Verification of Detectability for Unambiguous Weighted Automata using Self-Composition

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Detectability is an important property of discrete event systems (DESs), which requires determining the current and subsequent states of the system within a finite number of observations.

- It plays an important part in several related problems
  - State Estimation
  - Verification of Opacity
  - Verification of Diagnosability
  - Control Synthesis

Logical Finite State Automata **(Without weight information)**

- S. Shu, F. Lin, "Generalized detectability for DES," *Systems control & letters*, 2011.
- K. Zhang, "The problem of determining the weak (periodic) detectability of DES is PSPACE-Complete," *Automatica*, 2017.
- T. Masopust, "Complexity of deciding detectability in DES," *Automatica*, 2018.
- J. Balun and T. Masopust, "On verification of d-detectability for DES," *Automatica*, 2021.
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Weighted Automata **(With much higher complexity)**

- A. Lai, S. Lahaye, and A. Giua, "Verification of detectability for UWA," *IEEE Transactions on Automatic Control*, 2020.
- K. Zhang, "Detectability of labeled weighted automata over monoids," *Discrete Event Dynamic Systems*, 2022.
- W. Dong, X. Yin, K. Zhang, and S. Li, "On the verification of detectability for timed systems," in *Proc. American Control Conference*, 2022.

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Definition 1 (Semiring)

A semiring is a quintuple $\mathbb{S} = \langle \mathcal{D}, \oplus, \otimes, \varepsilon, e \rangle$ satisfying the following axioms:

- $\langle \mathcal{D}, \oplus, \varepsilon \rangle$ is a commutative monoid, i.e., $a \oplus b = b \oplus a$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, and $\varepsilon \oplus a = a \oplus \varepsilon = a$ hold for any $a, b, c \in \mathcal{D}$;
- $\langle \mathcal{D}, \otimes, e \rangle$ is a monoid, i.e., $(a \otimes b) \otimes c = a \otimes (b \otimes c)$, and $e \otimes a = a \otimes e = a$ hold for any $a, b, c \in \mathcal{D}$;
- the distributivity laws $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ hold for any $a, b, c \in \mathcal{D}$;
- $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$ for any $a \in \mathcal{D}$.

Example

- $\mathbb{S} = \langle \mathbb{N}, +, \times, 0, 1 \rangle$
- $\mathbb{R}_{max} = \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$

Definition 2 (Weighted Automata (WA))

A WA G over a semiring $\mathbb{S} = \langle \mathcal{D}, \oplus, \otimes, \varepsilon, e \rangle$ is defined as a tuple $G = (Q, E, t, Q_i, \varrho)$, where

- Q (resp. E) is the finite set of states (resp. events);
- $t : Q \times E \times Q \to \mathcal{D}$ is the transition function;
- $\varrho : Q \to \mathcal{D}$ is a function mapping the states to the initial delays, and specifying the set of initial states $Q_i = \{q \in Q \mid \varrho(q) \neq \varepsilon\}$.

Definition 3 (Unambiguity)

A WA G is unambiguous if $\forall q \in Q$, $\forall \omega \in E^*$, $|Q_i \overset{\omega}{\rightsquigarrow} \{q\}| \leq 1$.

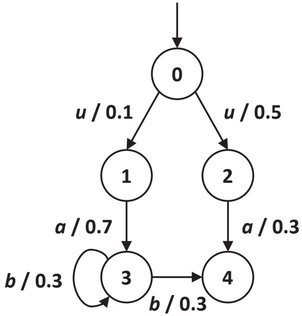

Figure 1: An (nondeterministic) unambiguous weighted automata $G = (Q, E, t, Q_i, \varrho)$.

- $\mathbb{R}_{max} = \langle \mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$
- State Set $Q = \{0, 1, 2, 3, 4\}$, initial state set $Q_i = \{0\}$
- Observable events $E_o = \{a, b\}$, unobservable event $E_{uo} = \{u\}$
- Transitions $t(0, u, 1) = 0.1$, $t(1, a, 3) = 0.7$, $\cdots$
- Initial delays $\rho(0) = 0$ and $\rho(i) = -\infty$ for $i = 1, \ldots, 4$
- Replace $t(0, u, 1) = 0.1$ with $t(0, b, 1) = 0.1$ $\Longrightarrow$ deterministic

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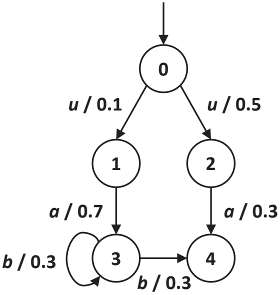
Definition 4 (Observation-Consistent States)

For an observed weighted sequence $\sigma_o \in P(L(G))$, the set of all σ_o-consistent states is defined as:

$$C(\sigma_o) = \{q \in Q \mid \exists \sigma \in L(G), \exists q_0 \in Q_0 : q_0 \overset{\sigma}{\rightsquigarrow} q, P(\sigma) = \sigma_o\}.$$

