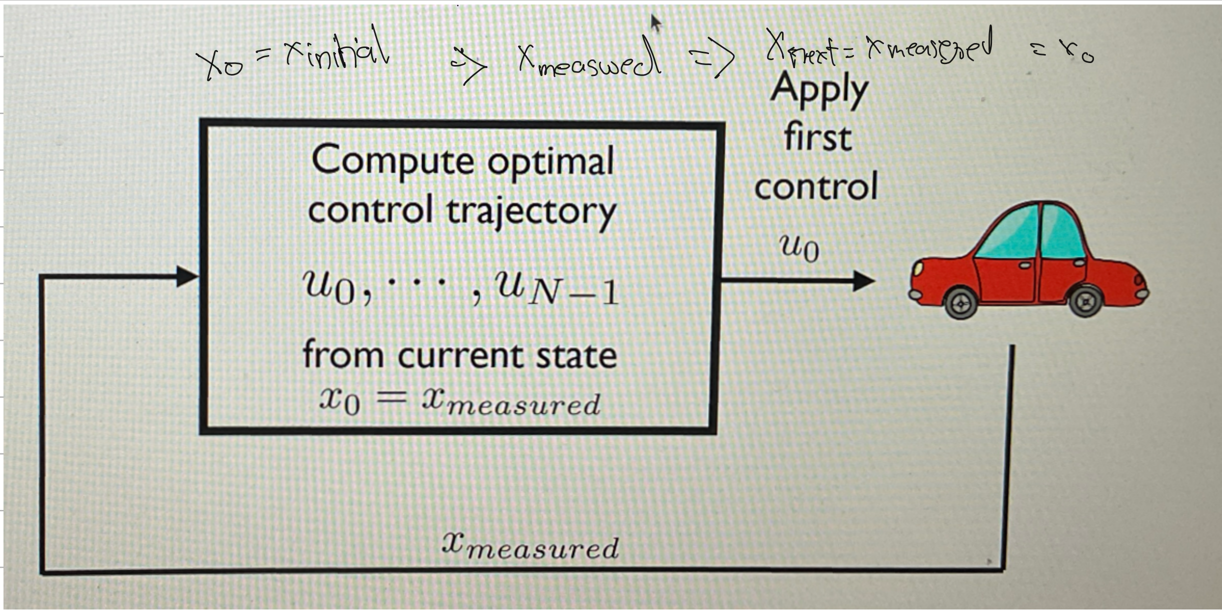
Model Predictive Control / Receding Horizon Control



Gradient estimation via Gaussian Smoothening:

Suppose we have a function $f(x)$, the gaussian smoothened form of $f(x)$ is given as;

$$f_\sigma(x) = \mathbb{E}_{u \sim N(0,\sigma^2)}[f(x+u)]$$

where $u \to$ noise vector
$0 \to$ mean
$\sigma^2 \to$ variance

ie; $f_\sigma(x)$ is the expected value of $f(x+u)$ where u is drawn from the gaussian distribution with 0 mean and σ^2 variance.

In integral form;

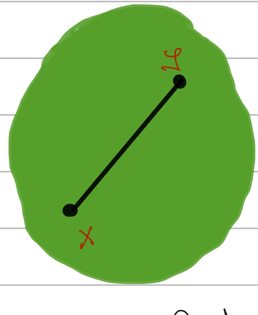
$$f_\sigma(x) = \int f(x+u)\frac{e^{-\frac{u^T u}{2\sigma^2}}}{\sqrt{(2\pi)^n}}du$$

Generally $f(x) \neq f_\sigma(x)$

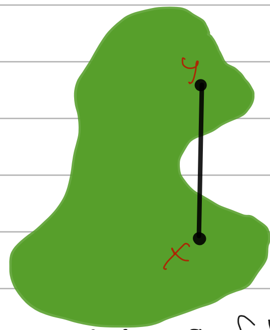
But if f(x) is a convex function then Jensen's inequality condition is valid; which states:

$$f(x) \leq f_\sigma(x)$$

The inequality indicates that the smoothed function has an upper bound to the original function if its convex.



convex function



non convex function.

Difference Boundaries:

The difference b/w $f(x)$ & $f_\sigma(x)$ is bounded by certain factors that determine how quickly the function changes:

Condition 1: $|f(x) - f(y)| \leq L_0 \|x-y\|$ then;

$$|f_\sigma(x) - f(x)| \leq \sigma . L_0 . \sqrt{n}$$

Condition 2: $|\nabla f(x) - \nabla f(y)| \leq L_1 \|x-y\|$ then;

$$|f_\sigma(x) - f(x)| \leq \frac{\sigma^2}{2} . L_1 . n$$

L_1 & L_0 → constants measure rate of change of f(x)
σ → smoothing control
n → dimension(x)

Policies have ∞ no. of variables as shown below

Policies $\longrightarrow$ computing optimal control for every initial state

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We will use this example through out the lecture

Let us have a state x;

$\boxed{X = \{0, 1, 2\}}$ & $\boxed{x_{n+1} = x_n + U_n}$

where u can be any thing & $x_{n+1} \in \{0,1,2\}$

So;

