Research Statement

Jishen Du

November 23, 2025

1 Background

I am interested in **vertex operator algebras** (VOA) and their representation theory, as the underlying algebraic/categorical structure for **two-dimensional conformal field theory** (CFT; we will omit 'two-dimensional' hereafter). In particular, I am interested in the construction of tensor-categorical structure in module categories of vertex operator algebras. Here, the 'modules' I consider could be 'twisted modules' in a certain sense. The theory of twisted modules is called 'orbifold' theory.

Very roughly speaking, a VOA is a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}} V_{(n)}$, together with a particular element $\mathbf{1} \in V$, and a binary operation $Y(-, x)-$, called the vertex operator map, whose outputs are formal Laurent series with coefficients in V, i.e., $Y(u, x)v \in V((x))$ for $u, v \in V$. It satisfies certain axioms, including the most important one, the **Jacobi identity**, which is an algebraic assertion involving **formal delta function** $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. Although the notion of VOA is highly sophisticated, it shares certain features with both Lie algebras and associative algebras. Moreover, its representation-theoretic behavior is extremely interesting.

1.1 CFTs and VOAs

Quantum field theory (QFT) has become a unifying framework in modern mathematics, providing deep conceptual bridges among geometry, topology, analysis, representation theory, and tensor category theory, which has been more and more significant in modern physics and mathematics. **Topological field theory** (TFT) and CFT are two types of QFTs that are mathematically well defined and studied. It has been a long-standing open problem, since the late 1980s, to construct chiral and full CFTs in the sense of Segal.

In TFT, the notions of vector space, Frobenius algebra, and modular tensor category provide the underlying algebraic/categorical structure for one, two, and three-dimensional TFT. Analogously, the algebra of **intertwining operators** among modules for a VOA provides the underlying algebraic structure for chiral CFT. Therefore, the study of CFTs can be largely converted to the study of VOAs and their representation theory.

Based on the results in [HL2], [HL3], [HL4], [H1], [H3], [H4], and [H5], Huang proved the following theorem in [H6]:

Theorem 1.1. *Let V be a simple VOA satisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$; $V_{(0)} = \mathbb{C}\mathbf{1}$; and as a V-module, V is equivalent to its contragredient V-module V'.*
2. *Every lower-bounded generalized V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then the category of V-modules has a natural structure of modular tensor category in the sense of Turaev [Tu1].

[0]For a longer version of my research statement, check https://jishendumath.github.io/Research%20description.pdf

1.2 Orbifold conformal field theory

Orbifold CFTs are CFTs constructed from known theories and their automorphisms. The first example of orbifold CFT is the the moonshine module VOA $V^\natural$ constructed by Frenkel, Lepowsky and Meurman [FLM1]-[FLM4] in mathematics. The automorphism group of $V^\natural$ is the Monster finite simple group $\mathbb{M}$. Their construction of $V^\natural$ proved the McKay-Tompson conjecture, profoundly relating number theory and finite group theory. Later, this VOA $V^\natural$ played a major role in Borcherds' proof of the rest of the Conway–Norton Conjectures. FLM's construction introduced a new string theory, which was later interpreted by physicists as an "orbifold" theory. In string theory, the more general systematic study of orbifold CFTs was started by Dixon, Harvey, Vafa and Witten [DHVW1] [DHVW2]. See [H14] for an exposition of general results, conjectures and open problems in the construction of orbifold CFTs using the approach of the representation theory of vertex operator algebras.

It is natural to expect that Theorem 1.1 has generalizations in orbifold CFT. In [K3], Kirillov Jr. stated that the category of g-twisted modules for a VOA V for all g in a finite subgroup G of the automorphism group of V is a G-equivariant fusion category. For general V, this is certainly not true. The VOA V must satisfy certain conditions. Here is a precise conjecture formulated by Huang in [H9]:

Conjecture 1.2. *Let V be a VOA satisfying conditions 1,2,3 in Theorem 1.1, and let G be a finite subgroup of* $\mathrm{Aut}(V)$*. Then the category of g-twisted V-modules for all $g \in G$ is a G-crossed braided tensor category.*

We also conjecture that the category of g-twisted V-modules for all $g \in G$ is a G-crossed modular tensor category in a suitable sense.

