Lanchester combat models

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1. The Lanchester aimed-fire model

As a simple, deterministic model^[1] of a battle, suppose that R(t) red and G(t) green units* begin fighting at t = 0, and that each unit destroys r or g (the **fighting effectiveness**) enemy units in one unit of time, so that

$$\frac{dR}{dt} = -gG , \qquad \frac{dG}{dt} = -rR . \tag{1}$$

Rather than solve directly, we eliminate the explicit t-dependence by dividing the second equation by the first and then separating variables: then

$$\int rR \, dR = \int gG \, dG \,, \tag{2}$$

and we see that

$$rR^2 - gG^2 = \text{constant.} (3)$$

This has remarkable implications. Since this quantity never changes sign, only one of R and G can ever be zero. If the initial value of (3) is positive, for example, only G can equal zero, and so only red can win[†]. So this quantity is key in determining who will win the battle, and a side's fighting strength – we shall call rR^2 and gG^2 the red and green forces' respective **fighting strengths** – varies as the units' fighting effectiveness times the *square* of their numbers.

Consider an example. Suppose red begins with twice as many units as green, $R_0 = 2G_0$, but the green units are three times as effective, g = 3r. Then

$$rR^{2} - gG^{2} = r(2G_{0})^{2} - 3rG_{0}^{2} = rG_{0}^{2} > 0, (4)$$

and (perhaps rather counter-intuitively) the reds win. The model assumes that the battle is a homogeneous mixing of forces, or, in other words, a constant bloody $m\hat{e}l\acute{e}e$, and in such engagements the tactical conclusion is simple: if your strength is in numbers, then you need to fight in this way, bringing all your units to engage with the enemy's as rapidly as possible. Conversely, if your units are fewer in number but more effective, then you need

^{*}We shall worry later on about what exactly constitutes a 'unit'.

[†]To show that red actually does win, we would need to solve the equations explicitly for R(t) and G(t), which is most easily done by differentiating one of (1), substituting into the other, and solving the resulting second order ODE.

the tactics which will prevent a $m\hat{e}l\acute{e}e$, allowing you to pick-off opponents, and preventing the enemy from bringing all his units to bear (– like Hannibal at Cannae).

This last point becomes clearer if we consider what would have happened had green been able to divide the reds into two equal forces and engage them sequentially. At the end of the first engagement, between G_0 greens and $R_0 = G_0$ reds, G_1 greens remain, where

$$rG_0^2 - 3rG_0^2 = -3rG_1^2 \qquad \Rightarrow \qquad G_1 = \sqrt{\frac{2}{3}}G_0 , \qquad (5)$$

and then in the second engagement

$$rG_0^2 - 3r_{\overline{3}}^2 G_0^2 = -rG_0^2 < 0. (6)$$

Green has now won, with $3rG_2^2 = rG_0^2$ and thus $\sqrt{\frac{1}{3}}$ or nearly 60% of its original forces left when all the reds have been destroyed – an amazing turnaround. This becomes even more striking with an N-fold (rather than two-fold) division of red forces: after the simple resulting iteration, green now wins with a final number G_F of units remaining, where

$$gG_F^2 = gG_0^2 - \frac{1}{N}rR_0^2 \,, (7)$$

so that the N-fold division has reduced red's fighting strength N-fold. Again, this is a classic military maxim: you should (almost) never divide your forces ‡ .

2. Background

The above is known as the Lanchester aimed- (or directed-) fire combat model, and it seems to me to exemplify all that's best in a simple mathematical model. Most modern warfare simulations are, of course, stochastic, heterogeneous and complex; and will, if their myriad assumptions are correct, give much better predictions. But the Lanchester model, in contrast^[2], has the virtues of simplicity: it makes strong simplifying assumptions, which nevertheless are (at least sometimes) close to being realizable, and the model brings out, through a subtle process, some stark conclusions.

