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Effect of Anisotropic Permeability on the Blink-Induced Motion of a Soft Contact Lens on the Eye

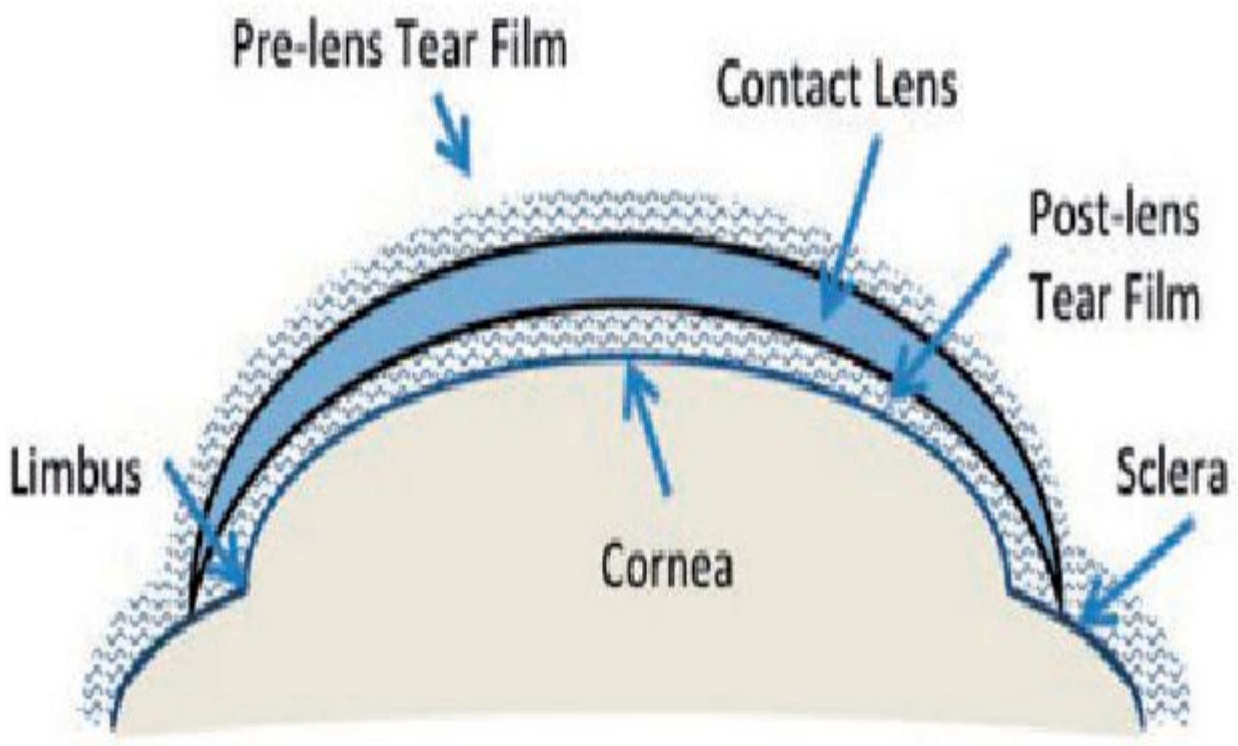
Jitendra Padmanabhuni

INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

DEPARTMENT OF MATHEMATICS

MOTIVATION

- Dry Eye Syndrome
- Development of Anisotropy in Contact Lenses



RELATED WORKS

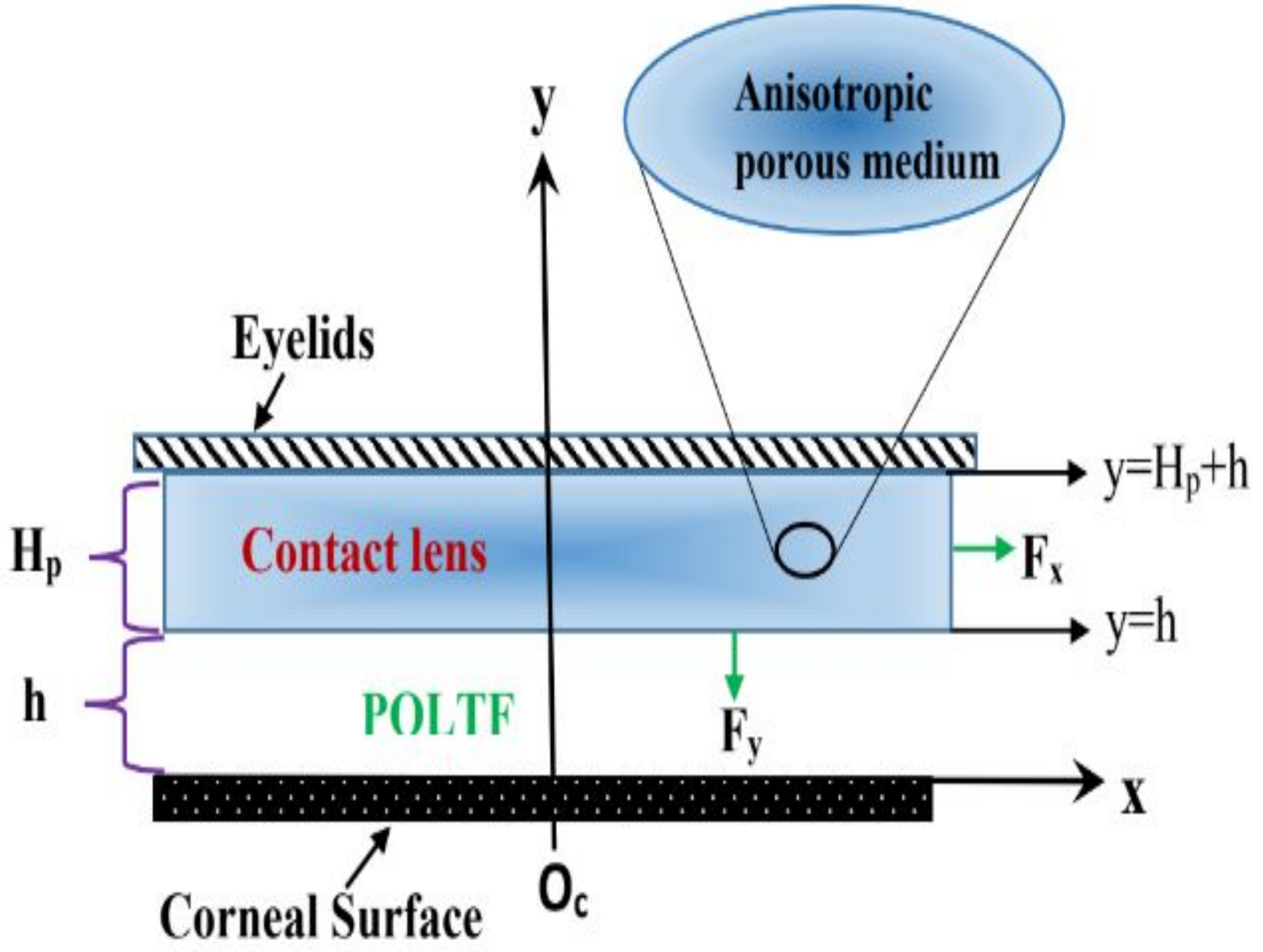
- Conway & Knoll: Investigated motion of lens when tear film thickness varies linearly with time [2]
- Raad & Sabau: Analysed motion of isotropic permeable lens [3]
- Nong & Anderson: Studied the thinning of Pre-lens Tear Film due to evaporation effects [4]
- Usha et al: Investigated rupture phenomena of PRLTF [5]

PROBLEM STATEMENT

- Analyse the effect of anisotropic permeability of the contact lens on the movement of the lens during blinking.
- Analysed while assuming:
  - Only POLTF exists
  - Tear Fluid is Newtonian and incompressible
  - Contact lens is planar, rigid , porous and anisotropic.
- Analysis done using:
  - Darcy Equation in Porous Region.
  - Navier-Stokes Equation in Fluid Region
  - Beavers-Joseph Slip Condition at Lens-Fluid interface.
- First effort in showing effect of anisotropic permeability on motion.

PROBLEM FORMULATION

- The eyelids are fully in contact with the lens, and only the lens surface is in contact with the POLTF
- The liquid-porous interface $y = h$ is assumed to be flat.
- POLTF fluid is assumed as a Newtonian fluid with viscosity μ, density ρ.
- During a blink, lens has a horizontal and vertical force on it which are coupled with the equations of motion of the fluid.
- Motion is taken to be two-dimensional.
- Calculations are done in the frame of reference of the contact lens.



