Research Statement

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1 Background

1.1 CFTs and VOAs

Quantum field theory (QFT) has become a unifying framework in modern mathematics, providing deep conceptual bridges among geometry, topology, analysis, representation theory, and tensor category theory, which has been more and more significant in modern physics and mathematics. **Topological field theory** (TFT) and two-dimensional **conformal field theory** (CFT; 'two-dimensional' will be omitted hereafter) are two types of QFT that are mathematically well-defined and well-studied. In 1987, Kontsevich and Segal independently gave a precise definition of full CFT using the properties of path integrals as axioms. Segal further introduced modular functors and weak conformal field theories. In 1988, Moore and Seiberg formulated certain basic hypotheses for rational CFT and derived some important consequences in [MS]. It has been a long-standing open problem to construct chiral and full CFTs in the sense of Kontsevich and Segal.

In TFT, the notions of vector space, Frobenius algebra, and modular tensor category provide the underlying algebraic/categorical structure for one, two, and three-dimensional TFT. Analogously, the algebra of **intertwining operators** among modules for a **vertex operator algebra** (VOA) is the underlying algebraic structure for two-dimensional chiral CFT. Therefore, the study of CFTs can be largely converted to the study of VOAs and their representation theory. The systematic study of VOAs and their representation theory was started by Frenkel, Lepowsky, Meurman ([FLM4]), and Borcherds ([B]). Highly roughly speaking, a vertex operator algebra is a $\mathbb{Z}$-graded vector space $V = \sum_{n\in\mathbb{Z}} V_{(n)}$, together with a particular element $\mathbf{1} \in V$, called the vacuum, and a binary product $Y(-, x)-$, called vertex operator, whose outputs are formal Laurent series, i.e., $Y(u, x)v \in V((x))$ for $u, v \in V$. It satisfies certain axioms, including the most important one, the **Jacobi identity** (See (2.1)).

Based on the results in [HL2], [HL3], [HL4], [H1], [H3], [H4], and [H5], Huang proved the following theorem in [H6]:

Theorem 1.1. *Let V be a simple vertex operator algebra satisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$; $V_{(0)} = \mathbb{C}\mathbf{1}$; and as a V-module, V is equivalent to its contragredient V-module V'.*
2. *Every lower-bounded generalized V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then the category of V-modules has a natural structure of modular tensor category in the sense of Turaev [Tu1].

(Remark: Without having a precise definition of VOA and of modular tensor category, modular tensor categories associated to conformal field theories were discovered first in physics by Moore and Seiberg [MS].)

1.2 Orbifold conformal field theory

Orbifold CFTs are CFTs constructed from known theories and their automorphisms. The first example of orbifold CFT is the the moonshine module VOA $V^\natural$ constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM3] [FLM4] in mathematics. The automorphism group of $V^\natural$ is the Monster finite simple group. Their construction of $V^\natural$ proved the Mckay-Tompson conjecture, profoundly relating number theory and finite group theory. Later, this VOA $V^\natural$ further played a significant role in Borcherds' proof of Conway–Norton Conjecture. Their construction introduced a new string theory, which was later interpreted by physicists as an "orbifold" theory. In string theory, the more generally systematic study of orbifold CFTs was started by Dixon, Harvey, Vafa and Witten [DHVW1] [DHVW2]. See [H14] for an exposition of general results, conjectures and open problems in the construction of orbifold CFTs using the approach of the representation theory of vertex operator algebras.

It is natural to expect that Theorem 1.1 has generalizations in orbifold CFT.

