1}^n\frac{h_i}{\omega_i^3\sin\omega_i T}\\
&\times\left[\sin\omega_i(T-t)\sin\omega_i s-4\sin\frac{\omega_i t}{2}\sin\frac{\omega_i}{2}(T-t)\sin\frac{\omega_i s}{2}\sin\frac{\omega_i}{2}(T-s)\right]\Bigg\}\,.
\end{aligned}
\tag{6.2.120}
$$

Here $h_i=\sum_{j=1}^n V_{ij}\Omega_j^2/m$, $(V_{ij})^{-1}=\Omega_i^2/(\omega_j^2-\Omega_i^2)$, where it is assumed without loss of generality that $\Omega_1^2<\omega_1^2<\Omega_2^2<\omega_2^2<\ldots<\Omega_n^2<\omega_n^2$, and the ω_i are the eigenvalues of the matrix $A_{ij}=\Omega_i^2\Big(\delta_{ij}+\kappa_j/m\Omega_j^2)\Big)$, determined by the equation $\sum_{j=1}^n\kappa_j/(\omega_i^2-\Omega_j^2)=m$.

6.2.1.39 Quadratic Action with Generalized Memory Kernel. [113,576,584, 843] ($d(T)=2(1-\cos\Omega T)-\Omega T\sin\Omega T$)

$$
\begin{aligned}
&\int\limits_{x(0)=x'}^{x(T)=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}(\dot{x}^2-\Omega^2x^2)\right)\mathrm{d}t+\frac{\mathrm{i}}{\hbar}\mu F\left(\frac{1}{\sigma}\int_0^T x(t)\,\mathrm{d}t\right)\right]\\
&=\frac{m\Omega^2}{2\pi\mathrm{i}\,\hbar\sqrt{d(T)}}\\
&\times\exp\left[\frac{\mathrm{i}\,m\Omega}{2\hbar d(T)}\Big((x'^2+x''^2)(\sin\Omega T-\Omega T\cos\Omega T)+2x'x''(\Omega T-\sin\Omega T)\Big)\right]\\
&\times\int_{\mathbb{R}}du\exp\left[\frac{\mathrm{i}\,m\Omega^2}{\hbar d(T)}\left((x'+x'')u(\cos\Omega T-1)-\frac{u^2\Omega^2}{2}\sin\Omega T\right)+\frac{\mathrm{i}\mu}{\hbar}F\left(\frac{u}{\sigma}\right)\right]\,.
\end{aligned}
\tag{6.2.121}
$$

6.2.1.40 General Quadratic Action with Two-Time Quadratic Action. [249]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_0^T \dot{x}^2\,\mathrm{d}t - \frac{m\Omega^2}{4T}\int_0^T \mathrm{d}s\int_0^T \mathrm{d}t\, G(t,s)\big(x(t)-x(s)\big)^2\right]$$

$$= \sqrt{\frac{m\dot{\lambda}(0)}{2\pi\,\mathrm{i}\,\hbar\lambda(T)}}\exp\left[\frac{\mathrm{i}\, m}{2\hbar}\frac{\dot{\lambda}(T)}{\lambda(T)}(x''-x')^2\right] . \tag{6.2.122}$$

Here $\lambda(t)$ is the solution of the system of equations

$$\dot{\lambda}(t) = P(t) + \lambda\int_t^T Q(t,s)\,\mathrm{d}s \ , \qquad \dot{P}(t) = \lambda\int_0^T \frac{P(s)Q(t,s)}{\lambda^2(s)}\mathrm{d}s \ , \tag{6.2.123}$$

$$\frac{\mathrm{d}\,Q(t,s)}{\mathrm{d}t} = -\frac{\Omega^2}{T}\lambda(t)G(t,t') + \lambda(t)\int_0^T \frac{Q(t,s)Q(s,t')}{\lambda^2(s)}\mathrm{d}s \ , \tag{6.2.124}$$

for $t < t'$ and the initial conditions $\lambda(0)=0, Q(0,t')=\dot{Q}(0,t')=0, P(0)=\dot{\lambda}(0), \dot{P}(0)=0$.

6.2.1.41 General Two-Time Action with Generalized Memory Kernel.

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t) \exp\Bigg[\frac{\mathrm{i}}{2}\int_0^T \dot{x}^2\mathrm{d}t + F\bigg(\int_0^T f(t)x(t)\,\mathrm{d}t\bigg)$$

$$- \frac{1}{2}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, G(t,s)\big(x(t)-x(s)\big)^2\Bigg]$$

$$= \exp\left[\frac{\mathrm{i}}{2}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, x_{\mathrm{Cl}}(t)G(t,s)x_{\mathrm{Cl}}(s)\right]\frac{K_\Omega(x'',x';T)}{\sqrt{2\pi\,\mathrm{i}\,DW_{22}}}$$

$$\times\int_{\mathbb{R}} \mathrm{d}u\,\exp\left\{\frac{\mathrm{i}}{2}\Big[W_{22}^{-1}\big(u^2+2W_{12}u-\det W\big)+2F(u+\alpha)\Big]\right\} , \tag{6.2.125}$$

where $K_\Omega(T)$ is the propagator of a harmonic oscillator with frequency $\Omega^2(t)=\int_0^T \mathrm{d}s\, G(t,s)$, $x_{\mathrm{Cl}}(t)$ is the corresponding classical path with $x_{\mathrm{Cl}}(t')=x', x_{\mathrm{Cl}}(t'')=x''$, and $\alpha=\int_0^T \mathrm{d}t f(t)x_{\mathrm{Cl}}(t)$; the quantities W_{ij} are given by

$$\left.\begin{aligned} W_{11} &= \int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, g(t)Q(t,s)g(s) \ , \\ W_{12}=W_{21} &= \int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, f(t)Q(t,s)g(s) \ , \\ W_{22} &= \int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, f(t)Q(t,s)f(s) \ , \end{aligned}\right\} \tag{6.2.126}$$

$\det W = \det(W_{ij})$, with $g(t) = \int_0^T \mathrm{d}s\, x_{\mathrm{Cl}}(s) G(t,s)$, and $Q(t,s)$ satisfies the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \Omega^2(t)\right) Q(t,s) = -\int_0^T G(t,r) Q(r,s)\, \mathrm{d}r = -\delta(t-s) \ . \tag{6.2.127}$$

The determinant D is given by

$$\frac{1}{D} = \det\left(\mathbb{1} - \int_0^T \mathrm{d}r\, Q(t,r) G(r,s)\right)^{-1} \ . \tag{6.2.128}$$

6.2.1.42 Effective Potential for an Electron Coupled to Radiation Field. ($\mathcal{D}\mu(\boldsymbol{\omega})$ is the Wiener measure in three dimensions) [802]

$$\begin{aligned}
&(2\pi\beta)^{-3/2} \exp\Big[-\beta V_{\mathrm{eff}}(\mathbf{x})\Big] \\
&= \int \mathcal{D}\mu(\boldsymbol{\omega}) \exp\left[-\frac{e^2}{16\pi^2}\int_{-\beta}^{\beta}\int_{\mathbb{R}} \frac{\mathrm{d}\boldsymbol{\omega}(t)\,\mathrm{d}\boldsymbol{\omega}(t') + \mathrm{d}t\,\mathrm{d}t''}{|\boldsymbol{\omega}(t)-\boldsymbol{\omega}(t')|^2 + (t-t')^2} - \int_0^{\beta} V(\boldsymbol{\omega}(t))\mathrm{d}t\right] \ .
\end{aligned} \tag{6.2.129}$$

6.2.1.43 N-Particle Effective Potential in Static Limit. ($\mathcal{D}\mu_1(\boldsymbol{\omega}_1)$ is the Wiener measure in three dimensions) [802]

$$\begin{aligned}
&\exp\Big[-\beta V_{\mathrm{eff}}(\mathbf{x}_1; \mathbf{x}_2, \ldots, \mathbf{x}_N)\Big] \\
&\qquad = \int \mathcal{D}\mu_1(\boldsymbol{\omega}_1) \exp\left[-\int_0^{\beta} \mathrm{d}t \sum_{j=2}^{N} \frac{e_1 e_j}{4\pi|\boldsymbol{\omega}_1(t) - \mathbf{x}_j|}\right] \ .
\end{aligned} \tag{6.2.130}$$

6.2.1.44 The Polaron Problem. [4,21,121,232,233,261,293,333,334,342,801]

$$\begin{aligned}
&\lim_{\alpha\to\infty}\frac{1}{\alpha^2}\lim_{T\to\infty}\frac{1}{T}\ln\int_{x(0)=x}^{x(T)=x} \mathcal{D}_E x(t) \exp\left(-\int_0^T \frac{\dot{\mathbf{x}}^2}{2} + \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s \frac{\alpha\, \mathrm{e}^{-|t-s|}}{|\mathbf{x}(t)-\mathbf{x}(s)|}\right) \\
&= \sup_{\phi\in L^2(\mathbb{R}^3)}\left[2\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{y}\frac{\phi^2(\mathbf{x})\phi^2(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2}\int \mathrm{d}\mathbf{x}|\nabla\phi|^2\right] \ .
\end{aligned} \tag{6.2.131}$$

6.2.1.45 Second Derivative Lagrangian in One Dimension. [194,351,607]

$$\int\limits_{\substack{x(t')=x'\\ v(t')=v'}}^{\substack{x(t'')=x''\\ v(t'')=v''}} \mathcal{D}_E x(t) \exp\left\{-\frac{1}{\hbar}\int_{t'}^{t''}\left[\frac{\kappa}{2}\ddot{x}^2(t)+\frac{m}{2}\dot{x}^2(t)+\frac{k}{2}x^2(t)-J(t)x(t)\right]\mathrm{d}t\right\}$$
$$=\frac{\sqrt{\omega_1\omega_2}}{2\pi\hbar}\frac{|\omega_1^2-\omega_2^2|}{\sqrt{(\omega_1^2+\omega_2^2)\sinh\omega_1 T\sinh\omega_2 T-2\omega_1\omega_2(\cosh\omega_1 T\cosh\omega_2 T-1)}}$$
$$\times\exp\left\{-\frac{1}{\hbar}(R_{|\mathbf{M}|,\mathrm{Cl}}+R_{J,\mathrm{Cl}})+\frac{1}{2\hbar}\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t''}\mathrm{d}s\,J(t)D_F(t,s)J(s)\right\}\ , \tag{6.2.132}$$

$$R_{|\mathbf{M}|,\mathrm{Cl}}[x]=\frac{1}{2|\mathbf{M}|}$$
$$\times\left\{(\omega_1^2-\omega_2^2)\Big[(\omega_1\cosh\omega_1 T\sinh\omega_2 T-\omega_2\sinh\omega_1 T\cosh\omega_2 T)({v'}^2+{v''}^2)\right.$$
$$-2(\omega_1\sinh\omega_2 T-\omega_2\sinh\omega_1 T)v'v''\Big]-2\omega_1\omega_2(v''x''-v'x')$$
$$\times\Big[(\omega_1^2+\omega_2^2)(\cosh\omega_1 T\cosh\omega_2 T-1)-2\omega_1\omega_2\sinh\omega_1 T\sinh\omega_2 T\Big]$$
$$+2\omega_1\omega_2(\omega_1^2-\omega_2^2)(\cosh\omega_1 T-\cosh\omega_2 T)(v''x'-v'x'')$$
$$+\omega_1\omega_2(\omega_1^2-\omega_2^2)(\omega_1\sinh\omega_1 T\cosh\omega_2 T-\omega_2\cosh\omega_1 T\sinh\omega_2 T)({x'}^2+{x''}^2)$$
$$\left.-2\omega_1\omega_2(\omega_1^2-\omega_2^2)(\omega_1\sinh\omega_1 T-\omega_2\sinh\omega_2 T)x'x''\right\}\ , \tag{6.2.133}$$

$$R_{J,\mathrm{Cl}}=-\int_{t'}^{t''}x_{\mathrm{Cl}}(t)J(t)\,\mathrm{d}t\ . \tag{6.2.134}$$

To specify the classical path we have introduced the matrix

$$\mathbf{M}=\begin{pmatrix}1&0&1&0\\ \cosh\omega_1 T&\sinh\omega_1 T&\cosh\omega_2 T&\sinh\omega_2 T\\ 0&\omega_1&0&\omega_2\\ \omega_1\sinh\omega_1 T&\omega_1\cosh\omega_1 T&\omega_2\sinh\omega_2 T&\omega_2\cosh\omega_2 T\end{pmatrix} \tag{6.2.135}$$

with determinant

$$|\mathbf{M}|=(\omega_1^2+\omega_2^2)\sinh\omega_1 T\sinh\omega_2 T-2\omega_1\omega_2(\cosh\omega_1 T\cosh\omega_2 T-1)\ ,$$

and $\omega_{1,2}$ are determined through $\omega_1^2+\omega_2^2=m\kappa^{-1/3}/m, \omega_1^2\omega_2^2=k\kappa^{1/3}$. Then

$$x(t)=\mathbf{M}^{-1}\begin{pmatrix}\cosh\omega_1(t-t')\\ \sinh\omega_1(t-t')\\ \cosh\omega_2(t-t')\\ \sinh\omega_2(t-t')\end{pmatrix}\ . \tag{6.2.136}$$

The function D_F (Feynman propagator) is given by

$$D_F(t,s) = -\frac{1}{\omega_1^2-\omega_2^2}\Bigg\{\bigg(\frac{1}{\omega_1\sinh\omega_1 T}\sinh\omega_1(t''-t)\sinh\omega_1(s-t')$$
$$-\frac{1}{\omega_2\sinh\omega_2 T}\sinh\omega_2(t''-t)\sinh\omega_2(s-t')\bigg)$$
$$+\frac{1}{2}\frac{1}{\omega_1\coth\frac{\omega_1 T}{2}-\omega_2\coth\frac{\omega_2 T}{2}}$$
$$\times\left[\frac{1}{\sinh\frac{\omega_1 T}{2}}\sinh\omega_1\left(t-\frac{t'+t''}{2}\right)-\frac{1}{\sinh\frac{\omega_2 T}{2}}\sinh\omega_2\left(t-\frac{t'+t''}{2}\right)\right]$$
$$\times\left[\frac{1}{\sinh\frac{\omega_1 T}{2}}\sinh\omega_1\left(s-\frac{t'+t''}{2}\right)-\frac{1}{\sinh\frac{\omega_2 T}{2}}\sinh\omega_2\left(s-\frac{t'+t''}{2}\right)\right]$$
$$+\frac{1}{2}\frac{1}{\omega_1\tanh\frac{\omega_1 T}{2}-\omega_2\tanh\frac{\omega_2 T}{2}}$$
$$\times\left[\frac{1}{\cosh\frac{\omega_1 T}{2}}\cosh\omega_1\left(t-\frac{t'+t''}{2}\right)-\frac{1}{\cosh\frac{\omega_2 T}{2}}\cosh\omega_2\left(t-\frac{t'+t''}{2}\right)\right]$$
$$\times\left[\frac{1}{\cosh\frac{\omega_1 T}{2}}\cosh\omega_1\left(s-\frac{t'+t''}{2}\right)-\frac{1}{\cosh\frac{\omega_2 T}{2}}\cosh\omega_2\left(s-\frac{t'+t''}{2}\right)\right]\Bigg\}. \tag{6.2.137}$$

The special case $k=J=0$ gives [194] ($\lambda=\sqrt{m/\kappa}$, $s=\sinh\lambda T$, $c=\cosh\lambda T$, $\rho=2(c-1)/\lambda Ts$, $T=t''-t'$, $\dot{x}'=v'$, $\dot{x}''=v''$)

$$\int\limits_{\substack{x(t')=x'\\ v(t')=v'}}^{\substack{x(t'')=x''\\ v(t'')=v''}}\mathcal{D}_E x(t)\exp\left[-\frac{1}{2\hbar}\int_{t'}^{t''}(\kappa\ddot{x}^2+m\dot{x}^2)\mathrm{d}t\right]$$
$$=\frac{\kappa\lambda^{3/2}}{2\pi[sT(1-\rho)]^{1/2}}$$
$$\times\exp\left\{-\frac{\mathrm{i}}{2\hbar}\left[\kappa(\dot{x}''\ddot{x}''-\dot{x}'\ddot{x}'-x''\dddot{x}''+x'\dddot{x}')+m(x''\dot{x}''-x'\dot{x}')\right]\right\}. \tag{6.2.138}$$

6.2.1.46 Second Derivative Lagrangian in Three Dimensions. [745,757] ($\mathbf{x} = (x, y, z) \in \mathbb{R}^3$)

$$\int\limits_{\mathbf{x}(s')=\mathbf{x}''}^{\mathbf{x}(s'')=\mathbf{x}''} \exp\left\{-\frac{1}{2}\int_{s'}^{s''}\left[\alpha\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right)^2 + \gamma\left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2}\right)^2\right]\mathrm{d}s\right\}$$
$$= \left(\frac{\gamma\Omega^3\beta}{2\pi D(s'', s'; L)}\right)^{3/2} \exp\left[-\frac{\gamma\Omega^3\beta}{2D(s'', s'; L)}(\mathbf{x}'' - \mathbf{x}')^2\right] . \qquad (6.2.139)$$

Here $\Omega = \sqrt{\alpha/\gamma}$, where s' and s'' denote the positions of two particles on a chain of entire length L, $|s'' - s'|$ is interpreted as an arc length, and

$$\begin{aligned} D(s'', s'; L) &= \Omega|s'' - s'| - \{1 + \cosh[\Omega(s' + s'')] - \cosh[\Omega(s'' - s')] \\ &\quad - \cosh[\Omega(s'' + s'')]\cosh[\Omega(s'' - s')]\}\tanh(\Omega L) \\ &\quad + \sinh[\Omega(s' + s'')] - \sinh(\Omega|s'' - s'|) - \sinh[\Omega(s' + s'')]\cosh[\Omega(s'' - s')] . \end{aligned} \qquad (6.2.140)$$

For $s' = 0$, $s'' = L$ one has

$$\int\limits_{\mathbf{x}(0)=x'}^{\mathbf{x}(L)=\mathbf{x}''} \mathcal{D}_E\mathbf{x}(s) \exp\left\{-\frac{1}{2}\int_0^L\left[\alpha\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right)^2 + \gamma\left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2}\right)^2\right]\mathrm{d}s\right\}$$
$$= \left(\frac{\gamma\Omega^3\beta}{2\pi(\Omega L - \tanh\Omega L)}\right)^{3/2} \exp\left[-\frac{\gamma\Omega^3\beta}{2(\Omega L - \tanh\Omega L)}(\mathbf{x}'' - \mathbf{x}')^2\right] . \qquad (6.2.141)$$

6.2.2 The Semiclassical Expansion About the Harmonic Approximation.

6.2.2.1 The Propagator. [237,238,245,712,770]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2 - V(x(t))\right)\mathrm{d}t\right]$$
$$= K_{\mathrm{GHO}}(x'', x'; T)\left(1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{\mathrm{i}}{\hbar}\right)^k \sum_{n_1=3}^{\infty}\cdots\sum_{n_k=3}^{\infty}\right.$$
$$\times \int_0^T \frac{\mathrm{d}t_1 \ldots \mathrm{d}t_k}{n_1! \ldots n_k!} V^{(n_1)}(t_1) \ldots V^{(n_k)}(t_k)$$
$$\left. \times\, (\mathrm{i}\,\hbar)^{N/2} \sum_{\{i_1, i_2, \ldots, i_{2m}\}} G_{ab}(t_{i_1}, t_{i_2}) \ldots G_{ab}(t_{i_{N-1}}, t_{i_N})\right) , \qquad (6.2.142)$$

for $N = \sum_{i=1}^{k} n_i$ even, and no contribution in (6.2.142) for N odd. Here $\{i_1, i_2, \dots, i_{2m}\}$ denotes all partitions which can be constructed from the $n_1 \cdot t_1 + \dots + n_k \cdot t_k$ different t_i (equal times $t_i = t_j$ included), and G_{ab} is as in (4.4.18).

6.2.2.2 The Semiclassical Feynman Kernel. [109,479,483,765] (For details see Sect. 5.2, $\mathbf{x} \in \mathbb{R}^D, \hbar \to 0, T > 0$)

$$\begin{aligned} K(\mathbf{x}'', \mathbf{x}'; T) &= \int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(T)=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} R[\mathbf{x}(t)] \right\} \\ &= \frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{D/2}} \sum_\gamma \left| \det\left(-\frac{\partial^2 R_\gamma}{\partial x''^a \partial x'^b} \right) \right| \\ &\quad \times \exp\left[\frac{\mathrm{i}}{\hbar} R_\gamma(\mathbf{x}'', \mathbf{x}'; T) - \mathrm{i}\,\frac{\pi}{2}\nu_\gamma \right] \cdot \Big(1 + O((\hbar)) \Big) \ . \end{aligned} \tag{6.2.143}$$

Here the sum over γ runs over all classical paths $x_\gamma(t)$ satisfying the end-point conditions $\mathbf{x}_\gamma(0) = \mathbf{x}'$, $\mathbf{x}_\gamma(T) = \mathbf{x}''$. Since the time T is fixed, but not the energy E of the classical paths, there usually will exist several solutions to Hamilton's principle. R_γ denotes the *classical action* evaluated along an actual path $\mathbf{x}_\gamma(t)$ of the system

$$R_\gamma = R_\gamma(\mathbf{x}'', \mathbf{x}'; T) := R[\mathbf{x}_\gamma(t)] \ , \tag{6.2.144}$$

where

$$R[\mathbf{x}(t)] := \int_0^T \left[\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x}) \right] dt \ . \tag{6.2.145}$$

The number ν_γ is known as the *Morse index* of the trajectory and has the geometric meaning of the number of *conjugate points* on the trajectory $x_\gamma(t), 0 \le t \le T$, conjugate to $\mathbf{x}'$ counted with multiplicities. Further references are [47,91,111,192,197,219,239,247,300,420,479,483,522,649,571,700,705,710,711, 720,723,765,770,791,804,828,833,906].

6.2.2.3 The Semiclassical Green Function. [479,483] ($\hbar \to 0$)

$$\begin{aligned} G(\mathbf{x}'', \mathbf{x}'; E) &= \bar{G}(\mathbf{x}'', \mathbf{x}'; E) + \frac{\mathrm{i}}{\hbar} \frac{1}{(2\pi\,\mathrm{i}\,\hbar)^{(D-1)/2}} \\ &\quad \times \sum_\gamma \sqrt{D_\gamma(\mathbf{x}'', \mathbf{x}'; E)} \exp\left[\frac{\mathrm{i}}{\hbar} S_\gamma(\mathbf{x}'', \mathbf{x}'; E) - \mathrm{i}\,\frac{\pi}{2}\mu_\gamma \right] \cdot \Big(1 + O(\hbar) \Big) \ . \end{aligned} \tag{6.2.146}$$

$\bar{G}$ denotes the contribution to the Green function from the stationary point at $T = 0$, the Maslov index μ_γ counts the number of points on $\mathbf{x}_\gamma$ conjugate to $\mathbf{x}'$ in energy E, see (5.3.8), and D_γ is defined in (5.3.7). Furthermore, $S_\gamma = \int_\gamma \mathbf{p} \cdot d\mathbf{x}$ denotes the classical action of the given trajectory γ with energy E.

6.2.2.4 The Generating Functional. [217,340,534,634,635,637,828] ($H_{0,1}(p,q)$ are two Hamiltonian functions, in the notation of (4.4.18-4.4.21))

$$\int\limits_{q(0)=q'}^{q(T)=q''} \mathcal{D}(q(t),p(t)) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^T \Big[p\dot{q} - H_0(p,q) + Jq + J^*p - H_1(p,q)\Big]\mathrm{d}t\right\}$$

$$= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T H_1\left(\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J^*(t)}, \frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J(t)}\right)\mathrm{d}t\right] Z_0[J,J^*] \ . \tag{6.2.147}$$

$$Z_0[J,J^*] = \int\limits_{q(0)=q'}^{q(T)=q''} \mathcal{D}(q(t),p(t)) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \Big[p\dot{q} - H_0(p,q) + Jq + J^*p\Big]\mathrm{d}t\right\}$$

$$= Z_0[0,0] \exp\left\{-\frac{\mathrm{i}}{2\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, J^*(t) G_p(t,s) J^*(s)\right.$$

$$\left. -\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, J^*(t)\bar{G}(t,s)J(s) - \frac{\mathrm{i}}{2\hbar}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}s\, J(t)G_{ab}(t,s)J(s)\right\} \ . \tag{6.2.148}$$

6.2.2.5 The Cylindrical Functional Formula. [123,700,701,704,801]

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) F(\langle\boldsymbol{\mu},q\rangle, \langle\boldsymbol{\nu},p\rangle)$$

$$= \int_{\mathbb{R}^{n+m}} \frac{\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}\, F(\mathbf{u},\mathbf{v})}{(2\pi\mathrm{i}\hbar)^{\frac{1}{2}(n+m)}\sqrt{\det\mathbf{W}\det\mathbf{S}}}$$

$$\times \exp\left\{\frac{\mathrm{i}}{2\hbar}\Big[(\mathbf{v}-\mathbf{b})^\dagger\mathbf{S}^{-1}(\mathbf{v}-\mathbf{b}) - 2(\mathbf{u}-\mathbf{a})^\dagger(\mathbf{W}^{-1}\mathbf{C}\mathbf{S}^{-1})(\mathbf{v}-\mathbf{b})\right.$$

$$\left. + (\mathbf{u}-\mathbf{a})^\dagger(\mathbf{W}^{-1} + \mathbf{W}^{-1}\mathbf{S}\mathbf{C}^{-1}\tilde{\mathbf{C}}\mathbf{W}^{-1})(\mathbf{u}-\mathbf{a})\Big]\right\} , \tag{6.2.149}$$

with the quantities ($i = 1,\dots,n, j = 1,\dots,m; G_{ab}, \bar{G}, G_p$ as in (4.4.18-4.4.21) in the following)

$$\mathbf{a} = \langle\boldsymbol{\mu},\bar{q}\rangle, \qquad \mathbf{b}_i = \langle\boldsymbol{\nu},\bar{p}\rangle \ , \tag{6.2.150}$$

$$\left.\begin{aligned} \mathbf{W} &= \int_0^T\int_0^T G_{ab}(t,t')\,\mathrm{d}\boldsymbol{\mu}(t)\otimes\mathrm{d}\boldsymbol{\mu}(t') \ , \\ \mathbf{C} &= \int_0^T\int_0^T \bar{G}(t,t')\,\mathrm{d}\boldsymbol{\mu}(t)\otimes\mathrm{d}\boldsymbol{\nu}(t') \ , \\ \mathbf{V} &= \int_0^T\int_0^T G_p(t,t')\,\mathrm{d}\boldsymbol{\nu}(t)\otimes\mathrm{d}\boldsymbol{\nu}(t') \ , \end{aligned}\right\} \tag{6.2.151}$$

$$\mathbf{S} = \mathbf{V} - \tilde{\mathbf{C}}\mathbf{W}^{-1}\mathbf{C} \ , \qquad \tilde{\mathbf{C}} = \mathbf{C}^t \ . \tag{6.2.152}$$

$\bar{q}(t), \bar{p}(t)$ describe the average over the classical path, e.g., $\bar{q}(t) = \int_0^t q_{\rm Cl}(s)\,\mathrm{d}s/t$. Special cases are

$$\begin{aligned}
&\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) F(\langle \boldsymbol{\mu}, q\rangle) \\
&= \int_{\mathbb{R}^n} \frac{\mathrm{d}\mathbf{u}\, F(\mathbf{u})}{(2\pi\,\mathrm{i}\,\hbar)^{n/2}\sqrt{\det \mathbf{W}}} \exp\left[\frac{\mathrm{i}}{2\hbar}(\mathbf{u}-\mathbf{a})^\dagger \mathbf{W}^{-1}(\mathbf{u}-\mathbf{a})\right] \ .
\end{aligned} \tag{6.2.153}$$

$$\begin{aligned}
&\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) F(\langle \boldsymbol{\nu}, p\rangle) \\
&= \int_{\mathbb{R}^m} \frac{\mathrm{d}\mathbf{v}\, F(\mathbf{v})}{(2\pi\,\mathrm{i}\,\hbar)^{m/2}\sqrt{\det \mathbf{V}}} \exp\left[\frac{\mathrm{i}}{2\hbar}(\mathbf{v}-\mathbf{b})^\dagger \mathbf{V}^{-1}(\mathbf{v}-\mathbf{b})\right] \ .
\end{aligned} \tag{6.2.154}$$

6.2.2.6 Moments Formulæ – Wick's Theorem. [201,347,666,700,701,704,781, 801,921] ($c_k = \langle \mu_k, \bar{q}\rangle$, $k = 1,\dots,n$; $c_k = \langle \nu_k, \bar{p}\rangle$, $k = n+1,\dots,n+m$)

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)\langle \boldsymbol{\mu}, q\rangle\langle \boldsymbol{\nu}, p\rangle = \mathrm{i}^{n+m}\,\mathcal{H}_n\left(\frac{c_1}{2\,\mathrm{i}},\dots,\frac{c_{n+m}}{2\,\mathrm{i}}\right) \ , \tag{6.2.155}$$

$$\begin{aligned}
&\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)\langle \boldsymbol{\mu}, \bar{q}\rangle \\
&= (\mathrm{i}\,\hbar)^{n/2} \sum_{\{i_1,i_2,\dots,i_m\}} G_{ab}(t_{i_1},t_{i_2})\dots G_{ab}(t_{i_{n-1}},t_{i_n}) \ ,
\end{aligned} \tag{6.2.156}$$

if n is even, and (6.2.156) is vanishing for n odd.

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t) = \bar{q}(t) \ , \qquad \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)p(t) = \bar{p}(t) \ . \tag{6.2.157}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t)p(s) = \bar{q}(t)\bar{p}(s) + \mathrm{i}\,\hbar\bar{G}(t,s) \ . \tag{6.2.158}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t)q(s) = \bar{q}(t)\bar{q}(s) + \mathrm{i}\,\hbar G_{ab}(t,s) \ . \tag{6.2.159}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)p(t)p(s) = \bar{p}(t)\bar{p}(s) + \mathrm{i}\,\hbar G_p(t,s) \ . \tag{6.2.160}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t_1)q(t_2)p(t_3) = \bar{q}(t_1)\bar{q}(t_2)\bar{p}(t_3)$$
$$+ \mathrm{i}\,\hbar\big[\tilde{G}(t_2,t_3)\bar{q}(t_1) + \tilde{G}(t_1,t_3)\bar{q}(t_2) + G_{ab}(t_1,t_2)\bar{p}(t_3)\big] \ . \tag{6.2.161}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t_1)q(t_2)p(t_3)p(t_4) = \bar{q}(t_1)\bar{p}(t_2)\bar{p}(t_3)\bar{p}(t_4)$$
$$+ \mathrm{i}\,\hbar\big[\tilde{G}(t_1,t_4)\bar{q}(t_2)\bar{p}(t_3) + \tilde{G}(t_2,t_4)\bar{q}(t_1)\bar{p}(t_3) + \tilde{G}(t_1,t_3)\bar{q}(t_2)\bar{p}(t_4)$$
$$+ \tilde{G}(t_2,t_3)\bar{q}(t_1)\bar{p}(t_4) + G_p(t_3,t_4)\bar{q}(t_1)\bar{q}(t_2) + G_{ab}(t_1,t_2)\bar{p}(t_3)\bar{p}(t_4)\big]$$
$$+ (\mathrm{i}\,\hbar)^2\big[G_{ab}(t_1,t_2)G_p(t_3,t_4) + \tilde{G}(t_2,t_3)\tilde{G}(t_1,t_4) + \tilde{G}(t_1,t_3)\tilde{G}(t_2,t_4)\big] \ . \tag{6.2.162}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q(t_1)q(t_2)q(t_3)q(t_4) = \bar{q}(t_1)\bar{q}(t_2)\bar{q}(t_3)\bar{q}(t_4)$$
$$+ \mathrm{i}\,\hbar\big[G_{ab}(t_1,t_2)\bar{q}(t_3)\bar{q}(t_4) + G_{ab}(t_1,t_3)\bar{q}(t_2)\bar{q}(t_4) + G_{ab}(t_1,t_4)\bar{q}(t_2)\bar{q}(t_3)$$
$$+ G_{ab}(t_2,t_3)\bar{q}(t_1)\bar{q}(t_4) + G_{ab}(t_2,t_4)\bar{q}(t_1)\bar{q}(t_3) + G_{ab}(t_3,t_4)\bar{q}(t_1)\bar{q}(t_2)\big]$$
$$+ (\mathrm{i}\,\hbar)^2\big[G_{ab}(t_1,t_2)G_{ab}(t_3,t_4) + G_{ab}(t_1,t_3)G_{ab}(t_2,t_4)$$
$$+ G_{ab}(t_2,t_3)G_{ab}(t_1,t_4)\big] \ . \tag{6.2.163}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)p(t_1)p(t_2)p(t_3)p(t_4) = \bar{p}(t_1)\bar{p}(t_2)\bar{p}(t_3)\bar{p}(t_4)$$
$$+ \mathrm{i}\,\hbar\big[G_p(t_1,t_2)\bar{p}(t_3)\bar{p}(t_4) + G_p(t_1,t_3)\bar{p}(t_2)\bar{p}(t_4) + G_p(t_1,t_4)\bar{p}(t_2)\bar{p}(t_3)$$
$$+ G_p(t_2,t_3)\bar{p}(t_1)\bar{p}(t_4) + G_p(t_2,t_4)\bar{p}(t_1)\bar{p}(t_3) + G_p(t_3,t_4)\bar{p}(t_1)\bar{p}(t_2)\big]$$
$$+ (\mathrm{i}\,\hbar)^2\big[G_p(t_1,t_2)G_p(t_3,t_4) + G_p(t_1,t_3)G_p(t_2,t_4) + 2G_p(t_2,t_3)G_p(t_1,t_4)\big] \ . \tag{6.2.164}$$

6.2.2.7 Special Moments Formulæ for Vanishing Average Classical Paths. [702]

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p)q^{2n}(t) = (\mathrm{i}\,\hbar)^n \frac{(2n)!}{2^n n!} G_{ab}^n(t,t) \ , \tag{6.2.165}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) q^2(t) q^2(t') = -\hbar^2 \big[G_{ab}(t,t) G_{ab}(t',t') + 2G_{ab}^2(t,t') \big] \ . \tag{6.2.166}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) q^3(t) q^3(t')$$
$$= -\mathrm{i}\,\hbar^3 \big[9G_{ab}(t,t) G_{ab}(t,t') G_{ab}(t',t') + 6G_{ab}^3(t,t') \big] \ . \tag{6.2.167}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) q^2(t) q^4(t')$$
$$= -\mathrm{i}\,\hbar^3 \big[12G_{ab}^2(t,t') G_{ab}(t',t') + 3G_{ab}^3(t,t) G_{ab}^2(t',t') \big] \ . \tag{6.2.168}$$

$$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}w(q,p) q^4(t) q^4(t')$$
$$= \hbar^4 \big[9G_{ab}^2(t,t) G_{ab}^2(t',t') + 24G_{ab}^4(t,t') + 72G_{ab}(t,t) G_{ab}(t',t') G_{ab}^2(t,t') \big] \, , \tag{6.2.169}$$

with G_{ab} as in (4.4.18).

6.2.2.8 Special Reflection Symmetry Functionals. [801] ($\inf x = \inf\{x(t) | t' \le t \le t''\}$, $\sup x = \sup\{x(t) | t' \le t \le t''\}$, $K_0(x,T) \equiv K_0(x,0;T)$ is the free particle kernel)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) F\big(\inf(x)\big) \exp\left(\frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \dot{x}^2 \mathrm{d}t \right)$$
$$= \int_{-\infty}^{\min(x'',x')} \mathrm{d}x \, F(x) \frac{\mathrm{d}}{\mathrm{d}x} K_0(x'' + x' - 2x; T) \ . \tag{6.2.170}$$

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) F\big(\sup(x)\big) \exp\left(\frac{\mathrm{i}\,m}{2\hbar} \int_{t'}^{t''} \dot{x}^2 \mathrm{d}t \right)$$
$$= -\int_{\max(x'',x')}^{\infty} \mathrm{d}x \, F(x) \frac{\mathrm{d}}{\mathrm{d}x} K_0(x'' + x' - 2x; T) \ . \tag{6.2.171}$$

6.2.2.9 Barrier Penetration. [499,720]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right]$$

$$\overset{(\hbar\to 0)}{\simeq} \left[2\pi\mathrm{i}\,k(x')k(x'')\int_{x'}^{x''}\frac{\mathrm{d}x}{k^3(x)}\right]^{-1/2} \exp\left(\mathrm{i}\int_{x'}^{x''}k(x)\mathrm{d}x - \frac{\mathrm{i}}{\hbar}ET\right) , \tag{6.2.172}$$

where the energy E is implicitly determined by $T = m\int_{x'}^{x''}\mathrm{d}x/\hbar k(x)$ with the wave number $k(x) = \sqrt{2m(E - V(x))}/\hbar$, and $V(x)$ describes a potential barrier.

6.2.2.10 The Quartic Oscillator. [702,800,856]

$$\int\limits_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T\left(\frac{m}{2}(\dot{x}^2 - \omega^2x^2) - \frac{\lambda}{4}x^4\right)\mathrm{d}t\right]$$

$$\begin{aligned} &= \left(\frac{1}{2\pi\mathrm{i}\,\hbar J(t',t'')}\right)^{1/2} \exp\left(\frac{\mathrm{i}}{\hbar}R_{\mathrm{Cl}}(x'',x')\right)\Bigg(1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{\mathrm{i}}{\hbar}\right)^k \\ &\qquad\times \sum_{n_1=3}^{4}\cdots\sum_{n_k=3}^{4}\int_0^T\frac{\mathrm{d}t_1\ldots\mathrm{d}t_k}{n_1!\ldots n_k!}V^{(n_1)}(x(t_1))\ldots V^{(n_k)}(x(t_k)) \\ &\qquad\times (\mathrm{i}\hbar)^{N/2}\sum_{\{i_1,i_2,\ldots,i_N\}} G(t_{i_1},t_{i_2})\ldots G(t_{i_{N-1}},t_{i_N})\Bigg) , \end{aligned} \tag{6.2.173}$$

$(V(x) = \lambda x^4/4)$ if $N = \sum_{i=1}^k n_i$ is even, and the sum over the partitions $\{i_1,\ldots,i_N\}$ is zero if N is odd. $x_{\mathrm{Cl}}(t)$ is the classical path, and $\{i_1,i_2,\ldots,i_N\}$ denote all partitions which can be constructed from the $n_1\cdot t_1 + \cdots + n_k\cdot t_k$ different t_i. We obtain for the classical action in $K_{(\mathrm{GHO})}(T)$

$$\begin{aligned} R_{\mathrm{Cl}}(x'',x') &= -\frac{2m\omega^3}{3\lambda\sqrt{1-2k^2}}E\left(\frac{\omega T}{\sqrt{1-2k^2}}\right) + \frac{m\omega^4(1-k^2)(2-3k^2)T}{3\lambda(1-2k^2)^2} \\ &\quad + \frac{2m\omega^3k^2}{3\lambda\sqrt{1-2k^2}}\Bigg[\operatorname{sn}u'\operatorname{sn}u''\operatorname{sn}\left(\frac{\omega T}{\sqrt{1-2k^2}}\right) \\ &\quad + \frac{1}{1-2k^2}\left(\operatorname{sn}u'\operatorname{cn}u'\operatorname{dn}u' - \operatorname{sn}u''\operatorname{cn}u''\operatorname{dn}u''\right)\Bigg] , \end{aligned} \tag{6.2.174}$$

where $u(t) \equiv \omega(t - t_0)/\sqrt{1-2k^2}$. Furthermore, $E(u)$ and $K(u)$ denote the complete elliptic integrals with module k, the latter implicitly being defined by

$$\pm\frac{\omega T}{\sqrt{1-2k^2}}\pm\mathrm{cn}^{-1}\left(\frac{x'}{x_0},k\right)\pm\mathrm{cn}^{-1}\left(\frac{x''}{x_0},k\right)+4NK(k)\ , \tag{6.2.175}$$

where $N\in\mathbb{N}$ and any combination of signs is permitted, $k'^2=1-k^2$. t_0, x_0 are constants of integration defined via ($t\in[t',t'']$)

$$x(t)=x_0\,\mathrm{cn}[\Omega(t-t_0),k]\ ,\qquad x(t')=x'\ ,\qquad x(t'')=x''\ , \tag{6.2.176}$$

where we have the relations $\Omega^2=\omega^2+\frac{\lambda x_0}{2m}$, $k^2=\frac{\lambda}{2m}\frac{x_0}{\Omega}$ yielding, e.g., $x_0=\sqrt{2mk^2\omega^2/\lambda(1-k^2)}$. The Feynman–Green function (propagator) for the moments integral (6.2.149) has the form

$$\begin{aligned}J(t',t'')&=\frac{(1-2k^2)^{3/2}}{\omega}\,\mathrm{sn}\,u'\,\mathrm{sn}\,u''\,\mathrm{dn}\,u'\,\mathrm{dn}\,u''\\&\quad\times\Bigg[-\frac{1}{1-2k^2}\left(\frac{\mathrm{cn}\,u''}{\mathrm{sn}\,u''\,\mathrm{dn}\,u''}-\frac{\mathrm{cn}\,u'}{\mathrm{sn}\,u'\,\mathrm{dn}\,u'}\right)+\frac{u''-u'}{1-2k^2}-\frac{E(u''-u')}{k'^2}\\&\qquad+\frac{k^2}{k'^2}\left(\frac{\mathrm{sn}\,u''\,\mathrm{cn}\,u''}{\mathrm{dn}\,u''}-\frac{\mathrm{sn}\,u'\,\mathrm{cn}\,u'}{\mathrm{dn}\,u'}-\mathrm{sn}\,u'\,\mathrm{sn}\,u''\,\mathrm{sn}(u''-u')\right)\Bigg]\ ,\end{aligned} \tag{6.2.177}$$

$$G_{ab}(s,t)=\frac{J(t,t')J(t'',s)\Theta(s-t)+J(s,t')J(t'',t)\Theta(t-s)}{J(t',t'')}\ . \tag{6.2.178}$$

6.2.2.11 Velocity-Dependent Quartic Oscillator. [702] ($q_l=q(t_l)$, $p_l=p(t_l)$ describe the classical path)

$$\begin{aligned}&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{\mathrm{MP}}x(t)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{x}^2+k^2x^4+2mkx^2\dot{x})\mathrm{d}t\right]\\&=\sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}}\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}|x''-x'|^2\right)\left[1-\frac{\mathrm{i}}{\hbar}\frac{mk}{3}(x'^3+x''^3)+O(k^2)\right]\end{aligned} \tag{6.2.179}$$

$$\begin{aligned}&=\sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}}\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}|x''-x'|^2\right)\sum_{j\in\mathbb{N}_0}k^j\frac{(-\mathrm{i}/\hbar)^j}{(2\pi\hbar)^{2j}}\\&\quad\times\int_{t'}^{t''}\mathrm{d}t_1\int_{t'}^{t_1}\cdots\int_{t'}^{t_{j-1}}\mathrm{d}t_j\prod_{l=1}^{j}\mathrm{d}q_l\,\mathrm{d}p_l\,\mathrm{d}u_l\,\mathrm{d}v_l\,(p_lq_l^2-\mathrm{i}\,\hbar q_l)\cdot E_j\ ,\end{aligned} \tag{6.2.180}$$

$$\begin{aligned}E_j=\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\Bigg[&\sum_{s=1}^{j}\left(q_su_s+p_sv_s+\frac{u_sv_s}{2}\right)\\&-\frac{1}{T}\sum_{r=1}^{j}\Big(u_r(t_r-t')x''+u_r(t''-t_r)x'+m(x''-x')v_r\Big)\end{aligned}$$

$$- \frac{1}{T} \sum_{r,s=1}^{j} \left(\frac{u_r u_s}{m} (t_r - t')(t'' - t_s) - m v_r v_s \right) \left(1 - \frac{1}{2} \delta_{rs} \right)$$
$$- \frac{1}{T} \left(\sum_{\substack{r,s=1 \\ r \le s}}^{j} u_r v_s (t'' - t_r) - \sum_{\substack{s=1,r=2 \\ r > s}}^{j} u_r v_s (t_r - t') \right) \Bigg] \Bigg\} \; . \tag{6.2.181}$$

6.2.3 Reformulation of the Radial Kernel. [380,384,385,663,664] ($x = \ln r$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 - \hbar^2 \frac{l(l+1)}{2mr^2} - V(r) \right) \mathrm{d}t \right]$$
$$= \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \mu_{l+1/2}[r^2] \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \dot{r}^2 - V(r) \right) \mathrm{d}t \right]$$
$$= \frac{1}{\sqrt{r'r''}} \int\limits_{x(t')=\ln r'}^{x(t'')=\ln r''} \mathcal{D}_{\mathrm{MP}} x(t)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2} \mathrm{e}^{2x} \dot{x}^2 - \hbar^2 \frac{(l+\frac{1}{2})^2}{2m\,\mathrm{e}^{2x}} - V(e^x) \right) \mathrm{d}t \right] \; . \tag{6.2.182}$$

6.2.4 Trace Formulæ.

6.2.4.1 The Gutzwiller Trace Formula. [479–483] (For details, see Sect. 5.4)

$$d(E) := \sum_{n=0}^{\infty} \delta(E - E_n) = \frac{1}{\pi} \Im \int_{\mathbb{R}^D} \mathrm{d}\mathbf{x}\, G(\mathbf{x},\mathbf{x};E)$$
$$= \bar{d}(E) + O(\hbar^\infty)$$
$$+ \frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{T_{\gamma_p}}{|\det(\mathbf{M}^k_{\gamma_p} - \mathbb{1})|^{1/2}} \cos\left(\frac{1}{\hbar} k S_{\gamma_p} - \frac{\pi}{2} k \tilde{\mu}_{\gamma p} \right) \cdot \left(1 + O(\hbar) \right) \; . \tag{6.2.183}$$

Here the sum runs over all *primitive periodic orbits* γ_p with *period* T_{γ_p}, *classical action* S_{γ_p}, and *Maslov index* $\tilde{\mu}_{\gamma_p}$ of γ_p, and $\bar{d}(E)$ denotes the Thomas–Fermi approximation to the level density $d(E)$.

6.2.4.2 The Periodic Orbit Formula for Two-Dimensional Billiards. [850,852]

$$\sum_n h(p_n) \overset{(\hbar\to 0)}{\simeq} 2\int_0^\infty \mathrm{d}p\, p h(p)\bar{d}(E) + Ch(0) + \frac{1}{\hbar}\sum_{\gamma_p}\sum_{k=1}^{\infty}\frac{l_{\gamma_p} g(kl_{\gamma_p}/\hbar)}{\sqrt{|\operatorname{Tr} M_{\gamma_p}^k - 2|}}\chi_{\gamma_p,k} \ . \tag{6.2.184}$$

This is the absolutely convergent version of the *Gutzwiller trace formula* (6.2.183) for two-dimensional Euclidean billiards. The energy levels E_n are parameterized in terms of the momenta, $p_n := \sqrt{E_n}$; units $2m = 1$ are used. $h(p)$ is an even test function, holomorphic in the strip $|\Im(p)| \leq \sigma_a + \epsilon, \epsilon > 0$, satisfying $h(p) = O(p^{-2-\delta}), |p| \to \infty, \delta > 0$. ($\sigma_a$ is the so-called "entropy barrier", see [37, 109, 866, 868, 869].) l_{γ_p} denotes the geometrical length of γ_p, and χ_{γ_p} is a phase factor depending on the Maslov index γ_p and the boundary conditions, e.g., Dirichlet- or Neumann boundary conditions. C is the constant in Weyl's improved law for billiards, see e.g., [45,49]. $g(u) = (1/\pi)\int_0^\infty h(p)\cos(up)\,\mathrm{d}p$.

6.2.4.3 Trace Formula for the Heat-Kernel on the Sphere – Orbifold Space-Time $\mathbb{R}\times S^{(2)}/\Gamma$. [174]

$$K_{S^{(2)}/\Gamma}(\tau) = \frac{2\pi}{|\Gamma|}\frac{\mathrm{e}^{\tau/4}}{(\pi\tau)^{3/2}}P\int_0^\infty \mathrm{d}\,\alpha\,\frac{\mathrm{e}^{-\alpha^2/\tau}}{\sin\alpha}\left[\alpha + \frac{\tau}{2}\sum_{\{\gamma\}} n_q(\cot\alpha - q\cot q\alpha)\right] \ . \tag{6.2.185}$$

Here $S^{(2)}/\Gamma$ is the fundamental domain of Γ (an elliptic triangle), q is the generic order of the rotation such that for each primitive $\gamma \in \Gamma$ we have $\gamma^q = \mathbb{1}$, where Γ is a finite subgroup of O(3) acting with fixed points, and n_q is the number of conjugate q-fold axes. The summation over all primitive conjugacy classes of elements $\gamma \in \Gamma$ is denoted by $\sum_{\{\gamma\}}$. The quantity $|\Gamma|$ is defined via $|\Gamma|\int_{S^{(2)}/\Gamma}\mathrm{d}s = 4\pi$ and describes the order of Γ.

6.2.4.4 Trace Formula on CP^N Manifolds. [355] ($Q = \sum_\alpha n_\alpha$ with n_α the occupation numbers, c_α are the matrix elements of the Hamiltonian $\underline{\mathrm{H}} = \underline{\mathrm{a}}^\dagger \operatorname{diag}(c_1,\dots,c_{N+1})\underline{\mathrm{a}}$ of the coherent states $\underline{\mathrm{a}} = (\underline{\mathrm{a}}_1,\dots,\underline{\mathrm{a}}_{N+1})$ with the commutation relations $[\underline{\mathrm{a}}_\alpha,\underline{\mathrm{a}}_\beta^\dagger] = \delta_{\alpha\beta}\,\mathbb{1}, [\underline{\mathrm{a}}_\alpha,\underline{\mathrm{a}}_\beta] = [\underline{\mathrm{a}}_\alpha^\dagger,\underline{\mathrm{a}}_\beta^\dagger] = 0, \forall_{\alpha,\beta}$)

$$Z(\beta) = \sum_{\alpha=1}^{N+1}\frac{\mathrm{e}^{-\mathrm{i}Qc_\alpha\beta}}{\prod_{\alpha\neq\beta}\left(1 - \mathrm{e}^{-\mathrm{i}(c_\beta - c_\alpha)\beta}\right)} \ . \tag{6.2.186}$$

6.2.4.5 Trace Formula for Zoll Surfaces. [825]

$$\sum_{n=0}^{\infty}\delta\left(p - \sqrt{E_n}\right) = \sum_{k\in\mathbb{Z}}\alpha_k(p)\,\mathrm{e}^{\mathrm{i}\pi\nu k/2}\,\mathrm{e}^{-2\pi\mathrm{i}kp} \ . \tag{6.2.187}$$

ν is the Maslov index, the d-dimensional Zoll surfaces ($= SC_{2\pi}$-manifolds) are manifolds all of whose geodesics are closed,and $\alpha_k(p) = \mathrm{vol}(SC_{2\pi})p^{d-1}/(4\pi)^{d/2}$ $\Gamma(d/2+1) + O(p^{d-2})$, as $p \to +\infty$, and $\alpha_k(p) = O(p^{-\infty})$, as $p \to -\infty$.

6.2.4.6 Periodic Orbit Formula for Integrable Systems. [92]

$$\sum_{\mathbf{m}} \delta(U-U(\mathbf{m})) = 1+\sum_{\mathbf{M}\neq 0} \frac{|\mathbf{m_M}\cdot\mathbf{M}|}{DU^{(D-1)/2D}|\mathbf{M}|^{(D-1)/2}} \frac{\mathrm{e}^{2\pi\,\mathrm{i}\,\mathbf{m}_M\cdot\mathbf{M}U^{1/D}-\frac{\mathrm{i}}{4}\pi(D-1)}}{\sqrt{\det\left|\frac{\partial^2\mathbf{m}\cdot\hat{\mathbf{M}}}{\partial\mu^a\partial\mu^b}\right|_{\mathbf{m}=\mathbf{m}_M}}} . \tag{6.2.188}$$

Here U is the total number of states below energy E, $U(\mathbf{m})$ denotes the scaled levels in the regular spectrum labelled by D quantum numbers $(m_1,\dots,m_D) =: \mathbf{m}$, $\mathbf{M}$ is a D-dimensional lattice of integers, and $\boldsymbol{\mu} = (\mu^1,\dots,\mu^{D-1})$ are $D-1$ curvilinear coordinates on the contours $U(\mathbf{m}) = 1$ in $\mathbf{m}$ space. The points $\mathbf{m_M} := \mathbf{m}(\boldsymbol{\mu}_\mathbf{M})$ are defined by the points $\boldsymbol{\mu}_\mathbf{M}$ where $\mathbf{M}\cdot\partial\mathbf{m}/\partial\mu^a = 0, 1\le a\le D-1, \hat{\mathbf{M}} = \mathbf{M}/|\mathbf{M}|$.

6.2.4.7 Periodic Orbit Formula for Perturbed Quadratic Lagrangians. [12]

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathrm{d}\mathbf{x}\,\phi(\mathbf{x}) \int\limits_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^T \left(\frac{\dot{\mathbf{x}}^2}{2} - \frac{\mathbf{x}^t\mathbf{A}\mathbf{x}}{2} - V(\mathbf{x})\right)\mathrm{d}t\right] \\
&\overset{(\hbar\to 0)}{\simeq} \frac{1}{\sqrt{2\pi\,\mathrm{i}\,\hbar}} \sum_{\gamma_p}\sum_{k=1}^{\infty} \exp\left(\frac{\mathrm{i}}{\hbar}kR\gamma_p - \mathrm{i}\frac{\pi}{2}k\nu_{\gamma_p}\right) \\
&\quad\times C_p\sqrt{k}\left|\frac{\mathrm{d}E_{\gamma_p}}{\mathrm{d}t}\right| \int_0^{T/k} \phi(\gamma_x^t(\tau))\,\mathrm{d}\tau \ ,
\end{aligned} \tag{6.2.189}$$

with the quantity

$$C_p = \frac{1}{\sqrt{|D_T|}}\left|\prod_{k=1}^{d-1} 4\cos^2\mathrm{i}\,\beta_k\frac{T}{n}\right|^{-1/2} . \tag{6.2.190}$$

Here $\mathbf{A}$ is a $d\times d$ strictly positive matrix and the potential V is such that the energy E is the only conserved quantity of the system, with γ_p the family of all primitive periodic orbits of period T/n, ν_{γ_p} a Maslov index for γ_p of the closed orbit γ_x^T, β_k the stability angles, $D_T = \det(\partial^2 T/\partial x_T^a\partial x_T^b)$, where $\mathbf{x}_{T,L}$ denote the transversal (longitudinal) coordinates with respect to the periodic orbit, and $\phi(\mathbf{x}) \in C_0^\infty(\mathbb{R}^d)$ is a suitable test function.

6.2.4.8 The Selberg Trace Formula. [489,844,866,910]

$$\sum_n h(p_n) = \frac{\mathcal{A}}{4\pi}\int_0^\infty \mathrm{d}p\,p\tanh\pi p h(p) + \sum_{\{\gamma\}\in\Gamma}\sum_{k=1}^{\infty} \frac{l_\gamma g(kl_\gamma)}{\sinh\frac{kl_\gamma}{2}} . \tag{6.2.191}$$

Here units $\hbar = 2m = 1$ are used, $g(u) = (1/\pi)\int_0^\infty h(p)\cos(up)\,\mathrm{d}p$, $E = p^2 + 1/4$. $\{\gamma\}\in\Gamma$ denotes the summation over all primitive conjugacy classes in the Fuchsian group Γ. l_γ denotes the length of a closed geodesic corresponding to a closed $\gamma\in\Gamma$. $h(p)$ is an even test function with $h(p) = O(p^{-2-\epsilon})$, as $|p|\to\infty$, and is analytic in the strip $|\Im(p)| \le 1/2+\delta, \delta > 0$. $\mathcal{A}$ is the area of the fundamental domain corresponding to Γ.

6.2.4.9 The Selberg Trace Formula for Compact Symmetric Space Forms of Rank One. [363]

$$\sum_{j\geq 0} n_j(\chi)h(\nu_j^+) = \chi(1)\frac{\mathcal{V}(\Gamma\backslash G)}{4\pi}\int_{\mathbb{R}}\frac{h(r)\mathrm{d}r}{|c(\mathrm{i}\,r)|^2} + \sum_{\{\gamma\}}|u_{\gamma_0}|C(h(\gamma))g(u_\gamma) \ , \tag{6.2.192}$$

$$\frac{1}{c(\mathrm{i}\,r)} = \frac{\Gamma(\frac{m_\alpha+m_\beta}{2})\Gamma(\mathrm{i}\,r+m_\alpha/2)\Gamma(\mathrm{i}\,r/2+m_\alpha/4+m_\beta/2)}{\Gamma(m_\alpha+m_\beta)\Gamma(\mathrm{i}\,r)\Gamma(\mathrm{i}\,r/2+m_\alpha/4)} \ , \tag{6.2.193}$$

$$C\big(h(\gamma)\big) = \epsilon_R^A\big(h(\gamma)\big)\xi\big(h_{\mathfrak{p}}(\gamma)\big)\prod_{\alpha\in P+}\frac{1}{1-1/\xi_\alpha\big(h(\gamma)\big)} \ . \tag{6.2.194}$$

$c(z)$ is the Harish–Chandra c-function. Further, for any α, ξ_α stands for the character of $A = A_{\mathfrak{k}}A_{\mathfrak{p}}$ defined by $\xi_\alpha(h) = \mathrm{e}^{\alpha(\log(h)}$, and $\epsilon_R^A(h)$ is, for $h\in A$, equal to the sign of $\prod_{\alpha\in\Phi_R^+}\big(1-1/\xi_\alpha(h)\big)$, Φ_R^+ being the set of roots of $(\mathfrak{g}^{\mathbb{C}},\mathfrak{a}^{\mathbb{C}})$, i.e., those that are real on $\mathfrak{a}$. $C(h)$ is a positive function on A, cf. Sect. 6.10.9, and [363]. The test function $h(r)$ must fulfill the requirements: $h(r)$ is holomorphic in the strip $|\Im(r)|\leq\varrho_0+\epsilon$, $\epsilon>0$, $h(r)$ has to decrease faster than $|r|^{-2}$ for $r\to\pm\infty$, $g(u)=\pi^{-1}\int_0^\infty h(r)\cos(\pi r)\,\mathrm{d}\,p$. See also 6.7.14 and 6.10.9.

6.2.4.10 The Selberg Trace Formula on D-Dimensional Hyperbolic Space. [156,363,364,909]

$$\sum_{n=0}^{\infty}h(p_n) = 2\mathcal{V}\int_0^\infty \mathrm{d}p\,h(p)\Phi_D(p) + 2\sum_{\{\gamma\}}\frac{l_\gamma g(l_\gamma)}{N_\gamma^{(D-1)/2}|\det(\mathbb{1}-S^{-1}K^{-1})|} \ , \tag{6.2.195}$$

$$\Phi_D(p) = \frac{\Omega(D)}{(2\pi)^2}\left|\frac{\Gamma(\mathrm{i}p+\frac{D-1}{2})}{\Gamma(\mathrm{i}p)}\right|^2 \ . \tag{6.2.196}$$

The Harish–Chandra function in this case is relatively simple and gives a polynomial of degree $\frac{D-1}{2}$ in p^2 if D is odd, and a polynomial of degree $\frac{D-2}{2}$ in p^2 times $p\tanh\pi p$, if D is even. The matrices $K\in\mathrm{O}(D-1)$ and S denote a rotation and a dilation, respectively, which arise in the evaluation of the trace formula corresponding to the conjugation procedure in order to obtain a convenient fundamental domain $\Gamma\backslash\mathcal{H}^{(D-1)}$. $\mathcal{V}$ is the volume of this fundamental domain.

6.2.4.11 The Selberg Super-Trace Formula. [51,428,447]

$$\sum_{n=0}^{\infty}\Big[h(\tfrac{1}{2}+\mathrm{i}\,p_n^{(B)})-h(\tfrac{1}{2}+\mathrm{i}\,p_n^{(F)})\Big]=-\frac{A(\mathcal{F})}{4\pi}\int_0^\infty\frac{g(u)-g(-u)}{\sinh\frac{u}{2}}\,\mathrm{d}u$$
$$+\sum_{\{\gamma\}}\sum_{k=1}^{\infty}\frac{l_\gamma}{2\sinh\frac{kl_\gamma}{2}}\Big[g(kl_\gamma)+g(-kl_\gamma)-\chi_\gamma^k\Big(g(kl_\gamma)\,\mathrm{e}^{-\frac{kl_\gamma}{2}}+g(-kl_\gamma)\,\mathrm{e}^{\frac{kl_\gamma}{2}}\Big)\Big]\ . \tag{6.2.197}$$

The test function h is required to have the following properties: $h(1/2+\mathrm{i}\,p)\in C^\infty(\mathbb{R})$, $h(p)$ vanishes faster than $1/|p|$ for $p\to\pm\infty$, $h(1/2+\mathrm{i}\,p)$ is holomorphic in the strip $|\Im(p)|\leq\frac{1}{2}+\epsilon$, $\epsilon>0$, and $g(u)$ is the Fourier transform of $h(1/2+\mathrm{i}\,p)$. (B) and (F) denote the bosonic and fermionic eigenvalues of the Laplacian $\Box=2(y+\theta\bar\theta/2)D\bar D$, $D=\theta\partial_z+\partial_\theta$ on the supersymmetric extension of the Poincaré upper half-plane. $\theta,\bar\theta$ are Grassmann variables. See 6.2.4.8 for further notation, and compare 6.16.3.2.

6.3 Discontinuous Potentials

6.3.1 Motion in Half-Space.

6.3.1.1 Dirichlet Boundary Conditions. [144,196,316,317,340,439,448,613,722, 785,801,828,865]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(x>0)}^{(D)}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot x^2\,\mathrm{d}t\right)$$
$$=\frac{m\sqrt{x'x''}}{\mathrm{i}\,\hbar T}\exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}(x'^2+x''^2)\right]I_{1/2}\left(\frac{mx'x''}{\mathrm{i}\,\hbar T}\right)\ , \tag{6.3.1}$$
$$=\sqrt{x'x''}\int_0^\infty k\,\mathrm{d}k\,J_{1/2}(kx'')J_{1/2}(kx')\,\mathrm{e}^{-\mathrm{i}\,\hbar Tk^2/2m}\ . \tag{6.3.2}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty\mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(x>0)}^{(D)}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot x^2\,\mathrm{d}t\right)$$
$$=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\left[\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''-x'|\right)-\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''+x'|\right)\right]\ . \tag{6.3.3}$$

6.3.1.2 Neumann Boundary Conditions. [448]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(N)}_{(x>0)}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\,\mathrm{d}t\right)$$

$$= \frac{m\sqrt{x'x''}}{\mathrm{i}\,\hbar T}\exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}(x'^2+x''^2)\right]I_{-1/2}\left(\frac{mx'x''}{\mathrm{i}\,\hbar T}\right) , \tag{6.3.4}$$

$$= \sqrt{x'x''}\int_0^\infty k\,\mathrm{d}k\,J_{-1/2}(kx'')J_{-1/2}(kx')\,\mathrm{e}^{-\mathrm{i}\,\hbar Tk^2/2m} . \tag{6.3.5}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(N)}_{(x>0)}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\,\mathrm{d}t\right)$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\left[\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''-x'|\right)+\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''+x'|\right)\right] . \tag{6.3.6}$$

6.3.1.3 Linear Potential in Half-Space. [399,439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(x>R)}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-kx\right)\mathrm{d}t\right]$$

$$\begin{aligned}
&= \frac{4}{3}\frac{m}{\hbar^2}\left[\left(x'-\frac{E}{k}\right)\left(x''-\frac{E}{k}\right)\right]^{1/2}\\
&\quad\times\left\{K_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x_>-\frac{E}{k}\right)^{3/2}\right]I_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x_<-\frac{E}{k}\right)^{3/2}\right]\right.\\
&\qquad -K_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x'-\frac{E}{k}\right)^{3/2}\right]K_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x''-\frac{E}{k}\right)^{3/2}\right]\\
&\qquad\left.\times\, I_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(R-\frac{E}{k}\right)^{3/2}\right]\Bigg/K_{1/3}\left[\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(R-\frac{E}{k}\right)^{3/2}\right]\right\} .
\end{aligned} \tag{6.3.7}$$

6.3.2 Particle in a Box.

6.3.2.1 Dirichlet–Dirichlet Boundary Conditions. [145,257,395,400,448,439, 487,530,541,613,765,828,858].

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DD)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\,\mathrm{d}t\right)$$

$$= \sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}} \sum_{n\in\mathbb{Z}} \left[\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''-x'+4nb)^2\right) - \exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''+x'+2(2n+1)b)^2\right)\right] , \tag{6.3.8}$$

$$= \frac{1}{4b}\left[\Theta_3\left(\frac{|x''-x'|}{4b}, -\frac{\pi\hbar T}{8mb^2}\right) - \Theta_3\left(\frac{x''+x'}{4b}+\frac{1}{2}, -\frac{\pi\hbar T}{8mb^2}\right)\right] , \tag{6.3.9}$$

$$= \frac{1}{b}\sum_{n=1}^{\infty} \sin\left(\frac{\pi n}{2b}(x''+b)\right) \sin\left(\frac{\pi n}{2b}(x'+b)\right) \exp\left(-\mathrm{i}\,\hbar T\frac{\pi^2 n^2}{8mb^2}\right) . \tag{6.3.10}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^{\infty}\mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DD)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\,\mathrm{d}t\right)$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\,\frac{\cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(|x''-x'|-2b)] - \cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(x''+x')]}{\sinh(\sqrt{-2mE}\,\frac{2b}{\hbar})} . \tag{6.3.11}$$

6.3.2.2 Neumann–Neumann Boundary Conditions. [448]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(NN)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\,\mathrm{d}t\right)$$

$$= \sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}} \sum_{n\in\mathbb{Z}} \left[\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''-x'+4nb)^2\right) + \exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''+x'+2(2n+1)b)^2\right)\right] , \tag{6.3.12}$$

$$= \frac{1}{4b}\left[\Theta_3\left(\frac{|x''-x'|}{4b}, -\frac{\pi\hbar T}{8mb^2}\right) + \Theta_3\left(\frac{x''+x'}{4b}+\frac{1}{2}, -\frac{\pi\hbar T}{8mb^2}\right)\right] , \tag{6.3.13}$$

$$= \frac{1}{b} \sum_{n \in \mathbb{N}} \cos\left(\frac{\pi n}{2b}(x''+b)\right) \cos\left(\frac{\pi n}{2b}(x'+b)\right) \exp\left(-\mathrm{i}\,\hbar T \frac{\pi^2 n^2}{8mb^2}\right) . \tag{6.3.14}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(NN)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''} \dot{x}^2\,\mathrm{d}t\right)$$
$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}} \frac{\cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(|x''-x'|-2b)] + \cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(x''+x')]}{\sinh(\sqrt{-2mE}\,\frac{2b}{\hbar})} . \tag{6.3.15}$$

6.3.2.3 Dirichlet-Neumann Boundary Conditions. [448]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DN)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''} \dot{x}^2\,\mathrm{d}t\right)$$
$$= \sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}} \sum_{n\in\mathbb{Z}} (-1)^n \left[\exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''-x'+4nb)^2\right)\right.$$
$$\left. - \exp\left(\frac{\mathrm{i}\,m}{2\hbar T}(x''+x'+2(2n+1)b)^2\right)\right] , \tag{6.3.16}$$

$$= \frac{1}{4b}\left[\Theta_2\left(\frac{|x''-x'|}{4b}, -\frac{\pi\hbar T}{8mb^2}\right) - \Theta_2\left(\frac{x''+x'}{4b}+\frac{1}{2}, -\frac{\pi\hbar T}{8mb^2}\right)\right] , \tag{6.3.17}$$

$$= \frac{1}{b}\sum_{n=0}^{\infty} \sin\left(\frac{\pi(n+\frac{1}{2})}{2b}(x''+b)\right) \sin\left(\frac{\pi(n+\frac{1}{2})}{2b}(x'+b)\right)$$
$$\times \exp\left(-\mathrm{i}\,\hbar T \frac{\pi^2(n+\frac{1}{2})^2}{8mb^2}\right) . \tag{6.3.18}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DN)}_{(|x|<b)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''} \dot{x}^2\,\mathrm{d}t\right)$$
$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}} \frac{\sinh[\frac{1}{\hbar}\sqrt{-2mE}\,(x''+x')] - \sinh[\frac{1}{\hbar}\sqrt{-2mE}\,(|x''-x'|-2b)]}{\cosh(\sqrt{-2mE}\,\frac{2b}{\hbar})} . \tag{6.3.19}$$

6.3.3 The Potential Step.

6.3.3.1 Feynman Kernel. [56,182,227,801] ($\alpha = \hbar T/2m$, $\beta = 2k\alpha$, $k_1 = \sqrt{2mE}/\hbar$, $k_2 = \sqrt{2m(E-V_0)}/\hbar \equiv \sqrt{k^2 - 2mV_0/\hbar^2} \equiv k_2(k)$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \Theta(x)V_0\right)\mathrm{d}t\right]$$
$$= \Theta(-x'')\Theta(-x')\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\left((x''-x')p - \frac{p^2}{2m}T\right)\right]$$
$$+\Theta(-x'')\Theta(x')\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\left(x''p - x'\sqrt{p^2-2mV_0} - \frac{p^2}{2m}T\right)\right]$$
$$+\Theta(x'')\Theta(-x')\,\mathrm{e}^{-\mathrm{i}V_0T/\hbar}\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\left(x''p - x'\sqrt{p^2+2mV_0} - \frac{p^2}{2m}T\right)\right]$$
$$+\Theta(x'')\Theta(x')\,\mathrm{e}^{-\mathrm{i}V_0T/\hbar}\int \frac{\mathrm{d}p}{2\pi\hbar}\exp\left[\frac{\mathrm{i}}{\hbar}\left((x''-x')p - \frac{p^2}{2m}T\right)\right], \tag{6.3.20}$$
$$= \sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left(\frac{\mathrm{i}m}{2\hbar T}|x''-x'|^2\right)\Bigg\{-\Theta(-x')\Theta(-x'')\sqrt{\frac{2m}{\mathrm{i}\pi\hbar T}}$$
$$\times\int_0^\infty \frac{\mathrm{d}k}{k}J_2\left(\sqrt{\frac{2V_0}{m}}kT\right)\exp\left[\frac{\mathrm{i}\hbar T}{2m}\left(k - \frac{m}{\hbar T}(x'+x'')\right)^2\right]$$
$$-\Theta(x')\Theta(x'')\sqrt{\frac{2m}{\mathrm{i}\pi\hbar T}}\,\mathrm{e}^{-\mathrm{i}V_0T/\hbar}$$
$$\times\int_0^\infty \frac{\mathrm{d}k}{k}I_2\left(\sqrt{\frac{2V_0}{m}}kT\right)\exp\left[\frac{\mathrm{i}\hbar T}{2m}\left(k + \frac{m}{\hbar T}(x'+x'')\right)^2\right]\Bigg\}$$
$$+\Theta(-x')\Theta(x'')\sqrt{\frac{\hbar T}{2\mathrm{i}\pi m}}\int_{(x''-x')/2\alpha}^\infty \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\mathrm{i}\hbar Tk^2/2m}$$
$$\times\int_{\mathbb{R}}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}[k_2(k)x''-k(\beta+x')]}\left[1 - 2\int_0^\infty \frac{\mathrm{d}y}{y}\,\mathrm{e}^{-\mathrm{i}ky}J_2\left(\sqrt{\frac{2V_0}{m}}y\right)\right]$$
$$+\Theta(x')\Theta(-x'')\sqrt{\frac{\hbar T}{2\mathrm{i}\pi m}}\int_{(x'-x'')/2\alpha}^\infty \frac{\mathrm{d}k}{2\pi}\,\mathrm{e}^{\mathrm{i}\hbar Tk^2/2m}$$
$$\times\int_{\mathbb{R}}\mathrm{d}k'\,\mathrm{e}^{\mathrm{i}[k_2(k')x'-k'(\beta+x'')]}\left[1 - 2\int_0^\infty \frac{\mathrm{d}y}{y}\,\mathrm{e}^{-\mathrm{i}k'y}J_2\left(\sqrt{\frac{2V_0}{m}}y\right)\right]. \tag{6.3.21}$$

6.3.3.2 Green Function. [214,224,438,654] ($k^2 = 2m(E+V_0)/\hbar^2$, $\chi^2 = -2mE/\hbar^2$, alternatively we can write $e^{2\,i\arctan(k/\chi)} = (\chi+i\,k)/\chi - i\,k)$; the relativistic case is similar [654]).

$$
\begin{aligned}
&\frac{i}{\hbar}\int_0^\infty dT\, e^{i\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-[\Theta(x-b)-1]V_0\right)dt\right]\\
&=\Theta(b-x')\Theta(b-x'')\frac{1}{\hbar}\sqrt{-\frac{m}{2(E+V_0)}}\\
&\qquad\times e^{-i\,k(x_<-b)}\left(e^{i\,k(x_>-b)}-\frac{\chi+i\,k}{\chi-i\,k}e^{-i\,k(x_>-b)}\right)\\
&+\Theta(x'-b)\Theta(x''-b)\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\,e^{-\chi(x_>-b)}\left(e^{\chi(x_<-b)}+\frac{\chi+i\,k}{\chi-i\,k}e^{-\chi(x_<-b)}\right)\\
&+\Theta(x_>-b)\Theta(b-x_<)\frac{1}{\hbar}\frac{\sqrt{2m}}{\sqrt{-E}+\sqrt{-E-V_0}}\,e^{-i\,k(x_<-b)}\,e^{-\chi(x_>-b)}\quad.
\end{aligned}
\tag{6.3.22}
$$

6.3.4 Finite Potential Well.

6.3.4.1 Feynman Kernel. [56,208] ($\tilde{\Theta}_1 = \Theta(x+a)-\Theta(x)$, $\tilde{\Theta}_2 = \Theta(-x-a)+\Theta(x)$)

$$
\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V_0\tilde{\Theta}_1(x)\right)dt\right]\\
&=\left[\tilde{\Theta}_2(x'')\tilde{\Theta}_2(x')+\tilde{\Theta}_1(x'')\tilde{\Theta}_1(x')\,e^{-i\,V_0T/\hbar}\right]\int\frac{dp}{2\pi\hbar}\,e^{i[((x''-x')p-p^2T/2m)/\hbar]}\\
&\quad+\tilde{\Theta}_2(x'')\tilde{\Theta}_1(x')\int\frac{dp}{2\pi\hbar}\,e^{i[(x''p-x'\sqrt{p^2-2mV_0}-p^2T/2m)/\hbar]}\\
&\quad+\tilde{\Theta}_1(x'')\tilde{\Theta}_2(x')\,e^{-i\,V_0T/\hbar}\int\frac{dp}{2\pi\hbar}\,e^{i[(x''p-x'\sqrt{p^2+2mV_0}-p^2T/2m)/\hbar]}\quad.
\end{aligned}
\tag{6.3.23}
$$

6.3.4.2 Green Function. [67,214,224] ($\tilde{\Theta}_1 = \Theta(x+a)-\Theta(x-a)$)

$$
\frac{i}{\hbar}\int_0^\infty dT\, e^{i\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V_0\tilde{\Theta}_1(x)\right)dt\right]
$$

$$
= \begin{cases} \dfrac{1}{\hbar}\sqrt{\dfrac{m}{2(E+V_0)}}\left\{ e^{i\alpha|x''-x'|} + 2iA\sin(\alpha|x''-x'|) \right. \\ \qquad\qquad \left. + 2iC\sin[\alpha(x''+x')]\right\} , & -a \le x'', x' \le a , \\ \dfrac{1}{\hbar}\sqrt{-\dfrac{m}{2E}}\left(e^{-i\beta|x''-x'|} - G\, e^{-i\beta|x''+x'|}\right) , & x'', x' > |a| , \\ \dfrac{1}{\hbar}\sqrt{-\dfrac{m}{2E}}\, e^{-i\beta x_>}\left(I\, e^{i\alpha x_<} + J\, e^{-i\alpha x_<}\right) , & x_> > a, |x_<| \le a , \\ \dfrac{1}{\hbar}\sqrt{-\dfrac{m}{2E}}\, K\, e^{-i\beta(x_> - x_<)} , & x_> > a, x_< < -a , \\ \dfrac{1}{\hbar}\sqrt{-\dfrac{m}{2E}}\, e^{i\beta x_<}\left(J\, e^{i\alpha x_>} + I\, e^{-i\alpha x_>}\right) , & x_< < -a, |x_>| \le a . \end{cases}
\tag{6.3.24}
$$

Here we have set

$$
\left.\begin{aligned}
A &= \frac{e^{-2ia\alpha}}{T}\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\beta}{\alpha}\right) , \quad C = \frac{1}{T}\left(\frac{\beta}{\alpha}-\frac{\alpha}{\beta}\right) , \\
G &= i\,\frac{e^{2ia\beta}}{2T}\sin(2a\alpha)\left(\frac{\beta}{\alpha}-\frac{\alpha}{\beta}\right) , \quad I = \frac{e^{ia(\alpha+\beta)}}{T}\left(1+\frac{\beta}{\alpha}\right) , \\
J &= \frac{e^{ia(\beta-\alpha)}}{T}\left(1-\frac{\beta}{\alpha}\right) , \qquad K = \frac{2}{T}e^{2ia\beta} , \\
T &= 2\cos(2a\alpha) + i\sin(2a\alpha)\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) ,
\end{aligned}\right\}
\tag{6.3.25}
$$

with $\beta^2 = 2mE/\hbar^2$ and $\alpha^2 = 2m(E+V_0)/\hbar^2$. The bound states are determined by the roots of the equation

$$
2\cos(2a\alpha) + i\sin(2a\alpha)\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) = 0 . \tag{6.3.26}
$$

6.3.5 Finite Radial Potential Well.

6.3.5.1 Feynman Kernel. [56]

$$
\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}(r(t), p_r(t)) \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}(\varphi(t), p_\varphi(t))
$$
$$
\times \exp\left[\!\left[\frac{i}{\hbar}\int_{t'}^{t''} \left\{ \dot{r}p_r + \dot{\varphi}p_\varphi - \left[\frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\varphi^2\right) + V_0\Theta(R-r)\right]\right\} dt \right]\!\right]
$$

$$
\begin{aligned}
&= \Theta(R-r'')\Theta(R-r')\frac{m}{2\pi\mathrm{i}\hbar T} \\
&\qquad\times\exp\left\{-\frac{\mathrm{i}}{\hbar}V_0T+\frac{\mathrm{i}m}{2\hbar T}\Big[r'^2+r''^2-2r'r''\cos(\varphi''-\varphi')\Big]\right\} \\
&+\Theta(r''-R)\Theta(r'-R)\frac{m}{2\pi\mathrm{i}\hbar T}\exp\left\{\frac{\mathrm{i}m}{2\hbar T}\Big[r'^2+r''^2-2r'r''\cos(\varphi''-\varphi')\Big]\right\} \\
&+\Theta(R-r'')\Theta(r'-R)\int\frac{p_r\,\mathrm{d}p_r\,\mathrm{d}p_\varphi}{(2\pi\hbar)^2} \\
&\qquad\times\exp\left[\frac{\mathrm{i}}{\hbar}\left(r''\sqrt{p_r^2-2mV_0}\sin(p_\varphi+\varphi'')-r'p_r\sin(p_\varphi+\varphi')-\frac{p_r^2T}{2m}\right)\right] \\
&+\Theta(r''-R)\Theta(R-r')\int\frac{p_r\mathrm{d}p_r\mathrm{d}p_\varphi}{(2\pi\hbar)^2} \\
&\qquad\times\exp\left[\frac{\mathrm{i}}{\hbar}\left(r''p_r\sin(p_\varphi+\varphi'')-r'\sqrt{p_r^2-2mV_0}\sin(p_\varphi+\varphi')-\frac{p_r^2T}{2m}\right)\right]\ .
\end{aligned}
\tag{6.3.27}
$$

6.3.5.2 Green Function. [438] [$\chi^2=-2mE/\hbar^2$, $k^2=2m(E+V_0)/\hbar^2$, alternatively we can write $\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}=(\chi+\mathrm{i}k)/\chi-\mathrm{i}k)$].