In the case that G is trivial, Conjecture 1.2 is true by Theorem 1.1. Thus the G-crossed modular tensor category conjecture is a natural generalization of Theorem 1.1 to the category of g-twisted V-modules for $g \in G$.

In the case that the fixed point subalgebra V^G of V under G satisfies the conditions in Theorem 1.1 above, the category of V^G-modules is a modular tensor category. In this case, Conjecture 1.2 can be proved using the modular tensor category structure on the category of V^G-modules and the results on tensor categories by Kirillov Jr. [K1] [K2] [K3] and Müger [Mü1] [Mü2]. For a general finite group G, the conjecture that the fixed point subalgebra V^G of V under G also satisfies the conditions in Theorem 1.1 is still open and seems to be a difficult problem. On the other hand, using twisted modules and twisted intertwining operators to construct G-crossed braided tensor categories seems to be a more conceptual and direct approach. If this approach works, we expect that the category of V^G-modules can also be studied using the G-crossed braided tensor category structure on the category of twisted V-modules.

2 What I have done

In brief, I have proved the associativity of twisted intertwining operators, under certain convergence and extension assumptions. This is equivalent to a construction of **the associativity isomorphism in the G-crossed vertex/braided tensor category**, which is a main step in proving Conjecture 1.2. (Another main step would be proving the assumptions I need; see [Ta].)

To achieve this, I have done the following:

• 2.1 Systematic development of a complex analytic approach to VOA theory (in [DH] and [D])

The classical study of VOAs and their representation theory uses an algebraic approach based on formal calculus, where formal series are studied, and the formal delta function $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$ plays a significant role. This algebraic approach has been fully developed in the last 40 years and been used to successfully solve many problems. The *Jacobi identity* ([FLM4]) in the definition of VOA is powerful enough to derive many important results.

In the classical (i.e., untwisted) theory, the algebraic formulation alone is not enough to study products and iterates of *more than one intertwining operator*. We can no longer have a Jacobi identity for such products and iterates, essentially because the correlation functions are multivalued *in all varibles*, which means that one can no longer obtain a single-valued meromorphic 1-form on $\hat{\mathbb{C}}$. Therefore, the coefficients in expansions at different singularities cannot have a relation (i.e., the Jacobi identity) by simply using the Cauchy formula. This is where Huang [H1] and Huang-Lepowsky-Zhang [HLZ4] had to introduce certain complex analytic assumptions to go further.

Despite the involvedness of these complex analytic assumptions, their work ([HL2]-[HL4], [H1], [H3]-[H6], and [HLZ1]-[HLZ7]) mainly used the algebraic approach. This is natural - one should always use the algebraic approach whenever there is a Jacobi identity, because although often lengthy and extremely technical, formal delta-function calculus offers an effective way to do computation and prove theorems, and therefore has been foundational to the VOA theory.

However, in the study of orbifold CFT, a systematic complex-analytic approach inevitably needs to be developed. This is because even for the vertex operator acting on modules, there are non-integer powers of x (and of $\log x$, for g-twisted modules for an infinite-order automorphism g). This causes an extra multivaluedness, which makes it impossible to write down a Jacobi identity as the definition of *twisted intertwining operators* (see Section 2.2). Geometrically, it is because we cannot have a single-valued meromorphic function even for products like $Y_W(v, z_1)\mathcal{Y}(w_1, z_2)w_2$. Instead, we have to use a certain notion of **duality** as the definition of twisted intertwining operator, which is a complex analytic statement. (In CFT and string theory, "duality" roughly means that inserting vertex operators in different ways gives different local expansions of the same correlation function on certain different regions.)

The "definition" of the "complex analytic approach to VOA (orbifold theory)" could be:

> *Start from the duality version of the definitions of (twisted) module and (twisted) intertwining operator, and develop the (twisted) representation theory of VOAs, without using the formal delta function, Jacobi identity, and Cauchy formula. In this way, one can push the boundary of the theory beyond what formal calculus can handle.*

Under this definition, because of its inevitability in the study of orbifold theory, we have systematically developed the complex analytic approach.

