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Counting

Combinatorial Analysis

→ It includes the study of permutations, combinations, and partitions. It is concerned with determining the number of logical possibilities of some event without necessarily identifying every case. It is the study of arrangements of objects.

There are 2 basic counting principles

- Sum Rule Principle
- Product Rule Principle

→ The fundamental 'counting principle' is a rule ~~that~~ used to count the total number of possible outcomes in a situation.

~~It states that if there are 'n' ways of doing something, and 'm' ways of doing another thing after that, then there are 'n × m' ways to perform both of these actions. In other words, when choosing an option for 'n' and an option for 'm', there are 'n × m' different ways to do both actions.~~

- **Sum Rule Principle** → Suppose some event 'E' can occur in 'n' ways, and a second event 'F' can occur in 'm' ways and suppose both events cannot occur simultaneously. Then, E or F can occur in 'm + n' ways.

In a general way, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, a third event E_3 can occur in n_3 ways, ---, and suppose no two of the events can occur at the same time. Then, one of the events can occur in $n_1 + n_2 + n_3 + \cdots$ ways.

Examples → (a) Suppose there are 8 male professors and 5 female professors teaching a calculus class. A student can choose a calculus professor in $8 + 5 = 13$ ways.

(b) Suppose E is the event of choosing a prime number less than 10, and suppose F is the event of choosing an even number less than 10. Then,

E can occur in four ways [2, 3, 5, 7], and F can occur in four ways [2, 4, 6, 8].

However, E or F cannot occur in $4+4=8$ ways since 2 is both a prime number less than 10 and an event less than 10. In fact, E or F can occur in only $4+4-1=7$ ways.

(c) Suppose E is the event of choosing a prime number between 10 and 20, and suppose F is the event of choosing an even number between 10 and 20. Then,

E can occur in 4 ways [11, 13, 17, 19], and F can occur in 4 ways [12, 14, 16, 18].

Then, E or F can occur in $4+4=8$ ways since now none of the even numbers is prime.

<u>Theorem</u> (Sum Rule)

If $A \cap B = \phi$ then $|A \cup B| = |A| + |B|$.

<u>Example</u> → Suppose that you are in a restaurant, and are going to have either soup or salad but not both. There are two soups and four salads on the menu. How many choices do you have?

<u>Soln</u> By Sum Rule, you have $2+4=6$ choices.

• <u>Product Rule Principle</u> → Suppose there is an event 'E' which can occur in 'm' ways, and independent of this event, there is a second event 'F' which can occur in 'n' ways. Then combinations of E and F can occur in m.n ways.

In a general way, suppose an event E_1 can occur in n_1 ways, and, following E_1, a second event E_2 can occur in n_2 ways, and following E_2, a third event E_3 can occur in n_3 ways, and so on. Then all the events can occur in the order indicated in $n_1 \cdot n_2 \cdot n_3 \cdots$ ways.

<u>Thm</u>. (Product Rule)

For any choice of sets A and B, $|A \times B| = |A||B|$.

<u>Examples</u> → (a) Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digit in 10 ways. Hence,

$$26 \cdot 26 \cdot 9 \cdot 10 \cdot 10 = 608400$$

different Plates can be printed.

(b) In How many ways can an organization containing 26 members elect a president, treasurer, and secretary (assuming no person is elected to more than one position)?

The president can be elected in 26 different ways; following this, the treasurer can be elected in 25 different ways (since the person chosen president is not eligible to be treasurer); and following this, the secretary can be elected in 24 different ways. So, there are $26 \cdot 25 \cdot 24 = 15600$ different ways.

Q: A boy lives at X and wants to go to School at Z. From his home X, he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

Sol → From X to Y, he can go in $3+2 = 5$ ways (Rule of Sum)
Thereafter, he can go Y to Z in $4+5 = 9$ ways (Rule of Sum).
Hence, from X to Z he can go in $5 \times 9 = 45$ ways (Rule of Product).

Q: In class, there are 4 boys and 10 girls if a boy and a girl have to be chosen for the class monitor, How many ways the students can choose the class monitor?