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty\mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2+\Theta(b-r)V_0\right]\mathrm{d}t\right\} \\
&=\Theta(b-r')\Theta(b-r'')k\Big[\mathrm{e}^{-\mathrm{i}k(r_<-b)} \\
&\times\Big(\mathrm{e}^{\mathrm{i}k(r_>-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\,\mathrm{e}^{-\mathrm{i}k(r_>-b)}\Big)-\Big(\mathrm{e}^{2\mathrm{i}k(a-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\Big)^{-1} \\
&\times\Big(\mathrm{e}^{\mathrm{i}k(r''-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\,\mathrm{e}^{-\mathrm{i}k(r''-b)}\Big) \\
&\times\Big(\mathrm{e}^{\mathrm{i}k(r'-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\,\mathrm{e}^{-\mathrm{i}k(r'-b)}\Big)\Big] \\
&+\Theta(r'-b)\Theta(r''-b)\Big[\chi\,\mathrm{e}^{-\chi(r_>-b)}\Big(\mathrm{e}^{\chi(r_<-b)}+\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\,\mathrm{e}^{-\chi(r_<-b)}\Big) \\
&\quad-\frac{2k}{(\chi+k)^2}\Big(\mathrm{e}^{2\mathrm{i}k(a-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\Big)^{-1}\mathrm{e}^{-\chi(r''-b)-\chi(r'-b)}\Big] \\
&+\frac{\Theta(r_>-b)\Theta(b-r_<)}{\chi+k}\Big[\mathrm{e}^{-\mathrm{i}k(r_<-b)}\,\mathrm{e}^{-\chi(r_>-b)} \\
&\quad-\Big(\mathrm{e}^{2\mathrm{i}k(a-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\Big)^{-1} \\
&\quad\times\Big(\mathrm{e}^{\mathrm{i}k(r_<-b)}-\mathrm{e}^{2\mathrm{i}\arctan(k/\chi)}\,\mathrm{e}^{-\mathrm{i}k(r_<-b)}\Big)\,\mathrm{e}^{-\chi(r_>-b)}\Big]\ .
\end{aligned}
\tag{6.3.28}
$$

6.4 The Radial Harmonic Oscillator

6.4.1 The General Radial Harmonic Oscillator.

See Sect. 3.3 and Refs. [26,134,274,291,343,378,464,518,521,528,609,633,663, 771,865] ($\Re(\lambda) > -1$; note that for $|\lambda| < \frac{1}{2}$ both signs of λ must be taken into account)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot r^2-\omega^2r^2)\mathrm{d}t\right]$$

$$\equiv \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-\frac{m}{2}\omega^2r^2-\hbar^2\frac{\lambda^2-\frac{1}{4}}{2mr^2}\right)\mathrm{d}t\right]$$

$$= \frac{m\omega\sqrt{r'r''}}{\mathrm{i}\,\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(r'^2+r''^2)\cot\omega T\right]I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sin\omega T}\right) , \tag{6.4.1}$$

$$= \frac{2m\omega}{\hbar}\sqrt{r'r''}\sum_{n\in\mathbb{N}_0}\frac{n!}{\Gamma(n+\lambda+1)}\left(\frac{m\omega}{\hbar}r'r''\right)^\lambda\exp\left(-\frac{m\omega}{2\hbar}(r''^2+r'^2)\right)$$
$$\times L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r''^2\right)L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r'^2\right)\mathrm{e}^{-\mathrm{i}\,\omega T(2n+\lambda+1)} . \tag{6.4.2}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot r^2-\omega^2r^2)\mathrm{d}t\right]$$

$$= \frac{\Gamma[\frac{1}{2}(1+\lambda-E/\hbar\omega)]}{\hbar\omega\sqrt{r'r''}\,\Gamma(1+\lambda)}W_{E/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}r_>^2\right)M_{E/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{\hbar}r_<^2\right) . \tag{6.4.3}$$

The time-dependent case has the form [126,251,320,402,578]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot r^2-\omega^2(t)r^2)\mathrm{d}t\right]$$

$$= \frac{m\sqrt{r'r''}}{\mathrm{i}\,\hbar\eta(T)}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(\frac{\xi(T)}{\eta(T)}r'^2+\frac{\dot\eta(T)}{\eta(T)}r''^2\right)\right]I_\lambda\left(\frac{mr'r''}{\mathrm{i}\,\hbar\eta(T)}\right) . \tag{6.4.4}$$

The quantities $\eta(T)$ and $\xi(T)$, respectively, are determined by the differential equations

$$\begin{aligned}&\ddot\eta+\omega^2(t)\eta=0 , &&\eta(t')=0 , &&\dot\eta(t')=1 ,\\ &\ddot\xi+\omega^2(t)\xi=0 , &&\xi(t')=1 , &&\dot\xi(t')=0 .\end{aligned} \tag{6.4.5}$$

6.4.1.1 The Repelling Radial Harmonic Oscillator. [664] ($\lambda > 0$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}(\dot r^2+\omega^2 r^2)\mathrm{d}t\right]$$

$$= \frac{m\omega\sqrt{r'r''}}{\mathrm{i}\,\hbar\sinh\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(r'^2+r''^2)\coth\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sinh\omega T}\right)\,, \tag{6.4.6}$$

$$= \frac{1}{\sqrt{r'r''}}\int_{\mathbb{R}} dE\,\frac{\left|\Gamma\left[\frac12(1+\lambda+\mathrm{i}\,E/\hbar\omega)\right]\right|^2}{2\pi\omega\hbar\Gamma^2(1+\lambda)}\,\mathrm{e}^{-\mathrm{i}\,ET/\hbar+\pi E/2\omega\hbar}$$

$$\times M_{+\mathrm{i}\,E/2\hbar\omega,\lambda/2}\left(\frac{m\omega}{2\,\mathrm{i}\,\hbar}r''^2\right)M_{-\mathrm{i}\,E/2\hbar\omega,\lambda/2}\left(\frac{\mathrm{i}\,m\omega}{2\hbar}r'^2\right)\,. \tag{6.4.7}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}(\dot r^2+\omega^2 r^2)\mathrm{d}t\right]$$

$$= \frac{\Gamma\left[\frac12(1+\lambda+\mathrm{i}\,E/\hbar\omega)\right]}{\mathrm{i}\,\hbar\omega\sqrt{r'r''}\,\Gamma(1+\lambda)}W_{-\mathrm{i}\,E/2\hbar\omega,\lambda/2}\left(\frac{\mathrm{i}\,m\omega}{\hbar}r_>^2\right)M_{-\mathrm{i}\,E/2\hbar\omega,\lambda/2}\left(\frac{\mathrm{i}\,m\omega}{\hbar}r_<^2\right)\,. \tag{6.4.8}$$

6.4.1.2 The $1/r^2$ Potential. [26,96,191,274,281,467,544,609,640,642,771,785] ($\lambda > 0$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\dot r^2\mathrm{d}t\right)$$

$$= \sqrt{r'r''}\frac{m}{\mathrm{i}\,\hbar T}\exp\left[\frac{\mathrm{i}\, m}{2\hbar T}(r'^2+r''^2)\right] I_\lambda\left(\frac{mr'r''}{\mathrm{i}\,\hbar T}\right)\,, \tag{6.4.9}$$

$$= \sqrt{r'r''}\int_0^\infty k\,\mathrm{d}k\,J_\lambda(kr'')J_\lambda(kr')\,\mathrm{e}^{-\mathrm{i}\,\hbar Tk^2/2m}\,. \tag{6.4.10}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2]\exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\dot r^2\mathrm{d}t\right)$$

$$= \frac{2m}{\hbar^2}\sqrt{r'r''}\,I_\lambda\left(\sqrt{-2mE}\,\frac{r_<}{\hbar}\right)K_\lambda\left(\sqrt{-2mE}\,\frac{r_>}{\hbar}\right)\,. \tag{6.4.11}$$

6.4.1.3 Attractive $1/r^2$ *Potential.* [447,696] ($\kappa > 0$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2+\hbar^2\frac{\kappa^2+\frac{1}{4}}{2mr^2}\right)\mathrm{d}t\right]$$

$$=\sqrt{r'r''}\int_0^\infty\frac{k\,\mathrm{d}k}{\pi^2}K_{\mathrm{i}\,\kappa}(-\mathrm{i}\,kr'')K_{\mathrm{i}\,\kappa}(\mathrm{i}\,kr')\,\mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}\ . \qquad (6.4.12)$$

6.4.1.4 The D-Dimensional Radial Free Particle. [33,464]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{r}^2\mathrm{d}t\right)$$

$$=\sqrt{r'r''}\,\frac{m}{\mathrm{i}\,\hbar T}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}(r'^2+r''^2)\right]I_{l+\frac{D-2}{2}}\left(\frac{mr'r''}{\mathrm{i}\,\hbar T}\right)\ . \qquad (6.4.13)$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\ \mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{r}^2\mathrm{d}t\right)$$

$$=\frac{2m}{\hbar^2}\sqrt{r'r''}\,K_{l+\frac{D-2}{2}}\left(\sqrt{-2mE}\,\frac{r_>}{\hbar}\right)I_{l+\frac{D-2}{2}}\left(\sqrt{-2mE}\,\frac{r_<}{\hbar}\right)\ . \qquad (6.4.14)$$

6.4.1.5 The D-Dimensional Radial Harmonic Oscillator. [402,464,771]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)\mathrm{d}t\right]$$

$$=\sqrt{r'r''}\,\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(r'^2+r''^2)\cot\omega T\right]I_{l+\frac{D-2}{2}}\left(\frac{m\omega r'r''}{\mathrm{i}\,\hbar\sin\omega T}\right)\ . \qquad (6.4.15)$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\ \mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+\frac{D-2}{2}}[r^2]\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)\mathrm{d}t\right]$$

$$=\frac{\Gamma[\frac{1}{2}(l+\frac{D}{2}-\frac{E}{\hbar\omega})]}{\hbar\omega(r'r'')^{1/2}\Gamma(l+D/2)}W_{\frac{E}{2\hbar\omega},\frac{1}{2}(l+\frac{D-2}{2})}\left(\frac{m\omega}{\hbar}r_>^2\right)M_{\frac{E}{2\hbar\omega},\frac{1}{2}(l+\frac{D-2}{2})}\left(\frac{m\omega}{\hbar}r_<^2\right)\ . \qquad (6.4.16)$$

6.4.1.6 The Potential $V(r)=(\hbar^2V_0^2/2m)(a/r-r/a)^2$. [421,771] $(V_0>0)$

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2-\frac{\hbar^2V_0^2}{2m}\left(\frac{a}{r}-\frac{r}{a}\right)^2\right]\mathrm{d}t\right\}$$
$$=\frac{V_0}{\mathrm{i}\,a}\frac{\sqrt{r'r''}}{\sin\left(\frac{\hbar V_0}{ma}T\right)}I_{\sqrt{V_0^2a^2+1/4}}\left(\frac{V_0\,r'r''}{\mathrm{i}\,a\sin\left(\frac{\hbar V_0}{ma}T\right)}\right)$$
$$\times\exp\left[\frac{\mathrm{i}\,\hbar V_0^2}{m}T-\frac{V_0}{2\,\mathrm{i}\,a}(r'^2+r''^2)\cot\left(\frac{\hbar V_0}{ma}T\right)\right]\ . \tag{6.4.17}$$

6.4.2 Morse Potential. [26,129,186,208,271,382,423,528,741,775,871] $(V_0>0)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty\mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot x^2-\frac{\hbar^2V_0^2}{2m}\left(\mathrm{e}^{2x}-2\alpha e^{x}\right)\right]\mathrm{d}t\right\}$$
$$=\frac{m\Gamma(\frac12+\sqrt{-2mE}/\hbar-\alpha V_0)}{\hbar^2V_0\Gamma(1+2\sqrt{-2mE})}\mathrm{e}^{-(x'+x'')/2}$$
$$\times W_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{x_>})M_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{x_<})\ , \tag{6.4.18}$$
$$=\sum_{n\in\mathbb{N}_0}\frac{2\alpha V_0-2n-1}{-\hbar^2(\alpha V_0-n-\frac12)^2/2m-E}\frac{n!(2V_0)^{2\alpha V_0-2n-1}}{\Gamma(2\alpha V_0-n)}$$
$$\times\exp\left[(x'+x'')(\alpha V_0-n-\tfrac12)-V_0(\mathrm{e}^{x'}+\mathrm{e}^{x''})\right]$$
$$\times L_n^{(2\alpha V_0-2n-1)}(2V_0\,\mathrm{e}^{x''})L_n^{(2\alpha V_0-2n-1)}(2V_0\,\mathrm{e}^{x'})$$
$$+\frac{1}{2\pi^2}\int_0^\infty\mathrm{d}k\frac{k\sinh2\pi k}{\hbar^2k^2/2m-E}\frac{|\Gamma(\mathrm{i}\,k-\alpha V_0+1/2)|^2}{V_0}\mathrm{e}^{-(x'+x'')/2}$$
$$\times W_{\alpha V_0,\mathrm{i}\,k}(2V_0\,\mathrm{e}^{x''})W_{\alpha V_0,\mathrm{i}\,k}(2V_0\,\mathrm{e}^{x'})\ . \tag{6.4.19}$$

6.4.3 Liouville Quantum Mechanics. [281,465] $(V_0>0)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty\mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot x^2-\frac{\hbar^2V_0^2}{2m}\mathrm{e}^{2x}\right)\mathrm{d}t\right]$$
$$=\frac{2m}{\hbar^2}I_{\sqrt{-2mE}/\hbar}(V_0\,\mathrm{e}^{x_<})K_{\sqrt{-2mE}/\hbar}(V_0\,\mathrm{e}^{x_>})\ , \tag{6.4.20}$$
$$=\frac{2}{\pi^2}\int_0^\infty\mathrm{d}k\,\frac{k\sinh\pi k}{\hbar^2k^2/2m-E}K_{\mathrm{i}\,k}(V_0\,\mathrm{e}^{x''})K_{\mathrm{i}\,k}(V_0\,\mathrm{e}^{x'})\ . \tag{6.4.21}$$

6.4.4 Inverted Liouville Potential. [447,696,893] ($k > 0, 0 < \alpha \leq 2$)

$$\int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{y}^2 + \frac{\hbar^2\kappa^2}{2m}\,\mathrm{e}^{2y}\right)\mathrm{d}t\right]$$
$$= \sum_{n\in\mathbb{N}} 2(2n+\alpha) J_{2n+\alpha}(\kappa\,\mathrm{e}^{-y'}) J_{2n+\alpha}(\kappa\,\mathrm{e}^{-y''})\ \mathrm{e}^{\mathrm{i}\,\hbar(4n^2-1)T/2m}$$
$$+ \int_0^\infty \frac{k\,\mathrm{d}k}{2\sinh\pi k}\,\mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}$$
$$\times \left[J_{\mathrm{i}\,k}(\kappa\,\mathrm{e}^{y''}) + J_{-\mathrm{i}\,k}(\kappa\,\mathrm{e}^{y''})\right]\left[J_{\mathrm{i}\,k}(\kappa\,\mathrm{e}^{y'}) + J_{-\mathrm{i}\,k}(\kappa\,\mathrm{e}^{y'})\right] . \tag{6.4.22}$$

6.4.5 Particle Inside a Sector (Sommerfeld Problem).

6.4.5.1 Free Particle Inside a Sector. [167,178,206,244,829,861,928,929].

$$\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t)\varrho \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}^{(DD)}_{(0<\varphi<\alpha)}\varphi(t)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{\varrho}^2 + \varrho^2\dot{\varphi}^2) + \frac{\hbar^2}{8m\varrho^2}\right)\mathrm{d}t\right]$$
$$= \frac{2m}{\mathrm{i}\,\alpha\hbar T}\sum_{\nu\in\mathbb{N}} \sin\left(\frac{\nu\pi}{\alpha}\varphi'\right)\sin\left(\frac{\nu\pi}{\alpha}\varphi''\right)\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}(\varrho'^2 + \varrho''^2)\right] I_{\pi\nu/\alpha}\left(\frac{m\varrho'\varrho''}{\mathrm{i}\,\hbar T}\right) \tag{6.4.23}$$
$$= \frac{2}{\alpha}\sum_{\nu\in\mathbb{N}}\int_0^\infty k\,\mathrm{d}k\,\mathrm{e}^{-\mathrm{i}\,\hbar Tk^2/2m}$$
$$\times \sin\left(\frac{\nu\pi}{\alpha}\varphi''\right)\sin\left(\frac{\nu\pi}{\alpha}\varphi'\right) J_{\pi\nu/\alpha}(k\varrho'')J_{\pi\nu/\alpha}(k\varrho') . \tag{6.4.24}$$

6.4.5.2 Radial Harmonic Oscillator Inside a Sector with Magnetic Field. [166–168,171,175] ($\Omega = \sqrt{\omega^2 + (eB/2mc)^2}, \lambda > 0$)

$$\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t)\varrho \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}_{(0<\varphi<\alpha)}\varphi(t)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{\varrho}^2 + \varrho^2\dot{\varphi}^2 - \omega^2\varrho^2) - \frac{eB}{2c}\varrho^2\dot{\varphi} - \hbar^2\frac{\lambda^2 - 1/4}{2m\varrho^2}\right)\mathrm{d}t\right]$$

$$= \frac{2}{\alpha}\sum_{\nu\in\mathbb{N}} \sin\left[\frac{\nu\pi}{\alpha}\left(\varphi'' - \frac{qB}{2mc}t''\right)\right]\sin\left[\frac{\nu\pi}{\alpha}\left(\varphi' - \frac{qB}{2mc}t'\right)\right]$$
$$\times \frac{m\Omega\sqrt{\varrho'\varrho''}}{\mathrm{i}\hbar\sin\Omega T}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}(\varrho'^2+\varrho''^2)\cot\Omega T\right] I_{\sqrt{(\pi\nu/\alpha)^2+\lambda^2}}\left(\frac{m\Omega\varrho'\varrho''}{\mathrm{i}\hbar\sin\Omega T}\right) . \tag{6.4.25}$$

6.4.6 The Calogero Model.

6.4.6.1 The Calogero Model. [315,402,577] ($\mathbf{x} = (x_1, x_2, x_3), \lambda > 0$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - \frac{\omega^2}{2}\sum_{i\neq j}(x_i-x_j)^2 - \hbar^2\frac{\lambda^2-1/4}{m(x_1-x_2)^2}\right)dt\right]$$

Cartesian Coordinates ($\tilde\omega = \sqrt{3/2}\,\omega$):

$$= \left(\frac{6\pi m}{\mathrm{i}\hbar T}\right)^{1/2}\left(\frac{m\tilde\omega}{2\pi\mathrm{i}\hbar\sin\tilde\omega T}\right)^{3/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right] I_\lambda\left(\frac{m\tilde\omega x'x''}{\mathrm{i}\hbar\sin\tilde\omega T}\right)$$
$$\times\sqrt{x'x''}\exp\left\{\frac{\mathrm{i}\,m\tilde\omega}{2\hbar\sin\tilde\omega T}\left[(x'^2+x''^2+y'^2+y''^2)\cos\tilde\omega T - 2y'y''\right]\right\} . \tag{6.4.26}$$

Circular Polar Coordinates ($\tilde\omega = \sqrt{3/2}\,\omega$):

$$= \frac{m\tilde\omega}{2\pi\mathrm{i}\hbar\sin\tilde\omega T}\left(\frac{3m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right]$$
$$\times\sum_{n\in\mathbb{N}_0}\frac{2^{2\lambda+1}(n+\lambda+\frac{1}{2})n!}{\Gamma(2\lambda+n+1)}\Gamma^2(\lambda+\tfrac{1}{2})C_n^{(\lambda+\frac{1}{2})}(\cos\varphi')C_n^{(\lambda+\frac{1}{2})}(\cos\varphi'')$$
$$\times(4\sin\varphi'\sin\varphi'')^{\lambda+\frac{1}{2}}\exp\left[\frac{\mathrm{i}\,m\tilde\omega}{2\hbar}(r'^2+r''^2)\cot\tilde\omega T\right] I_{\lambda+n+\frac{1}{2}}\left(\frac{m\tilde\omega r'r''}{\mathrm{i}\hbar\sin\tilde\omega T}\right) . \tag{6.4.27}$$

The coordinate transformation $(x_1, x_2, x_3) \mapsto (x, y, R)$ has the form

$$\left.\begin{aligned} x_1 &= R + \sqrt{\frac{1}{2}}\,x + \sqrt{\frac{1}{6}}\,y \\ x_2 &= R - \sqrt{\frac{1}{2}}\,x + \sqrt{\frac{1}{6}}\,y \\ x_3 &= R - \sqrt{\frac{2}{3}}\,y \end{aligned}\right\} \iff \left\{\begin{aligned} x_1 + x_2 + x_3 &= 3R \\ x_1 - x_2 &= \sqrt{2}\,x \\ x_1 + x_2 - 2x_3 &= \sqrt{6}\,y \end{aligned}\right. , \tag{6.4.28}$$

and $x = r\sin\varphi$, $y = r\cos\varphi$.

6.4.6.2 Three Particle Calogero–Moser Model. [577] ($\mathbf{x}=(x_1,x_2,x_3)$ as before, φ lies in the sectors $n\pi/3<\varphi<(n+1)\pi/3$, $n=0,1,\dots,5$; $\lambda>0$)

$$\begin{aligned}
&\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-\sum_{i\neq j}\left(\frac{m}{6}\omega^2(x_i-x_j)^2+\frac{\hbar^2}{m}\frac{\lambda^2-\frac{1}{4}}{(x_i-x_j)^2}\right)\right]\mathrm{d}t\right\}\\
&=\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega T}\left(\frac{3m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right](4\sin3\varphi'\sin3\varphi'')^{\lambda+\frac{1}{2}}\\
&\quad\times\sum_{n\in\mathbb{N}_0}\frac{3^{2\lambda+1}(n+\lambda+\frac{1}{2})n!}{\Gamma(2\lambda+n+1)}\Gamma^2(\lambda+\tfrac{1}{2})C_n^{(\lambda+\frac{1}{2})}(\cos3\varphi')C_n^{(\lambda+\frac{1}{2})}(\cos3\varphi'')\\
&\quad\times\exp\left[\frac{\mathrm{i}m\omega}{2\hbar}(r'^2+r''^2)\cot\omega T\right]I_{3(\lambda+n+\frac{1}{2})}\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)\ .
\end{aligned}\tag{6.4.29}$$

6.4.6.3 Three Particle Calogero–Marchioro Model. [579] ($\mathbf{x}=(x_1,x_2,x_3)$ as before, $x_{i+3}\equiv x_i$, $i=1,2,3$, φ lies in the sectors $n\pi/6<\varphi<(n+1)\pi/6$, $n=0,1,\dots,11$; $\lambda,\mu>0$)

$$\begin{aligned}
&\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-\sum_{i=1}^{3}\left(\frac{m}{6}\omega^2(x_i-x_{i+1})^2\right.\right.\right.\\
&\qquad\left.\left.\left.+\frac{\hbar^2}{m}\frac{\lambda^2-\frac{1}{4}}{(x_i-x_{i+1})^2}+3\frac{\mu^2-\frac{1}{4}}{(x_i+x_{i+1}-2x_{i+2})^2}\right)\right]\mathrm{d}t\right\}\\
&=3\left(\frac{3m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right]\sum_{n\in\mathbb{N}_0}\Phi_n^{(\lambda,\mu)}(3\varphi'')\Phi_n^{(\lambda,\mu)}(3\varphi')\\
&\quad\times\frac{m\omega}{\mathrm{i}\hbar\sin\omega T}\exp\left[\frac{\mathrm{i}m\omega}{2\hbar}(r'^2+r''^2)\cot\omega T\right]I_{3(\lambda+\mu+2n+1)}\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)\ .
\end{aligned}\tag{6.4.30}$$

6.4.6.4 Three Particle Singular Calogero–Marchioro Model. ($\mathbf{x}=(x_1,x_2,x_3)$ as before, $x_{i+3}\equiv x_i$, $i=1,2,3$, φ lies in the sectors $n\pi/6<\varphi<(n+1)\pi/6$, $n=0,1,\dots,11$; $\lambda,\alpha>0$; $\tilde{N}=N+\lambda+\frac{1}{2}$, $a=\hbar^2/m\alpha$, $\Lambda^2=9\tilde{N}^2-m^2\alpha^2/\hbar^4\tilde{N}^2$, $\sigma_N=1/3a\tilde{N}$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-\sum_{i=1}^{3}\left(\frac{m}{6}\omega^2(x_i-x_{i+1})^2\right.\right.\right.$$

$$+\frac{\hbar^2}{m}\frac{\lambda^2-\frac{1}{4}}{(x_i-x_{i+1})^2}+\frac{\alpha}{\sqrt{3}\,r^2}\frac{x_i+x_{i+1}-2x_{x+2}}{x_i-x_{i+1}}\Bigg)\Bigg]\mathrm{d}t\Bigg\}$$

$$=3\left(\frac{3m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right]\sum_{N\in\mathbb{N}_0}\Phi_N^{(\alpha)}(\varphi'')\Phi_N^{(\alpha)}(\varphi')$$
$$\times\frac{m\omega}{\mathrm{i}\hbar\sin\omega T}\exp\left[\frac{\mathrm{i}m\omega}{2\hbar}(r'^2+r''^2)\cot\omega T\right]I_\Lambda\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)\ , \tag{6.4.31}$$

$$\Phi_l^{(\alpha)}(\varphi)=\frac{3}{\Gamma(2\lambda+1)}\left[9\frac{\sigma_N^2+\tilde{N}^2}{\tilde{N}^2}\frac{\Gamma(\tilde{N}+\lambda+1/2)\Gamma(\mathrm{i}\sigma_N+\lambda+1/2)}{\Gamma(\tilde{N}-\lambda)\Gamma(\mathrm{i}\sigma_N-\lambda)}\right]^{1/2}$$
$$\times(2\sin3\varphi)^{\lambda+1/2}\exp\left[3\mathrm{i}\varphi(\mathrm{i}\sigma_N-N)\right]$$
$$\times{}_2F_1\Big(-N,\lambda+\frac{1}{2}+\mathrm{i}\sigma_N;2\lambda+1;1-\mathrm{e}^{6\mathrm{i}\varphi}\Big)\ . \tag{6.4.32}$$

6.4.6.5 Modified Three Particle Calogero–Marchioro Model. [579] ($\mathbf{x}=(x_1,x_2,x_3)$, $x_{i+3}\equiv x_i$, $i=1,2,3$, φ lies in the sectors $n\pi/6<\varphi<(n+1)\pi/6$, $n=0,1,\dots,11$, $A,B>0$, $\kappa=+3\sqrt{A+B+1/4},\lambda=+3\sqrt{A-B+1/4}$, $\Lambda=(\kappa+\lambda+2l+1)/4$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{m}{2}\dot{\mathbf{x}}^2-\sum_{i=1}^{3}\Bigg(\frac{m}{6}\omega^2(x_i-x_{i+1})^2$$
$$+\frac{\hbar^2}{m}\left(\frac{A}{(x_i-x_{i+1})^2}-\frac{B}{\sqrt{6}\,r}\frac{x_i+x_{i+1}-2x_{i+2}}{(x_i-x_{i+1})^2}\right)\Bigg)\Bigg]\mathrm{d}t\Bigg\}$$

$$=3\left(\frac{3m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\exp\left[-\frac{3m}{2\mathrm{i}\hbar T}(R''-R')^2\right]\sum_{l\in\mathbb{N}}\Phi_l(\varphi'')\Phi_l(\varphi')$$
$$\times\frac{m\omega}{\mathrm{i}\hbar\sin\omega T}\exp\left[\frac{\mathrm{i}m\omega}{2\hbar}(r'^2+r''^2)\cot\omega T\right]I_\Lambda\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right)\ , \tag{6.4.33}$$

$$\Phi_l(\varphi)=\left[\frac{(\kappa+\lambda+2n+1)n!\,\Gamma(\kappa+\lambda+n+1)}{\Gamma(\kappa+n+1)\Gamma(\lambda+n+1)}\right]^{1/2}$$
$$\times\left(\sin\frac{3\varphi}{2}\right)^{\kappa+1/2}\left(\cos\frac{3\varphi}{2}\right)^{\lambda+1/2}P_n^{(\kappa,\lambda)}(\cos3\varphi)\ . \tag{6.4.34}$$

6.4.7 Time-Dependent Centrifugal Potential. [170] ($m(t)g(t)\equiv K=$ time-independent)

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m(t)}{2}\dot{r}^2-\frac{m(t)}{2}\omega^2(t)r^2-\frac{g(t)}{r^2}\right)\mathrm{d}t\right]$$

$$
= \frac{\sqrt{m'm''\dot\mu'\dot\mu''r'r''}}{\mathrm{i}\hbar\sin(\mu''-\mu')} I_\nu\left(\frac{\sqrt{m'm''\dot\mu'\dot\mu''}\,r'r''}{\mathrm{i}\hbar\sin(\mu''-\mu')}\right)
$$

$$
\times \exp\left\{\frac{\mathrm{i}}{2\hbar}\left[(m'\dot\mu'r'^2+m''\dot\mu''r''^2)\cot(\mu''-\mu')+\frac{\dot m'r'^2-\dot m''r''^2}{2}+\frac{m''\dot s''r''^2}{s''}\right]\right\},
\tag{6.4.35}
$$

where $s(t)$ and $\mu(t)$ satisfy ($\eta(t)=\sqrt{m(t)}$)

$$
\ddot s(t)+\left(\omega^2(t)-\frac{\ddot\eta(t)}{\eta(t)}\right)s(t)=\frac{1}{s^3(t)},\qquad \ddot\mu(t)+\left(\omega^2(t)-\frac{\ddot\eta(t)}{\eta(t)}\right)\mu(t)=\frac{1}{\mu^3(t)}\ ,
\tag{6.4.36}
$$

with boundary conditions $s(t')=1, \dot s(t')=0, s^2(t)\dot\mu(t)=1$, and $\dot\mu(t')=1$, and $\nu=\frac{1}{2}\sqrt{1+8mg/\hbar^2}\equiv\sqrt{2K/\hbar^2+1/4}$.

6.4.8 Aharonov–Bohm Effect. [7]

6.4.8.1 Aharonov–Bohm Effect in the Plane. [79,86,87,289,290,362,387,515, 529,574,575,613,629,650,651,709,812,827,877,880,886,925,926,927]

$$
\int_{\mathbb{R}} d\varphi \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)r \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\,\delta\left(\varphi-\int_{t'}^{t''}\dot\varphi\,\mathrm{d}t\right)
$$

$$
\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot r^2+r^2\dot\varphi^2)+f\hbar\dot\varphi+\frac{\hbar^2}{8mr^2}\right)\mathrm{d}t\right]
$$

$$
=\frac{m}{2\pi\mathrm{i}\hbar T}\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}(\nu+f)(\varphi''-\varphi')}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(r'^2+r''^2)\right]I_{|\nu|}\left(\frac{mr'r''}{\mathrm{i}\hbar T}\right)
\tag{6.4.37}
$$

($f=e\Phi/2\pi c\hbar$, $\Phi=$ magnetic flux). The Green function is given by [877]

$$
G(x'',x';E)=\left\{\begin{matrix}1\\ \mathrm{e}^{2\pi\mathrm{i}f}\\ \mathrm{e}^{-2\pi\mathrm{i}f}\end{matrix}\right\}\frac{m}{\pi\hbar^2}K_0\left(\sqrt{-2mE}\,\frac{|\mathbf{x}''-\mathbf{x}'|}{\hbar}\right)
$$

$$
-\frac{\sin\pi f}{\pi}\frac{m}{\pi\hbar^2}\int_{\mathbb{R}}\mathrm{d}s\,K_0\left(\sqrt{-2mE}\,\frac{R(s)}{\hbar}\right)\frac{\mathrm{e}^{-f[(s-\mathrm{i}(\varphi''-\varphi')]}}{1+\mathrm{e}^{-s+\mathrm{i}(\varphi''-\varphi')}}\ ,
\tag{6.4.38}
$$

where the value in $\{\cdot\}$ is taken depending on whether $(\varphi''-\varphi')\in(-\pi,\pi)$, $(\pi,2\pi),(-2\pi,-\pi)$, respectively, and $R^2(s)=r'^2+r''^2+r'r''\cosh s$. Note that the claim of Ref. [83] to evaluate ring-shaped topological defects via toroidal coordinates cannot be seen as correct.

6.4.8.2 Aharonov–Bohm Effect with a Radial Harmonic Oscillator and a Magnetic Field. [87,166,168,172,387,529,771,886,925] ($\omega = eB/2mc$, $\Omega^2 = \omega^2 + \omega_0^2$, $f = e\Phi/2\pi\hbar c$, $\mu_\nu = \sqrt{(\nu - f)^2 + g}$, $g > 0$)

$$\int_{\mathbb{R}} \mathrm{d}\varphi \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\, \delta\left(\varphi - \int_{t'}^{t''} \dot\varphi\, \mathrm{d}t + \omega T\right)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot r^2 + r^2(\dot\varphi^2 + 2\omega\dot\varphi - \omega_0^2)\right) + f\hbar\dot\varphi - \hbar^2\frac{g-1/4}{2mr^2}\right]\mathrm{d}t\right\}$$

$$= \frac{m\Omega}{2\pi\mathrm{i}\hbar\sin\Omega T}\mathrm{e}^{\mathrm{i}f(\varphi''-\varphi'+\omega T)}\exp\left[-\frac{m\Omega}{2\mathrm{i}\hbar}(r'^2 + r''^2)\cot\Omega T\right]$$

$$\times \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi'+\omega T)} I_{\mu_\nu}\left(\frac{m\Omega r'r''}{\mathrm{i}\hbar\sin\Omega T}\right) . \tag{6.4.39}$$

6.4.9 Anharmonic Potentials.

6.4.9.1 Anharmonic Radial (Confinement) Potentials. [659,865] ($L = (2l+1)/(p+2)$, $p > -2$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{l+1/2}[r^2]\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot r^2 - gr^p)\mathrm{d}t\right]$$

$$= \frac{4m}{\hbar^2}\frac{\sqrt{r'r''}}{p+2}K_L\left(\frac{2}{p+2}\sqrt{\frac{2mg}{\hbar^2}}r_>^{(p+2)/2}\right)I_L\left(\frac{2}{p+2}\sqrt{\frac{2mg}{\hbar^2}}r_<^{(p+2)/2}\right) . \tag{6.4.40}$$

6.4.9.2 Modified Coulomb Potential (Conditionally Solvable Natanzon Potential). [449,450,452]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2 - V(r)\right)\mathrm{d}t\right]$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma(-\nu)$$

$$\times\left\{D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h_>(r) - \frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[-\sqrt{\frac{2m\omega}{\hbar}}\left(h_<(r) - \frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]\right.$$

$$- D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h(r') - \frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[\sqrt{\frac{2m\omega}{\hbar}}\left(h(r'') - \frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\right]$$

$$\times\, D_\nu\left(\sqrt{\frac{2m\omega}{\hbar}}\,\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\Bigg/ D_\nu\left(-\sqrt{\frac{2m\omega}{\hbar}}\,\frac{\hbar^2 g_1}{4m\sigma_1 E}\right)\Bigg\}\ , \tag{6.4.41}$$

$$V(r)=\frac{\hbar^2}{2m}\frac{g_2h^2+g_1h^3}{R(r)}+\frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2-2\frac{h'''}{h'}\right)\ . \tag{6.4.42}$$

Here $R=\sigma_2h^2+\sigma_1h^4$, and the function $h=h(r)$ is implicitly defined by the differential equation $h'(r)=2h(r)/\sqrt{R}$. Furthermore we have abbreviated ($\omega^2=-\sigma_1E/2m$)

$$\nu=-\frac{1}{2}+\frac{E'}{\omega\hbar}\ ,\qquad E'=\frac{1}{4}\left(\sigma_2E-\frac{\hbar^2g_2}{2m}+\frac{\hbar^4g_1^2}{16m^2\sigma_1E}\right)\ . \tag{6.4.43}$$

Table 6.6. Confluent conditionally solvable Natanzon potentials

$R(h)$	$V(h)$	Range
$R=\sigma_1h^4+\sigma_2h^2$	$\frac{\hbar^2}{2m}\frac{g_1h^3+g_2h^2}{R(h)}+\Delta V(h)$	$h>0$
$R=4h^2, h=r$	$\frac{\hbar^2}{8m}(g_1r+g_2)$	$r>0$
$R=16h^4, h=\sqrt{r}$	$\frac{\hbar^2}{32M}\left(\frac{g_1}{\sqrt{r}}+\frac{g_2}{r}-\frac{3}{r^2}\right)$	$r>0$
$R=\sigma_1h^3+\sigma_2h^2$	$\frac{\hbar^2}{2m}\frac{g_1h^4+g_2h^2}{R(h)}+\Delta V(h)$	$h>0$
$R=4h^2, h=r$	$\frac{\hbar^2}{8m}(g_1r^2+g_2)$	$r>0$
$R=9h^3, h=r^{2/3}$	$\frac{\hbar^2}{18M}\left(g_1r^{2/3}+g_2r^{-2/3}-\frac{5}{4r^2}\right)$	$r>0$

6.4.9.3 Radial Confinement Potential (Natanzon Potential). [449,450,452]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-V(r)\right)\mathrm{d}t\right]$$

$$=\left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{2\pi\hbar^3\omega}}\,\Gamma(-\nu)$$

$$\times\left\{D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h_>(r)-\frac{m\sigma_1}{\hbar^2g_1}E\right)\right]D_\nu\left[-\sqrt{\frac{m\omega}{\hbar}}\left(h_<(r)-\frac{m\sigma_1}{\hbar^2g_1}E\right)\right]\right.$$

$$- D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h(r') - \frac{m\sigma_1}{\hbar^2 g_1}E\right)\right] D_\nu\left[\sqrt{\frac{m\omega}{\hbar}}\left(h(r'') - \frac{m\sigma_1}{\hbar^2 g_1}E\right)\right]$$

$$\times D_\nu\left(\sqrt{\frac{m\omega}{\hbar}}\frac{m\sigma_1}{\hbar^2 g_1}E\right) \Big/ D_\nu\left(-\sqrt{\frac{m\omega}{\hbar}}\frac{m\sigma_1}{\hbar^2 g_1}E\right)\Bigg\} , \tag{6.4.44}$$

$$V(r) = \frac{\hbar^2}{2m}\frac{g_2 h^2 + g_1 h^4}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) . \tag{6.4.45}$$

Here $R = \sigma_2 h^2 + \sigma_1 h^3$, and the function $h = h(r)$ is implicitly defined by the differential equation $h'(r) = 2h(r)/\sqrt{R}$, and ($\omega = \hbar\sqrt{g_1}/2m$)

$$\nu = -\frac{1}{2} + \frac{E'}{\hbar\omega} , \qquad E' = \frac{1}{4}\left(\sigma_2 E + \frac{m\sigma_1^2 E^2}{2\hbar^2 g_1} - \frac{\hbar^2 g_2}{2m}\right) . \tag{6.4.46}$$

6.4.9.4 Sextic Potential – Quasi-Exactly Solvable Model. [458,659,903] ($\lambda^2 = (\beta^2 - 29)/16$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{x}^2 - \omega^2 x^6) + \frac{\hbar^2}{2m}\left(k^2 x^2 - \frac{\beta^2}{x^2}\right)\right]\mathrm{d}t\right\}$$

$$= (x'x'')^{1/2}\frac{4}{\hbar\omega}\frac{\Gamma(\frac{1}{2} + \lambda + \hbar k^2/8m\omega)}{\Gamma(1 + 2\lambda)}$$

$$\times W_{-\hbar k^2/8m\omega,\lambda}\left(\frac{m\omega}{2\hbar}z_>^4\right) M_{-\hbar k^2/8m\omega,\lambda}\left(\frac{m\omega}{2\hbar}z_<^4\right) . \tag{6.4.47}$$

6.4.9.5 Power-Confinement Potential. [71] ($\mathbf{x} \in \mathbb{R}^2$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2 - \left(\alpha r^{2d-2} - \beta r^{d-2}\right)\right]\mathrm{d}t\right\}$$

$$= \sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi}\frac{-2\mathrm{i}}{d\sqrt{2\alpha}}\frac{\Gamma\left(\frac{1}{2} + |\nu|/d - \beta/d\sqrt{2\alpha}\right)}{(r'r'')^{d/2}\Gamma(1 + 2|\nu|/d)}$$

$$\times W_{\beta/d\sqrt{2\alpha},|\nu|/d}\left(\frac{2\sqrt{2\alpha}}{d}r_>^d\right) M_{\beta/d\sqrt{2\alpha},|\nu|/d}\left(\frac{2\sqrt{2\alpha}}{d}r_<^d\right) . \tag{6.4.48}$$

6.4.10 Super-Integrable Potentials. [305,668]

6.4.10.1 Generalized Oscillator. ($\mathbf{x}=(x_1,x_2)$, $k_{1,2}>0$, [274,402,458,771,865])

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{\mathbf{x}}^2-\omega^2\mathbf{x}^2)-\frac{\hbar^2}{2m}\left(\frac{k_1^2-\frac14}{x_1^2}+\frac{k_2^2-\frac14}{x_2^2}\right)\right]\mathrm{d}t\right\}$$

Cartesian:

$$=\left(\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\right)^2\prod_{i=1}^{2}\sqrt{x_i'x_i''}\exp\left[\frac{\mathrm{i}\,m\omega}{2\hbar}(x_i'^2+x_i''^2)\cot\omega T\right]I_{k_i}\left(\frac{m\omega x_i'x_i''}{\mathrm{i}\,\hbar\sin\omega T}\right)\ . \tag{6.4.49}$$

Polar ($\lambda=k_1+k_2+2n+1$, [139–142,458]):

$$=\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\sum_{n\in\mathbb{N}_0}\Phi_n^{(k_1,k_2)}(\varphi'')\Phi_n^{(k_1,k_2)}(\varphi')$$
$$\times\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(\varrho'^2+\varrho''^2)\cot\omega T\right]I_\lambda\left(\frac{m\omega\varrho'\varrho''}{\mathrm{i}\,\hbar\sin\omega T}\right)\ . \tag{6.4.50}$$

6.4.10.2 Holt Potential. [274,340,402,458,771,865] ($\mathbf{x}=(x,y)$, $x\in\mathbb{R}, y>0$, $k_1\in\mathbb{R}, k_2>0$, $\tilde{x}=x+k_1/8m\omega^2$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(\dot{\mathbf{x}}^2-\omega^2(4x^2+y^2)\Big)-k_1x-\frac{\hbar^2}{2m}\frac{k_2^2-\frac14}{y^2}\right]\mathrm{d}t\right\}$$

$$=\sqrt{\frac{m\omega y'y''}{\mathrm{i}\,\pi\hbar\sin2\omega T}}\exp\left\{-\frac{m\omega}{\mathrm{i}\,\hbar\sin2\omega T}\Big[(\tilde{x}'^2+\tilde{x}''^2)\cos2\omega T-2\tilde{x}'\tilde{x}''\Big]\right\}$$
$$\times\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(y'^2+y''^2)\cot\omega T\right]I_{k_2}\left(\frac{m\omega y'y''}{\mathrm{i}\,\hbar\sin\omega T}\right)\ . \tag{6.4.51}$$

6.4.10.3 Generalized Oscillator. ($\mathbf{x}=(x_1,x_2,x_3)$, $k_{1,2,3}>0$, $\lambda_1=k_1+k_2+2n+1$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{\mathbf{x}}^2-\omega^2\mathbf{x}^2)-\frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{k_i^2-\frac14}{x_i^2}\right)\mathrm{d}t\right]$$

Cartesian [274,402,458,771,865]:

$$=\left(\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\right)^3\prod_{j=1}^{3}\sqrt{x_i'x_i''}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(x_i'^2+x_i''^2)\cot\omega T\right]I_{k_i}\left(\frac{m\omega x_i'x_i''}{\mathrm{i}\,\hbar\sin\omega T}\right). \tag{6.4.52}$$

Circular Polar ($\lambda_1 = k_1 + k_2 + 2n + 1$, [139–143,458]):

$$= \left(\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\right)^2 \sqrt{z'z''\varrho'\varrho''}\,\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}({z'}^2+{z''}^2)\cot\omega T\right] I_{k_3}\left(\frac{m\omega z'z''}{\mathrm{i}\,\hbar\sin\omega T}\right)$$
$$\times \sum_{n\in\mathbb{N}_0} \Phi_n^{(k_1,k_2)}(\varphi'')\Phi_n^{(k_1,k_2)}(\varphi')$$
$$\times \exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}({\varrho'}^2+{\varrho''}^2)\cot\omega T\right] I_{\lambda_1}\left(\frac{m\omega\varrho'\varrho''}{\mathrm{i}\,\hbar\sin\omega T}\right) . \tag{6.4.53}$$

Spherical ($\lambda_2 = 2m + \lambda_1 + 1$, [139–142]):

$$= (r'r''\sin\vartheta'\sin\vartheta'')^{-1/2}$$
$$\times \sum_{m,n\in\mathbb{N}_0} \Phi_n^{(k_1,k_2)}(\varphi'')\Phi_n^{(k_1,k_2)}(\varphi')\Phi_m^{(\lambda_1,k_3)}(\vartheta'')\Phi_m^{(\lambda_1,k_3)}(\vartheta')$$
$$\times \frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}({r'}^2+{r''}^2)\cot\omega T\right] I_{\lambda_2}\left(\frac{mr'r''}{\mathrm{i}\,\hbar\sin\omega T}\right) . \tag{6.4.54}$$

6.4.10.4 Ring Shaped Potential. ($k_{1,2,3} > 0, \lambda_\pm^2 = k_2^2 \pm k_1^2, \lambda_1 = n + (\lambda_+ + \lambda_- + 1)/2, \lambda_2 = 2m + \lambda_1 + k_3 + 1$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2 - \frac{\hbar^2}{2m}\left(\frac{k_1^2 x/y^2}{\sqrt{x^2+y^2}} + \frac{k_2^2-\frac14}{y^2} + \frac{k_3^2-\frac14}{z^2}\right)\right]\mathrm{d}t\right\} \tag{6.4.55}$$

Spherical [139,458]:

$$= \frac{1}{2}(r'r''\sin\vartheta'\sin\vartheta'')^{-1/2}\sum_{n\in\mathbb{N}_0}\Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi''}{2}\right)\Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi'}{2}\right)$$
$$\times \frac{m}{\mathrm{i}\,\hbar T}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}({r'}^2+{r''}^2)\right]\sum_{m\in\mathbb{N}_0}\Phi_m^{(\lambda_1,k_3)}(\vartheta'')\Phi_m^{(\lambda_1,k_3)}(\vartheta')I_{\lambda_2}\left(\frac{mr'r''}{\mathrm{i}\,\hbar T}\right) . \tag{6.4.56}$$

Circular Polar [458]:

$$= \left(\frac{m}{\mathrm{i}\,\hbar T}\right)^2\sqrt{z'z''}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}({z'}^2+{z''}^2+{\varrho'}^2+{\varrho''}^2)\right] I_{k_3}\left(\frac{mz'z''}{\mathrm{i}\,\hbar T}\right)$$
$$\times \frac{1}{2}\sum_{n\in\mathbb{N}_0}\Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi''}{2}\right)\Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi'}{2}\right) I_{\lambda_1}\left(\frac{m\varrho'\varrho''}{\mathrm{i}\,\hbar T}\right) . \tag{6.4.57}$$

Circular Parabolic ($\omega = \sqrt{-2E/m}$ [458]):

$$= \frac{m\sqrt{z'z''}}{\mathrm{i}\,\hbar T}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}({z'}^2+{z''}^2)\right] I_{k_3}\left(\frac{mz'z''}{\mathrm{i}\,\hbar T}\right)$$

$$\times \int_{\mathbb{R}} \mathrm{d}\zeta \int_0^\infty \frac{\mathrm{d}k}{k} \frac{\left|\Gamma\left(\frac{1+\lambda_+}{2} + \frac{\mathrm{i}\zeta}{2k}\right)\right|^2 \left|\Gamma\left(\frac{1+\lambda_-}{2} - \frac{\mathrm{i}\zeta}{2k}\right)\right|^2 \mathrm{e}^{\pi/ak}}{4\pi^2\sqrt{\xi'\xi''\eta'\eta''}\,\Gamma^2(1+\lambda_-)\Gamma^2(1+\lambda_+)}$$
$$\times M_{-\mathrm{i}\zeta/2k,\lambda_+/2}(-\mathrm{i}\,k\xi''^2) M_{\mathrm{i}\zeta/2k,\lambda_+/2}(\mathrm{i}\,k\xi'^2)$$
$$\times M_{\mathrm{i}\zeta/2k,\lambda_-/2}(-\mathrm{i}\,k\eta''^2) M_{-\mathrm{i}\zeta/2k,\lambda_-/2}(\mathrm{i}\,k\eta'^2)\,\mathrm{e}^{-\mathrm{i}\hbar k^2 T/2m} \quad . \tag{6.4.58}$$

6.4.10.5 Holt Potential. [139–143,458] ($\mathbf{x} = (x_1, x_2, x_3), k_{1,2} > 0$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{\mathbf{x}}^2 - \omega^2(x_1^2 + x_2^2 + 4x_3^2)\right) - \frac{\hbar^2}{2m}\left(\frac{k_1^2 - \frac14}{x_1^2} + \frac{k_2^2 - \frac14}{x_2^2}\right)\right]\mathrm{d}t\right\}$$
$$= \sqrt{\frac{m\omega}{\mathrm{i}\,\pi\hbar\sin 2\omega T}} \exp\left\{-\frac{m\omega}{\mathrm{i}\,\hbar\sin 2\omega T}\left[(x_3'^2 + x_3''^2)\cos 2\omega T - 2x_3'x_3''\right]\right\}$$
$$\times \left(\frac{m\omega}{\mathrm{i}\,\hbar\sin\omega T}\right)^2 \prod_{j=1}^{2} \sqrt{x_i'x_i''} \exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(x_i'^2 + x_i''^2)\cot\omega T\right] I_{k_i}\left(\frac{m\omega x_i'x_i''}{\mathrm{i}\,\hbar\sin\omega T}\right) . \tag{6.4.59}$$

6.4.10.6 Ring Shaped Potential plus Linear Term. [458] ($\mathbf{x} = (x_1, x_2, x_3)$, $x_1, x_2 > 0, x_3 \equiv z \in \mathbb{R}, k_{1,2} > 0, k_3 \in \mathbb{R}, \lambda_\pm^2 = k_2^2 \pm k_1^2, \lambda_1 = (n + (\lambda_+ + \lambda_- + 1)/2), \omega = \sqrt{-2E/m}$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2 - \frac{\hbar^2}{2m}\left(\frac{k_1^2 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2^2 - \frac14}{y^2}\right) - k_3 z\right]\mathrm{d}t\right\}$$

<u>Circular Polar Coordinates</u>:

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{1/2} \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2T}(z''-z')^2 - \frac{k_3 T}{2}(z'+z'') - \frac{k_3^2 T^3}{24m}\right)\right]$$
$$\times \frac{m}{2\,\mathrm{i}\,\hbar T}\sum_{n\in\mathbb{N}_0} \Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi''}{2}\right)\Phi_n^{(\lambda_+,\lambda_-)}\left(\frac{\varphi'}{2}\right)$$
$$\times \exp\left[-\frac{m}{2\,\mathrm{i}\,\hbar T}(\varrho'^2 + \varrho''^2)\right] I_{\lambda_1}\left(\frac{m\varrho'\varrho''}{\mathrm{i}\,\hbar T}\right) \quad . \tag{6.4.60}$$

<u>Circular Parabolic Coordinates</u> ($\omega = \sqrt{-2E/m}$):

$$= \left(\frac{m}{2\pi\,\mathrm{i}\,\hbar T}\right)^{1/2} \sqrt{\xi'\xi''\eta'\eta''} \exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2T}(z''-z')^2 - \frac{k_3 T}{2}(z'+z'') - \frac{k_3^2 T^3}{24m}\right)\right]$$

$$\times \int_{\mathbb{R}} \mathrm{d}\zeta \int_0^\infty \frac{\mathrm{d}k}{k} \frac{|\Gamma(\frac{1+\lambda_+}{2} + \frac{\mathrm{i}\zeta}{2k})|^2 |\Gamma(\frac{1+\lambda_-}{2} - \frac{\mathrm{i}\zeta}{2k})|^2 \,\mathrm{e}^{\pi/ak}}{4\pi^2\sqrt{\xi'\xi''\eta'\eta''}\,\Gamma^2(1+\lambda_-)\Gamma^2(1+\lambda_+)}$$
$$\times M_{-\mathrm{i}\zeta/2k,\lambda_+/2}(-\mathrm{i}\,k\xi''^2) M_{\mathrm{i}\zeta/2k,\lambda_+/2}(\mathrm{i}\,k\xi'^2)$$
$$\times M_{\mathrm{i}\zeta/2k,\lambda_-/2}(-\mathrm{i}\,k\eta''^2) M_{-\mathrm{i}\zeta/2k,\lambda_-/2}(\mathrm{i}\,k\eta'^2)\,\mathrm{e}^{-\mathrm{i}\hbar k^2 T/2m} \ . \tag{6.4.61}$$

6.4.11 The General Besselian Path Integral (Natanzon Potential). [451,719]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right]$$
$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E'}}\left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2} \frac{\Gamma(1/2+\lambda-\kappa)}{\Gamma(2\lambda+1)}$$
$$\times W_{\kappa,\lambda}\left(\sqrt{-8mE'}\frac{h_>(r)}{\hbar}\right) M_{\kappa,\lambda}\left(\sqrt{-8mE'}\frac{h_<(r)}{\hbar}\right) , \tag{6.4.62}$$

$$V(r) = \frac{\hbar^2}{2m}\frac{g_2h^2+g_1h+\eta}{R} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) , \tag{6.4.63}$$

where $R = \sigma_2 h^2 + \sigma_1 h + c_0$, and the function $h = h(r)$ is according to [719] defined implicitly by the differential equation $h'(r) = 2h(r)/\sqrt{R}$. Furthermore we have for the quantities λ, κ, E':

$$\left.\begin{aligned} &\lambda^2 = \frac{1}{4}\left(\eta + 1 - \frac{2mc_0E}{\hbar^2}\right) , \qquad E' = \frac{1}{4}\left(\sigma_2 E - \frac{g_2\hbar^2}{2m}\right) , \\ &\kappa = \frac{\sigma_1 E - g_1\hbar^2/2m}{4\hbar}\sqrt{-\frac{m}{2E'}} \ . \end{aligned}\right\} \tag{6.4.64}$$

6.5 The Pöschl–Teller Potential [780]

6.5.1 The Pöschl–Teller Potential.

6.5.1.1 General Pöschl–Teller Potential. [104,272,528,531,617] $(\alpha,\beta > 0)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}$$
$$\times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\alpha^2-\frac{1}{4}}{\sin^2 x} + \frac{\beta^2-\frac{1}{4}}{\cos^2 x}\right)\right]\mathrm{d}t\right\}$$

$$= \frac{m}{2\hbar^2}\sqrt{\sin 2x' \sin 2x''}\frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\sin x' \sin x''\right)^{m_1 - m_2}\left(\cos x' \cos x''\right)^{m_1 + m_2}$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 - m_2 + 1; \sin^2 x_<\right)$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 + m_2 + 1; \cos^2 x_>\right) \ , \qquad (6.5.1)$$

$$= \sum_{n\in\mathbb{N}_0} \frac{\Phi_n^{(\alpha,\beta)}(x'')\Phi_n^{(\alpha,\beta)}(x')}{E_n - E} \ , \qquad (6.5.2)$$

$$\Phi_n^{(\alpha,\beta)}(x) = \left[2(\alpha+\beta+2n+1)\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2}$$
$$\times (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x) \ , \qquad (6.5.3)$$

$$E_n = \frac{\hbar^2}{2m}(\alpha+\beta+2n+1)^2 \ , \qquad n \in \mathbb{N}_0 \ , \qquad (6.5.4)$$

with $m_{1/2} = \frac{1}{2}(\beta \pm \alpha), L_E = -\frac{1}{2} + \frac{1}{2}\sqrt{2mE}/\hbar$.

6.5.1.2 Symmetric Pöschl–Teller Potential. [344,526,527,742] ($L_E = -\frac{1}{2} + \sqrt{2mE}/\hbar, \lambda > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\frac{\lambda^2 - \frac{1}{4}}{\sin^2 x}\right)\mathrm{d}t\right]$$
$$= \frac{m}{\hbar^2}\sqrt{\sin x' \sin x''}\,\Gamma(\lambda - L_E)\Gamma(L_E + \lambda + 1)P_{L_E}^{-\lambda}(\cos x_<)P_{L_E}^{-\lambda}(-\cos x_>) \ , \qquad (6.5.5)$$

$$= \sum_{n\in\mathbb{N}_0} \frac{\Psi_n(x'')\Psi_n(x')}{\hbar^2\left(n+\lambda+\frac{1}{2}\right)^2/2m - E} \ , \qquad (6.5.6)$$

$$\Psi_n(x) = \left[(n+\lambda+\tfrac{1}{2})\frac{\Gamma(n+2\lambda+1)}{n!}\right]^{1/2}\sqrt{\sin x}\,P_{\lambda+n}^{-\lambda}(\cos x) \ . \qquad (6.5.7)$$

6.5.2 Scarf-Like Potential. [441] ($|A \pm B| + 1/4 > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}$$
$$\times \int\limits_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''} \mathcal{D}\alpha(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\alpha}^2 - \frac{\hbar^2}{2m}\left(A\cot^2\alpha - B\frac{\cot\alpha}{\sin\alpha}\right)\right]\mathrm{d}t\right\}$$
$$= \frac{m}{4\hbar^2}\sqrt{\sin\alpha' \sin\alpha''}\frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$

$$\times\left(\frac{1-\cos\alpha'}{2}\frac{1-\cos\alpha''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos\alpha'}{2}\frac{1+\cos\alpha''}{2}\right)^{(m_1+m_2)/2}$$
$$\times\,{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos\alpha_<}{2}\right)$$
$$\times\,{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos\alpha_>}{2}\right)\ , \tag{6.5.8}$$
$$=\frac{1}{2}\sum_{n\in\mathbb{N}_0}\frac{\Phi_n^{(\kappa,\lambda)}(\alpha''/2)\Phi_n^{(\kappa,\lambda)}(\alpha'/2)}{E_n-E}\ , \tag{6.5.9}$$
$$E=\frac{\hbar^2}{8m}(\kappa+\lambda+2n+1)^2-A\frac{\hbar^2}{2m}\ , \tag{6.5.10}$$

where $\kappa=+\sqrt{A+B+1/4}$, $\lambda=+\sqrt{A-B+1/4}$, $m_{1,2}=\frac{1}{2}(\lambda\pm\kappa)$, and $L_E=-\frac{1}{2}+\sqrt{A+2mE/\hbar^2}$.

