Multi-headed lattices and Green functions

Lin Jiu

@Number Theory Seminar & Discrete Math Seminar

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Acknowledgment: Joint Work with

Qipin Chen, Amazon

Shane Chern

Journal of Physics A: Mathematical and Theoretical <onbehalfof@manuscriptcentral.com> 18:06 (45分钟前)

发送至 qipinche、 chenxiaohang92、 我

Dear Dr Jiu,

Re: "Multi-headed lattices and Green functions"

Manuscript reference: JPhysA-121035.R1

We are pleased to tell you that we are provisionally accepting your Paper in Journal of Physics A: Mathematical and Theoretical. Any further comments from reviewer(s) can be found below and/or attached to this message.

We will now perform some final checks to ensure that we have everything we need to publish your Paper. These checks enable us to publish your Paper as quickly and efficiently as possible.

Lattice in $\mathbb{Z}^N$

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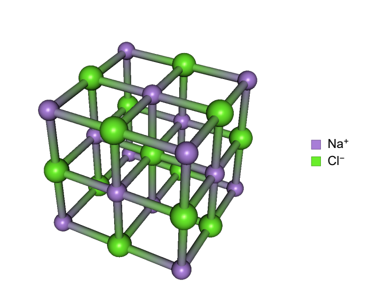
Bravais lattices

An arrangement of points in the three-dimensional space such that when viewed from each point the lattice appears exactly the same; from a mathematical perspective, it is a $\mathbb{Z}$-module generated by three linearly independent vectors in $\mathbb{R}^3$.

NaCl



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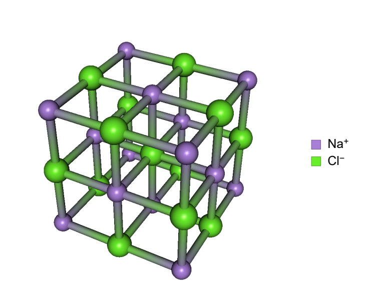

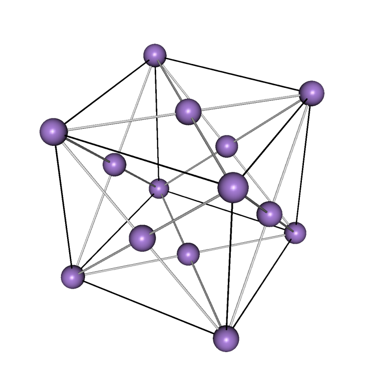

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$$\begin{aligned} \text{simple:} &\quad \mathcal{L}_{\mathcal{V}_{1,3}} \\ \text{face-centered:} &\quad \mathcal{L}_{\mathcal{V}_{2,3}} \\ \text{body-centered:} &\quad \mathcal{L}_{\mathcal{V}_{3,3}} \end{aligned}$$

$\mathcal{L}_{\mathcal{V}_{M,N}}$

The M-headed lattice in dimension N:

$$\mathcal{V}_{M,N} := \left\{(v_1, \ldots, v_N) \in \{-1, 0, 1\}^N : |v_1| + \cdots + |v_N| = M\right\}.$$

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$$(v_1, v_2, v_3) \in \{-1, 0, 1\}^3 : |v_1| + |v_2| + |v_3| = 2.$$

Main Results

Table 1: M-Headed lattice in dimension N

N	M	Differential Equation: Order	Differential Equation: Degree	Pólya Number
1	1	1	2	1
2	1	2	2	1
2	2	2	2	1
3	1	3	4	$1 - 2^4(\sqrt{3}+1)\pi^3\Gamma(\frac{1}{24})^{-2}\Gamma(\frac{11}{24})^{-2}$ (≈ 0.34054)
3	2	3	3	$1 - 2^{\frac{14}{3}}3^{-2}\pi^4\Gamma(\frac{1}{3})^{-6}$ (≈ 0.25632)
3	3	3	2	$1 - 2^2\pi^3\Gamma(\frac{1}{4})^{-4}$ (≈ 0.28223)
4	1	4	4	≈ 0.19313
4	2	4	7	≈ 0.09571
4	3			
4	4	4	2	≈ 0.10605
5	1	5	6	≈ 0.13517
5	2	6	13	≈ 0.04657
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4	3	8	32	≈ 0.04332
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5	1	5	6	≈ 0.13517
5	2	6	13	≈ 0.04657
		19	7	
5	3	14	110	≈ 0.01581
		69	16	
5	4	9	24	≈ 0.01560
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Definition

