Lecture :- 1

Date / /

Proposition/Statement :- It is a declarative statement which is either true or false but not both.

* The sentences which includes questions, exclamations & expressions of opinions are not propositions.

* Generally, the propositions are denoted by the lower case letters like p, q, r, etc.

examples:-

1) I am a doctor. P
2) Paris is in England. P
3) $2+3=5$ P
4) $5>4$ P
5) Earth is the only planet in universe to have life P
6) How nice! Not a P
7) Where are you going? Not a P
8) $x+y>0$ Not a P
9) Moon is made of cheese P
10) 42 is a perfect square P
11) Go to your room Not a P
12) $1+2+3+\cdots+n$ Not a P
13) Sum of 2 squares Not a P
14) He is a good person. Not a P
15) $x.0=0$ Not a P (because propositions don't contain variables)

Compound Proposition :- These are the statements constructed by combining 2 or more propositions.

eg:- It is a pen and that is a chair.

* To make compound propositions, we use several connectives.

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Basic Connectives / Basic Logical Operators :-

1) Conjunction / And / $\wedge$:-

eg:- 1) if p : This is a pen
q : This boy is intelligent
then $p \wedge q$: This is a pen and this boy is intelligent.

2) if p : $1+1=3$
q : A decade is 10 years
then $p \wedge q$: $1+1=3$ and a decade is 10 years.

* Sometimes the word 'but' is used ~~to~~ as a replacement of 'and'.

2) Disjunction / Or / $\vee$:- either, or, both

eg:- 1) if p : I am at home
q : It is raining
then $p \vee q$: I am at home or it is raining.

2) Students who have taken calculus or computer science can take this class.

3) Negation / Not / $\sim$ / $\neg$:-

eg:- 1) If p : It is cold
then $\sim p$: It is not cold.

2) $\sim p$ means "It is not the case that p happens or exists.

* exclusive or : either p or q (but not both)
$p \oplus q$

Conditional + Biconditional Statements :-

1) Implication / Conditional Statement / $\rightarrow$:-

if p + q are 2 statements then $p \rightarrow q$ is a conditional statement.

For $p \rightarrow q$; p is called antecedent / premise
q is called consequent / consequence.

* There need not be any connection between premise + consequence.

* The various ways to express/read the implication $p \rightarrow q$ are :

if p then q	p implies q
if p, q	p only if q
q if p	p is sufficient for q
q whenever p	q is necessary for p
q follows from p	
q when p	
q unless $\sim p$	

2) Biconditional Statement / $\leftrightarrow$:-

if p, q are two statements then $p \leftrightarrow q$ is a biconditional statement.

$p \leftrightarrow q$ can be read as p iff q (p if and only if q)
($p \rightarrow q$ and $q \rightarrow p$)
p is necessary + sufficient for q

example :- let p: It is below freezing
q: It is snowing.

Then

	Statement Form	Propositional form
1)	It is below freezing + snowing	$p \wedge q$
2)	It is below freezing or snowing	$p \vee q$

3) It is below freezing but not snowing $p \wedge \sim q$
4) It is neither below freezing nor it is snowing $\sim p \wedge \sim q$
5) If it is below freezing, it is also snowing $p \to q$
6) It is either below freezing or it is snowing but it is not snowing if it is below freezing
$(p \vee q) \wedge (p \to \sim q)$

Truth Table :-

1) Conjunction :-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2) Dis~~junct~~unction :-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

3) Negation :-

p	$\sim p$
T	F
F	T

4) Conditional / Implication:-

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

let p: You try
q: You succeed
$p \to q$: If you try then you succeed.

1) if p + q both are true $\Rightarrow$ $p \to q$ is true
2) if p is true but q is false then $p \to q$ can't be true as for if p is true then q has to be true
$\Rightarrow$ $p \to q$ is false
3) if p is not true but q is true ; we don't have any reason to say that $p \to q$ is false ; So $p \to q$ is vacously / by default true.
4) if p + q are false : then $(p \to q)$ is True
($\because$ $p \to q$ means q is necessary for p to happen)
here q is not true $\Rightarrow$ p can't be true which is given as well.

5) Biconditional :-

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Question:- If p: Maria learns discrete maths
q: Maria will find a good job.