6.5.3 The Symmetric Top. [424,547,740,826] ($d=|M-L|$, $s=|M+L|$)

$$\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sqrt{\frac{B}{A}}\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''}\mathcal{D}\psi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{A}{2}\dot\vartheta^2+\frac{1}{2}(A\sin^2\vartheta+B\cos^2\vartheta)\dot\psi^2\right.\right.$$
$$\left.\left.+\frac{B}{2}\dot\varphi^2+B\cos\vartheta\,\dot\psi\dot\varphi+\frac{\hbar^2}{8A}\left(1+\frac{1}{\sin^2\vartheta}\right)\right]\mathrm{d}t\right\}$$
$$=\sum_{n\in\mathbb{N}_0}\sum_{M\in\mathbb{Z}}\sum_{L\in\mathbb{Z}}\Psi_{n,L,M}(\vartheta'',\psi'',\varphi'')\Psi^*_{n,L,M}(\vartheta',\psi',\varphi')\,\mathrm{e}^{-\mathrm{i}TE_n/\hbar}\ , \tag{6.5.11}$$
$$\Psi_{n,L,M}(\vartheta,\psi,\varphi)=N_n\,\mathrm{e}^{-\mathrm{i}L\varphi}\,\mathrm{e}^{-\mathrm{i}M\psi}(1-\cos\vartheta)^{|M-L|/2}(1+\cos\vartheta)^{|M+L|/2}$$
$$\times P_n^{(|M-L|,|M+L|)}(\cos\vartheta)\ , \tag{6.5.12a}$$
$$N_{n,L,M}=\left[\sqrt{\frac{A}{B}}\frac{(d+s+2n+1)\,n!\,\Gamma(d+s+n+1)}{8\pi^2 2^{d+s+1}(|M-L|+n)!(M+L|+n)!}\right]^{1/2}\ , \tag{6.5.12b}$$
$$E_{n,L,M}=\hbar^2\frac{\left(n+\frac{d+s+1}{2}\right)^2-\frac{1}{4}}{2A}+\hbar^2\frac{L^2}{2}\left(\frac{1}{B}-\frac{1}{A}\right)\ . \tag{6.5.13}$$

6.5.4 The Magnetic Top. [58] ($\tilde{L} = L - gBI/\hbar$, $A = B = I$ in the notation of Sect. 6.5.3, $n \in \mathbb{N}$, $d = |M - \tilde{L}|$, $s = |M + \tilde{L}|$)

$$\Psi_{n,L,M}(\vartheta, \psi, \varphi) = N_n \, \mathrm{e}^{-\mathrm{i}L\varphi} \, \mathrm{e}^{-\mathrm{i}M\psi} \times (1 - \cos\vartheta)^{d/2}(1 + \cos\vartheta)^{s/2} P_n^{(d,s)}(\cos\vartheta) \ , \tag{6.5.14a}$$

$$N_{n,L,M} = \left[\frac{(d+s+2n+1)\,n!\,\Gamma(d+s+n+1)}{8\pi^2 2^{d+s+1}\Gamma(d+n+1)\Gamma(s+n+1)}\right]^{1/2} \ , \tag{6.5.14b}$$

$$E_{n,L,M} = \frac{\hbar^2}{2I}\left[\left(n + \frac{d+s+1}{2}\right)^2 - \frac{1}{4}\right] \ . \tag{6.5.15}$$

6.5.5 Higgs Oscillator on the Sphere. [459] ($\lambda_1 = k_1 + k_2 + 2n + 1$, $\lambda_2 = \lambda_1 + k_3 + 2m + 1$, $\lambda_3^2 = m^2\omega^2 R^4/\hbar^2 + \frac{1}{4}$, $k_{1,2,3} > 0$)

$$\begin{aligned}
&\frac{1}{R^3}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''} \mathcal{D}\chi(t)\sin^2\chi \int\limits_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \mathcal{D}\theta(t)\sin\theta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \\
&\times \exp\left[\!\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}R^2\Big[\dot\chi^2 + \sin^2\chi(\dot\vartheta^2 + \sin^2\vartheta\dot\varphi^2) - \omega^2\tan^2\chi\Big]\right.\right.\right. \\
&\left.\left.\left. - \frac{\hbar^2}{2mR^2}\left\{\frac{1}{\sin^2\chi}\left[\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2-\frac{1}{4}}{\sin^2\varphi} + \frac{k_2^2-\frac{1}{4}}{\cos^2\varphi} - \frac{1}{4}\right) + \frac{k_3^2-\frac{1}{4}}{\cos^2\vartheta} - \frac{1}{4}\right] - 1\right\}\right)dt\right]\!\!\right] \\
&= \frac{1}{R^3}(\sin^2\chi'\sin^2\chi''\sin\theta'\sin\theta'')^{-1/2}\sum_{n\in\mathbb{N}_0}\Phi_n^{(k_1,k_2)}(\varphi'')\Phi_n^{(k_1,k_2)}(\varphi') \\
&\times \sum_{m\in\mathbb{N}_0}\Phi_m^{(\lambda_1,k_3)}(\vartheta'')\Phi_m^{(\lambda_1,k_3)}(\vartheta')\sum_{l\in\mathbb{N}_0}\Psi_l^{(\lambda_2,\lambda_3)}(\chi'')\Psi_l^{(\lambda_2,\lambda_3)}(\chi')\,\mathrm{e}^{-\mathrm{i}TE_N/\hbar} \ ,
\end{aligned} \tag{6.5.16}$$

and the energy spectrum ($N = n + m + l \in \mathbb{N}_0$) is

$$E_N = \frac{\hbar^2}{2mR^2}\Big[(2N + 4 + k_1 + k_2 + k_3 + \lambda_3)^2 - 1\Big] - \frac{m}{2}\omega^2 R^2 \ . \tag{6.5.17}$$

6.6 The Modified Pöschl–Teller Potential [780]

6.6.1 The Modified Pöschl–Teller Potential. [104,528,551,617] ($\eta, \mu > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r} - \frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]\mathrm{d}t\right\}$$
$$= \frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times (\cosh r' \cosh r'')^{-(m_1-m_2)}(\tanh r' \tanh r'')^{m_1+m_2+1/2}$$
$$\times {}_2F_1\Big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; \cosh^{-2} r_<\Big)$$
$$\times {}_2F_1\Big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \tanh^2 r_>\Big) \ , \tag{6.6.1}$$
$$= \sum_{n=0}^{N_M}\frac{\Psi_n^{(\eta,\nu)}(r'')\Psi_n^{(\eta,\nu)\,*}(r')}{E_n - E} + \int_0^\infty \mathrm{d}k\, \frac{\Psi_k^{(\eta,\nu)}(r'')\Psi_k^{(\eta,\nu)\,*}(r')}{\hbar^2 k^2/2m - E} \tag{6.6.2}$$

$[m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2mE}/\hbar),\ L_\nu = \frac{1}{2}(\nu - 1)]$. The bound states are

$$\Psi_n^{(\eta,\nu)}(r) = N_n^{(\eta,\nu)}(\sinh r)^{\eta+1/2}(\cosh r)^{n-\nu+1/2}{}_2F_1(-n, \nu - n; 1 + \eta; \tanh^2 r) \ , \tag{6.6.3a}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(1+\eta)}\left[\frac{2(\nu - \eta - 2n - 1)\Gamma(n + 1 + \eta)\Gamma(\nu - n)}{\Gamma(\nu - \eta - n)n!}\right]^{1/2} \ , \tag{6.6.3b}$$

$$E_n = -\frac{\hbar^2}{2m}(2n + \eta - \nu - 1)^2 \ , \qquad n = 0, 1, \ldots, N_M < \tfrac{1}{2}(\nu - \eta - 1) \ . \tag{6.6.4}$$

The continuum states with $E_k = \hbar^2 k^2/2m, k \geq 0$ have the form

$$\Psi_k^{(\eta,\nu)}(r) = N_k^{(\eta,\nu)}(\cosh r)^{\mathrm{i}k}(\tanh r)^{\eta+1/2}$$
$$\times {}_2F_1\left(\frac{\nu + \eta + 1 - \mathrm{i}k}{2}, \frac{\eta - \nu + 1 - \mathrm{i}k}{2}; 1 + \eta; \tanh^2 r\right) \ , \tag{6.6.5a}$$

$$N_k^{(\eta,\nu)} = \frac{1}{\Gamma(1+\eta)}\sqrt{\frac{k \sinh \pi k}{2\pi^2}}\,\Gamma\left(\frac{\nu + \eta + 1 - \mathrm{i}k}{2}\right)\Gamma\left(\frac{\eta - \nu + 1 - \mathrm{i}k}{2}\right) \ . \tag{6.6.5b}$$

6.6.2 Special Cases.

6.6.2.1 The Special Case $V(r) = \frac{\hbar^2}{2m}(\lambda^2 - \frac{1}{4})/\sinh^2 r$. [104,466,617] ($\lambda > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\frac{\lambda^2-\frac{1}{4}}{\sinh^2 r}\right)\mathrm{d}t\right]$$

$$= \mathrm{e}^{-\mathrm{i}\pi\lambda}\frac{2m}{\hbar^2}\sqrt{\sinh r'\sinh r''}$$
$$\times \mathcal{P}^{-\lambda}_{-1/2-\sqrt{-2mE}/\hbar}(\cosh r_>)\mathcal{Q}^{\lambda}_{-1/2+\sqrt{-2mE}/\hbar}(\cosh r_<)\ , \tag{6.6.6}$$

$$= \frac{1}{\pi}\sqrt{\sinh r'\sinh r''}\int_0^\infty \frac{k\,\mathrm{d}k\sinh\pi k}{\hbar^2k^2/2m - E}$$
$$\times \left|\Gamma(\tfrac{1}{2}+\mathrm{i}\,k+\lambda)\right|^2 \mathcal{P}^{-\lambda}_{\mathrm{i}k-1/2}(\cosh r'')\mathcal{P}^{-\lambda}_{\mathrm{i}k-1/2}(\cosh r')\ . \tag{6.6.7}$$

6.6.2.2 The Special Case $V(x) = \frac{\hbar^2}{2m}(\lambda^2 + \frac{1}{4})/\cosh^2 x$. [9,427,617] (only continuous states, $k > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\frac{k^2+\frac{1}{4}}{\cosh^2 x}\right)\mathrm{d}t\right]$$

$$= \frac{m}{\hbar^2}\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} - \mathrm{i}\,\lambda + \frac{1}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} + \mathrm{i}\,\lambda + \frac{1}{2}\right)$$
$$\times P^{-\sqrt{-2mE}/\hbar}_{\mathrm{i}\lambda-1/2}(\tanh x_<)P^{-\sqrt{-2mE}/\hbar}_{\mathrm{i}\lambda-1/2}(-\tanh x_>)\ , \tag{6.6.8}$$

$$= \frac{1}{2}\sum_{\pm}\int_0^\infty \frac{k\,\mathrm{d}k\sinh\pi k}{\hbar^2k^2/2m - E}\,\frac{P^{\mathrm{i}k}_{\mathrm{i}\lambda-1/2}(\pm\tanh x'')P^{-\mathrm{i}k}_{\mathrm{i}\lambda-1/2}(\pm\tanh x')}{\cosh^2\pi\lambda + \sinh^2\pi p}\ . \tag{6.6.9}$$

6.6.2.3 The Special Case $V(x) = -\frac{\hbar^2}{2m}(\lambda^2 - \frac{1}{4})/\cosh^2 x$. (Bound and continuum states, $\lambda > 0$) [26,344] ([186,527,664,742] only discrete spectrum, $n = 0, 1, \dots, N_M < k - \frac{1}{2}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m}\frac{\lambda^2-\frac{1}{4}}{\cosh^2 x}\right)\mathrm{d}t\right]$$

$$= \frac{m}{\hbar^2}\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} - \lambda + \frac{1}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2mE} + \lambda + \frac{1}{2}\right)$$
$$\times P^{-\sqrt{-2mE}/\hbar}_{\lambda-1/2}(\tanh x_<)P^{-\sqrt{-2mE}/\hbar}_{\lambda-1/2}(-\tanh x_>)\ , \tag{6.6.10}$$

$$= \sum_{n=0}^{N_M} \left(n - \lambda - \frac{1}{2}\right) \frac{\Gamma(2\lambda - n)}{n!} \frac{P_{\lambda-1/2}^{n-\lambda+\frac{1}{2}}(\tanh x'') P_{\lambda-1/2}^{n-\lambda+\frac{1}{2}}(\tanh x')}{-\hbar^2(n - \lambda + \frac{1}{2})^2/2m - E}$$
$$+ \frac{1}{2} \sum_{\pm} \int_0^\infty \frac{k\,\mathrm{d}k \sinh \pi k}{\hbar^2 k^2/2m - E} \frac{P_{\lambda-1/2}^{\mathrm{i}k}(\pm \tanh x'') P_{\lambda-1/2}^{-\mathrm{i}k}(\pm \tanh x')}{\cos^2 \pi\lambda + \sinh^2 \pi k} . \tag{6.6.11}$$

6.6.3 Reflectionless Potential. [207,209] ($N \in \mathbb{N}$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m} \frac{N(N+1)}{\cosh^2 x}\right) \mathrm{d}t\right]$$
$$= \sqrt{\frac{m}{2\pi \mathrm{i} \hbar T}} \exp\left(\frac{\mathrm{i}\, m}{2\hbar T} |x'' - x'|^2\right) + \frac{1}{2} \sum_{n=0}^{N-1} \exp\left(\frac{\mathrm{i}\hbar T}{2m}(N-n)^2\right)$$
$$\times (N - n) \frac{(2N-n)!}{n!} P_N^{n-N}(\tanh x') P_N^{n-N}(\tanh x'')$$
$$\times \left[\operatorname{erf}\left(\sqrt{\frac{\mathrm{i}\hbar T}{2m}}(N-n) - (x'' - x')\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\right)\right.$$
$$\left. + \operatorname{erf}\left(\sqrt{\frac{\mathrm{i}\hbar T}{2m}}(N-n) + (x'' - x')\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\right)\right] , \tag{6.6.12}$$
$$= \sum_{n=0}^{N-1} \exp\left(\frac{\mathrm{i}\hbar T}{2m}(N-n)^2\right)(N-n) \frac{(2N-n)!}{n!} P_N^{n-N}(\tanh x'') P_N^{n-N}(\tanh x')$$
$$+ \sum_{\pm} \int_0^\infty \mathrm{e}^{-\mathrm{i}T\hbar k^2/2m} \frac{k\,\mathrm{d}k}{2\sinh \pi k} P_N^{-\mathrm{i}k}(\pm \tanh x'') P_N^{\mathrm{i}k}(\pm \tanh x') . \tag{6.6.13}$$

6.6.4 Rosen–Morse Potential. [424,613,617,653,807] ([275,553,742] only discrete spectrum, $A \in \mathbb{R}, B > 0$)

$$\frac{\mathrm{i}}{\hbar} \int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x''}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{m}{2}\dot{x}^2 - A \tanh x + \frac{B}{\cosh^2 x}\right) \mathrm{d}t\right]$$
$$= \frac{m}{\hbar^2} \frac{\Gamma(m_1 - L_B)\Gamma(L_B + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\frac{1 - \tanh x'}{2} \cdot \frac{1 - \tanh x''}{2}\right)^{\frac{m_1 - m_2}{2}} \left(\frac{1 + \tanh x'}{2} \cdot \frac{1 + \tanh x''}{2}\right)^{\frac{m_1 + m_2}{2}}$$
$$\times {}_2F_1\left(-L_B + m_1, L_B + m_1 + 1; m_1 + m_2 + 1; \frac{1 + \tanh x_>}{2}\right)$$

$$\times\, {}_2F_1\left(-L_B+m_1, L_B+m_1+1; m_1-m_2+1; \frac{1-\tanh x_<}{2}\right) \ , \tag{6.6.14}$$

$$= \sum_{n=0}^{N_M} \frac{\Psi_n(x'')\Psi_n^*(x')}{E_n - E} + \sum_{\pm}\int_0^\infty \mathrm{d}k\, \frac{\Psi_k^{(\pm)}(x'')\Psi_k^{(\pm)\,*}(x')}{A + \hbar^2k^2/2m - E} \ . \tag{6.6.15}$$

$L_B = -\frac{1}{2} + \frac{1}{2}\sqrt{8mB/\hbar^2+1}$, $m_{1,2} = \sqrt{m/2}\left(\sqrt{-A-E_k} \pm \sqrt{A-E_k}\right)/\hbar$. The wave functions and the energy spectrum are given by $\Big[s \equiv \sqrt{1+8mB/\hbar^2}$; $0,\dots,n \le N_M < \frac{1}{2}(s-1) - \sqrt{m|A|/2}/\hbar$, $k_1 = \frac{1}{2}(1+s)$, $k_2 = \frac{1}{2}\big(1+\frac{1}{2}(s-2n-1) - \frac{2mA}{\hbar(s-2n-1)}\big) > \frac{1}{2}\Big]$:

$$\begin{aligned}\Psi_n = &\left[\left(1 - \frac{4mA}{\hbar(s-2n-1)^2}\right)\frac{(s-2k_2-2n)n!\,\Gamma(s-n)}{\Gamma(s+1-n-2k_2)\Gamma(2k_2+n)}\right]^{1/2} 2^{n+(1-s)/2}\\ &\times (1-\tanh x)^{\frac{1}{2}s-k_2-n}(1+\tanh x)^{k_2-\frac{1}{2}} P_n^{(s-2k_2-2n,2k_2-1)}(\tanh x) \ ,\end{aligned} \tag{6.6.16}$$

$$E_n = -\left[\frac{\hbar^2(s-2n-1)^2}{8m} + \frac{2mA^2}{\hbar^2(s-2n-1)^2}\right] \ . \tag{6.6.17}$$

The wave functions and the energy spectrum of the continuum states are given by $(E_k = A + \hbar^2k^2/2m, k \ge 0)$

$$\begin{aligned}\Psi_k^{(\pm)}(x) = &\frac{1}{\Gamma(1+m_1\pm m_2)}\frac{\sqrt{m\sinh(\pi|m_1\pm m_2|)/2R}}{\hbar|\sin\pi(m_1+L_B)|}\\ &\times \left(\frac{1+\tanh x}{2}\right)^{(m_1+m_2)/2}\left(\frac{1-\tanh x}{2}\right)^{(m_1-m_2)/2}\\ &\times {}_2F_1\big(m_1+L_B+1, m_1-L_B; 1+m_1\pm m_2; 1\pm\tanh x\big) \ .\end{aligned} \tag{6.6.18}$$

6.6.5 The Wood–Saxon Potential. [424,654] ($m_{1,2}$ as before, $V_0 > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{V_0}{1+\mathrm{e}^x}\right)\mathrm{d}t\right]$$

$$\begin{aligned}= &\frac{2m}{\hbar^2}\frac{\Gamma(m_1)\Gamma(m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\ &\times \left(\frac{1-\tanh\frac{x'}{2}}{2}\frac{1-\tanh\frac{x''}{2}}{2}\right)^{\frac{m_1-m_2}{2}}\left(\frac{1+\tanh\frac{x'}{2}}{2}\frac{1+\tanh\frac{x''}{2}}{2}\right)^{\frac{m_1+m_2}{2}}\\ &\times {}_2F_1\left(m_1, m_1+1, m_1+m_2+1; \frac{1+\tanh\frac{x_>}{2}}{2}\right)\end{aligned}$$

$$\times\, {}_2F_1\left(m_1, m_1+1, m_1-m_2+1; \frac{1-\tanh\frac{x_<}{2}}{2}\right) , \tag{6.6.19}$$

$$= \sum_{\pm} \int_0^\infty \frac{\Psi_k^{(\pm)}(x'')\Psi_k^{(\pm)*}(x')}{\hbar^2k^2/2m - V_0 - E} \mathrm{d}k , \tag{6.6.20}$$

$$\Psi_k^{(\pm)}(x) = \frac{1}{\Gamma(1+m_1 \pm m_2)} \frac{\sqrt{m \sinh(\pi|m_1 \pm m_2|)/2}}{\hbar|\sin \pi m_1|}$$
$$\times \left(\frac{1+\tanh\frac{x}{2}}{2}\right)^{(m_1+m_2)/2} \left(\frac{1-\tanh\frac{x}{2}}{2}\right)^{(m_1-m_2)/2}$$
$$\times\, {}_2F_1\left(m_1+1, m_1; 1+m_1\pm m_2; 1\pm\tanh\frac{x}{2}\right) . \tag{6.6.21}$$

6.6.6 The Manning–Rosen Potential. [116,424,613,670]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2 + A\coth r - \frac{B}{\sinh^2 r}\right)\mathrm{d}t\right]$$
$$= \frac{m}{\hbar^2}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\left(\frac{2}{\coth r'+1}\cdot\frac{2}{\coth r''+1}\right)^{(m_1+m_2+1)/2}$$
$$\times\left(\frac{\coth r'-1}{\coth r'+1}\cdot\frac{\coth r''-1}{\coth r''+1}\right)^{(m_1-m_2)/2}$$
$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{\coth r_>-1}{\coth r_>+1}\right)$$
$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{2}{\coth r_<+1}\right) , \tag{6.6.22}$$

$$= \sum_{n=0}^{N_M} \frac{\Psi_n(r'')\Psi_n^*(r')}{E_n - E} + \int_0^\infty \mathrm{d}k \frac{\Psi_k(r'')\Psi_k^*(r')}{\hbar^2k^2/2m - A - E} , \tag{6.6.23}$$

where $A \in \mathbb{R}, B > 0$, $L_E = -\frac{1}{2} + \sqrt{2m(A-E)}/2$, and

$$m_{1,2} = \frac{1}{2}\left(\sqrt{1+\frac{8mB}{\hbar^2}} \pm \frac{1}{\hbar}\sqrt{-2m(A+E)}\right) . \tag{6.6.24}$$

The wave functions and the energy spectrum of the bound states read $[0,1,\dots,n \le N_M < \sqrt{mA/2}/\hbar - \frac{1}{2}(s+1)$, $s = \sqrt{1+8mB/\hbar^2}$, $k_2 = (1+s)/2$, $k_1 = [1+(s+2n+1)/2 + 2mA/\hbar^2(s+2n+1)]/2$, note $n+\frac{1}{2}-k_1 < 0]$:

$$\Psi_n(r) = \left[\left(\frac{1}{R} + \frac{4mA}{\hbar(s+2n+1)^2}\right)\frac{(2k_1-2n-s-2)n!\,\Gamma(2k_1-n-1)}{\Gamma(n+s+1)\Gamma(2k_1-s-n-1)}\right]^{1/2}$$
$$\times \left(1-e^{-2r}\right)^{(s+1)/2} e^{-2r(k_1-s/2-n-1)}\, P_n^{(2k_1-2n-s-2,s)}(1-2\,e^{-2r}) \ , \tag{6.6.25}$$

and the energy spectrum is

$$E_n = -\frac{\hbar^2(s+2n+1)^2}{8m} - \frac{2mA^2}{\hbar^2(s+2n+1)^2} \ . \tag{6.6.26}$$

The wave functions and the energy spectrum of the continuum states are given by $[k_2 \equiv \frac{1}{2}(1+\mathrm{i}\,\tilde{k})$, $\tilde{k} \equiv \sqrt{2m(E_k - A)}\,/\hbar > 0]$:

$$\Psi_k(r) = \frac{N_p^{(k_1,k_2)}}{\sqrt{R}}\left[\tfrac{1}{2}(\coth r - 1)\right]^{-\mathrm{i}\,k/2}\left[\tfrac{1}{2}(\coth r + 1)\right]^{-[\mathrm{i}\,k-(1+s)]/2}$$
$$\times\, {}_2F_1\left(\frac{1+s+\mathrm{i}(\tilde{k}-k)}{2}, \frac{1+s-\mathrm{i}(\tilde{k}+k)}{2}; s+1; \frac{2}{\coth r + 1}\right) \tag{6.6.27}$$

with energy spectrum $E_k = \hbar^2 k^2/2m - A$.

6.6.7 The Hulthén Potential. [125,208,506,613] ([275,528] only discrete spectrum, $V_0 \in \mathbb{R}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, e^{\mathrm{i}\,ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 + \frac{V_0}{2}\left(\coth\frac{r}{2a} - 1\right)\right]\mathrm{d}t\right\}$$
$$= \frac{mR}{\hbar}\frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times \left(\frac{1}{1+u'}\cdot\frac{1}{1+u''}\right)^{(m_1+m_2+1)/2}\left(\frac{u'}{1+u'}\cdot\frac{u''}{1+u''}\right)^{(m_1-m_2)/2}$$
$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{u_<}{1+u_<}\right)$$
$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{1}{1+u_>}\right) \ , \tag{6.6.28}$$

$$= \sum_{n=0}^{N_M}\frac{\Psi_n(r'')\Psi_n^*(r')}{E_n - E} + \int_0^\infty \mathrm{d}k\, \frac{\Psi_k(r'')\Psi_k^*(r')}{\hbar^2k^2/8ma^2 - V_0/2 - E} \ , \tag{6.6.29}$$

where $u = \frac{1}{2}(\coth\frac{r}{2a} - 1)$, $L_E = -\frac{1}{2} + a\sqrt{2m(V_0 - E)}/\hbar$, $m_{1,2} = \frac{1}{2} \pm a\sqrt{-2mE}/\hbar$, and $n = 0,1,2,3,\ldots,N_M < \sqrt{2ma^2V_0}/\hbar - 1$. The bound-state wave functions are

$$\Psi_n(r) = \left[\left(\frac{1}{2a} + \frac{maV_0}{\hbar^2(n+1)^2}\right)\frac{(2k_1-2n-3)(2k_1-n-2)}{n+1}\right]^{1/2}$$
$$\times \left(1-\mathrm{e}^{-r/a}\right)\mathrm{e}^{-r(k_1-n-3/2)/a} P_n^{(2k_1-2n-3,1)}\left(1-2\,\mathrm{e}^{-r/a}\right) , \tag{6.6.30}$$

and the energy spectrum has the form

$$E_n = -\frac{1}{2}\left[\frac{\hbar^2(n+1)^2}{4ma^2} + \frac{ma^2V_0^2}{(n+1)^2\hbar^2}\right] + \frac{V_0}{2} . \tag{6.6.31}$$

The continuum states are given by ($k \geq 0, E_k = \hbar^2k^2/8ma^2 - V_0/2$ and $\tilde{k}$ as in the previous example)

$$\Psi_k(r) = \sqrt{\frac{k\sinh\pi k}{4a\pi^2}}\,\Gamma\left(1+\frac{\mathrm{i}}{2}(\tilde{k}-k)\right)\Gamma\left(1+\frac{\mathrm{i}}{2}(\tilde{k}+k)\right)$$
$$\times u^{-\mathrm{i}k/2}(u-1)^{\mathrm{i}k/2-1}{}_2F_1\left(1+\frac{\mathrm{i}}{2}(\tilde{k}-k), 1-\frac{\mathrm{i}}{2}(\tilde{k}+k); 2; 1-\mathrm{e}^{-r/a}\right) . \tag{6.6.32}$$

6.6.8 Hyperbolic Scarf Potential. [441] ($V_1 \pm V_2 + 1/4 > 0$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}$$
$$\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(V_0 + V_1\coth^2 r + V_2\frac{\coth r}{\sinh r}\right)\right]\mathrm{d}t\right\}$$
$$= \frac{2m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times \left(\cosh\tfrac{r'}{2}\cosh\tfrac{r''}{2}\right)^{-(m_1-m_2)}\left(\tanh\tfrac{r'}{2}\tanh\tfrac{r''}{2}\right)^{m_1+m_2+1/2}$$
$$\times {}_2F_1\Big(-L_\nu+m_1, L_\nu+m_1+1; m_1-m_2+1; \cosh^{-2}\tfrac{r_<}{2}\Big)$$
$$\times {}_2F_1\Big(-L_\nu+m_1, L_\nu+m_1+1; m_1+m_2+1; \tanh^2\tfrac{r_>}{2}\Big) , \tag{6.6.33}$$

$$= \sum_{n=0}^{N_M}\frac{\Psi_n(r'')\Psi_n(r')}{E_n - E} + \int_0^\infty \mathrm{d}k\,\frac{\Psi_k(r'')\Psi_k^*(r')}{\hbar^2(k^2+V_0+V_1)/2m - E} , \tag{6.6.34}$$

with $m_{1,2} = \eta/2 \pm \sqrt{V_0+V_1-2mE/\hbar^2}$, where $\eta = \sqrt{V_1+V_2+1/4}$, $\nu = \sqrt{V_1-V_2+1/4}$, and $L_\nu = \frac{1}{2}(\nu-1)$. The bound-state wave functions and the energy spectrum are given by

$$\Psi_n(r) = \left[\frac{(2k_1 - 2k_2 - 2n - 1)n!\,\Gamma(2k_1 - n - 1)}{2\Gamma(2k_2 + n)\Gamma(2k_1 - 2k_2 - n)}\right]^{1/2} \left(\sinh\frac{r}{2}\right)^{2k_2 - 1/2}$$
$$\times \left(\cosh\frac{r}{2}\right)^{2n - 2k_1 + 3/2} P_n^{[2k_2 - 1, 2(k_1 - k_2 - n) - 1]}\left(\frac{2}{\cosh^2\frac{r}{2}} - 1\right) , \tag{6.6.35}$$

$$E_n = \frac{\hbar^2}{2m}(V_0 + V_1) - \frac{\hbar^2}{8m}\Big[2(k_1 - k_2 - n) - 1\Big]^2 . \tag{6.6.36}$$

Here we denote $n = 0, 1, \dots, N_M < k_1 - k_2 - 1/2$, $k_1 = \frac{1}{2}(1 + \sqrt{V_1 - V_2 + 1/4})$, $k_2 = \frac{1}{2}(1 + \sqrt{V_1 + V_2 + 1/4})$, and $\kappa = k_1 - k_2 - n$. In order that bound states can exist, it is required that $V_2 < 0$. The continuum states have the form $[\kappa = \frac{1}{2}(1 + 2\mathrm{i}\,k), E_k = \hbar^2(k^2 + V_0 + V_1)/2m]$

$$\Psi_k(r) = N_k\big(\cosh\tfrac{r}{2}\big)^{2n - 2k_1 + 3/2}\big(\sinh\tfrac{r}{2}\big)^{2k_2 - 1/2}$$
$$\times {}_2F_1\left(k_1 + k_2 - \kappa, k_2 - k_1 - \kappa + 1; 2k_2; \tanh^2\frac{r}{2}\right) , \tag{6.6.37a}$$

$$N_k = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{k\sinh 2\pi k}{2\pi^2}}\Big[\Gamma(k_1 + k_2 - \kappa)\Gamma(-k_1 + k_2 + \kappa)$$
$$\times \Gamma(k_1 + k_2 + \kappa - 1)\Gamma(-k_1 + k_2 - \kappa + 1)\Big]^{1/2} . \tag{6.6.37b}$$

6.6.9 Hyperbolic Barrier Potential. [441] ($V_{1,2,3} \in \mathbb{R}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}\,ET/\hbar}$$
$$\times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(V_0 + V_1\frac{\tanh x}{\cosh x} + V_2\tanh^2 x\right)\right]dt\right\}$$
$$= \frac{m}{\hbar^2}\frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times (\cosh r'\cosh r'')^{-(m_1 - m_2)}(\tanh r'\tanh r'')^{m_1 + m_2 + \frac{1}{2}}$$
$$\times {}_2F_1\Big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; \cosh^{-2} r_<\Big)$$
$$\times {}_2F_1\Big(-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \tanh^2 r_>\Big) , \tag{6.6.38}$$

$$= \sum_{n=0}^{N_M}\frac{\Psi_n(z'')\Psi_n(z')}{E_n - E} + \int_{\mathbb{R}} dk\,\frac{\Psi_k(z'')\Psi_k^*(z')}{(\hbar^2/2m)(k^2 + V_0 + V_2) - E} , \tag{6.6.39}$$

with $\cosh^2 r = \frac{1}{2}(1 + \mathrm{i}\sinh x)$, $\eta = \sqrt{V_2 - \mathrm{i}\,V_1 + 1/4}$, $\nu = \sqrt{V_2 + \mathrm{i}\,V_1 + 1/4}$, $L_\nu = \frac{1}{2}(\nu - 1)$, and $m_{1,2} = \eta/2 \pm \sqrt{V_0 + V_2 - 2mE/\hbar^2}$. Furthermore we have

$k_1 = \frac{1}{2}\sqrt{V_2 - \mathrm{i}\,V_1 + \frac{1}{4}} \equiv \frac{1}{2}(1+\lambda)$, $k_2 = \frac{1}{2}(1-\lambda^*)$, with the wave functions [discrete spectrum, $\lambda_{R,I} = (\Re,\Im)(\lambda)$, $n = 0,1,\ldots,N_M < \lambda_R - \frac{1}{2}$]

$$\begin{aligned}\Psi_n(x) &= \left[\frac{(2\lambda_R - 2n - 1)n!\,\Gamma(\lambda - n)}{2\Gamma(2\lambda_R - n)\Gamma(n+1-\lambda^*)}\right]^{1/2}\\ &\quad\times\left(\frac{1+\mathrm{i}\sinh x}{2}\right)^{\frac{1}{2}(\frac{1}{2}-\lambda)}\left(\frac{1-\mathrm{i}\sinh x}{2}\right)^{\frac{1}{2}(\frac{1}{2}-\lambda^*)} P_n^{(-\lambda^*,-\lambda)}(\mathrm{i}\sinh x)\ ,\end{aligned}\tag{6.6.40}$$

with the energy spectrum

$$E_n = \frac{\hbar^2}{2m}(V_0+V_2) - \frac{\hbar^2}{2m}\left\{n + \frac{1}{2} - \sqrt{\frac{1}{2}\left[\sqrt{\left(\frac{1}{4}+V_2\right)^2 + V_1^2} + \frac{1}{4} + V_2\right]}\right\}^2\ ,\tag{6.6.41}$$

and (continuous spectrum, $E_k = \hbar^2(k^2 + V_0 + V_2), k \in \mathbb{R}$)

$$\begin{aligned}\Psi_k(x) &= \frac{\Gamma(\frac{1}{2} - \lambda_R - \mathrm{i}\,k)|}{\pi\Gamma(1-\lambda^*)}\sqrt{k\sinh(2\pi k)\Gamma\Big(\tfrac{1}{2} + \mathrm{i}(k-\lambda_I)\Big)\Gamma\Big(\tfrac{1}{2} + \mathrm{i}(k+\lambda_I)\Big)}\\ &\quad\times {}_2F_1\left(\frac{1}{2} + \mathrm{i}(\lambda_I - k), \frac{1}{2} - \lambda_R - \mathrm{i}\,k; 1-\lambda^*; \frac{\mathrm{i}\sinh x - 1}{\mathrm{i}\sinh x + 1}\right)\ .\end{aligned}\tag{6.6.42}$$

6.6.10 Trigonometric Coulomb-Like Potential. [59] ($\lambda > 0, \alpha \in \mathbb{R}$, $\tilde{N} = N + \lambda + \frac{1}{2}, a = \hbar^2/m\alpha, \sigma_N = a/R\tilde{N}$)

$$\begin{aligned}&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar}\\ &\quad\times\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}R^2\dot{\chi}^2 - \frac{\hbar^2}{2mR^2}\frac{\lambda^2 - \frac{1}{4}}{\sin^2\chi} + \frac{\alpha}{R}\cot\chi\right)dt\right]\\ &= \sum_{N\in\mathbb{N}_0}\frac{\Psi_N^{(\alpha)}(\chi'')\Psi_N^{(\alpha)\,*}(\chi')}{E_N - E}\ ,\end{aligned}\tag{6.6.43}$$

$$\begin{aligned}\Psi_N^{(\alpha)}(\chi) &= \frac{1}{R\Gamma(2\lambda+1)}\left[\frac{\sigma_N^2 + \tilde{N}^2}{R^2\tilde{N}^2}\frac{\Gamma(\tilde{N}+\lambda+1/2)\Gamma(\mathrm{i}\sigma_N + \lambda + 1/2)}{\Gamma(\tilde{N}-\lambda)\Gamma(\mathrm{i}\sigma_N - \lambda)}\right]^{1/2}\\ &\quad\times(2\sin\chi)^{\lambda+1/2}\exp\left[\mathrm{i}\,\chi(\mathrm{i}\,\sigma_N - N)\right]\\ &\quad\times {}_2F_1\big(-N, \lambda + \tfrac{1}{2} + \mathrm{i}\,\sigma_N; 2\lambda+1; 1-\mathrm{e}^{2\mathrm{i}\chi}\big)\ ,\end{aligned}\tag{6.6.44}$$

$$E_N = \frac{\hbar^2\tilde{N}^2}{2mR^2} - \frac{m\alpha^2}{2\hbar^2\tilde{N}^2}\ .\tag{6.6.45}$$

6.6.11 Symmetric Natanzon Potentials. [183] ($b > 0, \lambda \geq 1, |\nu| \geq \frac{1}{2}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - \frac{\hbar^2 b^2}{2m} v(bx)\right)\mathrm{d}t\right]$$
$$= \frac{m}{\hbar^2\lambda^2 b}\left[\left(\lambda^2 + (1-\lambda^2)X''^2\right)\left(\lambda^2 + (1-\lambda^2)X'^2\right)\right]^{1/4}$$
$$\times \Gamma(\mu - \bar{\nu})\Gamma(\bar{\nu} + \mu + 1) P_{\bar{\nu}}^{-\mu}(-X_<) P_{\bar{\nu}}^{-\mu}(X_>) \ ,$$
$$= \sum_{n\in\mathbb{N}_0} \frac{\Psi_n(X'')\Psi_n^*(X')}{E_n - E} + \sum_{\pm}\int_0^\infty \mathrm{d}E_k \frac{\Psi_{E_k,\pm}(X'')\Psi_{E_k,\pm}^*(X')}{E_k - E} \ . \tag{6.6.46}$$

The symmetric Natanzon potentials $v(z)$ are defined by ($z \in \mathbb{R}$, $|y| \leq 1$)

$$v(z) = \left\{-\lambda^2\nu(\nu+1) + \frac{1-\lambda^2}{4}\left[5(1-\lambda^2)y^4 - (7-\lambda^2)y^2 + 2\right]\right\}(1-y^2) \ , \tag{6.6.47}$$

$$z = \frac{1}{2\lambda^2}\left[\ln\left(\frac{1+y}{1-y}\right) + \mathrm{i}\, a \ln\left(\frac{\mathrm{i}+ay}{\mathrm{i}-ay}\right)\right], \qquad a = \sqrt{\lambda^2 - 1} \ . \tag{6.6.48}$$

Furthermore denote $n = 0, 1, \dots N_M < [\nu]$, and

$$\mu^2 = -\frac{2mE}{\lambda^4\hbar^2 b^2} \equiv -\frac{k^2}{\lambda^4} \ , \qquad y = -\frac{\tanh\xi}{\sqrt{\lambda^2 + (1-\lambda^2)\tanh\xi}} \ , \tag{6.6.49a}$$

$$\bar{\nu} = -\frac{1}{2} + \sqrt{\left(\nu + \frac{1}{2}\right)^2 + \frac{\lambda^2 - 1}{\lambda^4}k^2} \ , \tag{6.6.49b}$$

$$X = -\tanh\xi = -\frac{1}{2}\left[(1-\lambda^2)y^2 + \sqrt{(1-\lambda^2)^2 y^4 + 4\lambda^2 y^2}\right] \ . \tag{6.6.49c}$$

The wave functions and the energy spectra have the form ($E_k = \hbar^2 b^2 k^2/2m$)

$$\Psi_n(X) = \sqrt{\frac{b\mu(n+\mu+\frac{1}{2})\Gamma(n+2\mu+1)}{n!\,(n+\lambda^2\mu+\frac{1}{2})}}\left[\lambda^2 + (1-\lambda^2)X^2\right]^{1/4} P_{n+\mu}^{-\mu}(X) \ , \tag{6.6.50}$$

$$E_n = \frac{\hbar^2 b^2}{2m}\left[-\lambda^2\left(\nu + \frac{1}{2}\right)^2 + (\lambda^2 - 2)\left(n + \frac{1}{2}\right)^2 \right.$$
$$\left. + (2n+1)\sqrt{\lambda^2\left(\nu + \frac{1}{2}\right)^2 + (1-\lambda^2)\left(n + \frac{1}{2}\right)^2}\right] \ , \tag{6.6.51}$$

$$\Psi_{E_k,\pm}(X) = \sqrt{\frac{m\sinh(\pi k/\lambda^2)}{2\lambda^2 b\hbar^2}} \frac{[\lambda^2 + (1-\lambda^2)X^2]^{1/4}}{|\sin\pi(\bar{\nu} - \mathrm{i}\,k/\lambda^2)|} P_{\bar{\nu}}^{\mathrm{i}\,k/\lambda^2}(\pm X) \ . \tag{6.6.52}$$

6.6.12 The General Legendrian Path Integral (Natanzon Potential). [451]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2-V(r)\right)\mathrm{d}t\right]$$
$$=\Big(R(r')R(r'')\Big)^{1/4}\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\Big((1-z(r'))(1-z(r''))\Big)^{(m_1-m_2)/2}\Big(z(r')z(r'')\Big)^{(m_1+m_2)/2}$$
$$\times\,{}_2F_1\Big(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;1-z_<^2(r)\Big)$$
$$\times\,{}_2F_1\Big(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;z_>^2(r)\Big)\ , \tag{6.6.53}$$

$$V(r)=\frac{\hbar^2}{2m}\frac{fz(z-1)+h_0(1-z)+h_1z}{R(z)}+\frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2-2\frac{z'''}{z'}\right)\ , \tag{6.6.54}$$

where $R(z)=az^2+b_0z+c_0$, and the function $z=z(r)$ is according to [719] defined via the differential equation $z'=2z(1-z)/\sqrt{R(z)}$. Furthermore denote $m_{1,2}=\frac{1}{2}(\eta\pm\sqrt{-2mE'}/\hbar)$, $L_\nu=\frac{1}{2}(\nu-1)$, where we have abbreviated

$$\eta^2=h_0+1-\frac{2mEc_0}{\hbar^2}\ ,\qquad \nu^2=f+1-\frac{2mEa}{\hbar^2}\ , \tag{6.6.55}$$

$$E'=(a+b_0+c_0)E-\frac{\hbar^2}{2m}(h_1+1)\ . \tag{6.6.56}$$

6.6.13 Higgs Oscillator on the Pseudosphere. [460–461] ($\lambda_1=k_2+2n+1$,$\lambda_2=\lambda_1+k_4+2m+1$, $\lambda_3^2=m^2\omega^2R^4/\hbar^2+\frac{1}{4}$, $k_{1,2,3}>0$)

$$\frac{1}{R^3}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^2\tau\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left[\!\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}R^2\Big[\dot\tau^2+\sinh^2\tau(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)-\omega^2\tanh^2\tau\Big]\right.\right.\right.$$
$$-\frac{\hbar^2}{2mR^2}\left\{1+\frac{1}{\sinh^2\tau}\left[\frac{1}{\sin^2\vartheta}\left(\frac{k_2^2-\frac14}{\sin^2\varphi}+\frac{k_3^2-\frac14}{\cos^2\varphi}-\frac14\right)\right.\right.$$
$$\left.\left.\left.\left.\left.+\frac{k_4^2-\frac14}{\cos^2\vartheta}-\frac14\right]\right\}\right)\mathrm{d}t\right]\!\!\right]$$

$$= \frac{1}{R^3}(\sinh^2\tau' \sinh^2\tau'' \sin\theta' \sin\theta'')^{-1/2} \sum_{n\in\mathbb{N}_0} \Phi_n^{(k_2,k_3)}(\varphi'')\Phi_n^{(k_2,k_3)}(\varphi')$$

$$\times \sum_{m\in\mathbb{N}_0} \Phi_m^{(\lambda_1,k_4)}(\vartheta'')\Phi_m^{(\lambda_1,k_4)}(\vartheta')\left\{ \sum_{N=0}^{N_M} \Psi_N^{(\lambda_2,\lambda)}(\tau'')\Psi_N^{(\lambda_2,\lambda)}(\tau')\,\mathrm{e}^{-\mathrm{i}TE_N/\hbar} \right.$$

$$\left. + \int_0^\infty \mathrm{d}k\, \Psi_k^{(\lambda_2,\lambda)}(\tau'')\Psi_k^{(\lambda_2,\lambda)*}(\tau')\,\mathrm{e}^{-\mathrm{i}E_k/\hbar} \right\} . \tag{6.6.57}$$

The energy spectra of the bound and continuum states have the form

$$E_N = -\frac{\hbar^2}{2mR^2}\Big[(2N+1-\lambda_3+\lambda_2)^2 - 1\Big] + \frac{m}{2}\omega^2 R^2 , \tag{6.6.58}$$

$$E_k = \frac{\hbar^2}{2mR^2}\left(k^2+\tfrac{1}{4}\right) + \tfrac{1}{2}m\omega^2R^2 , \qquad k \geq 0 . \tag{6.6.59}$$

6.6.14 Modified Rosen–Morse Potential. I (Conditionally Solvable Natanzon Potential). [449,450,452]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right]$$

$$= \frac{m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

$$\times \left(\frac{2\sqrt{z(r')}}{\sqrt{z(r')}+1}\cdot\frac{2\sqrt{z(r'')}}{\sqrt{z(r'')}+1}\right)^{(m_1+m_2+1)/2}$$

$$\times \left(\frac{1+\sqrt{z(r')}}{1-\sqrt{z(r')}}\cdot\frac{1+\sqrt{z(r'')}}{1-\sqrt{z(r'')}}\right)^{(m_1-m_2)/2}$$

$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{1-\sqrt{z_>(r)}}{1+\sqrt{z_>(r)}}\right)$$

$$\times\, {}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{2\sqrt{z_<(r)}}{1+\sqrt{z_<(r)}}\right) , \tag{6.6.60}$$

$$V(r) = \frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z)+h_0(1-z)+h_1z^{1/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) . \tag{6.6.61}$$

Here $R(z) = b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$, and $L_E = \frac{1}{2}(\sqrt{1 - h_1 - 2m(b_0+c_0)E/\hbar^2} - 1), \eta^2 = h_0 + 1 - 2mc_0E/\hbar^2$

$$m_{1,2} = \sqrt{h_0 + 1 - 2mc_0E/\hbar^2} \pm \frac{1}{2}\sqrt{h_1 + 1 - 2m(b_0+c_0)E/\hbar^2} \ . \quad (6.6.62)$$

6.6.15 Modified Rosen–Morse Potential. II (Conditionally Solvable Natanzon Potential). [449,450,452]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right]$$
$$= \frac{2m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \frac{\Gamma(m_1 - L_\nu)\Gamma(L_\nu + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left[\frac{1}{4}\left(1 + \frac{1}{\sqrt{1 - z(r')}}\right)\cdot\left(1 + \frac{1}{\sqrt{1 - z(r'')}}\right)\right]^{-(m_1 - m_2)/2}$$
$$\times \left(\frac{\sqrt{z(r')}}{1 + \sqrt{1 - z(r')}}\cdot\frac{\sqrt{z(r'')}}{1 + \sqrt{1 - z(r'')}}\right)^{(m_1 + m_2 + 1/2)/2}$$
$$\times {}_2F_1\left[-L_\nu + m_1, L_\nu + m_1 + 1; m_1 - m_2 + 1; 2\left(1 + \frac{1}{\sqrt{1 - z_<(r)}}\right)^{-1}\right]$$
$$\times {}_2F_1\left[-L_\nu + m_1, L_\nu + m_1 + 1; m_1 + m_2 + 1; \frac{z_>(r)}{(1 + \sqrt{1 - z_>(r)}\,)^2}\right] , \quad (6.6.63)$$

$$V(r) = \frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z) + h_0(1-z) + h_1(1-z)^{1/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) . \quad (6.6.64)$$

Here $R(z) = b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$. Furthermore $L_\eta = (\sqrt{h_0 - h_1 + 1 - 2mc_0E\hbar^2} - 1)/2$ and

$$m_{1,2} = \frac{1}{2}\sqrt{h_0 + h_1 + 1 - 2mc_0E/\hbar^2} \pm \sqrt{1 - 2m(b_0 + c_0)E/\hbar^2}) \ . \quad (6.6.65)$$

Table 6.7. Hypergeometric conditionally solvable Natanzon potentials

$R(z)$, $z \in (0,1)$	$V(z)$, $\quad \Delta V = \dfrac{\hbar^2}{2m}\left[\left(\dfrac{z''}{z'}\right)^2 - \dfrac{z'''}{z'}\right]$
$R = b_0 z + c_0$	$\dfrac{\hbar^2}{2m}\dfrac{3z(1-z)/4 + h_0(1-z) + h_1 z^{1/2}}{R(z)} + \Delta V(z)$
$R = 1$, $x \in \mathbb{R}$ $z = \frac{1}{2}(1+\tanh x)$	$\dfrac{\hbar^2}{2m}\left(h_0 + 1 - \dfrac{h_0 - 3/4}{1+\mathrm{e}^{-2x}} + \dfrac{h_1}{\sqrt{1+\mathrm{e}^{-2x}}} - \dfrac{3}{4(1+\mathrm{e}^{-2x})^2}\right)$
$R = z$, $r > 0$ $z = \tanh^2 r$	$\dfrac{\hbar^2}{2m}\left(\dfrac{h_0 + 3/4}{\sinh^2 r} + h_1 \coth r + 1\right)$
$R = 1 - z$, $r > 0$ $z = 1/\cosh^2 r$	$\dfrac{\hbar^2}{2m}\left(h_0 + 1 + \dfrac{3}{4\sinh^2 r} + h_1 \dfrac{\cosh r}{\sinh^2 r}\right)$
$R = b_0 z + c_0$	$\dfrac{\hbar^2}{2m}\dfrac{3z(1-z)/4 + h_0(1-z) + h_1(1-z)^{1/2}}{R(z)} + \Delta V(z)$
$R = 1$, $x \in \mathbb{R}$ $z = \frac{1}{2}(1+\tanh x)$	$\dfrac{\hbar^2}{2m}\left(h_0 + 1 - \dfrac{h_0 - 3/4}{1+\mathrm{e}^{-2x}} + \dfrac{h_1\,\mathrm{e}^{-x}}{\sqrt{1+\mathrm{e}^{-2x}}} - \dfrac{3}{4(1+\mathrm{e}^{-2x})^2}\right)$
$R = z$, $r > 0$ $z = \tanh^2 r$	$\dfrac{\hbar^2}{2m}\left(\dfrac{h_0 + 3/4}{\sinh^2 r} + h_1 \dfrac{\cosh r}{\sinh^2 r}\right)$
$R = 1 - z$, $r > 0$ $z = 1/\cosh^2 r$	$\dfrac{\hbar^2}{2m}\left(h_0 + 1 + \dfrac{3}{4\sinh^2 r} + h_1 \coth r\right)$
$R = a_0 z^2 + b_0 z$	$\dfrac{\hbar^2}{2m}\dfrac{fz(z-1) - 3(1-z)/4 + h_1 z^{3/2}}{R(z)} + \Delta V(z)$
$R = z\, r > 0$ $z = \tanh^2 r$	$\dfrac{\hbar^2}{2m}\left(h_1 \tanh r - \dfrac{f + 3/4}{\cosh^2 r} + 1\right)$
$R = z^2$, $r > 0$ $z = 1 - \mathrm{e}^{-2r}$	$\dfrac{\hbar^2}{2m}\left(f + 1 - \dfrac{f + 3/4}{1 - \mathrm{e}^{-2r}} + \dfrac{h_1}{\sqrt{1-\mathrm{e}^{-2r}}} - \dfrac{3}{4(1-\mathrm{e}^{-2r})^2}\right)$
$R = 4z(1-z)$ $z = \frac{1}{2}(1-\cos\varphi)$ $\varphi \in (0,\pi)$	$\dfrac{\hbar^2}{8m}\left[h_1 \dfrac{\sin(\varphi/2)}{\cos^2(\varphi/2)} - \left(f + \dfrac{1}{4}\right) + \dfrac{3}{4}\dfrac{1}{\cos^2(\varphi/2)}\right]$
$R = a_0 z^2 + b_0 z$	$\dfrac{\hbar^2}{2m}\dfrac{fz(z-1) - 3(1-z)/4 + h_1 z^{3/2}\sqrt{1-z}}{R(z)} + \Delta V(z)$
$R = z$, $r > 0$ $z = \tanh^2 r$	$\dfrac{\hbar^2}{2m}\left(h_1 \dfrac{\sinh r}{\cosh^2 r} - \dfrac{f + 3/4}{\cosh^2 r} + 1\right)$
$R = z^2$, $r > 0$ $z = 1 - \mathrm{e}^{-2r}$	$\dfrac{\hbar^2}{2m}\left(f + 1 - \dfrac{f + 3/4}{1 - \mathrm{e}^{-2r}} + \dfrac{h_1\,\mathrm{e}^{-r}}{\sqrt{1-\mathrm{e}^{-2r}}} - \dfrac{3}{4(1-\mathrm{e}^{-2r})^2}\right)$
$R = 4z(1-z)$ $z = \frac{1}{2}(1-\cos\varphi)$ $\varphi \in (0,\pi)$	$\dfrac{\hbar^2}{8m}\left[h_1 \tan\dfrac{\varphi}{2} - \left(f + \dfrac{1}{4}\right) + \dfrac{3}{4}\dfrac{1}{\cos^2(\varphi/2)}\right]$

6.7 Motion on Group Spaces and Homogeneous Spaces

6.7.1 General Formulæ.

6.7.1.1 Motion on a Group Manifold. [104,262,263,447,454,618,679,726,776]

$$\int\limits_{g(t')=g'}^{g(t'')=g''} \mathcal{D}g(t) \exp\left(-\frac{\mathrm{i}\, m}{2\hbar} \int_{t'}^{t''} \left\langle g^{-1}(\dot{g} + \mathrm{i}\,\varphi g), g^{-1}(\dot{g} + \mathrm{i}\,\varphi g) \right\rangle \mathrm{d}t \right)$$

$$= \int \mathrm{d}E_l \, d_l \chi^l(\varphi)\, \mathrm{e}^{-\mathrm{i}\, E_l T/\hbar} \ , \tag{6.7.1}$$

$$= \frac{(2\pi)^{p+l}\sqrt{\det D}}{\prod_{\alpha\in R_+}\langle\alpha,\rho\rangle} \frac{\mathrm{e}^{\mathrm{i}\,\hbar n T/48}}{(2\pi\mathrm{i}\,\hbar T)^{n/2}} \times \sum_{\substack{\nu\in T\\ \mathrm{e}^{2\pi\mathrm{i}\,\nu}=1}} \mathrm{e}^{\mathrm{i}\langle\varphi+2\pi\nu,\varphi+2\pi\nu\rangle/2\hbar T} \sum_{\alpha\in R_+} \frac{\langle\alpha,\varphi+2\pi\nu\rangle}{2\sin(\langle\alpha,\varphi+2\pi\nu\rangle/2)} \ . \tag{6.7.2}$$

Here d^l is the dimension of the representation l of the group G labeled by its highest weight l, with the eigenvalue $2E_l$ of the corresponding Casimir operator, $\chi_l = \mathrm{Tr}[D^l(g'^{-1})D^l(g'')] \equiv \mathrm{Tr}[D^l(\mathrm{e}^{\mathrm{i}\,\varphi})]$, with $D^l(g)$ the matrix representing g in the representation l. Furthermore α_i $(i = 1,\dots,l = \mathrm{rank}\, G)$ are the simple roots of G, $\langle\cdot,\cdot\rangle$ is the Cartan–Killing form, $n = \dim G$; R_+ is the set of positive roots, $\rho = \sum_{\alpha\in R_+}\alpha$, p is the number of positive roots, and D is an $l\times l$ matrix $D_{ij} = (\tilde{\alpha}_i, \tilde{\alpha}_j)$, where $\tilde{\alpha}_i = 2\alpha_i/\langle\alpha_i,\alpha_i\rangle$.

6.7.1.2 Path Integration in a Space $\mathcal{H}_\alpha$: $G \cong \mathcal{H}_\alpha$. [104] (In suitable spherical coordinates, with signature $(+p,-q)$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^{p+q-1} \int\limits_{g(t')=g'}^{g(t'')=g''} \mathcal{D}g(t) \times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 + r^2\,\mathrm{Tr}(g^{-1}\circ g) - V(r) - \Delta V(r)\right)\mathrm{d}t\right]$$

$$= (r'r'')^{(1-p-q)/2} \int \mathrm{d}E_l\, d_l \chi^l(g'^{-1}g'') \times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_l[r^2] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r) - \Delta V(r)\right)\mathrm{d}t\right] \ , \tag{6.7.3}$$

$$\mu_l[r^2] = \lim_{N\to\infty}\prod_{j=1}^N \mu_l[r_{j-1}r_j]$$

$$= \lim_{N\to\infty} \prod_{j=1}^{N} (-1)^q \left(\frac{m r_{j-1} r_j}{2\pi \mathrm{i}\epsilon\hbar}\right)^{(p+q-1)/2} \exp\left(\frac{\mathrm{i}\, m}{\epsilon\hbar} r_{j-1} r_j\right) \hat{f}^l\left(\frac{m r_{j-1} r_j}{2\,\mathrm{i}\,\epsilon\hbar}\right) , \tag{6.7.4a}$$

$$\hat{f}^l(z) = \frac{1}{d_l} \int_G \mathrm{d}g\, \mathrm{e}^{z\,\mathrm{Tr}(g)} \chi^{l\,*}(g) \ . \tag{6.7.4b}$$

6.7.1.3 Path Integration on a Homogeneous Space $\mathcal{H}_\alpha = G/H$. [104] (In suitable spherical coordinates, with signature $(+p, -q)$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^{p+q-1} \int\limits_{g(t')=g'}^{g(t'')=g''} \mathcal{D}g(t)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 + r^2\,\mathrm{Tr}(g^{-1}\circ g) - V(r) - \Delta V(r)\right)\mathrm{d}t\right]$$

$$= (r'r'')^{(1-p-q)/2} \int \mathrm{d}E_l\, d_l D^l_{00}(g'^{-1}g'')$$
$$\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_l[r^2] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r) - \Delta V(r)\right)\mathrm{d}t\right] , \tag{6.7.5}$$

with the zonal spherical functions D^l_{00}.

6.7.1.4 Path Integration on a Homogeneous Space $\mathcal{H}_\alpha = G/H$. [104,618] (Space with signature $(+p, -q)$, $g \in G$)

$$\int\limits_{\mathrm{q}(t')=\mathrm{q}'}^{\mathrm{q}(t'')=\mathrm{q}''} \mathcal{D}\,\mathrm{q}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathrm{q}}^2 \mathrm{d}t\right) ,$$
$$= \int \mathrm{d}E_l\, d_l\, \mathrm{e}^{\dot{\lambda}_l(0)T} D^l_{00}(g'^{-1}g'') \tag{6.7.6}$$
$$= \int \mathrm{d}E_l \sum_m \mathrm{e}^{-\mathrm{i}\, E_l T\hbar} \langle \mathrm{q}'' \,|lm\rangle\langle lm|\, \mathrm{q}'\rangle , \tag{6.7.7}$$

where $\langle \mathrm{q}\,|lm\rangle = \sqrt{d_l}\, D^l_{m0}(g)$, $E_l = \mathrm{i}\,\hbar\dot{\lambda}_l(0)$ in the notation of (3.4.33), and λ_l is the coefficient in the expansion of the short time kernel in terms of the zonal spherical functions D^l_{00}.

6.7.1.5 Parabolic Coordinates in $\mathbb{R}^2$. [444,447] ($\xi \in \mathbb{R}, \eta > 0$, see Sect. 2.11)

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2+\eta^2)\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)\,\mathrm{d}t\right]$$

$$= \int_{\mathbb{R}} \mathrm{d}\zeta \int_0^\infty \frac{\mathrm{d}k}{32\pi^4}\mathrm{e}^{-\mathrm{i}\,\hbar k^2 T/2m}$$

$$\times \begin{pmatrix} |\Gamma(\frac{1}{4}+\frac{\mathrm{i}\zeta}{2k})|^2 E^{(0)}_{-1/2+\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi'')E^{(0)}_{-1/2-\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta'') \\ |\Gamma(\frac{3}{4}+\frac{\mathrm{i}\zeta}{2k})|^2 E^{(1)}_{-1/2+\mathrm{i}\zeta/k}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2k}\,\xi'')E^{(1)}_{-1/2-\mathrm{i}\zeta/k}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2k}\,\eta'') \end{pmatrix}$$

$$\times \begin{pmatrix} |\Gamma(\frac{1}{4}+\frac{\mathrm{i}\zeta}{2k})|^2 E^{(0)}_{-1/2-\mathrm{i}\zeta/k}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2k}\,\xi')E^{(0)}_{-1/2+\mathrm{i}\zeta/k}(\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2k}\,\eta') \\ |\Gamma(\frac{3}{4}+\frac{\mathrm{i}\zeta}{2k})|^2 E^{(1)}_{-1/2-\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi')E^{(1)}_{-1/2+\mathrm{i}\zeta/k}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta') \end{pmatrix} . \tag{6.7.8}$$

6.7.1.6 Elliptic Coordinates in $\mathbb{R}^2$. [447,454] ($\mu > 0, \nu \in [-\pi,\pi), d > 0$)

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) d^2(\sinh^2\mu+\sin^2\nu)$$

$$\times \exp\left[\frac{\mathrm{i}\,m}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)\,\mathrm{d}t\right]$$

$$= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^\infty k\mathrm{d}k\ \mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}$$

$$\times \mathrm{me}_n(\nu'';\tfrac{d^2k^2}{4})\,\mathrm{me}_n^*(\nu';\tfrac{d^2k^2}{4})M_n^{(1)}(\mu'';\tfrac{dk}{2})M_n^{(1)*}(\mu';\tfrac{dk}{2}) . \tag{6.7.9}$$

6.7.1.7 Parabolic Coordinates in $\mathbb{R}^3$. [444,447] ($\eta > 0, \xi > 0, \varphi \in [0,2\pi)$)

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2+\eta^2)\xi\eta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\Big((\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\varphi^2\Big)+\frac{\hbar^2}{8m\xi^2\eta^2}\right)\mathrm{d}t\right]$$

$$= \sum_{n\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,n(\varphi''-\varphi')}}{2\pi}\int_{\mathbb{R}}\mathrm{d}\zeta\int_0^\infty\frac{\mathrm{d}k}{k}\frac{|\Gamma(\frac{1+|n|}{2}+\frac{\mathrm{i}\zeta}{2k})|^4}{4\pi^2\xi'\xi''\eta'\eta''\Gamma^4(1+|n|)}\mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}$$

$$\times M_{-\mathrm{i}\zeta/2k,|n|/2}(-\mathrm{i}\,k\xi''^2)M_{\mathrm{i}\zeta/2k,|n|/2}(\mathrm{i}\,k\xi'^2)$$

$$\times M_{\mathrm{i}\zeta/2k,|n|/2}(-\mathrm{i}\,k\eta''^2)M_{-\mathrm{i}\zeta/2k,|n|/2}(\mathrm{i}\,k\eta'^2) . \tag{6.7.10}$$

6.7.1.8 Prolate-Spheroidal Coordinates in $\mathbb{R}^3$. [444,447] ($\mu > 0, \nu \in (0,\pi), \varphi \in [0,2\pi), d > 0$, the case of oblate-spheroidal coordinates is similar)

$$\int\limits_{\mu(t')=\mu'}^{\mu(t'')=\mu''} \mathcal{D}\mu(t) \int\limits_{\nu(t')=\nu'}^{\nu(t'')=\nu''} \mathcal{D}\nu(t) d^3(\sinh^2\mu + \sin^2\nu)\sinh\mu\sin\nu \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}d^2\Big((\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)+\sinh^2\mu\sin^2\nu\dot\varphi^2\Big)\right.\right.$$
$$\left.\left. + \frac{\hbar^2}{8md^2\sinh^2\mu\sin^2\nu}\right]\mathrm{d}t\right\}$$

$$= \sum_{l\in\mathbb{N}_0}\sum_{n=-l}^{l} \mathrm{e}^{\mathrm{i}\,n(\varphi''-\varphi')}\frac{2l+1}{2\pi^2}\frac{(l-n)!}{(l+n)!}\int_0^\infty k^2\mathrm{d}k\ \mathrm{e}^{-\mathrm{i}\,\hbar k^2 T/2m}$$
$$\times\ \mathrm{ps}_l^n(\cos\nu''; k^2d^2)\,\mathrm{ps}_l^{n\,*}(\cos\nu'; k^2d^2) S_l^{n\,(1)}(\cosh\mu''; kd) S_l^{n\,(1)\,*}(\cosh\mu'; kd)\,. \tag{6.7.11}$$

6.7.1.9 Polar Coordinates in $\mathbb{R}^{(1,1)}$. [447,470] ($\varrho > 0, \tau \in \mathbb{R}$)

$$\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''} \mathcal{D}\varrho(t)\varrho \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\varrho^2-\varrho^2\dot\tau^2)+\frac{\hbar^2}{8m\varrho^2}\right]\mathrm{d}t\right\}$$
$$= \int_{\mathbb{R}}\frac{\mathrm{d}\nu}{2\pi}\mathrm{e}^{\mathrm{i}\,\nu(\tau''-\tau')}\int_0^\infty\frac{k\mathrm{d}k}{\pi^2}K_{\mathrm{i}\,\nu}(-\mathrm{i}\,\nu\varrho'')K_{\mathrm{i}\,\nu}(\mathrm{i}\,k\varrho')\ \mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}\ . \tag{6.7.12}$$

6.7.1.10 Elliptic Coordinates in $\mathbb{R}^{(1,1)}$. [447,470] ($a \in \mathbb{R}, b > 0, d > 0$)

$$\int\limits_{a(t')=a'}^{a(t'')=a''} \mathcal{D}a(t) \int\limits_{b(t')=b'}^{b(t'')=b''} \mathcal{D}b(t)d^2(\sinh^2 a - \sinh^2 b)$$
$$\times\exp\left[\frac{\mathrm{i}\,m}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2 a-\sinh^2 b)(\dot a^2-\dot b^2)\,\mathrm{d}t\right]$$
$$= \frac{1}{8\pi}\int_0^\infty k\mathrm{d}k\int_{\mathbb{R}}\mathrm{d}\nu\ \mathrm{e}^{-\pi\nu-\mathrm{i}\,\hbar k^2T/2m}$$
$$\times\ \mathrm{Me}_{\mathrm{i}\,\nu}(b''; \tfrac{k^2d^2}{4})\,\mathrm{Me}^*_{\mathrm{i}\,\nu}(b'; \tfrac{k^2d^2}{4})M^{(3)}_{\mathrm{i}\,\nu}(a''; \tfrac{kd}{2})M^{(3)\,*}_{\mathrm{i}\,\nu}(a'; \tfrac{kd}{2})\ . \tag{6.7.13}$$

6.7.1.11 Spheroidal Coordinates in $\mathbb{R}^{(2,1)}$. [447] $(\xi,\eta>0,\varphi\in[0,2\pi),d>0)$

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\sinh^2\xi-\sinh^2\eta)d^3\sinh\xi\sinh\eta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(d^2(\sinh^2\xi-\sinh^2\eta)(\dot\eta^2-\dot\xi^2)-\sinh^2\xi\sinh^2\eta\,\dot\varphi^2\Big)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8md^2\sinh^2\xi\sinh^2\eta}\right]dt\right\}$$

$$=\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi}\int_0^\infty\frac{d\mu}{\pi}\mu\sinh\pi\mu\big|\Gamma(\tfrac12+\mathrm{i}\,\mu+\nu)\big|^2\int_0^\infty\frac{k^2dk}{2\pi}\,\mathrm{e}^{-\mathrm{i}\hbar k^2T/2m-\pi k}$$
$$\times\,\mathrm{Ps}_{\mathrm{i}\,\mu-1/2}^{-\nu}(\cosh\eta'';k^2d^2)\,\mathrm{Ps}_{\mathrm{i}\,\mu-1/2}^{-\nu\,*}(\cosh\eta';k^2d^2)$$
$$\times\,S_{\mathrm{i}\,\mu-1/2}^{\nu\,(3)}(\cosh\xi'';kd)S_{\mathrm{i}\,\mu-1/2}^{\nu\,(3)\,*}(\cosh\xi';kd)\ . \tag{6.7.14}$$

6.7.1.12 Summation Formula for Path Integral on a Quotient Manifold (Mirror Principle, Method of Images). [340,541,676,679,828]

$$K_{\Gamma\backslash\,\mathbb{M}}(\mathbf{z}'',\mathbf{z}';T)=\sum_{\gamma\in\Gamma}K_{\mathbb{M}}(\mathbf{z}'',\gamma\mathbf{z}';T)\ . \tag{6.7.15}$$

6.7.2 Motion on the D-Dimensional Unit-Sphere. [25,26,104,105,136, 257,282,357,358,444,447,464,468,528,612,613,653,660,678,679,762,763,787,826]

$$\int\limits_{\mathbf{\Omega}(t')=\mathbf{\Omega}'}^{\mathbf{\Omega}(t'')=\mathbf{\Omega}''}\mathcal{D}\mathbf{\Omega}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{\Omega}}^2+\hbar^2\frac{(D-1)(D-3)}{8m}\right]dt\right\}$$

$$=\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\mathbf{\Omega}')S_l^\mu(\mathbf{\Omega}'')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}l(l+D-2)\right]\ , \tag{6.7.16}$$

$$=\frac{1}{\Omega(D)}\sum_{l\in\mathbb{N}_0}\frac{2l+D-2}{D-2}C_l^{\frac{D-2}{2}}(\cos\psi_{(\prime\prime,\prime)})\exp\left[-\frac{\mathrm{i}\hbar T}{2m}l(l+D-2)\right]\ , \tag{6.7.17}$$

$$S_l^\mu(\mathbf{\Omega})=\frac{1}{N^{S^{(D-1)}}}\,\mathrm{e}^{\mathrm{i}\,m_{D-2}\varphi}\prod_{k=0}^{D-3}(\sin\vartheta^{k+1})^{m_{k+1}}C_{m_k-m_{k+1}}^{m_{k+1}+\frac{D-2-k}{2}}(\cos\vartheta^{k+1})\ , \tag{6.7.18}$$

$$E_l^{S^{(D-1)}}=\hbar^2\frac{l(l+D-2)}{2m}\ , \tag{6.7.19}$$

where $\Omega(D) = 2\pi^{D/2}/\Gamma(D/2)$, $l = m_0 \geq m_1 \geq \ldots \geq m_{D-2} \geq 0$, $N^{S^{(D-1)}} = 2\pi \prod_{k=1}^{D-2} E_k(m_{k-1}, m_k)$ and

$$E_k(l,m) = \frac{\pi 2^{k-2m-(D-2)}\Gamma(l+m-k+D-1)}{(l+\frac{D-1-k}{2})(l-m)!\Gamma^2(m+\frac{D-1-k}{2})}\;, \tag{6.7.20}$$

and $\cos\psi_{('',')}$ denotes the quantity defined by (2.7.3) For D even one has

$$\int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''} \mathcal{D}\boldsymbol{\Omega}(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\boldsymbol{\Omega}}^2 + \hbar^2\frac{(D-1)(D-3)}{8m}\right]\mathrm{d}t\right\}$$
$$= \frac{1}{(2\pi)^{D/2}} \exp\left[\frac{\mathrm{i}\,\hbar T(D-2)^2}{8m}\right]\left(\frac{\mathrm{d}}{\mathrm{d}\cos\psi_{('',')}}\right)^{\frac{D-2}{2}} \Theta_3\left(\frac{\psi_{('',')}}{2}\middle| -\frac{\hbar T}{2\pi m}\right)\;. \tag{6.7.21}$$

The Green function for D even has the form

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''} \mathcal{D}\boldsymbol{\Omega}(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\boldsymbol{\Omega}}^2 + \hbar^2\frac{(D-1)(D-3)}{8m}\right]\mathrm{d}t\right\}$$
$$= \frac{1}{\hbar}\sqrt{\frac{m}{2E+\hbar^2(D-2)^2/4m}}$$
$$\times \left(\frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\cos\psi_{('',')}}\right)^{\frac{D-2}{2}} \frac{\cos\left[(\psi_{('',')}-\pi)\sqrt{2mE/\hbar^2+(D-2)^2/4}\right]}{\sin\left(\pi\sqrt{2mE/\hbar^2+(D-2)^2/4}\right)}\;, \tag{6.7.22}$$

$$= \frac{m}{2\hbar^2\sin\pi(a+\frac{1}{2})}\frac{\Gamma(a+\frac{D-1}{2})}{\Gamma(a+\frac{5-D}{2})}\left(\frac{-1}{2\pi\sin\psi_{('',')}}\right)^{\frac{D-3}{2}} P_a^{\frac{3-D}{2}}\left[\cos(\psi_{('',')}-\pi)\right] \tag{6.7.23}$$

$\left(a = -\frac{1}{2} + \sqrt{2mE/\hbar^2+(D-2)^2/4}\right)$; and for D odd

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''} \mathcal{D}\boldsymbol{\Omega}(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\boldsymbol{\Omega}}^2 + \hbar^2\frac{(D-1)(D-3)}{8m}\right]\mathrm{d}t\right\}$$

$$= \frac{m}{2\hbar^2 \sin(-a\pi)} \frac{\Gamma(a+\frac{D-1}{2})}{\Gamma(a+\frac{5-D}{2})} \left(\frac{1}{2\pi \sin\psi_{('',')}}\right)^{\frac{D-3}{2}} P_a^{\frac{3-D}{2}}(-\cos\psi_{('',')}) \ . \tag{6.7.24}$$

6.7.3 Bispherical Coordinates. $(\mu = l + (N-2)/2, \nu = \lambda + (M-2)/2)$ [104]

$$\begin{aligned}
&\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^{N+M-1} \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \sin^{N-1}\vartheta \cos^{M-1}\vartheta \\
&\quad\times \int\limits_{\Omega_\alpha^{N-1}(t')=\Omega_\alpha^{N-1\prime}}^{\Omega_\alpha^{N-1}(t'')=\Omega_\alpha^{N-1\prime\prime}} \mathcal{D}\Omega_\alpha^{N-1}(t) \int\limits_{\Omega_\beta^{M-1}(t')=\Omega_\beta^{M-1\prime}}^{\Omega_\beta^{M-1}(t'')=\Omega_\beta^{M-1\prime\prime}} \mathcal{D}\Omega_\beta^{M-1}(t) \\
&\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}\left(\dot r^2 + r^2\left(\dot\vartheta^2 + \sin^2\vartheta\dot{\Omega}_\alpha^2 + \cos^2\vartheta\dot{\Omega}_\beta^2\right)\right)\right.\right. \\
&\qquad\qquad\qquad \left.\left. + \frac{\hbar^2}{8mr^2}\left(1 + \frac{1}{\cos^2\vartheta} + \frac{1}{\sin^2\vartheta}\right) - V(r)\right]\mathrm{d}t\right\} \\
&= \sum_{l,\lambda\in\mathbb{N}_0} \sum_N S_l^N(\Omega_\alpha'')S_l^N(\Omega_\alpha') \sum_M S_\lambda^M(\Omega_\beta'')S_\lambda^M(\Omega_\beta') \\
&\quad\times (r'r''\sin\vartheta'\sin\vartheta'')^{\frac{2-N}{2}} (r'r''\cos\vartheta'\cos\vartheta'')^{\frac{2-M}{2}} \\
&\quad\times \sum_{J=\frac{\nu+\mu}{2}}^{\infty} (2J+1) D^J_{\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}}(\cos 2\vartheta'') D^{J\,*}_{\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}}(\cos 2\vartheta') \\
&\quad\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_{2J+1}[r^2] \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot r^2 - V(r)\right)\mathrm{d}t\right] \ .
\end{aligned} \tag{6.7.25}$$

6.7.4 Bispherical Coordinates on the Unit Sphere. $(\mu = l + (N-2)/2, \nu = \lambda + (M-2)/2)$ [104]

$$\begin{aligned}
&\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t) \sin^{N-1}\vartheta \cos^{M-1}\vartheta \\
&\quad\times \int\limits_{\Omega_\alpha^{N-1}(t')=\Omega_\alpha^{N-1\prime}}^{\Omega_\alpha^{N-1}(t'')=\Omega_\alpha^{N-1\prime\prime}} \mathcal{D}\Omega_\alpha^{N-1}(t) \int\limits_{\Omega_\beta^{M-1}(t')=\Omega_\beta^{M-1\prime}}^{\Omega_\beta^{M-1}(t'')=\Omega_\beta^{M-1\prime\prime}} \mathcal{D}\Omega_\beta^{M-1}(t)
\end{aligned}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{\vartheta}^2+\sin^2\vartheta\dot{\Omega}_\alpha^2+\cos^2\vartheta\dot{\Omega}_\beta^2\right)\right.\right.$$

$$\left.\left.+\frac{\hbar^2}{8m}\left((N+M-2)^2+\frac{1}{\cos^2\vartheta}+\frac{1}{\sin^2\vartheta}\right)\right]dt\right\}$$

$$=\sum_{l,\lambda\in\mathbb{N}_0}\sum_N S_l^N(\Omega_\alpha')S_l^N(\Omega_\alpha'')\sum_M S_\lambda^M(\Omega_\beta')S_\lambda^M(\Omega_\beta'')$$

$$\times\left(r'r''\sin\vartheta'\sin\vartheta''\right)^{\frac{2-n}{2}}\left(r'r''\cos\vartheta'\cos\vartheta''\right)^{\frac{2-m}{2}}$$

$$\times\sum_{J=\frac{\nu+\mu}{2}}^{\infty}(2J+1)D^J_{\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}}(\cos 2\vartheta'')D^{J\,*}_{\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}}(\cos 2\vartheta')$$

$$\times\exp\left[-\frac{\mathrm{i}\hbar T}{2m}(L+l+\lambda)(L+l+\lambda+N+M-2)\right]\ . \tag{6.7.26}$$

6.7.5 Motion on the D-Dimensional Pseudo-Sphere.

6.7.5.1 General Form of the Kernel. [104,466] ($\tau>0,\boldsymbol{\Omega}\in S^{(D-2)},\boldsymbol{\omega}\in\Lambda^{D-1}$, and cf. [444, 447] for details concerning spectral expansions into various coordinate space representations for $D=2,3$, $\mathrm{u}=(u_0,\mathbf{u})\in\Lambda^{D-1}$)

$$\int\limits_{\mathbf{u}(t')=\mathbf{u}'}^{\mathbf{u}(t'')=\mathbf{u}''}\frac{\mathcal{D}\mathbf{u}(t)}{u_0}\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathrm{u}}^2\,dt\right)$$

$$=\int_0^\infty dk\sum_{l,\mu}H^{(D)}_{k,l,\mu}(\boldsymbol{\omega}'')H^{(D)\,*}_{k,l,\mu}(\boldsymbol{\omega}')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(k^2+\frac{(D-2)^2}{4}\right)\right]\ , \tag{6.7.27}$$

$$=\frac{1}{2\pi}(2\pi\sinh r)^{\frac{3-D}{2}}\int_0^\infty dk\left|\frac{\Gamma(\mathrm{i}\,k+\frac{D-2}{2})}{\Gamma(\mathrm{i}\,k)}\right|^2$$

$$\times\mathcal{P}^{\frac{3-D}{2}}_{\mathrm{i}\,k-\frac{1}{2}}(\cosh r)\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(k^2+\frac{(D-2)^2}{4}\right)\right]\ , \tag{6.7.28}$$

$$H^{(D)}_{k,l,\mu}(\boldsymbol{\omega})=S^{(D-1)}_{l,\mu}(\boldsymbol{\Omega})\frac{\Gamma(\mathrm{i}\,k+l+\frac{D-2}{2})}{\Gamma(\mathrm{i}\,k)}(\sinh\tau)^{\frac{3-D}{2}}\mathcal{P}^{\frac{3-D}{2}-l}_{\mathrm{i}\,k-\frac{1}{2}}(\cosh r_{('','))})\ , \tag{6.7.29}$$

and $\cosh r$ is the hyperbolic distance as defined by

$$\cosh r_{(1,2)}=\cosh\tau_1\cosh\tau_2-\sinh\tau_1\sinh\tau_2\cdot\left(\cos\vartheta_1^{D-2}\cos\vartheta_2^{D-2}\right.$$

$$\left.+\sum_{m=1}^{D-3}\cos\vartheta_1^m\cos\vartheta_2^m\prod_{n=m+1}^{D-2}\sin\vartheta_1^n\sin\vartheta_2^n+\prod_{n=1}^{D-2}\sin\vartheta_1^n\sin\vartheta_2^n\right)\ . \tag{6.7.30}$$

The Green function is given by

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\boldsymbol{\omega}(t')=\boldsymbol{\omega}'}^{\boldsymbol{\omega}(t'')=\boldsymbol{\omega}''}\mathcal{D}\boldsymbol{\omega}(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\boldsymbol{\omega}}^2-\hbar^2\frac{(D-1)(D-3)}{8m}\right]\mathrm{d}t\right\}$$
$$=\frac{m}{\pi\hbar^2}\left(\frac{-1}{2\pi\sinh r}\right)^{(D-3)/2}\mathcal{Q}^{(D-3)/2}_{-\mathrm{i}\sqrt{2mE/\hbar^2-(D-2)^2/4}-1/2}(\cosh r_{('','))})\ . \tag{6.7.31}$$

6.7.5.2 Spectral Expansions in Special Coordinates. [104,427,435,444,466]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{u}(t')=\mathbf{u}'}^{\mathbf{u}(t'')=\mathbf{u}''}\frac{\mathcal{D}\mathbf{u}(t)}{u_0}\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathrm{u}}^2\,\mathrm{d}t\right)$$

Horocyclic, $\mathbf{x}\in\mathbb{R}^{D-2}, y>0$:

$$=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^{D-1}}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{\dot{\mathbf{x}}^2+\dot{y}^2}{y^2}-\frac{\hbar^2}{8m}(D-1)(D-3)\right)\mathrm{d}t\right]$$
$$=\frac{(y'y'')^{\frac{D-2}{2}}}{(2\pi)^{D-2}}\int_{\mathbb{R}^{D-2}}\mathrm{d}\boldsymbol{\nu}\,\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot(\mathbf{x}''-\mathbf{x}')}\,\frac{2}{\pi^2}\int_0^\infty\frac{k\mathrm{d}k\sinh\pi k}{E_k-E}K_{\mathrm{i}k}(|\boldsymbol{\nu}|y'')K_{\mathrm{i}k}(|\boldsymbol{\nu}|y')\ . \tag{6.7.32}$$

Spherical, $\tau>0, \mathbf{\Omega}\in S^{(D-2)}$:

$$=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^{D-2}\tau\int\limits_{\mathbf{\Omega}(t')=\mathbf{\Omega}'}^{\mathbf{\Omega}(t'')=\mathbf{\Omega}''}\mathcal{D}\mathbf{\Omega}(t)$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{\tau}^2+\sinh^2\tau\dot{\mathbf{\Omega}}^2)-\hbar^2\frac{(D-2)^2}{8m}\right)\mathrm{d}t\right]$$
$$=\int_0^\infty\mathrm{d}k\sum_{l,\mu}\frac{H^{(D)}_{k,l,\mu}(\mathbf{\Omega}'')H^{(D)\,*}_{k,l,\mu}(\mathbf{\Omega}')}{E_k-E}\ . \tag{6.7.33}$$

Equidistant, $\tau_1,\dots,\tau_{D-1}\in\mathbb{R}^{D-1}$, $\epsilon_i=\pm1$:

$$=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''}\mathcal{D}\tau_1(t)\cosh^{D-2}\tau_1\dots\int\limits_{\tau_{D-1}(t')=\tau_{D-1}'}^{\tau_{D-1}(t'')=\tau_{D-1}''}\mathcal{D}\tau_{D-1}(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{\tau}_1^2+\cosh^2\tau_1\dot{\tau}_2^2+\dots+(\cosh^2\tau_2\dots\cosh^2\tau_{D-2})\dot{\tau}_{D-1}^2\right)\right.\right.$$

$$-\frac{\hbar^2}{8m}\left((D-2)^2+\frac{1}{\cosh^2\tau_1}+\dots+\frac{1}{\cosh^2\tau_1\dots\cosh^2\tau_{D-2}}\right)\Bigg]\mathrm{d}t\Bigg\}$$

$$=\left(\cosh^{D-2}\tau_1'\cosh^{D-2}\tau_1''\times\dots\times\cosh\tau_{D-2}'\cosh\tau_{D-2}''\right)^{-1/2}$$
$$\times\int_{\mathbb{R}}\frac{\mathrm{d}k_0}{2\pi}\mathrm{e}^{\mathrm{i}k_0(\tau_{D-1}''-\tau_{D-1}')}\prod_{j=1}^{D-2}\sum_{\epsilon_j=\pm}\frac{1}{2}\int_0^\infty\frac{\mathrm{d}k_j\,k_j\sinh\pi k_j}{(\cosh^2\pi k_{j-1}+\sinh^2\pi k_j)}$$
$$\times\frac{P_{\mathrm{i}k_{j-1}-1/2}^{\mathrm{i}k_j}(\epsilon_{D-1-j}\tanh\tau_{D-1-j}'')P_{\mathrm{i}k_{j-1}-1/2}^{-\mathrm{i}k_j}(\epsilon_{D-1-j}\tanh\tau_{D-1-j}')}{E_{k_{D-2}}-E}\ . \qquad (6.7.34)$$

General expression for the Green function:

$$=\frac{m}{\pi\hbar^2}\left(\frac{\mathrm{e}^{-\mathrm{i}\pi}}{2\pi\sinh d}\right)^{(D-3)/2}Q_{-1/2-\mathrm{i}\sqrt{2m(E-E_0)}/\hbar}^{(D-3)/2}\left(\cosh d(\mathbf{q}',\mathbf{q}')\right)\ . \qquad (6.7.35)$$

$\left(\mathbf{x}=\{x_i\}\equiv(x_1,\dots,x_{D-2}),\ r^2=\sum_{i=1}^{D-2}x_i^2\right)$. In the equidistant system we identify $E_{k_{D-2}}\equiv E_k$ with $E_k=E_0+\hbar^2k^2/2m$, $E_0=\hbar^2(D-2)^2/8m$, the wave functions $H_{k,l,\mu}^{(D)}(\mathbf{\Omega})$ (6.7.29) from Sect. 6.7.5, and, e.g.,

$$\cosh d(\mathbf{q}'',\mathbf{q}')=\frac{|\mathbf{x}''-\mathbf{x}'|^2+y'^2+y''^2}{2y'y''}\ . \qquad (6.7.36)$$

6.7.6 Pseudo-Bispherical Coordinates. $(N=0,1,\dots,N_M<\frac{1}{2}(\nu-\mu-1),\ \mu=l+(N-2)/2,\nu=\lambda+(M-2)/2)$ [104]

$$\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^{N-1}\tau\cosh^{M-1}\tau$$
$$\times\int\limits_{\mathbf{\Omega}_\alpha^{N-1}(t')=\mathbf{\Omega}_\alpha^{N-1\prime}}^{\mathbf{\Omega}_\alpha^{N-1}(t'')=\mathbf{\Omega}_\alpha^{N-1\prime\prime}}\mathcal{D}\mathbf{\Omega}_\alpha^{N-1}(t)\int\limits_{\mathbf{\Omega}_\beta^{M-1}(t')=\mathbf{\Omega}_\beta^{M-1\prime}}^{\mathbf{\Omega}_\beta^{M-1}(t'')=\mathbf{\Omega}_\beta^{M-1\prime\prime}}\mathcal{D}\mathbf{\Omega}_\beta^{M-1}(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot\tau^2+\sinh^2\tau\dot{\mathbf{\Omega}}_\alpha^2+\cosh^2\tau\dot{\mathbf{\Omega}}_\beta^2\right)\right.\right.$$
$$\left.\left.-\frac{\hbar^2}{8m}\left((N+M-2)^2+\frac{1}{\cosh^2\tau}-\frac{1}{\sinh^2\tau}\right)\right]\mathrm{d}t\right\}$$
$$=\sum_{l,\lambda\in\mathbb{N}_0}\sum_N S_l^N(\mathbf{\Omega}_\alpha'')S_l^N(\mathbf{\Omega}_\alpha')\sum_M S_\lambda^M(\mathbf{\Omega}_\beta'')S_\lambda^M(\mathbf{\Omega}_\beta')$$

$$\times\left(\sum_{N=0}^{N_M}(\tau')\Psi_N^{(\mu,\nu)}(\tau'')\Psi_N^{(\mu,\nu)*}\,\mathrm{e}^{-\,\mathrm{i}\,E_NT/\hbar}\right.$$
$$\left.+\int_0^\infty \mathrm{d}k\,\Psi_k^{(\mu,\nu)}(\tau'')\Psi_k^{(\mu,\nu)*}(\tau')\,\mathrm{e}^{-\,\mathrm{i}\,E_kT/\hbar}\right)\ , \tag{6.7.37}$$

$$\Psi_N^{(\mu,\nu)}(\tau)=\left[\frac{2N!\nu\Gamma(\nu-N)}{\Gamma(1+\mu+N)\Gamma(\nu-\mu-N)}\right]^{1/2}(\cosh\tau)^{2N-\nu+1}$$
$$\times(\sinh\tau)^{1+\mu-\frac{N}{2}}P_N^{(\mu,\nu-\mu-2N-1)}\left(\frac{1-\sinh^2\tau}{\cosh^2\tau}\right)\ , \tag{6.7.38}$$

$$E_N=-\frac{\hbar^2}{2m}\left[(\nu-\mu-2N-1)^2-\frac{(N+M-2)^2}{4}\right]\ , \tag{6.7.39}$$

$$\Psi_k^{(\mu,\nu)}(\tau)=N_k^{(\mu,\nu)}(\cosh\tau)^{1+\nu}(\sinh\tau)^{1+\mu-\frac{N}{2}}$$
$$\times{}_2F_1\left(\frac{1+\nu+\mu-\mathrm{i}\,k}{2},\frac{1+\nu+\mu+\mathrm{i}\,k}{2};1+\mu;-\sinh^2\tau\right)\ , \tag{6.7.40}$$

$$E_k=\frac{\hbar^2}{2m}\left(k^2+\frac{(N+M-2)^2}{4}\right)\ . \tag{6.7.41}$$

Here $k_1=\frac{1}{2}(1+\nu)$, $k_2=\frac{1}{2}(1+\mu)$, and $N_k^{(\mu,\nu)}$ as in Sect. 3.4.5.2.