We say a chain $\mathfrak{w}$ of sites:

$$\mathfrak{w} : 0 = \boldsymbol{s}_0 \to \boldsymbol{s}_1 \to \boldsymbol{s}_2 \to \cdots$$

is a *walk* if there is a bond from $\boldsymbol{s}_j$ to $\boldsymbol{s}_{j+1}$ for every $j \geq 0$, and each move from $\boldsymbol{s}_j$ to $\boldsymbol{s}_{j+1}$ is called a *step*.

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$P_{\mathcal{L}_{\mathcal{V}}}(\boldsymbol{s}, n) :=$ the probability that a random walk ends at $\boldsymbol{s}$ after n steps

Green Function

Definition

The Green function of the lattice $\mathcal{L}_{\mathcal{V}}$ is the generating function of $P_{\mathcal{L}_{\mathcal{V}}}(\boldsymbol{s}, n)$:

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The *structure function*[1] is defined by, for $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_N)$,

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Fact

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main object—$r_{M,N}(n)$ and $\tilde{r}_{M,N}(n)$

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$r_{M,N}(n)$ and $\tilde{r}_{M,N}(n)$ (continued)

- $$P_{M,N}(0,z) = P_{\mathcal{L}_{\mathcal{V}}}(0,z) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_N}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)}$$

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- $$R_{M,N}(0,z) = \sum_{n\geq 0} r_{M,N}(n)z^n$$

- $$\widetilde{R}_{M,N}(0,z) := \sum_{n\geq 0} r_{M,N}(2n)z^n \quad \widetilde{r}_{M,N}(n) := r_{M,N}(2n).$$

$r_{M,N}(n)$ and $\tilde{r}_{M,N}(n)$ (continued)

- $$P_{M,N}(0,z) = P_{\mathcal{L}_{\mathcal{V}}}(0,z) = \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_N}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)}$$

- $$\begin{aligned} R_{M,N}(0,z) &:= P_{M,N}\left(0, 2^M \tbinom{N}{M} z\right) \\ &= \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_1 \cdots d\theta_N}{1 - z\, 2^M \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} \end{aligned}$$

- $$R_{M,N}(0,z) = \sum_{n \geq 0} r_{M,N}(n) z^n$$

- $$\widetilde{R}_{M,N}(0,z) := \sum_{n \geq 0} r_{M,N}(2n) z^n \quad \widetilde{r}_{M,N}(n) := r_{M,N}(2n).$$

Proposition (Q. Chen, S. Chern, and LJ)

We have $r_{M,N}(2n+1) = 0$ for all $n \geq 0$ if M is odd or $M = N$.

Table 1: M-Headed lattice in dimension N

†Entries with a gray background are newly computed in this work.

N	M	Differential Equation Order	Differential Equation Degree	Pólya Number
1	1	1	2	1
2	1	2	2	1
2	2	2	2	1
3	1	3	4	$1 - 2^4(\sqrt{3}+1)\pi^3\Gamma(\frac{1}{24})^{-2}\Gamma(\frac{11}{24})^{-2}$ (≈ 0.34054)
3	2	3	3	$1 - 2^{\frac{14}{3}}3^{-2}\pi^4\Gamma(\frac{1}{3})^{-6}$ (≈ 0.25632)
3	3	3	2	$1 - 2^2\pi^3\Gamma(\frac{1}{4})^{-4}$ (≈ 0.28223)
4	1	4	4	≈ 0.19313
4	2	4	7	≈ 0.09571
		11	5	
4	3	8	32	≈ 0.04332
		24	8	
4	4	4	2	≈ 0.10605
5	1	5	6	≈ 0.13517
5	2	6	13	≈ 0.04657
		19	7	
5	3	14	110	≈ 0.01581
		69	16	
5	4	9	24	≈ 0.01560
		33	6	
5	5	5	2	≈ 0.04473

