Portfolio Optimization Under Regime Switching Market

Term Project for Financial Mathematics (MA60225)
under the supervision of Prof. Nitin Gupta

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Single v/s Multi Period Optimization

- **Single period problem** - immediate timestamp.
- **Multi Period** - longer time horizon.
- No analytical solution for a general multi period problem.
- Transaction costs, taxes, and underlying market regimes are key factors.
- Multi-period financial models are more complex than single-period.

Multiperiod Portfolio Optimization

Problem Setup

A general multi-period portfolio optimization problem is,

$$
\begin{aligned}
\underset{\pi_0,\pi_1,\ldots,\pi_{T-1}\in\mathbb{R}^n}{\text{Maximize}}\ & Utility[Z_1, Z_2, \ldots] \\
\text{subject to } & \mathbf{1}^T\pi_t = 1 && \forall t = 0, \ldots, T-1 \\
& W_t^{\rightarrow} = W_t(\pi_t{}^T(1+r_t)) && \forall t = 0, \ldots, T-1 \\
& \pi_t{}^{\rightarrow} = \frac{\pi_t \odot (1+r_t)}{\pi_t{}^T(1+r_t)} && \forall t = 0, \ldots, T-1 \\
& W_{t+1} = W_t^{\rightarrow} - C(W_t^{\rightarrow}; \pi_t{}^{\rightarrow}, \pi_{t+1}) && \forall t = 0, \ldots, T-1
\end{aligned}
$$

The Market

- Market has different **regimes**.
- Regimes of the market follow Markov Property.
- Regime changes only at discrete time intervals.
- Prices of the assets follow correlated geometric brownian motion.
- **Returns of risky assets**: Multivariate normal distribution

$$r_t \sim N(\mu_{S_t}, \Omega_{S_t}) S_t \in \{1, 2, ..., N\}$$

- Risk free asset has a rate - r_f

The Investor

- **Objective:** Maximize the terminal wealth
- Can trade only at specific time intervals.
- **Objective Function**: Constant Relative Risk Aversion (CRRA) utility

$$U(W) = \begin{cases} \frac{W^\gamma}{\gamma}, & \gamma \neq 0 \\ log(W), & \gamma = 0 \end{cases}$$

Dynamic Programming

State Space

- **S** = { (p, τ) }, where p = [$p_1, .., p_N$] is probability distribution over regimes
- State space is discretized.

Action Space

- **A** = $\{\pi = [\pi_1, ..., \pi_n] | \pi_j \in [\pi_l, \pi_u] \forall j \in 1, ..., n\}$
- Action space is also discretized.

Value Function

- **V: S -> R**

$$V(s_i) = \underset{\pi}{\text{Maximize}} E[U(W_T)|s_i, W_i = 1] \forall s_i \in S$$

- **Bellman Equation:**

$$V(s_i) = \underset{\pi_i}{\text{Maximize}} \sum_{s_{i+1}, W_{i+1}} p(s_{i+1}, W_{i+1}|s_i, \pi_i)\mathbf{E}[W_{i+1}{}^{\gamma} V(s_{i+1})]$$

Algorithm

function UPDATEBELIEF(r,p)

$p_k{}^{new} = \sum_{i=1}^{N} pdf(r; \mu_i, \sigma_i) * p_i * tpm[i][k]$

$p^{new} = [p_1{}^{new}, ..., p_N{}^{new}]$

Normalize p^{new} so that the sum of probabilities is 1

return p^{new}

end function

function NEWWEALTH(r,π)