In [K3], Kirillov Jr. stated that the category of g-twisted modules for a vertex operator algebra V for all g in a finite subgroup G of the automorphism group of V is a G-equivariant fusion category (G-crossed braided (tensor) category in the sense of Turaev [Tu2]). For general V, this is certainly not true. The vertex operator algebra V must satisfy certain conditions. Here is a precise conjecture formulated by Huang in [H9]:

Conjecture 1.2. *Let V be a vertex operator satisfying the three conditions in Theorem 1.1 and let G be a finite group of automorphisms of V. Then the category of g-twisted V-modules for all $g \in G$ is a G-crossed braided tensor category.*

We also conjecture that the category of g-twisted V-modules for all $g \in G$ is a G-crossed modular tensor category in a suitable sense. Since the definitions of G-crossed modular tensor category in [K3] and [Tu2] are different, more work needs to be done to find out which definition is the correct one for the category of twisted modules for a vertex operator algebra. But we do believe that this stronger G-crossed modular tensor category conjecture should be true in a suitable sense.

In the case that G is trivial (the group containing only the identity), Conjecture 1.2 and even the stronger G-crossed modular tensor category conjecture is true by Theorem 1.1. Thus the G-crossed modular tensor category conjecture is a natural generalization of Theorem 1.1 to the category of category of g-twisted V-modules for $g \in G$.

In the case that the fixed point subalgebra V^G of V under G satisfies the conditions in Theorem 1.1 above, the category of V^G-modules is a modular tensor category. In this case, Conjecture 1.2 can be proved using the modular tensor category structure on the category of V^G-modules and the results on tensor categories by Kirillov Jr. [K1] [K2] [K3] and Müger [Mü1] [Mü2]. In the special case that G is a finite cyclic group and V satisfies the conditions in Theorem 1.1, Carnahan-Miyamoto [CM] proved that V^G also satisfies the conditions in Theorem 1.1. In the case that G is a finite cyclic group and V is in addition a holomorphic vertex operator algebra (meaning that the only irreducible V-module is V itself), Conjecture 1.2 can be obtained as a consequence of the results of van Ekeren-Möller-Scheithauer [EMS] and Möller [Mö] on the modular tensor category of V^G-modules. Assuming that G is a finite group containing the parity involution and that the category of grading-restricted V^G-modules has a natural structure of vertex tensor category structure in the sense of [HL1], McRae [Mc] constructed a nonsemisimple G-crossed braided tensor category structure on the category of grading-restricted (generalized) g-twisted V-modules.

For general finite group G, the conjecture that the fixed point subalgebra V^G of V under G also satisfies the conditions in Theorem 1.1 is still open and seems to be a difficult problem. On the other hand, using twisted modules and twisted intertwining operators to construct G-crossed braided tensor categories seems to be a more conceptual and direct approach. If this approach works, we expect that the category of V^G-modules can also be studied using the G-crossed braided tensor category structure on the category of twisted V-modules.

In the case that the vertex operator algebra V does not satisfy the three conditions in Theorem 1.1 and/or the group G is not finite, it is not even clear what the precise conjecture should be. This was proposed as an open problem in [H9].

2 What I have done

In brief, I proved the associativity of twisted intertwining operators, under certain convergence and extension assumption. This is equivalent to a construction of **the associativity isomorphism in the G-crossed vertex/braided tensor category**, which is a main difficulty in proving Conjecture 1.2. (Another main difficulty is to prove the assumptions I need; see [Ta].)

To achieve this, I have done the following:

- **2.1 Systematic development of a complex analytic approach to VOA theory (in [DH] and [D])**

  Starting from Frenkel, Lepowsky, Meurman ([FLM4]), the classical study of VOAs and their representation theory is based on an algebraic approach - starting with formal series (most generally, the exponents can be any element in a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} = 0$). This algebraic approach has been fully developed in the last 40 years and been used to successfully solve many problems. The *Jacobi identity* ([FLM4]) (in the definition of VOA),

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y\left(u,x_1\right)Y\left(v,x_2\right)-x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y\left(v,x_2\right)Y\left(u,x_1\right)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y\left(Y\left(u,x_0\right)v,x_2\right), \tag{2.1}$$

is powerful enough to derive many useful results. In (2.1) and throughout the whole theory of VOA, the formal delta function $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ plays a significant role. Importantly, not only VOA itself, but also the (generalized) modules of VOA and the notion of *intertwining operators* among (generalized) modules also can be defined using a Jacobi identity similar to (2.1). The Jacobi identity is the algebraic expression which is equivalent to the fact that the sum of residues of certain meromorphic functions on the Riemmann sphere $\hat{\mathbb{C}}$ at all singularities is zero (Cauchy formula).