$p \to q$: If Maria learns maths then she will find a good job
: Maria will find a good job when she learns math.
: Maria will find a good job unless she doesn't learn discrete maths.

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Inverse, Converse and Contrapositive of a Statement :-

For an implication $p \to q$

Converse	:	$q \to p$
Inverse	:	$\sim p \to \sim q$
Contrapositive	:	$\sim q \to \sim p$

eg:-1) If n is an even integer then 2n is divisible by 4
Converse :- if 2n is divisible by 4 then n is an even integer
Inverse :- if n is not an even integer then 2n is not divisible by
Contrapositive :- if 2n is not divisible by 4 then n is not an even integer.

eg:-2) The home team wins whenever it is raining.
Converse :- It is raining whenever the home team wins
Inverse :- The home team doesn't win whenever it is not raining
Contrapositive :- It is not raining whenever the home team doesn't wi

You can restate the above statement as if p then q.

eg:-3) If I do not go to cinema then I will study.
Converse :- If I will study then I do not go to cinema
inverse :- If I go to cinema then I will not study
Contrapositive :- if I will not study then I go to cinema

Lecture :- 2

Tautology :- A compound proposition is called Tautology if it is always true, irrespective to the truth values of its component propositions.

eg :- $p \to p$ is a tautology
$p \vee \sim p$ is a tautology.

~~Contradict~~ **Contradiction** :- A compound proposition which is always false is called contradiction, irrespective to the truth values of its compound propositions.

* Negation of a Tautology is a Contradiction and vice-versa.

Contingency :- A compound proposition which can be either true or false depending on the truth values of its component propositions is called contingency.

eg :- 1) $p \vee \sim p$

its truth table is

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

} all true's

It is a Tautology.

2) $p \wedge \sim p$

its truth table is

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

} all false

It is a Contradiction.

3) $(p \vee p) \leftrightarrow p$

its truth table is

p	$p \vee p$	$(p \vee p) \leftrightarrow p$
T	T	T
F	F	T

} all true

It is a Tautology.

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Precedence of Logical Operators :-

operator	precedence
$\sim / \neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \sim r$ means $(p \vee q) \rightarrow \sim r$

not $p \vee (q \rightarrow \sim r)$

eg:- Check for Tautology, Contradiction + Contingency.

1) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p) = X$ (say)

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$	X
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

all true

$\Rightarrow$ It is a Tautology.

2) $p \vee p \leftrightarrow p$

it means $(p \vee p) \leftrightarrow p$

p	$p \vee p$	$(p \vee p) \leftrightarrow p$
T	T	T
F	F	T

all true $\Rightarrow$ It is a Tautology

3) $(p \wedge \sim q) \vee (\sim p \wedge q) = X$ (say)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	X
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

Contingency

4) $\sim(p \vee q) \vee (\sim p \vee \sim q) = X$ (let)

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \vee q)$	X
T	T	T	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	T	F	T
F	F	F	T	T	T	T	T

Contingency

5) $\underbrace{(p \to q) \wedge (q \to r)}_{} \to (p \to r)$: $X \to Y$

p	q	r	$p \to q$	$q \to r$	X	$Y (p \to r)$	$X \to Y$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Tautology

6) $(p \vee q) \vee (\sim p \vee \sim q) = X$ (let)

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	X
T	T	T	F	F	F	T
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	T

Tautology

7) $(q \wedge \sim p) \leftrightarrow r = X$ (let)

Check yourself.

Ans:- Contingency.

Lecture :- 3

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Logical Equivalence :-

2 simple statements are logical equivalent iff they have the same truth value.

OR

2 compound statements X & Y are logically equivalent iff $X \leftrightarrow Y$ is a Tautology.

* equivalence is denoted by '≡'.

eg :- " Dogs bark & Cats Meow." and " Cats meow & Dogs bark" are logically equivalent.

		I	II			III	IV
p	q	$p \to q$	$q \to p$	$\sim p$	$\sim q$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

II, III, IV are converse, inverse & contrapositive of I.

Here I ≡ IV , II ≡ III

* ⇒ A conditional statement is equivalent to its contrapositive statement.

eg :- If p : 8 is a prime number

q : $3^2 + 2^2 = 3^2$

Since p & q both are false ⇒ they have same truth value ⇒ $p \equiv q$.

