MATH MINI PROJECT

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SEC :- D

BRANCH :- MECHANICAL

Introduction :-

In vector calculus, and more generally differential geometry, Stokes's theorem is statement about integration of differential forms on manifolds, which both simplifies and generalizes several theorems from vector calculus. Stokes's Theorems says that integral of differential form w over boundary of some orientable manifold Ω is equal to integral of its exterior derivative dw over the whole of Ω i.e.

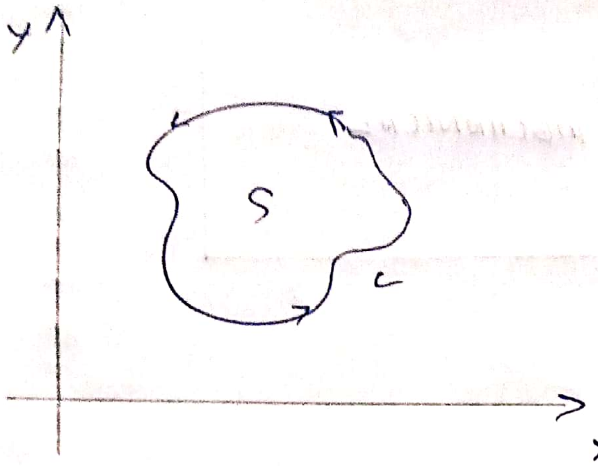
$$\left[\int_{\partial\Omega} w = \int_{\Omega} dw \right]$$

Defination :-

If $\vec{F}$ is any differential vector point function and S is a surface bounded by a curve C then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds$$

where $\hat{n}$ is unit normal vector at any point on S is drawn in the sence in which a right hand screw would move when rotated in the sense of description of C



Note :-

(I) where
$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(II) $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$

Formula :-

$$\boxed{\oint_C \vec{F}.d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}.\hat{n}\,ds}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is position vector of any point on S and $\hat{n}$ is unit outward normal vector at point whose P.V. is $\vec{r}$
[P.V. = Position vector]

Let $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$, $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$

where α, β, γ are direction angle of $\hat{n}$.

$$\therefore \vec{\nabla}\times\vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

$$\vec{F}.d\vec{r} = f\,dx + g\,dy + h\,dz$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\, ds$$

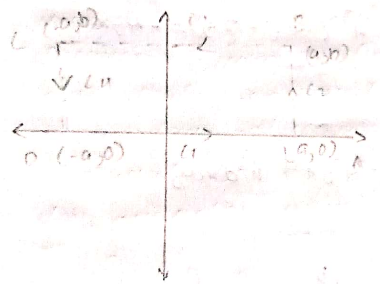
$$\oint_C (f\,dx + g\,dy + h\,dz) = \iint \Big[\Big(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial z}\Big) dy\,dz + \Big(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\Big) dz\,dx + \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\Big) dx\,dy\Big]$$

— x —

Examples :-

Q verify stoki theorem for $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$.

Ans



Here C is consist four side C_1, C_2, C_3, C_4 and S is surface bounded by curve C let $AB - C_2$, $BC = C_3$, $CD - C_4$, $OA - C_1$

By stokes theorem :-

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, ds \quad -(1)$$

since $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}$

Now, $\vec{F}\cdot d\vec{r} = [(x^2+y^2)\hat{i} - 2xy\hat{j}][dx\hat{i} + dy\hat{j}]$

