Chapter 6
Proof Techniques

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Direct Proofs

- A direct proof of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

- A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

- In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

Direct Proofs

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

- Solution: we assume that the hypothesis of this conditional statement is true, and assume that n is odd. By the definition of an odd integer $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2.

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

- By the definition of an odd integer, we can conclude that $n^2$is an odd integer. Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

Direct Proofs

1) Use a direct proof " if ***n*** is a even integer then n^3+n is also even integer."
2) Use a direct proof " if ***n*** is a odd integer then n^2-2n+3 is even integer."
3) Use a direct proof "if n is a even integer then $n^3 + 5$ is odd integer."
4) Use a direct proof "if n is a even integer then $3n + 2$ is even integer."
5) Use a direct proof "if **n** is a odd integer then $2n^2+5n+4$ must be odd integer."

Indirect Proofs

- Indirect Proofs: It does not start with the premises and end with the conclusion. An extremely useful type of indirect proof ***is known as proof by contraposition.***

- **Proofs by contraposition** make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

- In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

Indirect Proofs

Example: Prove that if n is an integer and 3n + 2 is odd, then n is odd.

- Solution: Assume that the conclusion of the conditional statement "If 3n + 2 is odd, then n is odd" is false; and assume that n is even. Then, by the definition of an even integer, n = 2k for some integer k. Substituting 2k for n,

$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

- This tells us that 3n + 2 is even and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If 3n + 2 is odd, then n is odd."

Indirect Proofs

Give an indirect proof, if x is a real number and $x^3 - 5x^2 + 6x - 30 = 0$, then $x = 5$ using proof of contrapositive.

Proof:

We assume that the conclusion of the statement is false. If $x^3 - 5x^2 + 6x - 30 = 0$ then $x = 5$ is false.

Assume $x \neq 5$, then

$$x^3 - 5x^2 + 6x - 30 = x^2(x-5) + 6(x-5) = (x^2+6)(x-5) \neq 0$$

$$x^2 + 6 \neq 0 \text{ or } x - 5 \neq 0$$

Therefore, we have proved that if $x^3 - 5x^2 + 6x - 30 = 0$ then $x = 5$.

Indirect Proofs

Show that if $x^3 - 7x^2 + x - 7 = 0$, then $x = 7$.

Solution

Assume $x^3 - 7x^2 + x - 7 = 0$, we want to show that $x = 7$. Suppose $x \neq 7$, then $x - 7 \neq 0$, and

$$0 = x^3 - 7x^2 + x - 7 = x^2(x-7) + (x-7) = (x^2+1)(x-7)$$

would have implied that $x^2 + 1 = 0$, which is impossible. Therefore, we must have $x = 7$.

Indirect Proofs

1) Prove that if n is an integer and n^2 is odd, then n is odd.
2) Given an indirect proof, "if x is a real number and $x^3 + 2x^2 + 8x + 16=0$, then $x=-2$ using proof of contrapositive."
3) Show that if n is an integer and $n^3 + 5$ is odd, then n is even using a proof by contrapositive.
4) Prove that if n is an integer and $3n + 2$ is even, then n is even using a proof by contrapositive.
5) Given an indirect proof, "if x is a real number and $x^3+6x^2+12x+8=0$, then $x=-2$ using proof of contrapositive."

Proofs by Contradiction

- Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true. Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

- ***Example: Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.***

  [A rational number is any number that can be expressed as fraction p/q of two integers, p and q, with the denominator q not equal to zero. All numbers that are not rational are considered irrational. An irrational number has endless non-repeating digits to the right of the decimal point.]

Proofs by Contradiction

- Solution: If $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors so that the fraction a/b is in lowest terms.
- $\sqrt{2} = a/b$, when both sides of this equation are squared, it follows that

$$2 = \frac{a^2}{b^2}$$

Hence, $2b^2 = a^2$

- By the definition of an even integer it follows that a^2 is even. We next use the fact that if a^2 is even, a must also be even.
- We can write $a = 2c$ for some integer c. Thus, $2b^2 = 4c^2$.
- Dividing both sides of this equation by 2 gives $b^2 = 2c^2$.
- Here b^2 is even.
- Again, using the fact that if the square of an integer is even, then the integer itself must be even, we conclude that b must be even as well.
- We have now shown that the assumption leads to the equation $\sqrt{2} = a/b$, where a and b have no common factors, but both a and b are even, that is, 2 divides both a and b.
- Note that the statement that $\sqrt{2} = a/b$, where a and b have no common factors, means, in particular, that 2 does not divide both a and b.
- Because our assumption leads to the contradiction hat 2 divides both a and b and 2 does not divide both a and b, the statement must be false.
- That is, the statement p, "$\sqrt{2}$ is irrational," is true.