At $\boxed{x_0 = 0}$

$$x_{n+1} \Rightarrow 0 = 0 + U_0 = 0$$
$$\Rightarrow 1 = 0 + U_0 = 1$$
$$\Rightarrow 2 = 0 + U_0 = 2$$

ie; $\boxed{u_0 = \{0, 1, 2\}.}$

$\boxed{x_0 = 1}$

$$x_{n+1} \Rightarrow 0 = 1 + U_0 \Rightarrow U_0 = -1$$
$$1 = 1 + U_0 \Rightarrow U_0 = 0$$
$$2 = 1 + U_0 \Rightarrow U_0 = 1$$

ie;

$\boxed{x_0 = 2}$

$$x_{n+1} \Rightarrow 0 = 2 + U_0 \Rightarrow U_0 = -2$$
$$1 = 2 + U_0 \Rightarrow U_0 = -1$$
$$2 = 2 + U_0 \Rightarrow U_0 = 0$$

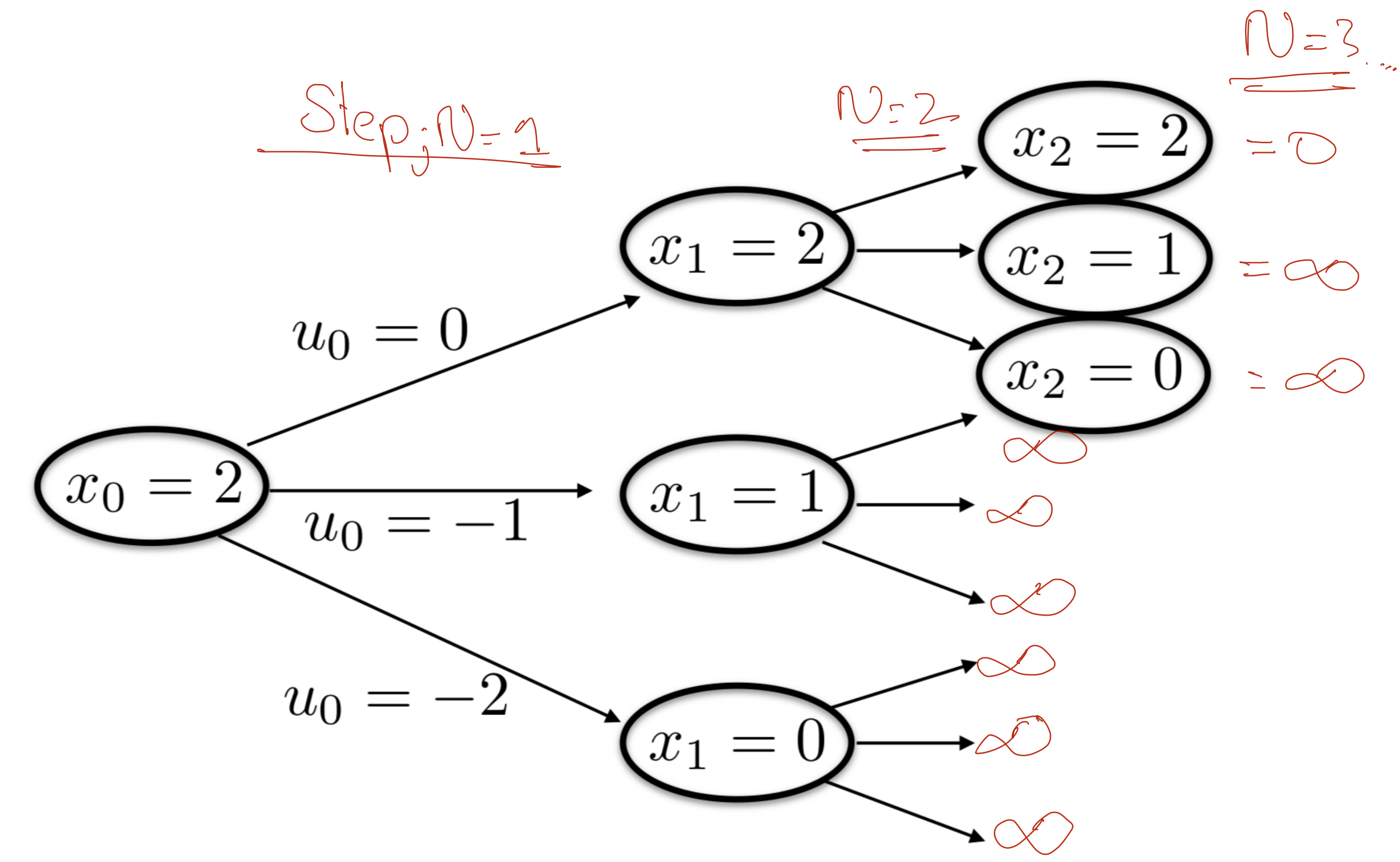
ie; $\boxed{U_0 = \{-2, -1, 0\}}$

Given function to minimize;