In classical (i.e., untwisted) theory, the algebraic formulation is not enough to study products and iterates of *more than one intertwining operator*, for example, $\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)$ and $\mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)$. The study of these objects is indispensable and vital for the construction of the vertex (therefore braided) tensor category. Essentially, we can no longer have a Jacobi identity in this case, because the correlation functions are multivalued *in both varibles* z_1 and z_2, which means that one can no longer get a single-valued meromorphic 1-form on $\hat{\mathbb{C}}$. Therefore, the coefficients in expansions at different singularities cannot have a relation (Jacobi identity) by simply using the Cauchy formula. This is where Huang ([H1]) and Huang, Lepowsky, Zhang ([HLZ4]) had to introduce certain complex analytic assumptions (convergence and extension properties) to go further. These assumptions must be satisfied for proving their result (i.e. the vertex tensor category and in particular, the braided tensor category structure on the module category), and also were proved (for nice VOAs and C_1-cofinite modules) using regular singular differential equation theory by Huang ([H3]).

Despite the involvedness of these complex analytic assumptions, Huang-Lepowsky's and Huang-Lepowsky-Zhang's work mainly used the algebraic approach. This is natural - one should always use algebraic approach whenever "one can", i.e., when there is a Jacobi identity to use, because although often lengthy and extremely technical, formal delta-function calculus offers an effective way to do computation and prove theorems, and therefore has been foundational to the VOA theory.

However, in the study of orbifold CFT, a systematic complex-analytic approach inevitably needs to be developed. This is because even for the vertex operator $Y_W(-, x)$ acting on modules, there are non-integer powers of x (and $\log x$ for g-twisted modules for an infinite-order automorphim g). This leads to an extra multivaluedness, which makes it impossible to write down a Jacobi identity as the definition of *twisted intertwining operators* (i.e., intertwining operators among three twisted modules). Geometrically speaking, this means we cannot have a single valued meromorphic 1-form on $\hat{\mathbb{C}}$ even for a product like $Y_W(v, z_1)\mathcal{Y}(w_1, z_2)w_2$. Instead, we have to use a certain notion of **duality** as the definition of twisted intertwining operator, which is a complex analytic statement. (In CFT, "duality" means that inserting vertex operators in different way gives different local expansions of the same correlation function on certain different but overlapping regions.)

The "definition" of the "complex analytic approach to VOA (orbifold theory)" could be:

> *Start from the duality version of the definitions of (twisted) module and (twisted) intertwining operator, develop the (twisted) representation theory of VOA, without using formal delta function, Jacobi identity, and Cauchy formula. In this way, one can push the boundary of the theory beyond what formal calculus can handle.*

Under this definition, because of its inevitability in the study of orbifold theory, we have systematically developed this complex analytic approach.

- **2.2 Introduction of the most general notion of twisted intertwining operator (in [DH] and [D])**

  Intertwining operators among twisted modules associated to commuting automorphisms of finite order appeared implicitly in the work [FFR] of Feingold, Frenkel and Ries and were introduced explicitly by Xu in [X] in terms of a generalization of the Jacobi identity for twisted modules. Xu's Jacobi identity works because in [X], only modules twisted by automorphisms in a *finite abelian group* are considered.

  In [H8], Huang introduced a definition of twisted intertwining operators among modules twisted by noncommuting automorphisms. In this definition, the correlation functions obtained from the products and iterates of a twisted intertwining operator and a twisted vertex operator are required to be of a special explicit form. It turns out that this definition is not general enough to study orbifold theory associated to a nonabelian group of automorphisms.

