A pair of dice is thrown and the sum of the numbers appearing on the dice is observed to be 7. Find the probability that the number 5 has appeared on atleast one die.

A: sum is 7

B: 5 has appeared at least on one die

$$A \cap B = \{(2,5), (5,2)\}$$

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{{}^{2}/_{36}}{6/36} = \frac{1}{3}$$

Find the product of the order and the degree of the differential equation

$$\left[\frac{d}{dx}(xy^2)\right] \cdot \frac{dy}{dx} + y = 0.$$

Given differential equation can be written as

$$2xy\left(\frac{dy}{dx}\right)^2 + y^2\frac{dy}{dx} + y = 0 \qquad \text{Order} = 1, \text{Degree} = 2$$

Order × degree = 1 × 2 = 2

Find :

$$\int \frac{\sin 3x}{\sin x}\, dx$$

$$\int \frac{\sin 3x}{\sin x} dx = \int \frac{3\sin x - 4\sin^3 x}{\sin x} dx$$

$$= \int [3 - 4\frac{(1-\cos 2x)}{2}] dx$$

$$= \int (1 + 2\cos 2x) dx$$

$$= x + \sin 2x + C$$

$\vec{a}$ and $\vec{b}$ are two unit vectors such that $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$.

$|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$

$\Longrightarrow |2\vec{a} + 3\vec{b}|^2 = |3\vec{a} - 2\vec{b}|^2$

$\Longrightarrow 4|\vec{a}|^2 + 12\,\vec{a}.\vec{b} + 9|\vec{b}|^2 = 9|\vec{a}|^2 - 12\,\vec{a}.\vec{b} + 4|\vec{b}|^2$

As $|\vec{a}| = |\vec{b}| = 1$

$\therefore 24\,\vec{a}.\vec{b} = 5|\vec{a}|^2 - 5|\vec{b}|^2 = 0 \Longrightarrow \vec{a}.\vec{b} = 0$

So, $\vec{a} \perp \vec{b}$ or Angle between them is $\frac{\pi}{2}$

Write the cartesian equation of the line PQ passing through point P(2, 2, 1) and Q(5, 1, – 2). Hence, find the y-coordinate of the point on th line PQ whose z-coordinate is – 2.

Required equation of line is given by

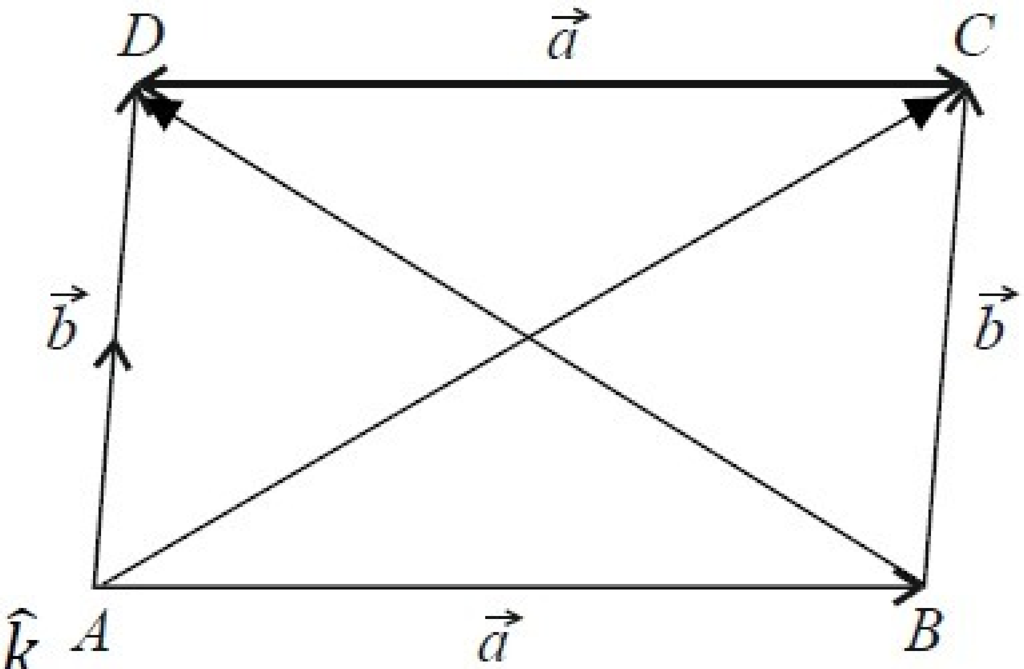
$$\frac{x-2}{3}=\frac{y-2}{-1}=\frac{z-1}{-3}$$