So how close a fit is the model to past battles? Lanchester originally applied it to Nelson's tactics at Trafalgar, and intended that it should describe aerial combat, but for detailed fits of data one has to look to battles such as Iwo Jima^[3], the Ardennes^[4] and Kursk^[5]. (Indeed my values and notation above are intended approximately to match Kursk, the great Russian-German tank battle of 1943.) In fact the fit seems not to be so

[‡]It is perhaps worth noting that if red and green both divided their forces, and fought two separate, simultaneous, identical battles, the outcome would necessarily be the same as for our single, original battle. The argument is rather like Galileo's for the falling cannonballs: making an imaginary or a real division of the battle makes no difference to its course or outcome.

good^[5], but my impression after exploring the literature is that this is to be expected: the conditions implied by the model, of constant tactically-blind slaughter, are hardly those which allow the collection of accurate time-series data; and conversely where good data exist they are more likely to describe a series of different small engagements. Those who teach military tactics, however, still value Lanchester's model and its generalizations^[2], because, above all, they provoke careful thought about the consequences of the conditions of engagement.

It seems to me that the model offers an excellent pedagogical tool for curriculum enrichment at least down to A-level (where simple separable first-order ODEs are in the core curriculum). Moreover, it is easily extended to suit students with no calculus (see section 3.1 below). Perhaps its main virtue is that so many of the natural questions one can ask about it have mathematically-tractable answers. We investigate some of these in the next section.

3. Further explorations

3.1. A calculus-free version

A model in the form of discrete recursion relations not only avoids calculus but is also essential in modelling battles between small numbers of units, or battles effectively fought as a sequence of discrete engagements (like salvos in a naval battle^[6]). The difference equations

$$R_{n+1} = R_n - gG_n$$
, $G_{n+1} = G_n - rR_n$, (8)

whose continuous-time limit is (1), lead to

$$rR_{n+1}^2 - gG_{n+1}^2 = (1 - rg)\left(rR_n^2 - gG_n^2\right). (9)$$

But $rg \ll 1$, and indeed $(1 - rg)^T \simeq 1$ for the duration T of the battle (which is $\sim 1/r$ or 1/g). Thus (3) remains approximately conserved, and the same conclusions apply.

3.2. The Lanchester unaimed-fire model

There are various scenarios in which a side's fighting strength is proportional to its numbers rather than their square. Indeed any model of the form

$$\frac{dR}{dt} = -F_G , \qquad \frac{dG}{dt} = -F_R , \qquad (10)$$

where F_G/F_R is independent of R and G, gives this result. Lanchester considered that F =constant modeled ('ancient') hand-to-hand combat without firearms, while a modern model with this property would be that of artillery fire or bombardment, in which, if R red

guns fire with effectiveness r at random into an area A_G in which there are known to be G green units (which therefore have density G/A_G per unit area), we have $F_R = -rRG/A_G$. Either way, it is now $\rho R - \gamma G$ (for some constants ρ and γ) which is constant, and the fighting strength of unaimed-fire units is proportional to their numbers and effectiveness but inversely proportional to the enemy's area of dispersal. Once again, there are some simple implications. For example, in a tank battle you may wish to engage the enemy with aimed-fire from well-dispersed and -disguised positions, forcing the enemy to use unaimed fire. Thus you might emplace your guns or tanks, perhaps within the edge of a wood. However, if the enemy later becomes able to aim his fire, he may be able to divide your forces; you will need to be able to regroup quickly.

3.3. Mixed forces

Of course fighting forces have always been composed of units of different effectiveness in different numbers. Suppose the green forces are composed of two types, so $G = G_1 + G_2$, with effectiveness g_1 and g_2 . Then

$$\frac{dR}{dt} = -g_1 G_1 - g_2 G_2, \qquad \frac{dG_1}{dt} = -rR \frac{G_1}{G}, \qquad \frac{dG_2}{dt} = -rR \frac{G_2}{G}.$$
(11)

Although this looks more complicated, in fact the outcome is surprisingly simple: one can check straighforwardly that

$$\frac{d}{dt}\left(rR^2 - (g_1G_1 + g_2G_2)(G_1 + G_2)\right) = 0,$$
(12)

so that the conserved quantity analogous to (3) is now

$$rR^2 - g_{\text{ave}}G^2 \,, \tag{13}$$

where

$$g_{\text{ave}} = \frac{g_1 G_1 + g_2 G_2}{G} \tag{14}$$

is the average effectiveness of the units. So to calculate the fighting strength of mixed units one should simply use the number of autonomous units with their average effectiveness (in contradiction with Lanchester's original claims, as has been pointed out by Lepingwell^[7], whose article is an excellent introduction to Lanchester models).