  We have introduced the most general notion of twisted intertwining operator, where no explicit form needs to be satisfied. In this definition, the correlation functions are multivalued functions whose single-valued branches are indexed by elements of the fundamental group of some configuration space, and are determined only by the image of an anti-homomorphism from the fundamental group to the group of automorphisms $\langle g_1, g_2\rangle$ generated by g_1, g_2 (if the twisted intertwining operator is of type $\binom{g_1 g_2}{g_1\ g_2}$). This definition of twisted intertwining operator is general enough for studying the orbifold theory associated to a nonabelian group of automorphisms. To give the correct notion of $P(z)$-tensor product of twisted modules, we need to use the most general twisted intertwining operators. If we use only a certain special set of twisted intertwining operators as in [H8] to define and construct the $P(z)$-tensor product bifunctor, we would obtain a quotient of the correct $P(z)$-tensor products. The notion of $P(z)$-tensor product has been foundational to the theory from the beginning.

  Moreover, based on our definition of twisted intertwining operator, we have proved some properties of twisted intertwining operators which are essential for the construction of G-braided vertex tensor categories. For example, we have constructed the skew-symmetry isomorphisms $\Omega_\pm$ and contragredient isomorphisms $A_\pm$ between spaces of twisted intertwining operators. Althrough these isomorphisms have occurred and played crucial roles in the untwisted theory, since we have used our most general notion of intertwining operator, the proof of their existence was new and used the complex analytic approach.

- **2.3 Construction of and an equivalent condition for a $P(z)$-tensor product (in [DH] and [D])**

For any $z \in \mathbb{C}^\times$, we have given a definition of a $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ of two twisted modules W_1 and W_2 using a natural universal property. We have also given an explicit construction of $W_1 \boxtimes_{P(z)} W_2$ by using twisted intertwining operators. After having a suitable definition of twisted intertwining operator, the definition and the explicit construction of $W_1 \boxtimes_{P(z)} W_2$ are just a straightforward generalization of their untwisted version in Huang-Lepowsky [HL2], [HL3], [HL4].

Based on the explicit construction of $W_1 \boxtimes_{P(z)} W_2$ mentioned above, in [DH], we have found an equivalent condition for a linear functional $\lambda \in (W_1 \otimes W_2)^*$ to be contained in $(W_1 \boxtimes_{P(z)} W_2)'$. (For any g-twisted module W, we can define its contragredient module W' which is a g^{-1}-twisted module.) If the module category considered is the category of grading-restricted generalized g-twisted modules for $g \in G$ (where $G \leq \mathrm{Aut}(V)$ is any finite group), then the statement is the following:

Theorem 2.1. *Let the module category considered be the category of grading-restricted generalized g-twisted modules for $g \in G$ (where $G \leq \mathrm{Aut}(V)$ is any finite group). Suppose $\lambda \in (W_1 \otimes W_2)^*$. Then $\lambda \in (W_1 \boxtimes_{P(z)} W_2)'$ if and only if λ satisfies a suitable $P(z)$-compatibility condition and a suitable $P(z)$-local-grading-restriction condition.*

(Remark: $W_1 \boxtimes_{P(z)} W_2$ is dependent on the module category that is considered.)

We note that in the untwisted case, a $P(z)$-compatibility condition and a $P(z)$-grading-restriction condition (see [HL4] and [HLZ3]) play an important role in the proof of associativity (operator product expansion) of intertwining operators and in the construction of the associativity isomorphisms for the vertex tensor category structure (see [H1] and [HLZ5]).

Generalizing those ideas of proving the associativity in the untwisted situation to our twisted case has many obstructions. This is mainly because the untwisted intertwining operator is defined using the Jacobi identity, which is algebraic. The $P(z)$-compatibility condition in [HLZ3] is a purely algebraic statement, which is invalid under our notion of twisted intertwining operator and the complex analytic setting. To solve this problem, we have introduced a new formulation of the $P(z)$-compatibility condition, which is a complex-analytic statement. It looks very different from the algebraic version of $P(z)$-compatibility condition in [HLZ3]. Whether they are equivalent when the twisted modules considered are actually untwisted is still unclear, which is an interesting unsolved problem.