$$\min_{X} \sum_{n=0} (|U_n| + x_n^2) + g_2(x_2)$$

not $x = 2$ but $x_2 = 0$

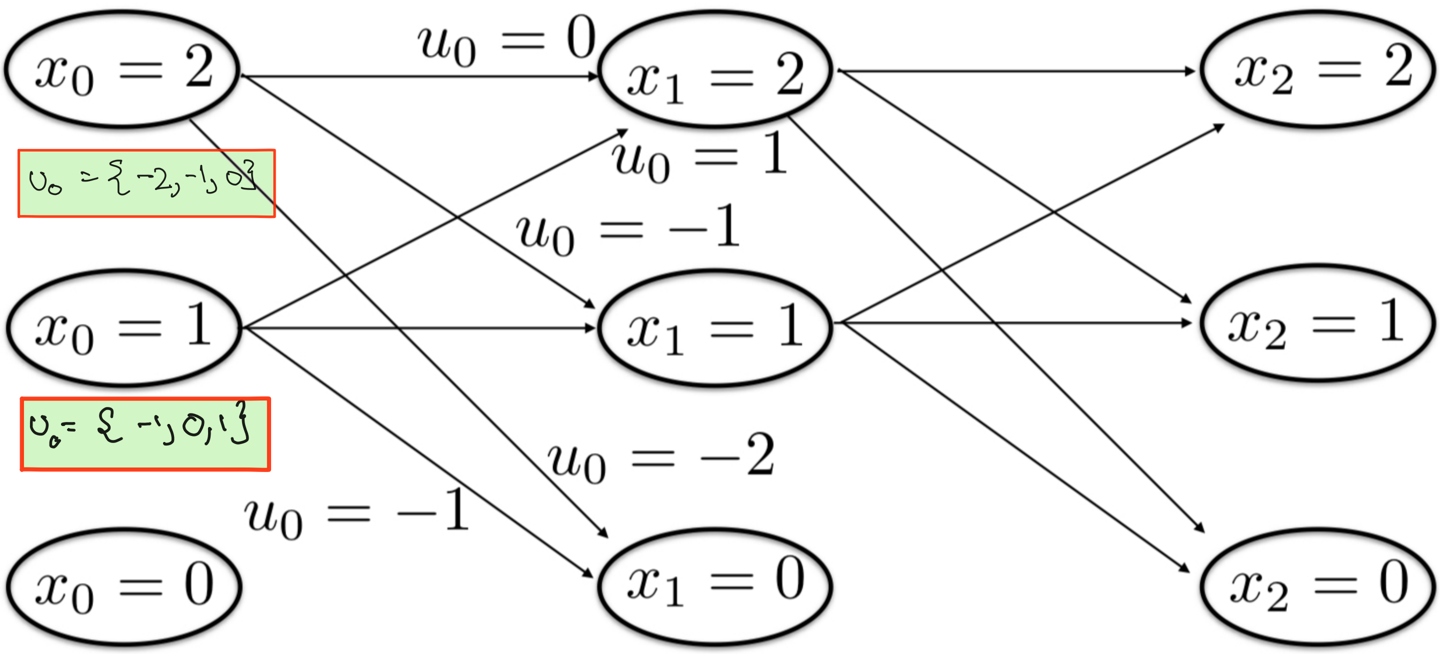
$$\& \quad g_2 = \begin{cases} 0 & \text{if } x_2 = 0 \\ \infty & \text{otherwise} \end{cases}$$



What will the outcomes be of them? →

Therefore the no. of variable span infinitely at the rate of 3^{N+1} (N: step).

Additionally there are also inter dependencies



Stage 0 — $f(x_0, u_0)$ → Stage 1 — $f(x_1, u_1)$ → Stage 2 →

So in conclusion the final stage is obtained by $\{(x_{N-1}, u_{N-1})\}$ where $N \in \infty$.

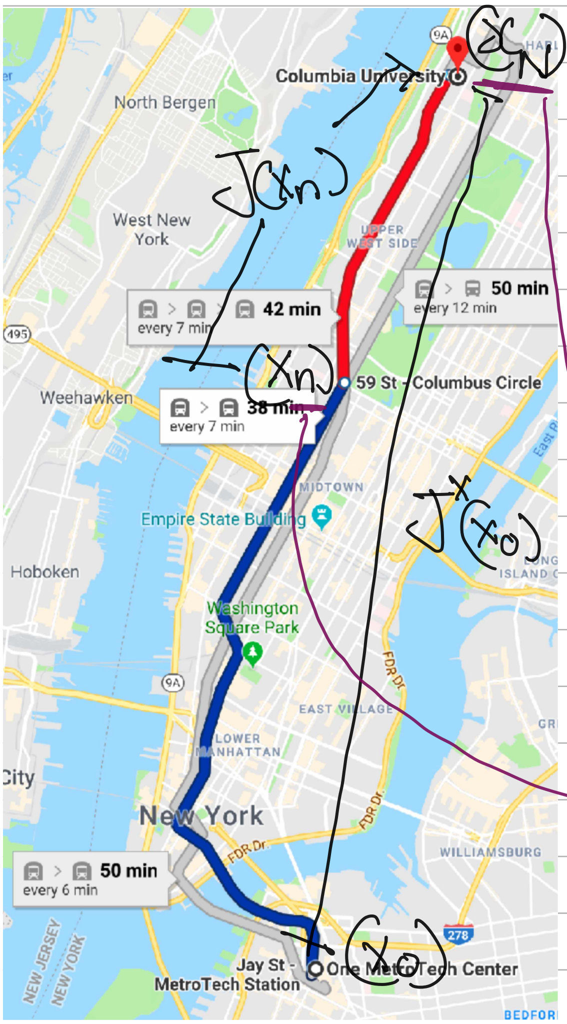
This leaves us with infinite horizon probleme and we have to find a way to navigate it.