GOVERNING EQUATIONS

- **Fluid Region:**
  - **Equation of Continuity**

    $$\nabla . V^f = 0$$

  - **Navier-Stokes Equation**

    $$\rho(\frac{\partial V^f}{\partial t} + V^f . \nabla V^f) = -\nabla p^f + \mu \nabla^2 V^f - \rho g cos\theta \hat{j} + \rho g sin\theta \hat{i}$$

    where

    $$V^f = (u^f, v^f)$$

- **Porous Region:**
  - **Equation of Continuity**

    $$\nabla . V^p = 0$$

  - **Darcy Equation**

    $$V^p = \frac{-K}{\mu}(\nabla p^p + \rho g cos\theta \hat{j} - \rho g sin\theta \hat{i})$$

    where $V^p = (u^p, v^p)$ **&** $K = \begin{pmatrix} K_1 sin^2\phi + K_2 cos^2\phi & (K_2 - K_1) sin\phi cos\phi \\ (K_2 - K_1) sin\phi cos\phi & K_2 sin^2\phi + K_1 cos^2\phi \end{pmatrix}$

GOVERNING EQUATIONS

- The equations are written in Cartesian form as,
- Fluid Region:

$$\frac{\partial u^f}{\partial x} + \frac{\partial v^f}{\partial y} = 0$$

$$\rho(\frac{\partial u^f}{\partial t} + u^f\frac{\partial u^f}{\partial x} + v^f\frac{\partial u^f}{\partial y}) = -\frac{\partial p^f}{\partial x} + \mu(\frac{\partial^2 u^f}{\partial x^2} + \frac{\partial^2 u^f}{\partial y^2}) + \rho g sin\theta$$

$$\rho(\frac{\partial v^f}{\partial t} + u^f\frac{\partial v^f}{\partial x} + v^f\frac{\partial v^f}{\partial y}) = -\frac{\partial p^f}{\partial y} + \mu(\frac{\partial^2 v^f}{\partial x^2} + \frac{\partial^2 v^f}{\partial y^2}) - \rho g cos\theta$$

- Porous Region:

$$\frac{\partial u^p}{\partial x} + \frac{\partial v^p}{\partial y} = 0$$

$$\mu u^p = -K_1(A_{xx}\frac{\partial p^p}{\partial x} + B_{xy}\frac{\partial p^p}{\partial y} + B_{xy}\rho g cos\theta - A_{xx}\rho g sin\theta)$$

$$\mu v^p = -K_1(B_{xy}\frac{\partial p^p}{\partial x} + C_{yy}\frac{\partial p^p}{\partial y} + C_{yy}\rho g cos\theta - B_{xy}\rho g sin\theta)$$

where, $A_{xx} = sin^2\phi + \lambda cos^2\phi,\ B_{xy} = (\lambda - 1)sin\phi cos\phi,\ C_{yy} = \lambda sin^2\phi + cos^2\phi$