putting $z=-2$, we get $\frac{y-2}{-1}=\frac{-3}{-3}=1$

$$y-2=-1 \Rightarrow y=1$$

ABCD is a parallelogram such that $\overrightarrow{AC} = \hat{i} + \hat{j}$ and $\overrightarrow{BD} = 2\hat{i} + \hat{j} + \hat{k}$

Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also, find the area of the parallelogram ABCD.



Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b} = \hat{\imath} + \hat{\jmath}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \vec{b} - \vec{a} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$

Adding we get, $2\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{BD} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\Longrightarrow \overrightarrow{AD} = \frac{3}{2}\hat{\imath} + \hat{\jmath} + \frac{1}{2}\hat{k}$

Subtracting, we get

$2\overrightarrow{AB} = \overrightarrow{AC} - \overrightarrow{BD} = -\hat{\imath} - \hat{k} \Longrightarrow \overrightarrow{AB} = -\frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{k}$

$$|\overrightarrow{AC} \times \overrightarrow{BD}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \hat{\imath} - \hat{\jmath} - \hat{k}$$

$$\text{Area} = \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$$

$$= \frac{\sqrt{3}}{2}$$

Find the particular solution of the differential equation $(y + 3x^2)\ \dfrac{dx}{dy} = x$,

given that $y = 1$, when $x = 1$.

Given differential equation can be written as

$$x\frac{dy}{dx} - y = 3x^2 \text{ or } \frac{dy}{dx} - \frac{1}{x}y = 3x$$

$$\text{I.F} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = -x^{-1} = \frac{1}{x}$$

Solution is $y \cdot \dfrac{1}{x} = \displaystyle\int 3x \frac{1}{x}\, dx + C$

$$\frac{y}{x} = 3x + C$$

Find the particular solution of the differential equation $(y + 3x^2)\dfrac{dx}{dy} = x$, given that y = 1, when x = 1.

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Solution is $y \cdot \dfrac{1}{x} = \int 3x\dfrac{1}{x}dx + C$

$$\frac{y}{x} = 3x + C$$

$x = 1,\ y = 1$ gives $C = -2$

Particular solution is $\dfrac{y}{x} = 3x - 2$ or $y = 3x^2 - 2x$

Find the distance between the lines $x = \frac{y-1}{2} = \frac{z-2}{3}$ and $x+1 = \frac{y+2}{2} = \frac{z-1}{3}$.

For lines $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and $\frac{x+1}{1} = \frac{y+2}{2} = \frac{z-1}{3}$

Let $\overrightarrow{a_1} = \hat{j} + 2\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{a_2} = -\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Clearly lines are parallel

Hence, Shortest distance or distance is given by

$$\frac{\left|(\overrightarrow{a_2} - \overrightarrow{a_1}) \times \vec{b}\right|}{\left|\vec{b}\right|}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = -\hat{i} - 3\hat{j} - \hat{k}$$

$$(\overrightarrow{a_2} - \overrightarrow{a_1}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -7\hat{i} + 2\hat{j} + \hat{k}$$

Required distance $= \frac{\sqrt{49+4+1}}{\sqrt{1+4+9}} = \frac{\sqrt{27}}{\sqrt{7}}$ or $\frac{3\sqrt{21}}{7}$