$W^{new} = \sum_{i=0}^{n} \pi^i * (1 + r^i)$

return W^{new}

end function

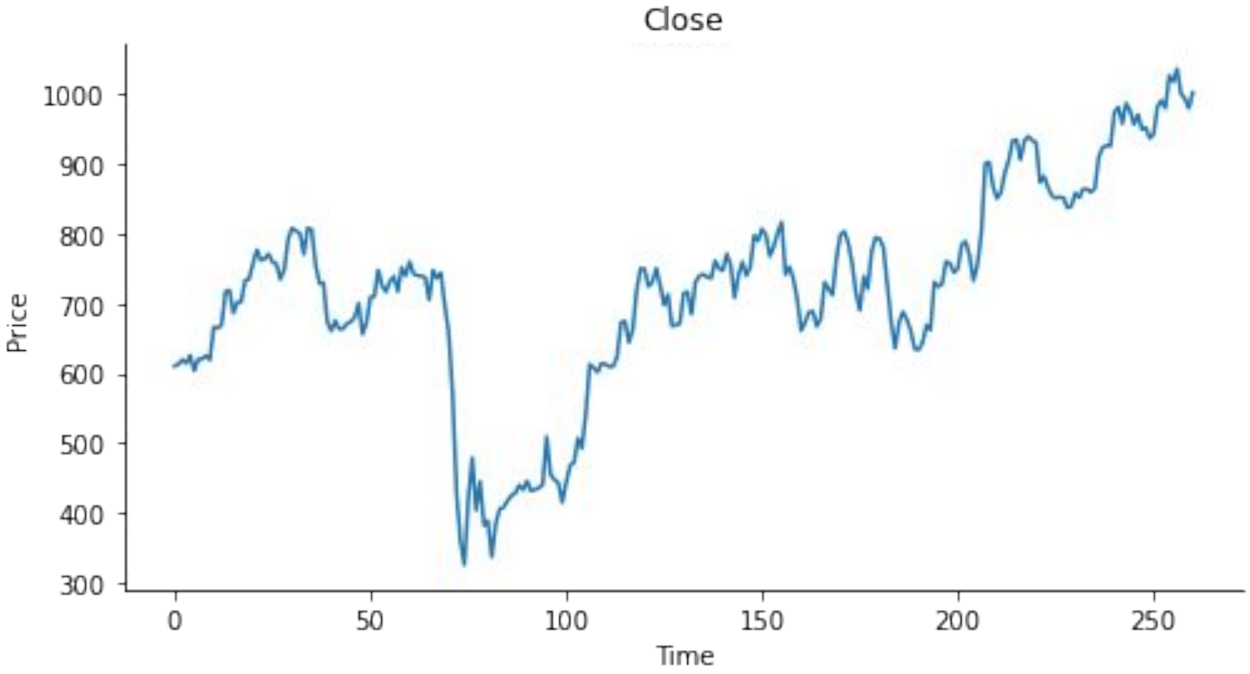
Algorithm

for each p = $[p_1,\ldots,p_N] \in$ P **do**
$\quad V(p,T) = \frac{1^\gamma}{\gamma}$
end for