and $\lambda = K_2/K_1$

BOUNDARY CONDITIONS

- **At y = h + Hp,**

$$v^p(x, y = h + H_p, t) = 0$$

$$p^p(x, y = h + H_p, t) = p_a$$

- **At y = 0,**

$$u^f = -U, v^f = -V$$

- **At y = h,**

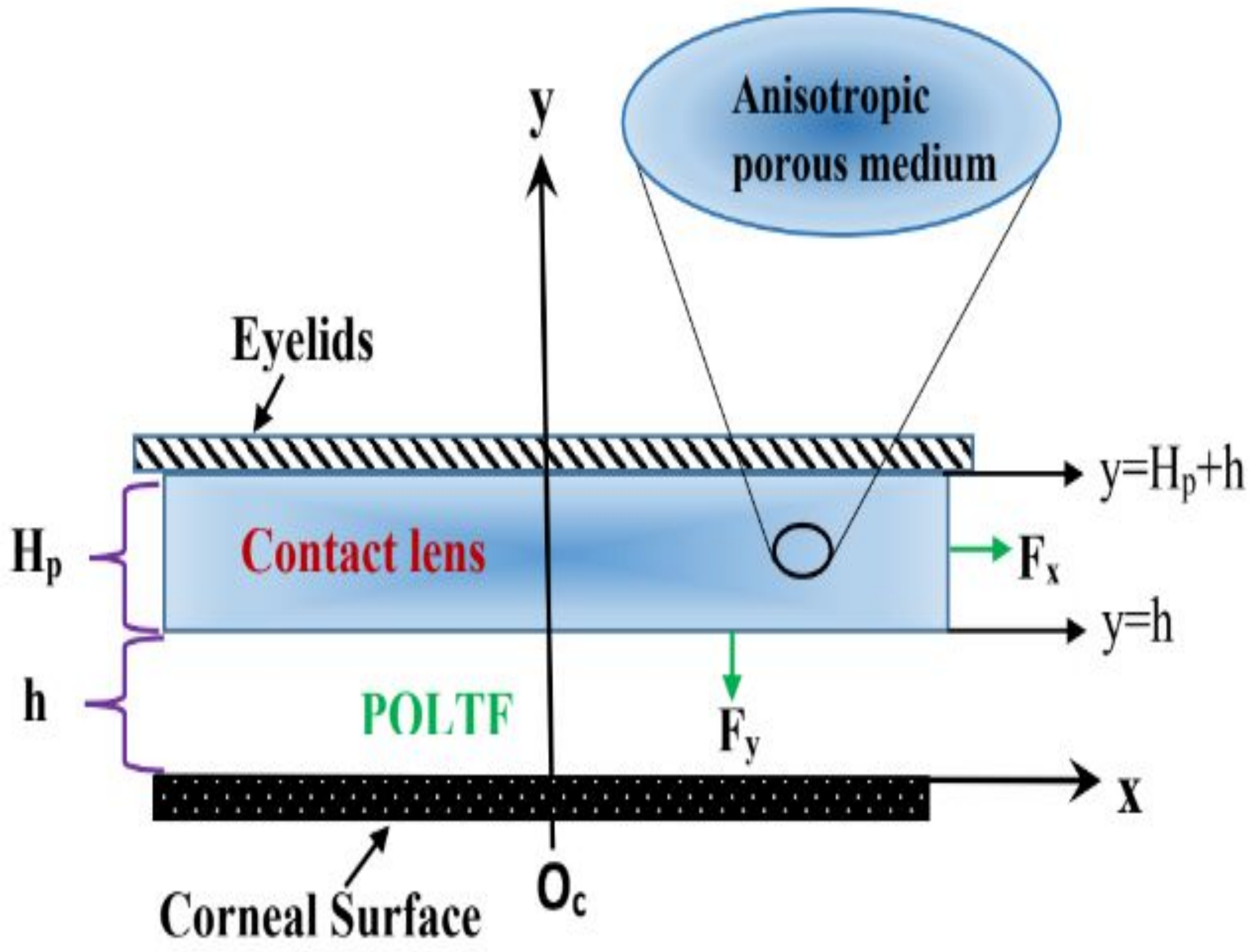
$$v^f - v^p = \frac{dh}{dt}$$

$$u^f - u^p = -\frac{\sqrt{K_1}}{\alpha}\frac{\partial u^f}{\partial y}$$

$$-p^f + 2\mu\frac{\partial v^f}{\partial y} = -p^p$$

- **At x = L and at x = -L,**

$$p^f(x = \pm L, y, t) = p_a$$



EQUATIONS OF MOTION OF

$$M\frac{dU}{dt} = F_x(t) - F_{drag}$$

$$M\frac{dV}{dt} = -F_y(t) + F_{lift}$$

where,

$$F_{lift} = L\int_{x=-L}^{L} (p^f - p_a)\, dx$$

$$F_{drag} = L\int_{x=-L}^{L} \mu\frac{\partial u^f}{\partial y}\, dx$$

NON-DIMENSIONALIZATIO

- The equations are non-dimensionalized using the variables,

$$u^* = \frac{u}{L/t_0}, v^* = \frac{v}{d/t_0}, x^* = \frac{x}{L}, y^* = \frac{y}{d}, p^* = \frac{p}{\mu L^2/t_0 d^2}, t^* = \frac{t}{t_0}, {H_p}^* = \frac{H_p}{d}, h^* = \frac{h}{d}$$