• 2.2 Introduction of the most general notion of twisted intertwining operator (in [DH] and [D])

Intertwining operators among twisted modules (i.e. what we called *twisted intertwining operators*) associated to commuting automorphisms of finite order appeared implicitly in [FFR] and were introduced explicitly in [X] in terms of a generalization of the Jacobi identity for twisted modules. This Jacobi identity works because in [X], only the case that the automorphism group is *finite abelian* is considered. However, what we want is a theory for nonabelian groups. In [H8], Huang introduced a definition of twisted intertwining operators among modules twisted by noncommuting automorphisms. However, it turns out that this definition is not general enough to study orbifold theory associated to a nonabelian group of automorphisms.

We have introduced the most general notion of twisted intertwining operator. This definition of twisted intertwining operator is general enough for studying the orbifold theory associated to a nonabelian group of automorphisms. In order to give the correct notion of $P(z)$-tensor product of twisted modules, we need to use the most general twisted intertwining operators. If we use only a certain special set of twisted intertwining operators as in [H8] to define and construct the $P(z)$-tensor products, we would obtain a quotient of the correct $P(z)$-tensor product structure.

Moreover, based on our definition of twisted intertwining operator, we have proved some properties of twisted intertwining operators that are essential for the construction of G-braided vertex tensor categories.

• 2.3 Construction of and an equivalent condition for a $P(z)$-tensor product (in [DH] and [D])

For any $z \in \mathbb{C}^\times$, we have given a definition of a $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$ of two twisted modules W_1 and W_2 using a natural universal property. We have also given an explicit construction of $W_1 \boxtimes_{P(z)} W_2$ by using the new notion of twisted intertwining operator.

Based on the explicit construction of $W_1 \boxtimes_{P(z)} W_2$ mentioned above, in [DH], we have found an equivalent condition for a linear functional $\lambda \in (W_1 \otimes W_2)^*$ to be contained in $(W_1 \boxtimes_{P(z)} W_2)'$. (For any g-twisted module W, we can define its contragredient module W' which is a g^{-1}-twisted module.) We denote by $\mathcal{GM}_{gr}(G)$ the category of grading-restricted generalized g-twisted modules for $g \in G$, where $G \leq \mathrm{Aut}(V)$.

Theorem 2.1. *Let the module category considered be $\mathcal{GM}_{gr}(G)$. Suppose $\lambda \in (W_1 \otimes W_2)^*$. Then $\lambda \in (W_1 \boxtimes_{P(z)} W_2)'$ if and only if λ satisfies a suitable $P(z)$-compatibility condition and a suitable $P(z)$-local-grading-restriction condition.*

(Remark: $W_1 \boxtimes_{P(z)} W_2$ depends on the module category that is considered.)

We note that in the untwisted case, a $P(z)$-compatibility condition and a $P(z)$-grading-restriction condition (see [HL4] and [HLZ3]) play important roles in the proof of associativity of intertwining operators (see [H1] and [HLZ5]).

Generalizing those ideas of proving the associativity in the untwisted situation to our twisted case has the following main obstruction. The $P(z)$-compatibility condition in [HLZ3] is purely an algebraic statement, which fits the definition of intertwining operator using the Jacobi identity, but is invalid under our notion of twisted intertwining operator and the complex analytic setting. To solve this problem, we have introduced a new formulation of the $P(z)$-compatibility condition, which is a complex-analytic statement. It looks very different from the algebraic version of the $P(z)$-compatibility condition in [HLZ3]. (Whether they are equivalent when the twisted modules considered are actually untwisted is still unclear, which is an interesting unsolved problem.) The complex-analytic version of the $P(z)$-compatibility condition serves the same function as the algebraic one, in the sense that we still can prove Theorem 2.1 in our complex analytic setting (See [DH]). Again, since our notions of twisted intertwining operator and of $P(z)$-compatibility condition are very different, the method of proving Theorem 2.1 is entirely new.