6.7.7 The Single-Sheeted Hyperboloid. [8,442,447] ($n=0,1,\ldots,N_M<l-\frac{1}{2}$, $\tilde{E}=E-\hbar^2(D-2)^2/8mR^2$, $\tau\in\mathbb{R}$, $\boldsymbol{\Omega}\in S^{(D-2)}$)

$$R^{1-D}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\cosh^{D-2}\tau\int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''}\mathcal{D}\boldsymbol{\Omega}(t)$$
$$\times\exp\left[\frac{\mathrm{i}\,mR^2}{2\hbar}\int_{t'}^{t''}\left(\dot{\tau}^2-\cosh^2\tau\dot{\boldsymbol{\Omega}}^2\right)\mathrm{d}t-\frac{\mathrm{i}\,\hbar T(D-2)^2}{8mR^2}\right]$$
$$=\frac{R^{3-D}}{\left(\cosh\tau'\cosh\tau''\right)^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\boldsymbol{\Omega}'')S_l^\mu(\boldsymbol{\Omega}')$$
$$\times\frac{m}{\hbar^2}\Gamma\left(\frac{1}{\hbar}\sqrt{-2mR^2\tilde{E}}-l-\frac{D-4}{2}\right)\Gamma\left(\frac{1}{\hbar}\sqrt{-2mR^2\tilde{E}}+l+\frac{D-2}{2}\right)$$
$$\times P_{l+(D-4)/2}^{-\sqrt{-2mR^2\tilde{E}}/\hbar}(\tanh\tau_<)P_{l+(D-4)/2}^{-\sqrt{-2mR^2\tilde{E}}/\hbar}(-\tanh\tau_>)\ , \tag{6.7.42}$$
$$=\frac{R^{1-D}}{\left(\cosh\tau'\cosh\tau''\right)^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\boldsymbol{\Omega}'')S_l^\mu(\boldsymbol{\Omega}')$$
$$\times\left[\sum_{n=0}^{N_M}\left(l+\frac{D-4}{2}-n\right)\frac{(2l+D-4-n)!}{n!}\right.$$

$$\times \frac{P^{n-l-(D-4)/2}_{l+(D-4)/2}(\tanh\tau'')P^{n-l-(D-4)/2}_{l+(D-4)/2}(\tanh\tau')}{-\hbar^2[(n-l-\frac{D-4}{2})^2-(D-2)^2/4]/2mR^2-E}$$
$$+\sum_{\pm}\int_{\mathbb{R}}\frac{k\mathrm{d}k\sinh\pi k}{\cos^2\pi(l+\frac{D-3}{2})+\sinh^2\pi k}$$
$$\left.\times\frac{P^{\mathrm{i}k}_{l+(D-4)/2}(\pm\tanh\tau'')P^{-\mathrm{i}k}_{l+(D-4)/2}(\pm\tanh\tau')}{\hbar^2\dfrac{k^2+(D-2)^2/4}{2mR^2}-E}\right] . \qquad (6.7.43)$$

6.7.8 Motion on SU(n).

6.7.8.1 Motion on SU(2) $\cong S^{(3)}$. [528,553,826]

$$\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''}\mathcal{D}\psi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{I}{2}(\dot\vartheta^2+\dot\psi^2+\dot\varphi^2+2\cos\vartheta\,\dot\psi\dot\varphi)+\frac{\hbar^2}{8I}\left(1+\frac{1}{\sin^2\vartheta}\right)\right]\mathrm{d}t\right\}$$
$$=\left(\frac{I}{2\pi\mathrm{i}\hbar T}\right)^{3/2}\sum_{n\in\mathbb{Z}}\frac{\Omega+4\pi n}{\sin(\Omega/2)}\exp\left(\frac{\mathrm{i}I(\Omega+4\pi n)^2}{2\hbar T}+\frac{\mathrm{i}\hbar T}{8I}\right) , \qquad (6.7.44)$$
$$=\frac{1}{16\pi^2}\sum_{2J\in\mathbb{N}_0}(2J+1)C^1_{2J}\left(\cos\frac{\Omega}{2}\right)\exp\left(-\frac{\mathrm{i}\hbar T}{2I}J(J+1)\right) , \qquad (6.7.45)$$

with

$$\cos\frac{\Omega}{2}=\cos\frac{\vartheta'}{2}\cos\frac{\vartheta''}{2}\cos\frac{\varphi''-\varphi+\psi''-\psi'}{2}$$
$$+\sin\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\cos\frac{\varphi''-\varphi-\psi''+\psi'}{2} . \qquad (6.7.46)$$

6.7.8.2 Motion on SU(n). [263,679]

$$\int\limits_{\mathrm{e}(t')=\mathrm{e}'}^{\mathrm{e}(t'')=\mathrm{e}''}\mathcal{D}_{\mathrm{SU}(n)}\mathrm{e}(t)\exp\left[\frac{\mathrm{i}m}{\hbar}\int_{t'}^{t''}(1-\dot{\mathrm{e}}^2)\mathrm{d}t\right]$$
$$=\frac{1}{|M|}\sum_l d^l\chi^l(q)\,\mathrm{e}^{-\mathrm{i}E^lT/\hbar} , \qquad (6.7.47)$$

with $|M|$ the volume of the group space SU(n), $q=(\mathrm{e}',\mathrm{e}'')$ (e some suitably chosen unit vector system), $\chi^l(q)$ its character, $2E^l$ the eigenvalue of the corresponding Casimir operator, and d^l the dimension of its representation l. (Note: This representation guarantees that in the path integral only corrections $\propto \hbar^2 \times$ constant may be present; they cannot be further specified because explicit representations cannot be stated.)

6.7.9 Motion on the SU(n)/SU($n-1$)-Sphere.

6.7.9.1 Motion on the SU(n)/*SU*(n−1)*-Sphere.* [433,679] ($\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_{n-1})$, $\boldsymbol{\varphi} = (\varphi_1, \ldots \varphi_n)$)

$$
\begin{aligned}
&K^{(n)}(\boldsymbol{\vartheta}'', \boldsymbol{\vartheta}', \boldsymbol{\varphi}'', \boldsymbol{\varphi}'; T)\\
&= \int\limits_{\boldsymbol{\vartheta}(t')=\boldsymbol{\vartheta}'}^{\boldsymbol{\vartheta}(t'')=\boldsymbol{\vartheta}''} \mathcal{D}\boldsymbol{\vartheta}(t) \left(\prod_{k=1}^{n-1} \cos\vartheta_k (\sin\vartheta_k)^{2k-1}\right) \int\limits_{\boldsymbol{\varphi}(t')=\boldsymbol{\varphi}'}^{\boldsymbol{\varphi}(t'')=\boldsymbol{\varphi}''} \mathcal{D}\boldsymbol{\varphi}(t)\\
&\times \exp\Bigg[\!\!\Bigg[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \Bigg\{\frac{m}{2}\Big[\dot\vartheta_{n-1}^2 + \cos^2\vartheta_{n-1}\dot\varphi_n^2 + \ldots\\
&\qquad + \sin^2\vartheta_2\big(\dot\vartheta_1^2 + \cos^2\vartheta_1\dot\varphi_2^2 + \sin^2\vartheta_1\dot\varphi_1^2\big)\ldots\Big] + \frac{\hbar^2}{8m}\Bigg[1 + \frac{1}{\cos^2\vartheta_{n-1}} + \ldots\\
&\qquad + \frac{1}{\sin^2\vartheta_2}\left((2n-2)^2 + \frac{1}{\cos^2\vartheta_1} + \frac{1}{\sin^2\vartheta_1}\right)\ldots\Bigg]\Bigg\}\mathrm{d}t\Bigg]\!\!\Bigg]\\
&= \sum_{\mathbf{L}} \Psi_{\mathbf{L}}^{(n)}(\boldsymbol{\vartheta}'', \boldsymbol{\varphi}'') \Psi_{\mathbf{L}}^{(n)*}(\boldsymbol{\vartheta}', \boldsymbol{\varphi}')\, \mathrm{e}^{\mathrm{i} E_L T/\hbar} \ . \qquad (6.7.48)
\end{aligned}
$$

The energy spectrum is

$$
E_L = \frac{\hbar^2}{2m} L(L+2n-2) \ , \qquad L \in \mathbb{N}_0 \ , \qquad (6.7.49)
$$

and the wave functions are ($\mathbf{L} = \{L_1, \ldots, L_n\}$):

$$
\begin{aligned}
\Phi_{\mathbf{L}}^{(n)}(\boldsymbol{\vartheta}, \boldsymbol{\varphi}) &= \left[(2\pi)^n \prod_{j=1}^{n-1} \cos\vartheta_j (\sin\vartheta_j)^{2j-1}\right]^{-1/2}\\
&\quad \times \exp\left(\mathrm{i}\sum_{j=1}^{n} k_i\varphi_i\right) \Phi_{n_1}^{(k_1,k_2)}(\vartheta_1) \times \ldots \times \Phi_{n_{n-1}}^{(L+n-2,k_n)}(\vartheta_{n-1}) \ . \qquad (6.7.50)
\end{aligned}
$$

The quantum numbers $\mathbf{L}$ are defined by

$$
\left.
\begin{aligned}
L_1 &= 2n_1 + |k_1| + |k_2| \ ,\\
L_2 &= 2n_2 + L_1 + |k_3| \ ,\\
&\ \vdots\\
L_i &= 2n_i + L_{i-1} + |k_{i+1}| \ , \qquad i = 2, \ldots, n-2 \ ,\\
L \equiv L_{n-1} &= 2n_{n-1} + L_{n-2} + |k_n| \ .
\end{aligned}
\right\} \qquad (6.7.51)
$$

6.7.9.2 Motion on the SU(3)/$SU(2)$*-Sphere.* [433] ($\boldsymbol{\vartheta}=(\vartheta_1,\vartheta_2),\boldsymbol{\varphi}=(\varphi_{1,2,3})$)

$$K^{(3)}(\boldsymbol{\vartheta}'',\boldsymbol{\vartheta}',\boldsymbol{\varphi}'',\boldsymbol{\varphi}';T)=\sum_{L\in\mathbb{N}_0}\sum_{k_3=-L}^{L}\sum_{L_1=0}^{L}\sum_{k_{1,2}=-L_1}^{L_1}\frac{(L+2)(L_1+1)}{2\pi^3}\,\mathrm{e}^{\mathrm{i}\,\mathbf{k}(\boldsymbol{\varphi}''-\boldsymbol{\varphi}')}$$
$$\times D^{\frac{L+1}{2}}_{\frac{k_3+L_1+1}{2},\frac{k_3-L_1-1}{2}}(\cos 2\vartheta_2'')D^{\frac{L+1}{2}*}_{\frac{k_3+L_1+1}{2},\frac{k_3-L_1-1}{2}}(\cos 2\vartheta_2')$$
$$\times D^{L_1/2}_{\frac{k_1+k_2}{2},\frac{k_1-k_2}{2}}(\cos 2\vartheta_1'')D^{L_1/2*}_{\frac{k_1+k_2}{2},\frac{k_1-k_2}{2}}(\cos 2\vartheta_1')\exp\left[-\frac{\mathrm{i}\hbar T}{2m}L(L+4)\right]. \tag{6.7.52}$$

6.7.9.3 Motion on the SU(2)*-Sphere.* [104,360,433,553,826] ($\boldsymbol{\varphi}=(\varphi_1,\varphi_2)$, $SU(2)$ is generated by $\mathbf{J}_1,\mathbf{J}_2,\mathbf{J}_3$, and J is the eigenvalue of $\mathbf{J}_3$; the path integral for spin)

$$K^{(2)}(\vartheta'',\vartheta',\boldsymbol{\varphi}'',\boldsymbol{\varphi}';T)=(\sin\vartheta'\sin\vartheta''\cos\vartheta'\cos\vartheta'')^{-1/2}\frac{1}{4\pi^2}\sum_{\mathbf{k}\in\mathbb{Z}^2}\mathrm{e}^{\mathrm{i}\,\mathbf{k}(\boldsymbol{\varphi}''-\boldsymbol{\varphi}')}$$
$$\times\sum_{N\in\mathbb{N}_0}\Phi_N^{(k_1,k_2)}(\vartheta'')\Phi_N^{(k_1,k_2)*}(\vartheta')\exp\left(-\frac{\mathrm{i}\hbar T}{2m}L(L+2)\right), \tag{6.7.53}$$

$$=\sum_{L\in\mathbb{N}_0}\sum_{k_1,k_2=-L}^{L}\frac{L+1}{2\pi^2}\exp\left(-\frac{\mathrm{i}\hbar T}{2m}L(L+2)\right)\mathrm{e}^{\mathrm{i}\,\mathbf{k}(\boldsymbol{\varphi}''-\boldsymbol{\varphi}')}$$
$$\times D^{L/2}_{\frac{k_1+k_2}{2},\frac{k_1-k_2}{2}}(\cos 2\vartheta'')D^{L/2*}_{\frac{k_1+k_2}{2},\frac{k_1-k_2}{2}}(\cos 2\vartheta'), \tag{6.7.54}$$

$$=\sum_{J=0,\frac12}^{\infty}\frac{2J+1}{2\pi^2}C^1_{2J}\left(\cos\frac{\Omega}{2}\right)\exp\left[-\frac{2\,\mathrm{i}\hbar T}{m}J(J+1)\right], \tag{6.7.55}$$

$$\cos\frac{\Omega}{2}=\sin\vartheta'\sin\vartheta''\cos(\varphi_1''-\varphi_1')+\cos\vartheta'\cos\vartheta''\cos(\varphi_2''-\varphi_2'). \tag{6.7.56}$$

The partition function is:

$$Z(T)=\mathrm{Tr}[\mathrm{e}^{-\mathrm{i}\,h\mathbf{J}_3T}]=\frac{\sin((J+1/2)hT)}{\sin(hT/2)}. \tag{6.7.57}$$

6.7.10 Free Motion in SU(n)/ SU($n-1$)-Spherical Polar Coordinates. [433] ($\boldsymbol{\vartheta}=(\vartheta_1,\dots,\vartheta_{n-1}),\boldsymbol{\varphi}=(\varphi_1,\dots,\varphi_n)$, $\Psi^{(n)}_{\mathbf{L}}$ as in (6.7.50))

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^{2n-1}\prod_{k=1}^{n-1}\int\limits_{\vartheta_k(t')=\vartheta_k'}^{\vartheta_k(t'')=\vartheta_k''}\mathcal{D}\vartheta_k(t)\cos\vartheta_k(\sin\vartheta_k)^{2k-1}\int\limits_{\boldsymbol{\varphi}(t')=\boldsymbol{\varphi}'}^{\boldsymbol{\varphi}(t'')=\boldsymbol{\varphi}''}\mathcal{D}\boldsymbol{\varphi}(t)$$

$$\times \exp\left[\!\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2}\Big[\dot{r}^2 + r^2\Big(\dot{\vartheta}_{n-1}^2 + \cos^2\vartheta_{n-1}\dot{\varphi}_n^2 + \dots\sin^2\vartheta_2(\dot{\vartheta}_1^2 + \cos^2\vartheta_1\dot{\varphi}_2^2 + \sin^2\vartheta_1\dot{\varphi}_1^2)\dots\Big)\Big] + \frac{\hbar^2}{8mr^2}\left[1 + \frac{1}{\cos^2\vartheta_{n-1}} + \dots + \frac{1}{\sin^2\vartheta_2}\left(1 + \frac{1}{\cos^2\vartheta_1} + \frac{1}{\sin^2\vartheta_1}\right)\dots\right]\right\}dt\right]\!\!\right]$$

$$= (r'r'')^{-n}\frac{m}{\mathrm{i}\hbar T}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(r'^2 + r''^2)\right] \times \sum_{\mathbf{L}}\Psi_{\mathbf{L}}^{(n)}(\boldsymbol{\vartheta}'',\boldsymbol{\varphi}'')\Psi_{\mathbf{L}}^{(n)*}(\boldsymbol{\vartheta}',\boldsymbol{\varphi}')I_{L+n-1}\left(\frac{mr'r''}{\mathrm{i}\hbar T}\right) . \tag{6.7.58}$$

6.7.11 Motion on SU(1,1). [104,386,551] (see Sect. 3.4.5)

$$\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh\tau\cosh\tau\int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''}\mathcal{D}\psi(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(-\dot{\tau}^2 - \sinh^2\tau\dot{\varphi}^2 + \cosh^2\tau\dot{\psi}^2\Big) + \frac{\hbar^2}{8m}\left(4 - \frac{1}{\sinh^2\tau} + \frac{1}{\cosh^2\tau}\right)\right]dt\right\}$$

$$= \frac{1}{4\pi^2}\sum_{\sigma}\left[\sum_{l=-\frac{1}{2},0}^{\infty}\sum_{m,n}\mathrm{e}^{\mathrm{i}m(\varphi''-\varphi')+\mathrm{i}n(\psi''-\psi')}\,\Psi_{l,mn,\sigma}^{\mathrm{SU}(1,1)}(\tau'')\Psi_{l,mn,\sigma}^{\mathrm{SU}(1,1)*}(\tau')\times\exp\left(-\frac{\mathrm{i}\hbar T}{2m}2l(2l+2)\right)\right.$$
$$\left.+\int_0^{\infty}dk\sum_{m,n}\mathrm{e}^{\mathrm{i}m(\varphi''-\varphi')+\mathrm{i}n(\psi''-\psi')}\times\Psi_{-\frac{1}{2}+\mathrm{i}k,mn,\sigma}^{\mathrm{SU}(1,1)}(\tau'')\Psi_{-\frac{1}{2}+\mathrm{i}k,mn,\sigma}^{\mathrm{SU}(1,1)*}(\tau')\exp\left(\frac{\mathrm{i}\hbar T}{2m}(k^2+1)\right)\right] . \tag{6.7.59}$$

6.7.12 Partition Function on SU(1,1) **(Discrete Series).** [104,360] (SU(1,1) is generated by $\underline{\mathrm{K}}_1, \underline{\mathrm{K}}_2, \underline{\mathrm{K}}_3$, and K is the eigenvalue of $\underline{\mathrm{K}}_3$)

$$Z(T) = \mathrm{Tr}[\mathrm{e}^{-\mathrm{i}\hbar\underline{\mathrm{K}}_3}] = \frac{\mathrm{e}^{-\mathrm{i}\hbar KT}}{1 - \mathrm{e}^{-\mathrm{i}KT}} . \tag{6.7.60}$$

6.7.13 Motion on the O(2,2) Hyperboloid (Anti-De Sitter Gravity). [447]

$$\begin{aligned}
&\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh\tau\cosh\tau\int\limits_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''}\mathcal{D}\varphi_1(t)\int\limits_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''}\mathcal{D}\varphi_2(t)\\
&\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\tau^2+\sinh^2\dot\varphi_1^2-\cosh^2\tau\dot\varphi_2^2)-\frac{\hbar^2}{2m}\left(1-\frac{1}{\sinh^2 2\tau}\right)\right]\mathrm{d}t\right\}\\
&=(\sinh\tau'\sinh\tau''\cosh\tau'\cosh\tau'')^{-1/2}\sum_{\nu_{1,2}\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,\nu_1(\varphi_1''-\varphi_1')+\mathrm{i}\,\nu_2(\varphi_2''-\varphi_2')}}{4\pi^2}\\
&\times\left\{\sum_{n=0}^{N_M}\Psi_n^{(\nu_1,\nu_2)}(\tau'')\Psi_n^{(\nu_1,\nu_2)}(\tau')\,\mathrm{e}^{\mathrm{i}\,\hbar T\left[\left(|\nu_1|-|\nu_2|-2n-\frac{1}{2}\right)^2-1/4\right]/2m}\right.\\
&\qquad\left.+\int_0^\infty\mathrm{d}k\,\Psi_k^{(\nu_1,\nu_2)}(\tau'')\Psi_k^{(\nu_1,\nu_2)*}(\tau')\,\mathrm{e}^{-\mathrm{i}\,\hbar k^2T/2m}\right\}\ ,
\end{aligned}\tag{6.7.61}$$

and cf. [447] for an expansion into horocyclic coordinates.

6.7.14 Motion on the SU$(n-1,1)/$SU$(n-1)$-Pseudosphere. [433,435]

$$\begin{aligned}
&\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\cosh\tau(\sinh\tau)^{2n-3}\\
&\times\prod_{k=1}^{n-2}\int\limits_{\vartheta_k(t')=\vartheta_k'}^{\vartheta_k(t'')=\vartheta_k''}\mathcal{D}\vartheta_k(t)\cos\vartheta_k(\sin\vartheta_k)^{2k-1}\prod_{k=1}^{n}\int\limits_{\varphi_k(t')=\varphi_k'}^{\varphi_k(t'')=\varphi_k''}\mathcal{D}\varphi_k(t)\\
&\times\exp\left[\!\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\left\{\dot\tau^2-\cosh^2\tau\dot\varphi_n^2+\sinh^2\tau\left[\dot\vartheta_{n-2}^2+\cos^2\vartheta_{n-2}\dot\varphi_{n-1}^2+\dots\right.\right.\right.\right.\right.\\
&\qquad\left.\left.+\sin^2\vartheta_2\left(\dot\vartheta_1^2+\cos^2\vartheta_1\dot\varphi_2^2+\sin^2\vartheta_1\dot\varphi_1^2\right)\dots\right]\right\}\\
&\qquad-\frac{\hbar^2}{8m}\left\{(2n-2)^2+\frac{1}{\cosh^2\tau}-\frac{1}{\sinh^2\tau}\left[1+\frac{1}{\cos^2\vartheta_{n-2}}+\dots\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.\dots+\frac{1}{\sin^2\vartheta_2}\left(1+\frac{1}{\cos^2\vartheta_1}+\frac{1}{\sin^2\vartheta_1}\right)\dots\right]\right\}\right)\mathrm{d}t\right]\!\!\right]\\
&=\left[(\sinh\tau'\sinh\tau'')^{n-2}\prod_{j=2}^{n-2}(\sin\vartheta_j'\sin\vartheta_j'')^{j-1}\right]^{-1/2}
\end{aligned}$$

$$\times \sum_{n_1,\dots,n_{n-2}} \Bigg[\Phi_{n_1}^{(k_1,k_2)}(\vartheta_1'')\Phi_{n_1}^{(k_1,k_2)}(\vartheta_1') \times \dots$$
$$\times \Phi_{n_{n-2}}^{(L_{n-3}+n-3,k_{n-1})}(\vartheta_{n-2}'')\Phi_{n_{n-2}}^{(L_{n-3}+n-3,k_{n-1})}(\vartheta_{n-2}') \Bigg]$$
$$\times \left(\sum_{N=0}^{N_M} \Phi_n^{(L_{n-2}+n-2,k_n)}(\tau'')\Phi_n^{(L_{n-2}+n-2,k_n)\,*}(\tau')\, \mathrm{e}^{-\mathrm{i}TE_N/\hbar} \right.$$
$$\left. + \int_0^\infty \mathrm{d}k\, \Phi_k^{(L_{n-2}+n-2,k_n)}(\tau'')\Phi_k^{(L_{n-2}+n-2,k_n)\,*}(\tau')\, \mathrm{e}^{-\mathrm{i}TE_k/\hbar} \right) , \tag{6.7.62}$$

with the discrete energy spectrum

$$E_N = -\frac{\hbar^2}{2m}\left(|k_n| - L_{n-2} - n - 2N + 1\right)^2 , \tag{6.7.63}$$

where $N = 0, 1, \dots, N_M < \frac{1}{2}(|k_n| - L_{n-2} - n + 1)$. The continuous spectrum reads as $E_k = \hbar^2[k^2 + (n-1)^2]/2m, k > 0$ with the largest lower bound $E_0 = \hbar^2(n-1)^2/2m$.

6.7.15 Motion on the SU$(n-v,v)/$SU$(n-1)$-Pseudosphere. For the quantum motion on a SU$(n-v,v)/$SU$(n-1)$-pseudosphere, say, we set $\tau \to \tau_v$ $\vartheta_{n-2} \to \tau_{v-1}, \dots, \vartheta_{n-v-1} \to \tau_1$. The appropriate polar coordinate system for the SU$(n-v,v)/$SU$(n-1)$-pseudosphere has the form

$$\begin{aligned}
z_n &= \mathrm{e}^{\mathrm{i}\varphi_n} \cosh \tau_v \\
z_{n-1} &= \mathrm{e}^{\mathrm{i}\varphi_{n-1}} \sinh \tau_v \cosh \tau_{v-1} \\
&\ \vdots \\
z_{n-v+1} &= \mathrm{e}^{\mathrm{i}\varphi_{n-v+1}} \sinh \tau_v \dots \sinh \tau_2 \cosh \tau_1 \\
z_{n-v} &= \mathrm{e}^{\mathrm{i}\varphi_{n-v}} \sinh \tau_v \dots \sinh \tau_2 \sinh \tau_1 \cos \vartheta_{n-v-1} \\
z_{n-v-1} &= \mathrm{e}^{\mathrm{i}\varphi_{n-v-1}} \sinh \tau_v \dots \sinh \tau_2 \sinh \tau_1 \sin \vartheta_{n-v-1} \cos \vartheta_{n-v-2} \\
&\ \vdots \\
z_2 &= \mathrm{e}^{\mathrm{i}\varphi_2} \sinh \tau_v \dots \sinh \tau_2 \sinh \tau_1 \sin \vartheta_{n-v-1} \dots \sin \vartheta_2 \cos \vartheta_1 \\
z_1 &= \mathrm{e}^{\mathrm{i}\varphi_2} \sinh \tau_v \dots \sinh \tau_2 \sinh \tau_1 \sin \vartheta_{n-v-1} \dots \sin \vartheta_2 \sin \vartheta_1
\end{aligned} \tag{6.7.64}$$

with $0 \leq \varphi_i \leq 2\pi$, $(i = 1, \dots, n)$, $0 \leq \vartheta_j \leq \frac{\pi}{2}$, $(j = 1, \dots, n-v-1)$, and $\tau_l > 0$, $(l = 1, \dots, v)$. The Lagrangian for the SU$(n-v,v)/$SU$(n-1)$-pseudosphere is constructed from that of the SU$(n)/$SU$(n-1)$-sphere as follows $(\boldsymbol{\tau} = (\tau_1, \dots, \tau_v), \boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_{n-v-1}), \boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_n)$:

$$\mathcal{L}^{(n)}(\vartheta_{n-1}, \dot\vartheta_{n-1}, \dots, \vartheta_{n-v}, \dot\vartheta_{n-v}, \vartheta_{n-v-1}, \dot\vartheta_{n-v-1}, \dots, \vartheta_1, \dot\vartheta_1, \boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}})$$

$$\to -\mathcal{L}^{(n)}(\mathrm{i}\,\tau_v,\mathrm{i}\,\dot{\tau}_v,\ldots,\mathrm{i}\,\tau_1,\mathrm{i}\,\dot{\tau}_1,\vartheta_{n-v-1},\dot{\vartheta}_{n-v-1},\ldots,\vartheta_1,\dot{\vartheta}_1,\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}})$$
$$\equiv \mathcal{L}^{(n-v,v)}(\boldsymbol{\tau},\dot{\boldsymbol{\tau}},\boldsymbol{\vartheta},\dot{\boldsymbol{\vartheta}},\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}}) \ , \qquad (6.7.65)$$
$$\Delta V^{(n)}(\vartheta_{n-1},\ldots,\vartheta_{n-v},\vartheta_{n-v-1},\ldots,\vartheta_1)$$
$$\to -\Delta V^{(n)}(\mathrm{i}\,\tau_v,\ldots,\mathrm{i}\,\tau_1,\vartheta_{n-v-1},\ldots,\vartheta_1) \equiv \Delta V^{(n-v,v)}(\boldsymbol{\tau},\boldsymbol{\vartheta}) \ . \qquad (6.7.66)$$

Therefore for the Feynman kernel

$$\int\limits_{\boldsymbol{\tau}(t')=\boldsymbol{\tau}'}^{\boldsymbol{\tau}(t'')=\boldsymbol{\tau}''} \mathcal{D}\boldsymbol{\tau}(t) \left(\prod_{j=1}^{v} \cosh\tau_j(\sinh\tau_j)^{2(n+j-v)-3}\right)$$
$$\times \int\limits_{\boldsymbol{\vartheta}(t')=\boldsymbol{\vartheta}'}^{\boldsymbol{\vartheta}(t'')=\boldsymbol{\vartheta}''} \mathcal{D}\boldsymbol{\vartheta}(t) \left(\prod_{i=1}^{n-v-1} \cos\vartheta_i(\sin\vartheta_i)^{2i-1}\right) \int\limits_{\boldsymbol{\varphi}(t')=\boldsymbol{\varphi}'}^{\boldsymbol{\varphi}(t'')=\boldsymbol{\varphi}''} \mathcal{D}\boldsymbol{\varphi}(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\mathcal{L}_{\mathrm{Cl}}^{(n-v,v)}(\boldsymbol{\tau},\dot{\boldsymbol{\tau}},\boldsymbol{\vartheta},\dot{\boldsymbol{\vartheta}},\boldsymbol{\varphi},\dot{\boldsymbol{\varphi}}) - \Delta V^{(n-v,v)}(\boldsymbol{\tau},\boldsymbol{\vartheta})\right]\mathrm{d}t\right\}$$
$$= \int \mathrm{d}E_{\mathbf{L}}\,\Psi_{\mathbf{L}}^{(n-v,v)}(\boldsymbol{\tau}'',\boldsymbol{\vartheta}'',\boldsymbol{\varphi}'')\Psi_{\mathbf{L}}^{(n-v,v)\,*}(\boldsymbol{\tau}',\boldsymbol{\vartheta}',\boldsymbol{\varphi}')\,\mathrm{e}^{-\mathrm{i}E_L T/\hbar} \ , \qquad (6.7.67)$$

with the quantum numbers (with $L_{n-v-1} := L$ from (6.7.51), etc.)

$$\tilde{L}_{n-v-2} = \begin{cases} 2n_{n-v-1} + L_{n-v-1} - |k_{n-v+1}| + n - v - 1 \ , \\ p_{n-v-1} \ , \end{cases} \qquad (6.7.68a)$$

$$\tilde{L}_{n-v-3} = \begin{cases} 2n_{n-v} + \tilde{L}_{n-v-2} - |k_{n-v+2}| + n - v - 2 \ , \\ p_{n-v} \ , \end{cases} \qquad (6.7.68b)$$

$$\vdots$$

$$\tilde{L} \equiv \tilde{L}_{n-1} = \begin{cases} 2n_{n-1} + \tilde{L}_{n-2} - |k_n| \ , \\ p_{n-1} \ . \end{cases} \qquad (6.7.68c)$$

where $\mathrm{d}E_{\mathbf{L}}$ denotes the integration, respectively, summation over all quantum numbers. The energy spectrum is

$$E_{\mathbf{L}} = \begin{cases} -\dfrac{\hbar^2}{2m}(2N + \tilde{L}_{n-2} + n - |k_n| - 1)^2 \ , \\ +\dfrac{\hbar^2}{2m}[k^2 + (n-1)^2] \ , \end{cases} \qquad (6.7.69)$$

$N = 0, 1, \ldots, N_M < \frac{1}{2}(|k_n| - \tilde{L}_{n-2} - n - 1)$, $k > 0$, for the discrete, respectively, the continuous spectrum. The wave functions are given by

$$\Psi_{\mathbf{L}}^{(n-v,v)}(\boldsymbol{\tau},\boldsymbol{\vartheta},\boldsymbol{\varphi})$$

$$= \left[(2\pi)^n \prod_{j=1}^{n-v-1} \cos\vartheta_j(\sin\vartheta_j)^{2j-1} \prod_{i=1}^{v} \cosh\tau_i(\sinh\tau_i)^{2(n+j-v)-3}\right]^{-\frac{1}{2}}$$
$$\times \mathrm{e}^{\mathrm{i}\mathbf{k}\boldsymbol{\varphi}} \Phi_{n_1}^{(k_1,k_2)}(\vartheta_1) \times \ldots \times \Phi_{n_{n-v-1}}^{(L_{n-v-2}+n-v-2,k_{n-v})}(\vartheta_{n-v-1})$$
$$\times \begin{pmatrix} \Psi_{n_{n-v}}^{(L_{n-v-1}+n-v-1,k_{n-v+1})}(\tau_1) \\ \Psi_{k_{n-v}}^{(L_{n-v-1}+n-v-1,k_{n-v+1})}(\tau_1) \end{pmatrix} \times \ldots \times \begin{pmatrix} \Psi_{n_{n-1}}^{(\tilde{L}_{n-2},k_n)}(\tau_v) \\ \Psi_{k_{n-1}}^{(\tilde{L}_{n-2},k_n)}(\tau_v) \end{pmatrix} . \tag{6.7.70}$$

6.7.16 The Free Klein–Gordon Feynman Kernel in Explicit Time Representation by Group Path Integration. [548,788] ($\gamma = (1-\beta^2)^{-1/2}$, $\beta = |\mathbf{x}'' - \mathbf{x}'|/T$)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \int \mathcal{D}\mathbf{p}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\mathbf{p}\dot{\mathbf{x}} - c\sqrt{\mathbf{p}^2 + m^2c^2}\right)\mathrm{d}t\right]$$
$$= 2\,\mathrm{i}\,cT\left(\frac{m\gamma}{2\pi\,\mathrm{i}\,\hbar T}\right)^{\frac{D+1}{2}} K_{\frac{D+1}{2}}\left(\frac{\mathrm{i}\,mc^2T}{\hbar\gamma}\right) . \tag{6.7.71}$$

6.7.17 Motion on the Sphere for Klein–Gordon Particle. [653]

$$\frac{1}{2m}\int_0^\infty \mathrm{d}S \int \mathcal{D}\rho(s)\delta(\rho(s)-1) \int\limits_{\boldsymbol{\Omega}(0)=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(S)=\boldsymbol{\Omega}''} \mathcal{D}_E\boldsymbol{\Omega}(s)$$
$$\times \exp\left\{-\frac{1}{\hbar}\int_0^{cS}\left[\frac{mR^2}{2\rho}\dot{\boldsymbol{\Omega}}^2 - \rho\left(\frac{E^2}{2mc^2} - \frac{mc^2}{2}\right)\right]\mathrm{d}s\right\}$$
$$= \sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M} \frac{c\hbar}{[m^2c^2 + (c\hbar/R)^2 l(l+D-2)] - E^2} S_l^\mu(\boldsymbol{\Omega}'')S_l^\mu(\boldsymbol{\Omega}') . \tag{6.7.72}$$

6.8 Coulomb Potentials

6.8.1 Pure Coulomb Potential in One Dimension. [187,345,691]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{e_1 e_2}{|x|}\right)\mathrm{d}t\right]$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\Gamma(1-\kappa) W_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{x_>}{\hbar}\right) M_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{x_<}{\hbar}\right) , \tag{6.8.1}$$

$$= \sum_{n\in\mathbb{N}_0} \frac{\Psi_n(x'')\Psi_n^*(x')}{E_n - E} + \int_{\mathbb{R}} \mathrm{d}k\, \Psi_k(x'')\frac{\Psi_k^*(x')}{k^2\hbar^2/2m - E} , \tag{6.8.2}$$

$$\Psi_n(x) = \left[\frac{n!}{a(n+1)!}\right]^{1/2} \frac{2x}{a(n+1)^2} \exp\left[-\frac{x}{a(n+1)}\right] L_n^{(1)}\left(\frac{2x}{a(n+1)}\right) , \tag{6.8.3}$$

$$E_n = -\frac{m(e_1e_2)^2}{2\hbar^2(n+1)^2} , \qquad n = 0,1,\ldots, \tag{6.8.4}$$

$$\Psi_k(x) = \frac{\Gamma(1-\mathrm{i}/ak)}{\sqrt{2\pi}} \exp\left(\frac{\pi}{2ak}\right) M_{\mathrm{i}/ak,1/2}(-2\,\mathrm{i}\,kx) \tag{6.8.5}$$

$(\kappa = e_1e_2\sqrt{-m/2E}/\hbar, a = \hbar^2/me_1e_2, E_k = k^2\hbar^2/2m)$.

6.8.2 Kratzer Potential. $(\lambda > 0, \kappa = e_1e_2\sqrt{-m/2E}/\hbar, a = \hbar^2/me_1e_2)$ [864]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 + \frac{e_1 e_2}{r} - \frac{\hbar^2}{2m}\frac{\lambda^2 - \frac{1}{4}}{r^2}\right)\mathrm{d}t\right]$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(\frac{1}{2}+\lambda-\kappa)}{\Gamma(2\lambda+1)} W_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) , \tag{6.8.6}$$

$$= \sum_{n\in\mathbb{N}_0} \frac{\Psi_n(r'')\Psi_n^*(r')}{E_n - E} + \int_{\mathbb{R}} \mathrm{d}k\, \frac{\Psi_k(r'')\Psi_k^*(r')}{\hbar^2k^2/2m - E} , \tag{6.8.7}$$

$$\Psi_n(r) = \frac{1}{n+\lambda+\frac{1}{2}}\left[\frac{n!}{a\Gamma(n+2\lambda+1)}\right]^{1/2}\left(\frac{2r}{a(n+\lambda+\frac{1}{2})}\right)^{\lambda}$$
$$\times \exp\left[-\frac{r}{a(n+\lambda+1/2)}\right] L_n^{(2\lambda)}\left(\frac{2r}{a(n+\lambda+1/2)}\right) , \tag{6.8.8}$$

$$E_n = -\frac{m(e_1e_2)^2}{2\hbar^2(n+\lambda+\frac{1}{2})^2} , \qquad n = 0,1,\ldots \tag{6.8.9}$$

$$\Psi_k(r) = \frac{\Gamma(\frac{1}{2}+\lambda-\mathrm{i}/ak)}{\sqrt{2\pi}\Gamma(2\lambda+1)} \exp\left(\frac{\pi}{2ak}\right) M_{\mathrm{i}/ak,\lambda}(-2\,\mathrm{i}\,kr) . \tag{6.8.10}$$

6.8.3 Pure Coulomb Potential in Two Dimensions.

6.8.3.1 Green Function. [280,356,514,564,613] ($\kappa = e_1 e_2 \sqrt{-m/2E}/\hbar$, $\mathbf{x} = \varrho(\cos\varphi, \sin\varphi) \in \mathbb{R}^2$)

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1 e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right] \\
&= \frac{1}{2\pi}\sum_{\nu\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\,\nu(\varphi''-\varphi')} \frac{1}{\sqrt{\varrho'\varrho''}}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{1}{|2\nu|!}\Gamma\left(\frac{1}{2}+|\nu|-\kappa\right) \\
&\qquad \times W_{\kappa,|\nu|}\left(\sqrt{-8mE}\,\frac{\varrho_>}{\hbar}\right) M_{\kappa,|\nu|}\left(\sqrt{-8mE}\,\frac{\varrho_<}{\hbar}\right) \ .
\end{aligned} \tag{6.8.11}$$

6.8.3.2 Polar Coordinates. [280,514] ($a = \hbar^2/m e_1 e_2$)

$$\begin{aligned}
&\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1 e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right] \\
&= \sum_{\nu\in\mathbb{Z}}\sum_{N=1}^{\infty} \Psi_{N,\nu}(\varrho'',\varphi'')\Psi_{N,\nu}^*(\varrho',\varphi')\,\mathrm{e}^{-\mathrm{i}\,\hbar T E_N/\hbar} \\
&\qquad + \sum_{\nu\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\, \Psi_{k,\nu}(\varrho'',\varphi'')\Psi_{k,\nu}^*(\varrho',\varphi')\,\mathrm{e}^{-\mathrm{i}\,\hbar k^2 T/2m} \ ,
\end{aligned} \tag{6.8.12}$$

$$\begin{aligned}
\Psi_{N,\nu}(\varrho,\varphi) &= \left[\frac{(N-|\nu|-1)!}{\pi a^2(N-\frac{1}{2})^3(N+|\nu|-1)!}\right]^{1/2}\left(\frac{2\varrho}{a(N-\frac{1}{2})}\right)^{|\nu|} \\
&\quad \times \exp\left[-\frac{\varrho}{a(N-\frac{1}{2})} - \mathrm{i}\,\nu\varphi\right] L_{N-|\nu|-1}^{(2|\nu|)}\left(\frac{2\varrho}{a(N-\frac{1}{2})}\right) \ ,
\end{aligned} \tag{6.8.13}$$

$$E_N = -\frac{m(e_1 e_2)^2}{2\hbar^2(N-\frac{1}{2})^2} \ , \qquad N = 1,2,\ldots, \tag{6.8.14}$$

$$\begin{aligned}
\Psi_{k,\nu}(\varrho,\varphi) &= \sqrt{\frac{1}{4\pi^2\varrho}}\frac{1}{(2|\nu|)!}\Gamma\left(\frac{1}{2}+|\nu|+\frac{\mathrm{i}}{ak}\right) \\
&\quad \times \exp\left(\frac{\pi}{2ak} - \mathrm{i}\,\nu\varphi\right) M_{\mathrm{i}/ak,|\nu|}(-2\,\mathrm{i}\,k\varrho) \ .
\end{aligned} \tag{6.8.15}$$

6.8.3.3 Parabolic Coordinates. [280,458] ($a = \hbar^2/me_1e_2$)

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2+\eta^2)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\frac{2e_1e_2}{\xi^2+\eta^2}\right)\mathrm{d}t\right]$$
$$= \sum_{N,n\in\mathbb{N}_0} \Psi_{N,n}(\xi'',\eta'')\Psi_{N,n}(\xi',\eta')\,\mathrm{e}^{-\mathrm{i}\hbar E_N T/\hbar}$$
$$+\sum_{\mathrm{e,o}}\int_{\mathbb{R}}\mathrm{d}\zeta\int_0^\infty \mathrm{d}k\,\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi'',\eta'')\Psi_{k,\zeta}^{(\mathrm{e,o})\,*}(\xi',\eta')\,\mathrm{e}^{-\mathrm{i}\hbar k^2T/2m}\ , \tag{6.8.16}$$

$$\Psi_{N,n}(\xi,\eta) = \frac{1}{a(N+\frac12)2^N\sqrt{2\pi(N+\frac12)}}$$
$$\times H_{2N+n}\left(\frac{\xi}{a(N+\frac12)}\right)H_{2N-n}\left(\frac{\eta}{a(N+\frac12)}\right)\exp\left(-\frac{\xi^2+\eta^2}{2a(N+\frac12)}\right)\ , \tag{6.8.17}$$

$$\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta) = \frac{\mathrm{e}^{\pi/2ak}}{\sqrt{2}\,4\pi^2}\begin{pmatrix}\Gamma[\frac14+\frac{\mathrm{i}}{2k}(1/a+\zeta)]E^{(0)}_{-\frac12+\frac{\mathrm{i}}{k}(1/a+\zeta)}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi)\\ \Gamma[\frac34+\frac{\mathrm{i}}{2k}(1/a+\zeta)]E^{(1)}_{-\frac12+\frac{\mathrm{i}}{k}(1/a+\zeta)}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\xi)\end{pmatrix}$$
$$\times\begin{pmatrix}\Gamma[\frac14+\frac{\mathrm{i}}{2k}(1/a-\zeta)]E^{(0)}_{-\frac12+\frac{\mathrm{i}}{k}(1/a-\zeta)}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta)\\ \Gamma[\frac34+\frac{\mathrm{i}}{2k}(1/a-\zeta)]E^{(1)}_{-\frac12+\frac{\mathrm{i}}{k}(1/a-\zeta)}(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2k}\,\eta)\end{pmatrix}\ . \tag{6.8.18}$$

$\sum_{\mathrm{e,o}}$ denotes the summation over the even and odd states, with the same energy spectrum as in (6.8.14).

6.8.4 Two-Dimensional Coulomb Potential in a Sector. [176] ($\kappa = e_1e_2\sqrt{-m/2E}/\hbar$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}^{(DD)}_{(0<\varphi<\alpha)}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2+\frac{e_1e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right]$$
$$=\frac{2}{\alpha}\sum_{l\in\mathbb{Z}}\sin\left(\frac{l\pi}{\alpha}\varphi''\right)\sin\left(\frac{l\pi}{\alpha}\varphi'\right)\frac{1}{\sqrt{\varrho'\varrho''}}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(1/2+l\pi/\alpha-\kappa)}{\Gamma(2l\pi/\alpha+1)}$$
$$\times W_{\kappa,l\pi/\alpha}\left(\sqrt{-8mE}\,\frac{\varrho_>}{\hbar}\right)M_{\kappa,l\pi/\alpha}\left(\sqrt{-8mE}\,\frac{\varrho_<}{\hbar}\right)\ . \tag{6.8.19}$$

6.8.5 Non-Isotropic Two-Dimensional Coulomb System. [436,437] ($\mathbf{x}=(x,y)\in\mathbb{R}^2$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)$$

$$\times\exp\Bigg\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg[\frac{m}{2}\dot{\mathbf{x}}^2+\frac{e^2}{|\mathbf{x}|}+\frac{\hbar^2}{2m}\bigg(\frac{\beta-\frac14}{y^2}-\frac{\gamma x}{y^2|\mathbf{x}|}\bigg)$$

$$+\frac{m}{16}(\omega_1^2+\omega_2^2)-\frac{m}{16}(\omega_1^2-\omega_2^2)\frac{x}{|\mathbf{x}|}\Bigg]dt\Bigg\}$$

$$=\frac{\mathrm{i}}{\hbar}\bigg(\frac{m}{\mathrm{i}\hbar}\bigg)^2\Omega_1\Omega_2\sqrt{\xi'\xi''\eta'\eta''}$$

$$\times\int_0^\infty\frac{ds''\,e^{4\mathrm{i}\alpha s''/\hbar}}{\sin\Omega_1 s''\sin\Omega_2 s''}I_{\lambda_1}\bigg(\frac{m\Omega_1\xi'\xi''}{\mathrm{i}\hbar\sin\Omega_1 s''}\bigg)I_{\lambda_2}\bigg(\frac{m\Omega_1\eta'\eta''}{\mathrm{i}\hbar\sin\Omega_2 s''}\bigg)$$

$$\times\exp\bigg\{\frac{\mathrm{i}m}{2\hbar T}\Big[\Omega_1(u'^2+u''^2)\cot\Omega_1 s''+\Omega_2(v'^2+v''^2)\cot\Omega_2 s''\Big]\bigg\}\;,\tag{6.8.20}$$

$$=\sum_{n_1,n_2\in\mathbb{N}_0}\frac{\Psi_{n_1,n_2}(\xi'',\eta'')\Psi_{n_1,n_2}(\xi',\eta')}{E_{n_1,n_2}-E}+\int_{\mathbb{R}}d\zeta\int_0^\infty dk\,\frac{\Psi_{k,\zeta}(\xi'',\eta'')\Psi^*_{k,\zeta}(\xi',\eta')}{\hbar^2k^2/2m-E}\;.\tag{6.8.21}$$

Here $\Omega_{1,2}=\sqrt{\omega_{1,2}-8E/m}$ and $\lambda_{1,2}=\sqrt{\beta\pm\gamma}$, with two-dimensional parabolic coordinates defined by $x=\frac12(\eta^2-\eta^2)$, $y=\xi\eta$. The bound-state wave functions are given by

$$\Psi_{n_1,n_2}(\xi,\eta)=\Bigg[\bigg(\frac{m}{\hbar}\bigg)^3\frac{(2\Omega_1\Omega_2)^2}{A_1\Omega_2+A_2\Omega_1}\cdot\frac{n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_2+1)}\Bigg]^{1/2}$$

$$\times\bigg(\frac{m\Omega_1}{\hbar}\xi^2\bigg)^{\lambda_1/2}\bigg(\frac{m\Omega_2}{\hbar}\eta^2\bigg)^{\lambda_2/2}\exp\bigg[-\frac{m}{2\hbar}(\Omega_1\xi^2+\Omega_2\eta^2)\bigg]$$

$$\times L_{n_1}^{(\lambda_1)}\bigg(\frac{m\Omega_1}{\hbar}\xi^2\bigg)L_{n_2}^{(\lambda_2)}\bigg(\frac{m\Omega_2}{\hbar}\eta^2\bigg)\;,\tag{6.8.22}$$

and the energy spectrum has the form

$$E_{n_1,n_2}=\frac{m/8}{(A_1^2-A_2^2)^2}\Bigg((A_1^2-A_2^2)(A_1^2\omega_1^2-A_2^2\omega_2^2)-\frac{16\alpha^2}{\hbar^2}(A_1^2+A_2^2)$$

$$+\frac{8e^2}{\hbar}A_1A_2\sqrt{(A_1^2-A_2^2)(\omega_1^2-\omega_2^2)+\frac{16\alpha^2}{\hbar^2}}\Bigg)\;.\tag{6.8.23}$$

with $A_{1/2}=2n_{1/2}+\lambda_{1/2}+1$, and

$$\Omega_{1/2} = \frac{1}{|A_1^2 - A_2^2|} \left| A_{2/1} \sqrt{(A_1^2 - A_2^2)(\omega_1^2 - \omega_2^2) + \frac{16\alpha^2}{\hbar^2}} - \frac{4e^2}{\hbar} A_{1/2} \right| , \tag{6.8.24}$$

where all quantities are valid for $A_1 \neq A_2$. The continuum functions $\Psi_{k,\zeta}(\xi,\eta)$ are given by $[k_{1,2} = (\omega_0/\Omega_{1,2})(1/a \pm \zeta)/2k$, $\omega_0 = \sqrt{-2E/m}$, $a = \hbar^2/me^2]$

$$\begin{aligned}\Psi_{k,\zeta}(\xi,\eta) = \frac{\Gamma(\frac{1+\lambda_1}{2} + \mathrm{i}\,k_1)\Gamma(\frac{1+\lambda_2}{2} + \mathrm{i}\,k_2)}{\sqrt{2\pi^2 k}\,\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)} \, \mathrm{e}^{\pi/2(1/k_1+1/k_2)} \\ \times M_{\mathrm{i}\,k_1,\lambda_1/2}(-\mathrm{i}\,k\xi^2) M_{\mathrm{i}\,k_2,\lambda_2/2}(-\mathrm{i}\,k\eta^2) \ .\end{aligned} \tag{6.8.25}$$

6.8.6 Pure Coulomb Potential in Three Dimensions.

6.8.6.1 Feynman Kernel. [102,407,876] ($\mathbf{x} = (x,y,z) \in \mathbb{R}^3$)

$$\begin{aligned}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e^2}{|\mathbf{x}|}\right)\mathrm{d}t\right] \\ = -\frac{m}{2\pi\hbar^2|\mathbf{x}''-\mathbf{x}'|}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) k(x,y;T) \ ,\end{aligned} \tag{6.8.26}$$

with $x = |\mathbf{x}'| + |\mathbf{x}''| + |\mathbf{x}' - \mathbf{x}''|$, $y = |\mathbf{x}'| + |\mathbf{x}''| - |\mathbf{x}' - \mathbf{x}''|$. Define

$$k_0(x,y;T) = \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}}\left(\mathrm{e}^{-m(x-y)^2/8\,\mathrm{i}\,\hbar T} - \mathrm{e}^{-m(x+y)^2/8\,\mathrm{i}\,\hbar T}\right) , \tag{6.8.27}$$

and $\xi = (x-y)/2$, $e_n = e/n$. We then have ($\beta = \mathrm{i}\,\hbar T/m$)

$$\begin{aligned}k(x,y;T) &= k_0(x,y;T) \\ &+ \frac{\partial}{\partial\beta}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{\beta e_n^2/2}}{e_n} M_{n,l+\frac{1}{2}}(e_n x) M_{n,l+\frac{1}{2}}(e_n y)\,\mathrm{erfc}\left(\frac{\xi - e_n\beta}{\sqrt{2\beta}}\right) \\ &+ \frac{2q}{\sqrt{\pi}}\frac{\partial}{\partial\beta}\mathrm{e}^{-\xi^2/2\beta}\sum_{n=2}^{\infty} n^{-3} \sum_{\substack{p,q=0 \\ p,q\neq 0,0}}^{n-1} \left(-\sqrt{2\beta}\right)^{-(p+q)} \frac{x^{p+1}}{p!}\frac{y^{q+1}}{q!} \\ &\times L_{n-p-1}^{(p+1)}(e_n x) L_{n-q-1}^{(q+1)}(e_n y) H_{p+q-1}\left(\frac{\xi - e_n\beta}{\sqrt{2\beta}}\right) \ .\end{aligned} \tag{6.8.28}$$

6.8.6.2 Green Function. [26,97,152,264,279,280,304,356,382,407,408,417,418, 455,495,496,504,505,516,528,564,608,613,640,642,671,674,741,798,841,864,873, 943] ($\kappa = (e_1e_2/\hbar)\sqrt{-m/2E}$, $\omega^2 = -2E/m$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right]$$

$$= \sum_{l\in\mathbb{N}_0}\sum_{n=-l}^{l} Y_l^n(\vartheta'',\varphi'')Y_l^{n\,*}(\vartheta',\varphi')\frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(1+l-\kappa)}{(2l+1)!}$$

$$\times W_{\kappa,l+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,l+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) , \tag{6.8.29}$$

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\,e_1e_2\tau/\hbar}\left(\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sin\omega\tau}\right)^2 \exp\left(\frac{2\,\mathrm{i}\,m\omega}{\hbar}(r'^2 + r''^2)\cot\omega\tau\right)$$

$$\times I_0\left(\frac{2\sqrt{2}\,\mathrm{i}\,m\omega}{\hbar\sin\omega\tau}\sqrt{r'r'' + \mathbf{x}'\mathbf{x}''}\right) , \tag{6.8.30}$$

$$= -\frac{m\Gamma(1-\kappa)}{2\pi\hbar^2|\mathbf{x}''-\mathbf{x}'|}\begin{vmatrix} W_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{|\mathbf{x}|_>}{\hbar}\right) & M_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{|\mathbf{x}|_<}{\hbar}\right) \\ W'_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{|\mathbf{x}|_>}{\hbar}\right) & M'_{\kappa,1/2}\left(\sqrt{-8mE}\,\frac{|\mathbf{x}|_<}{\hbar}\right)\end{vmatrix} . \tag{6.8.31}$$

6.8.6.3 Polar Coordinates. [26,152,264,279,280,304,408,417,418,495,516,608, 741,798,864,873,943] ($a = \hbar^2/me_1e_2$, $x_\gtrless = |\mathbf{x}'| + |\mathbf{x}''| \pm |\mathbf{x}''-\mathbf{x}'|$).

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right]$$

$$= \sum_{l\in\mathbb{N}_0}\left\{\sum_{N=1}^{\infty}\sum_{n=-l}^{l}\Psi_{N,l,n}(r'',\vartheta'',,\varphi'')\Psi^*_{N,l,n}(r',\vartheta',\varphi')\,\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\right.$$

$$\left. + \sum_{n\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\,\Psi_{k,l,n}(r'',\vartheta'',\varphi'')\Psi^*_{k,l,n}(r',\vartheta',\varphi')\,\mathrm{e}^{-\mathrm{i}\hbar k^2T/2m}\right\} , \tag{6.8.32}$$

$$\Psi_{N,l,n}(r,\vartheta,\varphi) = \frac{2}{N^2}\left[\frac{(N-l-1)!}{a^3(N+l)!}\right]^{1/2}$$

$$\times \exp\left(-\frac{r}{aN}\right)\left(\frac{2r}{aN}\right)^l L_{N-l-1}^{(2l+1)}\left(\frac{2r}{aN}\right)Y_l^n(\vartheta,\varphi) , \tag{6.8.33}$$

$$E_N = -\frac{m(e_1e_2)^2}{2\hbar^2N^2}, \qquad N = n_r + l + 1 = 1,2,3,\dots , \tag{6.8.34}$$

$$\Psi_{k,l,n}(r,\vartheta,\varphi) = \frac{1}{\sqrt{2\pi}\, r(2l+1)!}\Gamma\left(1+l+\frac{\mathrm{i}}{ak}\right)$$
$$\times \exp\left(\frac{\pi}{2ak}\right) M_{\mathrm{i}/ak,l+\frac{1}{2}}(-2\,\mathrm{i}\,kr) Y_l^n(\vartheta,\varphi) \ . \tag{6.8.35}$$

6.8.6.4 Parabolic Coordinates. [189,190,564] ($a = \hbar^2/me^2$)

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)(\xi^2+\eta^2)\xi\eta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big((\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\varphi^2\Big)+\frac{2e^2}{\xi^2+\eta^2}+\frac{\hbar^2}{8m\xi^2\eta^2}\right]\mathrm{d}t\right\}$$
$$= \sum_{\nu\in\mathbb{Z}}\left[\sum_{n_1,n_2\in\mathbb{N}_0}\Psi_{n_1,n_2,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{n_1,n_2,\nu}(\xi',\eta',\varphi')\,\mathrm{e}^{-\mathrm{i}E_N T/\hbar}\right.$$
$$\left. + \int_0^\infty \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}\zeta\, \Psi_{k,\zeta,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{k,\zeta,\nu}(\xi',\eta',\varphi')\,\mathrm{e}^{-\mathrm{i}\hbar k^2 T/2m}\right] \ , \tag{6.8.36}$$

$$\Psi_{n_1,n_2,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\nu\varphi}}{\sqrt{2\pi}}\left[\frac{2}{a^3N^4}\frac{n_1!n_2!}{\pi(n_1+|\nu|)!(n_2+|\nu|)!}\right]^{1/2}\left(\frac{\xi^2\eta^2}{(aN)^2}\right)^{|\nu|/2}$$
$$\times \exp\left(-\frac{\xi^2+\eta^2}{2aN}\right) L_{n_1}^{(|\nu|)}\left(\frac{\xi^2}{aN}\right) L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right) \ , \tag{6.8.37}$$

$$E_N = -\frac{me^4}{2\hbar^2N^2} \ , \qquad N = n_1+n_2+|\nu|+1 = 1,2,\dots, \tag{6.8.38}$$

$$\Psi_{k,\zeta,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\nu\varphi}}{\sqrt{4\pi^3 k}}\frac{\Gamma\left[\frac{1+|\nu|}{2}+\frac{\mathrm{i}}{2k}(1/a+\zeta)\right]\Gamma\left[\frac{1+|\nu|}{2}+\frac{\mathrm{i}}{2k}(1/a-\zeta)\right]}{\xi\eta(|\nu|!)^2}$$
$$\times\, \mathrm{e}^{\pi/2ak} M_{\mathrm{i}(1/a+\zeta)/2k,|\nu|/2}(-\mathrm{i}\,k\xi^2) M_{\mathrm{i}(1/a-\zeta)/2k,|\nu|/2}(-\mathrm{i}\,k\eta^2) \ . \tag{6.8.39}$$

6.8.6.5 Sphero-Conical Coordinates. [434]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^2 \int\limits_{\alpha(t')=\alpha'}^{\alpha(t'')=\alpha''} \mathcal{D}\alpha(t) \int\limits_{\beta(t')=\beta'}^{\beta(t'')=\beta''} \mathcal{D}\beta(t)(k^2\,\mathrm{cn}^2\,\alpha + k'^2\,\mathrm{cn}^2\,\beta)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\Big(\dot r^2 + r^2(k^2\,\mathrm{cn}^2\,\alpha + k'^2\,\mathrm{cn}^2\,\beta)(\dot\alpha^2+\dot\beta^2)\Big)+\frac{e^2}{r}\right]\mathrm{d}t\right\}$$
$$= \frac{1}{r'r''}\sum_{l\in\mathbb{N}_0}\sum_{\lambda}\sum_{p,q=\pm}\Lambda^p_{l,h}(\alpha'')\Lambda^{p\,*}_{l,h}(\alpha')\Lambda^q_{l,\tilde h}(\beta'')\Lambda^{q\,*}_{l,\tilde h}(\beta')$$

$$\times\sqrt{-\frac{m}{2E}}\frac{\Gamma(1+l-\kappa)}{\hbar(2l+1)!}W_{\kappa,l+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,l+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right)\ . \tag{6.8.40}$$

6.8.7 Generalized Kepler–Coulomb Potential.

6.8.7.1 Green Function. [139–143,434,860]

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot r^2+r^2\dot\vartheta^2+r^2\sin^2\vartheta\dot\varphi^2)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{e_1e_2}{r}-\frac{\hbar^2}{2mr^2}\left(\frac{b-\frac{1}{4}}{\sin^2\vartheta}-c\frac{\cos\vartheta}{\sin^2\vartheta}-\frac{1}{4}\right)\right]\mathrm{d}t\right\}\\
&=\frac{1}{2\pi}\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\,\nu(\varphi''-\varphi')}\sum_{l\in\mathbb{N}_0}\frac{(\lambda_1+\lambda_2+2l+1)}{2}\frac{\Gamma(\lambda_1+\lambda_2+l+1)l!}{\Gamma(\lambda_1+l+1)\Gamma(\lambda_2+l+1)}\\
&\quad\times\left(\sin\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\right)^{\lambda_1}\left(\cos\frac{\vartheta'}{2}\cos\frac{\vartheta''}{2}\right)^{\lambda_2}P_l^{(\lambda_1,\lambda_2)}(\cos\vartheta'')P_l^{(\lambda_1,\lambda_2)}(\cos\vartheta')\\
&\quad\times\frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma[\frac{1}{2}(\lambda_1+\lambda_2)+l+1-\kappa]}{\Gamma(\lambda_1+\lambda_2+2l+2)}\\
&\quad\times W_{\kappa,l+\frac{1}{2}(1+\lambda_1+\lambda_2)}(-2\,\mathrm{i}\,kr_>)M_{\kappa,l+\frac{1}{2}(1+\lambda_1+\lambda_2)}(-2\,\mathrm{i}\,kr_<)
\end{aligned}\tag{6.8.41}$$

[$k=\sqrt{2mE}/\hbar, \lambda_{1,2}=\sqrt{\nu^2+b\pm c}, \kappa=e_1e_2\sqrt{-m/2E}/\hbar$].

6.8.7.2 Spherical Coordinates. [139–143,434,860] ($a=\hbar^2/me_1e_2, \lambda=\frac{1}{2}(\lambda_1+\lambda_2+1)+l$ with $\lambda_{1,2}$ as in the previous example)

$$\begin{aligned}
&\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot r^2+r^2\dot\vartheta^2+r^2\sin^2\vartheta\dot\varphi^2)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{e_1e_2}{r}-\frac{\hbar^2}{2mr^2}\left(\frac{b-\frac{1}{4}}{\sin^2\vartheta}-\frac{c}{2m}\frac{\cos\vartheta}{\sin^2\vartheta}-\frac{1}{4}\right)\right]\mathrm{d}t\right\}\\
&=\sum_{l\in\mathbb{N}_0}\sum_{\nu\in\mathbb{Z}}\left\{\sum_{N\in\mathbb{N}_0}\Psi_{N,l,\nu}(r'',\vartheta'',\varphi'')\Psi^*_{N,l,\nu}(r',\vartheta',\varphi')\,\mathrm{e}^{-\mathrm{i}\,E_NT/\hbar}\right.
\end{aligned}$$

$$+\int_0^\infty dk\,\Psi_{k,l,\nu}(r'',\vartheta'',\varphi'')\Psi^*_{k,l,\nu}(r',\vartheta',\varphi')\,e^{-i\hbar k^2T/2m}\Bigg\}\ , \tag{6.8.42}$$

$$\Psi_{N,l,\nu}(r,\vartheta,\varphi)=\frac{e^{i\nu\varphi}}{\sqrt{2\pi}}\sqrt{\frac{(\lambda_1+\lambda_2+2l+1)}{2}\frac{\Gamma(\lambda_1+\lambda_2+l+1)l!}{\Gamma(\lambda_1+l+1)\Gamma(\lambda_2+l+1)}}$$
$$\times\left(\sin\frac{\vartheta}{2}\right)^{\lambda_1}\left(\cos\frac{\vartheta}{2}\right)^{\lambda_2}P_l^{(\lambda_1,\lambda_2)}(\cos\vartheta)$$
$$\times\frac{1}{r}\frac{1}{N+\lambda+\frac{1}{2}}\left[\frac{N!}{a\Gamma(N+2\lambda+2)}\right]^{1/2}\left(\frac{2r}{a(N+\lambda+\frac{1}{2})}\right)^{\lambda+1/2}$$
$$\times\exp\left(-\frac{r}{a(N+\lambda+\frac{1}{2})}\right)L_N^{(2\lambda)}\left(\frac{2r}{a(N+\lambda+\frac{1}{2})}\right)\ , \tag{6.8.43}$$

$$E_N=-\frac{me_1^2e_2^2}{2\hbar^2(N+\lambda+1/2)^2},\qquad N\in\mathbb{N}_0\ , \tag{6.8.44}$$

$$\Psi_{k,l,\nu}(r,\vartheta,\varphi)=\frac{e^{i\nu\varphi}}{\sqrt{2\pi}}\sqrt{\frac{(\lambda_1+\lambda_2+2l+1)}{2}\frac{\Gamma(\lambda_1+\lambda_2+l+1)l!}{\Gamma(\lambda_1+l+1)\Gamma(\lambda_2+l+1)}}$$
$$\times\left(\sin\frac{\vartheta}{2}\right)^{\lambda_1}\left(\cos\frac{\vartheta}{2}\right)^{\lambda_2}P_l^{(\lambda_1,\lambda_2)}(\cos\vartheta)$$
$$\times\sqrt{\frac{1}{2\pi}}\frac{\Gamma(\lambda+\frac{1}{2}-i/ak)}{r\Gamma(2\lambda+1)}\exp\left(\frac{\pi}{2ak}\right)M_{i/ak,\lambda}(-2i\,kr)\ . \tag{6.8.45}$$

6.8.7.3 Parabolic Coordinates. [188,189,434]

$$\frac{i}{\hbar}\int_0^\infty dT\,e^{iET/\hbar}\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2+\eta^2)\xi\eta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\varphi^2\right)\right.\right.$$
$$\left.\left.+\frac{2e_1e_2}{\xi^2+\eta^2}-\frac{\hbar^2}{2m\xi^2\eta^2}\left(b-\frac{1}{4}+c\frac{\eta^2-\xi^2}{\xi^2+\eta^2}\right)\right]dt\right\}$$
$$=\left(\frac{m\omega}{i\hbar}\right)^2\sum_{\nu\in\mathbb{Z}}\frac{e^{i\nu(\varphi''-\varphi')}}{2\pi}\frac{i}{\hbar}\int_0^\infty\frac{ds''}{\sin^2\omega s''}I_{\lambda_1}\left(\frac{m\omega\xi'\xi''}{i\hbar\sin\omega s''}\right)I_{\lambda_2}\left(\frac{m\omega\eta'\eta''}{i\hbar\sin\omega s''}\right)$$
$$\times\exp\left[\frac{2i\,e_1e_2}{\hbar}s''-\frac{m\omega}{2i\hbar}(\xi'^2+\xi''^2+\eta'^2+\eta''^2)\cot\omega s''\right] \tag{6.8.46}$$
$$=\sum_{\nu\in\mathbb{Z}}\Bigg[\sum_{n_1,n_2\in\mathbb{N}_0}\frac{\Psi_{n_1,n_2,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{n_1,n_2,\nu}(\xi',\eta',\varphi')}{E_{n_1,n_2}-E}$$

$$+\int_0^\infty \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}\zeta \, \frac{\Psi_{k,\zeta,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{k,\zeta,\nu}(\xi',\eta',\varphi')}{\hbar^2 k^2/2m - E}\Bigg] , \tag{6.8.47}$$

$$\Psi_{n_1,n_2,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi}}{\sqrt{2\pi}}\left[\frac{2}{a^2N^3}\cdot\frac{2\,n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_2+1)}\right]^{1/2}$$
$$\times\left(\frac{\xi}{aN}\right)^{\lambda_1}\left(\frac{\eta}{aN}\right)^{\lambda_2}\exp\left(-\frac{\xi^2+\eta^2}{2aN}\right)L_{n_1}^{(\lambda_1)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(\lambda_2)}\left(\frac{\eta^2}{aN}\right), \tag{6.8.48}$$

$$E_{n_1,n_2} = -\frac{me_1^2e_2^2}{\hbar^2[n_1+n_2+\frac{1}{2}(\lambda_1+\lambda_2)+1]^2} , \tag{6.8.49}$$

$$\Psi_{k,\zeta,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi+\pi/2ak}}{\sqrt{4\pi^3 p}}\frac{\Gamma[\frac{1+\lambda_1}{2}+\frac{\mathrm{i}}{2k}(1/a+\zeta)]\Gamma[\frac{1+\lambda_2}{2}+\frac{\mathrm{i}}{2k}(1/a-\zeta)]}{\xi\eta\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)}$$
$$\times M_{\frac{\mathrm{i}}{2k}(1/a+\zeta),\lambda_1/2}(-\mathrm{i}\,k\xi^2)M_{\frac{\mathrm{i}}{2k}(1/a-\zeta),\lambda_2/2}(-\mathrm{i}\,k\eta^2) \tag{6.8.50}$$

$[\omega = \sqrt{-2E/m}, N = n_1+n_2+1+\frac{1}{2}(\lambda_1+\lambda_2)]$.

