# Verification of Detectability for Unambiguous Weighted Automata Using Self-Composition

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***Abstract*—This paper aims to explore the problem of verifying detectability for unambiguous weighted automata (UWAs) through the utilization of modified self-composition. Specifically, we focus on two types of detectability: strong periodic detectability (SPD) and strong D-detectability (SDD). The problem involves periodically determining the current state or distinguishing certain state-pairs of the system, based on the occurrence of a finite number of observable events. We introduce a new polynomial-time algorithm different from the detector, called self-composition for UWAs, and prove that it can be used to verify the SPD and SDD for UWAs. Ultimately, we propose necessary and sufficient conditions based on modified self-composition techniques to verify the aforementioned detectabilities for the studied UWA.**

***Index Terms*—Discrete event system, unambiguous weighted automaton, self-composition, detectability, polynomial time.**

## I. Introduction

Detectability is an important property of discrete event systems (DESs) that requires estimating the states of a system based on observations. This property also plays an important part in several related problems, including state estimation, verification of opacity or diagnosability, and control synthesis [1], [2].

The verification problem of detectability for finite state automata (FSAs) has been extensively investigated in the literature [3]–[6]. Note that an FSA has no more real-world information than the occurrences of events, however, the weights of events may represent the time needed or something else, to better model the actual physical systems. In this context, the verification of various detectabilities is also studied in FSAs with weighted sequences, namely, weighted automata (WAs) [5], [7]–[9]. In addition, the work in [10] considers a more complicated model, called timed automata (TAs), which adds some constraints on weights when systems generate events. It is shown in [10] that strong detectability (SD) for TA is decidable in exponential time and weak detectability is undecidable. However, for UWA in this paper, some detectabilities can be checked efficiently within polynomial time.

The notion of detectability is first defined in [11] for deterministic finite-state automata (DFAs), and an observer is used to check detectability. Eight types of different detectabilities are originally studied in [3] for nondeterministic finite-state automata (NFAs). An observer-based exponential algorithm is proposed to verify all these detectabilities. Furthermore, a detector-based polynomial-time algorithm is introduced to check certain detectabilities, including SD, SPD, and SDD. In [4], the concept of delayed detectability is introduced, which allows us to estimate the states of the system within some delays. Different from the above detectabilities focusing on the current state of DESs, I-detectability (I for "initial") and its corresponding I-observer are proposed to estimate the initial state of a DES [12].

A method different from the work in [4], i.e., concurrent-composition is first proposed in [6], to check delayed detectability for FSAs, from the negation of detectability. Following up on the concurrent-composition method, [9] investigate the verification problem of SD for WAs over monoids, additionally, the observer and detector are reconsidered for verifying weak (periodic) detectability and SPD. It is worth noting that there is no work focus on D-detectability for WAs with polynomial-time algorithms, and different from [9], we verify the SPD using self-composition rather than observer or detector.

In a recent study [13], it is proven that the problem of verifying weak (periodic) detectability of DFAs is PSPACE-complete. This result implies that, unless P=PSPACE, there does not exist a polynomial-time algorithm to solve this problem. Furthermore, [14] proves that even for DFAs without non-trivial cycles, this problem remains practically intractable. In addition, the problem of verifying weak (periodic) D-detectability and strong periodic D-detectability are also proven to be PSPACE-complete [15], [16]. In conclusion, there are only three types of those detectabilities, namely SD, SPD, and SDD, which can potentially be verified in polynomial time.

This paper proposes a method to verify the detectability for UWAs. Later, it will be demonstrated that the detectabilities of a UWA can differ from those of the underlying FSA as the weights of events have an impact on it. In this paper, the studied automata are assumed to be reachable, which means that all states can be reached starting from an initial state. Additionally, we assume that the WAs are unambiguous, i.e., there is no more than one path leading to the same state from an initial state labeled with the same string, which makes the verification problem of detectability solvable.

This paper makes the following main contributions.

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1) The notion of strong D-detectability is defined for WAs.
2) For a given UWA $G$, its modified $\epsilon$-extended self-composition and modified self-composition are constructed, and two corresponding necessary and sufficient conditions are derived to check the SPD and SDD, respectively, with a polynomial-time complexity.

The structure of the present paper is outlined as follows. Section II offers an overview of some preliminaries pertaining to UWAs. In Section III, we present formal definitions of SPD and SDD for WAs. Section IV introduces necessary and sufficient conditions in the form of theorems, which are based on the construction of self-composition, to enable verification of the two detectabilities of a UWA. Finally, Section V summarizes the key findings of the study and outlines possible avenues for future research.

## II. Preliminaries

This section provides a brief overview of some preliminaries related to WAs [17].

*Definition 1:* A semiring is a mathematical structure defined by a tuple $\mathbb{S} = \langle \mathcal{D}, \oplus, \otimes, \varepsilon, e \rangle$, where $\mathcal{D}$ is a non-empty set, $\oplus$ and $\otimes$ are two associative binary operations defined on $\mathcal{D}$, and $\varepsilon$ and $e$ are two elements belonging to $\mathcal{D}$. The semiring satisfies the following properties:

- $\langle \mathcal{D}, \oplus, \varepsilon \rangle$ is a commutative monoid with a neutral element $\varepsilon$, that is, $a \oplus b = b \oplus a$ and $\varepsilon \oplus a = a \oplus \varepsilon = a$ hold for every $a, b \in \mathcal{D}$;
- $\langle \mathcal{D}, \otimes, e \rangle$ is a monoid with a neutral element $e$, that is, $e \otimes a = a \otimes e = a$ holds for every $a \in \mathcal{D}$;
- the distributivity laws $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ and $c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$ hold for every $a, b, c \in \mathcal{D}$;
- $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$ for every $a \in \mathcal{D}$.