- Fluid Region:

$$\frac{\partial u^f}{\partial x} + \frac{\partial v^f}{\partial y} = 0$$

$$\epsilon^2 Re(\frac{\partial u^f}{\partial t} + u^f \frac{\partial u^f}{\partial x} + v^f \frac{\partial u^f}{\partial y}) = -\frac{\partial p^f}{\partial x} + (\epsilon^2 \frac{\partial^2 u^f}{\partial x^2} + \frac{\partial^2 u^f}{\partial y^2}) + G sin\theta$$

$$\epsilon^3 Re(\frac{\partial v^f}{\partial t} + u^f \frac{\partial v^f}{\partial x} + v^f \frac{\partial v^f}{\partial y}) = -\frac{1}{\epsilon}\frac{\partial p^f}{\partial y} + \epsilon(\epsilon^2 \frac{\partial^2 v^f}{\partial x^2} + \frac{\partial^2 v^f}{\partial y^2}) - G cos\theta$$

where,

$Re = \frac{\rho L^2}{\mu t_0}$ and g is non-dimensionalized by $\frac{\mu L}{\rho t_0 d^2}$

- Porous Region:

$$\frac{\partial u^p}{\partial x} + \frac{\partial v^p}{\partial y} = 0$$

$$u^p = -Da(A_{xx}\frac{\partial p^p}{\partial x} + \frac{B_{xy}}{\epsilon}\frac{\partial p^p}{\partial y} + B_{xy} G cos\theta - A_{xx} G sin\theta)$$

$$v^p = -Da(\frac{B_{xy}}{\epsilon}\frac{\partial p^p}{\partial x} + \frac{C_{yy}}{\epsilon^2}\frac{\partial p^p}{\partial y} + \frac{1}{\epsilon}(C_{yy} G cos\theta - B_{xy} G sin\theta))$$

where, $Da = \frac{K_1}{d^2}$

- $\epsilon = \frac{d}{L} << 1$

LUBRICATION APPROXIMATION

- The lubrication approximation makes use of $\epsilon = \frac{d}{L} << 1$ to simplify equations.
- Fluid Region: Using $\epsilon << 1$ along with $\epsilon^2 Re << 1$ gives

$$\frac{\partial u^f}{\partial x} + \frac{\partial v^f}{\partial y} = 0$$

$$-\frac{\partial p^f}{\partial x} + \frac{\partial^2 u^f}{\partial y^2} + Gsin\theta = 0$$

$$-\frac{\partial p^f}{\partial y} - \epsilon Gcos\theta = 0$$

LUBRICATION APPROXIMATION

- **Porous Region: Solved using Perturbation method**

$$p^p = {p_0}^p + \epsilon {p_1}^p + \epsilon^2 {p_2}^p + O(\epsilon^3)$$

$$\epsilon u^p = -Da(\epsilon A_{xx}\frac{\partial({p_0}^p + \epsilon {p_1}^p + \epsilon^2 {p_2}^p)}{\partial x} + B_{xy}\frac{\partial({p_0}^p + \epsilon {p_1}^p + \epsilon^2 {p_2}^p)}{\partial y} + B_{xy}\epsilon G cos\theta - A_{xx}\epsilon G sin\theta)$$

$$O(1) : B_{xy}\frac{\partial {p_0}^p}{\partial y} + B_{xy}\epsilon G cos\theta = 0$$

$$O(\epsilon) : u^p = -Da(A_{xx}\frac{\partial {p_0}^p}{\partial x} + B_{xy}\frac{\partial {p_1}^p}{\partial y} - A_{xx} G sin\theta)$$

$$O(\epsilon^2) : A_{xx}\frac{\partial {p_1}^p}{\partial x} + B_{xy}\frac{\partial {p_2}^p}{\partial y} = 0$$

$$\epsilon^2 v^p = -Da(\epsilon B_{xy}\frac{\partial({p_0}^p + \epsilon {p_1}^p + \epsilon^2 {p_2}^p)}{\partial x} + C_{yy}\frac{\partial({p_0}^p + \epsilon {p_1}^p + \epsilon^2 {p_2}^p)}{\partial y} + \epsilon(C_{yy} G cos\theta - B_{xy} G sin\theta))$$

$$O(1) : C_{yy}\frac{\partial {p_0}^p}{\partial y} + C_{yy}\epsilon G cos\theta = 0$$

$$O(\epsilon) : B_{xy}\frac{\partial {p_0}^p}{\partial x} + C_{yy}\frac{\partial {p_1}^p}{\partial y} - B_{xy} G sin\theta = 0$$

$$O(\epsilon^2) : v^p = -Da(B_{xy}\frac{\partial {p_1}^p}{\partial x} + C_{yy}\frac{\partial {p_2}^p}{\partial y})$$