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5	5	5	2	≈ 0.04473

$$R_{3,4}(\mathbf{0},z)=\frac{1}{\pi^4}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(***)\,\mathrm{d}\,x_1\,\mathrm{d}\,x_2\,\mathrm{d}\,x_3\,\mathrm{d}\,x_4,$$

where the integrand (***) is

$$\frac{1}{(1-8z(x_1x_2x_3+x_2x_3x_4+x_3x_4x_1+x_4x_1x_2))\sqrt{1-x_1^2}\sqrt{1-x_2^2}\sqrt{1-x_3^2}\sqrt{1-x_4^2}}.$$

After around *one hour* of computation, we arrive at the following result.

Theorem 4.2. *The function $\widetilde{R}_{3,4}(\mathbf{0},z)$ satisfies a differential equation of order 8 and degree 16. The differential equation is of the form*

$$\begin{aligned}
&(1344z+\cdots+15102632328225392235237425633056978227953664 0z^{16})\\
&+(18816z+\cdots+136678822570439799728898701979165652962980659 2z^{16})\vartheta\\
&+(102144z+\cdots+530312153503047731237878603965203504943228518 4z^{16})\vartheta^2\\
&+(212352z+\cdots+115333768879881245369763140417778457067472814 08z^{16})\vartheta^3\\
&+(-42+\cdots+153916799302850393255173863625340965057053327 36z^{16})\vartheta^4\\
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&+(-1113+\cdots+666361699726478139623642413209658450480044441 6z^{16})\vartheta^6\\
&+(1512+\cdots+193313693801285020611039048103129321317806899 2z^{16})\vartheta^7\\
&+(-756+\cdots+241642117251606275763798810128911651647258624z^{16})\vartheta^8.
\end{aligned}$$

Accordingly, $\widetilde{r}_{3,4}(n)$ satisfies a recurrence of order 16.

Meanwhile, if we directly apply creative telescoping to the summation

$$\widetilde{r}_{3,4}(n)=\sum_{\substack{k_1,k_2,k_3,k_4\geq 0\\ k_1+k_2+k_3+k_4=n}}\binom{2n}{2k_1,2k_2,2k_3,2k_4}\binom{2(n-k_1)}{n-k_1}\binom{2(n-k_2)}{n-k_2}\binom{2(n-k_3)}{n-k_3}\binom{2(n-k_4)}{n-k_4},$$

we get the following recurrence in ONLY *ten minutes*! This fact to some extent indicates that our summation expression is superior in a computational sense.

Theorem 4.3. *The sequence $\widetilde{r}_{3,4}(n)$ satisfies a recurrence of order 4. The recurrence is of the form*

$$\begin{aligned}
0=&(2210867920322586633830 40+\cdots+1988330027074191360n^{20})\widetilde{r}_{3,4}(n)\\
&-(1235966488843576210882 56+\cdots+135920997944524800n^{20})\widetilde{r}_{3,4}(n+1)\\
&+(2413729498666800513024+\cdots+745214176788480n^{20})\widetilde{r}_{3,4}(n+2)\\
&-(956961744081283584 0+\cdots+1074030451200n^{20})\widetilde{r}_{3,4}(n+3)\\
&+(9051531325562880+\cdots+462944160n^{20})\widetilde{r}_{3,4}(n+4).
\end{aligned}$$

Accordingly, $\widetilde{R}_{3,4}(\mathbf{0},z)$ satisfies a differential equation of order 24 and degree 4.