*Example 1:* One of the most commonly used examples of a semiring is the set of natural numbers under the standard addition and multiplication, denoted as $\mathbb{S} = \langle \mathbb{N}, +, \times, 0, 1 \rangle$. Another notable semiring is the max-plus semiring, denoted as $\mathbb{Q}_{max} = \langle \mathbb{Q} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$, where $\mathbb{Q}$ is the set of rational numbers.

Let $E$ be a non-empty, finite set of labels that serves as the alphabet of a DES. The set $E^*$ denotes the set of all the finite-length strings formed by concatenating the labels from $E$, including the empty word $\epsilon$, which is the identity element of concatenation. The set $E^\omega$ denotes the set of all infinite-length strings formed by concatenating the labels from $E$.

*Definition 2:* A WA $G$ over a max-plus semiring $\mathbb{Q}_{max} = \langle \mathbb{Q} \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$ is defined as a tuple:

$$G = (Q, E, t, Q_0, \varrho),$$

where

- $Q$ is a non-empty finite state set and $E$ is a non-empty finite event set;
- The transition function $t : Q \times E \times Q \to \mathbb{Q}_{max}$ associates each state transition with a value from the set $\mathbb{Q}_{max}$. If $t(p, e, q) \neq \varepsilon$, it signifies the presence of a transition from state $p$ to state $q$ labeled with $e$, and the weight of this transition is given by $t(p, e, q)$. Particularly, if $t(p, e, q) = \varepsilon$, it indicates that there is no transition from state $p$ to state $q$ labeled with $e$;
- The function $\varrho : Q \to \mathbb{Q}_{max}$ maps the states to their corresponding initial delays. If $\varrho(q) \neq \varepsilon$, it signifies that the state $q$ is an initial state, and $\varrho(q)$ represents its initial weight. Conversely, if $\varrho(q) = \varepsilon$, it indicates that the state $q$ is not an initial state;
- $Q_0$ is the initial state set: $Q_0 = \{q \in Q \mid \varrho(q) \neq \varepsilon\}$.

*Definition 3:* Given a WA $G$, we define its path of length $k \in \mathbb{N}$ as a sequence of state transitions: $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$, where $q_i \in Q$ for $i = 0, \ldots, k$, $e_i \in E$ and $t(q_{i-1}, e_i, q_i) \neq \varepsilon$ for $i = 1, \ldots, k$.

A path $\pi$ can be considered a circuit if and only if the first state $q_0$ coincides with the last state $q_k$. The set of all paths labeled with string $\omega \in E^*$ originating from state $q_1$ leading to $q_2$ is denoted by $q_1 \overset{\omega}{\rightsquigarrow} q_2$. Note that we have $q_1 \overset{\epsilon}{\rightsquigarrow} q_2 = \emptyset$. For any subsets $Q_1, Q_2 \subseteq Q$, we use $Q_1 \overset{\omega}{\rightsquigarrow} Q_2$ to denote the union of the path set $q_1 \overset{\omega}{\rightsquigarrow} q_2$ for any $q_1 \in Q_1$ and $q_2 \in Q_2$. For a circuit $q \xrightarrow{\omega} q$, we denote by $q(\xrightarrow{\omega} q)^k$ the concatenation of $q$ and $k$ copies of $\xrightarrow{\omega} q$, where $k \in \mathbb{N}$.

*Definition 4:* For an arbitrary path $\pi$ of length $k$, denoted as $\pi = (q_0, e_1, q_1)(q_1, e_2, q_2) \cdots (q_{k-1}, e_k, q_k)$ of a WA $G$ with $q_0 \in Q_0$, the weighted sequence $\sigma(\pi) \in (E \times \mathbb{Q})^*$ generated by $\pi$ is defined as a sequence of ordered pairs: $\sigma(\pi) = (e_1, \tau_1)(e_2, \tau_2) \cdots (e_k, \tau_k)$, where $\tau_1 = \varrho(q_0) + t(q_0, e_1, q_1), \tau_i = \tau_{i-1} + t(q_{i-1}, e_i, q_i)$ for $i = 2, \ldots, k$.

In an intuitive sense, the weighted sequence $\sigma(\pi)$ is the concatenation of the ordered pairs of the form (event, weight). To represent the set of all weighted sequences $\sigma(\pi)$ that lead from state $q_0$ to $q_k$, we use the notation $q_0 \overset{\sigma(\pi)}{\rightsquigarrow} q_k$. For transitions $q_0 \xrightarrow{e_1} q_1$, $q_1 \xrightarrow{e_2} q_2$, $\ldots$, $q_{n-1} \xrightarrow{e_n} q_n$, we denote them by $q_0 \xrightarrow{e_1 e_2 \cdots e_n} q_n$ for simplicity.

*Definition 5:* A WA $G$ is considered unambiguous if $|Q_0 \overset{\omega}{\rightsquigarrow} \{q\}| \leq 1$, where $q \in Q$ and $\omega \in E^*$.

In simple terms, unambiguity in this context means that there can only be one path labeled with a given string $\omega$ that leads to a state $q$ from the initial state. There are some efficient algorithms to verify the degree of ambiguity for the underlying FSA in polynomial-time [18].

A UWA $G$ is considered to be deterministic if both the following two conditions are true:

1) $G$ possesses only one initial state;
2) For any state in $G$, each of its transitions is associated with a distinct label.