$= (x^2+y^2)dx - 2xy\,dy$

on line C_1 :- $y=0 \Rightarrow dy = 0$ and x varies from $x=a$ to $x=0$

on line C_2 : $x=a \Rightarrow dx = 0$ and y varies from $y=0$ to $y=b$

on line C_3 : $y=b \Rightarrow dy=0$ and x varies from $x=a$ to $x=-a$

on line C_4 : $x=-a \Rightarrow dx=0$ & y varies from $y=b$ to $y=0$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2 dx + \int_0^b -2ay\,dy + \int_a^{-a}(x^2+b^2)\,dx + \int_b^0 2ay\,dy$$

$$= \frac{1}{3}(a^3+a^3) - ab^2 + \frac{1}{3}(-a^3-a^3) + b^2(-a-a) + (a\cdot b)$$

$$= \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

$$= -4ab^2 \quad -(2)$$

Now,

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= i(0) - j(0) + \hat{k}(-2y-2y) = -4y\hat{k}$$

$$\therefore \iint_S \text{curl } \vec{F}\cdot\hat{n}\,ds = \int_{x=-a}^{a}\int_{y=0}^{b} (-4y\hat{k})\cdot\hat{k}\,dx\,dy$$

$$= -4\int_{-a}^{a}\left[\frac{y^2}{2}\right]_0^b dy = -2\int_{-a}^{a} b^2 dx$$

$$= -2b^2[x]_{-a}^{a} = -2b^2[a-(-a)]$$

$$= -4ab^2 \quad -(3)$$

from (2) = (3)

$\therefore$ stokes theorem verified.

Q Evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ where C is the curve $x^2+y^2=1$, $z=y^2$.

Ans

$\oint_C (yz\,dx + zx\,dy + xy\,dz)$

$= \oint (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

$= \oint \vec{F}\cdot d\vec{r}$ where $\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= \iint \vec{\nabla}\times\vec{F}\cdot\hat{n}\,ds$ $\quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

(Stoke's theorem)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zy & zx & xy \end{vmatrix} = \vec{0}$$

$\therefore \iint \vec{0}\cdot\hat{n}\,ds = 0$

$\therefore \oint_C (yz\,dx + xy\,dy + xy\,dz) = 0$

Q Use Stoke's theorem evaluate $\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ & $(0,0,6)$

Sol^n we have

$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\text{curl } \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i} + \hat{k}$$

Also eq^n of plane through A, B, C (Fig 2.18) is

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1 \quad \text{or} \quad 3x + 2y + z = 6$$

vector $\bar{N}$ normal to this plane is

$$\bar{\nabla}(3x + 2y + z - 6) = 3\hat{i} + 2\hat{j} + \hat{k}$$

$$\therefore \quad \hat{N} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$$

So, $\int_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz = \int_C \bar{F} \cdot d\bar{R}$

$= \int_C \text{curl } \bar{F} \cdot \hat{N}\, ds$ where S is triangle ABC

$$= \int_S (2\hat{i} + \hat{k}) \cdot \left(\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}\right) ds = \frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$= \frac{7}{\sqrt{14}} (\text{Area of } \Delta ABC) = \frac{7}{\sqrt{14}} \cdot 3\sqrt{14} = 21$$

Q If $\bar{F} = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$ and S is surface of the paraboloid $2z = x^2+y^2$ bounded by $z=2$, evaluate $\iint_S (\bar{\nabla}\times\bar{F}).ds$ using Stokes theorem

Solution

$$I = \iint_S \bar{\nabla}\times\bar{F} \cdot d\bar{S} = \oint_C \bar{F}.d\bar{R}$$

[by Stokes theorem]

$$I = \oint_C \bar{F}.d\bar{R} = \oint_C (3x\hat{i} - xz\hat{j} + yz^2\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (3y\,dx - xz\,dy + yz^2 dz)$$

$$= \int_0^{2\pi} [6\sin\theta(-2\cos\theta\,d\theta) - 4\cos\theta(2\cos\theta\,d\theta) + 8\sin\theta\,(0)]$$