Holonomic (D-finite)

Definition

A formal power series $F = F(z)$ over a field $\mathbb{K}$ is called *D-finite/holonomic* if there are polynomials $v_0(z), \ldots, v_K(z) \in \mathbb{K}[z]$ with $v_K(z)$ not identical to zero such that

$$\left(v_0(z) + v_1(z)D + \cdots + v_K(z)D^K\right)(F) = 0, \tag{1}$$

where $D = D_z := d/dz$ is the usual *derivation operator*.

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- K is the order;
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Fact

Let $F(z) = \sum_{n=0}^{\infty} f(n)z^n$, then (1) shows $f(n)$ is P-finite/holonomic:

$$h_0(n)f(n) + h_1(n)f(n-1) + \cdots + h_L(n)f(n-L) = 0,$$

for polynomials $h_0(n), \ldots, h_L(n)$ with $h_0(n)$ and $h_L(n)$ not identical to zero.

Simple Examples

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$$a_n := \sum \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_k F(n,k)$$

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It shows that

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2. Integration

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$$\int_{-\infty}^{\infty} H_n(x)H_k(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{n,k}$$

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$$\int_{-\infty}^{\infty} H_n(x) H_k(x) e^{-x^2}\, dx = \sqrt{\pi} 2^n n! \delta_{n,k}$$

   Hermite polynomial $H_n(x)$:

$$H_{n+2}(x) - 2x H_{n+1}(x) + (2+2n) H_n(x) = 0$$

$x_j = \cos\theta_j$ and $\vartheta = \vartheta_z := z(d/dz)$

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$$R_{3,4}(0,z) = \frac{1}{\pi^4}\int_{[-1,1]^4} \frac{dx_1\,dx_2\,dx_3\,dx_4}{(1-8z(x_1x_2x_3+x_2x_3x_4+x_3x_4x_1+x_4x_1x_2))\sqrt{1-x_1^2}\sqrt{1-x_2^2}\sqrt{1-x_3^2}\sqrt{1-x_4^2}}$$

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Theorem (Q. Chen, S. Chern, and LJ)

The function $\widetilde{R}_{3,4}(0,z)$ satisfies a differential equation of order 8 and degree 16.

$$\begin{aligned}
&(1344z + \cdots + 15102632328225392235237425633056978227953664 0z^{16}) \\
&+ (18816z + \cdots + 136678822570439799728898701979165652962980659 2z^{16})\vartheta \\
&+ (102144z + \cdots + 530312153503047731237878603965203504943228518 4z^{16})\vartheta^2 \\
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\end{aligned}$$

Accordingly, $\widetilde{r}_{3,4}(n)$ satisfies a recurrence of order 16.

$$\widetilde{r}_{3,4}(n) = \sum_{\substack{k_1,k_2,k_3,k_4 \geq 0 \\ k_1+k_2+k_3+k_4=n}} \binom{2n}{2k_1, 2k_2, 2k_3, 2k_4} \binom{2(n-k_1)}{n-k_1} \binom{2(n-k_2)}{n-k_2} \binom{2(n-k_3)}{n-k_3} \binom{2(n-k_4)}{n-k_4}$$

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Theorem (Q. Chen, S. Chern, and LJ)

The sequence $\widetilde{r}_{3,4}(n)$ satisfies a recurrence of order 4.

$$\begin{aligned}
0 &= (221086792032258663383040 + \cdots + 198833002707419136 0n^{20})\widetilde{r}_{3,4}(n)\\
&\quad - (123596648884357621088256 + \cdots + 13592099794452480 0n^{20})\widetilde{r}_{3,4}(n+1)\\
&\quad + (24137294986668005130 24 + \cdots + 745214176788480n^{20})\widetilde{r}_{3,4}(n+2)\\
&\quad - (9569617440812835840 + \cdots + 1074030451200n^{20})\widetilde{r}_{3,4}(n+3)\\
&\quad + (9051531325562880 + \cdots + 462944160n^{20})\widetilde{r}_{3,4}(n+4).
\end{aligned}$$