It is evident that a deterministic automaton guarantees unambiguity, but the reverse is not always true. Consequently, it can be inferred that the category of UWAs is more extensive than that of DFAs.

*Example 2:* Let us examine the UWA $G$ depicted in Fig. 1, where $Q_0 = \{0\}$, $Q = \{0, 1, 2, 3, 4, 5\}$, $E = E_o \cup E_{uo}$ with $E_o = \{a, b, c\}$, $E_{uo} = \{u\}$. Transitions $t(0, u, 1) = 0.1$, $t(0, u, 2) = 0.5$, $t(1, a, 3) = 0.7$, $t(2, a, 4) = 0.3$, $t(3, b, 4) = 0.1$, $t(4, c, 4) = 0.1$, $t(5, c, 5) = 0.1$, $t(3, u, 5) = \tau$, where $\tau \neq 0$. It should be noted that any values not explicitly listed in the transition function $t(p, e, q)$ indicate the lack of a transition. Initial delays are $\varrho(0) = 0$ and $\varrho(i) = \varepsilon$ for $i = 1, \ldots, 5$.

Note that this automaton $G$ is unambiguous, but nondeterministic. Now, if we replace transition $t(0, u, 1) = 0.1$ with $t(0, b, 1) = 0.1$, then the modified automaton is deterministic, and then unambiguous.

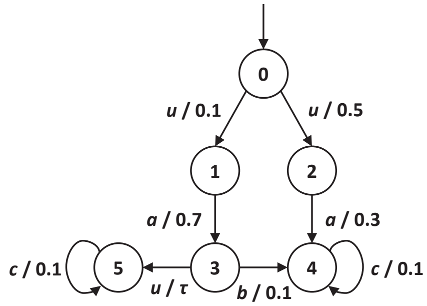


Fig. 1: A UWA $G$.

*Definition 6:* Given a UWA $G$, its generated weighted language $L(G)$ is defined as:

$$L(G) = \{s \in (E \times \mathbb{Q})^* \mid \exists p \in Q_i, \exists q \in Q, \\ \exists \omega \in E^*, \sigma(p \xrightarrow{\omega} q) = s\}.$$

We slightly abuse the notations and denote by $\epsilon_t$ the empty weighted sequence. For two weighted sequences $\sigma_1$ and $\sigma_2$, we denote by $\sigma_1 \cdot \sigma_2$ (or simply $\sigma_1\sigma_2$) the concatenation of $\sigma_1$ and $\sigma_2$. Specially, for any $\sigma \in (E \times \mathbb{Q})^*$, it holds that $\sigma\epsilon_t = \epsilon_t\sigma = \sigma$.

*Definition 7:* Given an event set $E = E_o \cup E_{uo}$, the natural projection $P : E^* \to E_o^*$ can be extended to the case of weighted sequences as $P : (E \times \mathbb{Q})^* \to (E_o \times \mathbb{Q})^*$. This extension is defined as follows:

$$P(\epsilon_t) = \epsilon_t, \quad P((e,\tau)) = \begin{cases} (e,\tau), & \text{if } e \in E_o \\ \epsilon_t, & \text{if } e \in E_{uo} \end{cases}$$
$$P(\sigma \cdot (e,\tau)) = P(\sigma)P((e,\tau))$$
$$\text{for } \sigma \in (E \times \mathbb{Q})^*, (e,\tau) \in (E \times \mathbb{Q}).$$

As a result, it follows that for any weighted sequence $\sigma = (e_1,\tau_1)(e_2,\tau_2)\cdots(e_k,\tau_k) \in (E \times \mathbb{Q})^*$, $P(\sigma)$ is defined as the weighted sequence that results from eliminating all ordered pairs in $\sigma$ that include an event in $E_{uo}$.

In the rest of this paper, we use $P(L(G))$ to denote the set of observations of UWA $G$, i.e., the projection of the weighted language.

## III. Problem Statement

We partition the alphabet $E$ into two distinct and separate parts as $E = E_o \cup E_{uo}$, where $E_o$ and $E_{uo}$ are the observable and the unobservable parts. All unobservable labels are assumed to be represented by symbol $u$, namely $E = E_o \cup \{u\}$. We can assume, without loss of generality, that the weights of all initial states in the UWA are equal to zero.

The following assumptions are made in this work to analyze the weighted automaton $G$. 1) The WA $G$ is unambiguous (see Def. 5); 2) There is no circuit that is only labeled with unobservable events in the system. However, even in a UWA, there can be cases where multiple paths can generate the same observed weighted sequence.

### A. Consistent State

In logical DESs, the set of all infinite-length event sequences that the system may generate is referred to as the $\omega$-language. It can be naturally extended to the WAs, for a WA $G$, the set of all infinite-length sequences of (event, weight) pairs that $G$ may generate is denoted by $L^\omega(G)$.

*Definition 8:* For an observed weighted sequence $\sigma_o \in P(L(G))$, the set of all $\sigma_o$-consistent states is defined as:

$$C(\sigma_o) = \{q \in Q \mid \exists \sigma \in L(G), \exists q_0 \in Q_0 : \\ q_0 \overset{\sigma}{\rightsquigarrow} q, P(\sigma) = \sigma_o\}.$$

In simple terms, a state $q$ is said to be consistent with observation $\sigma_o$ if there exists a weighted sequence $\sigma$ that leads to state $q$ and has the property that its projection coincides with $\sigma_o$.