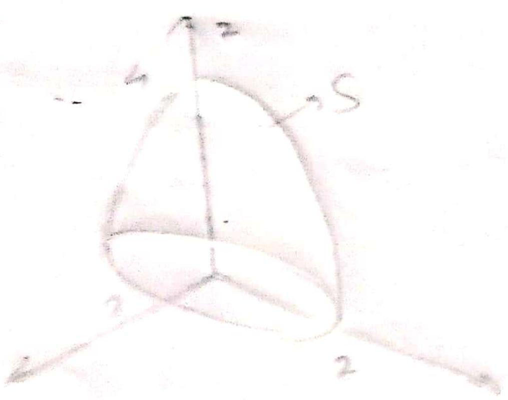
$$= -4\int_0^{2\pi} (12\sin^2\theta + 8\cos^2\theta)\,d\theta$$

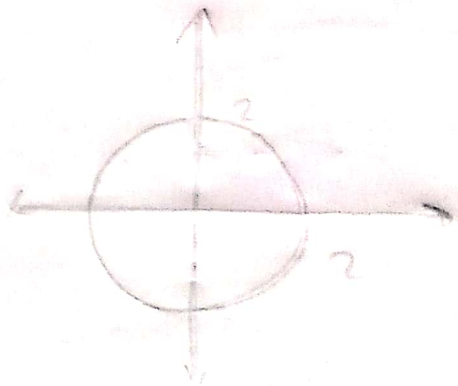
$$= -4\left[12\cdot\frac{1}{2}\frac{\pi}{2} + 8\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = -20\pi.$$

Q Let S be the portion of the paraboloid $z=4-x^2-y^2$ that lies above the plane $z=0$ and let $\bar{F}$ be the vector field $F = (z-y)\hat{i} + (x+z)\hat{j} - (e^{xyz}\cos y)\hat{k}$.

Q Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$ and let $\bar{F}$ be the vector field $\bar{F} = (z - y)\hat{i} + (x + z)\hat{j} - (e^{xyz}\cos y)\hat{k}$.

Ans





The curve in this case is circle of radius 2 centered at the origin on the xy plane.

we can parametrize as:

$$z = 4 - x^2 - y^2, \quad z = 0 \Rightarrow x^2 + y^2 = 4$$

$$C = \{ (x, y, z) : x^2 + y^2 = 4, z = 0 \}$$

$$r(\theta) = \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = 0 \end{cases} \quad 0 \le \theta \le 2\pi$$

$\therefore$ we have

$$\vec{F} = (z - y)\hat{i} + (x + z)\hat{j} - (e^{xyz}\cos y)\hat{k}$$

$$F(\bar{r}(\theta)) = -2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j} - \cos(2\sin\theta)\,\hat{k}$$

$$\bar{r}(\theta) = -2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j}$$

$$F(\bar{r}(\theta)) \cdot \bar{r}\,\theta = 4\sin^2\theta + 4\cos^2\theta = 4(1) = 4$$

Now,

$$I = \int_0^{2\pi} \vec{F}\,\vec{r}(\theta) \cdot \vec{r}(\theta)\,d\theta$$

$$= \int_0^{2\pi} 4\,d\theta$$

$$= [4\theta]_0^{2\pi} = 4[(2\pi) - (0)]$$

$$= 8\pi$$

Q Use Stoke's theorem to evaluate $\int \bar{F}\cdot d\bar{r}$ where $\bar{F} = z^2\hat{i} + y^2\hat{j} + x\hat{k}$ and C is the triangle with vertices (1,0,0), (0,1,0) and (0,0,1) with counter clockwise rotation.

Ans

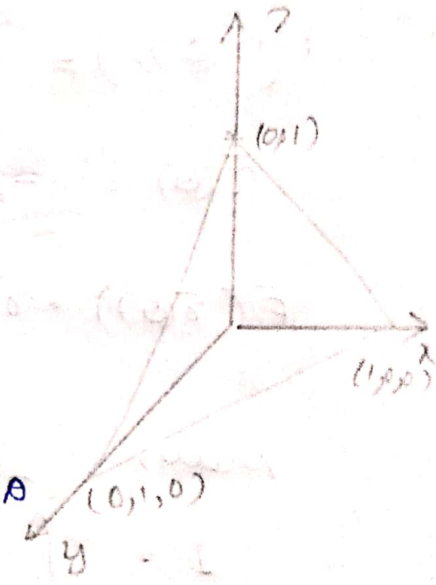
Now

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = 2z\hat{j} - \hat{j} = (2z-1)\hat{j}$$