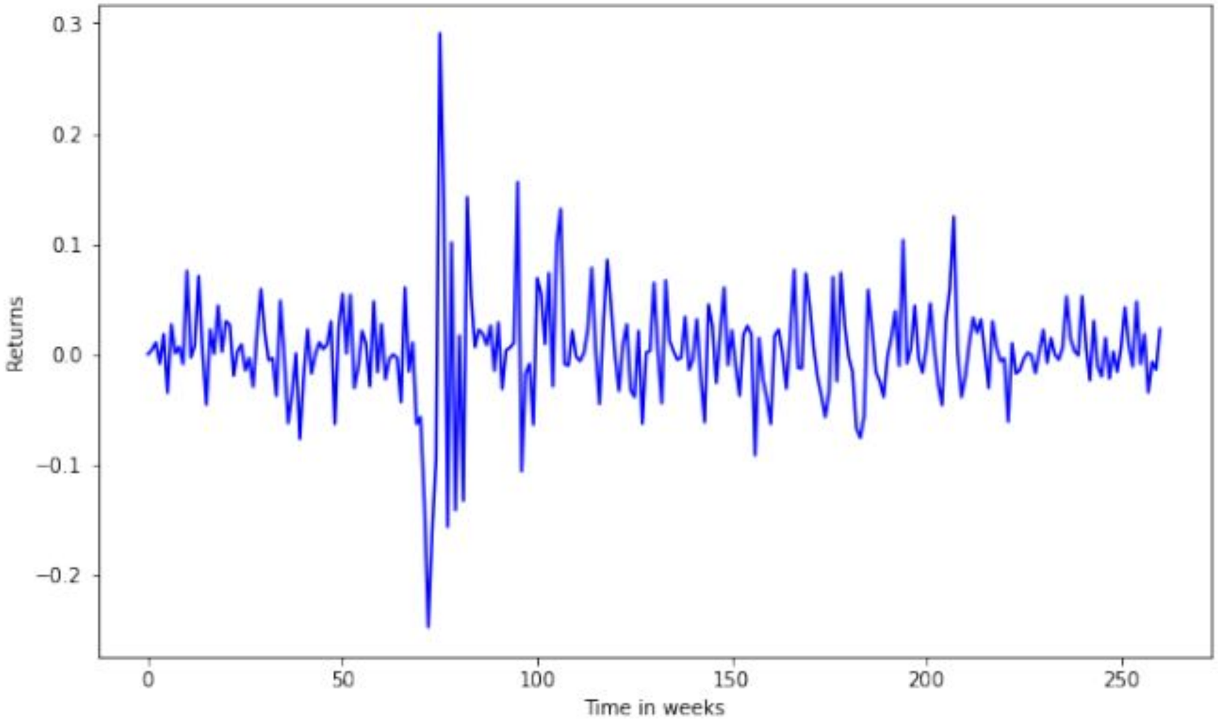
while t$\geq$0 **do**
$\quad$**for** each p = $[p_1,\ldots,p_N] \in$ P **do**
$\quad\quad$Simulate M return scenarios $r_1, \ldots, r_M \in \mathbb{R}^{n+1}$ in period t
$\quad\quad$based on probability of regimes p
$\quad\quad$$\pi^*(p,t) = argmax_\pi\left(\frac{sum_{s=1}^{M}\text{NEWWEALTH}(r_s,\pi)^\gamma V(\text{UPDATEBELEIF}(r_s,p)),t+1)}{M}\right)$
$\quad\quad$$\pi^*(p,t) = max(\pi_l, min(\pi_u, \pi^*(p,t)))$
$\quad\quad$$V(p,t) = \left(\frac{sum_{s=1}^{M}\text{NEWWEALTH}(r_s,\pi^*)^\gamma V(\text{UPDATEBELEIF}(r_s,p)),t+1)}{M}\right)$
$\quad$**end for**t = t-1
end while

Results

- **Dataset souce:** NSE NIFTY50 Assets (Taken from Yahoo Finance)
- Axis Bank stock prices are chosen for training.



- Returns plot is obtained as follows,



- Statistics of the returns

Feature	Description
count	261.000000
mean	0.003172
std	0.050405
min	-0.247275
25%	-0.017021
50%	0.002421
75%	0.022502
max	0.290982

- **Parameter Estimation:** Hidden Markov Model (HMM)