Laws of Logic

Result :- 1) $\sim(\sim p) \equiv p$ — Involution Law

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

same

2) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ — Demorgan's Law

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

same

Similarly

3) $\sim(p \wedge q) \equiv \sim p \vee \sim q$ — Demorgan's Law

4) $\sim(p \to q) \equiv p \wedge \sim q$ — Imp.

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

same

eg:- Write negation of "If I am ill then I can't go to university".

Ans :- I am ill and I can go to university.

eg:- Using Demorgan's law, write the negation of

p : Jim is tall and Jim is thin.

$\sim p$: Jim is not tall or Jim is not thin.

5) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ — Associative law

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(prove yourself).

* Identity Laws :- $p \vee F \equiv p$ $\quad p \wedge F \equiv F$ $\quad p \wedge \sim p \equiv F$

$p \wedge T \equiv p$ $\quad p \vee T \equiv T$ $\quad p \vee \sim p \equiv T$

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6) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ Distributive Laws

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(prove yourself)

7) $p \vee (p \wedge q) \equiv p$ Absorption Laws

$p \wedge (p \vee q) \equiv p$

(prove yourself)

8) $p \wedge p \equiv p$ Idempotent Laws.

$p \vee p \equiv p$

9) $p \wedge q \equiv q \wedge p$ Commutative Laws.

$p \vee q \equiv q \vee p$

eg:- Write negation of

"If my car is in repair shop then I cannot attend the class".

as $\sim(p \to q) \equiv p \wedge \sim q$

$\Rightarrow$ Ans:- My car is in repair shop and I can attend the class.

eg:- Write negation of

1) $q \vee \sim(p \wedge r)$

2) $(p \to r) \wedge (q \to p)$

Ans:- 1) $\sim(q \vee \sim(p \wedge r)) = \sim q \wedge (p \wedge r)$

2) $\sim((p \to r) \wedge (q \to p))$

$\equiv \sim(p \to r) \vee \sim(q \to p)$

$\equiv (p \wedge \sim r) \vee (q \wedge \sim p)$

eg:- Prove that

1) $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

2) $p \vee (p \wedge q) \equiv p$

sol^n 1) LHS $= (p \vee q) \wedge \sim p$
$\equiv \sim p \wedge (p \vee q)$ } Commutative
$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$ Distributive
$\equiv f \vee (\sim p \wedge q)$
$\equiv \sim p \wedge q$ Identity Law

2) LHS $= p \vee (p \wedge q)$
$\equiv (p \wedge T) \vee (p \wedge q)$ Identity Law
$\equiv p \wedge (T \vee q)$ Distributive Law
$\equiv p \wedge T$ Identity Law
$\equiv p =$ RHS

IIlly, we can prove that $p \wedge (p \vee q) \equiv p$ $\left\{ \because \text{LHS} = (p \vee F) \wedge (p \vee q) \right\} \equiv p \vee (F \wedge q) \equiv p \vee F \equiv p \}$

Satisfiability:- A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true.

If no such assignments exists (ie) when compound proposition is false for all assignments of truth values to its variables, the compound proposition is called unsatisfiable.

eg:- Check whether these propositions are satisfiable or unsatisfiable.

1) $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p) = X$ (say)

if p, q, r all all true then $p \vee \sim q$, $q \vee \sim r$, $r \vee \sim p$ are all true hence X is true

$\Rightarrow$ X is satisfiable as there is atleast one assignment of truth values of p, q, r that makes X true.

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2) $(p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r) = X$ (let)

if p is T + q is F

then $p \vee q \vee r$ is T + $\sim p \vee \sim q \vee \sim r$ is T

$\Rightarrow$ X is T

$\Rightarrow$ Satisfiable

3) $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p) \wedge (p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$

Note that given proposition is

1) $\wedge$ 2) where 1) + 2) are propositions discussed above

for 1) to be satisfiable all p, q, r must be true

for 2) " " " atleast one must be true + one must be false

which are contradictory.

Hence, proposition is unsatisfiable.