Evaluate :

$$\int_0^1 x(1-x)^n \, dx$$

$$I = \int_0^1 x(1-x)^n \, dx$$

$$= \int_0^1 (1-x)[1-(1-x)]^n \, dx \quad \text{[using property]}$$

$$= \int_0^1 x^n (1-x) \, dx$$

$$= \int_0^1 x^n \, dx - \int_0^1 x^{n+1} \, dx$$

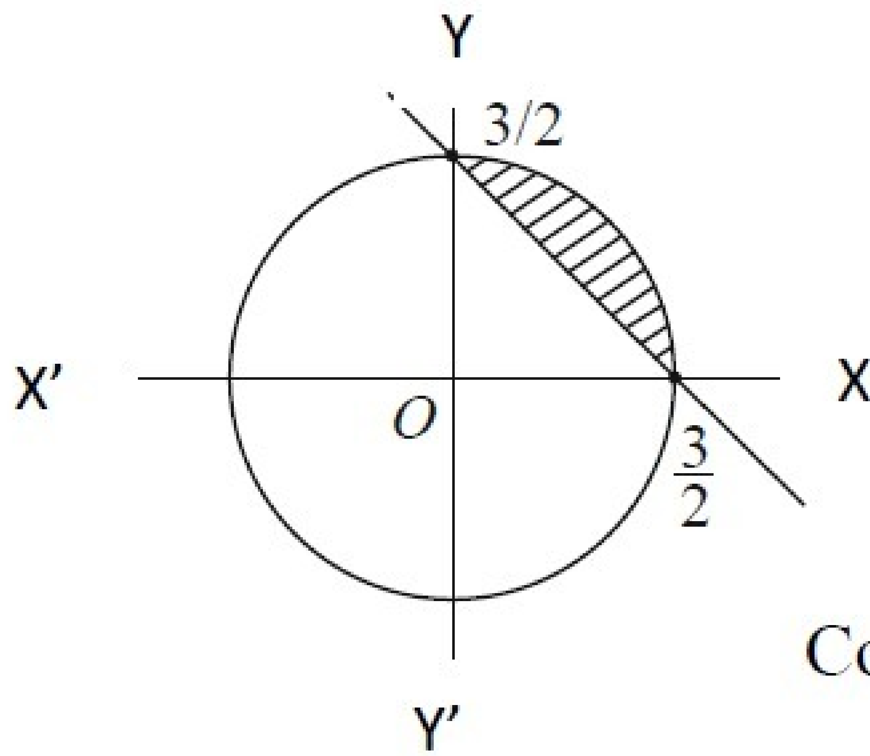
$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} \text{ Or } \frac{1}{(n+1)(n+2)}$$

Using integration, find the area of the smaller region enclosed by the curve $4x^2 + 4y^2 = 9$ and the line $2x + 2y = 3$.

Clearly point of intersection are

$$\left(\frac{3}{2}, 0\right) \;\&\; \left(0, \frac{3}{2}\right)$$



Correct Figure

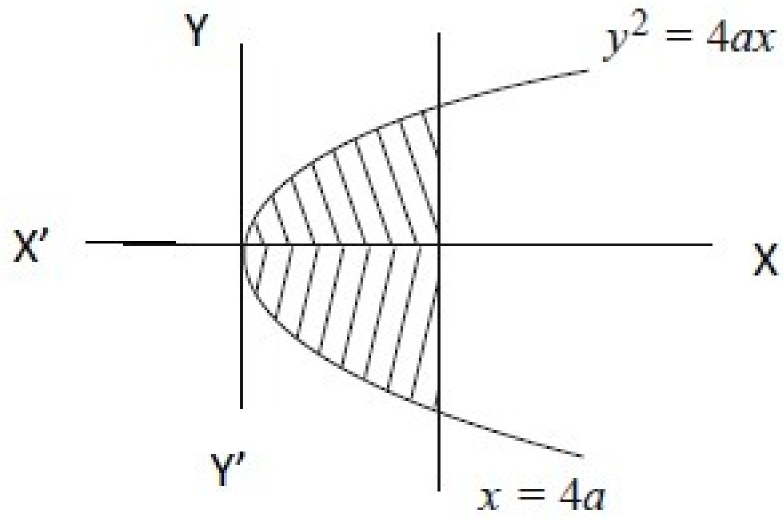
$$\text{Required area} = \int_0^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx - \int_0^{3/2} \left(\frac{3}{2} - x\right) dx$$