Control Law: A function denoted as $\mu_k(x_k)$ which tells us what to do at stage k. It maps every state in x_k to a control action u_k.

Control Policy: Sequence of superset of all control laws; ie;

$$\pi = \{\mu_0(x_0), \mu_1(x_1), \mu_2(x_2), \ldots\ldots\ldots, \mu_N(x_N)\}$$

Cost to go: Consider the following figure.



The path plotted is the optimal route from jay street to colombia Uni. This is also called optimal cost / $J^*(x_0)$.

Now any point taken along that optimal route is the most optimal route to go to Colombia Uni. This is also called optimal cost to go. Denoted as $J_n(x_n)$. It is defined as:

$$J_n(x_n) = \min_{x} \sum_{i=n}^{N-1} g_i(x_i, u_i) + g_N(x_N)$$

Nomenclature:

$\mu_k^* \longrightarrow$ optimal control law at stage k

$\pi^* \longrightarrow$ optimal control policy for the problem

Bellman's Principle Of Optimality :-

Let $\{u_0^*, u_1^*, \cdots, u_{N-1}^*\}$ be an optimal control sequence for the original optimal control problem and assume that when we are using the sequence of control, a given state x_k occurs at time k. Consider the subproblem where we start at x_k at time k and wish to minimize the *cost-to-go* from time k to time N

$$\sum_{i=k}^{N-1} g_i(x_i, u_i) + g_N(x_N)$$

Then the truncated control sequence $\{u_k^*, \cdots, u_{N-1}^*\}$ is optimal for this subproblem.

One way to tackle the infinite variable problems as we have seen above $\longrightarrow$ $\boxed{\text{DYNAMIC PROGRAMMING}}$

In dynamic programming : We start from end (stage N) of the problem and compute optimal policies of the subproblems.

Dynamic Programming :-

For every initial state x_0, the optimal cost $J^*(x_0)$ of the optimal control problem is equal to $J_0(x_0)$ (i.e. the optimal cost to go from x_0) which is computed backward in time from stage $N-1$ to stage 0 using the following recursion:

$$J_N(x_N) = g_N(x_N)$$
$$J_k(x_k) = \min_{u_k} g_k(x_k, u_k) + J_{k+1}(f(x_k, u_k))$$

Furthermore, if the control laws $u_k^* = \mu_k^*(x_k)$ minimize the cost-to-go for each x_k and k, the policy $\pi^* = \{\mu_0^*, \cdots, \mu_{N-1}^*\}$ is optimal.

$\pi^* \rightarrow$ set of optimal control vals $\longrightarrow \{u_0^*, u_1^*, \ldots u_{N-1}^*\}$

Note : $N-1$ not N

Deterministic Case

Filled in green

Fill the table backwards with DP:

Stage 0			Stage 1			Stage 2	
x_0	J_0^*	u_0^*	x_1	J_1^*	u_1^*	x_2	J_2^*
0	2	0	0	2	2	0	∞
1	3	0	1	2	1	1	∞
2	4	-2	2	4	0	2	0

Goal: $\min_{x} \sum_{n=0}^{N} (|u_n| + x_n^2) + g_n x_n$

where $g_n = 0$ at $x_2 = 0$ (inf otherwise)

Constraints: $x_{n+1} = x_n + u_n$

For each x_n, we
- enumerate possible u_n based on x_n
- calculate resulting x_2
- evaluate cost defined as (for each control):

$$\text{cost} = (|u_n| + x_n^2) + J_{n+1}^*(x_{n+1})$$

- choose the control that minimizes the cost and fill the table accordingly.

Solution:

$J_2^*(x_2 = 0) = \infty$ # By Function Definition
$J_2^*(x_2 = 1) = \infty$ #
$J_2^*(x_2 = 2) = 0$ #

At $x_1 = 0$; $u \in \{0, 1, 2\} \Rightarrow$

$$|0| + 0^2 + \overbrace{J_2(0+0)}^{\infty} = \infty$$
$$|1| + 0^2 + J_2(1+0) = \infty$$
$$|2| + 0^2 + \underbrace{J_2(2+0)}_{0} = 2$$