Lecture :- 4

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Predicate :- A predicate is a sentence that contains a finite no. of variables and becomes a statement when specific values are substituted for variables.

eg:- P : is greater than 3 ; predicate

then P(x) : x is greater than 3 ; propositional function

P(2) : 2 is greater than 3
P(4) : 4 is greater than 3 } these are propositions.

* By assigning value to variables in propositional function, we get propositions.
However, we can use some words (quantifying words) like all, some, many, none, few etc. Such words are called Quantifiers. & this process is called Quantification.

Types of Quantifiers :-

1) Universal Quantifiers :- Universal quantification tells us that a predicate is true for all elements under consideration eg:- for all x, P(x),
$\boxed{\forall}$
for every x, P(x)
for each x, P(x).

2) Existential Quantifiers :- Existential quantification tells us that there is one or more element under consideration for which the predicate is True. eg:- atleast, for some, there exists, etc.
$\boxed{\exists}$

Truth Set of P(x) :- let P(x) be a predicate & D is domain for x. Then

$\{x \in D \mid P(x) \text{ is true}\}$ is called truth set of P(x).

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eg:- "All humans are mortal"
Can be written as
$\forall x \in S$, x is mortal where S: set of human beings

Such statements are called universal statements

* A universal statement $P(x)$ is true iff $P(x)$ is true for every $x \in$ Domain
and it is false iff $P(x)$ is false for atleast one $x \in D$. (Such x will be counter example to universal statement).

eg:- $D = \{1, 2, 3, 4\}$ & let $P(x)$: $\forall x \in D$, $x^3 \geq x$

Since $P(x)$ is true for all $x \in D$
then universal statement $P(x)$ is true $\forall x \in D$.

but if $Q(x) = \forall x \in \mathbb{N}$, $x+2 > 8$
$Q(x)$ is not true $\forall x \in \mathbb{N}$ as
if $x = 6$ then $Q(x)$ is false
$\Rightarrow$ universal statement $Q(x)$ is false

eg:- let $P(x)$: $x+1 > x$. What is truth value of the quantification $\forall x \in P(x)$, where Domain $= \mathbb{R}$
sol^n:- Since $x+1 > x$ for all real numbers
$\Rightarrow$ the quantification, $\forall x \in P(x)$ is true.

$\forall$ Universal Quantifiers
for all, for every, for each, all of, given any, for arbitrary, for any

$\exists$ Existential Quantifiers
for some, atleast, There exists

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eg:- Let $Q(x): x < 2$. What is truth value of quantification $\forall x, Q(x)$, where Domain is $\mathbb{R}$.

sol^n $Q(x)$ is not true $\forall x \in \mathbb{R}$

as $Q(3): 3 < 2$ is false Then

$x = 3$ is counter example for $\forall x, Q(x)$.

Thus $\forall x, Q(x)$ is false.

The $\boxed{\text{existential quantifier}}$ of $P(x)$ is the proposition "there exists an element x in domain such that $P(x)$"

Notation :- $\exists x, P(x)$.

* Domain must be specified when a statement $\exists x, P(x)$ is used.

eg:- $P(x)$: "$x > 3$". The existential quantifier $\exists x, P(x)$ is : there exist an element x for which $x > 3$.

Thus for $\mathbb{R}$, $\exists x, P(x)$ is true.

* When all the elements in domain can be listed - say $x_1, x_2, \_\_, x_n$ then the universal quantification $\forall x, P(x)$ is same as conjunction $P(x_1) \wedge P(x_2) \_\_ \wedge P(x_n)$ because the conjunction $P(x_1) \wedge P(x_2) \wedge \_ \_ \wedge P(x_n)$ is true iff $P(x_1), P(x_2), \_\_, P(x_n)$ all are true.

However, when all the elements in domain can be listed - say $x_1, x_2, \_\_ x_n$ then existential quantification $\exists x, P(x)$ is same as disjunction $P(x_1) \vee P(x_2) \vee \_ \_ \vee P(x_n)$ because the disjunction $P(x_1) \vee P(x_2) \vee \_\_ \vee P(x_n)$ is true iff atleast one of $P(x_i)$ is true.