The complex analytic version of the $P(z)$-compatibility condition serves the same function as the algebraic one, in the sense that we still can prove Theorem 2.1 under our complex analytic setting (See [DH]). Again, since our notion of twisted intertwining operator and of $P(z)$-compatibility condition are very different, the method of proving Theorem 2.1 is entirely new.

In [D], I have introduced a $P(z)$-$\mathcal{C}$-embeddability condition, where $\mathcal{C}$ is the category of twisted V-modules. Then we have:

Theorem 2.2. *Denote by $\mathcal{C}$ the module category that is considered. Suppose $\lambda \in (W_1 \otimes W_2)^*$. Then $\lambda \in (W_1 \boxtimes_{P(z)} W_2)'$ if and only if λ satisfies both the $P(z)$-compatibility condition and the $P(z)$-$\mathcal{C}$-embeddability condition.*

Theorem 2.1 and 2.2 are crucial for proving the associativity of twisted intertwining operators. This is because they offer a feasible way to determine whether a functional $\lambda \in (W_1 \otimes W_2)^*$ is contained in the space $(W_1 \boxtimes_{P(z)} W_2)'$.

- **2.4 Proof of associativity of twisted intertwining operators (in [D])**

Using $P(z)$-compatibility, Theorems 2.1 and 2.2, and all other tools that had been developed, I have proved the associativity of twisted intertwining operators. The statement is roughly the following:

Theorem 2.3. *Fix $z_1, z_2 \in \mathbb{C}$ satisfying*

$$0 < |z_1 - z_2| < |z_2| < |z_1|, \tag{2.2}$$

$$|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}, \;\; |\arg(z_1 - z_2) - \arg(z_1)| < \frac{\pi}{2}. \tag{2.3}$$

Suppose that $G \leq \mathrm{Aut}(V)$, and $\mathcal{C}$ is a category of g-twisted generalized V-modules for $g \in G$. If $\mathcal{C}$ satisfies certain conditions, then for any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-, $g_1g_2g_3$-, g_2g_3-twisted modules W_1, W_2, W_3, W_4, M_1 in $\mathcal{C}$, and twisted intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, there exist a g_1g_2-twisted module M_2 in $\mathcal{C}$, and twisted intertwining operators $\mathcal{Y}_3$, $\mathcal{Y}_4$ of types $\binom{W_4}{M_2 W_3}$, $\binom{M_2}{W_1 W_2}$, such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle\,, \tag{2.4}$$

holds for any $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3, w_4' \in W_4'$.

Note that for the associativity of untwisted intertwining operators, the restriction (2.3) is not needed. However, due to the multivalued nature of orbifold theory, (2.3) is needed. When (2.3) does not hold, one can still find twisted intertwining operators $\mathcal{Y}_3$, $\mathcal{Y}_4$ such that (2.4) holds. But their types could be $\binom{\phi_{h_4}(W_4)}{M_2\ \phi_{h_3}(W_3)}$, $\binom{M_2}{\phi_{h_1}(W_1)\ \phi_{h_2}(W_2)}$, for some $h_i \in G$, $i = 1, 2, 3, 4$.

Theorem 2.3 directly leads to the natural associativity isomorphisms in the vertex tensor category:

Corollary 2.4. *Fix $z_1, z_2 \in \mathbb{C}$ satisfying*

$$0 < |z_1 - z_2| < |z_2| < |z_1|,$$
$$|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2},\ |\arg(z_1 - z_2) - \arg(z_1)| < \frac{\pi}{2}.$$

Suppose $\mathcal{C}$ is a category satisfying the conditions referred to in Theorem 2.3. For any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-twisted modules W_1, W_2, W_3 in $\mathcal{C}$, we have the isomorphism

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \longrightarrow (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3, \tag{2.5}$$

functorial in all three position.