3.4. The meaning of a 'unit'

This leads us to an interesting controversy in the military use of the model^[9]. Sometimes the use of Lanchester-type models is taught in which units' *numbers* are first weighted with a 'fighting power' – for instance, 1000 troops together with 20 tanks, the latter multiplied by a 'fighting power' of, say, 30, to give a total of 1600 units. As others have pointed out, and as we can see from the above, this cannot be correct – the Lanchester model's main

point is to distinguish the importance of numbers from that of fighting effectiveness. But it does lead us to a natural and fundamental question, which should by now be worrying the reader: what exactly do we mean by a 'unit'?

For example, consider an infantry engagement. Is the basic unit the individual soldier, or perhaps the section (about 10 men), platoon (3-4 sections), or company (3-4 platoons)? Does this choice change if (for example) each section fights with its own armoured personnel carrier (APC)? Above all, does this choice affect the outcome of the battle in our model? In fact it does not. For suppose we take green's basic unit to be N troops. Then its numbers are scaled by $1/N^2$ and its effectiveness by N. But red's effectiveness against green units is now scaled by 1/N, and the whole of the difference in fighting strengths (3) scales by 1/N. Its sign, and the course and outcome of the battle, do not change.

But N troops may gain some advantage by fighting as a group. It may be offensive, in which case their effectiveness scales better than N-fold, or defensive, in which case their opponents' effectiveness will scale worse than 1/N (for example, in hand-to-hand fighting, a small group fighting in a cluster and thus defending each other's backs). But of course we now rapidly run into the limitations of the model, for battles with different types of units are rarely homogeneous – indeed it is a tactical imperative to use units so as to maximise their fighting advantages. A section fighting in an APC may be better protected against other infantry, but it is also now participating in a different, armoured battle.

3.5. Support troops

During the battle for France in 1940, Churchill visited the French HQ and, while being shown on a map the location of the German breakthrough, asked 'Ou est la masse de manoeuvre?' ('Where is the strategic reserve?'). That there was none – 'aucune' – confirmed to him that the battle was lost. Tactical considerations, beyond the scope of Lanchester models, suggest that one should always maintain a reserve. But support and/or reserve troops may also increase the efficiency of fighting troops, and we must ask which is correct: to aggregate support troops within the overall numbers, or to deal with them separately. A thorough analysis clearly requires the latter.

Suppose that we split our N units into P fighting and N-P support units. We do so because support units increase the effectiveness of fighting units, so let us assume that this is proportional to the support ratio $\frac{N-P}{P}$, so that the P fighting units' effectiveness $f=f_0\frac{N-P}{P}$. Then the fighting strength fP^2 is maximized by P=N/2 – that is, half of our units should be in support roles. More generally, if we take $f=f_0\left(\frac{N-P}{P}\right)^\kappa$, then the fighting strength is maximized by $P=(1-\kappa/2)N$. For example, a ratio of one headquarters to three fighting companies, or P/N=3/4 and $\kappa=1/2$, might correspond to an implicit

belief that a 10% increase in such support per fighting unit leads to an approximate 5% increase in its effectiveness.

In the Ardennes campaign^[10], the Allies used a much greater support ratio than the Germans (about 0.8 as opposed to 0.5), but the fighting effectiveness of their total numbers – in our model above, $f_{\text{tot}} = f_0(\kappa/2)^{\kappa}(1-\kappa/2)^{2-\kappa}$ – turns out to have been about the same as the Germans'. Of course, this does not mean that the Allies should have reduced their support ratio. If we assume that this was optimal (perhaps because of more plentiful $mat\acute{e}riel$), so that $\kappa \simeq 0.89$, then to have reduced $\frac{N-P}{P}$ to that of the Germans would have reduced their f_{tot} by about 2%. It is not clear whether, in the model, this would have been within the margin of victory!