Sol → $4 \times 10 = 40$ ways

Q: Six friends A, B, C, D, E and F want to sit in a row at the cinema. If there are only six seats available, how many ways can we seat these friends?

Sol → $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways.

Q: My toy piano keyboard has 7 distinct white notes: letters A-G in English alphabet. I'm going to create a melody by playing three random notes. I am not allowed to repeat any notes and the melody cannot be ended with E, F or G. How many different melodies can I play?

Sol → There are 4 ways to choose the last note (we can't use E, F & G). Next, we can choose the second note. There are 6 ways of doing that, because we can't choose the one we used as the last one. Using the same reasoning, we can choose the first note from 5 remaining unused notes. In total, this gives $5 \times 6 \times 4 = 120$ melodies.

Q: Suppose a jar contains 15 red marbles, 20 blue marbles, 5 green marbles, and 16 yellow marbles. If you randomly select one marble from the jar, what is the probability that you will have a red or green marble?

Sol: $\frac{5+15}{5+15+16+20}$

Q: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Sol → $23+15+19 = 57$ ways.

Q: A history class contains 8 male students and 6 female students. Find the number n of ways that the class can elect: (a) 1 class representative;
(b) 2 class representative, 1 male and 1 female;
(c) 1 president and 1 vice president.

Sol → (a) $n = 8+6 = 14$; (b) $8 \cdot 6 = 48$; (C) $14 \times 13 = 182$

Substraction Rule (Inclusion - Exclusion for 2 sets)

Suppose that a task can be done in one of two ways, but some of the ways to do it are common to both ways. In this situation, we cannot use the sum rule to count the number of ways to do the task. If we add the number of ways to do the tasks in these two ways, we get an overcount of the total number of ways to do it, because the ways to do the task that are common to the two ways are counted twice. To correctly count the number of ways to do the two tasks, we must substract the number of ways that are counted twice. This leads us to the substraction rule.

"If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways."

The subtraction rule is also known as the principle of inclusion - exclusion, especially when it is used to count the number of elements in the union of two sets.

Suppose that A_1 and A_2 are sets,

$$\boxed{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|}$$

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Thm. For any finite sets A, B, C, we have

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - \left[|A \cap B| + |B \cap C| + |A \cap C|\right] + |A \cap B \cap C|}$$

In general,

$$\boxed{|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i_1<i_2<i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|}$$

Thm → Let $A_1, A_2, \ldots, A_r$ be subsets of a universal set U. Then the number m of elements which do not appear in any of the subsets $A_1, A_2, \ldots, A_r$ of U is:

$$m = n(A_1^c \cap A_2^c \cap \cdots \cap A_r^c) = |U| - s_1 + s_2 - s_3 + \cdots + (-1)^r s_r$$

where,

$$s_1 = \sum_i n(A_i)$$

$$s_2 = \sum_{i<j} n(A_i \cap A_j)$$

$$s_3 = \sum_{i_1<i_2<i_3} n(A_{i_1} \cap A_{i_2} \cap A_{i_3})$$

$$\vdots$$

$$s_r = n(A_1 \cap A_2 \cap \cdots \cap A_r)$$

Q: Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian, given the following data:

65 study French, 20 study French and German,
45 study German, 25 study French and Russian,
42 study Russian, 15 study German and Russian,
8 study all three languages.

Sol:

$$n(F \cup G \cup R) = n(F) + n(G) + n(R) - n(F \cap G) - n(F \cap R) - n(G \cap R) + n(R \cap F \cap G)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100$$

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Q: A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Sol: Let A_1 be the set of students who majored in computer science, & A_2 be the set of students who majored in business.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 220 + 147 - 51 = 316$$
$$\Rightarrow |A_1^c \cap A_2^c| = |U| - |A_1 \cup A_2| = 350 - 316 = 34$$

Q: Suppose among 32 people who save paper or bottles (or both) for recycling, there are 30 who save paper and 14 who save bottles. Find the number m of people who: (a) save both; (b) save only paper; (c) save only bottles.

Sol: Let P and B denote the sets of people saving paper and bottles, respectively.