$$= \left.\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{2x}{3}\right|_0^{3/2} + \left.\frac{\left(\frac{3}{2} - x\right)^2}{2}\right|_0^{3/2}$$

$$= \frac{9\pi}{16} - \frac{9}{8}$$

If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a, where $a > 0$.

Given area $= \frac{256}{3}$



Correct Figure

Area of Shaded region $= 2\int_0^{4a} \sqrt{4ax}\, dx$

$$= 8\sqrt{a}\left.\frac{x^{3/2}}{3}\right|_0^{4a}$$

$$= \frac{64a^2}{3}$$

$$\frac{64a^2}{3} = \frac{256}{3}$$

$\Rightarrow a^2 = 4$ gives $a = 2$ (as $a > 0$)

Evaluate :

$$\int_0^{\frac{1}{2}\log 3} \frac{e^x}{e^{2x}+1}\,dx$$

Let $e^x = t,\; e^x dx = dt$

$$\therefore \int_0^{\frac{1}{2}\log 3} \frac{e^x}{e^{2x}+1}\,dx = \int_1^{\sqrt{3}} \frac{dt}{t^2+1}$$

$$= \tan^{-1} t \Big|_1^{\sqrt{3}}$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Using vectors, find the value of 'b' if the points A(–1, –1, 2), B(2, b, 5) and C(3, 11, 6) are collinear . Also, determine the ratio in which the point B divides the line-segment AC internally.

Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points A, B, C respectively

$\vec{a} = -\hat{\imath} - \hat{\jmath} + 2\hat{k}$

$\vec{b} = 2\hat{\imath} + b\hat{\jmath} + 5\hat{k}$

$\vec{c} = 3\hat{\imath} + 11\hat{\jmath} + 6\hat{k}$

$\overrightarrow{AB} = \vec{b} - \vec{a} = 3\hat{\imath} + (b+1)\hat{\jmath} + 3\hat{k}$

$\overrightarrow{AC} = \vec{c} - \vec{a} = 4\hat{\imath} + 12\hat{\jmath} + 4\hat{k}$

As A, B, C are collinear

$$\frac{3}{4} = \frac{b+1}{12} = \frac{3}{4}$$

$$\Rightarrow b = 8$$

$$|\overrightarrow{AB}| = \sqrt{9+81+9} = \sqrt{99}$$
$$= 3\sqrt{11}$$

$$|\overrightarrow{AC}| = \sqrt{16+144+16}$$
$$= \sqrt{176} = 4\sqrt{11}$$

Here, B divides AC in the ratio 3 : 1

Evaluate :

$$\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$$

$$I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}\,dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx$$

$$= \frac{\pi}{2}\frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos x\frac{1}{\sqrt{2}} + \sin x\frac{1}{\sqrt{2}}}\,dx$$

$$= \frac{\pi}{2\sqrt{2}}\int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right)dx$$

$$I = \frac{\pi}{4\sqrt{2}} \log\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{2}\right)\right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log(\sqrt{2} + 1) - \log(\sqrt{2} - 1)\right]$$

Or $I = \dfrac{\pi}{4\sqrt{2}} \log\left(\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}\right)$

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$, then find the vector $\vec{c}$, given that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 4$.