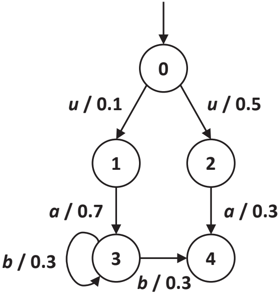


- $\sigma_o = (a, 0.8) \Longrightarrow C(\sigma_o) = \{3, 4\}$
- $\pi_1 = (0, u, 1)(1, a, 3)$
  $\pi_2 = (0, u, 2)(2, a, 4)$
- $\sigma(\pi_1) = (u, 0.1)(a, 0.8)$
  $\sigma(\pi_2) = (u, 0.5)(a, 0.8)$
- $P(\sigma(\pi_1)) = P(\sigma(\pi_2)) = \sigma_o$

Definition 5 (Strong Periodic Detectability)

A UWA G is strongly periodically detectable w.r.t. projection P if

$$(\exists k \in \mathbb{N})(\forall \sigma \in L^{\omega}(G))(\forall \sigma' \in \bar{\sigma})(\exists \sigma'' \in (E \times \mathbb{Q})^{*})$$
$$\sigma'\sigma'' \in \bar{\sigma} \wedge |P(\sigma'')| < k \wedge |C(P(\sigma'\sigma''))| = 1.$$

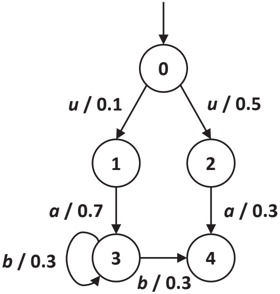


- $\sigma = (u, 0.1)(a, 0.8)(b, 0.8 + 0.3k)$
- $\sigma' = (u, 0.1)(a, 0.8)(b, 1.1)$
- $\sigma'' = (b, 0.3), |P(\sigma'')| = 1$
- $|C(P(\sigma'\sigma''))| = 2$
- G_1 is not strongly periodically detectable.

Definition 6 (Strong D-Detectability)

A UWA G is strongly D-detectable w.r.t. P and a specification Q_{spec} if

$$(\exists k \in \mathbb{N})(\forall \sigma \in L^{\omega}(G))(\forall \sigma' \in \bar{\sigma})|P(\sigma')| > k$$
$$\Rightarrow C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} = \emptyset.$$



- Let $Q_{spec} = \{(3, 4)\}$
- $\sigma = (u, 0.1)(a, 0.8)(b, 0.8 + 0.3k)$
- $\sigma' = (u, 0.1)(a, 0.8)(b, 1.1)$
- $C(P(\sigma')) = \{3, 4\}$
- G_1 is not strongly D-detectable with respect to Q_{spec}.

Definition 7 (Self-Composition)

Given a UWA G, its self-composition is defined as an NFA $CC(G) = (Q_c, E_c, Q_{0,c}, \Delta_c)$, where

- $Q_c = Q \times Q$; $E_c = E_o$; $Q_{0,c} = Q_0 \times Q_0$;
- $\Delta_c \subseteq Q_c \times E_o \times Q_c$ is the set of state transitions. We have $((q_1, q_2), e, (q_3, q_4)) \in \Delta_c$ for all $(q_3, q_4) \in C(\sigma_o)$ if there exist two paths in G as follows:

$$\pi_1 = q_1 \xrightarrow{\omega_1} q_1' \xrightarrow{e} q_3, \ \pi_2 = q_2 \xrightarrow{\omega_2} q_2' \xrightarrow{e} q_4,$$

generating two weighted sequences $\sigma_1, \sigma_2 \in (E \times \mathbb{Q})^*$ as follows:

$$\sigma_1 = (\omega_1, \tau_1)(e, \tau), \ \sigma_2 = (\omega_2, \tau_2)(e, \tau),$$

where $P(\sigma_1) = P(\sigma_2) = (e, \tau) = \sigma_o$.

Definition 8 (Modified Self-Composition)

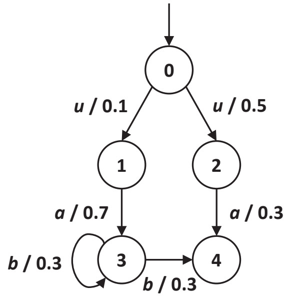
Given a UWA G, the modified self-composition $CC'(G) = (Q'_c, E_o, Q_{0,c}, \Delta'_c)$ is obtained from $CC(G) = (Q_c, E_o, Q_{0,c}, \Delta_c)$ by deleting every state-pair $(q_c(L), q_c(R))$ and its attached transitions if $t(p, u, q_c(L)) \neq 0$ or $t(p, u, q_c(R)) \neq 0$.