Then,

(a) $m = \cancel{REP}\ n(P \cap B) = n(P) + n(B) - n(P \cup B)$
$= 30 + 14 - 32 = 12$

(b) $m = n(P \setminus B) = n(P) - n(P \cap B) = 30 - 12 = 18$

(c) $m = n(B \setminus P) = n(B) - n(P \cap B) = 14 - 12 = 2$

Q: Let U be the set of positive integers not exceeding 1000. Then $|U| = 1000$. Find $|S|$ where S is the set of such integers which are not divisible by 3, 5, or 7.

Sol: Let A be the subset of integers which are divisible by 3, B which are divisible by 5, and C which are divisible by 7. Then, $S = A^c \cap B^c \cap C^c$ since each element of S is not divisible by 3, 5 or 7.

By integer division,

$$|A| = \left[\frac{1000}{3}\right] = 333, \quad |B| = \left[\frac{1000}{5}\right] = 200, \quad |C| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{15}\right] = 66, \quad |A \cap C| = \left[\frac{1000}{21}\right] = 47, \quad |B \cap C| = \left[\frac{1000}{35}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{105}\right] = 9$$

$$\Rightarrow |S| = 1000 - \left[(333+200+142) - (66+47+28) - 9\right] = 457$$

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Permutation and Combination

The study of permutations and combinations is concerned with determining the number of different ways of arranging and selecting objects out of a given number of objects, without actually listing them.

With Replacement (Independent) : It means the same item can be chosen more than once. (Repetition is allowed).

Without Replacement (Not Independent or dependent) : It means the same item cannot be selected more than once. (Repetition is not allowed).

Q: A pin code at your bank is made up of 4 digits, with replacement. How many combinations are possible?

Sol: $10 \times 10 \times 10 \times 10 = 10{,}000$

Q: A pin code at your bank is made up of 4 digits, without replacement. How many combinations are possible?

Sol: $10 \times 9 \times 8 \times 7 = 5040$

Q: Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Sol: $4 \times 3 \times 2 \times 1 = 24$

Q: In above question, replacement is allowed.

Sol: $4 \times 4 \times 4 \times 4 = 256$

Q: Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Sol: $4 \times 3 = 12$ for 2 flags

Q: $5 \times 4 \times 3 = 60$ for 3 flags if repetition is not allowed.

Q: How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

_ _

2 4

Sol: $5 \times 2 = 10$

Q: How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

_ _ _

2 4 6

Sol: $6 \times 6 \times 3 = 108$

Q: How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(i) repetition of the digits is allowed?

(ii) repetition of the digits is not allowed?

Sol: (i) $5 \times 5 \times 5 = 125$, (ii) $5 \times 4 \times 3 = 60$

Q: How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

6 7 _ _ _

Sol: $1 \times 1 \times 8 \times 7 \times 6 = 336$

Q: Three cards are chosen one after the other from a 52-card deck. Find the number 'm' of ways this can be done: (a) with replacement; (b) without replacement.

Sol: (a) $52 \times 52 \times 52$ - (upto 3) $= (52)^3$

(b) $52 \times 51 \times 50 = 52 \times (52-1) \times (52-2) \times \cancel{\ldots \times (52-(m-1))}$

$= {}^{52}P_3$

Permutation →

When selecting more than one item without replacement and order is important, it is called a permutation. In other words, a permutation is an arrangement of objects in a definite order.

- The number of permutations of n objects taken all at a time, denoted by the symbol ${}^nP_n$, is given by

$$\boxed{{}^nP_n = \lfloor n}$$

where $\lfloor n = n.(n-1) \cdots 3.2.1$, read as factorial n.

- The number of permutations of 'n' objects taken 'r' at a time, where $0 < r \leq n$, denoted by ${}^nP_r$, is given by

$$\Rightarrow P(n,r) = \boxed{{}^nP_r = \frac{\lfloor n}{\lfloor n-r}}$$

We assume that $\lfloor 0 = 1$.

- When repetition of objects is allowed: The number of permutations of n things taken all at a time, when repetition of objects is allowed, is $\boxed{n^r}$.