Proofs by Contradiction

Show that if $x^2 < 5$, then $|x| < \sqrt{5}$.

note: *if no set of numbers is specified, the default is the set of real numbers.*

Solution

Assume $x^2 < 5$. We want to show that $|x| < \sqrt{5}$. Suppose, on the contrary, we have $|x| \geq \sqrt{5}$.

By definition, $|x| = x$ or $|x| = -x$.

So either $x \geq \sqrt{5}$, or $-x \geq \sqrt{5}$.

The second case, $-x \geq \sqrt{5}$ is the same as $x \leq -\sqrt{5}$ (by multiplying both sides by negative 1).

If $x \geq \sqrt{5}$, then $x^2 \geq 5$, by algebra; note: since x is a positive number the inequality sign does not change.

If $x \leq -\sqrt{5}$, we again have $x^2 \geq 5$, by algebra; note: since x is a negative number the inequality sign reverses.

In either case, we have both $x^2 \geq 5$ and $x^2 < 5$ which is a contradiction.

Hence $|x| < \sqrt{5}$.

$\therefore$ if $x^2 < 5$, then $|x| < \sqrt{5}$.

Proofs by Contradiction

Use proof by contradiction to prove that "if $x^2 \le 144$, then $|x| \le 12$".

Proof:

If $x^2 \le 144$, then we need to show that $|x| \le 12$. Suppose on contrary we have $|x| > 12$.

Then either $x > 12$ or $x < -12$.

If $x > 12$ then $x^2 > 144$.

If $x < -12$ then $x^2 > 144$.

In either case, we have a contradiction. Hence $|x| \le 12$.

Therefore, we have proved that "if $x^2 \le 144$ then $|x| \le 12$".

Proofs by Contradiction: Prove that $\sqrt{3}$ is irrational.

- **Proof:** Once again we will prove this by contradiction. Suppose that there exists a rational number $r = \frac{a}{b}$ such that $r^2 = 3$. Let r be in lowest terms, that is the greatest common divisor of a and b is 1, or rather $\gcd(a, b) = 1$. And so:

$$\begin{aligned} r^2 &= 3 \\ \frac{a^2}{b^2} &= 3 \\ a^2 &= 3b^2 \end{aligned} \tag{1}$$

- We have two cases to consider now. Suppose that b is even. Then b^2 is even, and $3b^2$ is even which implies that a^2 is even and so a is even, but this cannot happen. If both a and b are even then $\gcd(a, b) \geq 2$ which is a contradiction.

- Now suppose that b is odd. Then b^2 is odd and $3b^2$ is odd which implies that a^2 is odd and so a is odd. Since both a and b are odd, we can write $a = 2m - 1$ and $b = 2n - 1$ for some $m, n \in \mathbb{N}$. Therefore:

$$\begin{aligned} a^2 &= 3b^2 \\ (2m-1)^2 &= 3(2n-1)^2 \\ 4m^2 - 4m + 1 &= 3(4n^2 - 4n + 1) \\ 4m^2 - 4m + 1 &= 12n^2 - 12n + 3 \\ 4m^2 - 4m &= 12n^2 - 12n + 2 \\ 2m^2 - 2m &= 6n^2 - 6n + 1 \\ 2(m^2 - m) &= 2(3n^2 - 3n) + 1 \end{aligned} \tag{2}$$

- We note that the lefthand side of this equation is even, while the righthand side of this equation is odd, which is a contradiction. Therefore there exists no rational number r such that $r^2 = 3$. $\blacksquare$

Proofs of Equivalence

- To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.
- The validity of this approach is based on the tautology

$$(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \land (q \rightarrow p).$$

Proofs of Equivalence or Contradiction : Questions

1) Use proof by contradiction to prove that "if $x^2 \geq 49$ then $|x| \geq 7$.
2) Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.
3) Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
4) Prove the theorem "If n is an integer, then n is odd if and only if n^2 is odd."

THANK YOU
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