Let $\vec{c} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\vec{a} \cdot \vec{c} = 4 \Rightarrow x - y + z = 4$$

$$\vec{a} \times \vec{c} = (-z - y)\hat{\imath} + (x - z)\hat{\jmath} + (x + y)\hat{k}$$

$$\vec{a} \times \vec{c} = \vec{b} \implies \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} = (2\vec{\hat{\imath}} - \hat{\jmath} - 3\hat{k})$$

$$\implies -(y + z)\hat{\imath} - (z - x)\hat{\jmath} + (y + x)\hat{k} = 2\vec{\hat{\imath}} - \hat{\jmath} - 3\hat{k}$$

$$\implies y + z = -2, z - x = 1, y + x = -3$$

Solving we get, $x = 0,\ \ y = -3,\ z = 1$

$$\therefore \vec{c} = -3\hat{\jmath} + \hat{k}$$

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the zx-plane.

Equations of line

$$\frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}$$

Any point on the line

$(-2\lambda + 5, 3\lambda + 1, -5\lambda + 6)$

The point lies on ZX- plane i.e., $y = 0$

$\therefore\ 3\lambda + 1 = 0$

$$\lambda = -\frac{1}{3}$$

$\therefore$ Point is $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$

Write the projection of the vector $(\vec{b} + \vec{c})$ on the vector $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

$$\vec{b} + \vec{c} = 3\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

Projection of $\vec{b} + \vec{c}$ on $\vec{a} = \frac{(\vec{b}+\vec{c}).\vec{a}}{|\vec{a}|}$

$$= \frac{(3\hat{i} + \hat{j} + 2\hat{k})\cdot(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{9}}$$

$$= \frac{6-2+2}{3} = \frac{6}{3} = 2$$

Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spade cards.

Let X denotes the number of spades

$$p = \frac{13}{52} = \frac{1}{4}, q = \frac{3}{4}$$

X	0	1	2
P(X)	$\frac{3}{4}.\frac{3}{4} = \frac{9}{16}$	$2.\frac{3}{4}.\frac{1}{4} = \frac{6}{16}$	$\frac{1}{4}.\frac{1}{4} = \frac{1}{16}$

Evaluate : $\int_0^{2\pi} \frac{\mathrm{d}x}{1+\mathrm{e}^{\sin x}}$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}} \qquad (1)$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin(2\pi-x)}}$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}}$$

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}} dx \qquad (2)$$

Adding (1) and (2)

$$2I = \int_0^{2\pi} 1\, dx \Rightarrow 2I = 2\pi$$

$$I = \pi$$

Find the general solution of the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y(\log y - \log x + 1)$.

$$x\frac{dy}{dx} = y[\log y - \log x + 1]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x} + 1\right]$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v[\log v + 1]$$

$$x\frac{dv}{dx} = v\log v$$

$$\int\frac{dv}{v\log v} = \int\frac{1}{x}dx$$

$$\log|\log v| = \log|x| + \log C$$

$$\log\left(\frac{y}{x}\right) = Cx$$

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that the vector $(\vec{a} + \lambda\vec{b})$ is perpendicular to vector $\vec{c}$, then find the value of λ.