Hence at $x_1 = 0$; $J_1^*(0) = 2$ & $u_1^* = 2$

At $x_1 = 1$, $u \in \{-1, 0, 1\}$ $\Rightarrow$ $|-1| + 1^2 + \overbrace{J_2(-1+1)}^{\infty} = \infty$

$$0 + 1^2 + J_2(0+1) = \infty$$

$$1 + 1^2 + \underbrace{J_2(1+1)}_{0} = 2$$

Hence at $x_1 = 1$; $J_1^*(1) = 2$ & $u_1^* = 1$

At $x_1 = 2$, $u \in \{-2, -1, 0\}$ $\Rightarrow$ $|-2| + (2)^2 + \overbrace{J_2(-2+2)}^{\infty} = \infty$

$$|-1| + 2^2 + J(-1+2) = \infty$$

$$|0| + 2^2 + J(0+2) = 4$$

Hence at $x_1 = 2$; $J_1^*(2) = 4$ & $u_1^* = 0$

At $x_0 = 0$; $u \in \{0, 1, 2\}$ $\Rightarrow$ $|0| + 0^2 + J_1(\overbrace{0+0}^{x_1 =}) = 2$

$$\Rightarrow |1| + 0^2 + J_1(0+1) = 3$$

$$|2| + 0^2 + J_1(0+2) = 6$$

Hence at $x_0 = 0$; $J_0^*(0) = 2$ & $u_0^* = 0$

At $x_0 = 1$; $u \in \{-1, 0, 1\}$ $\Rightarrow$ $|-1| + 1^2 + J_1(-1+1) = 4$

$$|0| + 1^2 + J_1(0+1) = 3$$

$$|1| + 1^2 + J_1(1+1) = 6$$

Hence at $x_0 = 1$; $J_0^*(1) = 3$ & $u_0^* = 0$

At $x_0 = 2$; $u \in \{-2, -1, 0\}$ $\Rightarrow$ $|-2| + 2^2 + J_1(0) = 4$

$$|-1| + 2^2 + J_1(1) = 5$$

$$|0| + 2^2 + J_1(2) = 8$$

Hence at $x_0 = 2$; $J_0^*(2) = 4$ & $u_0^* = -2$

Q2)

Consider the following dynamical system

$$x_{n+1} = \begin{cases} -x_n + 1 + u_n & \text{if } -2 \leq -x_n + 1 + u_n \leq 2 \\ 2 & \text{if } -x_n + 1 + u_n > 2 \\ -2 & \text{else} \end{cases}$$

where $x_n \in \{-2, -1, 0, 1, 2\}$ and $u_n \in \{-1, 0, 1\}$, and the cost function

$$J = \left(\sum_{k=0}^{2} 2|x_k| + |u_k| \right) + x_3^2 \tag{1}$$

Use the dynamic programming algorithm to solve the finite horizon optimal control problem that minimizes J. Show the different steps of the algorithms and present the results in a table including the cost to go and the optimal control at every stage.

	Stage 0			Stage 1			Stage 2			Stage3	
	x_0	J_0^*	u_0^*	x_1	J_1^*	u_1^*	x_2	J_2^*	u_2^*	x_3	J_3^*
-2	-2	10	0	-2	9	0	-2	8	0	-2	4
	-1	6	-1	-1	5	-1	-1	4	-1	-1	1
0	0	3	0/-1	0	2	0/-1	0	1	-1	0	0
	1	4	0	1	3	0	1	2	0	1	1
2	2	7	1	2	6	1	2	5	0/1	2	4

Given: $\left(\sum_{k=0}^{2} 2|x_k| + |u_k|\right) + x_3^2$ $\quad u \in \{-1, 0, 1\}$ $\quad x \in \{-2, -1, 0, 1, 2\}$

Now lets start with $J_3^*(x_3) = x_3^2 \therefore J_3^* = \{4, 1, 0, 1, 4\}$

$$\boxed{\text{Cost} = 2|x_k| + |u_k| + J_{k+1}^*(x_{k+1})}$$

At $x_2 = -2$ $\quad u \in \{-1, 0, 1\}$

$\text{cost}(x_2 = -2) = 2|(x_2)| + |u_2| + J_3^*(x_3)$

But what is x_3? $\quad -2$

At $u_2 = -1$ $\quad x_3 = -(-2) + \cancel{(-1)} + \cancel{1} = 2$ $\quad$ Hence $x_3 = 2$ at $(u_2 = -1)$

$u_2 = 0$ $\quad x_3 = -(-2) + 1 + 0 = 3 \Rightarrow 2$ $\quad x_3 = 2$ at $(u_2 = 0)$

$u_2 = 1$ $\quad x_3 = -(-2) + 1 + 1 = 4 \Rightarrow 2$ $\quad x_3 = 2$ at $(u_2 = 1)$

Now, cost $= 2|(-2)| + |(-1)| + J_3(2) = 4 + 1 + 4 = 9$

$= 2 \times |-2| + |0| + J_3(2) = 4 + 0 + 4 = 8$

$= 2 \times |(-2)| + |1| + J_3(2) = 4 + 1 + 4 = 9$

Hence at $x_2 = -2$ $\quad u_2^* = 0$ $\quad$ & $J_2^* = 8$

At $x_2 = -1$ $\quad u \in (-1, 0, 1)$

At $u_2 = -1$ $\quad x_3 = 1 + 1 - 1 = 1$ $\quad$ Hence $x_3 = 1$ at $(u_2 = -1)$