LUBRICATION APPROXIMATION

- **Boundary Conditions:**

at $y = h$:

$$v^f - v^p = \frac{dh}{dt}$$

$$u^f - u^p = -\frac{\sqrt{Da}}{\alpha}\frac{\partial u^f}{\partial y}$$

$$p^f = {p_0}^p, {p_1}^p = 0, {p_2}^p = -2\mu\frac{\partial v^f}{\partial y}$$

at $y = h + H_p$:

$$v^p = 0$$

$$p^p = p_a$$

at $y = 0$:

$$u^f = -U, v^f = -V$$

at $x = \pm 1$:

$$p^f = p_a$$

SOLUTIONS

- From the above equations and boundary conditions we get,

$$u^p = -Da(A_{xx} - \frac{{B_{xy}}^2}{C_{yy}})(\frac{\partial {p_0}^p}{\partial x} - Gsin\theta)$$

$$v^p = -Da(A_{xx} - \frac{{B_{xy}}^2}{C_{yy}})(\frac{\partial^2 {p_0}^p}{\partial x^2})(y-h)$$

$$u^f = (\frac{\partial p^f}{\partial x} - Gsin\theta)\frac{y^2}{2} + C_1(x,t)y + C_2(x,t)$$

where, $C_2(x,t) = -U$

$$C_1(x,t) = (\frac{1}{h + \frac{\sqrt{Da}}{\alpha}})(U + Da(A_{xx} - \frac{{B_{xy}}^2}{C_{yy}}) + \frac{h\sqrt{Da}}{\alpha})(Gsin\theta - \frac{\partial p^f}{\partial x}|_{y=h})$$

$$v^f = -\frac{y^2}{2}\frac{dC_1}{dt} + \frac{y^3}{6}\frac{\partial^2 p^f}{\partial x^2} - \frac{dh}{dt}$$

$$p^f = -\epsilon G cos\theta(y-h) + \frac{1}{\phi}\frac{dh}{dt}(x^2-1) + p_a$$

where,

$$\phi = \frac{h^2}{2}(\frac{1}{h + \frac{\sqrt{Da}}{\alpha}})(U + Da(A_{xx} - \frac{{B_{xy}}^2}{C_{yy}}) + \frac{h\sqrt{Da}}{\alpha}) + \frac{h^3}{6}$$

SOLUTIONS

- Finally the equations of motion of the lens in non-dimensional form are at y=h,

$$I\frac{d^2x}{dt^2} = F_x - C_{LD}\int_{x=-1}^{1}\frac{\partial u^f}{\partial y}\,dx$$

$$\epsilon I\frac{d^2h}{dt^2} = -F_y + \frac{C_{LD}}{\epsilon}\int_{x=-1}^{1}(p^f - p_a)\,dx$$

where, $I = \frac{ML}{{t_0}^2 F_{ye}}$ is the inertial coefficient and $C_{LD} = \frac{\mu L^3}{{t_0}^2 dF_{ye}}$

- The above equations upon simplification give,

$$\frac{d^2x}{dt^2} = \frac{F_x - 2C_{LD}(F(h) - h)Gsin\theta}{I}$$

$$\frac{d^2h}{dt^2} + \frac{4C_{LD}}{3\epsilon^2 I\phi}\frac{dh}{dt} + \frac{F_y}{\epsilon I} = 0$$

where $F(h) = (\frac{1}{h+\frac{\sqrt{Da}}{\alpha}})(U + Da(A_{xx} - \frac{{B_{xy}}^2}{C_{yy}}) + \frac{h\sqrt{Da}}{\alpha})$

CONCLUSION AND FUTURE WORK

- We can see that the equations of motion are second order ordinary differential equations which can now be solved via numerical techniques.
- We plan to examine how various material properties of the lens such as slip coefficients, permeability etc. affect its motion and finally determine suitable materials to make contact lenses.

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