Accordingly, $\widetilde{R}_{3,4}(0,z)$ satisfies a differential equation of order 24 and degree 4.

$$\widetilde{r}_{3,4}(n) = \sum_{\substack{k_1,k_2,k_3,k_4 \geq 0 \\ k_1+k_2+k_3+k_4=n}} \binom{2n}{2k_1, 2k_2, 2k_3, 2k_4} \binom{2(n-k_1)}{n-k_1} \binom{2(n-k_2)}{n-k_2} \binom{2(n-k_3)}{n-k_3} \binom{2(n-k_4)}{n-k_4}$$

Theorem (Q. Chen, S. Chern, and LJ)

The sequence $\widetilde{r}_{3,4}(n)$ *satisfies a recurrence of order* 4.

$$\begin{aligned} 0 &= (2210867920322586633830040 + \cdots + 1988330027074191360n^{20})\widetilde{r}_{3,4}(n) \\ &\quad - (1235966488843576210882556 + \cdots + 135920997944524800n^{20})\widetilde{r}_{3,4}(n+1) \\ &\quad + (241372949866680051302424 + \cdots + 7452141767884800n^{20})\widetilde{r}_{3,4}(n+2) \\ &\quad - (9569617440812835840 + \cdots + 1074030451200n^{20})\widetilde{r}_{3,4}(n+3) \\ &\quad + (9051531325562880 + \cdots + 462944160n^{20})\widetilde{r}_{3,4}(n+4). \end{aligned}$$

Accordingly, $\widetilde{R}_{3,4}(0,z)$ *satisfies a differential equation of order* 24 *and degree* 4.

Corollary (Q. Chen, S. Chern, and LJ)

The Pólya number is approximately $\boldsymbol{P}_{3,4}(0) \approx 0.04332$.

$$\widetilde{r}_{3,4}(n) = \sum_{\substack{k_1,k_2,k_3,k_4\geq 0\\ k_1+k_2+k_3+k_4=n}} \binom{2n}{2k_1,2k_2,2k_3,2k_4}\binom{2(n-k_1)}{n-k_1}\binom{2(n-k_2)}{n-k_2}\binom{2(n-k_3)}{n-k_3}\binom{2(n-k_4)}{n-k_4}$$

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The sequence $\widetilde{r}_{3,4}(n)$ satisfies a recurrence of order 4.

$$\begin{aligned}
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&\quad - (12359664888435762108 8256 + \cdots + 1359209979445248 00n^{20})\widetilde{r}_{3,4}(n+1)\\
&\quad + (24137294986668005130 24 + \cdots + 745214176788480n^{20})\widetilde{r}_{3,4}(n+2)\\
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\end{aligned}$$

Accordingly, $\widetilde{R}_{3,4}(0,z)$ satisfies a differential equation of order 24 and degree 4.

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The Pólya number is approximately $\boldsymbol{P}_{3,4}(0) \approx 0.04332$.

Fact (For the second method, it takes only 10 mins)

$R_{4,5}(0, z)$

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Remark

Using the integral

$$R_{4,5}(0,z) = \frac{1}{\pi^5}\int_{[-1,1]^5} (****)\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5$$

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Theorem (Q. Chen, S. Chern, and LJ)

For $n \geq 0$,

$$r_{N-1,N}(2n) = \sum_{\substack{k_1,\ldots,k_N \geq 0 \\ k_1+\cdots+k_N=n}} \binom{2n}{2k_1,\ldots,2k_N} \binom{2(n-k_1)}{n-k_1} \cdots \binom{2(n-k_N)}{n-k_N}.$$