Together with the parallel transport isomorphism introduced in [HLZ7], we have the associativity isomorphisms in the G-crossed braided tensor category:

Corollary 2.5. *Suppose $\mathcal{C}$ is a category satisfying the conditions referred to in Theorem 2.3. For any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-twisted modules W_1, W_2, W_3 in $\mathcal{C}$, we have a natural isomorphism*

$$W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3) \longrightarrow (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3, \tag{2.6}$$

functorial in all three positions.

3 Future Research Plan

- **3.1 Finish the construction of G-crossed vertex/braided tensor categories**

  To be a G-crossed vertex/braided tensor category, not only the are ingredients - associativity isomorphisms, G-action and grading, G-crossed braiding isomorphism, etc, - needed, but certain compatibility axioms including the pentagon/hexagon/triangle axioms also need to be satisfied. After Corollary 2.4 and 2.5 have been proved, we have all the ingredients. The next step is to prove these compatibility axioms.

  We plan to complete this work in the near future as a joint project with my advisor Yi-Zhi Huang, and Daniel Tan.

- **3.2 A generalized Jacobi identity for twisted intertwining operators**

  As mentioned in Section 2.1, contrast to the untwisted case, when studying correlation functions induced by $\langle w_3', Y(v, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$, because of the multivaluedness of $Y(v, z_1)$ one cannot get a **single-valued** meromorphic 1-form on $\hat{\mathbb{C}}$. This is the geometric obstruction to obtaining a Jacobi identity using the Cauchy theorem. However, one can have a branched covering space E of $\hat{\mathbb{C}}$, such that the multivalued function (1-form) on $\hat{\mathbb{C}}$ can be lifted to a single valued 1-form on E:

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "\text{single valued lifting}"] & \\ \hat{\mathbb{C}} \arrow[r, "\text{multivalued}"] & \mathbb{C} \end{tikzcd}$$

As a generalization of the Cauchy theorem, for a compact Riemann surface, one has the following theorem:

Theorem 3.1 (Global Residue Theorem)**.** *Let M be a compact Riemann surface, and let $S \subset M$ be a finite set of points in M. Let ω be a holomorphic 1-form on $M \setminus S$. Then we have*

$$\sum_{p \in S} \mathrm{Res}_p(\omega) = 0. \tag{3.7}$$

Suppose $\mathcal{Y}$ is of type $\binom{W_3}{W_1 W_2}$, where W_1, W_2, W_3, are g_1-, g_2-, $g_1 g_2$-twisted modules. If $\langle g_1, g_2 \rangle \leq \mathrm{Aut}(V)$ is a finite group, to study $\langle w_3', Y(v, z_1)\mathcal{Y}(w_1, z_2)w_2 \rangle$, the branched covering space E can be taken to be a compact Riemann surface. This means that we can get a single-valued correlation function on a compact Riemann surface so that we can use Theorem 3.1. In this way, one can get a "generalized" Jacobi identity for twisted intertwining operators for a finite automorphism group $\langle g_1, g_2 \rangle$.

However, although this idea looks clear and feasible, some difficulties seem to occur when $\langle g_1, g_2 \rangle$ is nonabelian. Details still need to be written down to examine the feasibility.

The importance of this work is that many results in untwisted VOA representation theory are proved using the algebraic approach. The proof of these results under our complex analytic setting is yet to be found. But once this work is done, it will be helpful for proving more results in the complex analytic setting.

For the one among these results most related to my program, see Section 3.3:

- **3.3 Proof of the convergence assumption**

The products and iterates of untwisted intertwining operators among C_1-cofinite modules are absolutely convergent and have the form of solutions of PDEs which have regular singularities at certain points. This result was proved by Huang in [H3] using the algebraic approach and regular singular differential equation theory. This is essentially the conditions referred to in Theorem 2.3.

The original proof by Huang heavily relies on the Jacobi identity, which means it is impossible to directly generalize to our twisted case. Once the work mentioned in Section 3.2 is done, the convergence assumption will hopefully be proved.