Now the plane from the vertices is,

$$x + y + z = 1 \Rightarrow z = g(x,y) = 1 - x - y$$

using stokes Theorem,



$$\int_C \bar{F}\cdot d\bar{r} = \iint \text{curl}\,\bar{F}\cdot d\bar{s}$$

$$= \iint_S (2z-1)\,\hat{j}\cdot d\bar{s}$$

$$= \iint_D (2z-1)\hat{j}\cdot\frac{\nabla f}{|\nabla f|}\,|\nabla f|\,dA$$

Now, gradient.

$$\nabla f = \hat{i} + \hat{j} + \hat{k} \quad \left(\because \nabla(f) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(z - 1 + x + y)$$

Now, D is the region in xy plane as show.

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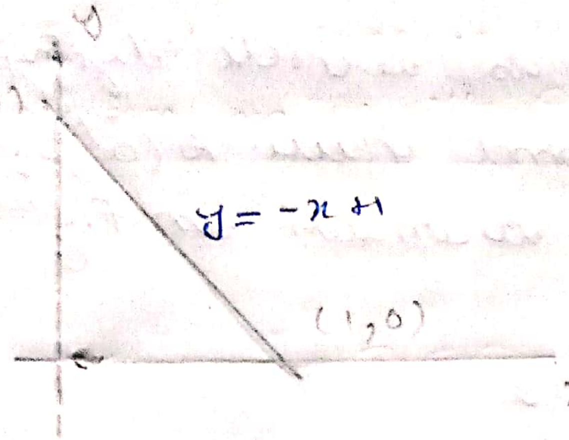


from this $0 \le x \le 1$ & $0 \le y \le -x+1$

The integral is then,

$$\int \bar{F} \cdot d\bar{r} = \iint (2z-1)\,\hat{j} \cdot (\hat{i}+\hat{j}+\hat{k})\,dA$$

$$= \int_0^1 \int_0^{-x+1} 2(1-x-y)-1 \, dy\, dx$$

$$= \int_0^1 \left(y - 2xy - y^2\right]_0^{1-x} dx$$

$$= \int_0^1 x^2 - x \, dx$$

$$= \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^1$$

$$= -\frac{1}{6}$$

Q Apply stoke's theorem to evaluate.

$\int (y\,dx + z\,dy + x\,dz)$ where C is The curve intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.

Q Ans The curve C is a circle lying in $x+z=a$ plane and have $A(a,0,0)$ & $B(0,0,a)$ as extremities of diameter in fig

$\therefore \int_C (y\,dx + z\,dy + x\,dz)$

$= \int (y\hat{i} + z\hat{j} + x\hat{k}) \cdot d\bar{R}$

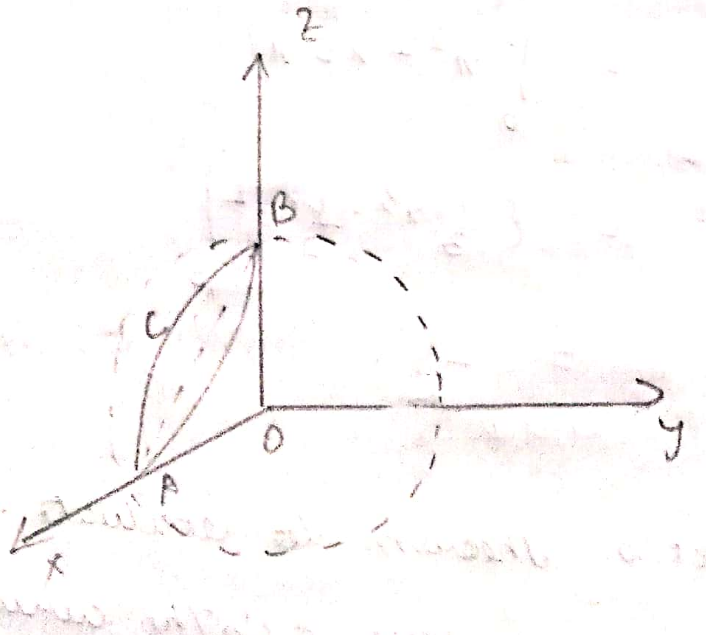
$= \int_S \text{curl}\,(y\hat{i} + z\hat{j} + x\hat{k}) \cdot \bar{N}\,ds$