Definition 9 (ϵ-extended Self-Composition)

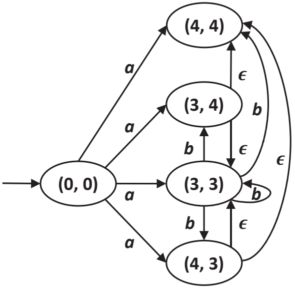
Given a UWA G, a variant from $CC(G) = (Q_c, E_o, Q_{0,c}, \Delta_c)$, called ϵ-extended self-composition is defined as:

$$CC^{\epsilon}(G) = (Q_c, E_o \cup \{\epsilon\}, Q_{0,c}, \Delta_c^{\epsilon}),$$

where $\Delta_c^{\epsilon} = \Delta_c \cup$
$\{((q_1, q_2), \epsilon, (q_1, q_1)) \mid \exists q_1, q_2 \in Q, \exists q_c \in Q_c, \exists e, e' \in E_o : ((q_1, q_1), e, q_c) \in \Delta_c \wedge ((q_1, q_2), e', q_c) \notin \Delta_c, q_1 \neq q_2\} \cup$
$\{((q_1, q_2), \epsilon, (q_2, q_2)) \mid \exists q_1, q_2 \in Q, \exists q_c \in Q_c, \exists e, e' \in E_o : ((q_2, q_2), e, q_c) \in \Delta_c \wedge ((q_1, q_2), e', q_c) \notin \Delta_c, q_1 \neq q_2\}$.



(a) Original system G.



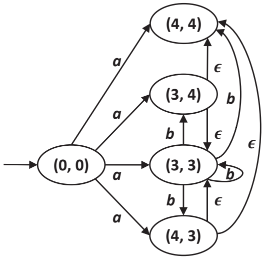
(b) Modified ϵ-extended self-composition $CC'^{-\epsilon}(G) = (Q'_c, E_o \cup \{\epsilon\}, Q_{0,c}, \Delta_c'^{-\epsilon})$ where $\tau \neq 0$.

- $CC'^{-\epsilon}(G) \equiv CC^{\epsilon}(G)$
- $(3,4) \xrightarrow{\epsilon} (3,3), (4,3) \xrightarrow{\epsilon} (3,3), (3,4) \xrightarrow{\epsilon} (4,4), (4,3) \xrightarrow{\epsilon} (4,4) \in \Delta_c$
- States $|Q'_c| \leq |Q|^2$; Transitions $|\Delta_c'^{-\varepsilon}| \leq |Q|^2 \times (|E_o| + 1) \times |Q|^2$

Theorem 1 (Criterion for Checking Strong Periodically Detectability)

A UWA G is not strongly periodically detectable if and only if in $CC^{\prime\prime-\epsilon}(G)$, at least one of the following two conditions is true.

1. There exists a state $q \in Q'_c$ such that $q(L) \neq q(R)$, and a path $q(L) \xrightarrow{\omega_1} q' \xrightarrow{\omega_2} q'$ in G, where $\omega_1 \in E^*_{uo}$, $\omega_2 \in E^*_{uo} \setminus \{\epsilon\}$, $q' \in Q$.
2. There exists a reachable circuit $q_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} q_{n+1}$ such that $q_1 = q_{n+1}$, $q_i(L) \neq q_i(R)$, and $\omega_i \in E_o$ for $i = 1, \ldots, n$.

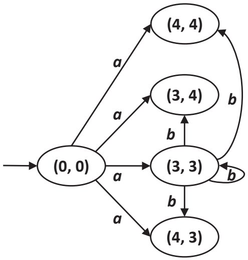


- There exists a reachable circuit $(3,4) \xrightarrow{\epsilon} (3,3) \xrightarrow{b} (3,4)$ such that $\epsilon b = b \in E_o$, and $(3,4) \xrightarrow{b} (3,4)$.
- G is not strongly periodically detectable.

Theorem 2 (Criterion for Checking Strong D-Detectability)

A UWA G is not strongly D-detectable w.r.t. Q_{spec} if and only if the following two conditions both hold.

1. There is a state q_2 such that $\{(q_2(L), q_2(R)), (q_2(R), q_2(L))\} \cap Q_{spec} \neq \emptyset$ and $q_2(L) \neq q_2(R)$, which is reachable from a circuit in $CC'(G)$.
2. There exists a circuit reachable from $q_2(L)$ or $q_2(R)$ in G.



- Let $Q_{spec} = \{(3,4)\}$, there exists a path $(0,0) \xrightarrow{a} (3,3) \xrightarrow{b} (3,3) \xrightarrow{b} (3,4)$ in $CC'(G_1)$.
- A circuit $3 \xrightarrow{b} 3$ reachable from state 3 in G_1.
- G is not strongly D-detectable w.r.t. Q_{spec}.

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Contributions

- We extend the notion of detectability from logical FSAs to UWAs.
- We construct the variants of self-composition and verify SPD and SDD with a polynomial-time complexity.

Further Extension

- To more general WAs.
- Other properties.
- Enforcement.

Reference

- **S. Miao**, A. Lai, X. Yu, S. Lahaye, and J. Komenda. Verification of Detectability for UWA using SC. in *Proc. 2023 9th International Conference on* ***Co****ntrol,* ***D****ecision and* ***I****nformation* ***T****echnologies*

Thank You!