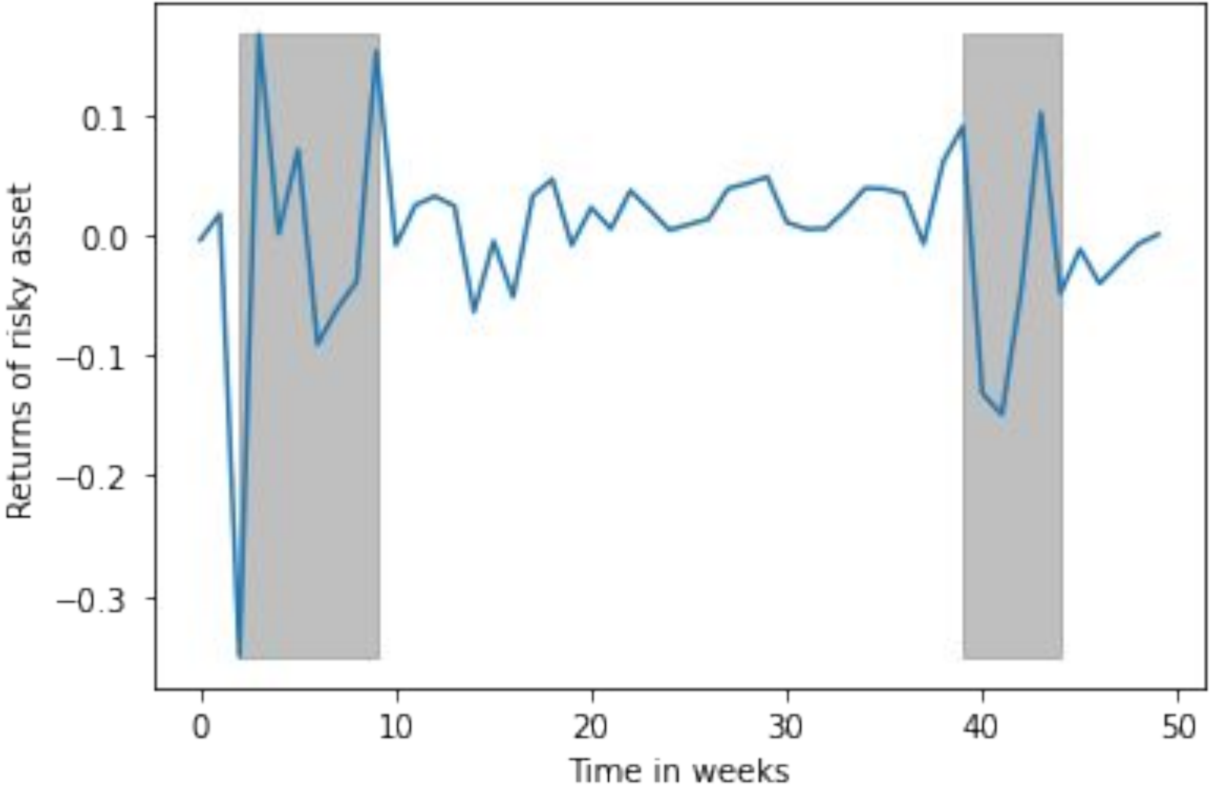
Regime 1: Mean: 0.00368291, Covariance: 0.00114034

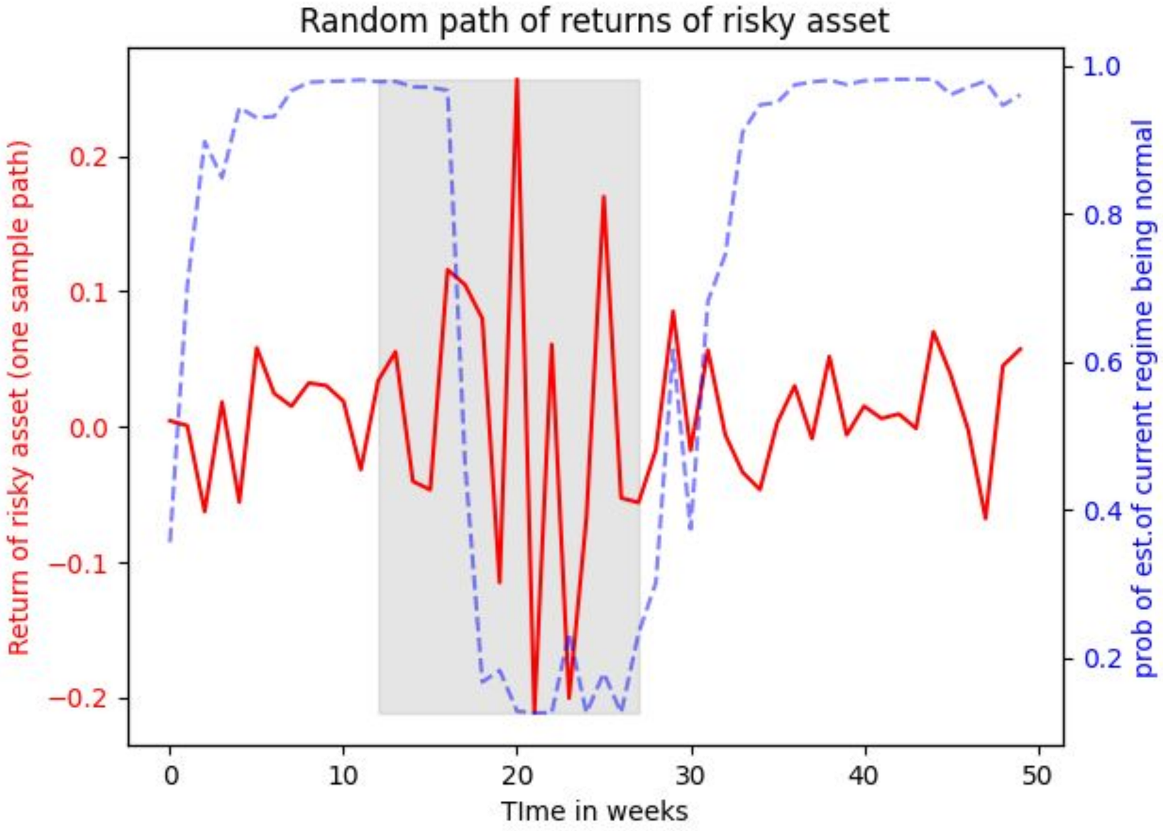
Regime 2: Mean: -0.00135955, Covariance: 0.01559686