$u_2 = 0$ $\quad x_3 = 1 + 1 + 0 = 2$ $\quad x_3 = 2$ at $(u_2 = 0)$

$u_2 = 1$ $\quad x_3 = 1 + 1 + 1 = 3 \Rightarrow 2$ $\quad x_3 = 2$ at $(u_2 = 1)$

Now, cost $= 2|(-1)| + |(-1)| + J_3(1) = 2 + 1 + 1 = 4$

$= 2|(-1)| + |(0)| + J_3(2) = 2 + 0 + 4 = 6$

$= 2|(-1)| + |1| + J_3(3) = 2 + 1 + 4 = 7$

Hence at $x_2 = -1$ $\quad u_2^* = -1$ $\quad$ & $J_2^* = 4$

At $x_2 = 0$ $\quad u \in \{-1, 0, 1\}$

At $u_2 = -1$ $\quad x_3 = 0 + 1 - 1 = 0$ $\quad$ Hence $x_3 = 0$ at $u_2 = -1$

$u_2 = 0$ $\quad x_3 = 0 + 1 + 0 = 1$ $\quad x_3 = 1$ at $u_2 = 0$

$u_2 = 1$ $\quad x_3 = 0 + 1 + 1 = 2$ $\quad x_3 = 2$ at $u_2 = 1$

Now, cost $= 2 \times 0 + |-1| + J_3(0) = 1$

$= 2 \times 0 + |0| + J_3(1) = 1$

$= 2 \times 0 + |1| + J_3(2) = 5$

Hence at $x_2 = 0$ $\quad u_2^* = -1$ $\quad$ & $J_2^* = 1$

At $x_2 = 1 \quad u \in \{-1, 0, 1\}$

At $u_2 = -1 \quad x_3 = -1 + 1 - 1 = -1$ Hence $x_3 = -1$ at $u_2 = -1$

$u_2 = 0 \quad x_3 = -1 + 1 + 0 = 0$ $x_3 = 0$ at $u_2 = 0$

$u_2 = 1 \quad x_3 = \cancel{-1} + \cancel{1} + 1 = 1$ $x_3 = 1$ at $u_2 = 1$

Now, cost $= 2|1| + |-1| + J_3(-1) = 2 + 1 + 1 = 4$

$= 2|1| + 0 + J_3(0) = 2 + 0 + 0 = 2$

$= 2|1| + |1| + J_3(1) = 2 + 1 + 1 = 4$

Hence at $x_2 = 1 \quad u_2^* = 0$ & $J_2^* = 2$

At $x_2 = 2 \quad u \in \{-1, 0, 1\}$

At $u_2 = -1 \quad x_3 = -(2) + \cancel{1} \cancel{-1} = -2$ Hence $x_3 = -2$ at $u_2 = -1$

$u_2 = 0 \quad x_3 = -(2) + 1 + 0 = -1$ $x_3 = -1$ at $u_2 = 0$

$u_2 = 1 \quad x_3 = \cancel{-(2)} + \cancel{1} + \cancel{1} = 0$ $x_3 = 0$ at $u_2 = 1$

Now, cost $= 2|2| + |-1| + J(-2) = 4 + 1 + 4 = 9$

$= 2|2| + |0| + J(-1) = 4 + 0 + 1 = 5$

$= 2|2| + |1| + J(0) = 4 + 1 + 0 = 5$

Hence at $x_2 = 2$; $u_2^* = 0/1$; $J_2^* = 5$

At $x_1 = -2 \quad u \in \{-1, 0, 1\}$

At $u_1 = -1 \quad x_2 = -(-2) + \cancel{1} \cancel{-1} = 2$ Hence $x_2 = 2$ at $u_1 = -1$

$u_1 = 0 \quad x_2 = -(-2) + 1 + 0 = 3 \Rightarrow 2$ $x_2 =$ at $u_1 = 0$

$u_1 = 1 \quad x_2 = -(-2) + 1 + 1 = 4 \Rightarrow 2$ $x_2 =$ at $u_1 = 1$

Now, cost $= 2|-2| + |-1| + J(2) = 4 + 1 + 5 = 10$

$= 2|-2| + |0| + J(2) = 4 + 0 + 5 = 9$

$= 2|-2| + |1| + J(2) = 4 + 1 + 5 = 10$

Hence at $x_1 = -2$; $u_1^* = 0$; $J_1^* = 9$

8
4
1
2
5

At $x_1 = -1$ $\quad u \in \{-1, 0, 1\}$

At $u_1 = -1$ $\quad x_2 = -(-1) + \cancel{1} \cancel{-1} = 1$ $\quad$ Hence $x_2 = 1$ at $u_1 = -1$

$u_1 = 0$ $\quad x_2 = -(-1) + 1 + 0 = 2$ $\quad x_2 = 2$ at $u_1 = 0$

$u_1 = 1$ $\quad x_2 = -(-1) + 1 + 1 = 3 \Rightarrow 2$ $\quad x_2 = 2$ at $u_1 = 1$