In [D], I have introduced a $P(z)$-$\mathcal{C}$-embeddability condition, where $\mathcal{C}$ is the category of twisted V-modules. We have:

Theorem 2.2. *Denote by $\mathcal{C}$ the module category that is considered. Suppose $\lambda \in (W_1 \otimes W_2)^*$. Then $\lambda \in (W_1 \boxtimes_{P(z)} W_2)'$ if and only if λ satisfies both the $P(z)$-compatibility condition and the $P(z)$-$\mathcal{C}$-embeddability condition.*

Theorems 2.1 and 2.2 are crucial for proving the associativity of twisted intertwining operators, because they offer a feasible way to determine whether a functional $\lambda \in (W_1 \otimes W_2)^*$ is contained in the space $(W_1 \boxtimes_{P(z)} W_2)'$.

• 2.4 Proof of associativity of twisted intertwining operators (in [D])

Using $P(z)$-compatibility, Theorems 2.1 and 2.2, and all other tools that had been developed, I have proved the associativity of twisted intertwining operators. The statement is roughly the following:

Theorem 2.3. *Fix $z_1, z_2 \in \mathbb{C}$ satisfying*

$$0 < |z_1 - z_2| < |z_2| < |z_1|, \tag{2.1}$$

$$|\arg(z_1) - \arg(z_2)| < \frac{\pi}{2}, \ |\arg(z_1 - z_2) - \arg(z_1)| < \frac{\pi}{2}. \tag{2.2}$$

Suppose that $G \leq \mathrm{Aut}(V)$, and $\mathcal{C}$ is a category of g-twisted generalized V-modules for all $g \in G$. If $\mathcal{C}$ satisfies certain conditions, then for any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-, $g_1g_2g_3$-, g_2g_3-twisted modules W_1, W_2, W_3, W_4, M_1 in $\mathcal{C}$, and twisted intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, there exist a g_1g_2-twisted module M_2 in $\mathcal{C}$, and twisted intertwining operators $\mathcal{Y}_3$, $\mathcal{Y}_4$ of types $\binom{W_4}{M_2 W_3}$, $\binom{M_2}{W_1 W_2}$, such that

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle, \tag{2.3}$$

holds for any $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3, w_4' \in W_4'$.

For the associtivity of untwisted intertwining operators, the restriction (2.2) is not needed. However, due to the multivalued nature of orbifold theory, (2.2) is needed, which is a new phenomenon. When (2.2) does not hold, one can still find twisted intertwining operators $\mathcal{Y}_3$, $\mathcal{Y}_4$ such that (2.3) holds. But their types could be $\binom{\phi_{h_4}(W_4)}{M_2 \ \phi_{h_3}(W_3)}$, $\binom{M_2}{\phi_{h_1}(W_1) \ \phi_{h_2}(W_2)}$, for some $h_i \in G$, $i = 1, 2, 3, 4$, where ϕ_g is the action of g on $\mathcal{C}$ required in the definition of G-crossed braided tensor category.

As a direct corollary, we have the desired natural associativity isomorphism in the vertex tensor category.

Corollary 2.4. *Fix $z_1, z_2 \in \mathbb{C}$ satisfying* (2.1) *and* (2.2). *Suppose $\mathcal{C}$ is a category satisfying the conditions referred to in Theorem 2.3. For any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-twisted modules W_1, W_2, W_3 in $\mathcal{C}$, we have a natural isomorphism*

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \longrightarrow (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3, \tag{2.4}$$

functorial in all three positions.

Together with the parallel transport isomorphisms introduced in [HLZ7], we have the associativity isomorphisms in the G-crossed braided tensor category:

Corollary 2.5. *Suppose $\mathcal{C}$ is a category satisfying the conditions referred to in Theorem 2.3. For any $g_1, g_2, g_3 \in G$, and g_1-, g_2-, g_3-twisted modules W_1, W_2, W_3 in $\mathcal{C}$, we have a natural isomorphism*

$$W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3) \longrightarrow (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3, \tag{2.5}$$

functorial in all three positions.

3 Future Research Plans

• 3.1 Finish the construction of G-crossed vertex/braided tensor categories

To be a G-crossed vertex/braided tensor category, not only are the ingredients - associativity isomorphisms, G-action and grading, G-crossed braiding isomorphisms, etc, - needed, certain compatibility axioms including the pentagon/hexagon/triangle axioms also need to be satisfied. Now that Corollaries 2.4 and 2.5 have been proved, we have all the ingredients. The next step is to prove these compatibility axioms.