Remark 3.2. *If it goes well, this work will be done by my colleague and friend, Daniel Tan. If it turns out to be much more difficult than expected, I will be interested in doing this.*

For a finite abelian group of automorphisms, Tan has already had this result due to the fact that a Jacobi identity can be derived in this case. The more difficult and also more interesting case is the finite (not necessarily abelian) group case.

- **3.4 Explore explicit examples of orbifold theory**

  Applying this whole theory to explicit examples will be very meaningful. In [H11], Huang came up with a construction theorem for twisted modules, used this theorem to construct a Verma-type twisted module $\widehat{M}_B^{[g]}$ (By "Verma-type", we mean that it satisfies a universal property similar to a classical Verma module). Moreover, under certain suitable conditions, this module has a unique maximal proper submodule and an irreducible quotient (See [H12]). These theorems make it possible to study twisted intertwining operators among these twisted modules.

  The first explicit example worth trying would be the affine VOA $V_{\hat{\mathfrak{g}}}(l, 0)$. In [H13], Huang explicit wrote down these Verma-type twisted modules for affine vertex (operator) algebras. Moreover, Huang proved that these modules are equivalent to suitable induced modules for the corresponding twisted affine Lie algebra, or quotients of such induced modules, by explicitly given submodules. Therefore, we have many useful tools to start explicitly studying the orbifold theory of affine VOAs, which will hopefully inspire the study of the general theory.

  Also, we need to explore non-abelian orbifold theories, since our theory works for modules twisted by a non-abelian group. Gemünden and Keller studied some orbifolds of holomorphic lattice vertex operator algebras for non-abelian finite automorphism groups G in [GK]. Their work on these examples offers a place where we can apply our theory.

- **3.5 Uniqueness conjecture of the moonshine module $V^\natural$**

  Once Conjecture 1.2 is proved, it offers a strategy to study Frenkel, Lepowsky, and Meurman's famous uniqueness conjecture for the moonshine module VOA $V^\natural$, which has a history of over 40 years, and also is the last piece of the classification program of holomorphic VOAs with central charge 24. This conjecture is the following:

Conjecture 3.3 (Uniqueness conjecture for $V^\natural$)**.** *Let V be a VOA satisfying the following three conditions:*

1. *V is the only irreducible module for itself.*
2. *$V_{(1)} = 0$.*
3. *The central charge of V is* 24.

Then $V \cong V^\natural$ (as VOAs).

As a weaker version of this conjecture, we have:

Conjecture 3.4. *Let V be a VOA satisfying the following conditions:*

1. *V satisfies conditions 1,2,3 in Conjecture 3.3.*
2. *V satisfies conditions 1,2,3 in Theorem 1.1 (and also Conjecture* (1.2)*).*

Then $V \cong V^\natural$ (as VOAs).

The analogy between the notion of even lattices and the notion of VOAs arose in the earliest literature [FLM1], [FLM4] in VOA theory, which was generalized to the analogy between (positive definite) lattices and completely-extendable conformal intertwining algebras (intertwining operator algebra) by Huang in [H15]. Under the philosophy of this analogy, Lepowsky suggested, at a workshop at American Institution of Mathematics in 2004, that Conway's proof of the uniqueness theorem for the Leech lattice Λ could be natural place to find inspiration. Conway's proof used the natural embedding $\Lambda \hookrightarrow \mathbb{R}^{24}$ of the lattice to Euclidean space. The technical feature of Conway's proof suggested potential analogous ideas for proving the uniqueness conjecture for $V^\natural$. For these ideas, one needs an analogue in VOA theory of an ambient space created from suitable internal structure of the VOA. **Therefore, to vaguely follow the same strategy as Conway's, the first step would be to find an ambient structure for V which plays a similar role as the ambient space $\mathbb{R}^{24}$ in Conway's proof.** Then, one should use VOA insight to carry out analogous steps in Conway's proof.