3.6. Bracken's generalized model

As we discussed in section two, attempts in the literature to fit either the aimed- or the unaimed-fire model to data have only been partly successful. If we are willing to move away from basing our model on clear assumptions (always a dangerous path), we can instead use a generalized model whose parameters we then fit to the data. In this spirit Bracken proposed^[10]

$$\frac{dR}{dt} = -gR^qG^p , \qquad \frac{dG}{dt} = -rR^pG^q , \qquad (15)$$

for p and q to be empirically determined. The conserved quantity (by eliminating t, separating and integrating) is

$$gG^{\alpha} - rR^{\alpha}, \tag{16}$$

where $\alpha = 1 + p - q$. (The Lanchester aimed-fire model (1,3) corresponds to p = 1, q = 0 and thus to $\alpha = 2$, the unaimed-fire model to p = q = 1 and thus $\alpha = 0$.)

Clearly if $\alpha > 1$ then numbers matter more than effectiveness, and conversely. It might be natural to assume that p > q (and thus $\alpha > 1$), so that one's rate of loss scales more quickly with the enemy's numbers than with one's own, but in fact an enormous variety of best-fitting p, q have been found^[4,5] by looking at various battles and with differing measures of numbers and effectiveness – which certainly indicates that this type of model is of little use in advance prediction of a particular battle's outcome! Hartley's analysis^[11] (as reported by Lucas and Dinges^[12]) of a range of battles suggests p = 0.45, q = 0.75.

3.7. Asymmetric warfare

This result, that one's own numbers can have at least as strong an effect on one's casualties as do enemy numbers, might lead us to add exponential decay terms in both equations of the original model. Instead, let us revisit a fundamental assumption in all of the foregoing:

that the correct model is symmetric under $R \leftrightarrow G$ and their associated parameters.

Suppose red has huge numerical superiority and green has vastly greater effectiveness – that is, $g \gg r$ but $G_0 \ll R_0$. If, further, $rR_0^2 \gg gG_0^2$ then the Lanchester aimed-fire model would give a clear win for red. However, in the early stages of the battle there will be many reds for each green, which will find it easy to acquire targets, and thus have a kill rate proportional to both R and G. The reds, meanwhile, will kill in proportion to G but will not be able to make their numbers tell. So let us take

$$\frac{dR}{dt} = -gGR/R_0 , \qquad \frac{dG}{dt} = -rG . \qquad (17)$$

Then G declines slowly and exponentially, $G = G_0 e^{-rt}$, and we can now solve for R to give

$$R = R_0 \exp\left\{\frac{gG_0}{rR_0}(e^{-rt} - 1)\right\}. \tag{18}$$

Now consider the situation when $e^{-rt} = \frac{1}{2}$. The greens have been reduced to half their original numbers, $G = \frac{1}{2}G_0$, while $R = R_0 \exp\left(-\frac{1}{2}\frac{gG_0}{rR_0}\right)$. So, if $gG_0 \gg rR_0$, – in other words, if green would clearly win in the unaimed-fire, linear-law model, – then it will win at least the early stages of this battle[§], even though it would lose under a conventional aimed-fire, square-law model. Thus, with technologically-superior forces in a 'target-rich environment', even under homogeneous, aimed-fire conditions, it may be a linear- rather than a square-law battle which is being fought.

4. Concluding remarks

Of course there have been many generalizations of the Lanchester models, but most of these lack the simplicity and force of the original. One can include exponential or linear growth or attrition^[13] in the original equations, corresponding to defence^[6], friendly fire or reinforcement, and all of these are easily soluble with A-level or first-year undergraduate techniques. A model which mixes aimed with unaimed fire would, however, be nonlinear (a simple Lotka-Volterra model), and perhaps not for pre-university students. Further, any of these properties can appear asymmetrically in the coupled equations. There are many websites dealing with various of the possibilities.

Lanchester models provide an excellent example of the strengths (and weaknesses) of simple mathematical modelling. Further, as we have seen, the basic model leads to many other 'What if...?' questions which can be easily investigated. Many more such questions can be asked, and of course once one begins numerical simulations the possibilities are endless. To model warfare can seem more politically challenging (not to say incorrect)

[§]Of course one then has to switch to a standard aimed-fire model for the later stages. Green may still lose the battle in the end!

than to use ecology or epidemiology, but a little understanding of how military planners arrive at their tactical conclusions can also strip away mystique and (for this author, at least) expose some of the subject's limitations!

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