6.8.7.4 Ring Potential. [127,142,143,177,179,859]
$(\lambda = \sqrt{\sigma^2\gamma^2+\nu^2}$, $\kappa = 2\gamma\sigma^2E_0 a\sqrt{-m/2E}/\hbar)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2+\gamma\sigma^2E_0\left(\frac{2a}{|\mathbf{x}|}-\frac{\gamma a^2}{x^2+y^2}\right)\right]\mathrm{d}t\right\}$$
$$=\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,\nu(\varphi''-\varphi')}}{2\pi}\sum_{l\in\mathbb{N}_0}\left(l+\lambda+\frac{1}{2}\right)\frac{\Gamma(l+2\lambda+1)}{l!}P_{l+\lambda}^{-\lambda}(\cos\vartheta'')P_{l+\lambda}^{-\lambda}(\cos\vartheta')$$
$$\times\frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(l+\lambda+1-\kappa)}{\Gamma(2l+2\lambda+2)}$$
$$\times W_{\kappa,l+\lambda+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,l+\lambda+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) . \tag{6.8.51}$$

In polar coordinates one has $(J = l+\lambda,\ \tilde{N} = n+J+1)$:

$$\Psi_{n,J,\nu}(r,\vartheta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi}}{\sqrt{2\pi}}\sqrt{\left(J+\frac{1}{2}\right)\frac{\Gamma(J+\lambda+1)}{l!}}P_J^{-\lambda}(\cos\vartheta)$$
$$\times\frac{2}{\tilde{N}^2}\sqrt{\frac{1}{a^3}\frac{n!}{\Gamma(n+2J+1)}}\left(\frac{2r}{a\tilde{N}}\right)^J\exp\left(-\frac{r}{a\tilde{N}}\right)L_n^{(2J+1)}\left(\frac{2r}{a\tilde{N}}\right) , \tag{6.8.52}$$

$$E_n = -\frac{\gamma^2\sigma^4E_0}{(n+J+1)^2} , \qquad n\in\mathbb{N} . \tag{6.8.53}$$

$$\Psi_{k,J,\nu}(r,\vartheta,\nu) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi}}{\sqrt{2\pi}}\sqrt{\left(J+\frac{1}{2}\right)\frac{\Gamma(J+\lambda+1)}{l!}}P_J^{-\lambda}(\cos\vartheta)$$
$$\times\frac{1}{r}\frac{\Gamma(J+1-\mathrm{i}\,\gamma\sigma^2/ak)}{\sqrt{2\pi}\,\Gamma(2J+2)}\exp\left(\frac{\pi\gamma\sigma^2}{2ak}\right)W_{\mathrm{i}\,\gamma\sigma^2/ak,J+\frac{1}{2}}(-2\,\mathrm{i}\,kr)\ . \tag{6.8.54}$$

In parabolic coordinates one has

$$\Psi_{n,l,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi}}{\sqrt{2\pi}}\beta^3\left[\frac{2n!l!}{N\Gamma(n+\lambda+1)\Gamma(l+\lambda+1)}\right]^{1/2}$$
$$\times(\beta\xi)^\lambda(\beta\eta)^\lambda\exp\left(-\beta^2\frac{\xi^2+\eta^2}{2}\right)L_n^{(\lambda)}(\beta^2\xi^2)L_l^{(\lambda)}(\beta^2\eta^2) \tag{6.8.55}$$

with $N = n+l+\lambda+1 \in \mathbb{N}$ and $\beta = (-2mE_N/\hbar^2)^{1/4} = \gamma\sigma^2/Na$. The wave functions of the continuous spectrum are

$$\Psi_{k,\zeta,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi}}{\sqrt{2\pi}}\frac{\Gamma\left[\frac{1+\lambda}{2}+\frac{\mathrm{i}}{2k}(\gamma\sigma^2/a+\zeta)\right]\Gamma\left[\frac{1+\lambda}{2}+\frac{\mathrm{i}}{2k}(\gamma\sigma^2/a-\zeta)\right]}{\sqrt{2\pi^2p}\,\xi\eta\Gamma^2(1+\lambda)}$$
$$\times\,\mathrm{e}^{\gamma\sigma^2\pi/2ak}\,M_{\mathrm{i}(\gamma\sigma^2/a+\zeta)/2k,\lambda/2}(-\,\mathrm{i}\,k\xi^2)M_{\mathrm{i}(\gamma\sigma^2/a-\zeta)/2k,\lambda/2}(-\,\mathrm{i}\,k\eta^2)\ . \tag{6.8.56}$$

6.8.8 Coulomb Problem with Aharonov–Bohm Potential. [179,860]. ($\kappa = e_1e_2\sqrt{-m/2E}/\hbar$, $\lambda = |\nu-(e_2\Phi/2\pi\hbar c)|$, $J = l+\lambda$)

$$\int_{\mathbb{R}}\mathrm{d}\varphi\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\delta\left(\varphi-\int_{t'}^{t''}\dot\varphi\,\mathrm{d}t\right)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot r^2+r^2\dot\vartheta^2+r^2\sin\vartheta^2\dot\varphi^2)\right.\right.$$
$$\left.\left.+\frac{e_1e_2}{r}+\frac{e_2\Phi}{2\pi c}\dot\varphi+\frac{\hbar^2}{8mr^2}\left(1+\frac{1}{\sin^2\vartheta}\right)\right]\mathrm{d}t\right\}$$
$$=\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\,\nu(\varphi''-\varphi')}}{2\pi\hbar r'r''}\sum_{l\in\mathbb{N}_0}\left(l+\lambda+\frac{1}{2}\right)\frac{\Gamma(l+2\lambda+1)}{l!}P_{l+\lambda}^{-\lambda}(\cos\vartheta'')P_{l+\lambda}^{-\lambda}(\cos\vartheta')$$
$$\times\sqrt{-\frac{m}{2E}}\frac{\Gamma(J+1+\kappa)}{\Gamma(2J+2)}W_{\kappa,J+\frac{1}{2}}\left(\sqrt{-2mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,J+\frac{1}{2}}\left(\sqrt{-2mE}\,\frac{r_<}{\hbar}\right)\ . \tag{6.8.57}$$

6.8.9 Non-Isotropic Three-Dimensional Coulomb System. [437]

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\\
&\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2+\frac{e^2}{|\mathbf{x}|}+\frac{\hbar^2}{2m}\left(\frac{b_1+b_2}{x^2+y^2}+\frac{(b_1-b_2)z}{(x^2+y^2)|\mathbf{x}|}\right)\right.\right.\\
&\qquad\qquad\left.\left.+\frac{m}{4}(\omega_1^2+\omega_2^2)-\frac{m}{4}(\omega_1^2-\omega_2^2)\frac{z}{|\mathbf{x}|}\right]\mathrm{d}t\right\}\\
&=\frac{\mathrm{i}}{2\pi\hbar}\left(\frac{m}{\mathrm{i}\,\hbar}\right)^2\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}\,\Omega_1\Omega_2\\
&\quad\times\int_0^\infty\frac{\mathrm{d}s''\,\mathrm{e}^{2\mathrm{i}\alpha s''/\hbar}}{\sin\Omega_1 s''\sin\Omega_2 s''}I_{\lambda_1}\left(\frac{m\Omega_1\xi'\xi''}{\mathrm{i}\,\hbar\sin\Omega_1 s''}\right)I_{\lambda_2}\left(\frac{m\Omega_1\eta'\eta''}{\mathrm{i}\,\hbar\sin\Omega_2 s''}\right)\\
&\quad\times\exp\left\{\frac{\mathrm{i}\,m}{2\hbar T}\left[\Omega_1({\xi'}^2+{\xi''}^2)\cot\Omega_1 s''+\Omega_2({\eta'}^2+{\eta''}^2)\cot\Omega_2 s''\right]\right\}\ .
\end{aligned}\tag{6.8.58}
$$

$$
\begin{aligned}
&=\sum_{\nu\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}\nu(\varphi''-\varphi')}}{2\pi}\left\{\sum_{n_1,n_2\in\mathbb{N}_0}\frac{\Psi_{n_1,n_2}(\xi'',\eta'')\Psi_{n_1,n_2}(\xi',\eta')}{E_{n_1,n_2}-E}\right.\\
&\qquad\qquad\left.+\int_{\mathbb{R}}\mathrm{d}\zeta\int_0^\infty\mathrm{d}k\,\frac{\Psi_{k,\zeta}(\xi'',\eta'')\Psi^*_{k,\zeta}(\xi',\eta')}{\hbar^2k^2/2m-E}\right\}\ .
\end{aligned}\tag{6.8.59}
$$

Here $\Omega_{1,2}=\sqrt{\omega_{1,2}-2E/m}$ and $\lambda_{1,2}=\sqrt{2b_{1,2}+\nu^2}$ with three-dimensional parabolic coordinates defined by $x=\xi\eta\cos\varphi$, $y=\xi\eta\sin\varphi$, $z=\frac{1}{2}(\eta^2-\xi^2)$. The bound-state wave functions are given by

$$
\begin{aligned}
\Psi_{n_1,n_2}(\xi,\eta)&=\left[\left(\frac{m}{\hbar}\right)^3\frac{(2\Omega_1\Omega_2)^2}{A_1\Omega_2+A_2\Omega_1}\cdot\frac{n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_2+1)}\right]^{1/2}\\
&\quad\times\left(\frac{m\Omega_1}{\hbar}\xi^2\right)^{\lambda_1/2}\left(\frac{m\Omega_2}{\hbar}\eta^2\right)^{\lambda_2/2}\exp\left[-\frac{m}{2\hbar}(\Omega_1\xi^2+\Omega_2\eta^2)\right]\\
&\quad\times L_{n_1}^{(\lambda_1)}\left(\frac{m\Omega_1}{\hbar}\xi^2\right)L_{n_2}^{(\lambda_2)}\left(\frac{m\Omega_2}{\hbar}\eta^2\right)\ .
\end{aligned}\tag{6.8.60}
$$

The energy spectrum has the form

$$
\begin{aligned}
E_{n_1,n_2}&=\frac{m/2}{(A_1^2-A_2^2)^2}\left((A_1^2-A_2^2)(A_1^2\omega_1^2-A_2^2\omega_2^2)-\frac{4\alpha^2}{\hbar^2}(A_1^2+A_2^2)\right.\\
&\qquad\left.+\frac{4e^2}{\hbar}A_1A_2\sqrt{(A_1^2-A_2^2)(\omega_1^2-\omega_2^2)+\frac{4\alpha^2}{\hbar^2}}\right)
\end{aligned}\tag{6.8.61}
$$

with $A_{1/2} = 2n_{1/2} + \lambda_{1/2} + 1$, and

$$\Omega_{1/2} = \frac{1}{|A_1^2 - A_2^2|} \left| A_{2/1} \sqrt{(A_1^2 - A_2^2)(\omega_1^2 - \omega_2^2) + \frac{4\alpha^2}{\hbar^2}} - \frac{2e^2}{\hbar} A_{1/2} \right| . \quad (6.8.62)$$

Here all quantities are valid for $A_1 \neq A_2$. The continuum functions $\Psi_{k,\zeta}(\xi,\eta)$ are given by ($k_{1,2} = (\omega_0/\Omega_{1,2})(1/a \pm \zeta)/2k$, $\omega_0 = \sqrt{-2E/m}$, $\Omega_{1,2} = \sqrt{\omega_{1,2} - 2E/m}$, $a = \hbar^2/me^2$)

$$\Psi_{k,\zeta}(\xi,\eta) = \frac{\Gamma(\frac{1+\lambda_1}{2} + \mathrm{i}\,k_1)\Gamma(\frac{1+\lambda_2}{2} + \mathrm{i}\,k_2)}{\sqrt{2\pi^2 p}\,\xi\eta\,\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)}\, \mathrm{e}^{\pi(1/k_1 + 1/k_2)/2} \times M_{\mathrm{i}\,k_1,\lambda_1/2}(-\mathrm{i}\,k\xi^2) M_{\mathrm{i}\,k_2,\lambda_2/2}(-\mathrm{i}\,k\eta^2) \ . \quad (6.8.63)$$

6.8.10 Pure Coulomb Potential in D Dimensions. [187,504]
($\kappa = e_1 e_2 \sqrt{-m/2E}/\hbar$, $a = \hbar^2/me_1e_2$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1e_2}{|\mathbf{x}|}\right)\mathrm{d}t\right]$$

$$= \sum_{l\in\mathbb{N}_0} S_l^{(D)}(\Omega'')S_l^{(D)}(\Omega')(r'r'')^{\frac{1-D}{2}} \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\, \frac{\Gamma\big(l + \frac{D-1}{2} - \kappa\big)}{(2l+D-2)!} \times W_{\kappa,l+\frac{D-2}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,l+\frac{D-2}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) , \quad (6.8.64)$$

$$= \sum_{N=1}^\infty \sum_{l\in\mathbb{N}_0} \frac{\Psi_{N,l}(r'',\Omega'')\Psi_{N,l}^*(r',\Omega')}{E_N - E} + \sum_{l\in\mathbb{N}_0}\int_0^\infty \mathrm{d}k\, \frac{\Psi_{k,l}(r'',\Omega'')\Psi_{k,l}^*(r',\Omega')}{\hbar^2k^2/2m - E} , \quad (6.8.65)$$

$$\Psi_{N,l}(r,\Omega) = \left[\frac{1}{2(N+\frac{D-3}{2})}\frac{(N-l-1)!}{(N+l+D-3)!}\right]^{1/2} \times \left(\frac{2}{a(N+\frac{D-3}{2})}\right)^{D/2}\left(\frac{2r}{a(N+\frac{D-3}{2})}\right)^l \times \exp\left(-\frac{r}{a(N+\frac{D-3}{2})}\right) L_{N-l-1}^{(2l+D-2)}\left(\frac{2r}{a(N+\frac{D-3}{2})}\right) S_l^{(D)}(\Omega) , \quad (6.8.66)$$

$$E_N = -\frac{me_1^2e_2^2}{2\hbar^2(N+\frac{D-3}{2})^2}, \qquad (N\in\mathbb{N}) , \quad (6.8.67)$$

$$\Psi_{k,l}(r,\Omega) = r^{(1-D)/2}\frac{\Gamma\big(l + \frac{D-1}{2} + \mathrm{i}/ak\big)}{\sqrt{2\pi}(2l+D-2)!} \times \exp\left(\frac{\pi me_1^2e_2^2}{2\hbar^2k}\right) M_{\mathrm{i}/ak,l+\frac{D-2}{2}}(-2\,\mathrm{i}\,kr) S_l^{(D)}(\Omega) \ . \quad (6.8.68)$$

6.8.11 Super-Integrable Potentials in Two Dimensions. [305,668]

6.8.11.1.1 Polar Coordinates [458] ($\mathbf{x}\in\mathbb{R}^2, k_{1,2}>0, \kappa=e^2\sqrt{-m/2E}/\hbar, \lambda=|\nu|+(k_1+k_2+1)/2, a=\hbar^2/me^2$)

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2+\frac{e^2}{\varrho}-\frac{\hbar^2}{4m\varrho}\left(\frac{k_1^2-\frac14}{\varrho+x}+\frac{k_2^2-\frac14}{\varrho-x}\right)\right]\mathrm{d}t\right\}\\
&=\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\varrho\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\varrho^2+\varrho^2\dot\varphi^2)+\frac{e^2}{\varrho}-\frac{\hbar^2}{8m\varrho^2}\left(\frac{k_1^2-\frac14}{\cos^2\frac{\varphi}{2}}+\frac{k_2^2-\frac14}{\sin^2\frac{\varphi}{2}}\right)\right]\mathrm{d}t\right\}\\
&=\frac12(\varrho'\varrho'')^{-1/2}\sum_{\nu\in\mathbb{N}_0}\Phi_\nu^{(k_1,k_2)}(\varphi'')\Phi_\nu^{(k_1,k_2)}(\varphi')\\
&\qquad\times\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(\frac12+\lambda-\kappa)}{\Gamma(2\lambda+1)}W_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{\varrho_>}{\hbar}\right)M_{\kappa,\lambda}\left(\sqrt{-8mE}\,\frac{\varrho_<}{\hbar}\right)
\end{aligned}\tag{6.8.69}
$$

$$
\begin{aligned}
&=\frac12\sum_{\nu\in\mathbb{N}_0}\Phi_\nu^{(k_1,k_2)}(\varphi'')\Phi_\nu^{(k_1,k_2)}(\varphi')\\
&\qquad\times\left\{\sum_{n\in\mathbb{N}_0}\frac{\Psi_{n,\nu}(\varrho'')\Psi_{n,\nu}(\varrho')}{E_n-E}+\int_{\mathbb{R}}\mathrm{d}k\,\frac{\Psi_{k,\nu}(\varrho'')\Psi^*_{k,\nu}(\varrho')}{\hbar^2k^2/2m-E}\right\}\ ,
\end{aligned}\tag{6.8.70}
$$

$$
\begin{aligned}
\Psi_{n,\nu}(\varrho)&=\left[\frac{2n!}{a^2(n+\lambda+1/2)^3\Gamma(n+2\lambda+1)}\right]^{1/2}\left(\frac{2\varrho}{a(n+\lambda+\frac12)}\right)^{\lambda}\\
&\qquad\times\exp\left[-\frac{\varrho}{a(n+\lambda+1/2)}\right]L_n^{(2\lambda)}\left(\frac{2\varrho}{a(n+\lambda+1/2)}\right)\ ,
\end{aligned}\tag{6.8.71}
$$

$$
E_n=-\frac{me^4}{2\hbar^2(n+\lambda+\frac12)^2}\ ,\tag{6.8.72}
$$

$$
\Psi_{k,\nu}(\varrho)=\frac{\Gamma(\frac12+\lambda-\mathrm{i}/ak)}{\sqrt{2\pi\varrho}\,\Gamma(2\lambda+1)}\exp\left(\frac{\pi}{2ak}\right)M_{\mathrm{i}/ak,\lambda}(-2\mathrm{i}k\varrho)\ .\tag{6.8.73}
$$

6.8.11.1.2 Parabolic Coordinates. [458] ($a = \hbar^2/me^2, N = n_1 + n_2 + (1 + k_1 + k_2)/2), \omega = \sqrt{-2E/m}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t)(\xi^2+\eta^2)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2) + \frac{2e^2}{\xi^2+\eta^2} - \frac{\hbar^2}{2m(\xi^2+\eta^2)}\left(\frac{k_1^2-\frac14}{\xi^2}+\frac{k_2^2-\frac14}{\eta^2}\right)\right]\mathrm{d}t\right\}$$

$$= \left(\frac{m}{\mathrm{i}\,\hbar}\right)^2 \sqrt{\xi'\xi''\eta'\eta''}\int_0^\infty \frac{\omega^2 \mathrm{d}s''}{\sin^2\omega s''}\mathrm{e}^{2\mathrm{i}\,e^2 s''/\hbar} I_{k_1}\left(\frac{m\omega\xi'\xi''}{\mathrm{i}\,\hbar\sin\omega s''}\right) I_{k_2}\left(\frac{m\omega\eta'\eta''}{\mathrm{i}\,\hbar\sin\omega s''}\right)$$

$$\times \exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(\xi'^2+\xi''^2+\eta'^2+\eta''^2)\cot\omega s''\right], \tag{6.8.74}$$

$$= \sum_{n_1,n_2\in\mathbb{N}_0}\frac{\Psi_{n_1,n_2}(\xi'',\eta'')\Psi^*_{n_1,n_2}(\xi',\eta')}{E_{n_1,n_2}-E} + \int_0^\infty \frac{\mathrm{d}k}{k}\int_{\mathbb{R}}\mathrm{d}\zeta \frac{\Psi_{k,\zeta}(\xi'',\eta'')\Psi^*_{k,\zeta}(\xi',\eta')}{\hbar^2k^2/2m-E}, \tag{6.8.75}$$

$$\Psi_{n_1,n_2}(\xi,\eta) = \left[\frac{2}{a^2N^3}\cdot\frac{n_1!n_2!}{\pi\Gamma(n_1+k_1+1)\Gamma(n_2+k_2+1)}\right]^{1/2}$$

$$\times\left(\frac{\xi}{aN}\right)^{k_1/2}\left(\frac{\eta}{aN}\right)^{k_2/2}\exp\left(-\frac{\xi^2+\eta^2}{2aN}\right)L_{n_1}^{(k_1)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(k_2)}\left(\frac{\eta^2}{aN}\right), \tag{6.8.76}$$

$$E_{n_1,n_2} = -\frac{me^4}{\hbar^2[n_1+n_2+\frac12(k_1+k_2)+1]^2}, \tag{6.8.77}$$

$$\Psi_{k,\zeta}(\xi,\eta) = \frac{\Gamma[\frac{1+k_1}{2}+\frac{\mathrm{i}}{2k}(1/a+\zeta)]\Gamma[\frac{1+k_2}{2}+\frac{\mathrm{i}}{2k}(1/a-\zeta)]}{\sqrt{2\pi^2 p\xi\eta}\,\Gamma(1+k_1)\Gamma(1+k_2)}\mathrm{e}^{\pi/2ak}$$

$$\times M_{\frac{\mathrm{i}}{2k}(1/a+\zeta),k_1/2}(-\mathrm{i}\,k\xi^2)M_{\frac{\mathrm{i}}{2k}(1/a-\zeta),k_2/2}(-\mathrm{i}\,k\eta^2). \tag{6.8.78}$$

6.8.11.2. Coulombic Potential [458] ($\beta_{1,2}\in\mathbb{R}$, $(\tilde\xi,\tilde\eta) = (\xi-\beta_1/E, \eta-\beta_2/E)$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)$$

$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2+\frac{e^2}{|\mathbf{x}|}+\frac{\beta_1\sqrt{|\mathbf{x}|+x}+\beta_2\sqrt{|\mathbf{x}|-x}}{2|\mathbf{x}|}\right)\mathrm{d}t\right]$$

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2) + 2\frac{e^2-(\beta_1\xi+\beta_2\eta)}{\xi^2+\eta^2}\right]\mathrm{d}t\right\}$$

$$= \sum_{n_1,n_2\in\mathbb{N}_0} \frac{\Psi_{n_1,n_2}(\xi'',\eta'')\Psi_{n_1,n_2}(\xi',\eta')}{E_{n_1,n_2}-E}$$

$$+ \sum_{\mathrm{e,o}}\int_{\mathbb{R}}\mathrm{d}\zeta\int_0^\infty \mathrm{d}k\, \frac{\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi'',\eta'')\Psi_{k,\zeta}^{(\mathrm{e,o})\,*}(\xi',\eta')}{\hbar^2k^2/2m - E}\,, \tag{6.8.79}$$

where the bound-state energy levels E_N are given by ($N = n_1+n_2+1$, $n_1, n_2 \in \mathbb{N}_0$)

$$E_N = E_{n_1,n_2} = -\frac{m}{2}\omega_N^2\,, \qquad \omega_N = u_1+u_2+\frac{2e^2}{3\hbar N}\,, \tag{6.8.80}$$

$$u_{1,2} = \sqrt[3]{\left(\frac{2e^2}{3\hbar N}\right)^3 + \frac{\beta_1^2+\beta_2^2}{mN\hbar} \pm \sqrt{\left(\frac{\beta_1^2+\beta_2^2}{mN\hbar}\right)^2 + 2\frac{\beta_1^2+\beta_2^2}{mN\hbar}\left(\frac{2e^2}{3\hbar N}\right)^3}}\,. \tag{6.8.81}$$

The corresponding bound-state wave functions have the form

$$\Psi_{n_1,n_2}(\xi,\eta) = \sqrt{\frac{m}{\pi\hbar}\cdot\frac{4}{n_1!n_2!2^{n_1+n_2}}}\left(\lim_{E\to E_N}\frac{-(\frac{m}{2}\omega_N^2+E)m\omega_N^4/N}{\omega^3-\frac{2e^2}{N\hbar}\omega^2-2\frac{\beta_1^2+\beta_2^2}{mN\hbar}}\right)^{1/2}$$

$$\times \exp\left[-\frac{m\omega_N}{2\hbar}(\tilde\xi^2+\tilde\eta^2)\right] H_{n_1}\left(\sqrt{\frac{m\omega_N}{\hbar}}\,\tilde\xi\right) H_{n_2}\left(\sqrt{\frac{m\omega_N}{\hbar}}\,\tilde\eta\right)\,. \tag{6.8.82}$$

The continuum wave functions $\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta)$ are given by $[\tilde a = \hbar^2/m\big(e^2 - m(\beta_1^2+\beta_2^2)/\hbar^2k^2\big)]$

$$\Psi_{k,\zeta}^{(\mathrm{e,o})}(\xi,\eta) = \frac{\mathrm{e}^{\pi/2ak}}{\sqrt{2}\,4\pi^2}\begin{pmatrix}\Gamma[\frac14+\frac{\mathrm{i}}{2k}(1/\tilde a+\zeta)]E^{(0)}_{-\frac12+\frac{\mathrm{i}}{k}(1/\tilde a+\zeta)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\tilde\xi)\\ \Gamma[\frac34+\frac{\mathrm{i}}{2k}(1/\tilde a+\zeta)]E^{(1)}_{-\frac12+\frac{\mathrm{i}}{k}(1/\tilde a+\zeta)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2k}\,\tilde\xi)\end{pmatrix}$$

$$\times\begin{pmatrix}\Gamma[\frac14+\frac{\mathrm{i}}{2k}(1/\tilde a-\zeta)]E^{(0)}_{-\frac12+\frac{\mathrm{i}}{k}(1/\tilde a-\zeta)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2p}\,\tilde\eta)\\ \Gamma[\frac34+\frac{\mathrm{i}}{2k}(1/\tilde a-\zeta)]E^{(1)}_{-\frac12+\frac{\mathrm{i}}{k}(1/\tilde a-\zeta)}(\mathrm{e}^{-\mathrm{i}\,\pi/4}\sqrt{2k}\,\tilde\eta)\end{pmatrix}\,. \tag{6.8.83}$$

6.8.12 Super-Integrable Potentials in Three Dimensions. [305,668]

$(\mathbf{x} \in \mathbb{R}^3, k_{1,2} > 0,\ \lambda_1 = k_1 + k_2 + 2n + 1,\ \lambda_2 = l + \lambda_1 + \frac{1}{2},\ \kappa = e^2\sqrt{-m/2E}/\hbar)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e^2}{r} - \frac{\hbar^2}{2m}\sum_{i=1}^{2}\frac{k_i^2 - \frac{1}{4}}{x_i^2}\right)\mathrm{d}t\right]$$

Spherical [139,817]:

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^2 \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\dot{\varphi}^2)\right.\right.$$
$$\left.\left. + \frac{e^2}{r} - \frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin^2\vartheta}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\cos^2\varphi} - \frac{1}{4}\right) - \frac{1}{4}\right)\right]\mathrm{d}t\right\}$$
$$= \sum_{n\in\mathbb{N}_0} \Phi_n^{(k_1,k_2)}(\varphi'')\Phi_n^{(k_1,k_2)}(\varphi')$$
$$\times \sum_{\nu\in\mathbb{N}_0} (\nu + \lambda_1 + \tfrac{1}{2})\frac{\Gamma(\nu + \lambda_1 + 1)}{\nu!} P_{\lambda_1+\nu}^{-\lambda_1}(\cos\vartheta'')P_{\lambda_1+\nu}^{-\lambda_1}(\cos\vartheta')$$
$$\times \frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\Gamma(\frac{1}{2} + \lambda_2 - \kappa)}{\Gamma(1 + 2\lambda_2)} W_{\kappa,\lambda_2}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,\lambda_2}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right), \tag{6.8.84}$$

Parabolic [458]:

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t) \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t)(\xi^2 + \eta^2)\xi\eta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \xi^2\eta^2\dot{\varphi}^2\right)\right.\right.$$
$$\left.\left. + \frac{2e^2}{\xi^2 + \eta^2} - \frac{2\hbar^2}{m\xi^2\eta^2}\left(\frac{k_1^2 - \frac{1}{4}}{\sin^2\varphi} + \frac{k_2^2 - \frac{1}{4}}{\cos^2\varphi} - \frac{1}{4}\right)\right]\mathrm{d}t\right\}$$
$$= \sum_{n\in\mathbb{N}_0}\left[\sum_{n_1,n_2\in\mathbb{N}_0} \frac{\Psi_{n,n_1,n_2}(\varphi'',\xi'',\eta'')\Psi_{n,n_1,n_2}(\varphi',\xi',\eta')}{E_N - E}\right.$$
$$\left. + \int_0^\infty \mathrm{d}k \int_{\mathbb{R}} \mathrm{d}\zeta\, \frac{\Psi_{k,\zeta,n}(\varphi'',\xi'',\eta'')\Psi_{k,\zeta,n}^*(\varphi',\xi',\eta')}{\hbar^2k^2/2m - E}\right] . \tag{6.8.85}$$

For parabolic coordinates the discrete state wave functions are $(a = \hbar^2/me^2)$

$$\Psi_{n,n_1,n_2}(\varphi,\xi,\eta) = \Phi_n^{(k_2,k_1)}(\varphi)\left[\frac{2}{a^2N^3}\cdot\frac{2n_1!n_2!}{\Gamma(n_1+\lambda_1+1)\Gamma(n_2+\lambda_1+1)}\right]^{1/2}$$
$$\times\left(\frac{\xi\eta}{(aN)^2}\right)^{\lambda_1}\exp\left(-\frac{\xi^2+\eta^2}{2aN}\right)L_{n_1}^{(\lambda_1)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(\lambda_1)}\left(\frac{\eta^2}{aN}\right) , \tag{6.8.86}$$

$$E_N = -\frac{me^4}{\hbar^2N^2} , \qquad N = n_1+n_2+\tfrac{1}{2}(2n_+k_1+k_2)+1 . \tag{6.8.87}$$

The continuum wave functions are

$$\Psi_{k,\zeta,n}(\varphi,\xi,\eta) = \Phi_n^{(k_2,k_1)}(\varphi)\frac{\Gamma[\frac{1+\lambda_1}{2}+\frac{\mathrm{i}}{2k}(1/a+\zeta)]\Gamma[\frac{1+\lambda_1}{2}+\frac{\mathrm{i}}{2k}(1/a-\zeta)]}{\sqrt{2\pi^2p}\,\xi\eta\Gamma(1+\lambda_1)\Gamma(1+\lambda_1)}$$
$$\times\,\mathrm{e}^{\pi/2ak}M_{\mathrm{i}(1/a+\zeta)/2k,\lambda_1/2}(-\mathrm{i}\,k\xi^2)M_{\mathrm{i}(1/a-\zeta)/2k,\lambda_1/2}(-\mathrm{i}\,k\eta^2) . \tag{6.8.88}$$

6.8.13 Green Function for the Coulomb Problem for a Klein-Gordon Particle. [613,615] ($\nu = \alpha/\sqrt{m^2c^4/E-1}$, $\kappa = E\alpha/\hbar\nu$, $\tilde{l} = (\sqrt{(2l+1)^2-4\alpha^2}-1)/2$, $\alpha = e^2/\hbar c$)

$$\int_0^\infty \mathrm{d}\tau \int\limits_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(\tau)=\mathbf{x}''}\mathcal{D}_E\mathbf{x}(s)\exp\left[-\frac{1}{\hbar}\int_0^\tau\left(\frac{m}{2}\dot{\mathbf{x}}^2-\frac{(E+e^2/|\mathbf{x}|)^2}{2mc^2}+\frac{mc^2}{2}\right)\mathrm{d}s\right]$$
$$=\frac{1}{8\,\mathrm{i}\,\kappa\pi r'r''}\sum_{l\in\mathbb{N}_0}(2l+1)P_l(\cos\psi_{('','})})$$
$$\times\frac{\Gamma(1-\nu-\tilde{l})}{\Gamma(2\tilde{l}+2)}W_{\nu,\tilde{l}+1/2}(2\kappa r_>)M_{\nu,\tilde{l}+1/2}(2\kappa r_<) . \tag{6.8.89}$$

6.8.14 Dirac Coulomb Problem. [502,508,525] (α^a,β are the Dirac matrices)

$$\hat{M} = -c\alpha^ap_a+\beta\left(\frac{E}{c}+\frac{Ze^2}{|\mathbf{x}|}\right) , \tag{6.8.90}$$

$$\hat{G}(E) := \frac{1}{mc^2-\hat{M}+\mathrm{i}\,\epsilon} = (mc^2+\hat{M})\cdot\frac{1}{mc^2-\hat{M}^2+\mathrm{i}\,\epsilon} \equiv (mc^2+\hat{M})\cdot\hat{g}(E) , \tag{6.8.91}$$

$$\langle\mathbf{x}''|\hat{G}(E)|\mathbf{x}'\rangle = [mc^2+\hat{M}(x'')]\langle\mathbf{x}''|\hat{g}(E)|\mathbf{x}'\rangle , \tag{6.8.92}$$

$$\langle\mathbf{x}''|\hat{g}(E)|\mathbf{x}'\rangle = \sum_\lambda\langle\vartheta'',\varphi''|\lambda\rangle\langle r''|\hat{g}_\lambda(E)|r'\rangle\langle\lambda|\vartheta',\varphi'\rangle , \tag{6.8.93}$$

$$\langle r''|\hat{g}_\lambda(E)|r'\rangle \equiv g_\lambda(r'',r';E)$$
$$= \frac{\mathrm{i}\hbar}{2mr'r''}\int_0^\infty \mathrm{d}u'' \int\limits_{r(0)=r'}^{r(u'')=r''} \mathcal{D}r(u)$$
$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{u''}\left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{\lambda(\lambda+1)}{2mr^2} + \frac{a}{|\mathbf{x}|} + \frac{k^2}{2m}\right)\mathrm{d}u\right]$$
$$= \frac{\Gamma(\lambda+1-p)}{2\,\mathrm{i}\,kr'r''\Gamma(2\lambda+2)}W_{p,\lambda+\frac{1}{2}}(-2\,\mathrm{i}\,kr_>)M_{p,\lambda+\frac{1}{2}}(-2\,\mathrm{i}\,kr_<)\ . \tag{6.8.94}$$

$p = \mathrm{i}(Ze^2/\hbar c)E/\sqrt{E^2-m^2c^4}$, and λ are the eigenvalues of the Martin–Glauber operator $\mathcal{L}$, defined as $S\mathcal{L}(\mathcal{L}+1)S^{-1}|\lambda>= \lambda(\lambda+1)|\lambda>$; furthermore

$$\langle\vartheta,\varphi|\lambda\rangle = \begin{cases} \langle\vartheta\varphi|j,\mu,\kappa,-1\rangle = \begin{pmatrix} 0 \\ \chi^\mu_{-\kappa}\end{pmatrix}\ , \\ \langle\vartheta\varphi|j,\mu,\kappa,1\rangle = \begin{pmatrix} \chi^\mu_\kappa \\ 0\end{pmatrix}\ . \end{cases} \tag{6.8.95}$$

χ^μ_κ denotes the two-component spinor

$$\chi^\mu_\kappa(\vartheta,\varphi) = \sum_\nu (l,\tfrac{1}{2},\mu,-\nu,\nu|j,\mu)Y_l^{\mu-\nu}(\vartheta,\varphi)\chi^\nu_{\frac{1}{2}}$$
$$\equiv \begin{pmatrix} -\dfrac{\kappa}{|\kappa|}\sqrt{\dfrac{\kappa+1/2-\mu}{2\kappa+1}}\,Y^{\mu-\frac{1}{2}}_{|\kappa+1/2|-1/2}(\vartheta,\varphi) \\ \sqrt{\dfrac{\kappa+1/2+\mu}{2\kappa+1}}\,Y^{\mu+1/2}_{|\kappa+1/2|-1/2}(\vartheta,\varphi)\end{pmatrix}\ . \tag{6.8.96}$$

6.9 Magnetic Monopole and Anyon Systems

6.9.1 Dirac Monopole. [128,253,431,524,683]

$$\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2 \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2}[\dot{r}^2 + r^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2)] - \frac{n\hbar}{2}(1-\cos\vartheta)\dot{\varphi}\right.\right.$$
$$\left.\left. - \frac{\hbar^2\lambda^2}{2mr^2} + \frac{\hbar^2}{8mr^2}\left(1+\frac{1}{\sin^2\vartheta}\right)\right\}\mathrm{d}t\right)$$
$$= \sum_{J=|n|/2}^{\infty}\sum_{M=-J}^{J}\frac{J+\frac{1}{2}}{2\pi}\,\mathrm{e}^{\mathrm{i}(M-n/2)(\varphi''-\varphi')}\,D^J_{M,n/2}(\cos\vartheta'')D^{J\,*}_{M,n/2}(\cos\vartheta')$$

$$\times \frac{m}{\mathrm{i}\hbar T}(r'r'')^{-\frac{1}{2}} \exp\left[\frac{\mathrm{i}m}{2\hbar T}(r'^2 + r''^2)\right] I_{\tilde{J}+\frac{1}{2}}\left(\frac{m}{\mathrm{i}\hbar T}r'r''\right) , \tag{6.9.1}$$

$$= \int_0^\infty \mathrm{d}k\, \Psi_{k,J,M}(r'',\vartheta'',\varphi'')\Psi^*_{k,J,M}(r',\vartheta',\varphi')\,\mathrm{e}^{-\mathrm{i}T\hbar k^2/2m} , \tag{6.9.2}$$

$$\Psi_{k,J,M}(r,\vartheta,\varphi) = \sqrt{\frac{p}{2\pi r}\left(J+\frac{1}{2}\right)}\;\mathrm{e}^{\mathrm{i}(M-\frac{n}{2})\varphi}\, D^J_{M,\frac{n}{2}}(\cos\vartheta) J_{\tilde{J}+\frac{1}{2}}(pr) . \tag{6.9.3}$$

Here $\tilde{J}^2 = (J+\frac{1}{2})^2 + \lambda^2 - n^2/4$ with $n \in \mathbb{N}$.

6.9.2 Schwinger Monopole. [524]

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^2 \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2}[\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)] + \frac{eg}{2c}(\pm 1 + \cos\vartheta)\dot{\varphi}\right.\right.$$

$$\left.\left. + \frac{\hbar^2}{8mr^2}\left(1 + \frac{1}{\sin^2\vartheta}\right)\right\}\mathrm{d}t\right)$$

$$= \sum_{J=|q|/2}^{\infty} \sum_{\alpha=\pm\frac{1}{2}} \sum_{\mu=-J}^{J} Y^{(\alpha)}_{q/2,J,\mu}(\vartheta'',\varphi''+\pi-2\pi\alpha) Y^{(\alpha)*}_{q/2,J,\mu}(\vartheta',\varphi'+\pi-2\pi\alpha)$$

$$\times \frac{m}{\mathrm{i}\hbar T}(r'r'')^{-\frac{1}{2}} \exp\left[\frac{\mathrm{i}m}{2\hbar T}(r'^2 + r''^2)\right] I_{\tilde{J}+\frac{1}{2}}\left(\frac{m}{\mathrm{i}\hbar T}r'r''\right) , \tag{6.9.4}$$

$$Y^{(\alpha)}_{\beta q,J,\mu}(\vartheta,\varphi) = \sqrt{\frac{2J+1}{4\pi}}\;\mathrm{e}^{-\mathrm{i}(\alpha q+\mu)\varphi}\, D^J_{\mu,\beta q}(\vartheta) , \tag{6.9.5}$$

and $\tilde{J}^2 = (J+1)^2 - q^2/4$, $q = eg/\hbar c$. Note that the configuration space around the Schwinger monopole must be divided into four simply connected sectors, the first by $\varphi \in [0, 2\pi)$, the second by $\varphi \in [2\pi, 4\pi)$, and the remaining two by reversing the sign of α.

6.9.3 Non-Relativistic Dyon. [84,128,184,267,524,606]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^2 \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot{r}^2 + r^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)\right) - \frac{e_1 g_2}{c}(1-\cos\vartheta)\dot{\varphi}\right.\right.$$

$$+\frac{e_1e_2}{|\mathbf{x}|}-\frac{\hbar^2\lambda^2}{2mr^2}+\frac{\hbar^2}{8mr^2}\left(1+\frac{1}{\sin^2\vartheta}\right)\Bigg]\mathrm{d}t\Bigg\}$$

$$=\sum_{J=|n|/2}^{\infty}\sum_{M=-J}^{J}\frac{J+\frac{1}{2}}{2\pi}\,\mathrm{e}^{\mathrm{i}(M-n/2)(\varphi''-\varphi')}\,D^J_{M,n/2}(\cos\vartheta'')D^{J\,*}_{M,n/2}(\cos\vartheta')$$

$$\times\frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\,\frac{\Gamma(\tilde{J}+1-\kappa)}{\Gamma(2\tilde{J}+2)}$$

$$\times W_{\kappa,\tilde{J}+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,\tilde{J}+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right)\,, \tag{6.9.6}$$

$$=\sum_{J=|Q|}^{\infty}\sum_{M=-J}^{J}\Bigg[\sum_{n\in\mathbb{N}_0}\frac{\Psi_{n,J,M}(r'',\vartheta'',\psi'')\Psi^*_{n,J,M}(r',\vartheta',\psi')}{E_n-E}$$

$$+\int_0^\infty\mathrm{d}k\,\frac{\Psi_{k,J,M}(r'',\vartheta'',\psi'')\Psi^*_{k,J,M}(r',\vartheta',\psi')}{\hbar^2k^2/2m-E}\Bigg]\,, \tag{6.9.7}$$

$$\Psi_{n,J,M}(r,\vartheta,\varphi)=\sqrt{\frac{J+\frac{1}{2}}{2\pi}}\,\mathrm{e}^{\mathrm{i}(M-|Q|)\varphi}\,D^J_{M,-|Q|}(\cos\vartheta)$$

$$\times\frac{2}{(n+\tilde{J}+1)^2}\left[\frac{n!}{a^3\Gamma(n+2\tilde{J}+2)}\right]^{1/2}\left(\frac{2r}{a(n+\tilde{J}+1)}\right)^{\tilde{J}}$$

$$\times\exp\left(-\frac{r}{a(n+\tilde{J}+1)}\right)L_n^{(2\tilde{J}+1)}\left(\frac{2r}{a(n+\tilde{J}+1)}\right)\,, \tag{6.9.8}$$

$$E_n=-\frac{me_1^2e_2^2}{2\hbar^2(n+\tilde{J}+1)^2}\,, \tag{6.9.9}$$

$$\Psi_{k,J,M}(r,\vartheta,\varphi)=\sqrt{\frac{J+\frac{1}{2}}{4\pi^2}}\,\mathrm{e}^{\mathrm{i}(M-|Q|)\varphi}\,D^J_{M,-|Q|}(\cos\vartheta)$$

$$\times\frac{\Gamma(\tilde{J}+1+\mathrm{i}/ak)}{\Gamma(2\tilde{J}+2)}\exp\left(\frac{\pi}{2ak}\right)W_{\mathrm{i}/ak,\tilde{J}+\frac{1}{2}}(-2\,\mathrm{i}\,kr) \tag{6.9.10}$$

[$\kappa=e_1e_2\sqrt{-m/2E}/\hbar$, $(\tilde{J}+\frac{1}{2})^2=(J+\frac{1}{2})^2+\lambda^2-Q^2$ with $n\in\mathbb{N}$ fixed by the quantization condition $Q=n/2$, $a=\hbar^2/me_1e_2$, $Q=g_2e_1/\hbar c$].

6.9.4 Modified Non-Relativistic Dyon.

6.9.4.1 Spherical Coordinates. [606]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty\mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^2\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\sin\vartheta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot r^2 + r^2(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2)\right) - \frac{e_1 g_2}{c}(1-\cos\vartheta)\dot\varphi \right.\right.$$
$$\left.\left. + \frac{e_1 e_2}{|\mathbf{x}|} - \frac{\hbar^2 Q^2}{2mr^2} + \frac{\hbar^2}{8mr^2}\left(1+\frac{1}{\sin^2\vartheta}\right)\right]\mathrm{d}t\right\}$$
$$= \sum_{J=|n|/2}^{\infty}\sum_{M=-J}^{J}\frac{J+\frac{1}{2}}{2\pi}\,\mathrm{e}^{\mathrm{i}(M-n/2)(\varphi''-\varphi')}\,D^J_{M,n/2}(\cos\vartheta'')D^{J\,*}_{M,n/2}(\cos\vartheta')$$
$$\times \frac{1}{r'r''}\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\,\frac{\Gamma(J+1-\kappa)}{\Gamma(2J+2)}$$
$$\times W_{\kappa,J+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,J+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) \tag{6.9.11}$$

[$\kappa = e_1 e_2\sqrt{-m/2E}/\hbar$, with $n\in\mathbb{N}$ fixed by the quantization condition $Q = n/2$, $a=\hbar^2/me_1e_2$, $Q = g_2 e_1/\hbar c$].

6.9.4.2 Parabolic Coordinates. [461] ($N = n_1+n_2+\frac{1}{2}(|\nu_1|+|\nu_2|+1)$, $\nu_1 = M+2Q, \nu_2 = M$, $a=\hbar^2/me^2$)

$$\int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathcal{D}\xi(t)\int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathcal{D}\eta(t)(\xi^2+\eta^2)\xi\eta\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)$$
$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2)+\xi^2\eta^2\dot\varphi^2\right)+\frac{2e^2}{\xi^2+\eta^2}\right.\right.$$
$$\left.\left.+\frac{2\hbar^2Q^2}{m(\xi^2+\eta^2)^2} - \frac{e_1g_2}{c}\left(1-\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\dot\varphi\right)+\frac{\hbar^2}{8m\xi^2\eta^2}\right]\mathrm{d}t\right\}$$
$$=\sum_{\nu\in\mathbb{Z}}\left[\sum_{n_1,n_2\in\mathbb{N}_0}\Psi_{n_1,n_2,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{n_1,n_2,\nu}(\xi',\eta',\varphi')\,\mathrm{e}^{-\mathrm{i}E_NT/\hbar}\right.$$
$$\left.+\int_0^\infty \mathrm{d}k\int_{\mathbb{R}}\mathrm{d}\zeta\,\Psi_{k,\zeta,\nu}(\xi'',\eta'',\varphi'')\Psi^*_{k,\zeta,\nu}(\xi',\eta',\varphi')\,\mathrm{e}^{-\mathrm{i}\hbar k^2T/2m}\right], \tag{6.9.12}$$

$$\Psi_{n_1,n_2,\nu}(\xi,\eta,\varphi) = \frac{\mathrm{e}^{\mathrm{i}\nu\varphi}}{\sqrt{2\pi}}\left[\frac{2}{a^3N^4}\frac{n_1!n_2!}{\pi(n_1+|\nu_1|)!(n_2+|\nu_2|)!}\right]^{1/2}$$
$$\times\left(\frac{\xi^2}{aN}\right)^{|\nu_1|/2}\left(\frac{\eta^2}{aN}\right)^{|\nu_2|/2}\exp\left(-\frac{\xi^2+\eta^2}{2aN}\right)L_{n_1}^{(|\nu_1|)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(|\nu_2|)}\left(\frac{\eta^2}{aN}\right), \tag{6.9.13}$$

$$E_N = -\frac{me^4}{2\hbar^2N^2}, \tag{6.9.14}$$

$$\Psi_{k,\zeta,\nu}(\xi,\eta,\varphi) = \frac{e^{i\nu\varphi}}{\sqrt{4\pi^3 k}} \frac{\Gamma\left[\frac{1+|\nu_1|}{2} + \frac{i}{2k}(1/a+\zeta)\right]\Gamma\left[\frac{1+|\nu_2|}{2} + \frac{i}{2k}(1/a-\zeta)\right]}{\xi\eta|\nu_1|!|\nu_2|!}$$
$$\times e^{\pi/2ak} M_{i(1/a+\zeta)/2k,|\nu_1|/2}(-i k\xi^2) M_{i(1/a-\zeta)/2k,|\nu_2|/2}(-i k\eta^2) \ . \tag{6.9.15}$$

6.9.5 Non-Relativistic Dyon and Aharonov–Bohm Field. [267,496,606] $(\kappa = e^2 Z\sqrt{-m/2E}/\hbar,\ Q = 2q^2/c\hbar)$.

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{\mathbb{R}} d\varphi$$
$$\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) r^2 \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\delta\left(\varphi - \int_{t'}^{t''}\dot\varphi\, dt\right)$$
$$\times \exp\left(\frac{i}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2}[\dot r^2 + r^2(\dot\vartheta^2 + \sin^2\vartheta\dot\varphi^2)] - \frac{2q^2}{c}\left(\sin^2\frac{\vartheta}{2} - 2\mu\right)\dot\varphi\right.\right.$$
$$\left.\left. + \frac{e^2 Z}{|\mathbf{x}|} - \frac{\hbar^2 q^2}{2mr^2} + \frac{\hbar^2}{8mr^2}\left(1 + \frac{1}{\sin^2\vartheta}\right)\right\}dt\right)$$
$$= e^{-2i\mu q(\varphi''-\varphi')}\sum_{J\in\mathbb{N}_0}\sum_{M=-J}^{J}\frac{J+|Q|+\frac{1}{2}}{2\pi}e^{i(M-|Q|)(\varphi''-\varphi')}$$
$$\times D^{J+|Q|}_{M,-Q}(\cos\vartheta'')D^{J+|Q|\,*}_{M,-Q}(\cos\vartheta')\frac{1}{\hbar}\frac{1}{r'r''}\sqrt{-\frac{m}{2E}}\frac{\Gamma(J+|Q|+1-\kappa)}{\Gamma(2J+2|Q|+2)}$$
$$\times W_{\kappa,J+|Q|+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,J+|Q|+\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) \ . \tag{6.9.16}$$

6.9.6 Monopole Inside a Sphere. [28] $(\alpha = |M|, \beta = |M+2f|, f = qg/\hbar c)$

$$\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''} \mathcal{D}\vartheta(t)\sin\vartheta \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\vartheta^2 + \sin^2\vartheta\dot\varphi^2)\right.\right.$$
$$\left.\left. - f\hbar(1-\cos\vartheta)\dot\varphi + \frac{\hbar^2}{8m}\left(1 + \frac{1}{\sin^2\vartheta}\right)\right]dt\right\}$$
$$= \frac{1}{2\pi}\sum_{M\in\mathbb{Z}} e^{iM(\varphi''-\varphi')}\sum_{n\in\mathbb{N}_0}\frac{(\alpha+\beta+2n+1)\,n!\,\Gamma(\alpha+\beta+n+1)}{2\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}$$
$$\times\left(\sin\frac{\vartheta'}{2}\sin\frac{\vartheta''}{2}\right)^\alpha\left(\cos\frac{\vartheta'}{2}\cos\frac{\vartheta''}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\vartheta'')P_n^{(\alpha,\beta)}(\cos\vartheta')$$

$$\times \exp\left\{-\mathrm{i}\frac{\hbar T}{2m}\left[\left(n+\frac{\alpha+\beta+1}{2}\right)^2-\left(f^2+\frac14\right)\right]\right\} . \tag{6.9.17}$$

6.9.7 The Kaluza–Klein Monopole System.

6.9.7.1 Kaluza–Klein Monopole in Polar Coordinates. [81,520] [$\Lambda(r)=1/(1+4m/r)$, $q=4mk$, $\sqrt{g}=4mr^2\sin\vartheta/\Lambda(r)$]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)\int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''}\mathcal{D}\psi(t)\sqrt{g}$$

$$\times\exp\left[\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2\Lambda(r)}\left\{\dot r^2+r^2\dot\vartheta^2+\left[r^2\sin^2\vartheta+(4m\Lambda(r))^2(1-\cos\vartheta)^2\right]\dot\varphi^2\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+(4m\Lambda(r))^2\dot\psi^2+2(4m\Lambda(r))^2(1-\cos\vartheta)\dot\varphi\dot\psi\right\}\right)\mathrm{d}t\right]\!\right]$$

$$\begin{aligned}
&=\frac{1}{32\pi^2m^2}\sum_{J=|k|}^\infty\sum_{\nu=-J}^{J}\sum_{\mu\in\mathbb{Z}}\left(J+\frac12\right)\\
&\quad\times\mathrm{e}^{\mathrm{i}(\nu-|\mu|)(\varphi''-\varphi')+\mathrm{i}\,\mu(\psi''-\psi')}\,D^J_{\nu-\mu,-\mu}(\cos\vartheta'')D^{J\,*}_{\nu-\mu,-\mu}(\cos\vartheta')\\
&\quad\times\frac{2}{\omega r'r''}\frac{\Gamma(J+1-\kappa)}{\Gamma(2J+2)}W_{\kappa,J+\frac12}\left(\frac{m\omega}{\hbar}r_>\right)M_{\kappa,J+\frac12}\left(\frac{m\omega}{\hbar}r_<\right) ,
\end{aligned}\tag{6.9.18}$$

$$\begin{aligned}
&=\sum_{J=s}^\infty\sum_{\nu=-J}^{J}\sum_{\mu\in\mathbb{Z}}\left[\sum_{N=s}^\infty\frac{\Psi_{N,J,\nu,\mu}(r'',\vartheta'',\varphi'',\psi'')\Psi^*_{N,J,\nu,\mu}(r',\vartheta',\varphi',\psi')}{E_N-E}\right.\\
&\qquad\left.+\int_0^\infty\mathrm{d}k\,\frac{\Psi_{k,J,\nu,\mu}(r'',\vartheta'',\varphi'',\psi'')\Psi^*_{k,J,\nu,\mu}(r',\vartheta',\varphi',\psi')}{E_k-E}\right]
\end{aligned}\tag{6.9.19}$$

[$\omega^2=(2/M)(\hbar^2q^2/2M-E)$]. The wave functions have the form

$$\begin{aligned}
\Psi_{N/k,J,\nu,\mu}(r,\vartheta,\varphi,\psi)&=\frac{\mathrm{e}^{\mathrm{i}k\psi}}{\sqrt{16\pi m}}\sqrt{\frac{1}{8\pi m}\left(J+\frac12\right)}\\
&\times\mathrm{e}^{\mathrm{i}(\nu-\mu)\varphi}\,D^J_{\nu-\mu,-\mu}(\cos\vartheta)\times\begin{cases}\Psi_N(r) & \text{(bound states)} ,\\ \Psi_k(r) & \text{(continuum states)} .\end{cases}
\end{aligned}\tag{6.9.20}$$

The radial bound-state wave functions are

$$\Psi_N(r)=\left[\frac{(N-J-1)!}{(N+J)!}\frac{4}{a^3N^3\sqrt{N^2-s^2}}\right]^{1/2}\times\left(\frac{2r}{aN}\right)^J\exp\left(-\frac{r}{aN}\right)L^{(2J+1)}_{N-J-1}\left(\frac{2r}{aN}\right) , \tag{6.9.21}$$

where $a = 1\Big/\left(N\sqrt{q^2 - 2E_N M/\hbar^2}\right) = |4m|\Big/\left[N(N - \sqrt{N^2 - s^2})\right]$; the energy levels are

$$E_N = \frac{\hbar^2}{(4m)^2 M}\sqrt{N^2 - s^2}\left(N - \sqrt{N^2 - s^2}\right) , \qquad (N \geq s) . \tag{6.9.22}$$

The continuous spectrum has the form $E_k = \hbar^2(k^2 + q^2)/2M$, and the radial continuous wave functions $\Psi_k(r)$ are given by

$$\begin{aligned}\Psi_k(r) = &\frac{1}{\sqrt{2\pi}}\frac{|\Gamma[J + 1 + 2\,\mathrm{i}\,|m|(k^2 - q^2)/p]|}{\Gamma(2J + 2)r}\\ &\times \exp\left[\frac{\pi|m|}{p}(k^2 - q^2)\right] M_{2\,\mathrm{i}\,|m|(k^2-q^2)/k, J+\frac{1}{2}}(-2\,\mathrm{i}\,kr) .\end{aligned} \tag{6.9.23}$$

6.9.7.2 Kaluza-Klein Monopole in Parabolic Coordinates. [432] ($\Lambda(r) = 1/(1 + 4m/r)$, $q = 4mk$)

$$\begin{aligned}&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\xi(t')=\xi'}^{\xi(t'')=\xi''} \mathcal{D}\xi(t) \int\limits_{\eta(t')=\eta'}^{\eta(t'')=\eta''} \mathcal{D}\eta(t)\frac{(\xi^2 + \eta^2)4|m|\xi\eta}{\Lambda} \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)\\
&\quad\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{M}{2\Lambda}\Big((\xi^2 + \eta^2)(\dot\xi^2 + \dot\eta^2) + \xi^2\eta^2\dot\varphi^2\Big)\right.\right.\\
&\qquad\qquad\left.\left. + (4m)^2\Lambda\left(\dot\psi + \left(1 - \frac{\xi^2 - \eta^2}{\xi^2 + \eta^2}\right)\dot\varphi\right) + \frac{\hbar^2\Lambda}{8M\xi^2\eta^2}\right]\mathrm{d}t\right\}\\
&= \sum_{\nu\in\mathbb{Z}}\sum_{n_1\in\mathbb{N}_0}\sum_{n_2\in\mathbb{N}_0}\sum_{\mu\in\mathbb{Z}} \frac{\Psi_{n_1,n_2,\nu,\mu}(\xi'', \eta'', \varphi'', \psi'')\Psi^*_{n_1,n_2,\nu,\mu}(\xi', \eta', \varphi', \psi')}{E_N - E}\\
&\qquad + \sum_{\nu\in\mathbb{Z}}\sum_{\mu\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\int_{\mathbb{R}}\mathrm{d}\zeta\frac{\Psi_{k,\zeta,\nu,\mu}(\xi'', \eta'', \varphi''.\psi'')\Psi^*_{k,\zeta,\nu,\mu}(\xi', \eta', \varphi', \psi')}{E_k - E} ,\end{aligned} \tag{6.9.24}$$

with the discrete wave functions

$$\begin{aligned}&\Psi_N(\xi, \eta, \varphi, \psi) \equiv \Psi_{n_1,n_2,\nu,\mu}(\xi, \eta, \varphi, \psi)\\
&= \frac{\mathrm{e}^{\mathrm{i}\,\nu\varphi+\mathrm{i}\,k\psi}}{4\pi|m|\sqrt{2}}\left[\frac{2}{a^3n^3\sqrt{N^2 - s^2}}\frac{n_1!n_2!}{\Gamma(n_1 + |\nu - 2\mu| + 1)\Gamma(n_2 + |\nu| + 1)}\right]^{1/2}\\
&\quad\times \left(\frac{\xi^2}{aN}\right)^{\frac{|\nu-2\mu|}{2}}\left(\frac{\eta^2}{aN}\right)^{\frac{|\nu|}{2}}\exp\left(-\frac{\xi^2 + \eta^2}{2aN}\right)L_{n_1}^{(|\nu-2\mu|)}\left(\frac{\xi^2}{aN}\right)L_{n_2}^{(|\nu|)}\left(\frac{\eta^2}{aN}\right) ,\end{aligned} \tag{6.9.25}$$

and the energy spectrum as in the previous example. The continuum states are ($\beta_{1,2} = \frac{1}{4}\big[|4m|(p-q^2/p) \pm 2\zeta/k\big]$ and the spectrum $E_k = \hbar^2(k^2+q^2)/2M$)

$$\Psi_{k,\zeta,\nu,\mu}(\xi,\eta,\varphi,\psi) = \frac{e^{i(\nu\varphi+\mu\psi)}}{4\pi\sqrt{2|m|}}\exp\left[\pi|m|\left(p-\frac{q^2}{p}\right)\right]$$
$$\times \frac{|\Gamma(\frac{1}{2}+|\frac{\nu}{2}-\mu|+i\beta_1)\Gamma(\frac{1+|\nu|}{2}+i\beta_2)|}{\sqrt{2\pi^2 p}\,\xi\eta\,|\nu-2\mu|!\,|\nu|!} M_{i\beta_1,\frac{|\nu-2\mu|}{2}}\left(\frac{p\xi^2}{i}\right) M_{i\beta_2,\frac{|\nu|}{2}}\left(\frac{p\eta^2}{i}\right). \tag{6.9.26}$$

6.9.8 Electromagnetic and Gravitational Anyon.

6.9.8.1 Free Electromagnetic and Gravitational Anyon. [82,137] (σ spin, μ mass of the anyon, Φ magnetic flux, $\alpha = (\sigma E - e\Phi)/2\pi$, $\kappa^2 = E^2 - e^2 - (M/\hbar)^2$)

$$\frac{i}{2\hbar}\int_0^\infty dT\, e^{-iM^2T/2\hbar}\int_{\mathbb{R}} d\varphi \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\varrho$$
$$\times \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\delta\left(\varphi-\int_{t'}^{t''}\dot\varphi\, dt\right)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{1}{2}\left(-\left(\dot\tau+\frac{\sigma}{2\pi}\dot\varphi\right)^2+\dot\varrho^2 + (1-\mu)^2\varrho^2\dot\varphi^2+\left(\dot\vartheta+\frac{\Phi}{2\pi}\dot\varphi\right)^2\right)+\frac{\hbar^2}{8\varrho^2}\right]dt\right\}$$
$$= \frac{1}{(2\pi i\hbar)^2}\int_{\mathbb{R}} dE \sum_{e,m\in\mathbb{Z}} e^{-iE(\tau''-\tau')+ie(\vartheta''-\vartheta')+im(\varphi''-\varphi')}$$
$$\times I_{|\frac{m+\alpha}{1-\mu}|}(-i\kappa\varrho_<)K_{|\frac{m+\alpha}{1-\mu}|}(-i\kappa\varrho_>)\ , \tag{6.9.27}$$
$$= i\int_{\mathbb{R}} dE\int_0^\infty kdk\sum_{e,m\in\mathbb{Z}}\frac{\Psi_{Ekme}(\tau'',\varrho'',\vartheta'',\varphi'')\Psi^*_{Ekme}(\tau',\varrho',\vartheta',\varphi')}{e^2+k^2+M^2/\hbar^2-E^2}\ , \tag{6.9.28}$$

with the wave functions given by

$$\Psi_{Ekme}(\tau,\varrho,\vartheta,\varphi) = \frac{1}{2\pi\hbar}\exp\Big[i(-E\tau+e\vartheta+m\varphi)\Big]J_{|\frac{m+\alpha}{1-\mu}|}(k\varrho)\ . \tag{6.9.29}$$

6.9.8.2 Gravitational Anyon in a Uniform Magnetic Field. [82] (σ spin, μ mass of the anyon, Φ magnetic flux, $\omega = \beta/(1-\mu)$, $\beta = e\hbar B/4$, $\kappa^2 = E^2 - e^2 - M^2/\hbar^2 + 2\omega(m+\alpha)/\hbar(1-\mu)$, $\alpha = (\sigma E - e\Phi)/2\pi$, $l = |(m+\alpha)/(1-\mu)|$)

$$\frac{\mathrm{i}}{2\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{-\mathrm{i}M^2T/2\hbar}\int_{\mathbb{R}}\mathrm{d}\varphi \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\int\limits_{\varrho(t')=\varrho'}^{\varrho(t'')=\varrho''}\mathcal{D}\varrho(t)\varrho$$

$$\times \int\limits_{\vartheta(t')=\vartheta'}^{\vartheta(t'')=\vartheta''}\mathcal{D}\vartheta(t)\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\delta\left(\varphi - \int_{t'}^{t''}\dot\varphi\,\mathrm{d}t\right)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{1}{2}\left(-\left(\dot\tau + \frac{\sigma}{2\pi}\dot\varphi\right)^2 + \dot\varrho^2\right.\right.\right.$$

$$\left.\left.\left. + (1-\mu)^2\varrho^2\dot\varphi^2 + \left[\dot\vartheta + \left(\frac{\Phi}{2\pi} + \frac{1}{2}B\varrho^2\right)\dot\varphi\right]^2\right) + \frac{\hbar^2}{8\varrho^2}\right]\mathrm{d}t\right\}$$

$$= \frac{1}{\mathrm{i}\hbar}\int_{\mathbb{R}} dE \sum_{e,m\in\mathbb{Z}}\sum_{n_r\in\mathbb{N}_0}\frac{\Psi_{En_rme}(\tau'',\varrho'',\vartheta'',\varphi'')\Psi^*_{En_rme}(\tau',\varrho',\vartheta',\varphi')}{\kappa^2 - 2\omega(2n_r + l + 1)/\hbar}, \tag{6.9.30}$$

$$\Psi_{En_\varrho me}(\tau,\varrho,\vartheta,\varphi) = \frac{1}{2\pi\hbar}\exp\left[\mathrm{i}(-E\tau + e\vartheta + m\varphi)\right]\left(\frac{2n_\varrho!\,\omega}{\Gamma(n_\varrho + l + 1)}\right)^{1/2}$$

$$\times \left(\frac{\omega\varrho}{\hbar}\right)^{l/2}\exp\left(-\frac{\omega\varrho^2}{2\hbar}\right)L_{n_\varrho}^{(l)}\left(\frac{\omega\varrho^2}{\hbar}\right), \tag{6.9.31}$$

$$E = \pm\sqrt{eB\frac{n_\varrho + \frac{1}{2}}{1-\mu} + e^2 + \frac{M^2}{\hbar^2}}\ . \tag{6.9.32}$$

6.9.9 Pair Production in Magnetic Monopole Field. [277] (usual spherical coordinates are used for the space part $\mathbb{R}^3$, $\alpha = m - eg$, $(\mathbf{A})_\nu = (g(1-\cos\vartheta)/r\sin\vartheta)\mathbf{e}_\varphi$, $\lambda = l + \frac{1}{2}(|m| + |m - 2eg|)$, $\gamma = \sqrt{(\lambda + 1/2)^2 - e^2g^2}$)

$$\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\mu^2\tau}\int\limits_{\mathrm{x}(0)=\mathrm{x}'}^{\mathrm{x}(\tau)=\mathrm{x}''}\mathcal{D}\mathrm{x}(s)\exp\left[\frac{\mathrm{i}}{4}\int_0^\tau(-\dot\tau^2 + \dot{\mathbf{x}}^2 + 4eA^\nu\dot x_\nu)\mathrm{d}s\right]$$

$$= \frac{1}{8r'r''}\int_{\mathbb{R}}\frac{dE}{2\pi\mathrm{i}}\mathrm{e}^{\mathrm{i}E\tau}\sum_{l\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}m(\varphi''-\varphi')}$$

$$\times P_l^{(|m|,|m-2eg|)}\left(\cos\frac{\vartheta''}{2}\right)P_l^{(|m|,|m-2eg|)}\left(\cos\frac{\vartheta'}{2}\right)$$

$$\times H_\gamma^{(2)}\left(\sqrt{E^2-\mu^2}\,r_>\right)J_\gamma\left(\sqrt{E^2-\mu^2}\,r_<\right)\ . \tag{6.9.33}$$

6.9.10 Relativistic Dyon and Aharonov–Bohm Field. [655,656] ($\mathbf{A}\cdot\dot{\mathbf{x}} = -2g(x_2\dot{x}_1 - x_1\dot{x}_2)/r^2\,, \alpha = e^2/\hbar c\,, \beta_0 = -2eg/\hbar c$)

$$\int_0^\infty dT \int\limits_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}} \mathcal{D}_E\mathbf{x}(t)$$
$$\times \exp\left\{ -\frac{1}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - \mathrm{i}\frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}} - \frac{(E+e^2/r)^2}{2mc^2} + \frac{mc^2}{2}\right)\Big]dt\right\}$$
$$= \frac{mc}{4\pi r' r''\sqrt{m^2c^4 - E^2}}$$
$$\times \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{N}_0} e^{\mathrm{i}k(\varphi''-\varphi')}\left(\cos\tfrac{\vartheta'}{2}\cos\tfrac{\vartheta''}{2}\sin\tfrac{\vartheta'}{2}\sin\tfrac{\vartheta''}{2}\right)^{|k+\beta_0|}$$
$$\times \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{(2n+2|k+\beta_0|+1)^2 - 4\alpha^2} - E\alpha/\sqrt{m^2c^4-E^2}\right)}{\Gamma\left(\sqrt{(2n+2|k+\beta_0|+1)^2 - 4\alpha^2}+1\right)}$$
$$\times \frac{n!(2n+2|k+\beta_0|+1)}{\Gamma(n+2|k+\beta_0|+1)} P_n^{(|k+\beta_0|,|k+\beta_0|)}(\cos\vartheta'')\,P_n^{(|k+\beta_0|,|k+\beta_0|)}(\cos\vartheta')$$
$$\times W_{E\alpha/\sqrt{m^2c^4-E^2},\sqrt{(2n+2|k+\beta_0|+1)^2-4\alpha^2}/2}\left(\frac{2}{\hbar c}\sqrt{m^2c^2-E^2}\,r_>\right)$$
$$\times M_{E\alpha/\sqrt{m^2c^4-E^2},\sqrt{(2n+2|k+\beta_0|+1)^2-4\alpha^2}/2}\left(\frac{2}{\hbar c}\sqrt{m^2c^2-E^2}\,r_<\right) . \tag{6.9.34}$$

In two dimensions we must set $n = 0$, and for the Dyon we must replace $\alpha^2 \mapsto \alpha^2 + q^2$, $2g \mapsto -2\hbar q$, where $q = -(e_1g_2 - e_2g_1)/\hbar c$, $e^2 = -(e_1e_2 + g_1g_2)$, and furthermore $n + |k+\beta_0| \mapsto l$, and for the angular wave functions we have the monopole harmonics.