However, what makes the uniqueness conjecture for $V^\natural$ difficult to handle is its nature of **having no (easy) ambient structure**.

First, the philosophy "adding modules to enlarge the algebra" is invalid now - every module is just direct sum of copies of V itself. If we regard the vertex tensor category V-Mod (or one can say the corresponding intertwining operator algebra) as an ambient structure of V, this ambient structure is as small as V itself.

Second, suppose that V is a VOA such that $V_{(n)} = 0$ if $n < 0$, and $V_{(0)} = \mathbb{C}\mathbf{1}$ (condition 1. in Conjecture 1.2). If $V_{(1)} \neq 0$, then $V_{(1)}$ with $[a, b] := a_0 b$ forms a Lie algebra $\mathfrak{g}$, and the equation $a_1 b = \langle a, b\rangle_V \mathbf{1}$ defines a invariant bilinear form, from which the affine Lie algebra $\hat{\mathfrak{g}}$ is induced. Moreover, there is a homomorphism

$$V_{\hat{\mathfrak{g}}}(\ell, 0) \to V,$$

where $V_{\hat{\mathfrak{g}}}(\ell, 0)$ is the affine VOA induced by $\mathfrak{g}$ with level ℓ, and ℓ is some particular number determined by V. Therefore, we know that V is a $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module, which means the well-studied representation category, $V_{\hat{\mathfrak{g}}}(\ell, 0)$-Mod (or the corresponding intertwining operator algebra), can be regarded as an ambient structure of V. We can use affine VOA (and its representation theory) to study V. This has been a powerful tool in prove the uniqueness theorem of the other 70 VOAs in the classification conjecture for holomorphic VOAs with central charge 24. (The classification conjecture said that there are 71 such VOAs. All of the other 70, except for $V^\natural$, have been proved to satisfy certain uniqueness theorems.) For other 70 VOAs, their weight 1 space is nonzero, i.e. $V_{(1)} \neq 0$. However, in Conjecture 3.3 and 3.4, we have $V_{(1)} = 0$, which makes this strategy invalid. This was the original reason for the depth of FLM's uniqueness conjecture for $V^\natural$.

To overcome this difficulty, our strategy is the following: Roughly speaking, we want to use the category of twisted modules for V as an ambient structure. Although the category V-Mod is trivial now, the category of g-twisted modules for some $g \in G \leq \mathrm{Aut}(V)$ should be a nontrivial G-crossed vertex tensor category. Since the moonshine module $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$ is constructed using the Leech lattice VOA V_Λ and its twisted module V_Λ^T, if we consider the category of g-twisted $V^\natural$-modules for $g \in \mathbb{M} = \mathrm{Aut}(V^\natural)$, it should contain V_Λ, by a particular procedure of orbifolding. Therefore, if we start with an abstract VOA V satisfying the conditions in Conjecture 3.4, the first step would be to try to recover V_Λ in the $\mathbb{M}$-crossed vertex tensor category guaranteed by Conjecture 1.2. This is hopeful to be done by using the uniqueness theorem for the Leech lattice VOA (Any VOA satisfying conditions 1,2,3 in Theorem 1.1, conditions 1,3 in Conjecture 3.4, and also the condition that V_1 forms an abelian Lie algebra of rank 24, is isomorphic to V_Λ). In this way, it would be possible to get, from the abstract VOA V, an explicit VOA V_Λ using twisted V-modules. After realizing V_Λ using the twisted V-module category, V is possible to be realized using twisted V_Λ-modules by reversing the orbifolding procedure mentioned above. Then it is hopeful to use the well-understood orbifold theory of V_Λ to study the uniqueness conjecture 3.4.

However, one of the hard problems for this strategy is to prove the existence of **even one** nontrivial automorphism, because automorphisms are a piece of data we need for building an orbifold theory. (Similar as above, if $V_{(1)} \neq 0$ as dicussed above, it is easy to construct some automorphisms. However, this is not the case for Conjecture 3.3/3.4.)

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