We plan to complete this work in the near future as a joint project with my advisor Yi-Zhi Huang, and Daniel Tan.

• 3.2 A generalized Jacobi identity for twisted intertwining operators

As mentioned in Section 2.1, in contrast with the untwisted case, when studying correlation functions induced by $\langle w_3', Y(v, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$, because of the multivaluedness of $Y(v, z_1)$ one cannot obtain a **single-valued** meromorphic 1-form on $\hat{\mathbb{C}}$. This is the geometric obstruction to obtaining a Jacobi identity using the Cauchy theorem. However, one can have a branched covering space E of $\hat{\mathbb{C}}$, such that the multivalued function (1-form) on $\hat{\mathbb{C}}$ can be lifted to a single valued 1-form on E:

$$\begin{tikzcd} E \arrow[d] \arrow[dr, "\text{single valued lifting}"] & \\ \hat{\mathbb{C}} \arrow[r, "\text{multivalued}"] & \mathbb{C} \end{tikzcd}$$

As a generalization of the Cauchy theorem, for a compact Riemann surface one has the following theorem:

Theorem 3.1 (Global Residue Theorem)**.** *Let M be a compact Riemann surface, let $S \subset M$ be a finite set of points, and let ω be a holomorphic 1-form on $M \setminus S$. Then*

$$\sum_{p\in S} \mathrm{Res}_p(\omega) = 0. \tag{3.6}$$

Suppose $\mathcal{Y}$ is of type $\binom{W_3}{W_1 W_2}$, where W_1, W_2, W_3, are g_1-, g_2-, g_1g_2-twisted modules. If $\langle g_1, g_2\rangle \leq \mathrm{Aut}(V)$ is a finite group, to study $\langle w_3', Y(v, z_1)\mathcal{Y}(w_1, z_2)w_2\rangle$, the branched covering space E can be taken to be a compact Riemann surface. This means that we can get a single-valued correlation function on a compact Riemann surface so that we can use Theorem 3.1. In this way, one can get a "generalized" Jacobi identity for twisted intertwining operators for finite automorphism group $\langle g_1, g_2\rangle$. However, although this idea looks clear and feasible, some difficulties seem to occur when $\langle g_1, g_2\rangle$ is nonabelian. Details still need to be written down to examine the feasibility.

The importance of this work is that many results in untwisted VOA representation theory are proved using algebraic approach. These results (on untwisted modules) are expected to be generalized to the twisted module case. However, due to their algebraic nature, the proofs of these results cannot be directly generalized to the twisted case. But once this work is done, it will be helpful for proving more results in the complex-analytic setting. For the one among these results most related to my program, see Section 3.3:

• 3.3 Proof of the convergence assumption

The products and iterates of untwisted intertwining operators among C_1-cofinite modules are absolutely convergent and have the form of solutions of PDEs which have regular singularities at certain points. This result was proved by Huang in [H3] using the algebraic approach and regular singular differential equation theory. This is essentially the conditions referred to in Theorem 2.3.

The original proof by Huang heavily relies on the Jacobi identity, which means it is impossible to directly generalize to our twisted case. Once the work in Section 3.2 is done, the convergence assumption will hopefully be proved.

Remark 3.2. *If it goes well, this work will be done by my colleague and friend, Daniel Tan. If it turns out to be much more difficult than expected, I will be interested in doing this. For finite abelian group of automorphisms, Tan has already had this result due to the fact that a Jacobi identity can be derived in this case. The more difficult and also more interesting case is finite (not necessarily abelian) group case.*

• **3.4 Explore explicit examples of orbiford theory**

Applying this whole theory to explicit examples will be very interesting. One of the explicit examples worth trying would be affine VOAs $V_{\hat{\mathfrak{g}}}(l,0)$. In [H13], Huang explicitly constructed certain Verma-type twisted modules for affine VOAs, and proved that these modules are equivalent to suitable induced modules for the corresponding twisted affine Lie algebra; or quotients of such induced modules by explicitly given submodules. Therefore, we have many useful tools to start explicitly studying the orbifold theory of affine VOA.