*Example 3:* Consider the UWA $G$ depicted in Fig. 1. Given $\sigma_o = (a, 0.8)$, it follows that the set of $\sigma_o$-consistent states is given by $C(\sigma_o) = \{3, 4, 5\}$. It is worth noting that there exist two distinct paths originating from an initial state that generate weighted sequences consistent with $\sigma_o$, i.e., $\pi_1 = (0, u, 1)(1, a, 3)$ and $\pi_2 = (0, u, 2)(2, a, 4)$. Considering $\pi_2$, we have $\sigma_2 = \sigma(\pi_2) = (u, 0.5)(a, 0.8)$ and $0 \xrightarrow{\sigma_2} 4$. Consequently, state 4 is $\sigma_o$-consistent. Similarly, if we consider $\pi_1$, state 3 is also $\sigma_o$-consistent. Since the transition $(3, u, 5)$ is unobservable, the state 5 is also a consistent state.

### B. Detectability of Unambiguous Weighted Automata

In this subsection, we expand the scope of the detectability problem from its original definition for NFAs in [3], to encompass the framework of UWAs, as described in [7]. For every weighted sequence $\sigma$, we use $\bar{\sigma}$ to denote the set of all prefixes of $\sigma$.

*Definition 9 (Strong Periodic Detectability):* A UWA $G$ is strongly periodically detectable w.r.t. a projection $P$ if it is possible to periodically ascertain the current state of the automaton for all weighted sequences of infinite length, that is,

$$(\exists k \in \mathbb{N})(\forall \sigma \in L^\omega(G))(\forall \sigma' \in \bar{\sigma}) \\ (\exists \sigma'' \in (E \times \mathbb{Q})^*)\sigma'\sigma'' \in \bar{\sigma} \\ \wedge |P(\sigma'')| < k \wedge |C(P(\sigma'\sigma''))| = 1.$$

The SPD requires that through a finite number of observations, the set of observation-consistent states will periodically become a singleton set. However, in some cases, it may be too strict, and a more relaxed detectability is considered, namely, D-detectability (D for "distinguish").

D-detectability is another property that imposes the requirement that a finite number of observations is sufficient to distinguish any state-pair belonging to a given specification $Q_{spec} \subseteq Q \times Q$. It is useful for instance in supervisory control with partial observations, where the possibility of achieving a prescribed closed-loop behavior depends on the capability to differentiate some state-pairs based on system observation.

*Definition 10 (Strong D-Detectability):* A UWA $G$ is strongly D-detectable w.r.t. a projection $P$ and a specification $Q_{spec}$ if it is possible to consistently differentiate between state-pairs within $Q_{spec}$, through a finite number of observations, for all weighted sequences of infinite length, that is,

$$(\exists k \in \mathbb{N})(\forall \sigma \in L^\omega(G))(\forall \sigma' \in \bar{\sigma})|P(\sigma')| > k \\ \Rightarrow C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} = \emptyset.$$

When $Q_{spec} = \{(q_1, q_2) \in Q \times Q \mid q_1 \neq q_2\}$, strong D-detectability is equivalent to strong detectability.

## IV. Detectability Verification for unambiguous weighted automata

In this section, we draw inspiration from the previous work of [6], [9], a modified self-composition of a UWA $G$ is constructed to establish necessary and sufficient conditions for checking the detectabilities mentioned above in a UWA.

### A. Construction of Self-Composition

In this subsection, we recall a notion of concurrent composition for a UWA $G$ and itself, namely self-composition of $G$ in [9]. For any $Q_{c1}, Q_{c2} \subseteq Q_c$, we denote $(Q_{c1}, e, Q_{c2})$ the set of $(q_{c1}, e, q_{c2})$ for any $q_{c1} \in Q_{c1}$ and $q_{c2} \in Q_{c2}$.

*Definition 11:* Given a UWA $G$, its self-composition is defined as an FSA

$$CC(G) = (Q_c, E_c, Q_{0,c}, \Delta_c),$$

where

- $Q_c = Q \times Q$; $E_c = E_o$; $Q_{0,c} = Q_0 \times Q_0$;
- $\Delta_c \subseteq Q_c \times E_o \times Q_c$ is the set of state transitions. We have $(\{(q_1, q_2)\}, e, C(\sigma_o) \times C(\sigma_o)) \subseteq \Delta_c$ if and only if there exist two paths in $G$ as follows:

$$\pi_1 = q_1 \xrightarrow{\omega_1} q_1' \xrightarrow{e} q_3,$$
$$\pi_2 = q_2 \xrightarrow{\omega_2} q_2' \xrightarrow{e} q_4,$$

  where state-pairs $(q_1, q_2), (q_1', q_2'), (q_3, q_4) \in Q_c$, and two weighted sequences $\sigma_1, \sigma_2 \in (E \times \mathbb{Q})^*$ are generated by $\pi_1, \pi_2$ as follows:

$$\sigma_1 = (\omega_1, \tau_1)(e, \tau),$$
$$\sigma_2 = (\omega_2, \tau_2)(e, \tau),$$

  where $\sigma_1$ and $\sigma_2$ have the same observation $\sigma_o$, i.e., $P(\sigma_1) = P(\sigma_2) = (e, \tau) = \sigma_o$, $\omega_1, \omega_2 \in E_{uo}^*$ are unobservable event sequences, $e \in E_o$ is the unique observable event in the weighted sequences $\sigma_1$ and $\sigma_2$. Note that we have $q_3, q_4 \in C(\sigma_o)$.