6.10 Motion in Hyperbolic Geometry

6.10.1 Free Motion on the Hyperbolic Plane. ($\mathrm{u} = (u_0, \mathbf{u}) \in \Lambda^{(2)}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{u}(t')=\mathbf{u}'}^{\mathbf{u}(t'')=\mathbf{u}''} \frac{\mathcal{D}\mathbf{u}(t)}{u_0}\exp\left(\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\dot{\mathrm{u}}^2\,dt\right)$$

Pseudosphere, $\tau > 0$, $\varphi \in [0, 2\pi)$ [104,215,427,444,466]:

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}ET/\hbar} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh\tau \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{\tau}^2 + \sinh^2\tau\,\dot{\varphi}^2) - \frac{\hbar^2}{8m}\left(1 - \frac{1}{\sinh^2\tau}\right)\right]dt\right\}$$

$$= \int_0^\infty dk \sum_{l \in \mathbb{Z}} \frac{e^{i l(\varphi''-\varphi')}}{2\pi^2} \frac{k \sinh \pi k}{\hbar^2(k^2+\frac{1}{4})/2m - E}$$

$$\times \left| \Gamma\left(\frac{1}{2} + ik + l\right) \right|^2 \mathcal{P}^{-l}_{ik-\frac{1}{2}}(\cosh \tau'') \mathcal{P}^{-l}_{ik-\frac{1}{2}}(\cosh \tau') \ . \tag{6.10.1}$$

Poincaré Disc, $0 \le r < 1$, $\psi \in [0, 2\pi)$:

$$= \frac{i}{\hbar} \int_0^\infty dT\, e^{iET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \frac{4r}{(1-r^2)^2} \int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''} \mathcal{D}\psi(t)$$

$$\times \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(2m\frac{\dot r^2 + r^2\dot\psi^2}{(1-r^2)^2} + \hbar^2 \frac{(1-r^2)^2}{32mr^2}\right)dt\right]$$

$$= \frac{1}{2\pi^2}\int_0^\infty dk \sum_{l \in \mathbb{Z}} e^{i l(\psi''-\psi')} \frac{k \sinh \pi k}{\hbar^2(k^2+\frac{1}{4})/2m - E}$$

$$\times \left| \Gamma\left(\frac{1}{2} + ik + l\right) \right|^2 \mathcal{P}^{-l}_{ik-\frac{1}{2}}\left(\frac{1+r''^2}{1-r''^2}\right) \mathcal{P}^{-l}_{ik-\frac{1}{2}}\left(\frac{1+r'^2}{1-r'^2}\right) \ . \tag{6.10.2}$$

Hyperbolic Strip, $X \in \mathbb{R}$, $|Y| < \pi/2$ [215,427]:

$$= \frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}$$

$$\times \int\limits_{X(t')=X'}^{X(t'')=X''} \mathcal{D}X(t) \int\limits_{Y(t')=Y'}^{Y(t'')=Y''} \frac{\mathcal{D}Y(t)}{\cos^2 Y} \exp\left(\frac{im}{2\hbar}\int_{t'}^{t''} \frac{\dot X^2 + \dot Y^2}{\cos^2 Y} dt\right)$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}} d\nu \int_{\mathbb{R}} \frac{k dk \sinh \pi k}{\cosh^2 \pi\nu + \sinh^2 \pi k} \frac{1}{\hbar^2(k^2+\frac{1}{4})/2m - E}$$

$$\times \sqrt{\cos Y' \cos Y''}\, e^{i\nu(X''-X')}$$

$$\times \left[P^{ik}_{i\nu-1/2}(\sin Y'') + P^{ik}_{i\nu-1/2}(-\sin Y'')\right]$$

$$\times \left[P^{-ik}_{i\nu-1/2}(\sin Y') + P^{-ik}_{i\nu-1/2}(-\sin Y')\right] \ . \tag{6.10.3}$$

Poincaré Upper Half-Plane, $x \in \mathbb{R}$, $y > 0$ [215,427,465,481,624,821]:

$$= \frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \exp\left(\frac{im}{2\hbar}\int_{t'}^{t''} \frac{\dot x^2 + \dot y^2}{y^2} dt\right)$$

$$= \frac{\sqrt{y'y''}}{\pi^3}\int_{\mathbb{R}} dk_1\, e^{ik_1(x''-x')} \int_0^\infty \frac{dk\, k \sinh \pi k}{\hbar^2(k^2+\frac{1}{4})/2m - E} K_{ik}(|k_1|y'')\, K_{ik}(|k_1|y') \ . \tag{6.10.4}$$

General expression for the Green function [435,465,466,481]:

$$= \frac{m}{\pi\hbar^2} \mathcal{Q}_{-1/2-\mathrm{i}\sqrt{2mE/\hbar^2-1/4}}\left(\cosh d(\mathbf{q}'',\mathbf{q}')\right) \;, \tag{6.10.5}$$

where $\mathcal{Q}_\mu^\nu$ is a Legendre-function of the second kind, and $d(\mathbf{q}'',\mathbf{q}')$ is the hyperbolic distance invariant with respect to the group action in any of the coordinate systems on $\Lambda^{(2)}$. Cf. [444,447] for further details in expanding the path integral on $\Lambda^{(2)}$ and $\Lambda^{(3)}$ in coordinate systems which separate the path integral.

6.10.2 Motion in the Hyperbolic Plane with an Oscillator-Like Potential. [429]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2} - y^2\left(V(x)+\frac{m}{2}\omega^2 y^2\right)\right]\mathrm{d}t\right\}$$

$$= \int \mathrm{d}E_\lambda \Psi_\lambda(x'')\Psi_\lambda^*(x') \frac{\Gamma[\frac{1}{2}(1+\nu+E_\lambda/\hbar\omega)]}{\sqrt{y'y''}\hbar\omega\Gamma(1+\nu)}$$

$$\times W_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar}y_>^2\right) M_{-E_\lambda/2\hbar\omega,\nu/2}\left(\frac{m\omega}{\hbar}y_<^2\right) \;, \tag{6.10.6}$$

$$= \int \mathrm{d}E_\lambda \left[\sum_{n=0}^{N_M} \frac{\Psi_{n,\lambda}(x'',y'')\Psi_{n,\lambda}^*(x',y')}{E_n - E} + \int_0^\infty \mathrm{d}k \frac{\Psi_{k,\lambda}(x'',y'')\Psi_{k,\lambda}^*(x',y')}{E_k - E}\right] \;, \tag{6.10.7}$$

$\nu = \sqrt{1/4 - 2mE/\hbar^2}$, and with $\Psi_\lambda(x)$ and E_λ the wave functions and the energy spectrum, respectively, of the Euclidean one-dimensional path integral problem with potential $V(x)$. The energy levels are given by ($E_\lambda < 0$)

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m}\left(\frac{|E_\lambda|}{\hbar\omega} - 2n - 1\right)^2 \tag{6.10.8}$$

with $n = 0,1,2,\ldots,N_M < |E_\lambda|/2\hbar\omega - \frac{1}{2}$, and the wave functions are

$$\Psi_{n,\lambda}(x,y) = \sqrt{\frac{2n!(|E_\lambda|/\hbar\omega - 2n - 1)y}{\Gamma(|E_\lambda|/\hbar\omega - n)}} \left(\frac{m\omega}{\hbar}y^2\right)^{|E_\lambda|/\hbar\omega - n - 1/2}$$

$$\times \exp\left(-\frac{m\omega}{2\hbar}y^2\right) L_n^{(|E_\lambda|/\hbar\omega - 2n - 1)}\left(\frac{m\omega}{\hbar}y^2\right)\Psi_\lambda(x) \;. \tag{6.10.9}$$

The normalized wave functions of the continuum states with $E_k = \hbar^2(k^2 + 1/4)/2m$ are given by

$$\Psi_{k,\lambda}(x,y) = \sqrt{\frac{\hbar}{m\omega}\frac{k\sinh\pi k}{2\pi^2 y}}\,\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}\,k+\frac{E_\lambda}{\hbar\omega}\right)\right] \times W_{-E_\lambda/2\hbar\omega,\mathrm{i}\,k/2}\left(\frac{m\omega}{\hbar}y^2\right)\Psi_\lambda(x) \ . \tag{6.10.10}$$

6.10.3 Motion in the Hyperbolic Plane with a Coulomb-Like Potential. [429]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2}$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot x^2+\dot y^2}{y^2} - y^2\left(V(x)+\frac{\alpha}{2my}\right)\right]\mathrm{d}t\right\}$$
$$= \int \mathrm{d}E_\lambda\, \Psi_\lambda(x'')\Psi_\lambda^*(x')\frac{\Gamma(\frac{1}{2}+\nu+\alpha/2\hbar\sqrt{2mE_\lambda})}{\sqrt{y'y''}\,\Gamma(1+\nu)\sqrt{2mE_\lambda}/\hbar}$$
$$\times W_{-\alpha/2\hbar\sqrt{2mE_\lambda},\nu}\left(\frac{2y_>}{\hbar}\sqrt{2mE_\lambda}\right) M_{-\alpha/2\hbar\sqrt{2mE_\lambda},\nu}\left(\frac{2y_<}{\hbar}\sqrt{2mE_\lambda}\right) \ , \tag{6.10.11}$$
$$= \int \mathrm{d}E_\lambda\left[\sum_{n=0}^{N_M}\frac{\Psi_{n,\lambda}(x'',y'')\Psi_{n,\lambda}^*(x',y')}{E_n-E} + \int_0^\infty \mathrm{d}k\frac{\Psi_{\lambda,p}(x'',y'')\Psi_{\lambda,p}^*(x',y')}{E_k-E}\right] \tag{6.10.12}$$

$\left(\nu=\sqrt{1/4-2mE/\hbar^2}\right)$ with $\Psi_\lambda(x)$ and E_λ the wave functions and the energy spectrum, respectively, of the Euclidean one-dimensional path integral problem with potential $V(x)$. The wave functions for the continuum states with $E_k=\hbar^2(k^2+1/4)/2m$ are

$$\Psi_{k,\lambda}(x,y) = \sqrt{\frac{\hbar k\sinh 2\pi k}{2\pi^2\sqrt{2mE_\lambda}}}\,\Gamma\left(\frac{1}{2}+\mathrm{i}\,k+\frac{\alpha}{2\hbar\sqrt{2mE_\lambda}}\right) \times W_{-\alpha/2\hbar\sqrt{2mE_\lambda},\mathrm{i}\,k}\left(\frac{2y}{\hbar}\sqrt{2mE_\lambda}\right)\Psi_\lambda(x) \ . \tag{6.10.13}$$

For the bound states one has $\left[n=0,1,\dots,N_M<\frac{1}{2}(|\alpha|/\hbar\sqrt{2mE_\lambda}-1)\right]$:

$$\Psi_{n,\lambda}(x,y) = \sqrt{\frac{(|\alpha|\hbar/\sqrt{2mE_\lambda}-2n-1)n!}{4\pi|k|\Gamma(|\alpha|\hbar/\sqrt{2mE_\lambda}-n)}}\left(\frac{2y}{\hbar}\sqrt{2mE_\lambda}\right)^{|\alpha|\hbar/\sqrt{2mE_\lambda}-n}$$
$$\times \mathrm{e}^{-y\sqrt{2mE_\lambda}/\hbar}\, L_n^{(|\alpha|\hbar/\sqrt{2mE_\lambda}-2n-1)}\left(\frac{2y}{\hbar}\sqrt{2mE_\lambda}\right)\Psi_\lambda(x) \ , \tag{6.10.14}$$

$$E_n = \frac{\hbar^2}{8m} - \frac{\hbar^2}{2m}\left(\frac{|\alpha|\hbar}{\sqrt{2mE_\lambda}} - n - \frac{1}{2}\right)^2 . \tag{6.10.15}$$

Bound states can only exist if $\alpha < 0$ and $E_\lambda > 0$.

6.10.4 Motion in the Hyperbolic Plane with a Magnetic Field.

6.10.4.1 The Poincaré Upper Half-Plane. [323,423] ($b = emB/2c\hbar > 0$)

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \\
&\quad\times \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2} - b\hbar\frac{\dot{x}}{y}\right)\mathrm{d}t\right] \\
&= \frac{m}{2\pi\hbar}\frac{\Gamma(\frac{1}{2}-b_\nu-\mathrm{i}\,k)\sqrt{y'y''}}{\Gamma(1-2\,\mathrm{i}\,k)} \\
&\quad\times \int_{\mathbb{R}} \frac{\mathrm{d}\nu}{|\nu|} e^{\mathrm{i}\,\nu(x''-x')} W_{b_\nu,-\mathrm{i}\,k}(2|\nu|y_>) M_{b_\nu,-\mathrm{i}\,k}(2|\nu|y_>) \ , \qquad (6.10.16) \\
&= \frac{m}{2\pi\hbar}\frac{\Gamma(\frac{1}{2}+b_\nu-\mathrm{i}\,k)\Gamma(\frac{1}{2}-b_\nu-\mathrm{i}\,k)}{\Gamma(1-2\,\mathrm{i}\,k)} \exp\left[-\frac{2\,\mathrm{i}\,b_\nu}{\hbar}\,\mathrm{artanh}\left(\frac{x'-x''}{y'+y''}\right)\right] \\
&\quad\times \left(\cosh\frac{r}{2}\right)^{-2b_\nu}\left(\sinh\frac{r}{2}\right)^{2(b_\nu-\frac{1}{2}+\mathrm{i}\,k)} \\
&\quad\times {}_2F_1\left(\frac{1}{2}-b_\nu-\mathrm{i}\,k, \frac{1}{2}-b_\nu-\mathrm{i}\,k; 1-2\,\mathrm{i}\,k; \frac{2}{1-\cosh r}\right) \ , \qquad (6.10.17) \\
&= \sum_{n=0}^{N_M}\int_0^\infty \mathrm{d}\nu \frac{\Psi_{n,\nu}(x'',y'')\Psi^*_{n,\nu}(x',y')}{E_{n,\nu}-E} \\
&\qquad\qquad + \int_{\mathbb{R}}\mathrm{d}k\int_0^\infty \mathrm{d}\nu\, \frac{\Psi_{k,\nu}(x'',y'')\Psi^*_{k,\nu}(x',y')}{E_k - E} \ , \qquad (6.10.18)
\end{aligned}$$

with $\cosh r$ the invariant hyperbolic distance, $k = \sqrt{2mE/\hbar^2 - b^2 - 1/4}$. Wave functions and energy spectrum of the discrete spectrum are

$$E_{n,\nu} = \frac{\hbar^2}{2m}\left[b^2 + \frac{1}{4} - \left(b_\nu - n - \frac{1}{2}\right)^2\right] \ , \tag{6.10.19}$$

$$\Psi_{n,\nu}(x,y) = \sqrt{\frac{(2b_\nu - 2n - 1)n!}{4\pi\nu\Gamma(2b_\nu - n)}}\, \mathrm{e}^{\mathrm{i}\,\nu x}\, \mathrm{e}^{-\nu y}(2\nu y)^{b_\nu - n} L_n^{(2b_\nu - 2n - 1)}(2\nu y) \tag{6.10.20}$$

($n = 0, \ldots, N_M = b_\nu - \frac{1}{2}, k > 0, b_\nu = b\nu/|\nu|$). The wave-functions with $E_k = \hbar^2(b^2 + k^2 + 1/4)/2m$ of the continuous spectrum are

$$\Psi_{k,\nu}(x,y) = \sqrt{\frac{k\sinh 2\pi k}{4\pi^3|\nu|}}\Gamma\big(\mathrm{i}\,k - b_\nu + \tfrac{1}{2}\big)W_{b_\nu,\mathrm{i}\,k}(2|\nu|y)\,\mathrm{e}^{\mathrm{i}\,\nu x} \quad . \qquad (6.10.21)$$

6.10.4.2 The Pseudosphere and the Poincaré Disc. [425] ($b_l = bl/|l|, r, E_k$ as in the previous example)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\ \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t)\sinh\tau \int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t)$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\tau^2+\sinh^2\tau\dot\varphi^2)\right.\right.$$

$$\left.\left. - b\hbar(\cosh\tau-1)\dot\varphi - \frac{\hbar^2}{8m}\left(1-\frac{1}{\sinh^2\tau}\right)\right]\mathrm{d}t\right\}$$

$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\ \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\frac{4r}{(1-r^2)^2} \int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''} \mathcal{D}\psi(t)$$

$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(2m\frac{\dot r^2+r^2\dot\psi^2}{(1-r^2)^2} - 2b\hbar\frac{r^2}{1-r^2}\dot\psi + \hbar^2\frac{(1-r^2)^2}{32mr^2}\right)\mathrm{d}t\right]$$

$$= \frac{m}{2\pi\hbar}\mathrm{e}^{-2\,\mathrm{i}\,b\Phi}\left(\frac{\mathrm{i}-\zeta'^*}{\mathrm{i}-\zeta''^*}\cdot\frac{\zeta''+\mathrm{i}}{\zeta'+\mathrm{i}}\right)^{-b}\frac{\Gamma(\frac{1}{2}+b-\mathrm{i}\,k)\Gamma(\frac{1}{2}-b-\mathrm{i}\,k)}{\Gamma(1-2\,\mathrm{i}\,k)}$$

$$\times\left(1-\tanh^2\frac{r}{2}\right)^{\frac{1}{2}+\mathrm{i}\,k} {}_2F_1\left(\frac{1}{2}+b-\mathrm{i}\,k,\frac{1}{2}-b_l-\mathrm{i}\,k;1-2\,\mathrm{i}\,k;\frac{1}{\cosh^2\frac{r}{2}}\right), \qquad (6.10.22)$$

$$= \sum_{l\in\mathbb{Z}}\sum_{n=0}^{N_M}\frac{\Psi_{n,l}^{\Lambda^2_B}(\tau'',\varphi'')\Psi_{n,l}^{\Lambda^2_B\,*}(\tau',\varphi')}{E_n-E}$$

$$+\sum_{l\in\mathbb{Z}}\int_0^\infty \mathrm{d}k\,\frac{\Psi_{k,l}^{\Lambda^2_B}(\tau'',\varphi'')\Psi_{k,l}^{\Lambda^2_B\,*}(\tau',\varphi')}{E_k-E} \quad . \qquad (6.10.23)$$

Here we have

$$\Phi = \operatorname{artanh}\left(\frac{x'-x''}{y'+y''}\right) \quad , \qquad (6.10.24)$$

and the coordinates $\zeta = x+\mathrm{i}\,y$ are related to the pseudo-spherical coordinates (τ,φ) via

$$\frac{\zeta-\mathrm{i}}{\zeta+\mathrm{i}} = \tanh\frac{\tau}{2}\,\mathrm{e}^{-\,\mathrm{i}(\varphi+\pi/2)} \quad . \qquad (6.10.25)$$

Furthermore denote $k=\sqrt{2mE/\hbar^2-b^2-1/4}$. The bound-state wave functions and the energy spectrum have the form ($n=0,1,\ldots < N_M = b_l-\frac{1}{2}$)

$$\Psi_{n,l}^{A_B^2}(\tau,\varphi)=\left[\frac{n!(2b_l+|l|)\Gamma(2b_l-n+|l|)}{4\pi(n+|l|)!\Gamma(2b_l-n)}\right]^{1/2}$$
$$\times\,\mathrm{e}^{\mathrm{i}l\varphi}\left(\tanh\frac{\tau}{2}\right)^{|l|}\left(1-\tanh^2\frac{\tau}{2}\right)^{b_l-n}P_n^{(|l|,2b_l-2n-1)}\left(1-2\tanh^2\frac{\tau}{2}\right)\ , \tag{6.10.26}$$

$$E_n=\frac{\hbar^2}{2m}\Big[b^2+\tfrac14-\big(b_l-n-\tfrac12\big)^2\Big]\ . \tag{6.10.27}$$

The continuous spectrum is given by ($k\geq 0, E_k$ as in the previous example)

$$\Psi_{k,l}^{A_B^2}(\tau,\varphi)=\frac{1}{\pi(|l|!)}\sqrt{\frac{k\sinh 2\pi k}{4\pi}}\,\Gamma\Big(\frac{1+\mathrm{i}\,k}{2}+b_l+|l|\Big)\Gamma\Big(\frac{1+\mathrm{i}\,k}{2}-b_l\Big)$$
$$\times\,\mathrm{e}^{\mathrm{i}l\varphi}\left(1-\tanh^2\frac{\tau}{2}\right)^{\frac12+\mathrm{i}k}\left(\tanh\frac{\tau}{2}\right)^{|l|}$$
$$\times\,{}_2F_1\big(\tfrac12-\mathrm{i}\,k+b_l+|l|,\tfrac12+\mathrm{i}\,k-b_l;1+|l|;\tanh^2\tfrac{\tau}{2}\big)\ . \tag{6.10.28}$$

6.10.4.3 The Hyperbolic Strip. [425] (Φ, b_ν, r, E_k as in the previous example)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{X(t')=X'}^{X(t'')=X''}\mathcal{D}X(t)\int\limits_{Y(t')=Y'}^{Y(t'')=Y''}\frac{\mathcal{D}Y(t)}{\cos^2 Y}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\frac{\dot X^2+\dot Y^2}{\cos^2 Y}-b\hbar\tan Y\cdot\dot X\right)\mathrm{d}t\right]$$
$$=\frac{m}{2\pi\hbar}\mathrm{e}^{\mathrm{i}\,b_\nu(Y'-Y''-\Phi)}\,\frac{\Gamma(\frac12+b_\nu-\mathrm{i}\,k)\Gamma(\frac12-b_\nu-\mathrm{i}\,k)}{\Gamma(1-2\,\mathrm{i}\,k)}$$
$$\times\left(1-\tanh^2\frac{r}{2}\right)^{\frac12+\mathrm{i}\,k}{}_2F_1\left(\frac12+b_\nu-\mathrm{i}\,k,\frac12-b_\nu-\mathrm{i}\,k;1-2\,\mathrm{i}\,k;\frac{1}{\cosh^2\frac{r}{2}}\right), \tag{6.10.29}$$

$$=\sum_{n=0}^{N_M}\int_{\mathbb{R}}\mathrm{d}k\,\frac{\Psi_n^{S_B}(X'',Y'')\Psi_n^{S_B\,*}(X',Y')}{E_n-E}$$
$$+\int_{\mathbb{R}}\mathrm{d}k\int_{\mathbb{R}}\mathrm{d}\nu\,\frac{\Psi_k^{S_B}(X'',Y'')\Psi_k^{S_B\,*}(X',Y')}{E_k-E}\ . \tag{6.10.30}$$

The discrete spectrum is given by

$$\Psi_n^{S_B}(X,Y)=\left[\frac{n!(b_\nu+\mathrm{i}\,\nu)\Gamma(b_\nu+\mathrm{i}\,\nu-n)}{\pi\Gamma(1+n+\mathrm{i}\,\nu-b_\nu)\Gamma(2b_\nu-n)}\right]^{1/2}2^{n-b_\nu}$$
$$\times\,\mathrm{e}^{\mathrm{i}\nu X}\left(\mathrm{i}\,\mathrm{e}^{-\mathrm{i}Y}\right)^{\mathrm{i}\nu-n}(\cos Y)^{n-b_\nu+1}P_n^{(\mathrm{i}\nu-b_\nu,2b_\nu-2n-1)}\left(1+\mathrm{e}^{-2\,\mathrm{i}\,Y}\right)\ , \tag{6.10.31}$$

$$E_n = \frac{\hbar^2}{2m}\Big[b^2 + \tfrac{1}{4} - \big(b_\nu - n - \tfrac{1}{2}\big)^2\Big] \quad , \ n = 0,1,\dots,N_M < b_\nu - \tfrac{1}{2} \ . \tag{6.10.32}$$

The continuous spectrum has the form

$$\begin{aligned}\Psi_k^{S_B}(X,Y) &= N_{k,\nu}^{S_B}\,\mathrm{e}^{\mathrm{i}\nu X + \mathrm{i}bY}(\mathrm{i}\cos Y)^{-\mathrm{i}\nu}\\ &\quad\times {}_2F_1\bigg(\tfrac{1}{2} + \mathrm{i}(\nu - k), \tfrac{1}{2} + \mathrm{i}(\nu + k); 1 + \mathrm{i}\nu - b_\nu; \frac{1}{1 + \mathrm{e}^{2\mathrm{i}Y}}\bigg) \ ,\end{aligned} \tag{6.10.33a}$$

$$N_{k,\nu}^{S_B} = \frac{1}{\pi\Gamma(1 + \mathrm{i}\nu - b_\nu)}\sqrt{\frac{k\sinh 2\pi k}{4\pi}}\,\Gamma(\tfrac{1}{2} + \mathrm{i}k - b_\nu)\Gamma(\tfrac{1}{2} + \mathrm{i}\nu - \mathrm{i}k) \ . \tag{6.10.33b}$$

6.10.4.4 The Single-Sheeted Hyperboloid. [215,442,447] ($\tau \in \mathbb{R}$, $\varphi \in [0, 2\pi)$)

$$\begin{aligned}&\frac{\mathrm{e}^{-i\hbar T/8mR^2}}{R^2}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh\tau\int\limits_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''}\mathcal{D}\varphi(t)\\ &\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}R^2(\dot\tau^2 - \cosh^2\tau\dot\varphi^2) - \mathrm{i}\hbar b\sinh\tau\dot\varphi - \frac{\hbar^2}{8mR^2\cosh^2\tau}\right)\mathrm{d}t\right]\\ &= \frac{1}{2\pi R^2}(\cosh\tau'\cosh\tau'')^{-1/2}\sum_{l\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}l(\varphi''-\varphi')}\\ &\quad\times\frac{1}{2}\Bigg\{\sum_{n=0}^{N_M}\Psi_n^{(\mathcal{H}_{-1}^{(3)},b)}(\tau'')\Psi_n^{(\mathcal{H}_{-1}^{(3)},b)*}(\tau')\,\mathrm{e}^{-\mathrm{i}TE_n^{(\mathcal{H}_{-1}^{(3)},b)}/\hbar}\\ &\qquad + \int_{\mathbb{R}}\mathrm{d}k\,\Psi_k^{(\mathcal{H}_{-1}^{(3)},b)}(\tau'')\Psi_k^{(\mathcal{H}_{-1}^{(3)},b)*}(\tau')\,\mathrm{e}^{-\mathrm{i}TE_k^{(\mathcal{H}_{-1}^{(3)},b)}/\hbar}\Bigg\} \ .\end{aligned} \tag{6.10.34}$$

The energy spectrum and the wave functions are given by, respectively ($n = 0,1,2,\dots,N_M < \frac{1}{2}(|l + b_l| - |l - b_l| - 1), b_l = bl/|l|$)

$$E_n^{(\mathcal{H}_{-1}^{(3)},b)} = \frac{\hbar^2}{2mR^2}\left[\frac{1}{4} + b^2 - (n + \tfrac{1}{2} - \tfrac{1}{2}|l + b_l| + \tfrac{1}{2}|l - b_l|)^2\right] \ , \tag{6.10.35}$$

$$\begin{aligned}\Psi_{n,l}^{(\mathcal{H}_{-1}^{(3)},b)}(\tau) &= \left[\frac{(|l + b_l| - |l - b_l| - n - 1)n!\Gamma(|l + b_l| - n)}{\Gamma(|l + b_l| - |l - b_l| - n)\Gamma(|l - b_l| + n + 1)}\right]^{1/2}\\ &\quad\times\left(\frac{\mathrm{i}\sinh\tau - 1}{2}\right)^{\frac{1}{2}(\frac{1}{2} + |l - b_l|)}\left(\frac{\mathrm{i}\sinh\tau + 1}{2}\right)^{\frac{1}{2}(\frac{1}{2} - |l + b_l|)}\\ &\quad\times P_n^{(|l - b_l|, -|l + b_l|)}(\mathrm{i}\sinh\tau) \ .\end{aligned} \tag{6.10.36}$$

The continuum states with $E_k^{(\mathcal{H}_{-1}^{(3)},b)} = \hbar^2(k^2 + b^2 + 1/4)/2mR^2$ have the form

$$\Psi_{k,l}^{(\mathcal{H}_{-1}^{(3)},b)}(\tau) = \frac{\sqrt{k\sinh 2\pi k}}{\pi\Gamma(1+|l-b_l|)}$$
$$\times\left|\Gamma\left[\tfrac{1}{2}(1+|l-b_l|+|l+b_l|)-\mathrm{i}\,k\right]\Gamma\left[\tfrac{1}{2}(1+|l-b_l|-|l+b_l|)-\mathrm{i}\,k\right]\right|$$
$$\times\left(\frac{\mathrm{i}\sinh\tau+1}{2}\right)^{\frac{1}{2}(\frac{1}{2}+|l-b_l|)}\left(\frac{\mathrm{i}\sinh\tau-1}{2}\right)^{\mathrm{i}\,k-\frac{1}{2}(\frac{1}{2}+|l-b_l|)}$$
$$\times\,{}_2F_1\biggl(\frac{1}{2}(1+|l-b_l|+|l+b_l|)-\mathrm{i}\,k,$$
$$\frac{1}{2}(1+|l-b_l|-|l+b_l|)-\mathrm{i}\,k;1+|l-b_l|;\frac{\mathrm{i}\sinh\tau-1}{\mathrm{i}\sinh\tau+1}\biggr)\ . \tag{6.10.37}$$

6.10.4.5 Motion in $(D-1)$-Dimensional Hyperbolic Space with a Magnetic Field. [435] ($b = emB/2c\hbar$, $A_y = B_y \equiv 0$ gauge)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^{D-1}}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{\dot{\mathbf{x}}^2+\dot{y}^2}{y^2}-\frac{\mathbf{b}\cdot\dot{\mathbf{x}}}{y}-\frac{\hbar^2}{8m}(D-1)(D-3)\right)\mathrm{d}t\right]$$
$$=\sum_{\boldsymbol{\nu}}\sum_{n=0}^{N_M}\Psi_{n,\boldsymbol{\nu}}(\mathbf{x}'',y'')\Psi_{n,\boldsymbol{\nu}}^*(\mathbf{x}',y')\,\mathrm{e}^{-\mathrm{i}\,E_nT/\hbar}$$
$$+\sum_{\boldsymbol{\nu}}\int_0^\infty \mathrm{d}k\,\Psi_{k,\boldsymbol{\nu}}(\mathbf{x}'',y'')\Psi_{k,\boldsymbol{\nu}}^*(\mathbf{x}',y')\,\mathrm{e}^{-\mathrm{i}\,E_kT/\hbar} \tag{6.10.38}$$

($\mathbf{x} = (x^1,\dots,x^{D-2})$ and similarly for $\mathbf{b}$ and $\boldsymbol{\nu}$). The wave functions and the energy spectrum for the discrete spectrum are

$$E_n = -\frac{\hbar^2}{2m}\left[\left(\alpha-n-\frac{1}{2}\right)^2+\mathbf{b}^2+\frac{(D-2)^2}{4}\right]\ , \tag{6.10.39}$$

$$\Psi_{n,\boldsymbol{\nu}}(\mathbf{x},y) = \frac{\mathrm{e}^{\mathrm{i}\,\boldsymbol{\nu}\cdot\mathbf{x}}}{(2\pi)^{\frac{D-2}{2}}}\sqrt{\frac{n!(2\alpha-2n-1)}{2|\boldsymbol{\nu}|\Gamma(2\alpha-n)}}$$
$$\times(2|\mathbf{k}|y)^{\alpha-n}\,\mathrm{e}^{-|\mathbf{k}|y}\,L_n^{(2\alpha-2n-1)}(2|\boldsymbol{\nu}|y)\ , \tag{6.10.40}$$

with $n = 0,\dots,N_M < \alpha-\frac{1}{2}$, $\alpha = \boldsymbol{\nu}\cdot\mathbf{b}/|\boldsymbol{\nu}|$. For the continuous spectrum we have

$$E_k = \frac{\hbar^2}{2m}\left[k^2+\mathbf{b}^2+\frac{(D-2)^2}{4}\right] \tag{6.10.41}$$

$$\Psi_{k,\boldsymbol{\nu}}(\mathbf{x},y)=\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{x}}}{(2\pi)^{\frac{D-2}{2}}}\Gamma\left(\mathrm{i}k-\alpha+\frac{1}{2}\right)\sqrt{\frac{k\sinh\pi k}{2\pi^2|\boldsymbol{\nu}|}}\,W_{\alpha,\mathrm{i}k}(2|\boldsymbol{\nu}|y)\ . \tag{6.10.42}$$

6.10.5 Kepler Problem in a Space of Constant Negative Curvature. [60,426,460,462,463,824] ($\tau>0$, $\boldsymbol{\Omega}\in S^{(D-2)}$)

$$\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\,\mathrm{e}^{-\mathrm{i}\hbar T(D-2)^2/8mR^2}\int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathcal{D}\tau(t)\sinh^{D-2}\tau\int\limits_{\boldsymbol{\Omega}(t')=\boldsymbol{\Omega}'}^{\boldsymbol{\Omega}(t'')=\boldsymbol{\Omega}''}\mathcal{D}\boldsymbol{\Omega}(t)\\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{mR^2}{2}\dot\tau^2+\sinh^2\tau\dot{\boldsymbol{\Omega}}^2)+\frac{e_1e_2}{R}(\coth\tau-1)\right)\mathrm{d}t\right]\\
&=\frac{R^2}{\big(\sinh\alpha'\sinh\alpha''\big)^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\boldsymbol{\Omega}'')S_l^\mu(\boldsymbol{\Omega}')\\
&\quad\times\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\quad\times\left(\frac{1}{1+u'}\cdot\frac{1}{1+u''}\right)^{\frac{1}{2}(m_1+m_2+1)}\left(\frac{u'}{1+u'}\cdot\frac{u''}{1+u''}\right)^{\frac{1}{2}(m_1-m_2)}\\
&\quad\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{u_<}{1+u_<}\right)\\
&\quad\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1}{1+u_>}\right)\ ,
\end{aligned}\tag{6.10.43}$$

$$\begin{aligned}
&=\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}\left[\sum_{n=0}^{N_M}\frac{\Psi_{n,l,\mu}(\alpha'',\boldsymbol{\Omega}'')\Psi_{n,l,\mu}(\alpha',\boldsymbol{\Omega}')}{E_n-E}\right.\\
&\qquad\qquad\left.+\int_0^\infty\mathrm{d}k\,\frac{\Psi_{k,l,\mu}(\alpha'',\boldsymbol{\Omega}'')\Psi^*_{k,l,\mu}(\alpha',\boldsymbol{\Omega}')}{E_k-E}\right]\ .
\end{aligned}\tag{6.10.44}$$

Here we denote

$$\left.\begin{aligned}
L_E&=-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2mR^2}{\hbar^2}\left(\frac{e_1e_2}{R}-E\right)}\ ,\\
m_{1,2}&=l+\frac{D-3}{2}\pm\frac{1}{2}\sqrt{-\frac{2mR^2}{\hbar^2}\left(\frac{e_1e_2}{R}+E\right)}\ ,
\end{aligned}\right\}\tag{6.10.45}$$

and $u=\frac{1}{2}(\coth\alpha-1)$. The bound-state wave functions are given by ($\tilde N=N+(D-4)/2$, $N=n+l+1$, $n=0,1,\dots,N_M<\sqrt{R/a}-\frac{D-4}{2}$, $\sigma_N=R/a\tilde N, a=\hbar^2/me_1e_2$)

$$\Psi_{N,l,\mu}(\alpha,\Omega)$$
$$= \frac{2^{l+\frac{D-2}{2}}R^{\frac{2-D}{2}}}{(2l+D-3)!}\left[\frac{\sigma_N^2-\tilde{N}^2}{R^2\tilde{N}^2}\frac{(N+l+D-4)!\,\Gamma(\sigma_N+\frac{D-2}{2}+l)}{(N-l-1)!\,\Gamma(\sigma_N-l-\frac{D-4}{2})}\right]^{1/2}$$
$$\times \sinh^l\alpha\exp\big[-\alpha(\sigma_N+l+D-3-N)\big]$$
$$\times\,{}_2F_1\bigg(-N+l+1,l+\frac{D-2}{2}+\sigma_N;2l+D-2;\frac{2}{1+\coth\alpha}\bigg)S_l^\mu(\Omega)\ . \tag{6.10.46}$$

The energy levels are

$$E_N=\frac{e_1e_2}{R}-\hbar^2\frac{\tilde{N}^2-(\frac{D-2}{2})^2}{2mR^2}-\frac{me_1^2e_2^2}{2\hbar^2\tilde{N}^2}\ . \tag{6.10.47}$$

The continuum states read ($\tilde{k}=\sqrt{2mR^2(E_k-e_1e_2\hbar^2/R)}/\hbar$)

$$\Psi_{k,l,\mu}(\alpha,\Omega)=S_l^\mu(\Omega)\frac{2^{(\mathrm{i}/2)(k-\tilde{k})+l+(D-2)/2}}{\pi(2l+D-3)!}\sqrt{\frac{k\sinh\pi k}{2R^{D-1}}}$$
$$\times\,\Gamma\bigg(l+\frac{D-2}{2}+\frac{\mathrm{i}}{2}(\tilde{k}-k)\bigg)\Gamma\bigg(l+\frac{D-2}{2}-\frac{\mathrm{i}}{2}(\tilde{k}+k)\bigg)$$
$$\times\sinh^l\alpha\exp\bigg[\alpha\bigg(\frac{\mathrm{i}}{2}(\tilde{k}+k)-l-\frac{D-2}{2}\bigg)\bigg]$$
$$\times\,{}_2F_1\bigg(l+\tfrac{D-2}{2}+\tfrac{\mathrm{i}}{2}(\tilde{k}-k),l+\tfrac{D-2}{2}-\tfrac{\mathrm{i}}{2}(\tilde{k}+k);2l+D-2;\frac{2}{1+\coth\alpha}\bigg)\ , \tag{6.10.48}$$

with the energy spectrum

$$E_k=\frac{\hbar^2}{2mR^2}\bigg(k^2+\frac{(D-2)^2}{4}\bigg)\ . \tag{6.10.49}$$

6.10.6 Kepler Problem in a Space of Constant Positive Curvature. [59,459] ($\chi\in(0,\pi)$, $\Omega\in S^{(D-2)}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\chi(t')=\chi'}^{\chi(t'')=\chi''}\mathcal{D}\chi(t)(\sin\chi)^{D-2}\int\limits_{\Omega(t')=\Omega'}^{\Omega(t'')=\Omega''}\mathcal{D}\Omega(t)$$
$$\times\exp\bigg[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\bigg(\frac{mR^2}{2}(\dot{\chi}^2+\sin^2\chi\dot{\Omega}^2)+\frac{e_1e_2}{R}\cot\chi\bigg)\mathrm{d}t+\frac{\mathrm{i}\hbar T(D-2)^2}{8mR^2}\bigg]$$
$$=\frac{R^2}{\big(\sin\chi'\sin\chi''\big)^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\Omega'')S_l^\mu(\Omega')$$

$$\times \frac{m}{\hbar^2} \frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times \left(\frac{1}{1+u'} \cdot \frac{1}{1+u''}\right)^{(m_1+m_2+1)/2} \left(\frac{u'}{1+u'} \cdot \frac{u''}{1+u''}\right)^{(m_1-m_2)/2}$$
$$\times {}_2F_1\left(- L_E + m_1, L_E + m_1 + 1; m_1 - m_2 + 1; \frac{u_<}{1+u_<}\right)$$
$$\times {}_2F_1\left(- L_E + m_1, L_E + m_1 + 1; m_1 + m_2 + 1; \frac{1}{1+u_>}\right) , \tag{6.10.50}$$
$$= \sum_{l\in\mathbb{N}_0} \sum_{\mu=1}^{M} \sum_{N=0}^{\infty} \frac{\Psi_{N,l,\mu}(\chi'', \Omega'')\Psi^*_{N,l,\mu}(\chi', \Omega')}{E_N - E} , \tag{6.10.51}$$

with

$$\left.\begin{aligned} L_E &= -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2mR^2}{\hbar^2}\left(\mathrm{i}\,\frac{e_1 e_2}{R} - E\right)} , \\ m_{1,2} &= l + \frac{D-3}{2} \pm \frac{1}{2}\sqrt{-\frac{2mR^2}{\hbar^2}\left(\mathrm{i}\,\frac{e_1 e_2}{R} + E\right)} , \end{aligned}\right\} \tag{6.10.52}$$

and $u = \frac{1}{2}(\mathrm{i}\cot\chi - 1)$. The complete wave functions are given by ($\tilde{N} = N + (D-4)/2, \sigma_N = R/a\tilde{N}, a = \hbar^2/me_1e_2$)

$$\Psi_{N,l,\mu}(\chi, \Omega)$$
$$= \frac{2^{l+\frac{D-2}{2}} R^{\frac{2-D}{2}}}{(2l+D-3)!} \left[\frac{\sigma_N^2 + \tilde{N}^2}{R^2\tilde{N}^2} \frac{(N+l+D-4)!\,\Gamma(\mathrm{i}\sigma_N + \frac{D-2}{2} + l)}{(N-l-1)!\,\Gamma(\mathrm{i}\sigma_N - l - \frac{D-4}{2})}\right]^{1/2}$$
$$\times \sin^l\chi \exp\left[\mathrm{i}\chi(\mathrm{i}\sigma_N + l + D - 3 - N)\right]$$
$$\times {}_2F_1\left(- N + l + 1, l + \tfrac{D-2}{2} + \mathrm{i}\sigma_N; 2l + D - 2; 1 - \mathrm{e}^{2\mathrm{i}\chi}\right) S_l^\mu(\Omega) . \tag{6.10.53}$$

The energy spectrum is

$$E_N = \hbar^2 \frac{(N-1)(N+D-3)}{2mR^2} - \frac{me_1^2e_2^2}{2\hbar^2(N + \frac{D-4}{2})^2} . \tag{6.10.54}$$

6.10.7 Kepler-Like Problem on the Single-Sheeted Hyperboloid.

[442] ($\tau \in \mathbb{R}$, $\Omega \in S^{(D-2)}$)

$$R^{1-D}\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \cosh^{D-2}\tau \int\limits_{\Omega(t')=\Omega'}^{\Omega(t'')=\Omega''} \mathcal{D}\Omega(t)$$

$$\times \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}R^2(\dot{\tau}^2-\cosh^2\tau\dot{\Omega}^2)+\frac{e^2}{R}\tanh\tau\right)dt-\frac{\mathrm{i}}{\hbar}T\frac{\hbar^2(D-2)^2}{8mR^2}\right]$$

$$=\frac{R^{3-D}}{(\cosh\tau'\cosh\tau'')^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\Omega'')S_l^\mu(\Omega')$$

$$\times\frac{m}{\hbar^2}\frac{\Gamma(m_1-L_B)\Gamma(L_B+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

$$\times\left(\frac{1-\tanh\tau'}{2}\frac{1-\tanh\tau''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\tanh\tau'}{2}\frac{1+\tanh\tau''}{2}\right)^{(m_1+m_2)/2}$$

$$\times{}_2F_1\left(-L_B+m_1,L_B+m_1+1;m_1+m_2+1;\frac{1+\tanh\tau_>}{2}\right)$$

$$\times{}_2F_1\left(-L_B+m_1,L_B+m_1+1;m_1-m_2+1;\frac{1-\tanh\tau_<}{2}\right)\ ,\tag{6.10.55}$$

$$=\frac{R^{1-D}}{(\cosh\tau'\cosh\tau'')^{(D-2)/2}}\sum_{l\in\mathbb{N}_0}\sum_{\mu=1}^{M}S_l^\mu(\Omega')S_l^\mu(\Omega')$$

$$\times\left[\sum_{n=0}^{N_M}\frac{\Psi_{n,l}^{(e^2)*}(\tau')\Psi_{n,l}^{(e^2)}(\tau'')}{E_n^{(e^2)}-E}+\int_0^\infty dk\,\frac{\Psi_{k,l}^{(e^2)}(\tau'')\Psi_{k,l}^{(e^2)*}(\tau')}{E_k-E}\right]\ .\tag{6.10.56}$$

Here we denote $L_B=l+\frac{D-4}{2}$, $m_{1,2}=\sqrt{\frac{m}{2}}\,R\Big(\sqrt{-e^2/R-E}\pm\sqrt{e^2/R-E}\Big)/\hbar$, and $\tau_{<,>}$ denotes the smaller/larger of τ',τ'', respectively. The wave functions and the energy spectrum are given by $\Big[s\equiv 2l+D-3$, $n=0,\dots,N_M<l+\frac{D-4}{2}-\sqrt{R/a}$ with $a=\hbar^2/me^2$ the Bohr radius, $k_1=\frac{1}{2}(1+s)$, $k_2=\frac{1}{2}\big[1+\frac{1}{2}(s-2n-1)-\frac{2me^2R}{\hbar(s-2n-1)}\big]$, $u=\frac{1}{2}(1+\tanh\tau)$, note $k_2-\frac{1}{2}>0\Big]$:

$$\Psi_{n,l}^{(e^2)}(\tau)=\left[\left(\frac{1}{R}+\frac{4me^2}{\hbar^2(s-2n-1)^2}\right)\frac{(s-2k_2-2n)n!\,\Gamma(s-n)2^{2n+1-s}}{\Gamma(s+1-n-2k_2)\Gamma(2k_2+n)}\right]^{1/2}$$
$$\times(1-\tanh\tau)^{\frac{1}{2}s-k_2-n}(1+\tanh\tau)^{k_2-\frac{1}{2}}P_n^{(s-2k_2-2n,2k_2-1)}(\tanh\tau)\ ,\tag{6.10.57}$$

$$E_n^{(e^2)}=-\left[\hbar^2\frac{(\frac{s-1}{2}-n)^2-\frac{(D-2)^2}{4}}{2mR^2}+\frac{me^4}{2\hbar^2(\frac{s-1}{2}-n)^2-\frac{(D-2)^2}{4}}\right]\ .\tag{6.10.58}$$

The wave functions and the energy spectrum of the continuum states are given by $[k_2\equiv\frac{1}{2}(1+\mathrm{i}\,\tilde{k})$ $\kappa=\frac{1}{2}(1+\mathrm{i}\,k)$, $\tilde{k}\equiv\sqrt{2mR^2(-2e^2/R+\hbar^2k^2/2mR^2)}/\hbar>0]$:

$$\Psi_{k,l}^{(e^2)}(\tau)=N_k^{(\eta,\nu)}(1-u)^{-\mathrm{i}\,k/2}u^{\mathrm{i}\,\tilde{k}/2}$$

$$\times\, {}_2F_1\Big(\tfrac{1}{2}[1+s+\mathrm{i}(\tilde{k}-k)], \tfrac{1}{2}[1-s+\mathrm{i}(\tilde{k}-k)]; 1+\mathrm{i}\,\tilde{k}; u\Big) \ , \tag{6.10.59}$$

$$N_k^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\sqrt{\frac{k\sinh\pi k}{2\pi^2}}\Big[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)$$

$$\times\, \Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)\Big]^{1/2} \ , \tag{6.10.60}$$

$$E_k = \frac{\hbar^2}{2mR^2}\left(k^2+\frac{(D-2)^2}{4}\right)-\frac{e^2}{R} \ . \tag{6.10.61}$$

6.10.8 Motion in the Hyperbolic Space $\mathrm{SU}(n,1)/\,\mathrm{S}[\mathrm{U}(1)\times\mathrm{U}(n)]$.

In the space $S_2 \simeq \mathrm{SU}(n,1)/\,\mathrm{S}[\mathrm{U}(1)\times\mathrm{U}(n)]$ we have for the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}y^2}{y^2}+\frac{1}{y^2}\sum_{k=2}^{n}\mathrm{d}z_k\mathrm{d}z_k^* + \frac{1}{y^4}\left(\mathrm{d}x_1+\Im\sum_{k=2}^{n} z_k^*\mathrm{d}z_k\right)^2 \ , \tag{6.10.62}$$

$z_k = x_k + \mathrm{i}\, y_k \in \mathbb{C}$ $(k=2,\dots,n)$, $x_1\in\mathbb{R}$, $y>0$, with the hyperbolic distance given by

$$\cosh d(\mathbf{q}'',\mathbf{q}') = \frac{\Big[\big((\mathbf{x}''-\mathbf{x}')^2+y'^2+y''^2\big)^2+4\big(x_1''-x_1'+(\mathbf{x}''\mathbf{y}'-\mathbf{y}''\mathbf{x}')\big)^2\Big]}{4(y'y'')^4} \ . \tag{6.10.63}$$

The symmetry properties of the space give rise to two important coordinate systems, in which the problem is separable, namely $(n-1)$-fold two-dimensional polar coordinates according to

$$\begin{aligned} x_k &= r_k\cos\varphi_k \\ y_k &= r_k\sin\varphi_k \end{aligned} \qquad (r_k>0, 0\le\varphi_k\le 2\pi, k=2,\dots,n) \ , \tag{6.10.64}$$

respectively, $(2n-1)$-dimensional $\mathrm{SU}(n-1)/\,\mathrm{SU}(n-2)$ polar coordinates $(0\le\varphi_i\le 2\pi, i=2,\dots,n; 0\le\vartheta_j\le\pi, j=2,\dots,n-1; r>0)$

$$\left.\begin{aligned} z_n &= r\,\mathrm{e}^{-\varphi_n}\cos\vartheta_{n-1} \\ z_{n-1} &= r\,\mathrm{e}^{-\varphi_{n-1}}\sin\vartheta_{n-1}\cos\vartheta_{n-2} \\ z_{n-2} &= r\,\mathrm{e}^{-\varphi_{n-2}}\sin\vartheta_{n-1}\sin\vartheta_{n-2}\cos\vartheta_{n-3} \\ &\ \vdots \\ z_3 &= r\,\mathrm{e}^{-\varphi_3}\sin\vartheta_{n-1}\dots\sin\vartheta_3\cos\vartheta_2 \\ z_2 &= r\,\mathrm{e}^{-\varphi_2}\sin\vartheta_{n-1}\dots\sin\vartheta_3\sin\vartheta_2 \ . \end{aligned}\right\} \tag{6.10.65}$$

6.10.8.1 Motion in $(n-1)$-Fold Two-Dimensional Polar Coordinates. [435]
$(\mathbf{x}=\{x_k\}_{k=2}^n, \mathbf{y}=\{y_k\}_{k=2}^n, \mathbf{r}=\{r_k\}_{k=2}^n, \boldsymbol{\varphi}=\{\varphi_k\}_{k=2}^n)$

$$\begin{aligned}
&K^{S_2}(\mathbf{x}'',\mathbf{y}'',\mathbf{x}',\mathbf{y}',x_1'',x_1',y'',y';T)=K^{S_2}(\mathbf{r}'',\mathbf{r}',\boldsymbol{\varphi}'',\boldsymbol{\varphi}',x_1'',x_1',y'',y';T)\\
&=\exp\bigg(-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\bigg)\\
&\quad\times\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^{2n+1}}\int\limits_{x_1(t')=x_1'}^{x_1(t'')=x_1''}\mathcal{D}x_1(t)\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}\mathbf{x}(t)\int\limits_{\mathbf{y}(t')=\mathbf{y}'}^{\mathbf{y}(t'')=\mathbf{y}''}\mathcal{D}\mathbf{y}(t)\\
&\quad\times\exp\Bigg[\!\!\Bigg[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg(\frac{m}{2}\Bigg\{\frac{1}{y^2}\bigg(\dot y^2+\sum_{k=2}^n(\dot x_k^2+\dot y_k^2)\bigg)\\
&\qquad+\frac{1}{y^4}\Bigg[\dot x_1^2+2\dot x_1\sum_{k=2}^n(x_k\dot y_k-y_k\dot x_k)+\bigg(\sum_{k=2}^n(x_k\dot y_k-y_k\dot x_k)\bigg)^2\Bigg]\Bigg\}\Bigg)\mathrm{d}t\Bigg]\!\!\Bigg]\\
&=\exp\bigg(-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\bigg)\\
&\quad\times\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^{2n+1}}\int\limits_{x_1(t')=x_1'}^{x_1(t'')=x_1''}\mathcal{D}x_1(t)\prod_{k=2}^n\int\limits_{r_k(t')=r_k'}^{r_k(t'')=r_k''}\mathcal{D}r_k(t)r_k\int\limits_{\boldsymbol{\varphi}(t')=\boldsymbol{\varphi}'}^{\boldsymbol{\varphi}(t'')=\boldsymbol{\varphi}''}\mathcal{D}\boldsymbol{\varphi}(t)\\
&\quad\times\exp\Bigg[\!\!\Bigg[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\Bigg(\frac{m}{2}\Bigg\{\frac{1}{y^2}\bigg(\dot y^2+\sum_{k=2}^n(\dot r_k^2+r_k^2\dot\varphi_k^2)\bigg)\\
&\qquad+\frac{1}{y^4}\Bigg[\dot x_1^2+2\dot x_1\sum_{k=2}^n r_k^2\dot\varphi_k+\bigg(\sum_{k=2}^n r_k^2\dot\varphi_k\bigg)^2\Bigg]\Bigg\}+\frac{\hbar^2y^2}{8m}\sum_{k=2}^n\frac{1}{r_k^2}\Bigg)\mathrm{d}t\Bigg]\!\!\Bigg]\\
&=\int_{\mathbb{R}}\mathrm{d}k_1\prod_{k=2}^n\sum_{l_k\in\mathbb{Z}}\sum_{n_k\in\mathbb{N}_0}\int_0^\infty\mathrm{d}k\\
&\quad\times\Psi_{k,k_1,\mathbf{l},\mathbf{n}}(\mathbf{r}'',\boldsymbol{\varphi}'',x_1'',y'')\Psi^*_{k,k_1,\mathbf{l},\mathbf{n}}(\mathbf{r}',\boldsymbol{\varphi}',x_1',y')\,\mathrm{e}^{-\mathrm{i}E_kT/\hbar}\ ,
\end{aligned}\tag{6.10.66}$$

with the energy spectrum

$$E_k=\frac{\hbar^2}{2m}(k^2+n^2)\ ,\tag{6.10.67}$$

and the wave functions

$$\Psi_{k,k_1,\mathbf{l},\mathbf{n}}(\mathbf{r},\boldsymbol{\varphi},x_1,y)=\frac{\mathrm{e}^{\mathrm{i}(k_1x_1+l_k\varphi_k)}}{(2\pi)^{n/2}}\frac{1}{\sqrt{r_k}}R_{n_k}^{l_k}(r_k)\Phi_k(y)\ .\tag{6.10.68}$$

Here we have defined

$$R_n^l(r) = \sqrt{\frac{2|k_1|n!}{\Gamma(n+|l|+1)}}\,(|k_1|r)^{|l|}\exp(-|k_1|r)L_n^{(|l|)}(|k_1|r^2) \ , \tag{6.10.69}$$

$$\Phi_k(y) = \sqrt{\frac{k\sinh\pi k}{2\pi^2|k_1|}}\,\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}\,k+\frac{mE_\lambda}{|k_1|}\right)\right]y^{n-1}W_{-mE_\lambda/2|k_1|,\mathrm{i}\,k/2}(|k_1|y^2) \ , \tag{6.10.70}$$

$$E_\lambda = \frac{\hbar^2}{m}\sum_{k=2}^n \left[|k_1|(2n_k+1)+|k_1l_k|-k_1l_k\right] \ . \tag{6.10.71}$$

The Green function can be written as follows ($\mathbf{x}=\{x_k\}_{k=2}^n, \mathbf{y}=\{y_k\}_{k=2}^n, \boldsymbol{\vartheta}=\{\vartheta_k\}_{k=2}^n, \boldsymbol{\varphi}=\{\varphi_k\}_{k=2}^n$)

$$\begin{aligned}
G^{S_2}&(\mathbf{x}'',\mathbf{y}'',\mathbf{x}',\mathbf{y}',x_1'',x_1',y'',y';E) = \frac{m}{\hbar}(y'y'')^{2-n}\int_{\mathbb{R}}\frac{\mathrm{d}k_1}{k_1}\,\mathrm{e}^{\mathrm{i}\,k_1(x_1''-x_1')}\\
&\times\int_0^\infty \mathrm{d}u\left(\frac{|k_1|}{2\pi\,\mathrm{i}\sin u}\right)^n I_{-\mathrm{i}\sqrt{2mE/\hbar^2-n^2}}\left(\frac{y'y''|k_1|}{\mathrm{i}\sin u}\right)\\
&\times\exp\left[\!\!\left[-\frac{|k_1|}{2\,\mathrm{i}}\left\{\left[\sum_{k=2}^n\left((x_k''-x_k')^2+(y_k''-y_k')^2\right)+y'^2+y''^2\right]\cot u\right.\right.\right.\\
&\left.\left.\left.\qquad\qquad+2\sum_{k=2}^n\left(x_k'y_k''-x_k''y_k'\right)\right\}\right]\!\!\right] \ .
\end{aligned}\tag{6.10.72}$$

6.10.8.2 Motion in SU$(n)/$SU$(n-1)$*-Spherical Polar Coordinates.* [435]

$$\begin{aligned}
&K^{S_2}(\mathbf{x}'',\mathbf{y}'',\mathbf{x}',\mathbf{y}',x_1'',x_1',y'',y';T)\\
&= K^{S_2}(r'',r',\boldsymbol{\vartheta}'',\boldsymbol{\vartheta}',\boldsymbol{\varphi}'',\boldsymbol{\varphi}',x_1'',x_1',y'',y';T) = \frac{y'y''}{2\pi}\int_{\mathbb{R}}\mathrm{d}k_1\,\mathrm{e}^{\mathrm{i}\,k(x_1''-x_1')}\\
&\times\exp\left(-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right)\int\limits_{y(t')=y'}^{y(t'')=y''}\frac{\mathcal{D}y(t)}{y^{2n-1}}\int\limits_{r(t')=r'}^{r(t'')=r''}\mathcal{D}r(t)r^{2n-3}\\
&\times\int\limits_{\boldsymbol{\vartheta}(t')=\boldsymbol{\vartheta}'}^{\boldsymbol{\vartheta}(t'')=\boldsymbol{\vartheta}''}\mathcal{D}\boldsymbol{\vartheta}(t)\prod_{k=2}^n\left(\cos\vartheta_k(\sin\vartheta_k)^{2k-3}\right)\int\limits_{\boldsymbol{\varphi}(t')=\boldsymbol{\varphi}'}^{\boldsymbol{\varphi}(t'')=\boldsymbol{\varphi}''}\mathcal{D}\boldsymbol{\varphi}(t)\\
&\times\exp\left[\!\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2y^2}\left\{\dot{y}^2+\dot{r}^2+r^2\left[\dot{\vartheta}_{n-1}^2+\cos^2\vartheta_{n-1}\dot{\varphi}_n^2+\dots\right.\right.\right.\right.\right.\\
&\qquad\qquad\left.\left.\dots+\sin^2\vartheta_3\left(\dot{\vartheta}_3^2+\cos^2\vartheta_3\dot{\varphi}_3^2+\sin^2\vartheta_2\dot{\varphi}_2^2\right)\dots\right]\right\}\\
&\quad-\frac{\hbar^2k_1^2}{2m}y^4+\hbar k_1r^2\left\{\dot{\varphi}_n\cos^2\vartheta_{n-1}+\sin^2\vartheta_{n-1}\left[\dot{\varphi}_{n-1}\sin^2\vartheta_{n-2}+\dots\right.\right.
\end{aligned}$$

$$\ldots + \sin^2\vartheta_3\big(\dot\varphi_3\cos^2\vartheta_2 + \dot\varphi_2\sin^2\vartheta_2\big)\ldots]\big\} + \frac{\hbar^2 y^2}{8mr^2}\bigg[1 + \frac{1}{\cos^2\vartheta_{n-1}}$$

$$+ \ldots + \frac{1}{\sin^2\vartheta_3}\bigg(1 + \frac{1}{\cos^2\vartheta_2} + \frac{1}{\sin^2\vartheta_2}\bigg)\ldots\bigg]\bigg)dt\bigg]\bigg]$$

$$= \int_{\mathbb{R}} dk_1 \sum_{\mathbf{L}} \sum_{N\in\mathbb{N}_0} \int_0^\infty dk\, e^{-iE_kT/\hbar}$$

$$\times \Psi_{k,k_1,\mathbf{L},N}(x_1'', \boldsymbol{\vartheta}'', \boldsymbol{\varphi}'', r'', y'')\Psi^*_{k,k_1,\mathbf{L},N}(x_1', \boldsymbol{\vartheta}', \boldsymbol{\varphi}', r', y') \ , \tag{6.10.73}$$

with the same energy spectrum as in the previous example, and the wave functions

$$\Psi_{k,k_1,\mathbf{L},N}(x_1, \boldsymbol{\vartheta}, \boldsymbol{\varphi}, r, y) = \frac{e^{ik_1x_1}}{\sqrt{2\pi}}\Psi_{\mathbf{L}}^{(n-1)}(\boldsymbol{\vartheta}, \boldsymbol{\varphi})R_N^{L+n-2}(r)\Phi_k(y) \tag{6.10.74}$$

in the notation of the previous example for $R_n^l(r)$ and $\Phi_k(y)$, respectively.

6.10.9 Motion in Hyperbolic Spaces of Rank One. [435]
It is possible to push the path integral analysis even further for all hyperbolic spaces of rank one. The crucial observation is that a separation in polar and angular variables is always possible. For our purposes it is sufficient to consider the relevant Laplacian which can be cast into the form

$$\Delta_{\mathrm{LB}}^{G/K} = \frac{\partial^2}{\partial\tau^2} + (m_\alpha\coth\tau + 2m_{2\alpha}\coth 2\tau)\frac{\partial}{\partial\tau}$$

$$- \left[\frac{\mathcal{L}^{(\mu^+)}}{\sinh^2\tau} + \left(\frac{1}{\sinh^2 2\tau} - \frac{1}{\sinh^2\tau}\right)\mathcal{L}^{(2\mu)}\right] \ . \tag{6.10.75}$$

The operators $\mathcal{L}^{(\mu^+)}$ and $\mathcal{L}^{(2\mu)}$ act on the space of root systems $\mathfrak{g}(\alpha^+)$ (all positive roots) and $\mathfrak{g}(2\alpha)$, respectively. Theses subspaces have dimension m_α and $m_{2\alpha}$, respectively. Here it is understood that $X = G/K$ is a quotient space of the Gelfand pair (G, K) and $\mathfrak{g}$ denotes the algebra on G. For more details on notation and relevant references see Ref. [435]. It is found that the operators $\mathcal{L}^{(\mu^+)}$ and $\mathcal{L}^{(2\mu)}$ have eigenvalues $4l(l + m_{2\alpha} - 1) + lm_\alpha$ and $4l(l+m_{2\alpha}-1)$, respectively, with common quantum number $l \in \mathbb{N}_0$. Therefore we have by means of the path integral solution for the modified Pöschl–Teller potential the path integral solution in the hyperbolic polar coordinates $\tau > 0$ [435]

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \\
&\times \int\limits_{\tau(t')=\tau'}^{\tau(t'')=\tau''} \mathcal{D}\tau(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{8m}\left(\frac{(2l+m_\alpha+m_{2\alpha}-1)^2-1}{\sinh^2\tau}\right.\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\left.\left. + \frac{(2l+m_{2\alpha}-1)^2-1}{\cosh^2\tau}\right)\right]\mathrm{d}t\right\} \\
&= \frac{m}{\hbar^2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\
&\quad \times \big(\cosh r'\cosh r''\big)^{-(m_1-m_2)}\big(\tanh r'\tanh r''\big)^{m_1+m_2+1/2} \\
&\quad \times {}_2F_1\left(-L_\nu+m_1, L_\nu+m_1+1; m_1-m_2+1; \frac{1}{\cosh^2 r_<}\right) \\
&\quad \times {}_2F_1\left(-L_\nu+m_1, L_\nu+m_1+1; m_1+m_2+1; \tanh^2 r_>\right) \ ,
\end{aligned}
\tag{6.10.76}
$$

$$
= \int_0^\infty \mathrm{d}k\, \frac{\Psi_k^{G/K}(\tau'')\Psi_k^{G/K\,*}(\tau')}{E_k^{G/K}-E} \ , \tag{6.10.77}
$$

($m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2mE}/\hbar$, $L_\nu = \frac{1}{2}(\nu-1)$, $\eta = 2l+m_\alpha+m_{2\alpha}-1$, $\nu = 2l+m_{2\alpha}-1$), with energy spectrum

$$
E_k^{G/K} = \frac{\hbar^2}{2m}k^2 + E_0^{G/K} \ , \qquad E_0^{G/K} = \frac{\hbar^2}{8m}(m_\alpha+2m_{2\alpha})^2 , \tag{6.10.78}
$$

and the wave functions are given by

$$
\begin{aligned}
\Psi_k^{G/K}(\tau) &= N_k^{G/K}(\tanh\tau)^{l+\frac{1}{2}(m_\alpha+m_{2\alpha})}(\cosh\tau)^{\mathrm{i}k} \\
&\times {}_2F_1\left[l+\tfrac{1}{2}\left(\tfrac{m_\alpha}{2}+m_{2\alpha}-\mathrm{i}k\right), \tfrac{1}{2}\left(\tfrac{m_\alpha}{2}+1-\mathrm{i}k\right); \tfrac{1}{2}(m_\alpha+m_{2\alpha}+1); \tanh^2\tau\right] ,
\end{aligned}
\tag{6.10.79a}
$$

$$
N_k^{G/K} = \sqrt{\frac{k\sinh\pi k}{2\pi^2}}\,\frac{\Gamma[l+\frac{1}{2}(\frac{m_\alpha}{2}+m_{2\alpha}+\mathrm{i}k)]\Gamma[\frac{1}{2}(\frac{m_\alpha}{2}+1+\mathrm{i}k)]}{\Gamma[l+\frac{1}{2}(m_\alpha+m_{2\alpha}+1)]} \ . \tag{6.10.79b}
$$

6.11 Explicit Time-Dependent Problems

6.11.1 Transformation Formulæ.

6.11.1.1 General Transformation Formulæ. [126,180,251,259,294,440,640,643, 737,847,870] ($\varrho'' = \varrho(t'')$, $\alpha'' = \alpha(t'')$, etc.; we consider $y = f(t)(x - \alpha(t))$)

$$\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V_0(x)\right)dt\right] \\
&= \frac{1}{\sqrt{\varrho(t'')\varrho(t')}} \exp\left\{-\frac{i}{\hbar}\int_{t'}^{t''}\left[h(t) + \frac{m}{2}\left(\dot{\alpha}(t) - \frac{\dot{\varrho}(t)}{\varrho(t)}\alpha(t)\right)dt\right]\right\} \\
&\quad\times \exp\left\{\frac{i\,m}{\hbar}\left[\left(\dot{\alpha}'' - \frac{\dot{\varrho}''}{\varrho''}x''\right) - \left(\dot{\alpha}' - \frac{\dot{\varrho}'}{\varrho''}x'\right)\right] + \frac{1}{2}\left(\frac{\dot{\varrho}''}{\varrho''}x''^2 - \frac{\dot{\varrho}'}{\varrho'}x'^2\right)\right]\right\} \\
&\quad\times K^{(V)}\left(\frac{x'' - \alpha''}{\varrho''}, \frac{x' - \alpha'}{\varrho'}; \tau(t''), \tau(t')\right) \ .
\end{aligned} \tag{6.11.1}$$

Here we have introduced the notation

$$K^{(V)}(y'', y'; \tau'', \tau') = \int\limits_{y(\tau')=y'}^{y(\tau'')=y''} \mathcal{D}y(t) \exp\left[\frac{i}{\hbar}\int_{\tau'}^{\tau''}\left(\frac{m}{2}\dot{y}^2 - V(y,t)\right)dt\right] \ , \tag{6.11.2}$$

$$V(y,t) = \frac{1}{\varrho^2(t)} V_0\left(\frac{x - \alpha(t)}{\varrho(t)}\right) - F(t)x + \frac{m}{2}\omega^2(t)x^2 + h(t) \ , \tag{6.11.3}$$

$$\omega^2(t) = \frac{f(t)\ddot{f}(t) - 2\dot{f}^2(t)}{f^2(t)} \ , \quad f(t) = \frac{1}{\varrho(t)} \ , \quad \tau(t) = \int_0^t \frac{d\sigma}{\varrho^2(\sigma)} \ , \tag{6.11.4}$$

$$F(t) = m\left(\ddot{\alpha}(t) + \alpha(t)\frac{f(t)\ddot{f}(t) - 2\dot{f}^2(t)}{f^2(t)}\right) \ , \tag{6.11.5}$$

$$h(t) = m\left(\frac{f(t)\ddot{f}(t) - 2\dot{f}^2(t)}{2f^2(t)}\alpha^2(t) + \ddot{\alpha}(t)\alpha(t)\right) \ . \tag{6.11.6}$$

6.11.1.2 Explicit Time-Dependent Potentials. [180,251,259,440]
For the explicit time-dependence we take $\zeta(t) = \sqrt{at^2 + 2bt + c}$ ($\zeta' = \zeta(t')$, $\zeta'' = \zeta(t'')$, etc.)