where S is circle on AB as diameter &

$\bar{N} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}k$

$= \int_S -(\hat{i}+\hat{j}+\hat{k}) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\right) ds$

$= \frac{-2}{\sqrt{2}} \int_S ds = -\frac{2}{\sqrt{2}}\,\pi \left(\frac{a}{\sqrt{2}}\right)^2 = \frac{-\pi a^2}{\sqrt{2}}$



Q using Stoke's Theorem evaluate $\int xy\,dx + xy^2\,dy$ where C is a square in xy plane with vertices $(1,0), (-1,0), (0,1)$ & $(0,-1)$.

Soln

$$\vec{F} = xy\hat{i} + xy^2\hat{j}$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

$$= i[0-0] - \hat{j}(0-0) + k(\cancel{2}xy^2 - x)$$

$$= \hat{k}(y^2 - x)$$

In xy plane $\hat{n} = \hat{k}$

using Stoke's theorem

$$\oint F\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

$$= \int xy\,dx + xy^2\,dy$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1} \hat{n}(y^2 - x)\cdot k\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} (y^2 - x)\,dx\,dy$$

$$= \int_{-1}^{1}\left[\frac{y^3}{3} - xy\right]_{-1}^{1} dx = \int_{-1}^{1}\left[\frac{1}{3} - x - \left[\frac{-1}{3} + x\right]\right] dx$$

$$= \int_{-1}^{1} (2/3 - 2x)\,dx$$

$$= \int_{-1}^{1} \left(\frac{2}{3} - 2x\right) dx$$

$$= \left[\frac{2}{3}x - x^2\right]_{-1}^{1} dx = \frac{2}{3} - 1 - \left(-\frac{2}{3} - 1\right)$$

$$= \frac{4}{3}$$

Q Verify the Stoke theorem for vector field $F = (2x-y)\hat{i} - yz\hat{j} - y^2z\hat{k}$ over upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy plane

Ans

By line integral, in xy plane

$z = 0$

$dz = 0$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_C (2x-y)\,dx - y\cancel{z}^{0}\,dy$$

Now we have

$x^2+y^2=1$ ($\because z=0$ in xy plane)

Let $x = \cos\alpha$

$y = \sin\alpha$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = -\int_0^{2\pi} (2\cos\alpha - \sin\alpha)\sin\alpha\, d\alpha$$

$$= -\int_0^{2\pi} (\sin 2\alpha - \sin^2\alpha)\, d\alpha.$$

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$$= -\int_0^{2\pi} \sin 2\theta \, d\theta + \int_0^{2\pi} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \left(\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right)_0^{2\pi}$$

$$= \left(\frac{1-1}{2} + \frac{2\pi}{2} - 0 \right)$$

$$= \pi$$

By stoke theorem

$$\bar{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & y^2z \end{vmatrix}$$

$$= \hat{i}(-2yz + 2yz) - \hat{j}(0-0) + k(0+1)$$

$$= 1$$

In x-y plane $\hat{n} = k$

By stoke Theorem

$$\int F \cdot dr = \iint \text{curl } F \cdot \hat{n} \, ds$$

$$= \iint \hat{k} \cdot \hat{k} \, dx \, dy$$

$$= \iint dx \, dy$$

$$= \pi (1)^2 \qquad [\text{ from } x^2 + y^2 = 1)]$$

$$= \pi$$