Also, we need to explore non-abelian orbifold theories, since our theory works for modules twisted by a non-abelian group. Gemünden and Keller studied some orbifolds of holomorphic lattice vertex operator algebras for non-abelian finite automorphism groups G in [GK]. Their work on these examples possibly offers a place where we can apply our theory.

• **3.5 Uniqueness conjecture for the moonshine module $V^\natural$**

Once Conjecture 1.2 is proved, it offers a strategy to study Frenkel, Lepowsky, and Meurman's famous uniqueness conjecture for the moonshine module VOA $V^\natural$, which has a history of over 40 years, and also is the last piece of the classification program for holomorphic VOAs with central charge 24. Their conjecture is the following:

Conjecture 3.3 (Uniqueness conjecture for $V^\natural$)**.** *Let V be a VOA satisfying the following three conditions:*

1. *V is the only irreducible module for itself.*
2. *$V_{(1)} = 0$.*
3. *The central charge of V is* 24.

Then $V \cong V^\natural$ (as VOAs).

As a weaker version of this conjecture, we have:

Conjecture 3.4. *Let V be a VOA that satisfies conditions 1,2,3 in Conjecture 3.3, and conditions 1,2,3 in Theorem 1.1. Then $V \cong V^\natural$ (as VOAs).*

The analogy between the notion of positive definite even lattice and the notion of VOA arose in the earliest literature [FLM1], [FLM4] in VOA theory, which was generalized to the analogy between (positive definite) lattices and completely-extendable conformal intertwining algebras (intertwining operator algebra) by Huang in [H15]. Under the philosophy of this analogy, Lepowsky suggested, at a workshop at the American Institute of Mathematics in 2004, that Conway's proof of the uniqueness theorem for the Leech lattice Λ could be a natural place to find inspiration. The technical feature of Conway's proof suggested potential analogous ideas for proving the uniqueness conjecture for $V^\natural$. For these ideas, one needs an analogue in VOA theory of an ambient space created from suitable internal structure of the VOA. **Therefore, to vaguely follow the same strategy as Conway's, the first step would be to find an ambient structure for V which plays a similar role to the ambient space $\mathbb{R}^{24}$ in Conway's proof.** Then, one should use VOA insight to carry out analogous steps.

However, one thing that makes the uniqueness conjecture for $V^\natural$ difficult to handle is its nature of **having no (easy) ambient structure**.

First, the philosophy "adding modules to enlarge the algebra" is invalid now - every module is just direct sum of copies of V itself. If we regard the vertex tensor category V-Mod as an ambient structure of V, the ambient structure is as small as V itself. This issue has been well understood from the beginning.

Second, suppose that V is a VOA satisfying condition 1. in Theorem 1.1. If $V_{(1)} \neq 0$, then there is a natural Lie algebra $\mathfrak{g}$ structure on $V_{(1)}$ with a natural invariant bilinear form. Moreover, there is a VOA homomorphism $V_{\hat{\mathfrak{g}}}(\ell, 0) \to V$, where $V_{\hat{\mathfrak{g}}}(\ell, 0)$ is the affine VOA, and ℓ is some particular number determined by V. Therefore, we know that V is a $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module, which means $V_{\hat{\mathfrak{g}}}(\ell, 0)$-Mod can be regarded as an ambient structure of V. We can use the well-studied affine VOA and its representation theory to study V. This, too, has been well understood from the beginning and been a powerful tool in proving the uniqueness theorem for the other 70 VOAs in the classification conjecture for holomorphic VOAs with central charge 24. However, in Conjecture 3.3 and 3.4, we have $V_{(1)} = 0$, which makes this strategy invalid. This was the original reason for the depth of FLM's uniqueness conjecture for $V^\natural$.