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2 - \frac{1}{\zeta^2(t)}V\left(\frac{\mathbf{x}}{\zeta(t)}\right)\right]dt\right\}$$

$$= (\zeta''\zeta')^{-D/2} \exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\mathbf{x}''^2\frac{\dot\zeta''}{\zeta''} - \mathbf{x}'^2\frac{\dot\zeta'}{\zeta'}\right)\right] K_{\omega',V}\left(\frac{\mathbf{x}''}{\zeta''},\frac{\mathbf{x}'}{\zeta'};\int_{t'}^{t''}\frac{\mathrm{d}t}{\zeta^2(t)}\right) \tag{6.11.7}$$

$$= (\zeta''\zeta')^{-D/2} \exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\mathbf{x}''^2\frac{\dot\zeta''}{\zeta''} - \mathbf{x}'^2\frac{\dot\zeta'}{\zeta'}\right)\right]$$
$$\times \int \mathrm{d}E_\lambda \Psi_\lambda\left(\frac{\mathbf{x}''}{\zeta''}\right)\Psi_\lambda^*\left(\frac{\mathbf{x}'}{\zeta'}\right)\exp\left(-\frac{\mathrm{i}E_\lambda}{\hbar}\int_{t'}^{t''}\frac{\mathrm{d}t}{\zeta^2(t)}\right) , \tag{6.11.8}$$

where $\int \mathrm{d}E_\lambda$ denotes a Lebesgue–Stieltjes integral to include bound and scattering states Ψ_λ with energy E_λ of the corresponding time-independent problem, and with the path integral $K_{\omega',V}$ given by

$$K_{\omega',V}(\mathbf{z}'',\mathbf{z}';s'') = \int\limits_{\mathbf{z}(0)=\mathbf{z}'}^{\mathbf{z}(s'')=\mathbf{z}''} \mathcal{D}\mathbf{z}(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\mathbf{z}}^2 - \frac{m}{2}\omega'^2\mathbf{z}^2 - V(\mathbf{z})\right]\mathrm{d}s\right\} \tag{6.11.9}$$

$\omega'^2 = ac - b^2$ and $s'' = \tau(t'')$, where

$$\tau(t'') = \int_{t'}^{t''}\frac{dt}{\zeta^2(t)}\left\{\begin{array}{ll} = \dfrac{1}{\omega'}\arctan\dfrac{at+b}{\omega'}\Bigg|_{t'}^{t''} & (\omega'^2 > 0)\ , \\ = -\dfrac{1}{|\omega'|}\operatorname{artanh}\dfrac{at+b}{|\omega'|}\Bigg|_{t'}^{t''} & (\omega'^2 < 0)\ , \\ = \dfrac{a}{b}\dfrac{t}{at+b}\Bigg|_{t'}^{t''} & (\omega'^2 = 0)\ . \end{array}\right\} \tag{6.11.10}$$

6.11.1.3 Moving Potentials (Extended Galilean Transformation). [276,322,440, 497] ($q' = x' - f'$, $f' = f(t')$, etc.)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot x^2 - V(x-f(t))\right)\mathrm{d}t\right]$$
$$= \exp\left[\frac{\mathrm{i}m}{\hbar}\left(\dot f''(x''-f'') - \dot f'(x'-f') + \frac{1}{2}\int_{t'}^{t''}\dot f^2(t)\,\mathrm{d}t\right)\right]$$
$$\times \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot q^2 - V(q) - m\ddot f(t)q\right)\mathrm{d}t\right] . \tag{6.11.11}$$

6.11.2 Examples.

6.11.2.1 Time-Dependent Harmonic Oscillator. [180,259,440]
($\zeta(t) = \sqrt{at^2 + 2bt + c}$, $\Omega^2 = \omega^2 + \omega'^2$, τ, ω' as in (6.11.10))

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\left(\dot{x}^2 - \frac{\omega^2}{\zeta^4(t)}x^2\right)\mathrm{d}t\right]$$
$$= (\zeta'\zeta'')^{-1/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\left(x''^2\frac{\dot{\zeta}''}{\zeta''} - x'^2\frac{\dot{\zeta}'}{\zeta'}\right)\right]\left(\frac{m\Omega}{2\pi\,\mathrm{i}\,\hbar\sin\Omega\tau(t'')}\right)^{1/2}$$
$$\times \exp\left\{-\frac{m\Omega}{2\,\mathrm{i}\,\hbar}\left[\left(\frac{x''^2}{\zeta''^2} + \frac{x'^2}{\zeta'^2}\right)\cot\Omega\tau(t'') - \frac{2x'x''}{\zeta'\zeta''\sin\Omega\tau(t'')}\right]\right\} . \tag{6.11.12}$$

6.11.2.2 Time-Dependent Radial Harmonic Oscillator. [251,259,440] (τ as in (6.11.10), $\zeta(t) = \sqrt{at^2 + 2bt + c}$, $\tau(t)$ and Ω as in the previous example)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\lambda[r^2] \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\left(\dot{r}^2 - \frac{\omega^2}{\zeta^4(t)}r^2\right)\mathrm{d}t\right]$$
$$= \left(\frac{r'r''}{\zeta'\zeta''}\right)^{1/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\left(r''^2\frac{\dot{\zeta}''}{\zeta''} - r'^2\frac{\dot{\zeta}'}{\zeta'}\right)\right]\left(\frac{m\Omega}{\mathrm{i}\,\hbar\sin\Omega\tau(t'')}\right)^{1/2}$$
$$\times \exp\left[-\frac{m\Omega}{2\,\mathrm{i}\,\hbar}\left(\frac{r''^2}{\zeta''^2} + \frac{r'^2}{\zeta'^2}\right)\cot\Omega\tau(t'')\right] I_\lambda\left(\frac{m\Omega r'r''}{\mathrm{i}\,\hbar\zeta'\zeta''\sin\Omega\tau(t'')}\right) . \tag{6.11.13}$$

6.11.2.3 Time-Dependent δ-Function Perturbation. [259,440]
We consider a time-dependent δ-function perturbation according to $V(x) = -\gamma\delta(x)/\zeta(t)$, $\zeta(t) = \sqrt{at^2 + 2bt + c}$, however with $\omega'^2 = ac - b^2 = 0$. We have the path integral identity (note $\tau(t) = t/\zeta(t)$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{\gamma}{\zeta(t)}\delta(x)\right)\mathrm{d}t\right]$$
$$= (\zeta'\zeta'')^{-1/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar}\left(x''^2\frac{\dot{\zeta}''}{\zeta''} - x'^2\frac{\dot{\zeta}'}{\zeta'}\right)\right]$$
$$\times \left\{\left(\frac{m}{2\pi\,\mathrm{i}\,\hbar\tau(t'')}\right)^{1/2} \exp\left[\frac{\mathrm{i}\, m}{2\hbar\tau(t'')}\left(\frac{x''}{\zeta''} - \frac{x'}{\zeta'}\right)^2\right]\right.$$

$$+\frac{m\gamma}{2\hbar^2}\exp\left[-\frac{m\gamma}{\hbar^2}\left(\frac{|x''|}{\zeta''}+\frac{|x'|}{\zeta'}\right)+\frac{\mathrm{i}}{\hbar}\frac{m\gamma^2}{2\hbar^2}\tau(t'')\right]$$
$$\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar\tau(t'')}}\left(\frac{|x''|}{\zeta''}+\frac{|x'|}{\zeta'}-\frac{\mathrm{i}}{\hbar}\gamma\tau(t'')\right)\right]\Bigg\}\ . \tag{6.11.14}$$

6.11.2.4 Hard Wall Potential. [180,259]
We consider the example of a half-line (HL), i.e., $L(t)\le x<\infty$, with the boundary moving according to $L(t)=L_0\zeta(t)$. The result then has the form [$\omega'=0$, τ as in (6.11.10)]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(\mathrm{HL},\zeta(t))}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{x}^2\mathrm{d}t\right)$$
$$=\frac{(\zeta'\zeta'')^{-1/2}}{2L_0}\sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar\tau(t'')}}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(x''^2\frac{\dot\zeta''}{\zeta''}-x'^2\frac{\dot\zeta'}{\zeta'}\right)\right]$$
$$\times\left\{\exp\left[\frac{\mathrm{i}\,m}{2\hbar\tau(t'')}\left(\frac{x''}{\zeta''}-\frac{x'}{\zeta'}\right)^2\right]-\exp\left[\frac{\mathrm{i}\,m}{2\hbar\tau(t'')}\left(\frac{x''}{\zeta''}+\frac{x'}{\zeta'}\right)^2\right]\right\} \tag{6.11.15}$$
$$=\frac{(\zeta'\zeta'')^{-1/2}}{2L_0}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(x''^2\frac{\dot\zeta''}{\zeta''}-x'^2\frac{\dot\zeta'}{\zeta'}\right)\right]$$
$$\times\frac{2}{\pi}\int_0^\infty\mathrm{d}k\sin\left(k\frac{x''}{\zeta''}\right)\sin\left(k\frac{x'}{\zeta'}\right)\exp\left(-\,\mathrm{i}\,\hbar\tau(t'')\frac{k^2}{2m}\right)\ . \tag{6.11.16}$$

6.11.2.5 Rigid Box with One Wall Moving. [213,259]
We consider the example of the infinite well (IW) with one boundary fixed at $x=0$, and the other moving according to $L(t)=L_0\zeta(t)$. The result then has the form [$\omega'=0$, $\Theta_3(z,\tau)$ denotes a Jacobi theta function, τ as in (6.11.10)]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(\mathrm{IW},\zeta(t))}x(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_0^{t''}\dot{x}^2\mathrm{d}t\right)$$
$$=\frac{(\zeta'\zeta'')^{-1/2}}{2L_0}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(x''^2\frac{\dot\zeta''}{\zeta''}-x'^2\frac{\dot\zeta'}{\zeta'}\right)\right]$$
$$\times\left[\Theta_3\left(\frac{x''/\zeta''-x'/\zeta'}{2L_0},-\frac{\pi\hbar\tau(t'')}{2mL_0^2}\right)-\Theta_3\left(\frac{x''/\zeta''+x'/\zeta'}{2L_0},-\frac{\pi\hbar\tau(t'')}{2mL_0^2}\right)\right] \tag{6.11.17}$$

$$= \frac{(\zeta'\zeta'')^{-1/2}}{2L_0}\exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(x''^2\frac{\dot\zeta''}{\zeta''}-x'^2\frac{\dot\zeta'}{\zeta'}\right)\right]$$

$$\times\sum_{n\in\mathbb{N}}\sin\left(\frac{\pi n}{L_0}\frac{x''}{\zeta''}\right)\sin\left(\frac{\pi n}{L_0}\frac{x'}{\zeta'}\right)\exp\left(-\mathrm{i}\,\hbar\frac{\pi^2n^2}{2mL_0^2}\tau(t'')\right)\ . \tag{6.11.18}$$

6.11.2.6 Moving δ-Function Perturbation. [276,440]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot x^2+\gamma\delta(x-vt)\right]\mathrm{d}t\right\}$$

$$= \sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}}\exp\left[\frac{\mathrm{i}\,m}{2\hbar T}(x''-x')^2\right]$$

$$+\frac{m\gamma}{2\hbar^2}\exp\left\{\frac{\mathrm{i}}{\hbar}\left[v(x''-vt'')-v(x'-vt')+\frac{m}{2}v^2T\right]\right\}$$

$$\times\exp\left[-\frac{m\gamma}{\hbar^2}(|x''-vt''|+|x'-vt'|)+\frac{\mathrm{i}}{\hbar}\frac{m\gamma^2}{2\hbar}T\right]$$

$$\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-vt''|+|x'-vt'|-\frac{\mathrm{i}}{\hbar}\gamma T\right)\right]\ , \tag{6.11.19}$$

$$= \frac{m\gamma}{\hbar^2}\exp\left[-\frac{m\gamma}{\hbar^2}\Big(|x''-vt''|+|x'-vt'|\Big)\right.$$

$$\left.-\frac{\mathrm{i}\,mv}{\hbar}(x'-vt')+\frac{\mathrm{i}\,mv}{\hbar}(x''-vt'')+\frac{\mathrm{i}\,m}{2\hbar}\left(\frac{\gamma^2}{\hbar^2}+v^2\right)T\right]$$

$$+\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}k\,\exp\left(-\frac{\mathrm{i}\,k^2\hbar T}{2m}\right)\left\{\mathrm{e}^{\mathrm{i}\,k(x''-x')}\right.$$

$$\left.-\frac{\exp\left[\mathrm{i}\,k(|x''-vt''|+|x'-vt'|)+\dfrac{\mathrm{i}\,mv}{\hbar}\left(x''-x'-\dfrac{vT}{2}\right)\right]}{1+\mathrm{i}\,\dfrac{k\hbar^2}{m\gamma}}\right\}\ . \tag{6.11.20}$$

6.11.2.7 Time Dependent Hydrogen-Like System. [874] ($z'=2x'\,\mathrm{e}^{-\alpha t'}$, etc.)

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left(\dot x^2+\frac{\alpha^2}{4}x^2-\alpha x\dot x\right)-g\frac{\mathrm{e}^{-\alpha t/2}}{x}\right]\mathrm{d}t\right\}$$

$$= \alpha\sqrt{z'z''}\exp\left(-\alpha\frac{t'+t''}{4}\right)$$

$$\times\left\{\int_{-\infty}^{0}\frac{d\nu}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\nu\left(e^{-\alpha t''}-e^{-\alpha t'}\right)\right]\left(2g+\hbar\sqrt{-\frac{2\nu\alpha}{m}}(2n+2)\right)^{-1}\right.$$

$$\times 2\left(\frac{m}{\hbar}\sqrt{-\frac{2\nu\alpha}{m}}\right)^{2}(z'z'')^{3/2}\exp\left[-\frac{m}{2\hbar}\sqrt{-\frac{2\nu\alpha}{m}}\,(z'^2+z''^2)\right]$$

$$\times\sum_{n\in\mathbb{N}_0}\frac{1}{n+1}L_n^{(1)}\left(\frac{m}{\hbar}\sqrt{-\frac{2\nu\alpha}{\hbar}}\,z'^2\right)L_n^{(1)}\left(\frac{m}{\hbar}\sqrt{-\frac{2\nu\alpha}{\hbar}}\,z''^2\right)$$

$$\left.+\int_{0}^{\infty}\frac{d\nu}{2\pi\hbar}\exp\left[\frac{i}{\hbar}\nu\left(e^{-\alpha t''}-e^{-\alpha t'}\right)\right]\int_{\mathbb{R}}d\tilde{E}\frac{\Psi_{\tilde{E}}(z'')\Psi^*_{\tilde{E}}(z')}{2g+\tilde{E}}\right\}. \qquad (6.11.21)$$

The wave functions $\Psi_{\tilde{E}}(z)$ ($c=(\hbar^2/2\nu\alpha m)^{1/4}$, $\lambda = m\tilde{E}/\hbar\sqrt{2\nu\alpha m}$) are

$$\Psi_{\tilde{E}}(z) = \frac{c}{\hbar}\sqrt{\frac{m}{2\pi z}}\,e^{i\pi+\pi\lambda/4}\,2^{i\lambda/2}\Gamma(1+\tfrac{i}{2}\lambda)W_{i\lambda/2,1/2}(-i z^2/c^2)\ . \qquad (6.11.22)$$

6.12 Point Interactions

6.12.1 One-Dimensional Case. [17,371,430,801] ($G^{(V)}$ denotes G for the case $\gamma = 0$)

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)+\gamma\delta(x-a)\right)dt\right]$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)-1/\gamma}\ . \qquad (6.12.1)$$

6.12.1.1.1 The Free Particle: Feynman Kernel. [101,208,371,404,640,641]

$$\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\gamma\delta(x-a)\right)dt\right]$$

$$= \sqrt{\frac{m}{2\pi i\hbar T}}\exp\left[\frac{i m}{2\hbar T}(x''-x')^2\right]$$

$$+\frac{m\gamma}{2\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{i}{\hbar}\frac{m\gamma^2}{2\hbar^2}T\right)$$

$$\times\,\mathrm{erfc}\left[\sqrt{\frac{m}{2i\hbar T}}\left(|x''-a|+|x'-a|-\frac{i}{\hbar}\gamma T\right)\right]\ , \qquad (6.12.2)$$

$$= \frac{m\gamma}{\hbar^2}\exp\left[-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{\mathrm{i}}{\hbar}\frac{m\gamma^2}{2\hbar^2}T\right]$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}k\exp\left(-\mathrm{i}\frac{k^2\hbar}{2m}T\right)$$
$$\times\left(\sin kx''\sin kx'+\cos kx''\cos kx'-\frac{\mathrm{e}^{\mathrm{i}k(|x''-a|+|x'-a|)}}{1+\mathrm{i}k\hbar^2/m\gamma}\right) . \tag{6.12.3}$$

6.12.1.1.2 The Free Particle: Green Function. [65,404-406,430,641,642,672]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\gamma\delta(x-a)\right)\mathrm{d}t\right]$$
$$=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''-x'|\right)$$
$$+\frac{m\gamma}{2\hbar^2}\frac{\exp\left[-\frac{\sqrt{-2mE}}{\hbar}(|x''-a|+|a-x'|)\right]}{\sqrt{-E}\left(\sqrt{-E}-\frac{\gamma}{\hbar}\sqrt{\frac{m}{2}}\right)} . \tag{6.12.4}$$

6.12.1.2 The Free Particle on the Half-Line. [430]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(x>0)}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\gamma\delta(x-a)\right)\mathrm{d}t\right]$$
$$=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\left(\mathrm{e}^{-|x''-x'|\sqrt{-2mE}/\hbar}-\mathrm{e}^{-|x''+x'|\sqrt{-2mE}/\hbar}\right)$$
$$+\frac{m\gamma}{2\hbar^2}\left(\mathrm{e}^{-|x''-a|\sqrt{-2mE}/\hbar}-\mathrm{e}^{-|x''+a|\sqrt{-2mE}/\hbar}\right)$$
$$\times\left(\mathrm{e}^{-|a-x'|\sqrt{-2mE}/\hbar}-\mathrm{e}^{-|a+x'|\sqrt{-2mE}/\hbar}\right)$$
$$\times\left\{\sqrt{-E}\left[\sqrt{-E}-\frac{\gamma}{\hbar}\sqrt{\frac{m}{2}}\left(1-\mathrm{e}^{-2a\sqrt{-2mE}/\hbar}\right)\right]\right\}^{-1} . \tag{6.12.5}$$

6.12.1.3 The Infinite Potential Well. [430]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{(|x|<b)}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\gamma\delta(x-a)\right)\mathrm{d}t\right]$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(x''-x'-2b)] - \cosh[\frac{1}{\hbar}\sqrt{-2mE}\,(x''+x')]}{\sinh(\sqrt{-2mE}\,\frac{2b}{\hbar})}$$
$$- \frac{m\gamma}{2E\hbar^2}\left[\cosh\left(\sqrt{-2mE}\,\frac{x''-a-2b}{\hbar}\right) - \cosh\left(\sqrt{-2mE}\,\frac{x''+a}{\hbar}\right)\right]$$
$$\times\left[\cosh\left(\sqrt{-2mE}\,\frac{a-x'-2b}{\hbar}\right) - \cosh\left(\sqrt{-2mE}\,\frac{a+x'}{\hbar}\right)\right]$$
$$\times\left[\!\!\left[\sinh\left(\sqrt{-2mE}\,\frac{2b}{\hbar}\right)\left\{\sinh\left(\sqrt{-2mE}\,\frac{2b}{\hbar}\right)\right.\right.\right.$$
$$\left.\left.\left. - \frac{\gamma}{\hbar}\sqrt{-\frac{m}{2E}}\left[\cosh\left(\sqrt{-2mE}\,\frac{2b}{\hbar}\right) - \cosh\left(\sqrt{-2mE}\,\frac{2a}{\hbar}\right)\right]\right\}\right]\!\!\right]^{-1} . \tag{6.12.6}$$

6.12.1.4 The Reflectionless Potential. [430]
We set

$$G^{(RL)}(x'',x';E)$$
$$= \frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, e^{\mathrm{i}TE/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m}\frac{N(N+1)}{\cosh^2 x}\right)dt\right]$$
$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\exp\left(-|x''-x'|\frac{\sqrt{-2mE}}{\hbar}\right)$$
$$+\frac{1}{2}\sum_{n=0}^{N-1}\frac{(N-n)(2N+1-n)!}{-\hbar^2(N-n)^2/2m - E}\frac{1}{n!}P_N^{n-N}(\tanh x')P_N^{n-N}(\tanh x'')$$
$$\times\left\{1 - \left(1 - \frac{\hbar(N-n)}{\sqrt{-2mE}}\right)\cosh\left[|x''-x'|\left(N-n+\frac{\sqrt{-2mE}}{\hbar}\right)\right]\right\} , \tag{6.12.7}$$

$$\stackrel{(N=1)}{=} \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\exp\left(-\frac{|x''-x'|}{\hbar}\sqrt{-2mE}\right) - \frac{\hbar}{2\cosh x'\cosh x''}\frac{1}{\hbar^2/2m+E}$$
$$\times\left\{1 - \left(1 - \frac{\hbar}{\sqrt{-2mE}}\right)\cosh\left[|x''-x'|\left(1+\frac{\sqrt{-2mE}}{\hbar}\right)\right]\right\} . \tag{6.12.8}$$

Hence, we obtain for the total Green function with point interaction

$$G(x'',x';E) = G^{(RL)}(x'',x';E) + \frac{G^{(RL)}(x'',a;E)G^{(RL)}(a,x';E)}{1/\gamma - G^{(RL)}(a,a;E)} . \tag{6.12.9}$$

It is possible to obtain an explicit solution of the path integral for $N=1$ due to the fact that in this case the eigenvalue equation is cubic. Let us denote by E_l ($l=1,2,3$) the roots of the equation

$$\frac{\hbar}{\gamma} = \sqrt{-\frac{m}{2E_l}} - \frac{\hbar^2}{2\cosh^2 a\sqrt{-2mE_l}\,(\hbar^2/2m + E_l)} \ . \tag{6.12.10}$$

Then we have

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{\hbar^2}{2m}\frac{2}{\cosh^2 x} + \gamma\delta(x-a)\right)\mathrm{d}t\right]$$
$$= \sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}} \exp\left[-\frac{m}{2\mathrm{i}\hbar T}(x''-x')^2\right] + \sum_{k,l=1}^{3} K^{(k,l)}(x'',x';T) \ . \tag{6.12.11}$$

For $K^{(1,l)}(T)$ one obtains:

$$\begin{aligned}
&K^{(1,l)}(x'',x';T)\\
&= \frac{m}{2\mathrm{i}\hbar T}\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\left(|x''-a| + |a-x'| + \frac{\mathrm{i}}{\hbar}T\sqrt{\frac{-2E_l}{m}}\right)\\
&\qquad\times \exp\left[-\frac{m}{2\mathrm{i}\hbar T}\big(|x''-a| + |a-x'|\big)^2\right]\\
&+ \left(\frac{-mE_l}{2\hbar^2} - \frac{1}{4}\right)\exp\left[-\frac{\sqrt{-2mE_l}}{\hbar}\big(|x''-a| + |a-x'|\big) - \frac{\mathrm{i}}{\hbar}E_lT\right]\\
&\qquad\times \operatorname{erfc}\left[\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left(\big(|x''-a| + |a-x'|\big) - \frac{\mathrm{i}}{\hbar}T\sqrt{-2mE_l}\right)\right] \ .
\end{aligned} \tag{6.12.12}$$

Similarly for $K^{(2,l)}(T)$:

$$\begin{aligned}
&K^{(2,l)}(x'',x';T)\\
&= \frac{1}{2\cosh a\cosh x'}\left[\!\!\left[\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\left\{\exp\left[-\frac{m}{2\mathrm{i}\hbar T}(x''-a)^2\right]\right.\right.\right.\\
&\quad - \frac{1}{2}\exp\left[|x'-a| - \frac{m}{2\mathrm{i}\hbar T}\big(|x''-a| + |x'-a|\big)^2\right]\\
&\qquad\left. - \frac{1}{2}\exp\left[|x'-a| - \frac{m}{2\mathrm{i}\hbar T}\big(|x''-a| - |x'-a|\big)^2\right]\right\}\\
&\quad + \sqrt{\frac{-mE_l}{2\hbar^2}}\exp\left(|x''-a|\frac{\sqrt{-2mE_l}}{\hbar} - \frac{\mathrm{i}E_lT}{\hbar}\right)\\
&\qquad\times \operatorname{erfc}\left[\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left(|x''-a| - \mathrm{i}T\sqrt{\frac{-2E_l}{\hbar}}\right)\right]\\
&\quad + \frac{1}{4}\left(1 + \frac{\sqrt{-2mE_l}}{\mathrm{i}\hbar T}\right)
\end{aligned}$$

$$\times \exp\left[|x'-a| - (|x''-a| + |a-x'|)\frac{\sqrt{-2mE_l}}{\hbar} - \frac{\mathrm{i}\,E_l T}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-a| + |a-x'| - \mathrm{i}\,T\sqrt{\frac{-2E_l}{\hbar}}\right)\right]$$
$$+\frac{1}{4}\left(1+\frac{\sqrt{-2mE_l}}{\mathrm{i}\,\hbar T}\right)$$
$$\times \exp\left[-|x'-a| - (|x''-a| - |a-x'|)\frac{\sqrt{-2mE_l}}{\hbar} - \frac{\mathrm{i}\,E_l T}{\hbar}\right]$$
$$\times \operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-a| - |a-x'| - \mathrm{i}\,T\sqrt{\frac{-2E_l}{\hbar}}\right)\right] + (x'' \leftrightarrow x')\Bigg]\Bigg] \ . \tag{6.12.13}$$

In order to calculate $K^{(3,l)}(T)$ one splits the corresponding $G^{(3,l)}(s)$ into three contributions according to

$$\begin{aligned} G^{(3,l)}(x'',x';E) &= \frac{-1}{\hbar^2/2m + E_l}\frac{G^{(3,l,1)}(x'',x';E)}{\sqrt{-E}-\sqrt{-E_l}} \\ &+ \frac{1}{2}\left(1+\frac{\sqrt{-2mE_l}}{\hbar}\right)\frac{G^{(3,l,2)}(x'',x';E)}{\sqrt{-E}-\hbar/\sqrt{2m}} \\ &+ \frac{1}{2}\left(1-\frac{\sqrt{-2mE_l}}{\hbar}\right)\frac{G^{(3,l,3)}(x'',x';E)}{\sqrt{-E}+\hbar/\sqrt{2m}} \ . \end{aligned} \tag{6.12.14}$$

Thus

$$K^{(3,l)}(x'',x';T) = \sum_{j=1}^{3} K^{(3,l,j)}(x'',x';T) \ . \tag{6.12.15}$$

This yields for each l, where $E_j = -E_l, \pm\hbar^2/2m$ $(j = 1,2,3)$, respectively:

$$\begin{aligned} &K^{(3,l,j)}(x'',x';T) \\ &= \frac{1}{4\cosh^2 a \cosh x' \cosh x''}\Bigg[\Bigg[\mathrm{e}^{\mathrm{i}\,E_j T/\hbar}\operatorname{erfc}\left(-\sqrt{\frac{\mathrm{i}\,p_j T}{\hbar}}\right) + \frac{1}{2}\sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}} \\ &\quad\times\Bigg[\left(\frac{\hbar}{m} - \frac{|x''-a|}{\mathrm{i}\,T} + \sqrt{\frac{2E_j}{m}}\right)\exp\left(|x''-a| - \frac{m}{2\,\mathrm{i}\,\hbar T}(x''-a)^2\right) \\ &\quad+\left(\frac{\hbar}{m} + \frac{|x''-a|}{\mathrm{i}\,T} + \sqrt{\frac{2E_j}{m}}\right)\exp\left(-|x''-a| - \frac{m}{2\,\mathrm{i}\,\hbar T}(x''-a)^2\right) \\ &\quad+\left(\frac{\hbar}{m} - \frac{|x'-a|}{\mathrm{i}\,T} + \sqrt{\frac{2E_j}{m}}\right)\exp\left(|x'-a| - \frac{m}{2\,\mathrm{i}\,\hbar T}(x'-a)^2\right) \\ &\quad+\left(\frac{\hbar}{m} + \frac{|x'-a|}{\mathrm{i}\,T} + \sqrt{\frac{2E_j}{m}}\right)\exp\left(-|x'-a| - \frac{m}{2\,\mathrm{i}\,\hbar T}(x'-a)^2\right)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\sqrt{\frac{E_j}{2m}}-\frac{E_j}{\hbar}\right)\left\{\exp\left[|x''-a|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\right.\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-a|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[-|x''-a|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(-|x''-a|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[|x'-a|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x'-a|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[-|x'-a|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\left.\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(-|x'-a|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\right\}\\
&+\sqrt{\frac{m}{2\pi\,\mathrm{i}\,\hbar T}}\left\{\left(\frac{1}{4}\sqrt{\frac{2E_j}{m}}+\frac{|x''-a|+|a-x'|}{4\,\mathrm{i}\,T}-\frac{\hbar}{m}\right)\right.\\
&\qquad\times\exp\left[|x''-a|+|a-x'|-\frac{m}{2\,\mathrm{i}\,\hbar T}\big(|x''-a|+|a-x'|\big)^2\right]\\
&\quad+\left(\frac{1}{4}\sqrt{\frac{2E_j}{m}}+\frac{|x''-a|-|a-x'|}{4\,\mathrm{i}\,T}-\frac{\hbar}{m}\right)\\
&\qquad\times\exp\left[|x''-a|-|a-x'|-\frac{m}{2\,\mathrm{i}\,\hbar T}\big(|x''-a|-|a-x'|\big)^2\right]\\
&\quad+\left(\frac{1}{4}\sqrt{\frac{2E_j}{m}}-\frac{|x''-a|+|a-x'|}{4\,\mathrm{i}\,T}-\frac{\hbar}{m}\right)\\
&\qquad\times\exp\left[-|x''-a|-|a-x'|-\frac{m}{2\,\mathrm{i}\,\hbar T}\big(|x''-a|+|a-x'|\big)^2\right]\\
&\quad+\left(\frac{1}{4}\sqrt{\frac{2E_j}{m}}-\frac{|x''-a|-|a-x'|}{4\,\mathrm{i}\,T}-\frac{\hbar}{m}\right)\\
&\qquad\left.\times\exp\left[-|x''-a|+|a-x'|-\frac{m}{2\,\mathrm{i}\,\hbar T}\big(|x''-a|-|a-x'|\big)^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\hbar}{2m}+\frac{E_j}{4\hbar}+\sqrt{\frac{E_j}{2m}}\right)\\
&\quad\times\left\{\exp\left[|x''-a|+|a-x'|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\right.\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-a|+|a-x'|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[|x''-a|-|a-x'|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(|x''-a|-|a-x'|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[-|x''-a|-|a-x'|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\times\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(-|x''-a|-|a-x'|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\\
&\quad+\exp\left[-|x''-a|+|a-x'|\left(1+\frac{\sqrt{2mE_j}}{\hbar}\right)+\frac{\mathrm{i}}{\hbar}E_jT\right]\\
&\qquad\times\left.\operatorname{erfc}\left[\sqrt{\frac{m}{2\,\mathrm{i}\,\hbar T}}\left(-|x''-a|+|a-x'|-\mathrm{i}\,T\sqrt{\frac{2E_j}{m}}\right)\right]\right\}\Bigg]\ .
\end{aligned}
\tag{6.12.16}
$$

6.12.1.5 Periodic δ-Function Perturbations. [371,406].

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\gamma\sum_{n\in\mathbb{Z}}\delta(x-na)\right)\mathrm{d}t\right]\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\mathrm{d}k\,\mathrm{e}^{\mathrm{i}\,k(x''-x')}}{k^2\hbar^2/2m-E}+\frac{\gamma}{2\pi\hbar}\int_{\mathbb{R}}\frac{\mathrm{d}k\,\mathrm{e}^{\mathrm{i}\,k(x''-x')}}{\hbar^2k^2/2m-E}\left[\sinh\left(\frac{a}{\hbar}\sqrt{-2mE}\right)\right]^{-1}\\
&\quad\times\left\{\frac{1}{\hbar}\sqrt{-2mE}\left[\cosh\left(\frac{a}{\hbar}\sqrt{-2mE}\right)-\cos(ak)\right]+\frac{\gamma}{\hbar}\sinh\left(\frac{a}{\hbar}\sqrt{-2mE}\right)\right\}^{-1}\\
&\quad\times\left\{\sin\left(\frac{\left(k+\frac{\mathrm{i}}{\hbar}\sqrt{-2mE}\right)a}{2}\right)\mathrm{e}^{-\,\mathrm{i}[(k-\mathrm{i}\sqrt{-2mE})x''/\hbar-(n+\frac{1}{2})a]}\right.\\
&\qquad\left.-\sin\left(\frac{\left(k-\frac{\mathrm{i}}{\hbar}\sqrt{-2mE}\right)a}{2}\right)\mathrm{e}^{-\,\mathrm{i}[(k+\mathrm{i}\sqrt{-2mE})x''/\hbar-(n+\frac{1}{2})a]}\right\}
\end{aligned}
$$

$$\times\left\{\sin\left(\frac{\left(k+\frac{\mathrm{i}}{\hbar}\sqrt{-2mE}\right)a}{2}\right)\mathrm{e}^{-\mathrm{i}[(k-\mathrm{i}\sqrt{-2mE})x'/\hbar-(n+\frac{1}{2})a]}\right.$$
$$\left.-\sin\left(\frac{\left(k-\frac{\mathrm{i}}{\hbar}\sqrt{-2mE}\right)a}{2}\right)\mathrm{e}^{-\mathrm{i}[(k+\mathrm{i}\sqrt{-2mE})x'/\hbar-(n+\frac{1}{2})a]}\right\}\ . \tag{6.12.17}$$

6.12.1.6 N-Fold δ-Function Perturbations. [17,439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}$$
$$\times\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)+\sum_{j=1}^{N}\gamma_j\delta(x-a_j)\right)\mathrm{d}t\right]$$
$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}(x'',a_1;E) & \dots & G^{(V)}(x'',a_N;E)\\ G^{(V)}(a_1,x';E) & G^{(V)}(a_1,a_1;E)-\dfrac{1}{\gamma_1} & \dots & G^{(V)}(a_1,a_N;E)\\ \vdots & \vdots & \ddots & \vdots\\ G^{(V)}(a_N,x';E) & G^{(V)}(a_N,a_1) & \dots & G^{(V)}(a_N,a_N;E)-\dfrac{1}{\gamma_N}\end{vmatrix}}{\begin{vmatrix} G^{(V)}(a_1,a_1;E)-\dfrac{1}{\gamma_1} & \dots & G^{(V)}(a_1,a_N;E)\\ \vdots & \ddots & \vdots\\ G^{(V)}(a_N,a_1;E) & \dots & G^{(V)}(a_N,a_N;E)-\dfrac{1}{\gamma_N}\end{vmatrix}}, \tag{6.12.18}$$

$$=G^{(V)}(x'',x';E)-\sum_{j,j'=1}^{N}\left(\Gamma^{(V)}_{\boldsymbol{\gamma},\mathbf{a}}(E)\right)^{-1}_{j,j'}G^{(V)}(x'',a_j;E)G^{(V)}(a_{j'},x';E)\ , \tag{6.12.19}$$

with the matrix $\Gamma^{(V)}_{\boldsymbol{\gamma},\mathbf{a}}(E)$ given by $\left(\boldsymbol{\gamma}=\{\gamma_k\}_{k=1}^N,\ \mathbf{a}=\{a_k\}_{k=1}^N\right)$

$$\left(\Gamma^{(V)}_{\boldsymbol{\gamma},\mathbf{a}}(E)\right)_{j,j'}=G^{(V)}(a_j,a_{j'};E)-\frac{\delta_{jj'}}{\gamma_j}\ . \tag{6.12.20}$$

6.12.2 δ′-Function Perturbations.

6.12.2.1 General One-Dimensional Case. [17,446]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)+\beta\delta'(x-a)\right)\mathrm{d}t\right]$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}_{,x'}(x'',a;E)G^{(V)}_{,x''}(a,x';E)}{\widehat{G}^{(V)}_{,x'x''}(a,a;E)+1/\beta} \ , \tag{6.12.21}$$

$$\widehat{G}^{(V)}_{,xy}(a,a;E) = \left(\frac{\partial^2}{\partial x\partial y}G^{(V)}(x,y;E) - \frac{2m}{\hbar^2}\delta(x-y)\right)\bigg|_{x=y=a} \ . \tag{6.12.22}$$

6.12.2.2.1 The Free Particle: Feynman Kernel. [445,446]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\beta\delta'(x-a)\right)dt\right]$$

$$= \sqrt{\frac{m}{2\pi i\hbar T}}\exp\left(\frac{i m}{2\hbar T}|x''-x'|^2\right) + \operatorname{sgn}(x''-a)\operatorname{sgn}(x'-a)$$
$$\times\left(\sqrt{\frac{m}{2\pi i\hbar T}}\exp\left[\frac{i m}{2\hbar T}(|x''-a|+|x'-a|)^2\right]\right.$$
$$+\frac{\hbar^2}{2m\beta}\exp\left[-\frac{\hbar^2}{m\beta}\Big(|x''-a|+|x'-a|\Big)+\frac{i}{\hbar}\frac{\hbar^6}{2m^3\beta^2}T\right]$$
$$\left.\times\operatorname{erfc}\left\{\sqrt{\frac{m}{2i\hbar T}}\left[\Big(|x''-a|+|x'-a|\Big)-\frac{i\hbar^3 T}{m^2\beta}\right]\right\}\right) \ , \tag{6.12.23}$$

$$= \frac{\hbar^2}{m\beta}\exp\left[-\frac{\hbar^2}{m\beta}\Big(|x''-a|+|x'-a|\Big)+\frac{i}{\hbar}\frac{\hbar^6 T}{2m^3\beta^2}\right]\operatorname{sgn}(x''-a)\operatorname{sgn}(x'-a)$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}}dk\exp\left(-i\frac{k^2\hbar}{2m}T\right)\left(\sin kx''\sin kx'+\cos kx''\cos kx'\right.$$
$$\left.+\frac{i m p\beta/\hbar^2}{1+i k m\beta/\hbar^2}e^{i k(|x''-a|+|x'-a|)}\operatorname{sgn}(x''-a)\operatorname{sgn}(x'-a)\right) \ . \tag{6.12.24}$$

6.12.2.2.2 The Free Particle: Green function. [17,445,446]

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2+\beta\delta'(x-a)\right)dt\right]$$

$$= \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|x''-x'|\right)$$
$$-\frac{m^2}{\hbar^4}\frac{\exp\left[-\dfrac{\sqrt{-2mE}}{\hbar}(|x''-a|+|a-x'|)\right]}{\dfrac{1}{\beta}-\dfrac{m}{\hbar^3}\sqrt{-2mE}}\operatorname{sgn}(x''-a)\operatorname{sgn}(x'-a) \ . \tag{6.12.25}$$

6.12.2.3 N-Fold δ′-Function Perturbation. [17,446]

$$
\begin{aligned}
&\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\\
&\quad\times\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)+\sum_{k=1}^{N}\beta_k\delta'(x-a_k)\right)\mathrm{d}t\right]\\
&=\frac{\begin{vmatrix}
G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',a_1;E) & \dots & G^{(V)}_{,x'}(x'',a_N;E)\\
G^{(V)}_{,x''}(a_1,x';E) & \widehat{G}^{(V)}_{,x'x''}(a_1,a_1;E)+\beta_1^{-1} & \dots & G^{(V)}_{,x'x''}(a_1,a_N;E)\\
\vdots & \vdots & \ddots & \vdots\\
G^{(V)}_{,x''}(a_N,x';E) & G^{(V)}_{,x'x''}(a_N,a_1) & \dots & \widehat{G}^{(V)}(a_N,a_N;E)+\beta_N^{-1}
\end{vmatrix}}{\begin{vmatrix}
\widehat{G}^{(V)}_{,x'x''}(a_1,a_1;E)+\beta_1^{-1} & \dots & G^{(V)}_{,x'x''}(a_1,a_N;E)\\
\vdots & \ddots & \vdots\\
G^{(V)}_{,x'x''}(a_N,a_1;E) & \dots & \widehat{G}^{(V)}_{,x'x''}(a_N,a_N;E)+\beta_N^{-1}
\end{vmatrix}}
\end{aligned}
\tag{6.12.26}
$$

$$
=G^{(V)}(x'',x';E)-\sum_{j,j'=1}^{N}\left(\Gamma^{(V)}_{\boldsymbol{\beta},\mathbf{a}}(E)\right)^{-1}_{j,j'}G^{(V)}_{,x'}(x'',a_j;E)G^{(V)}_{,x''}(a_{j'},x';E)\ ,
\tag{6.12.27}
$$

with the matrix $\Gamma^{(V)}_{\boldsymbol{\beta},\mathbf{a}}(E)$ given by $(\boldsymbol{\beta}=\{\beta_k\}_{k=1}^N,\ \mathbf{a}=\{a_k\}_{k=1}^N)$

$$
\left(\Gamma^{(V)}_{\boldsymbol{\beta},\mathbf{a}}(E)\right)_{j,j'}=\delta_{jj'}\frac{1}{\beta_j}+\widehat{G}^{(V)}_{,x'x''}(a_i,a_j;E)\ .
\tag{6.12.28}
$$

6.12.3 δ-Function Perturbations along Perpendicular Lines and Planes. [439,491]

$$
\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\int\limits_{y(t')=y'}^{y(t'')=y''}\mathcal{D}y(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{x}^2+\dot{y}^2)-V(x)-V(y)\right.\right.\\
&\qquad\qquad\left.\left.+\sum_{k=1}^{N_1}\gamma_{1,k}\delta(x-a_k)+\sum_{k=1}^{N_2}\gamma_{2,k}\delta(y-b_k)\right)\mathrm{d}t\right]\\
&=\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)+\sum_{k=1}^{N_1}\gamma_{1,k}\delta(x-a_k)\right)\mathrm{d}t\right]
\end{aligned}
$$

$$\times \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{y}^2 - V(y) + \sum_{k=1}^{N_2}\gamma_{2,k}\delta(y-b_k)\right)\mathrm{d}t\right]$$
$$\equiv K_x^{(N_1)}(x'',x';T)\cdot K_y^{(N_2)}(y'',y':T) = K^{(N_1,N_2)}(x'',x',y'',y';T) \ . \tag{6.12.29}$$

The solution of each kernel is given in terms of its corresponding Green function. Now call these Green functions $G_x^{(N_1)}(E)$ and $G_y^{(N_2)}(E)$, respectively. The convolution theorem of the Fourier transformation now states that the Green function corresponding to the total kernel $G^{(N_1,N_2)}(x'',x',y'',y';E)$ is given by the convolution of $G_x^{(N_1)}(E)$ and $G_y^{(N_2)}(E)$, i.e.,

$$G^{(N_1,N_2)}(x'',x',y'',y';E) = \int_{\mathbb{R}} G_x^{(N_1)}(x'',x';z)G_y^{(N_2)}(y'',y';E-z)\,\mathrm{d}z \ . \tag{6.12.30}$$

The generalization to D dimensions is analogous yielding a $(D-1)$-fold convolution of kernels $G_{x_i}^{(N_i)}(x_i'',x_i';E)$ $(i=1,\dots,D)$.

6.12.4 δ-Function Perturbation Along a Line. [439]

One introduces centre of mass and relative coordinates R and r, respectively

$$\left.\begin{aligned} r &= ax - by - c \ , & \mu &= \frac{m}{a^2+b^2} \ , \\ R &= \frac{ab^2x + a^2(by+c)}{a^2+b^2} \ , & M &= m\frac{a^2+b^2}{a^2b^2} \ , \end{aligned}\right\} \tag{6.12.31}$$

and has the identity

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}(\dot{x}^2+\dot{y}^2) + \gamma\delta(ax-by-c)\right)\mathrm{d}t\right]$$
$$= ab \int\limits_{R(t')=R'}^{R(t'')=R''} \mathcal{D}R(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{R}^2\,\mathrm{d}t\right)$$
$$\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{\mu}{2}\dot{r}^2 + \gamma\delta(r)\right)\mathrm{d}t\right]$$
$$= \sqrt{\frac{M}{2\pi\mathrm{i}\,\hbar T}} \exp\left[-\frac{M}{2\hbar\mathrm{i}\,T}(R''-R')^2\right]$$
$$\times \left[\frac{\mu\gamma}{\hbar^2}\exp\left(-\frac{\mu\gamma}{\hbar^2}(r'+r'') + \frac{\mathrm{i}}{\hbar}\frac{\mu\gamma^2}{2\hbar^2}T\right) + \frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}k\exp\left(-\mathrm{i}\frac{k^2\hbar T}{2\mu}\right)\right.$$

$$\times \left(\sin kr'' \sin kr' + \cos kr'' \cos kr' - \frac{e^{\mathrm{i}\, k(r''+r')}}{1 + \mathrm{i}\, k\hbar^2/\mu\gamma} \right) \Bigg] \ . \tag{6.12.32}$$

6.12.5 δ-Function Perturbation in Two and Three Dimensions. [17,443,535,760]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, e^{\mathrm{i}\, ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\mathrm{d}t\right]$$
$$= G^{(V)}(\mathbf{x}'', \mathbf{x}'; E) + \left(\Gamma^{(V)}_{\alpha,\mathbf{a}}(E)\right)^{-1} G^{(V)}(\mathbf{x}'', \mathbf{a}; E) G^{(V)}(\mathbf{a}, \mathbf{x}'; E) \ , \tag{6.12.33}$$

$$\left.\begin{aligned} &\Gamma_{\alpha,\mathbf{a}} = \alpha g_{0,D} - g_{1,D} \ , \\ &g_{0,\lambda} = \lim_{r\to 0+} \frac{g(r)}{G_\lambda^{(0)}(r)} \ , \quad g_{1,\lambda} = \lim_{r\to 0+} \frac{g(r) - g_{0,\lambda} G_\lambda^B(r)}{r^\lambda} \ , \\ &g(r) = r^{(D-1)/2} G(r, r; E) \ , \end{aligned}\right\} \tag{6.12.34}$$

where $G^{(V)}(r'', r'; E)$ is the radial Green function of the corresponding unperturbed problem. $G_\lambda^B(r)$ denotes the asymptotic expansion of the irregular solution $G_\lambda(r)$ of the unperturbed problem up to order r^t, $t \le 2\lambda - 1$, $r = |\mathbf{x}|$. Generally one has $G_\lambda^B(r) = G_\lambda^{(0)}(r)$ + additional terms. In two and three dimensions, respectively, they have the form

$$G^B_{\frac{1}{2}}(r) = G^{(0)}_{\frac{1}{2}}(r) = -\frac{m}{\pi\hbar^2}\sqrt{r}\,\ln r \ , \tag{6.12.35}$$

$$G^B_1(r) = \frac{m}{2\pi\hbar^2}\left(1 - \frac{m\eta}{\hbar^2} r - \frac{2m\eta}{\hbar^2} r \ln r\right) \ . \tag{6.12.36}$$

η denotes a Coulomb coupling, i.e., it corresponds to a potential $V(r) = \eta/r$, and α is the (regularized) coupling of the self-adjoint extension. In three dimensions one also has

$$\Gamma_{\alpha,\mathbf{a}} = \alpha + \frac{\partial}{\partial r_{12}} r_{12} G^{(V)}(\mathbf{x}'', \mathbf{x}'; E)\Bigg|_{\substack{r_{12}=|\mathbf{x}''-\mathbf{x}'|=0 \\ \mathbf{x}', \mathbf{x}''=\mathbf{a}}} \ , \tag{6.12.37}$$

provided an explicit expression for $G^{(V)}(\mathbf{x}'', \mathbf{x}'; E)$ exists. $\mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t)$ symbolizes the regularization of the point interaction at $\mathbf{x} = \mathbf{a}$ with regularized coupling α.

6.12.5.1 The Free Particle in Two Dimensions: Feynman Kernel. [15,443]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2 \mathrm{d}t\right)$$

$$= \frac{m}{2\pi\,\mathrm{i}\,\hbar T}\exp\left[\frac{\mathrm{i}\, m}{2\hbar T}(\mathbf{x}''-\mathbf{x}')^2\right]$$

$$+\frac{m}{\pi\beta^2\hbar^2}\int_0^\infty \frac{\mathrm{d}v}{\Gamma(v)}\int_0^{\mathrm{i}T}\frac{\mathrm{d}u}{u}\left(\frac{T-u}{\beta^2}\right)^{v-1} \mathrm{e}^{-(a^2+b^2)/4u} K_0\left(\frac{ab}{2u}\right) , \tag{6.12.38}$$

with the abbreviations

$$a = \frac{\sqrt{2m}}{\hbar}|\mathbf{x}''-\mathbf{a}| , \qquad b = \frac{\sqrt{2m}}{\hbar}|\mathbf{x}'-\mathbf{a}| , \qquad \beta = \sqrt{\frac{m}{2\hbar^2}}\,\mathrm{e}^{\pi\alpha\hbar^2/m+\gamma} . \tag{6.12.39}$$

Here $\gamma = 0.577\,215\,664\,901\,532\,86061\ldots$ denotes Euler's constant.

6.12.5.2 The Free Particle in Two Dimensions. Green Function. [17,443]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2 \mathrm{d}t\right)$$

$$= \frac{m}{\pi\hbar^2}K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{x}''-\mathbf{x}'|\right)$$

$$+\left(\frac{m}{\pi\hbar^2}\right)^2 \frac{K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{x}''-\mathbf{a}|\right)K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{a}-\mathbf{x}'|\right)}{\alpha+\frac{m}{\pi\hbar^2}\left[\ln\frac{\sqrt{-2mE}}{2\hbar}+\gamma\right]} , \tag{6.12.40}$$

$$= \frac{m}{\pi\hbar^2}K_0\left(\sqrt{-2mE}\,\frac{r_>}{\hbar}\right)I_0\left(\sqrt{-2mE}\,\frac{r_<}{\hbar}\right)$$

$$+\left(\frac{m}{\pi\hbar^2}\right)^2 \frac{K_0\left(\sqrt{-2mE}\,\frac{r'}{\hbar}\right)K_0\left(\sqrt{-2mE}\,\frac{r'}{\hbar}\right)}{\alpha+\frac{m}{\pi\hbar^2}\left[\ln\frac{\sqrt{-2mE}}{2\hbar}+\gamma\right]}$$

$$+\frac{m}{\pi\hbar^2}\sum_{l\in\mathbb{N}}\mathrm{e}^{\mathrm{i}\,l(\varphi''-\varphi')}K_l\left(\sqrt{-2mE}\,\frac{r_>}{\hbar}\right)I_l\left(\sqrt{-2mE}\,\frac{r_<}{\hbar}\right) , \tag{6.12.41}$$

where $G_2^{(0)}(\mathbf{x};0) = -(m/\pi\hbar^2)\ln|\mathbf{x}|$ ($\mathbf{x}\in\mathbb{R}^2\backslash\{0\}$), and polar coordinates about $|\mathbf{x}-\mathbf{a}|$ have been used in the last line. The single bound-state wave function

$$\Psi^{(\alpha)} = \frac{1}{\sqrt{\pi}} K_0\left(|\mathbf{x} - \mathbf{a}|\, e^{-2[\alpha\pi\hbar^2/m+\gamma]}\right) \tag{6.12.42}$$

has the energy

$$E^{(\alpha)} = -\frac{2\hbar^2}{m}\, e^{-2[\alpha\pi\hbar^2/m+\gamma]} \quad . \tag{6.12.43}$$

6.12.5.3 Harmonic Oscillator in Two Dimensions. [443] ($\mathbf{x} = (x, y) \in \mathbb{R}^2$)

$$\begin{aligned}
&\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,a}}\mathbf{x}(t) \exp\left[\frac{i\,m}{2\hbar}\int_{t'}^{t''}(\dot{\mathbf{x}}^2 - \omega^2\mathbf{x}^2)dt\right] \\
&= \frac{\Gamma(\frac{1}{2} - E/2\hbar\omega)}{2\pi\hbar\omega r' r''} W_{E/2\hbar\omega,0}\left(\frac{m\omega}{\hbar}r_>^2\right) M_{E/2\hbar\omega,0}\left(\frac{m\omega}{\hbar}r_<^2\right) \\
&\quad + \frac{m\Gamma^2(\frac{1}{2} - E/2\hbar\omega)}{4\pi^2\hbar^3\omega r' r''} \frac{W_{E/2\hbar\omega,0}\left(\frac{m\omega}{\hbar}r''^2\right) W_{E/2\hbar\omega,0}\left(\frac{m\omega}{\hbar}r'^2\right)}{\alpha + \frac{m}{2\pi\hbar^2}\left[\Psi\left(\frac{1}{2} - \frac{E}{2\hbar\omega}\right) + \ln\frac{m\omega}{\hbar} + 2\gamma\right]} \\
&\quad + \sum_{l=1}^\infty e^{i\,l(\varphi''-\varphi')} \frac{\Gamma[\frac{1}{2}(l + 1 - E/\hbar\omega)]}{2\pi\hbar\omega l!\, r' r''} \\
&\qquad \times W_{E/2\hbar\omega,l/2}\left(\frac{m\omega}{\hbar}r''^2\right) M_{E/2\hbar\omega,l/2}\left(\frac{m\omega}{\hbar}r'^2\right) \quad ,
\end{aligned} \tag{6.12.44}$$

where $G_2^{(0)}(\mathbf{x}; 0) = G_2^{(0)}(r; 0) = -(m/\pi\hbar^2)\ln r$ $(r > 0)$.

6.12.5.4 Coulomb Potential in Two Dimensions. [443] ($\mathbf{x} = (x, y) \in \mathbb{R}^2$)

$$\begin{aligned}
&\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,a}}\mathbf{x}(t) \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e_1 e_2}{|\mathbf{x}|}\right)dt\right] \\
&= \sqrt{-\frac{m}{2E}} \frac{\Gamma(\frac{1}{2} - \kappa)}{2\pi\hbar\sqrt{r' r''}} W_{\kappa,0}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,0}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) \\
&\quad + \left(\frac{m}{\pi\hbar^2}\right)^2 \frac{\Gamma^2(\frac{1}{2} - \kappa)}{\sqrt{r' r''}} \frac{W_{\kappa,0}\left(\sqrt{-8mE}\,\frac{r'}{\hbar}\right) W_{\kappa,0}\left(\sqrt{-8mE}\,\frac{r''}{\hbar}\right)}{\alpha + \frac{m}{2\pi\hbar^2}\left[\Psi\left(\frac{1}{2} - \kappa\right) + \ln\frac{\sqrt{-8mE}}{\hbar} + 2\gamma\right]} \\
&\quad + \frac{1}{2\pi\hbar}\sqrt{-\frac{m}{2E}} \sum_{l=1}^\infty e^{i\,l(\phi''-\phi')} \frac{\Gamma(l + \frac{1}{2} - \kappa)}{(2l)!\,\sqrt{r' r''}} \\
&\qquad \times W_{\kappa,l}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right) M_{\kappa,l}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) \quad .
\end{aligned} \tag{6.12.45}$$

6.12.5.5 The Free Particle in Three Dimensions: Feynman Kernel. [15,820]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2 \mathrm{d}t\right)$$

$$= K^{(0)}(x'',x';T) + \frac{1}{|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}\int_0^\infty \mathrm{e}^{-2\pi\alpha\hbar u/m}(u+|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|)$$
$$\times K^{(0)}(u+|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|,0;T)\,\mathrm{d}u\ , \tag{6.12.46a}$$

$$= K^{(0)}(\mathbf{x}'',\mathbf{x}';T) + \frac{\mathrm{i}\,\hbar T}{m|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}K^{(0)}(|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|,0;T)\ , \tag{6.12.46b}$$

$$= K^{(0)}(\mathbf{x}'',\mathbf{x}';T) + \Psi^{(\alpha)}(\mathbf{x}')\Psi^{(\alpha)}(\mathbf{x}'')\,\mathrm{e}^{\mathrm{i}\,E^{(\alpha)}T/\hbar}$$
$$+\frac{1}{|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}\int_0^\infty \mathrm{e}^{-2\pi|\alpha|\hbar u/m}(u-|\mathbf{a}-\mathbf{x}'|-|\mathbf{x}''-\mathbf{a}|)$$
$$\times K^{(0)}(u-|\mathbf{a}-\mathbf{x}'|-|\mathbf{x}''-\mathbf{a}|,0;T)\,\mathrm{d}u\ , \tag{6.12.46c}$$

$$K^{(0)}(\mathbf{x},\mathbf{y};T) = \left(\frac{m}{2\pi\mathrm{i}\,\hbar T}\right)^{3/2}\exp\left(-\frac{m}{2\,\mathrm{i}\,\hbar T}|\mathbf{x}-\mathbf{y}|^2\right)\ , \tag{6.12.47}$$

for $\alpha>0$, $\alpha=0$ and $\alpha<0$, respectively, the bound-state wave function is

$$\Psi^{(\alpha)}(\mathbf{x}) = \sqrt{-\frac{\alpha\hbar^2}{m}}\,\frac{\mathrm{e}^{-2\pi\alpha\hbar|\mathbf{x}-\mathbf{a}|/m}}{|\mathbf{x}-\mathbf{a}|}\ , \tag{6.12.48}$$

with energy $E^{(\alpha)}$

$$E^{(\alpha)} = -2\pi^2\frac{\alpha^2\hbar^6}{m^3}\ . \tag{6.12.49}$$

6.12.5.6 The Free Particle in Three Dimensions: Green Function. [17,443] $(\mathbf{x}=(x,y,z)\in\mathbb{R}^3)$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2\mathrm{d}t\right)$$

$$= \frac{m}{2\pi\hbar^2|\mathbf{x}''-\mathbf{x}'|}\exp\left(-\frac{\sqrt{-2mE}}{\hbar}|\mathbf{x}''-\mathbf{x}'|\right)$$
$$+\left(\frac{m}{2\pi\hbar^2}\right)^2\frac{1}{|\mathbf{x}''-\mathbf{a}|\,|\mathbf{a}-\mathbf{x}'|}\frac{\exp\left[-\dfrac{\sqrt{-2mE}}{\hbar}(|\mathbf{x}''-\mathbf{a}|+|\mathbf{a}-\mathbf{x}'|)\right]}{\alpha+\dfrac{m}{2\pi\hbar^3}\sqrt{-2mE}}\ . \tag{6.12.50}$$

Here $G_3^{(0)}(\mathbf{x};0)=m/2\pi\hbar^2|\mathbf{x}|$ $(\mathbf{x}\in\mathbb{R}^3\setminus\{0\})$.

6.12.5.7 Harmonic Oscillator in Three Dimensions. [443] ($\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, $\nu = -\frac{1}{2} + E/\hbar\omega$, $a = |\mathbf{a}|$, $\tilde{a} = \sqrt{2m\omega/\hbar}\, a$, $G^{(\omega)}(E)$ denotes the three-dimensional Green function of the harmonic oscillator, cf. Sect. 6.2.2.7)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{\mathbf{x}}^2 - \omega^2\mathbf{x}^2)\mathrm{d}t\right]$$

$$= G^{(\omega)}(\mathbf{x}'', \mathbf{x}'; E) + \frac{G^{(\omega)}(\mathbf{x}'', \mathbf{a}; E) G^{(\omega)}(\mathbf{a}, \mathbf{x}'; E)}{\Gamma^{(\omega)}_{\alpha,\mathbf{a}}(E)} , \tag{6.12.51a}$$

$$\Gamma_{\alpha,\mathbf{a}}(E) = \alpha - \frac{m\Gamma(-\nu)}{2(2\pi)^{3/2}\hbar^2}\left\{\frac{1}{a}\Big[D_\nu(\tilde{a})D'_\nu(-\tilde{a}) - D'_\nu(\tilde{a})D_\nu(-\tilde{a})\Big]\right.$$
$$\left. + \sqrt{\frac{2m\omega}{\hbar}}\Big[D''_\nu(\tilde{a})D_\nu(-\tilde{a}) + 2D'_\nu(\tilde{a})D'_\nu(-\tilde{a}) + D_\nu(\tilde{a})D''_\nu(-\tilde{a})\Big]\right\} . \tag{6.12.51b}$$

The case $\mathbf{a} = \mathbf{0}$ gives:

$$= \frac{\Gamma[\frac{1}{2}(3/2 - E/\hbar\omega)]}{4\pi\hbar\omega(r'r'')^{3/2}} W_{E/2\hbar\omega,1/4}\left(\frac{m\omega}{\hbar}r_>^2\right) M_{E/2\hbar\omega,1/4}\left(\frac{m\omega}{\hbar}r_<^2\right)$$
$$+ \frac{m^2\Gamma^2[\frac{1}{2}(3/2 - E/\hbar\omega)]2^{-(E/\hbar\omega - \frac{1}{2})}}{4\pi^3\hbar^4 r'r''}$$
$$\times \frac{D_{E/\hbar\omega - \frac{1}{2}}\left(\sqrt{\frac{2m\omega}{\hbar}}r''\right) D_{E/\hbar\omega - \frac{1}{2}}\left(\sqrt{\frac{2m\omega}{\hbar}}r'\right)}{\alpha - \frac{m}{2\pi\hbar^2}\sqrt{\frac{m\omega}{\hbar}}\frac{\Gamma[\frac{1}{2}(3/2 - E/\hbar\omega)]}{\Gamma[\frac{1}{2}(1/2 - E/\hbar\omega)]}}$$
$$+ \sum_{l=1}^\infty \frac{\Gamma[\frac{1}{2}(l + 3/2 - E/\hbar\omega)]}{\hbar\omega\Gamma(l + 3/2)(r'r'')^2} \sum_{n=-l}^{l} Y_l^n(\vartheta'', \phi'') Y_l^{n\,*}(\vartheta', \phi')$$
$$\times W_{E/2\hbar\omega,\frac{1}{2}(l+\frac{1}{2})}\left(\frac{m\omega}{\hbar}r''^2\right) M_{E/2\hbar\omega,\frac{1}{2}(l+\frac{1}{2})}\left(\frac{m\omega}{\hbar}r'^2\right) . \tag{6.12.52}$$

6.12.5.8 Coulomb Potential in Three Dimensions. [17,443] ($a = |\mathbf{a}|$, $\tilde{a} = a\sqrt{-8mE}/\hbar$, $G^{(C)}(E)$ denotes the three-dimensional Coulomb Green function, cf. Sect. 6.8.6, $\kappa = e^2\sqrt{-m/2E}/\hbar$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 + \frac{e^2}{|\mathbf{x}|}\right)\mathrm{d}t\right]$$

$$= G^{(C)}(\mathbf{x}'', \mathbf{x}'; E) + \frac{G^{(C)}(\mathbf{x}'', \mathbf{a}; E) G^{(C)}(\mathbf{a}, \mathbf{x}'; E)}{\Gamma^{(C)}_{\alpha,\mathbf{a}}(E)} , \tag{6.12.53a}$$

$$\Gamma_{\alpha,\mathbf{a}}(E) = \alpha - \frac{m\Gamma(1-\kappa)}{2\pi\hbar^3}\sqrt{-8mE}$$
$$\times\left[2W'_{\kappa,\frac{1}{2}}(2\tilde{a})M'_{\kappa,\frac{1}{2}}(2\tilde{a}) - W_{\kappa,\frac{1}{2}}(2\tilde{a})M''_{\kappa,\frac{1}{2}}(2\tilde{a}) - M_{\kappa,\frac{1}{2}}(2\tilde{a})W''_{\kappa,\frac{1}{2}}(2\tilde{a})\right] . \tag{6.12.53b}$$

The case $\mathbf{a} = \mathbf{0}$ gives:

$$= \sqrt{-\frac{m}{2E}}\frac{\Gamma(1-\kappa)}{4\pi\hbar r'r''}W_{\kappa,\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right)$$
$$+\frac{\left(\frac{m}{2\pi\hbar^2}\right)^2\frac{\Gamma^2(1-\kappa)}{r'r''}W_{\kappa,\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r'}{\hbar}\right)W_{\kappa,\frac{1}{2}}\left(\sqrt{-8mE}\,\frac{r''}{\hbar}\right)}{\alpha + \frac{me^2}{2\pi\hbar^2}\left[\frac{2m}{\hbar^2}\left(\Psi(1)+\Psi(2)-\Psi(1-\kappa)+\ln\frac{\hbar e_1 e_2}{\sqrt{-8mE}}\right) - \frac{\sqrt{-2mE}}{\hbar e^2}\right]}$$
$$+\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\sum_{l=1}^{\infty}\frac{\Gamma(l+1-\kappa)}{(2l+1)!\,r'r''}\sum_{n=-l}^{l}Y_l^n(\vartheta'',\phi'')Y_l^{n*}(\vartheta',\phi')$$
$$\times W_{\kappa,l+1/2}\left(\sqrt{-8mE}\,\frac{r_>}{\hbar}\right)M_{\kappa,l+1/2}\left(\sqrt{-8mE}\,\frac{r_<}{\hbar}\right) . \tag{6.12.54}$$

6.12.6 Multiple δ-Function Perturbation in Two and Three Dimensions. (δ-perturbations located at points $\{\mathbf{a}\} = \{\mathbf{a}_j\}$ with strengths α_j) [17,443]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}_{\Gamma_{\alpha,\{\mathbf{a}\}}}\mathbf{x}(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right)\mathrm{d}t\right]$$
$$= \frac{\begin{vmatrix} G^{(V)}(\mathbf{x}'',\mathbf{x}';E) & G^{(V)}(\mathbf{x}'',\mathbf{a}_1;E) & \dots & G^{(V)}(\mathbf{x}'',\mathbf{a}_N;E) \\ G^{(V)}(\mathbf{a}_1,\mathbf{x}';E) & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{1,1} & \dots & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ G^{(V)}(\mathbf{a}_N,\mathbf{x}';E) & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{N,1} & \dots & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{N,N}\end{vmatrix}}{\begin{vmatrix}\left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{1,1} & \dots & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{1,N} \\ \vdots & \ddots & \vdots \\ \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{N,1} & \dots & \left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)_{N,N}\end{vmatrix}} , \tag{6.12.55}$$

$$= G^{(V)}(\mathbf{x}'',\mathbf{x}';E) + \sum_{j,j'=1}^{N}\left(\Gamma_{\alpha,\{\mathbf{a}\}}\right)^{-1}_{j,j'}G^{(V)}(\mathbf{x}'',\mathbf{a}_j;E)G^{(V)}(\mathbf{a}_{j'},\mathbf{x}';E) , \tag{6.12.56}$$

$$\left(\Gamma^{(V)}_{\boldsymbol{\alpha},\{\mathbf{a}\}}(E)\right)_{j,j'} = \begin{cases} \alpha_j g_{0,D} - g_{1,D} & j = j' \ , \\ -G^{(V)}(\mathbf{a}_j, \mathbf{a}_{j'}; E) & j \neq j' \ . \end{cases} \tag{6.12.57}$$

6.12.6.1 Multiple δ-Function Perturbation in Two Dimensions: Free Particle. (δ-perturbations located at points $\{\mathbf{a}\} = \{\mathbf{a}_j\}$ with strengths α_j) [17,443]

$$\begin{aligned} &\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathbf{x}}^2 \mathrm{d}t\right) \\ &= \frac{m}{\pi\hbar^2} K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{x}'' - \mathbf{x}'|\right) + \left(\frac{m}{\pi\hbar^2}\right)^2 \sum_{j,j'=1}^{N} \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}(E)\right)^{-1}_{j,j'} \\ &\qquad \times K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{x}'' - \mathbf{a}_j|\right) K_0\left(\frac{\sqrt{-2mE}}{\hbar}|\mathbf{a}_{j'} - \mathbf{x}'|\right) \ . \end{aligned} \tag{6.12.58}$$

$$\left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}(E)\right)_{j,j'} = \begin{cases} \alpha_j + \gamma + \dfrac{m}{\pi\hbar^2}\ln\dfrac{\sqrt{-2mE}}{2\hbar} & j = j' \ , \\ -G_2^{(0)}(\mathbf{a}_j, \mathbf{a}_{j'}; E) & j \neq j' \ . \end{cases} \tag{6.12.59}$$

6.12.6.2 Multiple δ-Function Perturbation in Three Dimensions: Free Particle. (δ-perturbations located at set $\{\mathbf{a}\} = \{\mathbf{a}_j\}$ with strengths α_j) [17,443]

$$\begin{aligned} &\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}}\mathbf{x}(t) \exp\left(\frac{\mathrm{i}\, m}{2\hbar}\int_{t'}^{t''} \dot{\mathbf{x}}^2 \mathrm{d}t\right) \\ &= \frac{\begin{vmatrix} G_3^{(0)}(\mathbf{x}'', \mathbf{x}'; E) & G_3^{(0)}(\mathbf{x}'', \mathbf{a}_1; E) & \dots & G_3^{(0)}(\mathbf{x}'', \mathbf{a}_N; E) \\ G_3^{(0)}(\mathbf{a}_1, \mathbf{x}'; E) & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{1,1} & \dots & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ G_3^{(0)}(\mathbf{a}_N, \mathbf{x}'; E) & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{N,1} & \dots & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{N,N} \end{vmatrix}}{\begin{vmatrix} \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{1,1} & \dots & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{1,N} \\ \vdots & \ddots & \vdots \\ \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{N,1} & \dots & \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)_{N,N} \end{vmatrix}} \ , \end{aligned} \tag{6.12.60}$$

$$= G_3^{(0)}(\mathbf{x}'',\mathbf{x}';E) + \sum_{j,j'=1}^{N} \left(\Gamma_{\boldsymbol{\alpha},\{\mathbf{a}\}}\right)^{-1}_{j,j'} G_3^{(0)}(\mathbf{x}'',\mathbf{a}_j;E) G_3^{(0)}(\mathbf{a}_{j'},\mathbf{x}';E) \ , \tag{6.12.61}$$

$$\left(\Gamma^{(0)}_{\boldsymbol{\alpha},\{\mathbf{a}\}}(E)\right)_{j,j'} = \begin{cases} \alpha_j + \dfrac{m}{2\pi\hbar^3}\sqrt{-2mE} & j = j' \ , \\ -G_3^{(0)}(\mathbf{a}_j,\mathbf{a}_{j'};E) & j \neq j' \ . \end{cases} \tag{6.12.62}$$

6.12.7 D-Dimensional Radial Case. [430]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}\mathbf{x}(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{\mathbf{x}}^2 - V(r) + \gamma\delta(r-a)\right)\mathrm{d}t\right]$$

$$= \sum_{l\in\mathbb{N}_0} S_l^\mu(\boldsymbol{\Omega}'') S_l^{\mu *}(\boldsymbol{\Omega}') \left(G_l^{(V)}(r'',r';E) + \frac{G_l^{(V)}(r'',a;E) G_l^{(V)}(a,r';E)}{a^{1-D}/\gamma - G_l^{(V)}(a,a;E)} \right) \ . \tag{6.12.63}$$

6.12.8 Point Interaction for Dirac Particle.

6.12.8.1 Electron δ-Function Perturbation. [446,508]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}\nu(t)$$

$$\times \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \mathbf{V}(x)\mathrm{d}t + \frac{\mathrm{i}}{\hbar}\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \int_{t'}^{t''} \delta(x-a)\,\mathrm{d}t \right)$$

$$= \begin{pmatrix} G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'',x';E) \\ G_{21}^{(V)}(x'',x';E) & G_{22}^{(V)}(x'',x';E) \end{pmatrix} + \frac{1}{1/\alpha - G_{11}^{(V)}(a,a;E)}$$

$$\times \begin{pmatrix} G_{11}^{(V)}(a,x';E) G_{11}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,x';E) G_{12}^{(V)}(x'',a;E) \\ G_{21}^{(V)}(a,x';E) G_{11}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,x';E) G_{12}^{(V)}(x'',a;E) \end{pmatrix} \ . \tag{6.12.64}$$

The support property of the measure $\mathcal{D}\nu$ [508] is defined in such a way that the motion which it is describing selects paths of N steps each of length $c\epsilon$ that start at x' in the direction α, and end at x'' in the direction β, where α and β take the values "right" and "left". $\Phi(R)$ is the number of possible paths with R corners. The path integration is then a summation over all reversings of directions. Note the different conventions used in the literature. $\mathbf{V}$ is a matrix valued potential, i.e., $\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$.

6.12.8.2 Positron δ-Function Perturbation. [446]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}\nu(t)$$
$$\times\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathbf{V}(x)\mathrm{d}t-\frac{\mathrm{i}}{\hbar}\beta\begin{pmatrix}0&0\\0&1\end{pmatrix}\int_{t'}^{t''}\delta(x-a)\,\mathrm{d}t\right)$$
$$=\begin{pmatrix}G_{11}^{(V)}(x'',x';E)&G_{12}^{(V)}(x'',x';E)\\G_{21}^{(V)}(x'',x';E)&G_{22}^{(V)}(x'',x';E)\end{pmatrix}-\frac{1}{\hbar^2/4m^2c^2\beta+G_{22}^{(V)}(a,a;E)}$$
$$\times\begin{pmatrix}G_{12}^{(V)}(a,x';E)G_{21}^{(V)}(x'',a;E)&G_{12}^{(V)}(a,x';E)G_{22}^{(V)}(x'',a;E)\\G_{22}^{(V)}(a,x';E)G_{21}^{(V)}(x'',a;E)&G_{22}^{(V)}(a,x';E)G_{22}^{(V)}(x'',a;E)\end{pmatrix}.\tag{6.12.65}$$

6.12.8.3 Electron δ-Function Perturbation and the Free Particle. [17,446] ($\zeta=(E+mc^2)/ck\hbar$, $ck\hbar=\sqrt{E^2-m^2c^4}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}\nu(t)\exp\left(\frac{\mathrm{i}}{\hbar}\alpha\begin{pmatrix}1&0\\0&0\end{pmatrix}\int_{t'}^{t''}\delta(x-a)\,\mathrm{d}t\right)$$
$$=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta&\operatorname{sgn}(x''-x')\\\operatorname{sgn}(x''-x')&1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}$$
$$-\frac{\alpha\,\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{2c\hbar(2c\hbar-\mathrm{i}\alpha\zeta)}\begin{pmatrix}\zeta^2&\zeta\operatorname{sgn}(x''-a)\\\zeta\operatorname{sgn}(a-x')&\operatorname{sgn}(x''-a)\operatorname{sgn}(a-x')\end{pmatrix}.\tag{6.12.66}$$

6.12.8.4 Positron δ-Function Perturbation and the Free Particle. [17,446] ($\zeta=(E+mc^2)/ck\hbar$, $ck\hbar=\sqrt{E^2-m^2c^4}$)

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}\nu(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\beta c^2\begin{pmatrix}0&0\\0&1\end{pmatrix}\int_{t'}^{t''}\delta(x-a)\,\mathrm{d}t\right)$$
$$=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta&\operatorname{sgn}(x''-x')\\\operatorname{sgn}(x''-x')&1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}$$
$$+\frac{2m^2\beta\,\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{\hbar(2\hbar^3+4\mathrm{i}m^2c\beta/\zeta)}\begin{pmatrix}\operatorname{sgn}(x''-a)\operatorname{sgn}(a-x')&\operatorname{sgn}(a-x')/\zeta\\\operatorname{sgn}(x''-a)/\zeta&1/\zeta^2\end{pmatrix}.\tag{6.12.67}$$

6.13 Boundary Value Problems

6.13.1 General Formulæ.

6.13.1.1 Dirichlet Boundary Conditions in the Half-Space $x \gtrless a$. [196,438,439, 610,628]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(x=a)} x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right]$$
$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', a; E) G^{(V)}(a, x'; E)}{G^{(V)}(a, a; E)} \ . \tag{6.13.1}$$

6.13.1.2 Neumann Boundary Conditions in the Half-Space $x \gtrless a$. [196,445,446, 610]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(N)}_{(x=a)} x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right]$$
$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}_{,x'}(x'', a; E) G^{(V)}_{,x''}(a, x'; E)}{\widehat{G}^{(V)}_{,x'x''}(a, a; E)} \ , \tag{6.13.2}$$
$$\widehat{G}^{(V)}_{,xy}(a, a; E) = \left(\frac{\partial^2}{\partial x \partial y} G^{(V)}(x, y; E) - \frac{2m}{\hbar^2}\delta(x-y)\right)\Bigg|_{x=y=a} \ . \tag{6.13.3}$$

6.13.1.3 Dirichlet–Dirichlet Boundary Conditions in the Box $a < x < b$. [445,446]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DD)}_{(a<x<b)} x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(x)\right)\mathrm{d}t\right]$$
$$= \frac{\begin{vmatrix} G^{(V)}(x'', x'; E) & G^{(V)}(x'', b; E) & G^{(V)}(x'', a; E) \\ G^{(V)}(b, x'; E) & G^{(V)}(b, b; E) & G^{(V)}(b, a; E) \\ G^{(V)}(a, x'; E) & G^{(V)}(a, b; E) & G^{(V)}(a, a; E) \end{vmatrix}}{\begin{vmatrix} G^{(V)}(b, b; E) & G^{(V)}(b, a; E) \\ G^{(V)}(a, b; E) & G^{(V)}(a, a; E) \end{vmatrix}} \ . \tag{6.13.4}$$

The quantization rule for the bound states is given by ($n \in \mathbb{N}_0$)

$$\begin{vmatrix} G^{(V)}(b, b; E_n) & G^{(V)}(b, a; E_n) \\ G^{(V)}(a, b; E_n) & G^{(V)}(a, a; E_n) \end{vmatrix} = 0 \ . \tag{6.13.5}$$

6.13.1.4 Neumann-Neumann Boundary Conditions in the Box $a < x < b$. [445,446]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(NN)}_{(a<x<b)}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right]$$

$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}_{,x'}(x'',a;E)\\ G^{(V)}_{,x''}(b,x';E) & \widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x''}(a,x';E) & G^{(V)}_{,x'x''}(a,b;E) & \widehat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}}{\begin{vmatrix} \widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x'x''}(a,b;E) & \widehat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}}\ . \tag{6.13.6}$$

6.13.1.5 Dirichlet-Neumann Boundary Conditions in the Box $a < x < b$. [445,446]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(DN)}_{(a<x<b)}x(t)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2-V(x)\right)\mathrm{d}t\right]$$

$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}(x'',a;E)\\ G^{(V)}_{,x''}(b,x';E) & \widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x''}(b,a;E)\\ G^{(V)}(a,x';E) & G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}{\begin{vmatrix} \widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x''}(b,a;E)\\ G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}\ . \tag{6.13.7}$$

6.13.1.6 Harmonic Oscillator in the Box $a < x < b$ with Dirichlet–Dirichlet boundary conditions. [445] The bound-state energy levels are determined by ($n \in \mathbb{N}_0$)

$$D_{-\frac{1}{2}+\frac{E_n}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}\,a\right)D_{-\frac{1}{2}+\frac{E_n}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,b\right)$$
$$=D_{-\frac{1}{2}+\frac{E_n}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}\,b\right)D_{-\frac{1}{2}+\frac{E_n}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}\,a\right)\ . \tag{6.13.8}$$

6.13.1.7 The Wood-Saxon Potential in Half-Space. [439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(x>a)}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 + \frac{V_0}{1+\mathrm{e}^{(x-b)/R}}\right)\mathrm{d}t\right]$$

$$= \frac{2mR}{\hbar^2}\frac{\Gamma(m_1)\Gamma(m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

$$\times \left(\frac{1-\tanh\frac{x_<-b}{2R}}{2}\right)^{\frac{m_1-m_2}{2}}\left(\frac{1+\tanh\frac{x_<-b}{2R}}{2}\right)^{\frac{m_1+m_2}{2}}$$

$$\times \left(\frac{1-\tanh\frac{x_>-b}{2R}}{2}\right)^{\frac{m_1-m_2}{2}}\left(\frac{1+\tanh\frac{x_>-b}{2R}}{2}\right)^{\frac{m_1+m_2}{2}}$$

$$\times \left\{ {}_2F_1\left(m_1, m_1+1; m_1-m_2+1; \frac{1-\tanh\frac{x_>-b}{2R}}{2}\right)\right.$$

$$\times {}_2F_1\left(m_1, m_1+1; m_1+m_2+1; \frac{1+\tanh\frac{x_<-b}{2R}}{2}\right)$$

$$- \frac{{}_2F_1\left(m_1, m_1+1; m_1+m_2+1; \frac{1+\tanh\frac{a-b}{2R}}{2}\right)}{{}_2F_1\left(m_1, m_1+1; m_1-m_2+1; \frac{1-\tanh\frac{a-b}{2R}}{2}\right)}$$

$$\times {}_2F_1\left(m_1, m_1+1; m_1-m_2+1; \frac{1-\tanh\frac{x'-b}{2R}}{2}\right)$$

$$\left.\times {}_2F_1\left(m_1, m_1+1; m_1-m_2+1; \frac{1-\tanh\frac{x''-b}{2R}}{2}\right)\right\} \tag{6.13.9}$$

($m_{1,2} = \sqrt{2m}\,R(\sqrt{-E-V_0} \pm \sqrt{-E}\,)/\hbar$), and similarly for $x < a$.