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Or in other words

$$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

eg:- 1) $\forall x<0\ (x^2>0)$

$\equiv \forall x\ (x<0 \rightarrow x^2>0)$

2) $\forall\ y\neq 0\ (y^3\neq 0)$

$\equiv \forall y\ (y\neq 0 \rightarrow y^3\neq 0)$

3) $\exists\ z>0\ (z^2=2)$

$\equiv \exists z\ (z>0 \wedge z^2=2)$

Negation of Quantified Expressions:-

1) Every student in your class has taken a course in calculus.

Means

$\forall x, P(x)$ where $P(x)$: x has taken a course in calculus

& Domain: student in your class.

Its negation will be

$\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$ i.e.

There is a student in your class who has not taken a course in calculus.

or It is not the case that every student in your class has taken a course in calculus.

$$\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$$

Imp. ||ly $\sim(\exists x\, Q(x)) \equiv \forall x, \sim Q(x)$

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Q39 Translate each of the following statements into logical expressions using predicates, quantifiers & logical connectives.

1) No one is perfect
2) Not everyone is perfect
3) All of your friends are perfect
4) Atleast one of your friends is perfect
5) Everyone is your friend & is perfect
6) Not everyone is your friend or someone is not perfect

Solⁿ let $P(x)$: "x is perfect"
$Q(x)$: "x is your friend"
Domain : all people

Ans :- 1) No one is perfect.
$\equiv$ Everyone is not perfect.
$\Rightarrow \forall x \sim P(x)$

2) Not everyone is perfect.
$\equiv \sim \forall x \, P(x)$

3) $\forall x \, (Q(x) \rightarrow P(x))$ i.e.
$\equiv \forall x$ (if x is your friend then x is perfect)

4) $\exists x \, (Q(x) \wedge P(x))$

5) $\forall x \, (Q(x) \wedge P(x))$

6) Not everyone is your friend or someone is not perfect.

$(\neg \forall x \, Q(x)) \vee (\exists x \sim P(x))$.

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* We know that

$$\sim(\forall x, P(x)) \equiv \exists x, \sim P(x)$$

&

$$\sim(\exists x, P(x)) \equiv \forall x, \sim P(x)$$

So, write the negations of following.

$\exists x, P(x)$ ① "Some drivers do not obey the speed limit".

its negation is

$\forall x, \sim P(x)$ "all drivers obey the speed limit".

$\forall x, P(x)$ ② "All swedish movies are serious".

its negation is

$\exists x, \sim P(x)$ "Some swedish movies are not serious"

or $\exists$ "there are some swedish movies which are not series"

$\exists x, P(x)$ ③ "There is someone in the class who does not have good attitude".

its negation is

$\forall x, \sim P(x)$ "everyone in the class has a good attitude"

$\forall x, P(x)$ ④ "Every bird can fly"

$\exists x, \sim P(x)$ its negation is "some birds can't fly"

$\exists x, P(x)$ ⑤ "Some birds can talk".

its negation is

$\forall x, \sim P(x)$ "no bird can talk"

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☆ Every student in this class has taken a course in Java.

1) if domain D : students in this class

& $P(x)$: x has taken a course in Java

then sentence $\equiv \forall x, P(x)$

2) if domain D : all people

& $P(x)$: x has taken a course in Java

$Q(x)$: x is a student in this class

then sentence $\equiv \forall x \left(Q(x) \rightarrow P(x) \right)$

☆ Some student in this class has taken a course in Java.

HW

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Well formed Formula :-

1) A statement variable standing alone is a wff.
eg:- p, q
2) If p is a wff $\Rightarrow \sim p$ is a wff.
3) If p + q are wff $\Rightarrow p \wedge q, p \vee q, p \rightarrow q, p \leftrightarrow q$ are wff.
4) A string of symbols consisting of statement variables, connectives, + parenthesis is wff iff it can be produced by using Rules 1) 2) 3) finite time

eg:- 1) (p) is not a wff as p is placed inside $(\;)$.

2) $\sim p \wedge q$ is not wff as it can be read as $\sim(p \wedge q)$ or $(\sim p) \wedge q$.

3) $((p \rightarrow q))$ is not wff because of extra $(\;)$.

4) $((p \wedge q) \wedge) q)$ is not wff.

5) $((p \wedge q) \wedge pq)$ is not wff.

6) $(p \vee q) \Rightarrow (\wedge q)$ is not wff.