Now, cost $= 2\overset{2}{|-1|} + \overset{1}{|-1|} + J^2(1) = 2+1+2 = 5$

$= 2\overset{2}{|-1|} + \overset{0}{|0|} + J^5(2) = 2+0+5 = 7$

$= 2\overset{2}{|-1|} + \overset{1}{|1|} + J^5(2) = 2+1+5 = 8$

Hence at $x_1 = -1$; $u_1^* = -1$; $J_1^* = 5$

At $x_1 = 0$ $\quad u \in \{-1, 0, 1\}$

At $u_1 = -1$ $\quad x_2 = -(\cancel{0}) + \cancel{1} \cancel{-1} = 0$ $\quad$ Hence $x_2 = 0$ at $u_1 = -1$

$u_1 = 0$ $\quad x_2 = -(\cancel{0}) + 1 + 0 = 1$ $\quad x_2 = 1$ at $u_1 = 0$

$u_1 = 1$ $\quad x_2 = -(\cancel{0}) + 1 + 1 = 2$ $\quad x_2 = 2$ at $u_1 = 1$

Now, cost $= 2\overset{0}{|0|} + \overset{1}{|-1|} + J^1(0) = 0+1+1 = 2$

$= 2\overset{0}{|0|} + \overset{0}{|0|} + J^2(1) = 0+2 = 2$

$= 2\overset{0}{|0|} + \overset{1}{|1|} + J^5(2) = 0+1+5 = 6$

Hence at $x_1 = 0$; $u_1^* = -1/0$; $J_1^* = 2$

At $x_1 = 1$ $\quad u \in \{-1, 0, 1\}$

At $u_1 = -1$ $\quad x_2 = -(1) + \cancel{1} \cancel{-1} = -1$ $\quad$ Hence $x_2 = -1$ at $u_1 = -1$

$u_1 = 0$ $\quad x_2 = -(\cancel{1}) + \cancel{1} + 0 = 0$ $\quad x_2 = 0$ at $u_1 = 0$

$u_1 = 1$ $\quad x_2 = -(\cancel{1}) + \cancel{1} + 1 = 1$ $\quad x_2 = 1$ at $u_1 = 1$

Now, cost $= 2\overset{2}{|1|} + \overset{1}{|-1|} + J^4(-1) = 2+1+4 = 7$

$= 2\overset{2}{|1|} + \overset{0}{|0|} + J^1(0) = 2+0+1 = 3$

$= 2\overset{2}{|1|} + \overset{1}{|1|} + J^2(1) = 2+1+2 = 5$

Hence at $x_1 = 1$; $u_1^* = 0$; $J_1^* = 3$

At $x_1 = 2$ $\quad u \in \{-1, 0, 1\}$

At $u_1 = -1$ $\quad x_2 = -(2) \cancel{+1} \cancel{-1} = -2$ $\quad$ Hence $x_2 = -2$ at $u_1 = -1$

$u_1 = 0$ $\quad x_2 = -(2) + 1 + 0 = -1$ $\quad x_2 = -1$ at $u_1 = 0$

$u_1 = 1$ $\quad x_2 = \cancel{-(2)} + \cancel{1} + \cancel{1} = 0$ $\quad x_2 = 0$ at $u_1 = 1$

Now, cost $= \overset{4}{2|2|} + \overset{1}{|-1|} + J^8(-2) = 4+1+8 = 13$

$= \overset{4}{2|2|} + \overset{0}{|0|} + J^4(-1) = 4+0+4 = 8$

$= \overset{4}{2|2|} + \overset{1}{|1|} + J^1(0) = 4+1+1 = 6$

Hence at $x_1 = 2$; $u_1^* = 1$; $J_1^* = 6$

8

4

1

2

5

At $x_0 = -2$ $\quad u \in \{-1, 0, 1\}$

At $u_0 = -1$ $\quad x_1 = -(-2) \cancel{+1} \cancel{-1} = 2$ $\quad$ Hence $x_1 = 2$ at $u_0 = -1$

$u_0 = 0$ $\quad x_1 = -(-2) + 1 + 0 = 3 \Rightarrow 2$ $\quad x_1 = 2$ at $u_0 = 0$

$u_0 = 1$ $\quad x_1 = -(-2) + 1 + 1 = 4 \Rightarrow 2$ $\quad x_1 = 2$ at $u_0 = 1$

Now, cost $= 2|-2| + |-1| + J(2) = 11$

$= 2|-2| + |0| + J(2) = 10$ $\quad$ $(-2)\ 9$

$= 2|-2| + |1| + J(2) = 11$ $\quad$ $(-1)\ 5$

Hence at $x_0 = -2$; $u_0^* = 0$; $J_0^* = 10$ $\quad$ $(0)\ 2$

$(1)\ 3$

At $x_0 = -1$ $\quad u \in \{-1, 0, 1\}$

At $u_0 = -1$ $\quad x_1 = -(-1) + \cancel{1} \cancel{-1} = 1$ $\quad$ Hence $x_1 = 1$ at $u_0 = -1$ $\quad$ $(2)\ 6$