To overcome this difficulty, our strategy is the following: Roughly speaking, we want to use the category of twisted modules for V as an ambient structure. Although the category V-Mod is trivial now, the category of g-twisted modules for $g \in G \leq \mathrm{Aut}(V)$ should be a nontrivial G-crossed vertex tensor category. Since the moonshine module $V^\natural = V_\Lambda^+ \oplus (V_\Lambda^T)^+$ was constructed using the Leech lattice VOA V_Λ and its twisted module V_Λ^T, if we consider the category of g-twisted $V^\natural$-modules for $g \in \mathbb{M} = \mathrm{Aut}(V^\natural)$, it should contain V_Λ, by a particular procedure of orbifolding. Therefore, if we start with an abstract VOA V satisfying the conditions in Conjecture 3.4, a key step is to try to recover V_Λ in the $\mathbb{M}$-crossed vertex tensor category expected from Conjecture 1.2. This can hopefully be done by using the uniqueness theorem for the Leech lattice VOA. In this way, it would be possible to get, from the abstract VOA V, an explicit VOA V_Λ using twisted V-modules. After realizing V_Λ using the twisted V-module category, V is possible to be realized using twisted V_Λ-modules by reversing the orbifolding procedure mentioned above. Then one can hopefully use the well understood orbifold theory of V_Λ to study the uniqueness conjecture 3.4.

However, one of the hard problems for this strategy probably is to prove the existence of **even one** nontrivial automorphism, because automorphisms are a piece of data we need for building an orbifold theory. (If $V_{(1)} \neq 0$, as discussed above, it is easy to construct some automorphisms of V. However, this is not the case for Conjecture 3.3/3.4.)

References

[B] R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83 (10)** (1986), 3068-3071.

[CM] S. Carnahan, M. Miyamoto, Regularity of fixed-point vertex operator subalgebras, to appear; arXiv:1603.05645.

[DHVW1] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds, *Nucl. Phys.* **B261** (1985), 678–686.

[DHVW2] L. Dixon, J. Harvey, C. Vafa and E. Witten, Strings on orbifolds, II, *Nucl. Phys.* **B274** (1986), 285–314.

[D] J. Du, Ph.D. thesis, Rutgers University, in preparation.

[DH] J. Du, Y.-Z. Huang, Twisted intertwining operators and tensor products of (generalized) twisted modules. arXiv:2501.15003

[EMS] J. van Ekeren, S. Möller, N. R. Scheithauer, Construction and classification of holomorphic vertex operator algebras. *J. Reine Angew. Math.* **759** (2020), 61–99.

[FFR] A. Feingold, I. Frenkel, J. Ries, Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$, *Contemp. Math.*, vol. 121, Amer. Math. Soc., Providence, 1991.

[FLM1] I. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function J as character, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3256–3260.

[FLM2] I. Frenkel, J. Lepowsky, A. Meurman. A Moonshine Module for the Monster. in: *Vertex Operators in Mathematics and Physics.* ed. by J. Lepowsky, S. Mandelstam, I.M. Singer, Mathematical Sciences Research Institute Publications, vol 3. Springer, New York, NY (1985).

[FLM3] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in: *Mathematical Aspects of String Theory, Proc. 1986 Conference, San Diego,* ed. by S.-T. Yau, World Scientific, Singapore, 1987, 150–188.

[FLM4] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.

[H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* 100 (1995) 173–216.

[H2] Y.-Z. Huang, Generalized rationality and a Jacobi identity for intertwining operator algebras, *Selecta Math.* **6** (2000), 225–267.

[H3] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H4] Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.* **7** (2005), 649–706.

[H5] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, *Comm. Contemp. Math.* **10** (2008), 103–154.

[H6] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.* **10** (2008), 871–911.

[H7] Y.-Z. Huang, Generalized twisted modules associated to general automorphisms of a vertex operator algebra, *Comm. Math. Phys.* **298** (2010), 265–292.

[H8] Y.-Z. Huang, Intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms, *J. Alg.* **493** (2018), 346–380.

[H9] Y.-Z. Huang, Some open problems in mathematical two-dimensional conformal field theory, in: *Proceedings of the Conference on Lie Algebras, Vertex Operator Algebras, and Related Topics, held at University of Notre Dame, Notre Dame, Indiana, August 14-18, 2015*, ed. K. Barron, E. Jurisich, H. Li, A. Milas, K. C. Misra, Contemp. Math, Vol. 695, American Mathematical Society, Providence, RI, 2017, 123–138.