- Transition Probability Matrix: $\begin{pmatrix} 0.98581142 & 0.01418858 \\ 0.12509587 & 0.87490413 \end{pmatrix}$

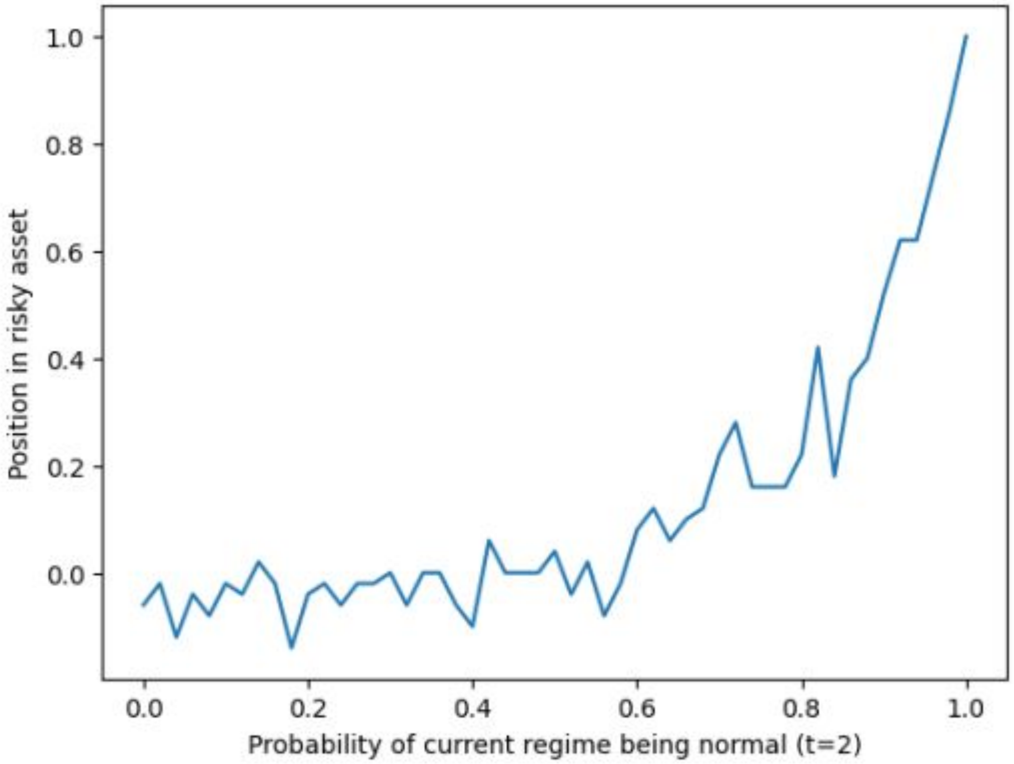
- **Parameters**:

  rf = 5%
  allocation range = [-100%,100%]
  CRRA coefficient = -1


0.1
0.0
−0.1
−0.2
−0.3
Returns of risky asset
0
10
20
30
40
50
Time in weeks

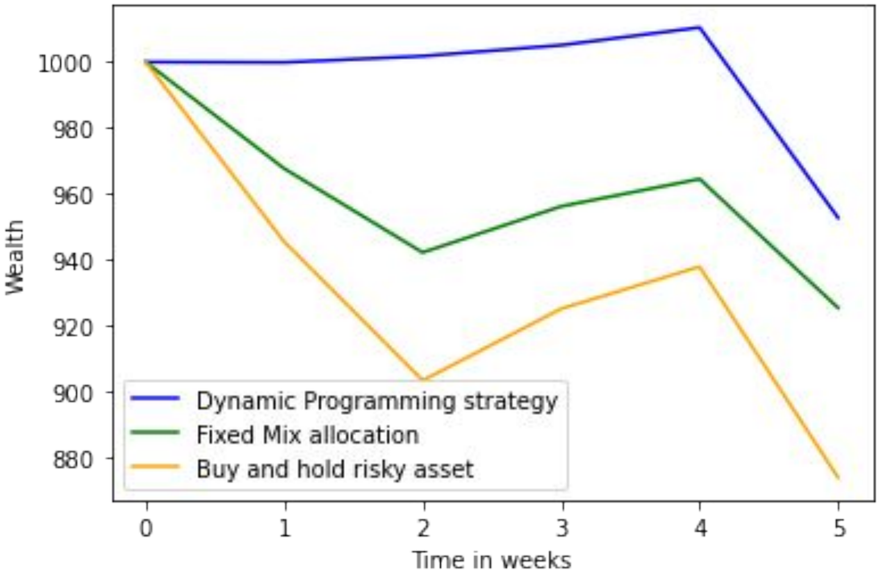

Random path of returns of risky asset
Return of risky asset (one sample path)
prob of est.of current regime being normal
TIme in weeks
0.2
0.1
0.0
−0.1
−0.2
1.0
0.8
0.6
0.4
0.2
0
10
20
30
40
50

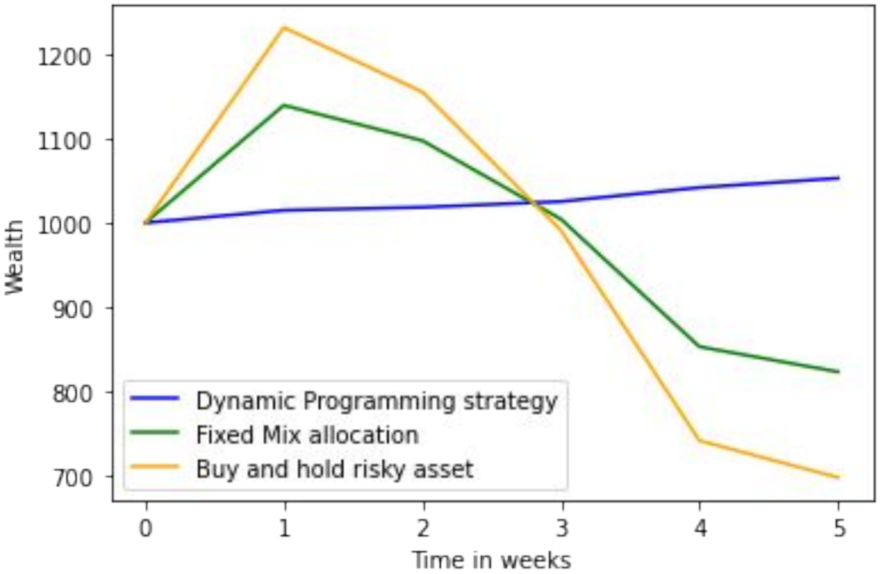
- T = 5 weeks
- State space discretization = 0.02
- No. of Samples, M = 10000



Comparison with different allocation strategies:

- Strategy 1: 100 % risky asset
- Strategy 2: 60 % risky asset and 40% risk-free asset





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Thank You!