Intuitively, we construct a transition $((q_1, q_2), e, (q_3, q_4))$ in $CC(G)$ if and only if there exist two paths that are labeled with the unique observable event $e$, which start from $q_1$ and $q_2$, through some identical weights, then lead to $q_3$ and $q_4$ in $C(\sigma_o)$, respectively. Since we are incapable of observing the exact weight at whim, between the current unique observable event $e$ occurs and the next observable event arrives, it is necessary to consider all the $\sigma_o$-consistent states after the occurrence of $e$ no matter how the weight is.

For any state $q_c \in Q_c$, we denote $q_c = (q_c(L), q_c(R))$. In addition, a state $q_c$ is called a singleton state if $q_c(L) = q_c(R)$, and called a non-singleton state otherwise.

*Definition 12:* Given a UWA $G$, a modified self-composition $CC'(G)$ obtained from $CC(G) = (Q_c, E_o, Q_{0,c}, \Delta_c)$ is defined by deleting each state-pair $(q_c(L), q_c(R))$ and the transitions attached to it if $t(p, u, q_c(L)) \neq 0$ or $t(p, u, q_c(R)) \neq 0$ as follows:

$$CC'(G) = (Q_c', E_o, Q_{0,c}, \Delta_c'),$$

where

- $Q_c' = Q_c \setminus \{q_c \in Q_c \mid \exists p, q_c(L), q_c(R) \in Q, \exists u \in E_{uo} : t(p, u, q_c(L)) \neq 0 \vee t(p, u, q_c(R)) \neq 0\}$;
- $\Delta_c' = \Delta_c \setminus \{(p_c, e, q_c) \in \Delta_c \mid \exists p_c, q_c \in Q_c, \exists u \in E_{uo} : t(p_c(L), u, q_c(L)) \neq 0 \vee t(p_c(R), u, q_c(R)) \neq 0\}$.

*Definition 13:* Given a UWA $G$, a variant from $CC(G) = (Q_c, E_o, Q_{0,c}, \Delta_c)$, called $\epsilon$-extended self-composition is defined as:

$$CC^{\epsilon}(G) = (Q_c, E_o \cup \{\epsilon\}, Q_{0,c}, \Delta_c^{\epsilon}),$$

where

- $\Delta_c^{\epsilon} = \Delta_c \cup \{((q_1, q_2), \epsilon, (q_1, q_1)) \mid \exists q_1, q_2 \in Q, \exists q_c \in Q_c, \exists e, e' \in E_o : ((q_1, q_1), e, q_c) \in \Delta_c \wedge ((q_1, q_2), e', q_c) \notin \Delta_c, q_1 \neq q_2\} \cup \{((q_1, q_2), \epsilon, (q_2, q_2)) \mid \exists q_1, q_2 \in Q, \exists q_c \in Q_c, \exists e, e' \in E_o : ((q_2, q_2), e, q_c) \in \Delta_c \wedge ((q_1, q_2), e', q_c) \notin \Delta_c, q_1 \neq q_2\}$.

Intuitively, for a singleton state $(q_1, q_1)$, the non-singleton state $(q_1, q_2)$ has a common component $q_1$ with it. Note that the term "an arbitrary state $q_c$" can refer to either a singleton state or a non-singleton state. If there exists an observable transition from the singleton state $(q_1, q_1)$ to the arbitrary state $q_c$, but there exists no observable transition from the non-singleton state $(q_1, q_2)$ to $q_c$, i.e., $((q_1, q_1), e, q_c) \in \Delta_c \wedge ((q_1, q_2), e', q_c) \notin \Delta_c$, then we add an $\epsilon$-extended transition from the non-singleton to the singleton state, i.e., $((q_1, q_2), \epsilon, (q_2, q_2))$, then we get a path from the non-singleton leading to the arbitrary state.

Without loss of generality, we obtain a modified $\epsilon$-extended self-composition $CC'^{-\epsilon}(G) = (Q_c', E_o \cup \{\epsilon\}, Q_{0,c}, \Delta_c'^{-\epsilon})$ by deleting the state-pairs as in Def. 12 and then adding the $\epsilon$-extended transitions as in Def. 13.

*Remark 1:* According to Def. 11, $CC(G)$ assembles every pair of transitions of $G$ that are labeled with a unique observable event with identical weights. The number of states and transitions of $CC(G)$ are at most $|Q|^2$ and $|Q|^2(|Q|^2 \times |E|)$, respectively. For any two states in the "unambiguous" WA $G$, there are at most $|Q|$ sequences $e, ue, uue, \cdots, u^k e$ from one to another, due to the absence of unobservable circuits, where $k = |Q| - 1$ and $e \in E_o$. Hence, for each state in $G$, there are at most $|Q|(|Q| - 1)|E_o|$ observable transitions with different weights. As a result, the complexity of constructing self-composition is $\mathcal{O}(|Q|^2(|Q|^2 \times |Q|(|Q| - 1)|E_o|)) = \mathcal{O}(|Q|^6|E_o|)$.

### B. Criterion for Checking Detectability

This subsection derives necessary and sufficient conditions from modified self-composition to check the SPD and SDD of a UWA $G$.

*Theorem 1:* A UWA $G$ is deemed not strongly periodically detectable iff in its modified $\epsilon$-extended self-composition $CC'^{-\epsilon}(G)$, at least one of the following two conditions is true.