$u_0 = 0$ $\quad x_1 = -(-1) + 1 + 0 = 2$ $\quad x_1 = 2$ at $u_0 = 0$

$u_0 = 1$ $\quad x_1 = -(-1) + 1 + 1 = 3 \Rightarrow 2$ $\quad x_1 = 2$ at $u_0 = 1$

Now, cost $= 2|-1| + |-1| + J(1) = 6$

$= 2|-1| + |0| + J(2) = 8$

$= 2|-1| + |1| + J(2) = 9$

Hence at $x_0 = -1$; $u_0^* = -1$; $J_0^* = 6$

At $x_0 = 0$ $\quad u \in \{-1, 0, 1\}$

At $u_0 = -1$ $\quad x_1 = -(0) \cancel{+1} \cancel{-1} = 0$ $\quad$ Hence $x_1 = 0$ at $u_0 = -1$

$u_0 = 0$ $\quad x_1 = -(0) + 1 + 0 = 1$ $\quad x_1 = 1$ at $u_0 = 0$

$u_0 = 1$ $\quad x_1 = -(0) + 1 + 1 = 2$ $\quad x_1 = 2$ at $u_0 = 1$

Now, cost $= 2|0| + |-1| + J(0) = 3$

$= 2|0| + |0| + J(1) = 3$

$= 2|0| + |1| + J(2) = 7$ $\quad 9$

Hence at $x_0 = 0$; $u_0^* = -1/0$; $J_0^* = 3$ $\quad 5$

At $x_0 = 1$ $\quad u \in \{-1, 0, 1\}$ $\quad 2$

At $u_0 = -1$ $\quad x_1 = -(1) \cancel{+1} \cancel{-1} = -1$ $\quad$ Hence $x_1 = -1$ at $u_0 = -1$ $\quad 3$

$u_0 = 0$ $\quad x_1 = \cancel{-(1)} \cancel{+1} + 0 = 0$ $\quad x_1 = 0$ at $u_0 = 0$

$u_0 = 1$ $\quad x_1 = \cancel{-(1)} \cancel{+1} + 1 = 1$ $\quad x_1 = 1$ at $u_0 = 1$ $\quad 6$

Now, cost $= 2|1| + |-1| + J(-1) = 8$

$= 2|1| + |0| + J(0) = 4$

$= 2|1| + |1| + J(1) = 6$

Hence at $x_0 = 1$; $u_0^* = 0$; $J_0^* = 4$

At $x_0 = 2$ $\qquad u \in \{-1, 0, 1\}$

At $u_0 = -1$ $\qquad x_1 = -(2) + \cancel{1} - \cancel{1} = -2$

$u_0 = 0$ $\qquad x_1 = -(2) + 1 + 0 = -1$

$u_0 = 1$ $\qquad x_1 = -\cancel{(2)} + \cancel{1} + \cancel{1} = 0$

Hence $x_1 = -2$ at $u_0 = -1$

$x_1 = -1$ at $u_0 = 0$

$x_1 = 0$ at $u_0 = 1$

9

5

2

3

6

Now, cost $= 2\overset{4}{|2|} + \overset{1}{|-1|} + \overset{9}{J(-2)} = 14$

$= 2\overset{4}{|2|} + \overset{0}{|0|} + \overset{5}{J(-1)} = 9$

$= 2\overset{4}{|2|} + \overset{1}{|1|} + \overset{2}{J(0)} = 7$

Hence at $x_0 = 2$; $u_0^* = 1$; $J_0^* = 7$

Finite-Horizon optimal Control:

$$x_{n+1} = f(x_n, u_n, w_n)$$ Uncertainity

Then the minimisation ~~becomes an~~ of the cost has to be rewritten as the ~~expectation~~ expected value of the minimisation of the cost.

$$\min_{u_0, u_1, \dots, u_{N-1}} \sum_{i=0}^{N-1} g_i(x_i, u_i) + g_N(x_N)$$

$$\min_{\mu_0(x_0), \mu_1(x_1) \dots \mu_{N-1}(x_{N-1})} \mathbb{E}\left(\sum_{i=0}^{N-1} g_i(x_i, u_i, w_i) + g_N(x_N)\right)$$

$x_n \longrightarrow$ coffee in week n (in hand)

$u_n \longrightarrow$ coffee ~~to be~~ ordered ~~for week $n+1$~~ at week n.

$w_n \longrightarrow$ client demand / coffee used.

(non negative) inventory $\longrightarrow \max(0, x_n + u_n - w_n)$

constraint (max storage capacity) = 2 ie; $x_n + u_n \leq 2$

storage cost for period n. $(x_n + u_n - w_n)^2$

$\pi^* = \{u_0^*, u_1^*, \ldots, u_{N-1}^*\}$ | $\mathbb{E}\left(\sum_{i=k}^{N-1} g_i(x_i, u_i, w_i) + g_N(x_N)\right)$

$$J_k(x_k) = \min_{u_k} \mathbb{E}_{w_k}\left(g_k(x_k, u_k) + J_{k+1}(f(x_k, u_k))\right)$$