[H10] Y.-Z. Huang, Twist vertex operators for twisted modules, *J. Alg.* **539** (2019), 53–83.

[H11] Y.-Z. Huang, A construction of lower-bounded generalized twisted modules for a grading-restricted vertex (super)algebra, *Comm. Math. Phys.* **377** (2020), 909-945.

[H12] Y.-Z. Huang, Generators, spanning sets and existence of twisted modules for a grading-restricted vertex (super)algebra, *Selecta Math.* **26** (2020), Paper no. 62.

[H13] Y.-Z. Huang, Lower-bounded and grading-restricted geralized twisted modules for affine vertex (operator) algebras, *J. Pure Appl. Alg.* **225** (2021), Paper no. 106618.

[H14] Y.-Z. Huang, Representation theory of vertex operator algebras and orbifold conformal field theory, in: *Lie groups, number theory, and vertex algebras*, ed. by D. Adamovic, A. Dujella, A. Milas and P. Pandzic, Contemp. Math., Vol. 768, Amer. Math. Soc., Providence, RI, 2021, 221–252.

[H15] Y.-Z. Huang, Conformal-field-theoretic analogues of codes and lattices, in: *Kac-Moody Lie Algebras and Related Topics, Proc. Ramanujan International Symposium on Kac-Moody Lie algebras and applications*, ed. N. Sthanumoorthy and K. C. Misra, Contemp. Math., Vol. 343, Amer. Math. Soc., Providence, 2004, 131–145.

[HL1] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant,* ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.

[HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica, New Series*, **1** (1995), 757–786.

[HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure. Appl. Alg.*, **100** (1995), 141–171.

[HLZ1] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators, arXiv:1012.4196.

[HLZ2] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, III: Intertwining maps and tensor product bifunctors, arXiv: 1012.4197.

[HLZ3] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, IV: Constructions of tensor product bifunctors and the compatibility conditions, arXiv:1012.4198.

[HLZ4] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, V: Convergence condition for intertwining maps and the corresponding compatibility condition, arXiv:1012.4199.

[HLZ5] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, arXiv:1012.4202.

[HLZ6] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, arXiv:1110.1929.

[HLZ7] Y.-Z. Huang, J, Lepowsky and L. Zhang, Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra, arXiv:1110.1931.

[GK] T. Gemünden, C. Keller, Non-Abelian Orbifolds of Lattice Vertex Operator Algebras, *J. Algebra* **585** (2021), 656–696.

[K1] A. Kirillov, Jr., Modular categories and orbifold models, *Comm. Math. Phys.* **229** (2002), 183–227.

[K2] A. Kirillov, Jr., On modular categories and orbifold models II; arXiv:math/0110221.

[K3] A. Kirillov, Jr, On G-equivariant modular categories; arXiv:math/0401119.

[Mc] R. McRae, Twisted modules and G-equivariantization in logarithmic conformal field theory, *Comm. in Math. Phys.* **383** (2021), 1939–2019.

[Mö] S. Möller. A cyclic orbifold theory for holomorphic vertex operator algebras and applications, Ph.D. thesis, Technische Universität Darmstadt, 2016; arXiv:1704.00478.

[MS] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[Mü1] M. Müger, Galois extensions of braided tensor categories and braided crossed G-categories, *J. Algebra* **277** (2004), 256–281.

[Mü2] M. Müger, Conformal orbifold theories and braided crossed G-categories, *Comm. Math. Phys.* **260** (2005), 727–762.

[RT] N. Reshetikhin and V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–598.

[Ta] D. Tan, Differential equations for intertwining operators among untwisted and twisted modules, arXiv:2510.14860.

[Tu1] V. Turaev, Modular categories and 3-manifold invariants, *Int. J. Mod. Phys.* **B6** (1992), 1807–1824.

[Tu2] V. Turaev, Homotopy field theory in dimension 3 and crossed group-categories, arxiv:math.GT/0005291.

[X] X. Xu, Intertwining operators for twisted modules for a colored vertex operator superalgebra, *J. Algebra* **175** (1995), 241–273.