1) There exists a reachable non-singleton state $q \in Q_c'$ such that $q(L) \neq q(R)$, and there exists a path $q(L) \xrightarrow{\omega_1} q' \xrightarrow{\omega_2} q'$ in $G$, where $\omega_1 \in E_{uo}^*$, $\omega_2 \in E_{uo}^* \setminus \{\epsilon\}$, $q' \in Q$.
2) There exists a reachable circuit of non-singleton states, i.e., $q_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} q_{n+1}$ for $n \in \mathbb{N}^+$ such that $q_1(L) = q_{n+1}(L), q_1(R) = q_{n+1}(R)$, $q_i(L) \neq q_i(R)$, $q_i \in Q_c'$, and $\omega_i \in E_o$ for $i = 1, \ldots, n$.

*Proof:* According to Def. 9, $G$ is not strongly periodically detectable iff for any $k \in \mathbb{N}$, there are $\sigma \in L^{\omega}(G)$ and $\sigma' \in \bar{\sigma}$ such that, for any $\sigma'' \in (E \times \mathbb{Q})^*$ where $|P(\sigma'')| < k$ and $\sigma'\sigma'' \in \bar{\sigma}$, the condition $|C(P(\sigma'\sigma''))| > 1$ holds true.

"if": Suppose that there is a reachable state $q$ in $CC'^{-\epsilon}(G)$ with $q(L) \neq q(R)$, and there is a path $q(L) \xrightarrow{\omega_1} q' \xrightarrow{\omega_2} q'$ in $G$. Choose $\sigma = \sigma(q_0 \xrightarrow{\omega'} q(L) \xrightarrow{\omega_1} q'(\xrightarrow{\omega_2} q')^\omega)$, where $q_0 \in Q_0$, $q' \in Q$, and $\sigma' = \sigma(\omega')$. Let $\sigma'' = \sigma(\omega_1(\omega_2)^{k-1})$, where $\omega_1 \in E_{uo}^*$, $\omega_2 \in E_{uo}^* \setminus \{\epsilon\}$. For any $\sigma''$, we can always find a $k$ that $|P(\sigma'')| < k$, then we have $\sigma'\sigma'' \in \bar{\sigma}$, $\sigma \in L^\omega(G)$, and $|C(P(\sigma'\sigma''))| = |C(P(\sigma'))| = |\{q(L), q(R)\}| > 1$. Thus, $G$ violates the definition of SPD.

Assume for $n \in \mathbb{N}^+$, there exists a reachable circuit $q_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} q_{n+1}$, where $q_1 = q_{n+1}$, $q_i(L) \neq q_i(R)$ and $q_i \in Q'_c$ for $i = 1, \ldots, n$. Choose $\sigma = \sigma(q_0 \xrightarrow{\omega'} q_1(\xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} q_{n+1})^\omega)$ such that $q_0 \in Q_0$, $\sigma' = \sigma(\omega')$. Let $\sigma'' = \sigma(\omega_1 \cdots \omega_n)$, then we have $|P(\sigma'')| < k$, $\sigma'\sigma'' \in \bar{\sigma}$, $\sigma \in L^\omega(G)$, and $|C(P(\sigma'\sigma''))| > 1$. Therefore, we know that $G$ is not strongly periodically detectable.

"only if": If $G$ violates the definition of SPD, according to Def. 9, we choose $k > |Q|^2$, and prove the two above conditions hold, respectively.

If there is no reachable circuit consisting of non-singleton states, then since $|C(P(\sigma'\sigma''))| > 1$, there exists a non-singleton state $q$ with $q(L) \neq q(R)$. Choose $\sigma'' = \sigma(\omega_1\omega_2)$, according to Pigeonhole Principle, there necessarily exists a path as $q_0 \xrightarrow{\omega'} q_1 \xrightarrow{\omega_1} q' \xrightarrow{\omega_2} q'$ where $\omega_1 \in E_{uo}^*$, $\omega_2 \in E_{uo}^* \setminus \{\epsilon\}$.

If there is no unobservable circuit from a component of the non-singleton state, since $|C(P(\sigma'\sigma''))| > 1$, there exists more than one non-singleton state $q_1, q_2, \cdots$. Then choose $\sigma'' = \sigma(\omega_1 \cdots \omega_n)$, according to Pigeonhole Principle, there necessarily exists a reachable circuit $q_1 \xrightarrow{\omega_1} \cdots \xrightarrow{\omega_n} q_{n+1}$ such that $q_1(L) = q_{n+1}(L), q_1(R) = q_{n+1}(R)$, $q_i(L) \neq q_i(R)$, $q_i \in Q'_c$ and $\omega_i \in E_o$ for $i = 1, \ldots, n$. ■

*Theorem 2:* A UWA $G$ is deemed not strongly D-detectable iff in its modified self-composition $CC'(G)$ the following two conditions both hold.

1) There is a non-singleton state having a non-empty intersection with $Q_{spec}$, and is reachable from a state-pair in a circuit, i.e., there exists a path

$$q_0 \xrightarrow{\omega_1} q_1 \xrightarrow{\omega_2} q_1 \xrightarrow{\omega_3} q_2, \tag{1}$$

where

$$\begin{aligned} &q_0 \in Q_{0,c};\ q_1, q_2 \in Q'_c; \\ &\omega_1, \omega_2, \omega_3 \in E_o^*;\ q_2(L) \neq q_2(R); \\ &\{(q_2(L), q_2(R)), (q_2(R), q_2(L))\} \cap Q_{spec} \neq \emptyset; \end{aligned} \tag{2}$$

2) In addition, in $G$, there exists a circuit reachable from $q_2(L)$, i.e., there exists a path

$$\cdots \longrightarrow q_2(L) \xrightarrow{\omega_4} q_3 \xrightarrow{\omega_5} q_3, \tag{3}$$

where $\omega_4, \omega_5 \in E_o^*$ and $q_3 \in Q$.