6.13.1.8 The Rotating Morse Oscillator (S-waves). [439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}$$

$$\times \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \frac{\hbar^2 V_0^2}{2m}\left(\mathrm{e}^{-2r} - 2\alpha e^{-r}\right)\right)\mathrm{d}t\right]$$

$$= \frac{m}{4\pi\hbar^2 r'r''}\frac{\Gamma(\frac{1}{2}+\sqrt{-2mE}/\hbar - \alpha V_0)}{\Gamma(1+2\sqrt{-2mE})}\,\mathrm{e}^{(r'+r'')/2}$$

$$\times \left\{ W_{\alpha V_0, \sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{-r_<}) M_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{-r_>})\right.$$

$$- \frac{M_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0)}{W_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0)} M_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{-r'}) M_{\alpha V_0,\sqrt{-2mE}/\hbar}(2V_0\,\mathrm{e}^{-r''}) \Bigg\} . \tag{6.13.10}$$

6.13.2 Free Motion in Half-Space with General Boundary Conditions. [196,448]

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(\beta)}_{(x>0)} x(t) \exp\left(\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''} \dot{x}^2 \mathrm{d}t\right)$$

$$= \sqrt{\frac{m}{2\pi\mathrm{i}\,\hbar T}}\left[\exp\left(-\frac{m}{2\hbar\mathrm{i}\,T}(x''-x')^2\right) + \exp\left(-\frac{m}{2\hbar\mathrm{i}\,T}(x''+x')^2\right)\right]$$

$$- \frac{1}{\beta}\exp\left[\frac{\mathrm{i}\,\hbar T}{2m\beta^2} + \frac{x'+x''}{\beta}\right] \mathrm{erfc}\left[\sqrt{\frac{\mathrm{i}\,\hbar T}{2m\beta^2}} + \frac{x'+x''}{2\beta}\sqrt{\frac{2m\beta^2}{\mathrm{i}\,\hbar T}}\right] , \tag{6.13.11}$$

$$= \frac{2}{\pi}\int_0^\infty \mathrm{d}k\,\cos(kx''+\varphi_k)\cos(kx'+\varphi_k)\,\mathrm{e}^{-\mathrm{i}\,\hbar k^2 T/2m} \equiv K^{(\beta)}(T) , \tag{6.13.12}$$

with $\varphi_k = -\arctan(1/k\beta)$. The kernel (6.13.12) $K^{(\beta)}(T)$ satisfies the boundary conditions

$$\begin{aligned} K^{(\beta)}(x'',x';0) &= \delta(x''-x') \\ K^{(\beta)}(0,x';T) &= \beta\frac{\partial K^{(\beta)}}{\partial x''}(0,x';T) . \end{aligned} \tag{6.13.13}$$

For $\varphi_k \to 0$ ($\beta \to \infty$) we have Neumann boundary conditions at $x = 0$, and for $\varphi \to \frac{\pi}{2}$ ($\beta \to 0$) we have the usual total reflecting case (destructive interference at the origin, i.e., Dirichlet boundary conditions).

6.13.3 Motion Inside and Outside a Radial Box.

6.13.3.1 Motion in the Radial Space $r \gtrless a$. [191,439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(r=R)} x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(r)\right)\mathrm{d}t\right]$$

$$= \sum_{l\in\mathbb{N}_0} G^{(D)}_{(r=R)}(r'',r';E) S_l^\mu(\Omega'') S_l^\mu(\Omega') , \tag{6.13.14}$$

where the radial Green function is given by

$$G^{(D)}_{(r=R)}(r'',r';E) = G_l^{(V)}(r'',r';E) - \frac{G_l^{(V)}(r'',R;E)G_l^{(V)}(R,r';E)}{G_l^{(V)}(R,R;E)} . \tag{6.13.15}$$

6.13.3.2 Motion in the Radial Ring $a < r < b$.

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}^{(DD)}_{(a<r<b)} r(t)\mu_{l+\frac{D-2}{2}} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - V(r)\right)\mathrm{d}t\right]$$
$$= \frac{\begin{vmatrix} G_l^{(V)}(r'',r';E) & G_l^{(V)}(r'',b;E) & G_l^{(V)}(r'',a;E) \\ G_l^{(V)}(b,r';E) & G_l^{(V)}(b,b;E) & G_l^{(V)}(b,a;E) \\ G_l^{(V)}(a,r';E) & G_l^{(V)}(a,b;E) & G_l^{(V)}(a,a;E) \end{vmatrix}}{\begin{vmatrix} G_l^{(V)}(b,b;E) & G_l^{(V)}(b,a;E) \\ G_l^{(V)}(a,b;E) & G_l^{(V)}(a,a;E) \end{vmatrix}} \ . \tag{6.13.16}$$

The bound states are given by the quantization rule ($n \in \mathbb{N}_0$)

$$\begin{vmatrix} G_l^{(V)}(b,b;E_n) & G_l^{(V)}(b,a;E_n) \\ G_l^{(V)}(a,b;E_n) & G_l^{(V)}(a,a;E_n) \end{vmatrix} = 0 \ . \tag{6.13.17}$$

The incorporation of Neumann boundary conditions can be done in a similar way as in Sect. 6.13.1.

6.13.3.2.1 Free Particle with Dirichlet-Dirichlet Boundary Conditions. The bound states for a free particle in a radial ring $a < r < b$ are determined by ($n \in \mathbb{N}_0$)

$$I_{l+\frac{D-2}{2}}\left(\sqrt{-2mE_n}\,\frac{a}{\hbar}\right)K_{l+\frac{D-2}{2}}\left(\sqrt{-2mE_n}\,\frac{b}{\hbar}\right)$$
$$I_{l+\frac{D-2}{2}}\left(\sqrt{-2mE_n}\,\frac{b}{\hbar}\right)K_{l+\frac{D-2}{2}}\left(\sqrt{-2mE_n}\,\frac{a}{\hbar}\right) \ . \tag{6.13.18}$$

6.13.3.2.2 Radial Harmonic Oscillator with Dirichlet-Dirichlet Boundary Conditions. The bound states for the radial harmonic oscillator in a radial ring $a < r < b$ are determined by ($n \in \mathbb{N}_0$)

$$M_{\frac{E_n}{2\hbar\omega},\lambda/2}\left(\frac{m\omega}{\hbar}a^2\right)W_{\frac{E_n}{2\hbar\omega},\lambda/2}\left(\frac{m\omega}{\hbar}b^2\right)$$
$$= W_{\frac{E_n}{2\hbar\omega},\lambda/2}\left(\frac{m\omega}{\hbar}a^2\right)M_{\frac{E_n}{2\hbar\omega},\lambda/2}\left(\frac{m\omega}{\hbar}b^2\right) \ . \tag{6.13.19}$$

6.13.3.2.3 Coulomb Potential with Dirichlet-Dirichlet Boundary Conditions. The bound states for the Coulomb potential in three dimensions in a radial ring $a < r < b$ are determined by ($\kappa = e_1e_2\sqrt{-m/2E_n}/\hbar, n \in \mathbb{N}_0$)

$$M_{\kappa,l+1/2}\left(\sqrt{-8mE_n}\,\frac{a}{\hbar}\right)W_{\kappa,l+1/2}\left(\sqrt{-8mE_n}\,\frac{b}{\hbar}\right)$$
$$= W_{\kappa,l+1/2}\left(\sqrt{-8mE_n}\,\frac{a}{\hbar}\right)M_{\kappa,l+1/2}\left(\sqrt{-8mE_n}\,\frac{b}{\hbar}\right) \ . \tag{6.13.20}$$

6.13.4 Potentials with Absolute-Value Dependence.

6.13.4.1 General Formula. [439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - V(|x|)\right)\mathrm{d}t\right]$$
$$= G^{(V)}(x'',x';E)$$
$$- \frac{G^{(V)}(x'',0;E)G^{(V)}(0,x';E)}{2G^{(V)}(0,0;E)} - \frac{G^{(V)}_{,x'}(x'',0;E)G^{(V)}_{,x''}(0,x';E)}{2G^{(V)}_{,x'x''}(0,0;E)} \ . \tag{6.13.21}$$

6.13.4.2 The Linear Potential. [439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{x}^2 - k|x|\right)\mathrm{d}t\right]$$
$$= \frac{4}{3}\frac{m}{\hbar^2}\left[\left(x'-\frac{E}{k}\right)\left(x''-\frac{E}{k}\right)\right]^{1/2}$$
$$\times\left\{K_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x_> - \frac{E}{k}\right)^{3/2}\right) I_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x_< - \frac{E}{k}\right)^{3/2}\right)\right.$$
$$- \frac{1}{2}K_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x' - \frac{E}{k}\right)^{3/2}\right) K_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(x'' - \frac{E}{k}\right)^{3/2}\right)$$
$$\times\left[I_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(-\frac{E}{k}\right)^{3/2}\right)\Big/K_{1/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(-\frac{E}{k}\right)^{3/2}\right)\right.$$
$$\left.\left. - 2\pi I_{2/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(-\frac{E}{k}\right)^{3/2}\right)\Big/K_{2/3}\left(\frac{2}{3}\frac{\sqrt{2mk}}{\hbar}\left(-\frac{E}{k}\right)^{3/2}\right)\right]\right\} . \tag{6.13.22}$$

6.13.4.3 The Double Oscillator. [439]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\left[\dot{x}^2 - \omega^2(|x|-a)^2\right]\mathrm{d}t\right\}$$
$$= \sqrt{\frac{m}{\pi\hbar^3\omega}}\,\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega}\right)$$

$$\times \left\{ D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}(x_> - a)\right) D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}(x_< - a)\right) \right.$$

$$- D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}(x' - a)\right) D_{\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}(x'' - a)\right)$$

$$\left. \times \frac{1}{2}\left[\frac{D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}a\right)}{D_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}a\right)} - \frac{D'_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(\sqrt{\frac{2m\omega}{\hbar}}a\right)}{D'_{-\frac{1}{2}+\frac{E}{\hbar\omega}}\left(-\sqrt{\frac{2m\omega}{\hbar}}a\right)}\right]\right\} . \tag{6.13.23}$$

6.14 Coherent States

6.14.1 The Coherent State Path Integral. [534,595,814,835] ($\hbar = 1$)

$$\left\langle a'' \middle| \exp\left[-\mathrm{i}T\underline{\mathrm{H}}(\underline{a}^\dagger, \underline{a})\right] \middle| a' \right\rangle = \lim_{N\to\infty} \int \prod_{j=1}^{N-1} \frac{\mathrm{d}a_j\,\mathrm{d}a_j^*}{2\pi\,\mathrm{i}}$$

$$\times \exp\left[a_N^* a_N + \mathrm{i}\sum_{j=1}^{N}\left(\mathrm{i}\,a_j^*(a_j - a_{j-1}) - \epsilon H(a_j^*, a_j)\right)\right] \tag{6.14.1}$$

$$\equiv \int\limits_{a(t')=a'}^{a^*(t'')=a''^*} \mathcal{D}a(t)\mathcal{D}a^*(t) \exp\left[a''^* a(t'') + \mathrm{i}\int_{t'}^{t''}\left(\mathrm{i}\,a^*\dot{a} - H(a^*, a)\right)\mathrm{d}t\right] . \tag{6.14.2}$$

6.14.1.1 The Harmonic Oscillator. [80,217,566,595,752,828]

$$\int\limits_{a(t')=a'}^{a^*(t'')=a''^*} \mathcal{D}a(t)\mathcal{D}a^*(t) \exp\left[a''^* a(t'') + \mathrm{i}\int_{t'}^{t''}(\mathrm{i}\,a^*\dot{a} - \omega a^* a)\,\mathrm{d}t\right] = \mathrm{e}^{a''^*\,\mathrm{e}^{-\mathrm{i}\omega T}\,a'} . \tag{6.14.3}$$

6.14.1.2 General Quadratic Hamiltonian. [493,686]

$$\int\limits_{a(t')=a'}^{a^*(t'')=a''^*} \mathcal{D}a(t)\mathcal{D}a^*(t) \exp\left[a''^* a(t'') + \mathrm{i}\int_{t'}^{t''}\left(\mathrm{i}\,a^*\dot{a} - H(a^*, a)\right)\mathrm{d}t\right]$$

$$= \exp\left\{-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\left[2f(t)X(t) + f(t)Z^2(t) + g(t)Z(t)\right]\right.$$

$$+Y(t'')a''^*a' + X(t'')(a'^*)^2 + Z(t'')a''^*$$
$$-\mathrm{i}(a')^2\int_{t'}^{t''}\mathrm{d}t\, f(t)Y^2(t) - \mathrm{i}\,a'\int_{t'}^{t''}\mathrm{d}t\Big[g(t)+2f(t)Z(t)\Big]Y(t)\Big\} \ , \tag{6.14.4}$$

$$\underline{\mathrm{H}}(\underline{\mathrm{a}}^\dagger,\underline{\mathrm{a}};t) = \omega(t)\underline{\mathrm{a}}^\dagger\underline{\mathrm{a}} + f(t)\underline{\mathrm{a}}^2 + f(t)^*\underline{\mathrm{a}}^{\dagger\,2} + g(t)\underline{\mathrm{a}} + g^*(t)\underline{\mathrm{a}}^\dagger \ , \tag{6.14.5}$$

$$\dot X(t) = -2\,\mathrm{i}\,\omega(t)X(t) - 4\,\mathrm{i}\,f(t)X^2(t) - \mathrm{i}\,f^*(t) \ , \qquad X(t')=0 \ , \tag{6.14.6}$$

$$Y(t) = \exp\left[-\mathrm{i}\int_{t'}^{t}\Big(\omega(s)+4f(s)X(s)\Big)\mathrm{d}s\right] \ , \tag{6.14.7}$$

$$Z(t) = -\mathrm{i}\int_{t'}^{t}\mathrm{d}s\Big(g^*(s)+2g(s)X(s)\Big)\exp\left[-\mathrm{i}\int_{s}^{t}\Big(\omega(\tau)+4f(\tau)X(\tau)\Big)\,\mathrm{d}\tau\right] \ . \tag{6.14.8}$$

6.14.1.3 Degenerate Parametric Amplifier. [493,534,917]

$$\int\limits_{a(t')=a'}^{a^*(t'')=a''^*}\mathcal{D}a(t)\mathcal{D}a^*(t)\exp\left[a''^*a(t'') + \mathrm{i}\int_{t'}^{t''}\Big(\mathrm{i}\,a^*\dot a - H(a^*,a)\Big)\mathrm{d}t\right]$$
$$= \sqrt{\operatorname{sech}(2\kappa T)}\exp\Big[a''^*a'\,\mathrm{e}^{-\mathrm{i}\omega T}\operatorname{sech}(2\kappa T)$$
$$-\frac{\mathrm{i}}{2}(a''^*)^2\,\mathrm{e}^{-2\,\mathrm{i}\,\omega t''}\tanh(2\kappa T) - \frac{\mathrm{i}}{2}(a')^2\,\mathrm{e}^{2\,\mathrm{i}\,\omega t'}\tanh(2\kappa T)\Big] \ , \tag{6.14.9}$$

$$H(\underline{\mathrm{a}}^\dagger,\underline{\mathrm{a}};t) = \omega\underline{\mathrm{a}}^\dagger\underline{\mathrm{a}} + \kappa(\mathrm{e}^{2\,\mathrm{i}\,\omega t}\,\underline{\mathrm{a}}^2 + \mathrm{e}^{-2\,\mathrm{i}\,\omega t}\,\underline{\mathrm{a}}^{\dagger\,2}) \ . \tag{6.14.10}$$

6.14.1.4 Time-Dependent Forced Harmonic Oscillator. [534]

$$\int\limits_{a(t')=a'}^{a^*(t'')=a''^*}\mathcal{D}a(t)\mathcal{D}a^*(t)\exp\left[a''^*a(t'') + \mathrm{i}\int_{t'}^{t''}\Big(\mathrm{i}\,a^*\dot a - H(a^*,a)\Big)\mathrm{d}t\right]$$
$$= \exp\Big[a''^*\,\mathrm{e}^{-\,\mathrm{i}\,\omega T}\,a' + \mathrm{i}\int_{t'}^{t''}\mathrm{d}t\big[a''^*\,\mathrm{e}^{-\,\mathrm{i}\,\omega(t''-t)}\,f(t) + f^*(t)\,\mathrm{e}^{-\,\mathrm{i}\,\omega(t-t')}\,a'$$
$$-\int_{t'}^{t''}\int_{t'}^{t''}\mathrm{d}t\,\mathrm{d}s\Theta(t-s)f^*(t)\,\mathrm{e}^{-\,\mathrm{i}\,\omega(t-s)}\,f(s)\Big] \ , \tag{6.14.11}$$

$$\underline{\mathrm{H}}(\underline{\mathrm{a}}^\dagger,\underline{\mathrm{a}}) = \omega\underline{\mathrm{a}}^\dagger\underline{\mathrm{a}} - f(t)\underline{\mathrm{a}}^\dagger - f^*(t)\underline{\mathrm{a}} \ . \tag{6.14.12}$$

6.14.1.5 Multidimensional General Quadratic Hamiltonian. [493] (matrix multiplication understood)

$$\int\limits_{\mathbf{a}(t')=\mathbf{a}'}^{\mathbf{a}^*(t'')=\mathbf{a}''^*} \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{a}^*(t)\exp\left[\mathbf{a}''^*\mathbf{a}(t'')+\mathrm{i}\int_{t'}^{t''}\Big(\mathrm{i}\,\mathbf{a}^*\dot{\mathbf{a}}-H(\mathbf{a}^*,\mathbf{a})\Big)\mathrm{d}t\right]$$
$$=\exp\left[-2\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\mathrm{Tr}[\mathbf{f}(t)\mathbf{X}(t)]+\mathbf{a}^{*\prime\prime}\mathbf{Y}(t'')\mathbf{a}'\right.$$
$$\left.+\,\mathbf{a}^{*\prime\prime}\mathbf{X}(t'')\mathbf{a}^{*\prime\prime}-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\mathbf{a}'\mathbf{Y}(t)\mathbf{f}(t)\mathbf{Y}(t)\mathbf{a}'\right]\ , \tag{6.14.13}$$

$$\underline{\mathrm{H}}(\underline{\mathbf{a}}^\dagger,\underline{\mathbf{a}};t)=\underline{\mathbf{a}}^\dagger\boldsymbol{\omega}\underline{\mathbf{a}}+\underline{\mathbf{a}}\mathbf{f}(t)\underline{\mathbf{a}}+\underline{\mathbf{a}}^\dagger\mathbf{f}^*(t)\underline{\mathbf{a}}^\dagger\ , \tag{6.14.14}$$

$$\dot{\mathbf{X}}(t)=-\mathrm{i}(\{\boldsymbol{\omega}(t),\mathbf{X}(t)\}+4\mathbf{X}(t)\mathbf{f}(t)\mathbf{X}(t)+\mathbf{f}^*(t)\ , \tag{6.14.15}$$

$$\mathbf{Y}(t)=T\exp\left[-\mathrm{i}\int_{t'}^{t}\Big(\boldsymbol{\omega}(s)+4\mathbf{X}(s)\mathbf{f}(s)\Big)\mathrm{d}s\right]\ . \tag{6.14.16}$$

6.14.1.6 Coupled Parametric Amplifier. [493] ($\omega_3=\omega_1+\omega_2$)

$$\int\limits_{\mathbf{a}(t')=\mathbf{a}'}^{\mathbf{a}^*(t'')=\mathbf{a}''^*} \mathcal{D}\mathbf{a}(t)\mathcal{D}\mathbf{a}^*(t)\exp\left[\mathbf{a}''^*\mathbf{a}(t'')+\mathrm{i}\int_{t'}^{t''}\Big(\mathrm{i}\,\mathbf{a}^*\dot{\mathbf{a}}-H(\mathbf{a}^*,\mathbf{a})\Big)\mathrm{d}t\right]$$
$$=\sqrt{\mathrm{sech}(\kappa T)}\exp\left[-\frac{\mathrm{i}}{2}\mathbf{a}''^*\sigma_1\mathbf{a}''^*\,\mathrm{e}^{-\mathrm{i}\omega_3t''}\tanh(\kappa T)\right.$$
$$\left.-\frac{\mathrm{i}}{2}\mathbf{a}'\sigma_1\mathbf{a}'\,\mathrm{e}^{\mathrm{i}\omega_3t'}\tanh(\kappa T)+\mathrm{sech}(\kappa T)\mathbf{a}''^*\,\mathrm{diag}(\mathrm{e}^{-\mathrm{i}\omega_1T},\mathrm{e}^{-\mathrm{i}\omega_2T})\mathbf{a}'\right]\ , \tag{6.14.17}$$

$$\underline{\mathrm{H}}(\underline{\mathbf{a}}^\dagger,\underline{\mathbf{a}};t)=\sum_{i=1,2}\omega_i\underline{\mathrm{a}}_i^\dagger\underline{\mathrm{a}}_i+\kappa(\mathrm{e}^{\mathrm{i}\omega_3t}\,\underline{\mathrm{a}}_1\underline{\mathrm{a}}_2+\mathrm{e}^{-\mathrm{i}\omega_3t}\,\underline{\mathrm{a}}_1^\dagger\underline{\mathrm{a}}_2^\dagger)\ . \tag{6.14.18}$$

6.14.1.7 Feynman Kernel for a Spinning Particle in a Magnetic Field via Coherent States. [114,115] ($\mathbf{x}\in\mathbb{R}^3$, $m_{i,f}$ denote the initial and final spin states, and $\underline{\mathrm{P}}_s$ the projection onto a well-defined value of a spin s)

$$K_s(m_f,\mathbf{x}_f,m_i,\mathbf{x}_i;T)=\int\frac{\mathrm{d}\boldsymbol{\alpha}_f}{\pi}\frac{\mathrm{d}\boldsymbol{\beta}_f}{\pi}\frac{\mathrm{d}\boldsymbol{\alpha}_i}{\pi}\frac{\mathrm{d}\boldsymbol{\beta}_i}{\pi}\langle m_f|\boldsymbol{\alpha}_f,\boldsymbol{\beta}_f\rangle$$
$$\times\langle\mathbf{x}_f,\boldsymbol{\alpha}_f,\boldsymbol{\beta}_f|\,\mathrm{e}^{-\mathrm{i}T(\underline{\mathrm{p}}^2/2m-\gamma\underline{\mathbf{S}}\mathbf{B})}\,\underline{\mathrm{P}}_s|\mathbf{x}_i,\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i\rangle\langle\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i|m_i\rangle$$
$$=\frac{1}{(m_i-m_f)!}\sqrt{\frac{(s+m_i)!(s-m_f)!}{(s+m_f)!(s-m_i)!}}\int\limits_{\mathbf{x}(0)=\mathbf{x}_i}^{\mathbf{x}(T)=\mathbf{x}_f}\mathcal{D}\mathbf{x}(t)\exp\left(\frac{\mathrm{i}m}{2}\int_0^T\dot{\mathbf{x}}^2dt\right)$$

$$\times \left[R_{11}(\mathbf{x},T)\right]^{s+m_f}\left[R_{11}^*(\mathbf{x},T)\right]^{s-m_i}\left[R_{12}^*(\mathbf{x},T)\right]^{m_i-m_f}$$
$$\times\, {}_2F_1\left(-s+m_i,-s-m_f;m_i-m_f+1;-\left|\frac{R_{12}(\mathbf{x},T)}{R_{11}(\mathbf{x},T)}\right|^2\right) \ , \tag{6.14.19}$$

$$R_{11}(\mathbf{x},T) = \mathrm{e}^{\mathrm{i}\int_0^T \omega(\mathbf{x},s)\,\mathrm{d}s} + \sum_{n=1}^{\infty}\left[(\mathrm{i})^{2n}\int_0^T \mathrm{d}s_1\cdots\int_0^{s_{2n-1}} \mathrm{d}s_{2n}\right.$$
$$\times\, \mathrm{e}^{\mathrm{i}\int_{s_1}^T \omega(\mathbf{x},s)\,\mathrm{d}s}\, u(\mathbf{x}(s_1),s_1)\,\mathrm{e}^{-\mathrm{i}\int_{s_2}^{s_1}\omega(\mathbf{x},s)\,\mathrm{d}s}\, u^*(\mathbf{x}(s_2),s_2)\dots$$
$$\left.\times \dots \mathrm{e}^{-\mathrm{i}\int_{s_{2n}}^{s_{2n-1}}\omega(\mathbf{x},s)\,\mathrm{d}s}\, u^*(\mathbf{x}(s_{2n}),s_{2n})\,\mathrm{e}^{-\mathrm{i}\int_0^{s_{2n}}\omega(\mathbf{x},s)\,\mathrm{d}s}\right] \ , \tag{6.14.20}$$

$$\left.\begin{aligned}
&R_{12}(\mathbf{x},t) = \mathrm{i}\int_0^T \mathrm{d}s_1\,\mathrm{e}^{\mathrm{i}\int_{s_1}^T \omega(\mathbf{x},s)\,\mathrm{d}s}\, u(\mathbf{x}(s_1),s_1)R_{22}(\mathbf{x}(s_1),s_1) \ ,\\
&R_{22}(\mathbf{x},T) = R_{11}^*(\mathbf{x},T) \ , \qquad R_{21}(\mathbf{x},T) = -R_{12}^*(\mathbf{x},T) \ ,\\
&u(\mathbf{x},t) = \frac{\gamma}{2}(B_x(\mathbf{x},t)-\mathrm{i}\,B_y(\mathbf{x},t)) \ , \qquad \omega(\mathbf{x},t) = \frac{\gamma}{2}B_z(\mathbf{x},t) \ ,\\
&\gamma\underline{\mathrm{S}}\mathbf{B}(\mathbf{x},t) = u(\mathbf{x},t)\underline{\mathrm{a}}^\dagger\, b + u^*(\mathbf{x},t)\underline{\mathrm{b}}^\dagger\underline{\mathrm{a}} - \omega(\mathbf{x},t)(\underline{\mathrm{a}}^\dagger\underline{\mathrm{a}} - \underline{\mathrm{b}}^\dagger\underline{\mathrm{b}}) \ .
\end{aligned}\right\} \tag{6.14.21}$$

6.14.1.8 Rotating Magnetic Field. [114] The model $\mathbf{B}(\mathbf{x},t) = \delta(\mathbf{x})\mathbf{B}(t)$ with $B(t) = B_0 + B_1(\mathrm{e}_1\cos\omega t + \mathrm{e}_2\sin\omega t)$ is exactly solvable ($\omega_{0,1} = \gamma B_{0,1}/2$, the Green functions $G^{\pm}(s)$ are the Green functions for the $\pm\omega_0\delta(q)$ potential):

$$G_{1/2}(\uparrow,q'',\uparrow,q';s) = \int_0^\infty \mathrm{d}T\,\mathrm{e}^{-sT}\,K_{1/2}(u,q'',u,q';s)$$
$$= G^+(q'',q';s) - \omega_1^2\frac{G^+(q'',0;s)G^+(0,q';s)}{1+\omega_1^2 G^+(0,0;s)G^-(0,0;s-\mathrm{i}\,\omega)} \ . \tag{6.14.22}$$

6.14.1.9 Generalized Jaynes-Cummings Model. [115] (Notation as in previous section. $\Delta = (m-n)\omega_0 - \omega, \omega_{mn} = g^2[(k+n)!]^2/(k!(k+n-m)!), \Omega_1 = \sqrt{\Delta^2 + 4\omega_{mn}^2}\,, \Omega_2 = \sqrt{\Delta^2+4\omega_{nm}^2}$, and for $m_{i,f}$ spins up $\uparrow$ and down $\downarrow$ are evaluated explicitly)

$$K_{\uparrow\uparrow}(Z_f,Z_i;T) = \mathrm{e}^{\mathrm{i}(m-n-1)\omega T/2}\sum_{k=0}^{\infty}\frac{(Z_f^*\,\mathrm{e}^{-\mathrm{i}\omega T}\,Z_i)^k}{k!}$$
$$\times\left(\cos\frac{\Omega_1 T}{2} - \mathrm{i}\frac{\Delta}{\Omega_1}\sin\frac{\Omega_1 T}{2}\right) \ , \tag{6.14.23}$$
$$K_{\uparrow\downarrow}(Z_f,Z_i;T) = -\mathrm{i}\,g\,\mathrm{e}^{\mathrm{i}(m-n-1)\omega T/2}(Z_f^*)^{n-m}$$
$$\times\sum_{k=0}^{\infty}\frac{(Z_f^*\,\mathrm{e}^{-\mathrm{i}\omega T}\,Z_i)^k}{k!}\frac{(k+n)!}{(k+n-m)!}\frac{\sin\frac{\Omega_1 T}{2}}{\Omega_1/2} \ , \tag{6.14.24}$$
$$K_{\downarrow\downarrow}(Z_f,Z_i;T) = \mathrm{e}^{\mathrm{i}(m-n-1)\omega T/2}$$

$$\times \sum_{k=0}^{\infty} \frac{(Z_f^* \, e^{-i\omega T} \, Z_i)^k}{k!} \left(\cos \frac{\Omega_2 T}{2} + i \frac{\Delta}{\Omega_2} \sin \frac{\Omega_2 T}{2} \right) , \tag{6.14.25}$$

$$K_{\downarrow\uparrow}(Z_f, Z_i; T) = -i \, g \, e^{-i(m-n+1)\omega T/2} (Z_i)^{n-m}$$
$$\times \sum_{k=0}^{\infty} \frac{(Z_f^* \, e^{-i\omega T} \, Z_i)^k}{k!} \frac{(k+n)!}{(k+n-m)!} \frac{\sin \frac{\Omega_1 T}{2}}{\Omega_1/2} , \tag{6.14.26}$$

$$\underset{\sim}{H} = \omega(\underline{a}^\dagger \underline{a} + \tfrac{1}{2}) + \tfrac{\omega_0}{2} \sigma_z + g \Big[(\underline{a}^\dagger)^m (\underline{a})^n \sigma_+ + (\underline{a}^\dagger)^n (\underline{a})^m \sigma_- \Big] , \qquad (n \geq m) . \tag{6.14.27}$$

6.14.2 The Regularized Coherent State Path Integral. [220,512,596–604,603, 625]

$$\Big\langle p'', q'' \Big| \exp \Big[-i T \underset{\sim}{H}(\underset{\sim}{p}, \underset{\sim}{q}) \Big] \Big| p', q' \Big\rangle$$
$$= \lim_{\nu \to \infty} 2\pi \, e^{\nu T/2} \int\limits_{\substack{p(t')=p' \\ q(t')=q'}}^{\substack{p(t'')=p'' \\ q(t'')=q''}} \mathcal{D}\mu^\nu(p,q) \exp \left[i \int_{t'}^{t''} \left(\frac{1}{2}(p dq - q dp) - H(p,q) \right) dt \right] \tag{6.14.28}$$

$$\int\limits_{\substack{p(t')=p' \\ q(t')=q'}}^{\substack{p(t'')=p'' \\ q(t'')=q''}} \mathcal{D}\mu^\nu(p,q) = \frac{1}{2\pi\nu T} \exp \left[- \frac{(p''-p')^2 + (q''-q')^2}{2\nu T} \right] , \tag{6.14.29}$$

and $H(\underset{\sim}{p}, \underset{\sim}{q})$ may be any polynomial Hamiltonian.

6.14.3 Path Integral for Spin System. [124,486] ($\mathbf{n}$ is a unit vector on S^2, $\mathbf{A}$ is defined via $\nabla_{\mathbf{n}} \wedge \mathbf{A}(\mathbf{n}) = \mathbf{n}$ such that $\int_0^\beta dt \mathbf{A}\dot{\mathbf{n}} = \int_0^\beta dt \int_0^1 d\tau \mathbf{n} \cdot \dot{\mathbf{n}} \times \mathbf{n}$)

$$\oint \mathcal{D}_E \mathbf{n}(s) \delta(|\mathbf{n}|^2 - 1) \exp \left[\int_0^{\beta\hbar} \Big(\mathbf{A}(\mathbf{n})\dot{\mathbf{n}} + \mathbf{n} \cdot \mathbf{B} \Big) s ds \right] = \sum_{m=-s}^{s} e^{-\beta|\mathbf{B}|m} . \tag{6.14.30}$$

6.14.4 Spin Quantization Phase-Space Model. [355,360,486,725] ($x = \lambda \cos\vartheta$, and $\cos\vartheta_j$ intermediate in time between φ_{j-1} and φ_j)

$$K(\varphi'', \varphi'; T) := \lim_{N\to\infty} \sum_{n \in \mathbb{Z}} \int \prod_{j=1}^{N} \frac{dx_j \, d\varphi_j}{2\pi} \delta(\varphi'' - \varphi_N - 2\pi n)$$
$$\times \exp \left[i \left(\epsilon (x_j - \lambda) \frac{\Delta\varphi_j}{\epsilon} + \epsilon\mu B x_j \right) \right] . \tag{6.14.31}$$

6.14.5 Spin in a Magnetic Field. [302]

$$\int\limits_{\Omega(t')=\Omega'}^{\Omega(t'')=\Omega''} \mathcal{D}\Omega(t) \exp\left[\mathrm{i}s\int_0^T \Big(\cos\vartheta\dot\varphi + \tfrac{\mathrm{i}}{4}\alpha(\dot\vartheta^2+\sin^2\vartheta\dot\varphi^2) - B\cos\vartheta\Big)\mathrm{d}t\right]\Bigg|_{\alpha=0}$$
$$= \left[\cos\frac{\vartheta''}{2}\cos\frac{\vartheta'}{2}\,\mathrm{e}^{\mathrm{i}(\varphi''-BT-\varphi')/2}\sin\frac{\vartheta''}{2}\sin\frac{\vartheta'}{2}\,\mathrm{e}^{-\mathrm{i}(\varphi''-BT-\varphi')/2}\right]^{2s} . \tag{6.14.32}$$

6.14.6 Coherent State Path Integral on Flag Manifold. [592]

$$\int\limits_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}^*(t'')=\mathbf{z}''^*} \mathcal{D}\mathbf{z}(t)\mathcal{D}\mathbf{z}^*(t)$$
$$\times\exp\left[m\log L_1(\mathbf{z}''^*,\mathbf{z}(t''))+m\log L_2(\mathbf{z}''^*,\mathbf{z}(t''))+\mathrm{i}\int_{t'}^{t''} L(\mathbf{z}^*,\mathbf{z},\mathbf{z}^*,\mathbf{z})\,\mathrm{d}t\right]$$
$$=\Big(1+z_1''^*\xi_1'\,\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)T}+z_2''^*\xi_2'\,\mathrm{e}^{\mathrm{i}(\omega_1+\omega_2)T}\Big)^m$$
$$\times\Big(1+z_3''^*z_3'\,\mathrm{e}^{-\mathrm{i}\omega_2 T}+(z_2''^*-z_1''^*z_3''^*)(z_2'-z_1'z_3')\,\mathrm{e}^{\mathrm{i}\omega_1 T}\Big)^n , \tag{6.14.33}$$

$$L_1(\mathbf{z}''^*,\mathbf{z}(t''))=(1+z_1''^*z_1''+z_2''^*z_2'') \tag{6.14.34}$$

$$L_2(\mathbf{z}''^*,\mathbf{z}(t''))=(1+z_3''^*z_3''+(z_2''^*-z_1''^*z_3''^*)(z_2''-z_1''z_3'') , \tag{6.14.35}$$

$$L(\mathbf{z}^*,\mathbf{z})=\mathrm{i}m\frac{z_1^*\dot z_1+z_2^*\dot z_2}{1+|z_1|^2+|z_2|^2}+\mathrm{i}n\frac{z_3^*\dot z_3+(z_2^*-z_1^*z_3^*)(\dot z_2-\dot z_1 z_3-z_1\dot z_3)}{1+|z_3|^2+|z_2-z_1z_3|^2}$$
$$-(\omega_1 Q_1(\mathbf{z})+\omega_2 Q_2(\mathbf{z})) , \tag{6.14.36}$$

$$Q_1(\mathbf{z})=m\frac{|z_1|^2+|z_2|^2}{1+|z_1|^2+|z_2|^2}+n\frac{|z_2-z_1z_3|^2}{1+|z_3|^2+|z_2-z_1z_3|^2} , \tag{6.14.37}$$

$$Q_2(\mathbf{z})=m\frac{|z_1|^2}{1+|z_1|^2+|z_2|^2}-n\frac{|z_2|^2}{1+|z_3|^2+|z_2-z_1z_3|^2} . \tag{6.14.38}$$

6.14.7 Coherent State Path Integral for SU(n). [360,733] ($\{\mu\}$ are real constants, the Hamiltonian $\underset{\sim}{H}$ is defined via isospin functions Q^m, J is the magnitude of the (classical) isospin, and $\boldsymbol{\xi}$ are coordinates on the complex projective space $\mathbb{M}=CP(N)$ with $|\boldsymbol{\xi}|^2=\sum_{m=1}^N \xi_m^*\xi_m=\boldsymbol{\xi}^*\boldsymbol{\xi}$)

$$\int\limits_{\boldsymbol{\xi}(t')=\boldsymbol{\xi}'}^{\boldsymbol{\xi}^*(t'')=\boldsymbol{\xi}''^*} \mathcal{D}\boldsymbol{\xi}(t)\mathcal{D}\boldsymbol{\xi}^*(t)$$

$$\times \exp\left[2J\log(1+\boldsymbol{\xi}''^*\boldsymbol{\xi}(t'')) + \mathrm{i}\int_{t'}^{t''}\left(\mathrm{i}\frac{2J\boldsymbol{\xi}^*\dot{\boldsymbol{\xi}}}{1+|\boldsymbol{\xi}|^2} - H(\boldsymbol{\xi}^*,\boldsymbol{\xi})\right)\mathrm{d}t\right]$$

$$= \left(1+\sum_{m=1}^{n}\xi_m^{*}{}''\xi_m'\,\mathrm{e}^{-\mathrm{i}\,\omega_m T}\right)^{2J}\exp\left(\mathrm{i}\,J\sum_{m=1}^{n}\frac{\mu_m}{\sqrt{m(m+1)/2T}}\right), \tag{6.14.39}$$

$$H(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = \sum_{m=1}^{N}\mu_m Q^m(\boldsymbol{\xi}^*,\boldsymbol{\xi}), \tag{6.14.40}$$

$$Q^m(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = \frac{-2J}{\sqrt{2m(m+1)}}\sum_{k=0}^{m-1}(u_k^*u_k - m u_m^* u_m)\Big|_{\substack{u_0=(1+|\xi|^2)^{-1/2}\\ u_m=u_0\xi_m}}, \tag{6.14.41}$$

$$\omega_k = \sum_{m=1}^{n}\frac{\mu_m}{\sqrt{2m(m+1)}}\Big[(m+1)\delta_{km} + \Theta(k-m)\Big]. \tag{6.14.42}$$

6.14.8 Generalized Coherent States for SU(2). [298,619] ($a(t), b(t)$ are determined via $\dot{a} = -\mathrm{i}\,Aa + \mathrm{i}\,fb^*, \dot{b}^* = -\mathrm{i}\,Ab - \mathrm{i}\,fa^*$ with the boundary conditions $a(0)=1, b(0)=0$)

$$\int\limits_{z(0)=z_2}^{z^*(T)=z_1^*} \frac{2J+1}{2\pi\mathrm{i}}\frac{\mathcal{D}z\mathcal{D}z^*}{(1+|z|^2)^2}\frac{(1+z_1^*z(T))^J(1+z^*(0)z_2)^J}{(1+|z_1|^2)(1+|z_2|^2)}$$

$$\times\exp\left(J\int_0^T\frac{\dot{z}^*z - z^*\dot{z}}{1+z^*z}\mathrm{d}t - \mathrm{i}\int_0^T H(z^*,z)\,\mathrm{d}s\right)$$

$$= \frac{\left(a^*(T) - b^*(T)z_2 + b(T)z_1^* + a(T)z_1^*z_2\right)^{2J}}{(1+|z_1|^2)^J(1+|z_2|^2)^J}, \tag{6.14.43}$$

$$H(z^*,z) = 2J\frac{-A(T)(1-|z|^2) + f(t)z^* + f^*(t)z}{1+|z|^2}. \tag{6.14.44}$$

6.14.9 Generalized Coherent States for SU(1,1). [298,383,619]

$$\int\limits_{z(0)=z_2}^{z^*(T)=z_1^*} \frac{2k-1}{2\pi\mathrm{i}}\frac{\mathcal{D}z\mathcal{D}z^*}{(1-|z|^2)^2}\frac{(1-z_1^*z(T))^k(1-z^*(0)z_2)^k}{(1-|z_1|^2)(1-|z_2|^2)}$$

$$\times\exp\left(k\int_0^T\frac{\dot{z}^*z - z^*\dot{z}}{1-z^*z}\mathrm{d}t - \mathrm{i}\int_0^T H(z^*,z)\,\mathrm{d}s\right)$$

$$= \frac{(1-|z_1|^2)^k(1-|z_2|^2)^k}{\big(a^*(T)+b^*(T)z_2-b(T)z_1^*-a(T)z_1^*z_2\big)^{2k}} \ , \tag{6.14.45}$$

$$\underset{\sim}{H} = 2A(t)\underset{\sim}{K}_0 + f(t)\underset{\sim}{K}_+ + f^*(t)\underset{\sim}{K}_- \ . \tag{6.14.46}$$

Here $a(t), b(t)$ are determined via $\dot a = -\mathrm{i}\,Aa - \mathrm{i}\,fb^*, \dot b^* = -\mathrm{i}\,Ab - \mathrm{i}\,fa^*$ with the boundary conditions $a(0)=1, b(0)=0$. $\underset{\sim}{K}_0, \underset{\sim}{K}_\pm$ span the $su(1,1)$ algebra $[\underset{\sim}{K}_0, \underset{\sim}{K}_\pm] = \pm\underset{\sim}{K}_\pm, [\underset{\sim}{K}_+, \underset{\sim}{K}_-] = -2\underset{\sim}{K}_0$.

6.14.10 Coherent State Path Integral for Anyons. [475] (With the transformation $\beta = \sqrt{2\mu}\zeta/\sqrt{1-|\zeta|^2}$; and μ describes via $\theta/\pi = 2\mu - 1/2$ the statistical behaviour, where $\theta = 0$ corresponds to bosons and $\theta = \pi$ to fermions, respectively; $|\zeta| < 1$)

$$\int\limits_{\zeta(0)=\zeta'}^{\zeta^*(T)=\zeta''^*} \frac{2\mu-1}{\pi(1-|\zeta|^2)}\mathcal{D}\zeta^*(t)\mathcal{D}\zeta(t)$$
$$\times \exp\left[\mu\int_0^T\left(\frac{\dot\zeta^*\zeta-\zeta^*\dot\zeta}{1-|\zeta|^2} - 2\,\mathrm{i}\,\frac{1+|\zeta|^2}{1-|\zeta|^2}\right)\mathrm{d}t\right] = \mathrm{e}^{\beta^*(T)\,\mathrm{e}^{-2\,\mathrm{i}(1+\mu)T}\,\beta(0)} \ . \tag{6.14.47}$$

6.14.11 Supercoherent State Path Integral for Osp(1|2;$\mathbb{R}$). [155,157, 657,822] ($\underset{\sim}{K}_\pm, \underset{\sim}{K}_0, \underset{\sim}{F}_\pm$ are the generators of $osp(1|2;\mathbb{R})$ with $[\underset{\sim}{K}_0, \underset{\sim}{K}_+] = \underset{\sim}{K}_+$, $[\underset{\sim}{K}_0, \underset{\sim}{K}_-] = -\underset{\sim}{K}_-, [\underset{\sim}{K}_+, \underset{\sim}{K}_-] = \pm 2\underset{\sim}{K}_0, [\underset{\sim}{K}_0, \underset{\sim}{F}_\pm] = \pm\underset{\sim}{F}_\pm/2$ with a sign ambiguity in, e.g., $\underset{\sim}{K}_-$; $\phi, \bar\phi, \theta, \bar\theta, \chi, \bar\chi$ are Grassmann variables; τ is the Casimir index)

$$\left\langle z'', \phi'' \left| \exp\left[-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\,\underset{\sim}{H}(t)\right]\right| z', \phi' \right\rangle$$
$$= \bigg(u^*(t'',t') \pm v(t'',t')z'^* \mp u^*(t'',t')z'' \pm u(t'',t')z'^*z'' \pm \frac{\phi''}{\sqrt2}\theta(t'',t')z'^*$$
$$\mp \frac{\bar\chi(t'',t')}{\sqrt2}\bar\phi' \mp \frac{\chi(t'',t')}{\sqrt2}\bar\phi' z'' - \frac{\phi''}{\sqrt2}\theta(t'',t') - \frac{\lambda(t'',t')}{2}\bar\phi'\phi''\bigg)^{\pm 2\tau}$$
$$\times \left(1 \pm z''^*z'' - \tfrac12\bar\phi'\phi''\right)^{\mp\tau}\left(1 \pm z'^*z' - \tfrac12\bar\phi'\phi'\right)^{\pm\tau} \ . \tag{6.14.48}$$

$$\underset{\sim}{H}(t) = A(t)\underset{\sim}{K}_0 + f(t)\underset{\sim}{K}_+ + f^*(t)\underset{\sim}{K}_- + \theta(t)\underset{\sim}{F}_+ - \bar\theta(t)\underset{\sim}{F}_- \ . \tag{6.14.49}$$

The coefficients $u(t), v(t), \chi(t)$ are determined by solving the coupled equations

$$\left.\begin{aligned}
\dot{u}(t) &= -\mathrm{i}\left(A(t)u(t) \pm f(t)v^*(t) \pm \frac{\vartheta(t)}{\sqrt{2}}\chi(t)\right) \ , \\
\dot{v}^*(t) &= \mathrm{i}\left(f(t)u^*(t) + \tfrac{1}{2}A(t)v^*(t) \pm \frac{\bar{\vartheta}(t)}{\sqrt{2}}\chi(t)\right) \ , \\
\dot{\chi}(t) &= \frac{i}{\sqrt{2}}\left(\bar{\vartheta}(t)u(t) - \theta(t)v^*(t)\right) \ .
\end{aligned}\right\} \qquad (6.14.50)$$

6.15 Fermions

6.15.1 The Fermionic Path Integral.

6.15.1.1 The General Fermionic Path Integral Via Coherent States. [217,313, 675,686,734,855,883]

$$\begin{aligned}
&\left\langle \eta'' \middle| \exp\left(-\mathrm{i}\,H(\underline{\mathrm{a}}^\dagger, \underline{\mathrm{a}}; t)\right) \middle| \eta' \right\rangle \\
&= \lim_{N\to\infty} \prod_{j=1}^{N-1} \int \mathrm{d}\bar{\eta}_j \, \mathrm{d}\eta_j \\
&\quad \times \exp\left\{ \bar{\eta}_N \eta_N + \mathrm{i} \sum_{j=1}^{N} \left[\mathrm{i}\,\bar{\eta}_j(\eta_j - \eta_{j-1}) - \epsilon H(\bar{\eta}_j, \eta_j; t) \right] \right\} \qquad (6.15.1) \\
&= \int\limits_{\eta(t')=\eta'}^{\bar{\eta}(t'')=\bar{\eta}''} \mathcal{D}\bar{\eta}(t)\mathcal{D}\eta(t) \exp\left[\bar{\eta}''\eta(t'') + \mathrm{i} \int_{t'}^{t''} \left(\mathrm{i}\,\bar{\eta}(t)\dot{\eta}(t) - H(\bar{\eta}, \eta; t) \right) \mathrm{d}t \right] .
\end{aligned}$$

(6.15.2)

6.15.1.2 Constant Magnetic Field. [534]

$$\begin{aligned}
&\int\limits_{\eta(t')=\eta'}^{\bar{\eta}(t'')=\bar{\eta}''} \mathcal{D}\bar{\eta}(t)\mathcal{D}\eta(t) \exp\left[\bar{\eta}''\eta(t'') + \mathrm{i} \int_{t'}^{t''} \left(\mathrm{i}\,\bar{\eta}(t)\dot{\eta}(t) - \mu B(2\bar{\eta}\eta - 1) \right) \mathrm{d}t \right] \\
&= \exp\left[\mathrm{i}\,\mu B T + \bar{\eta}''\eta'\, \mathrm{e}^{-2\mathrm{i}\,\mu B T} \right] \ . \qquad (6.15.3)
\end{aligned}$$

6.15.1.3 Forced Harmonic Oscillator: The Generating Functional. [369,490, 730,855]

$$\int\limits_{\eta(t')=\eta'}^{\bar\eta(t'')=\bar\eta''} \mathcal{D}\bar\eta(t)\mathcal{D}\eta(t)$$

$$\times \exp\left\{\bar\eta''\eta(t'') + \mathrm{i}\int_{t'}^{t''}\left[\mathrm{i}\,\bar\eta(t)\dot\eta(t) - \Big(B(t)(\bar\eta\eta - \tfrac{1}{2}) - \bar J(t)\eta - \bar\eta J(t)\Big)\right]\mathrm{d}t\right\}$$

$$= K_{00} + K_{11}\bar\eta''\eta' + \bar\eta''K_{10} + K_{01}\eta' \ , \tag{6.15.4}$$

$$K_{00} = \exp\left(\frac{\mathrm{i}}{2}\int_{t'}^{t''} B(t)\,\mathrm{d}t\right)\exp\left(-\int_{t'}^{t''}\mathrm{d}t\int_{t'}^{t''}\mathrm{d}s\bar J(t)D_F(t,s)J(s)\right) \ , \tag{6.15.5a}$$

$$D_F(t,s) = \Theta(t-s)\exp\left(-\mathrm{i}\int_s^t \mathrm{d}\tau\, B(\tau)\right) \ , \tag{6.15.5b}$$

$$K_{11} = \exp\left(-\mathrm{i}\int_{t'}^{t''} B(t)\,\mathrm{d}t\right)K_{00} \ , \tag{6.15.5c}$$

$$K_{10} = \mathrm{i}\int_{t'}^{t''}\mathrm{d}t\, J(t)\exp\left(-\mathrm{i}\int_t^{t''} B(s)\,\mathrm{d}s\right)K_{00} \ , \tag{6.15.5d}$$

$$K_{01} = \mathrm{i}\int_{t'}^{t''}\mathrm{d}t\, \bar J(t)\exp\left(-\mathrm{i}\int_{t'}^{t} B(s)\,\mathrm{d}s\right)K_{00} \ . \tag{6.15.5e}$$

6.15.2 Time-Ordered Correlation Function – Generating Functional. [76,217,340,397,534,790,799] (D_F is the Feynman propagator)

$$\Big\langle T\big(\Psi(t_1)\bar\Psi(t_2)\big)\Big\rangle = \frac{(-\mathrm{i})^2}{Z[\Theta,\bar\Theta]}\frac{\delta^2 Z[\Theta,\bar\Theta]}{\delta\bar\Theta(t_1)\delta\Theta(t_2)}\bigg|_{\Theta=\bar\Theta=0} = \mathrm{i}\,D_F(t_1-t_2) \ , \tag{6.15.6}$$

$$Z[\Theta,\bar\Theta] = \int\mathcal{D}\Psi\mathcal{D}\bar\Psi\exp\left[\mathrm{i}\int_{\mathbb{R}}\big(\mathcal{L}(\Psi,\bar\Psi) + \bar\Theta\Psi + \bar\Psi\Theta\big)\mathrm{d}t\right] \ , \tag{6.15.7}$$

$$= \int\mathcal{D}\Psi\mathcal{D}\bar\Psi\exp\left[\mathrm{i}\int_{\mathbb{R}}\mathcal{L}(\Psi,\bar\Psi)\mathrm{d}t - \frac{\mathrm{i}}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathrm{d}t\,\mathrm{d}t'\,\bar\Theta(t)D_F(t-t')\Theta(t')\right] . \tag{6.15.8}$$

6.16 Supersymmetric Quantum Mechanics

6.16.1 Path Integral Representation for Supersymmetric Quantum Mechanics. [23,225,428,476,550,729,730]

$$\begin{aligned}
&\left\langle x'', \psi'' \left| \mathrm{e}^{-\mathrm{i}HT/\hbar} \right| x', \psi' \right\rangle = \lim_{N\to\infty} \int \prod_{j=1}^{N-1} \mathrm{d}x_j \, \mathrm{d}\psi_j \int \prod_{j=1}^{N} \mathrm{d}p_j \, \mathrm{d}\psi_j^* \\
&\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left[p_i \Delta x_i + \mathrm{i}\, \psi_j^* \Delta\psi_j - H(x_i, p_i, \psi_j, \psi_j^*) \right] \right\} \\
&\equiv \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\mathrm{MP}}(x(t), p(t)) \int\limits_{\psi(t')=\psi'}^{\psi(t'')=\psi''} \mathcal{D}_{\mathrm{MP}}(\psi(t), \psi^*(t)) \\
&\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[p\dot{x} + \mathrm{i}\, \psi^* \dot{\psi} - H(x, p, \psi, \psi^*) \right] \mathrm{d}t \right\} , \qquad (6.16.1)
\end{aligned}$$

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu - \mathrm{i}\, k_\mu g^{\mu\nu} \Gamma^\alpha_{\nu\beta} \psi^*_\alpha \psi^\beta - \frac{1}{2} g^{\mu\nu} \Gamma^\sigma_{\mu\varrho} \Gamma^\alpha_{\nu\beta} \psi^*_\sigma \psi^\varrho \psi^*_\alpha \psi^\beta + V(x, \psi, \psi^*) \ , \qquad (6.16.2)$$

with the on-shell superpotential

$$V(x, \psi, \psi^*) = \frac{1}{2} g^{\mu\nu} \partial_\mu W \partial_\nu W + \psi^{*\mu} \psi^\nu D_\mu D_\nu W + \frac{1}{2} R_{\mu\nu\rho\sigma} \psi^\mu \psi^{*\nu} \psi^\rho \psi^{*\sigma} \ . \qquad (6.16.3)$$

$W(x)$ is the scalar superpotential, $R_{\mu\nu\rho\sigma}$ the curvature tensor, and D_μ the covariant derivative.

6.16.2 Quasi-Classical Approximation (CBC-Formula). [204,223,283, 522,550]

$$\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left(\frac{1}{2}\dot{x}^2 - \Phi^2(x) \mp \frac{1}{2}\hbar\Phi'(x) \right) \mathrm{d}t \right] \\
&\overset{(\hbar\to 0)}{\simeq} \frac{\exp\left\{ \mp \frac{\mathrm{i}}{2} [\mathrm{sgn}(\dot{x}')a(x') - \mathrm{sgn}(\dot{x}'')a(x'')] \right\}}{\mathrm{i}\hbar\sqrt{|\dot{x}'\dot{x}''|}} \\
&\quad \times \sum_{x_{\mathrm{qc}}}^{\text{fixed } E} \exp\left[\frac{\mathrm{i}}{\hbar} W[x_{\mathrm{qc}}] - \mathrm{i}\, n_R \left(\frac{\pi}{2} \pm a(x_R) \right) - \mathrm{i}\, n_L \left(\frac{\pi}{2} \mp a(x_L) \right) \right] \ , \qquad (6.16.4)
\end{aligned}$$

where $W[x_{\mathrm{qc}}] = \int_{x_{\mathrm{qc}}} dx \sqrt{2E - \Phi^2(x)}$ is taken along the quasi-classical path $x = x_{\mathrm{qc}}(t)$ defined as a solution of the differential equation $\ddot{x} = -\Phi(x)\Phi'(x)$,

$a(x) = \arcsin(\Phi(x)/\sqrt{2E})$, and n_R and n_L the number of right and left turning points along x_{qc}, respectively. The quasi-classical quantization rule in one dimension has the form (CBC-formula)

$$\pi n\hbar = \int_{x_L}^{x_R} \sqrt{2E - \Phi^2(x)}\,\mathrm{d}x \ , \qquad n \in \mathbb{N}_0 \ . \tag{6.16.5}$$

6.16.3 Path Integral Representation on (m, n)-dimensional Super-Riemann Manifolds.

6.16.3.1 General Form of the Path Integral. ($\Gamma_a = \frac{1}{2}\partial_a \ln\sqrt{G}$ where $G = |\,\mathrm{sdet}({}_aG_b)|$, and in $(-1)^a$ the quantity a takes on the values $0, 1$ depending whether a represents an even respectively an odd (Grassmann) variable, together with a basis $\mathbf{Q} = (\mathbf{q}, \boldsymbol{\xi})$, $\bar{\boldsymbol{\xi}}$ conjugate of $\boldsymbol{\xi}$, and $\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}$ are Grassmann variables) [103,428,687,803,898]

$$\left\langle \mathbf{Q}''\left| \exp\left[\frac{\mathrm{i}}{\hbar}\frac{\hbar^2}{2m}\Big({}^aG^b(\mathbf{Q})\partial_b\partial_a + (-1)^a G^{-1/2}\Big(\partial_a G^{1/2\,a}G^b(\mathbf{q})\Big)\partial_b\Big)\right]\right|\mathbf{Q}'\right\rangle$$
$$= [G(\mathbf{Q}')G(\mathbf{Q}'')]^{-1/4} \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{mN/2}\left(-\frac{\epsilon\hbar}{m}\right)^{nN/2}\int\prod_{j=1}^{N-1}\mathrm{d}\mathbf{Q}_j$$
$$\times\prod_{j=1}^{N}\sqrt{G(\bar{\mathbf{Q}}_j)}\exp\left[\frac{\mathrm{i}}{\hbar}\left(\frac{m}{2\epsilon}\Delta Q_j^a{}_aG_b(\bar{\mathbf{Q}}_j)\Delta Q_j^b + \epsilon V(\bar{\mathbf{Q}}_j) + \epsilon\Delta V(\bar{\mathbf{Q}}_j)\right)\right] \tag{6.16.6}$$

$$\equiv [G(\mathbf{Q}')G(\mathbf{Q}'')]^{-1/4}\int\limits_{\mathbf{Q}(t')=\mathbf{Q}'}^{\mathbf{Q}(t'')=\mathbf{Q}''}\mathcal{D}_{\mathrm{MP}}\mathbf{Q}(t)\sqrt{G(\mathbf{Q})}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{Q}^a{}_aG_b(\mathbf{Q})\dot{Q}^b - V(\mathbf{Q}) - \Delta V_{\mathrm{MP}}(\mathbf{Q})\right)\mathrm{d}t\right] \ , \tag{6.16.7}$$

$$\Delta V_{\mathrm{MP}} = \frac{\hbar^2}{8m}\left[{}^aG^b\Gamma_b\Gamma_a + (-1)^a(\partial_a{}^aG^b\Gamma_b) + (-1)^{a+b}(\partial_b\partial_a{}^aG^b)\right] \ , \tag{6.16.8}$$

$$= [G(\mathbf{Q}')G(\mathbf{Q}'')]^{-1/4}\int\limits_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathcal{D}_{\mathrm{MP}}\mathbf{q}(t)\sqrt{G(\mathbf{q})}$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}g_{ab}\dot{q}^a\dot{q}^b - V(\mathbf{q}) - \Delta V_{\mathrm{MP}}(\mathbf{q})\right)\mathrm{d}t\right]$$
$$\times\lim_{N\to\infty}\left(-\frac{\epsilon\hbar}{m}\right)^{nN/2}\prod_{j=1}^{N-1}\frac{\partial}{\partial\xi_j^1}\cdots\frac{\partial}{\partial\xi_j^n}\prod_{j=1}^{N}\sqrt{1 + \frac{G(\bar{\mathbf{Q}}_j) - G(\bar{\mathbf{q}}_j)}{G(\bar{\mathbf{q}}_j)}}$$

$$\times \exp\left[\frac{\mathrm{i}m}{2\epsilon\hbar}\Big(g_{ab,\mu}(\bar{\mathbf{q}}_j)\xi^{*\mu}_{j,j-1}\Delta q^a_j\Delta q^b_j + 2H_{\alpha\beta,\nu}(\bar{\mathbf{q}}_j)\xi^{*\nu}_{j,j-1}\Delta\xi^\beta_j\Delta q^a_j \right.$$
$$+ K_{\alpha\beta,\rho}(\bar{\mathbf{q}}_j)\xi^{*\rho}_{j,j-1}\Delta\xi^\beta_j\Delta\xi^\alpha_j\Big)$$
$$\left. - \epsilon\frac{\mathrm{i}}{\hbar}\Big(V_{j,\sigma}(\bar{\mathbf{q}}_j)\xi^{*\sigma}_{j,j-1} + \Delta V_{j,\sigma}(\bar{\mathbf{q}}_j)\xi^{*\sigma}_{j,j-1}\Big)\right] , \tag{6.16.9}$$

$$\Delta Q^a G_{ab}\Delta Q^b = g_{ab}(\bar{\mathbf{q}}_j)\Delta q^a_j\Delta q^b_j + 2H_{a\beta}(\bar{\mathbf{q}}_j)\Delta\xi^\beta_j\Delta q^a_j + K_{\alpha\beta}(\bar{\mathbf{q}}_j)\Delta\xi^\beta_j\Delta\xi^\alpha_j \ . \tag{6.16.10}$$

6.16.3.2 Super-Poincaré Upper Half-Plane. [29,103,428,687,739,898]

$$\int\limits_{Z(t')=Z'}^{Z(t'')=Z''} \frac{\mathcal{D}_{\mathrm{MP}}Z(t)}{Y}$$
$$\times \exp\left\{\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}\frac{dt}{Y}\Big[\dot{z}\dot{z}^* - \mathrm{i}\bar{\theta}\dot{z}^*\dot{\theta} - \mathrm{i}\theta\dot{z}\dot{\bar{\theta}} - (2Y + \theta\bar{\theta}\dot{\theta}\dot{\bar{\theta}})\Big]\right\}$$
$$= \frac{1}{2\sqrt{Y'Y''}}g(d,T) + \mathrm{i}\,\Delta\bar{\Delta}h(d,T) \ , \tag{6.16.11}$$

$$g(d,T) = -\frac{1}{\pi}\left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{1/2}\int_d^\infty \frac{du\sinh\frac{u}{2}}{\sqrt{2(\cosh u - \cosh d)}}\,\mathrm{e}^{-mu^2/2\mathrm{i}\hbar T} , \tag{6.16.12}$$

$$h(d,T) = -\frac{\sinh d}{2\pi}\left(\frac{m}{2\pi\mathrm{i}\hbar T}\right)^{3/2}\int_d^\infty \frac{du\cosh\frac{u}{2}}{\cosh\frac{d}{2}}\frac{\mathrm{e}^{-mu^2/2\mathrm{i}\hbar T}\sinh\frac{u}{2}}{\sqrt{2(\cosh u - \cosh d)}} , \tag{6.16.13}$$

$$\Delta = \frac{\theta z_{12} - \nu z_{\bar{2}1} + \bar{\nu}z_{12} + \theta\nu\bar{\nu}}{\sqrt{z_{12}z_{2\bar{2}}z_{\bar{2}1}}} , \qquad \cosh d = 1 - 2\frac{z_{ab}z_{b^*b^*}}{z_{a^*a^*}z_{bb^*}} \ . \tag{6.16.14}$$

$z_{ab} = z_a - z_b - \theta_a\theta_b$, where $a,b = z, z^*, \theta, \bar{\theta}$, $Z = (z,\theta)$, $z = x + \mathrm{i}y$, $Y = y + \frac{1}{2}\theta\bar{\theta}$. θ and $\bar{\theta}$ are Grassmann variables with $\{\theta,\bar{\theta}\} = 0$, respectively $\{\theta,\theta\} = \{\bar{\theta},\bar{\theta}\} = 0$.

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List of Symbols

$a = \hbar^2/me_1e_2$: Bohr radius.
action: see Hamilton principal function R and classical action S.
$\mathrm{Ai}(z)$: Airy function.

β: $(k_B \cdot \text{temperature})^{-1}$.
β_k: Morse index.
$\mathrm{Bi}(z)$: Airy function.

c: velocity of light.
$\mathbb{C}$: complex numbers.
$C_n^{(\lambda)}(x)$: Gegenbauer polynomials.

(D): Dirichlet boundary condition.
Δ: Laplacian.
$\delta(x)$: delta-function.
$\delta'(x)$: $\mathrm{d}\,\delta(x)/\,\mathrm{d}x$.
dE_λ: integration measure with respect to (discrete or continuous) energy spectrum.
Δ_{LB}: Laplace–Beltrami operator.
ΔV: quantum potential of order $\hbar^2$.
$D_{mn}^J(\cos\vartheta)$: Wigner polynomials.
$D_\nu(z)$: parabolic cylinder functions.
$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)$: standard Feynman path integral in Cartesian coordinates in $\mathbb{R}^D$.
$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}_E q(t)$: Euclidean path integral.
$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}_{\mathrm{MP}} q(t)$: path integral in *mid-point* (MP) formulation.
$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}_{\mathrm{PF}} q(t)$: path integral in *product form* (PF), usually the index PF is dropped.
$\int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}_{\mathrm{SR}} q(t)$: path integral in *symmetric rule* formulation.
$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(x>a)}^{(D)} x(t)$: path integral in half-space $x > a$ with Dirichlet boundary condition at $x = a$.
$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{(x>a)}^{(N)} x(t)$: path integral in half-space $x > a$ with Neumann boundary

condition at $x = a$.

$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DD)}_{(a<x<b)} x(t)$: path integral in the box $a < x < b$ with Dirichlet boundary conditions at $x = a$ and $x = b$.

$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(NN)}_{(a<x<b)} x(t)$: path integral in the box $a < x < b$ with Neumann boundary conditions at $x = a$ and $x = b$.

$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DN)}_{(a<x<b)} x(t)$: path integral in the box $a < x < b$ with Dirichlet boundary condition at $x = a$ and Neumann boundary condition at $x = b$.

$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{\Gamma_{\alpha,a}} x(t)$: path integral in $\mathbb{R}^2$ or $\mathbb{R}^3$ with point interaction at $x = a$ with strength α.

e: electric charge.
E: energy.
ϵ: infinitesimal time interval in the path integral $\epsilon = T/N$.
E_λ: energy level corresponding to (discrete or continuous) level λ.
$E_\nu^{(0,1)}(z)$: even/odd parabolic cylinder function.
$\mathrm{erf}(x), \mathrm{erfc}(x)$: error functions.

${}_1F_1(a;c;z)$: confluent hypergeometric function.
${}_2F_1(a,b;c;z)$: hypergeometric function.
$\Phi_n^{(\alpha,\beta)}(x)$: Pöschl–Teller wave functions.

g_{ij}, g^{ij}: metric tensor and its inverse.
$g = \det(g_{ij})$: determinant of the metric tensor.
$G(q'', q'; E)$: Green function (in energy).
$G_0(q'', q'; E)$: Green function (in energy) of free particle.
$G^{(V)}(q'', q'; E)$: Green function (in energy) of a particle moving in the potential V.
$\gamma = -\Psi(z)$: Euler's constant.
Γ: Fuchsian group.
$\Gamma_a = \partial_a \ln\sqrt{g} = \Gamma^a_{aa}$ (contracted Christoffel symbol).
$\Gamma(z)$: Gamma function.
$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g_{\alpha\sigma}(\partial g_{\beta\sigma}/\partial q_\gamma + \partial g_{\gamma\sigma}/\partial q_\beta - \partial g_{\beta\gamma}/\partial q_\sigma)$: Christoffel symbol.
$\Gamma_{\alpha,a}$: coupling for regularized point interaction.

H: Hamiltonian.
$\hbar$: Planck's constant $\hbar = h/2\pi$.
$H_n(x)$: Hermite polynomials.
$H^{(D)\,*}_{k,l,\mu}(\Omega)$: wave functions on the hyperboloid.
$H_\nu^{(1,2)}(z)$: Hankel function of the first, second kind.
$\mathcal{H}^{(d)}_{-1}$: single-sheeted hyperboloid of dimension d.

$\mathrm{i} = \sqrt{-1}$: the imaginary unit.
$\Im(z)$: imaginary part of complex number z.
$I_\nu(z)$: modified Bessel functions.

$J(t)$: external source, current.
$J_\nu(z)$: Bessel functions.

$k = \sqrt{2mE}/\hbar$: wavenumber.
k_B: Boltzmann's constant.
$K(q'', q'; T)$: Green function (in time), the Feynman kernel, "propagator".
$K_0(q'', q'; T)$: Green function (in time), Feynman kernel of free particle.
$K^{(V)}(q'', q'; T)$: Green function (in time), Feynman kernel of a particle moving in the potential V.
K_{GHO}: Feynman kernel for the generalized harmonic oscillator.
$K_\nu(z)$: modified Bessel functions.

$\mathcal{L}$: Lagrangian.
$L^2(\mathbb{M})$: Hilbert space over $\mathbb{M}$.
$\Lambda^{(d)}$: d-dimensional hyperboloid (pseudosphere).
$L_n^{(\lambda)}(x)$: Laguerre polynomials.
$\Lambda_{l,h}^{p}(\alpha)$: Lamé polynomials.

m: mass of a particle.
$\mathbb{M}$: Riemannian space.
$\boldsymbol{M}$: monodromy matrix.
$M_{\nu,\mu}(z)$: Whittaker functions.
$M_n^{(1)}(\mu; \frac{dp}{2}), M_{\mathrm{i}\,k}^{(3)}(a; \frac{pd}{2}), \mathrm{me}_n(\nu; \frac{d^2p^2}{4}), \mathrm{Me}_{\mathrm{i}\,k}(b; \frac{p^2d^2}{4})$: Mathieu functions.
$\mu_\gamma, \tilde{\mu}_\gamma$: Morse index.
$\mu_\nu[r^2]$: functional weight in radial path integrals with angular momentum number ν.

$n!$: factorial of the integer n.
$|n>$: state vector.
N: number of discretization steps of the time lattice.
$\mathbb{N}$: natural numbers.
(N): Neumann boundary conditions.
N_M: maximal number of bound states.

$\boldsymbol{\omega}$: unit vector on hyperboloids.
$\boldsymbol{\Omega}$: unit vector on spheres.
$\Omega(d)$: volume of the d-dimensional sphere.

p: momentum variable.
$\underline{\mathrm{p}}_q = -\mathrm{i}\,\hbar(\partial_q + \Gamma_q/2)$: Hermitian momentum operator, with $\Gamma_q = \partial_q \ln\sqrt{g} = \Gamma_{qa}^{a}$ in Riemannian space with metric (g_{ab}).
φ: circular polar variable $\varphi \in [0, 2\pi)$.
Φ: magnetic flux.
$\mathrm{ps}_l^n(\cos\nu; p^2d^2), \mathrm{Ps}_{\mathrm{i}\,k-1/2}^{-\nu}(\cosh\eta; p^2d^2)$: spheroidal functions.
$\Psi(z)$: $\Psi(z) = \Gamma'(z)/\Gamma(z)$.
$\Psi_n^{(\alpha,\beta)}(x)$: discrete modified Pöschl–Teller wave functions.
$\Psi_p^{(\alpha,\beta)}(x)$: continuous modified Pöschl–Teller wave functions.
$P_n^{(\alpha,\beta)}(x)$: Jacobi polynomials.
Ψ_λ: wave function of (discrete or continuous) level λ.

$P_n^\nu(x)$: Legendre polynomials.
$\mathcal{P}_\mu^\nu(z)$: Legendre functions.

$\boldsymbol{q}$: coordinates in a Riemannian space with positive definite metric.
q: coordinates in a Riemannian space with indefinite metric.
$\mathcal{Q}_\mu^\nu(z)$: Legendre function of the second kind.

R: scalar curvature.
$R[x]$: Hamilton's principal function $\equiv$ classical action.
$R_E[x]$: Euclidean action.
ϱ: two-dimensional radial polar variable.
$R_{\mu\nu\rho\sigma}$: curvature tensor.
$\rho(x'', x'; \beta)$: density matrix.
$\mathbb{R}$: real numbers.
$\mathbb{R}^n$: n-dimensional Euclidean space.
$\Re(z)$: real part of complex number z.

$S = \det(\Phi_{ij})$: Stäckel determinant.
$S_\gamma(\boldsymbol{x}'', \boldsymbol{x}'; E) := \int_\gamma \boldsymbol{p} \cdot d\boldsymbol{x}$: classical action evaluated along a classical path γ.
$S^{(d)}$: d-dimensional sphere.
$S_l^\mu(\boldsymbol{\Omega})$: hyperspherical harmonics.
$S_l^{n\,(1)}(\cosh\mu; pd), S_{\mathrm{i}\,k-1/2}^{\nu\,(3)}(\cosh\xi; pd)$: spheroidal functions.
$\boldsymbol{s}$: coordinate on spheres.
$\sigma_x, \sigma_y, \sigma_z$: Pauli matrices.
s-lim: strong operator limit, i.e., limit with respect to an operator norm.

ϑ: azimuthal polar variable $\vartheta \in (0, \pi)$.
$\Theta(t)$: Heaviside step function.
$\Theta_{2,3}(z, \tau)$: Jacobi theta functions.
T: $T = t'' - t'$: evolution time.

$\mathbf{1}$: unit tensor.
$\underset{\sim}{\mathrm{U}}(t'', t')$: time evolution operator.

$W_{\mu,\nu}(z)$: Whittaker functions.

$[x]$: integer part of x.
$|\boldsymbol{x}|$: absolute value of $\boldsymbol{x}$.
$x_\gtrless$: larger, smaller of two variables x', x''.
$\boldsymbol{x}$: coordinates in $\mathbb{R}^n$.
x: coordinates in Minkowski space-time.

$Y_l^m(\boldsymbol{\Omega})$: spherical harmonics.

$\mathbb{Z}$: integers.
$\zeta(s)$: Riemann's zeta function.
$Z(s)$: Selberg's zeta function.

Subject Index

Author Index

The numbers in brackets refer to the entries within the reference list (pp. 367–423) where a particular author may be found.

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