The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is $\boxed{n^r}$.

- Permutations when objects are not distinct: The number of permutations of n objects of which p_1 are of one kind, p_2 are of second kind, ---, p_k are of k^{th} kind and the rest if any, are of different kinds is $\boxed{\frac{n!}{p_1!\,p_2!\cdots p_k!}}$

Q: Find the number m of ways that 7 people can arrange themselves:

a) In a row of chairs;

b) Around a circular table.

Sol: (a) Here, $m = P(7,7) = \underline{|7}$ ways.

(b) One person can sit at any place at the table. The other 6 people can arrange themselves in $\underline{|6}$ ways around the table; i.e., $m = \underline{|6}$.

Note: Above ~~example~~ question is an example of a circular permutation. In general, n objects can be arranged in a circle in $(n-1)!$ ways.

Q: Find the number n of distinct permutations that can be ~~found~~ formed from all letters of each word:

(a) THOSE ; (b) UNUSUAL ; (c) SOCIOLOGICAL.

Sol: (a) $n = 5!$; (b) $n = \dfrac{7!}{3!}$; (c) $n = \dfrac{12!}{3!\,2!\,2!\,2!}$

Q: Find n if $P(n,2) = 72$.

Sol: $P(n,2) = n(n-1) = n^2 - n = 72$

$\Rightarrow n^2 - n - 72 = 0$

$\Rightarrow (n-9)(n+8) = 0$

Since, $n > 0$ then $n = 9$.

Combinations :

On many occasions we are not interested in arranging but only in selecting r objects from given n objects. A combination is a selection of some or all of a number of different objects where the order of selection is not important. The number of selections of r objects from the given n objects is denoted by ^nC_r, and is given by

$$C(n,r) = \boxed{^nC_r = \frac{n!}{r!\,(n-r)!}}$$

Remarks ① Use permutations if a problem calls for the number of arrangements of objects and order is important.

② Use combinations if a problem calls for the number of ways of selecting objects and order is not important.

① Let n and r be positive integers such that $r \le n$. Then

(i) $^nC_r = {^nC_{n-r}}$

(ii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

Q: A box has 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where : (a) There are no restrictions

(b) They are different colors

(c) They are the same color.

Sol: (a) $^{10}C_2$ (b) $^6C_1 \times {^4C_1}$ (c) $^6C_2 + {^4C_2}$ or $^{10}C_2 - (^6C_1 \times {^4C_1})$

Q: Find the number of automobile license plates where:

(a) Each plate contains 2 different letters followed by 3 different digits.

(b) The first digit cannot be 0.

Sol: (a) $26 \times 25 \times 10 \times 9 \times 8$ (b) $26 \times 25 \times 9 \times 9 \times 8$

~~#~~ Let n and $n_1, n_2, \ldots, n_r$ be non-negative integers such that $n_1 + n_2 + \cdots + n_r = n$. The multinomial coefficients are denoted and defined by:

$$\boxed{\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}}$$

Q: Compute the following multinomial coefficients:

(a) $\binom{6}{3,2,1}$, (b) $\binom{8}{4,2,2,0}$, (c) $\binom{10}{5,3,2,2}$

Sol: (a) $\binom{6}{3,2,1} = \frac{6!}{3! \times 2! \times 1!} = 60$

(b) $\binom{8}{4,2,2,0} = \frac{8!}{4! \times 2! \times 2! \times 0!} = 420$

(c) $\binom{10}{5,3,2,2}$ has no meaning, since $5+3+2+2 \neq 10$.

Q: A class contains 9 men and 3 women. Find the number of ways a teacher can select a committee of 4 from the class where there is:

a) no restrictions; b) 2 men & 2 women

c) exactly one woman; d) atleast one woman.

Sol: a) ${}^{12}C_4$; b) ${}^{9}C_2 \cdot {}^{3}C_2 = 108$; c) ${}^{9}C_3 \cdot {}^{3}C_1 = 252$

d) ${}^{12}C_4 - {}^{9}C_4 = 369$