*Proof:* According to Def. 10, $G$ is not strongly D-detectable iff for any $k \in \mathbb{N}$, there exists $\sigma \in L^\omega(G)$ and $\sigma' \in \bar{\sigma}$ such that $|P(\sigma')| > k$ and $C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} \neq \emptyset$.

"if": Given an arbitrary parameter $k \in \mathbb{N}$, according to (1) and (2), there is a path $\sigma' = \sigma(q_0(L) \xrightarrow{\omega_1} q_1(L)(\xrightarrow{\omega_2} q_1(L))^k \xrightarrow{\omega_3} q_2(L))$ in $G$ such that $C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} = (q_2(L), q_2(R))$ or $(q_2(R), q_2(L))$; According to (3), choose

$$\sigma = \sigma(q_0(L) \xrightarrow{\omega_1} q_1(L)(\xrightarrow{\omega_2} q_1(L))^k \xrightarrow{\omega_3} q_2(L) \xrightarrow{\omega_4} q_3(\xrightarrow{\omega_5} q_3)^\omega),$$

then we have $\sigma \in L^\omega(G)$ and $\sigma' \in \bar{\sigma}$ satisfies $|P(\sigma')| \geqslant k+2 > k$, and $C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} \neq \emptyset$. Therefore, $G$ violates the definition of SDD.

"only if": Suppose that $G$ violates the definition of SDD. Let $k > |Q|^2$, $\sigma \in L^\omega(G)$ and $\sigma' \in \bar{\sigma}$ such that $|P(\sigma')| > k$ and $C(P(\sigma')) \times C(P(\sigma')) \cap Q_{spec} \neq \emptyset$. Thus, there necessarily exist two distinct paths $\pi_1$ and $\pi_2$ in $G$ that originate from the initial state and lead to distinct states such that $\sigma(\pi_1) = \sigma(\pi_2) \in \bar{\sigma}$. According to Def. 11, from $\pi_1$ and $\pi_2$ we can construct a path of $CC'(G)$ as in (1), according to Pigeonhole Principle, since $CC'(G)$ has at most $|Q|^2$ states, there necessarily exists a circuit reachable from $q_2(L)$. ■

*Example 4:* Let us revisit the UWA $G$ illustrated in Fig. 1, and construct its modified $\epsilon$-extended self-composition $CC'^{-\epsilon}(G)$ in Fig. 2. The modified self-composition $CC'(G)$ is obtained from the self-composition $CC(G)$ by deleting five transitions, e.g., $(0,0) \xrightarrow{a} (3,5)$, $(0,0) \xrightarrow{a} (5,3)$, $(0,0) \xrightarrow{a} (4,5)$, $(0,0) \xrightarrow{a} (5,4)$, $(0,0) \xrightarrow{a} (5,5)$, since $t(3,u,5) = \tau \neq 0$. Then $CC'^{-\epsilon}(G)$ is obtained from $CC'(G)$ by adding four $\epsilon$-extended transitions, $(3,4) \xrightarrow{\epsilon} (3,3)$, $(4,3) \xrightarrow{\epsilon} (3,3)$, $(3,4) \xrightarrow{\epsilon} (4,4)$, and $(4,3) \xrightarrow{\epsilon} (4,4)$.

By Theorem 1, there exists no path in $CC'^{-\epsilon}(G)$ that satisfies one of the two conditions, that is $G$ is strongly periodically detectable.

Choose $Q_{spec} = \{(3,4)\}$, then by Theorem 2, similar to the above, there exists no path in $CC'(G)$ satisfying the two conditions, thus $G$ is strongly D-detectable.

Construct the observer or detector of the underlying FSA of $G$ as in [3], we find that the underlying FSA of $G$ lacks both SPD and SDD, i.e., the detectability of a UWA is changed due to the influence of the weights of transitions.

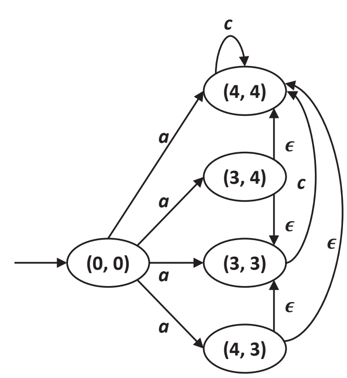


Fig. 2: Modified $\epsilon$-extended self-composition $CC'^{-\epsilon}(G)$ of $G$ in Fig. 1 where $\tau \neq 0$.

*Example 5:* Replace the transition $t(3,u,5) = \tau$ with $t(3,u,5) = 0$ in the UWA $G$ shown in Fig. 1, and construct its modified self-composition $CC'(G)$ shown in Fig. 3 (the $CC'^{-\epsilon}(G)$ is obtained from $CC'(G)$ by adding 12 $\epsilon$-extended transitions, such that $(3,4) \xrightarrow{\epsilon} (3,3)$, $(4,3) \xrightarrow{\epsilon} (3,3)$, $(3,5) \xrightarrow{\epsilon} (3,3)$, $(5,3) \xrightarrow{\epsilon} (3,3)$, $(4,3) \xrightarrow{\epsilon} (4,4)$, $(3,4) \xrightarrow{\epsilon} (4,4)$, $(4,5) \xrightarrow{\epsilon} (4,4)$, $(5,4) \xrightarrow{\epsilon} (4,4)$, $(5,3) \xrightarrow{\epsilon} (5,5)$,

$(3,5) \xrightarrow{\epsilon} (5,5)$, $(5,4) \xrightarrow{\epsilon} (5,5)$, and $(4,5) \xrightarrow{\epsilon} (5,5)$). Since $t(3,u,5) = 0$, the structure of $CC'(G)$ and $CC'^{\text{-}\epsilon}(G)$ are the same as that of $CC(G)$ and $CC^{\epsilon}(G)$, respectively.