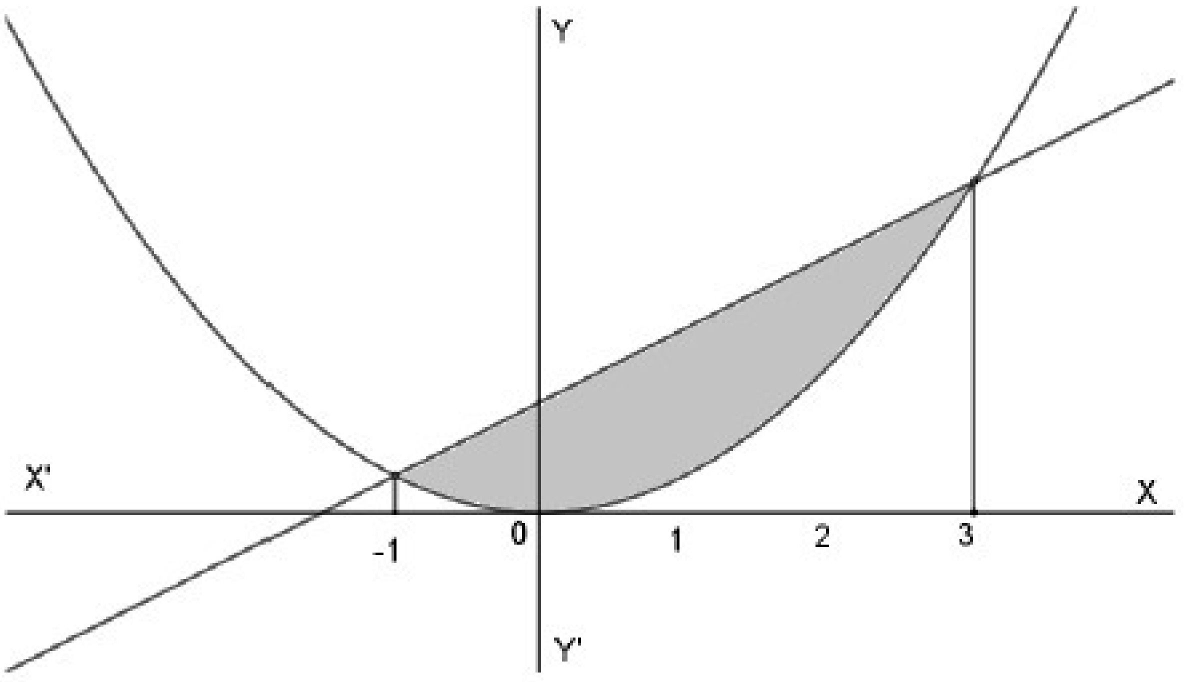
$$(\vec{a} + \lambda\vec{b}).\vec{c} = 0 \Rightarrow \left[(2-\lambda)\hat{\imath} + (2+2\lambda)\hat{\jmath} + (3+\lambda)\hat{k}\right].(3\hat{\imath} + \hat{\jmath}) = 0$$

$$\Rightarrow 3(2-\lambda) + (2+2\lambda)\cdot 1 = 0$$

$$\Rightarrow -3\lambda + 2\lambda + 6 + 2 = 0$$

$$\Rightarrow \lambda = 8$$

Find the area of the region bounded by curve $4x^2 = y$ and the line $y = 8x + 12$, using integration.



Correct Figure

Point of intersection

$4x^2 = 8x + 12$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, -1$

Area $= \int_{-1}^{3}[(8x+12) - 4x^2]dx$

$= 4x^2 + 12x - \frac{4}{3}x^3\Big]_{-1}^{3}$

$= 36 + 36 - 36 - \left(4 - 12 + \frac{4}{3}\right)$

$= 44 - \frac{4}{3} = \frac{128}{3}$

Find : $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$

$$\int \frac{x^2 dx}{(x^2+1)(3x^2+4)}$$

Let $x^2 = t$

$$\frac{t}{(t+1)(3t+4)} = \frac{-1}{t+1} + \frac{4}{3t+4} \quad \text{(by Partial fraction)}$$

$$\int \frac{x^2}{(x^2+1)(3x^2+4)}dx = \int -\frac{1}{x^2+1}dx + \int \frac{4}{3x^2+4}dx$$

$$= -\tan^{-1}x + \frac{4}{3} \times \frac{\sqrt{3}}{2} \tan^{-1}\frac{\sqrt{3}x}{2} + C$$

$$= -\tan^{-1}x + \frac{2}{\sqrt{3}} \tan^{-1}\frac{\sqrt{3}x}{2} + C$$

Evaluate : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (\sin|x| + \cos|x|)\, dx$

$$I = \int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|) dx$$

$$f(x) = \sin|x| + \cos|x|$$

$f(x)$ is an even function

$$I = 2\int_{0}^{\pi/2} (\sin|x| + \cos|x|) dx$$

$$= 2\int_{0}^{\pi/2} (\sin x + \cos x) dx$$

$$= 2[[-\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2}]$$

$$= 2[1+1]$$

$$= 4$$