By Theorem 1, there exists a reachable circuit $(3,5) \xrightarrow{c} (3,5)$ (or $(5,3) \xrightarrow{c} (5,3)$) in $CC'^{\text{-}\epsilon}(G)$ that satisfies the condition 2), thus $G$ is not strongly periodically detectable.

Choose $Q_{spec} = \{(4,5)\}$, then by Theorem 2, there exists a path $(0,0) \xrightarrow{a} (4,5) \xrightarrow{c} (4,5) \xrightarrow{c} (4,5)$ in $CC'(G)$, and in $G$, there exists a circuit $4 \xrightarrow{b} 4$ reachable from state $4$. We know that $G$ is not strongly D-detectable.

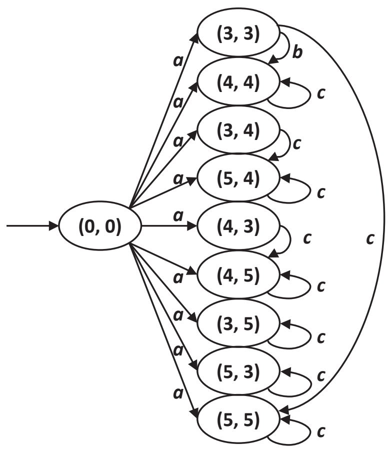


Fig. 3: Modified self-composition $CC'(G)$ of $G$ in Fig. 1 where $\tau = 0$.

The role of $\epsilon$-extended transitions might be unclear in the above examples, however, it is necessary in some cases (see Example 6).

*Example 6:* Consider another UWA $G_1$ and construct its modified $\epsilon$-extended self-composition $CC'^{\text{-}\epsilon}(G_1)$ in Fig. 4 (the $CC'(G_1)$ is obtained from $CC'^{\text{-}\epsilon}(G_1)$ by deleting the four $\epsilon$-extended transitions, i.e., $(3,4) \xrightarrow{\epsilon} (3,3)$, $(4,3) \xrightarrow{\epsilon} (3,3)$, $(3,4) \xrightarrow{\epsilon} (4,4)$, and $(4,3) \xrightarrow{\epsilon} (4,4)$), then the effect of $\epsilon$-extended transitions will be shown.

By Theorem 1, there exists a reachable circuit $(3,4) \xrightarrow{\epsilon} (3,3) \xrightarrow{b} (3,4)$ such that $\epsilon b = b \in E_o$ in $CC'^{\text{-}\epsilon}(G_1)$, that is, $G_1$ is not strongly periodically detectable.

Choose $Q_{spec} = \{(3,4)\}$, then by Theorem 2, there exists a path $(0,0) \xrightarrow{a} (3,3) \xrightarrow{b} (3,3) \xrightarrow{b} (3,4)$ in $CC'(G_1)$, and in $G_1$, there exists a circuit $3 \xrightarrow{b} 3$ reachable from state $3$. Therefore, $G_1$ is not strongly D-detectable.

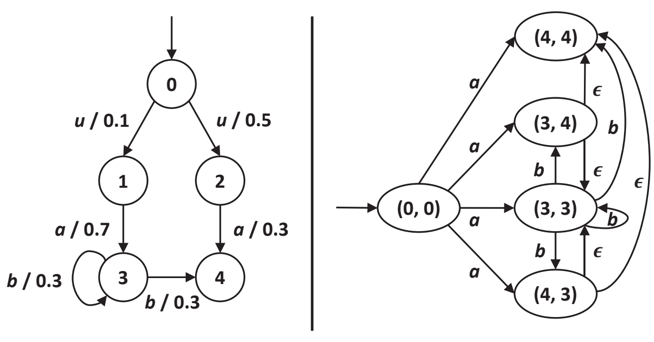


Fig. 4: A UWA $G_1$ (left) and its modified $\epsilon$-extended self-composition $CC'^{\text{-}\epsilon}(G_1)$ (right).

*Remark 2:* A circuit is considered elementary if none of its vertices, except for the first and last, are repeated. Two elementary circuits are regarded as distinct if one cannot be obtained from the other by a cyclic permutation. The above detectabilities can be verified by searching a non-singleton state belonging to the elementary circuit, with a complexity that is linear in the size of $CC(G)$, i.e., $\mathcal{O}(|Q_c| \times (|Q|^2 + |Q|^4|E|)) = \mathcal{O}(|Q|^2 \times (|Q|^2 + |Q|^4|E|)) = \mathcal{O}(|Q|^6|E|)$, according to Remark 1.

## V. Conclusion and Future Work

This paper focuses on studying the problem of verifying the SPD and SDD for UWAs using a new polynomial-time algorithm different from the detector, i.e., self-composition. For this purpose, we defined the notion of SDD for WAs and proposed a necessary and sufficient condition by modified self-composition for a UWA. In addition, we construct a structure called modified $\epsilon$-extended self-composition for a UWA to check SPD and derive its corresponding necessary and sufficient condition. As part of future research, we aim to explore the detectability and other related properties of more general WAs.

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