Also, if N is even, $r_{N-1,N}(2n+1) = 0$;and if N is odd,

$$r_{N-1,N}(2n+1) = \sum_{\substack{k_1,\ldots,k_N \geq 0 \\ k_1+\cdots+k_N=n+\frac{1-N}{2}}} \binom{2n+1}{2k_1+1,\ldots,2k_N+1} \binom{2(n-k_1)}{n-k_1} \cdots \binom{2(n-k_N)}{n-k_N}.$$

Theorem

The function $R_{4,5}(0,z)$ satisfies a differential equation of order 9 and degree 24:

$$
\begin{aligned}
&(\mathbf{333047697408}z + \cdots + \mathbf{3587135914162316664577589182300422144000000000000}z^{\mathbf{24}})\\
&+ (\mathbf{47239200} + \cdots + \mathbf{874671469605846758999618028704303677440000000000}z^{\mathbf{24}})\vartheta\\
&+ (\mathbf{291308400} + \cdots + \mathbf{9453516646138192392531605664037066506240000000000}z^{\mathbf{24}})\vartheta^{\mathbf{2}}\\
&+ (\mathbf{740080800} + \cdots + \mathbf{5943864920522147046229253977370938834944000000000}z^{\mathbf{24}})\vartheta^{\mathbf{3}}\\
&+ (\mathbf{1027452600} + \cdots + \mathbf{2395763624685018201440807167653926928384000000000}z^{\mathbf{24}})\vartheta^{\mathbf{4}}\\
&+ (\mathbf{861131250} + \cdots + \mathbf{641926999294401201636284114667753701376000000000}z^{\mathbf{24}})\vartheta^{\mathbf{5}}\\
&+ (\mathbf{449264475} + \cdots + \mathbf{114331247240822245206661000977438474240000000000}z^{\mathbf{24}})\vartheta^{\mathbf{6}}\\
&+ (\mathbf{143193825} + \cdots + \mathbf{13051386184451845257422171891507920896000000000}z^{\mathbf{24}})\vartheta^{\mathbf{7}}\\
&+ (\mathbf{25587900} + \cdots + \mathbf{866423326586435904331316912053026816000000000}z^{\mathbf{24}})\vartheta^{\mathbf{8}}\\
&+ (\mathbf{1968300} + \cdots + \mathbf{25483039017248114833274026825089024000000000}z^{\mathbf{24}})\vartheta^{\mathbf{9}}.
\end{aligned}
$$

Accordingly, $r_{4,5}(n)$ satisfies a recurrence of order 24.

Theorem

The sequence $r_{4,5}(n)$ satisfies a recurrence of order 6:

$$\begin{aligned}
0 = {} & (236482206192589127006772264960000 + \cdots + 312808771118086225920n^{27})r_{4,5}(n) \\
& + (880540948213763261498004602880000 + \cdots + 22881382331785936896n^{27})r_{4,5}(n+1) \\
& - (664078540666702251488371015680000 + \cdots + 5976795675008958464n^{27})r_{4,5}(n+2) \\
& + (36337840931616555318702833664000 + \cdots + 159149910074064896n^{27})r_{4,5}(n+3) \\
& + (1737772868400007324872130560000 + \cdots + 3900964176134144n^{27})r_{4,5}(n+4) \\
& - (36446102109669030849285120000 + \cdots + 51561082388480n^{27})r_{4,5}(n+5) \\
& - (154404486709237819219968000 + \cdots + 138110042112n^{27})r_{4,5}(n+6).
\end{aligned}$$

Accordingly, $R_{4,5}(0, z)$ satisfies a differential equation of order 33 and degree 6.

Theorem

The sequence $r_{4,5}(n)$ satisfies a recurrence of order 6:

$$\begin{aligned}
0 = {} & (23648220619258912700677226496000000 + \cdots + 312808771118086225920n^{27})r_{4,5}(n) \\
& + (880540948213763261498004602880000 + \cdots + 22881382331785936896n^{27})r_{4,5}(n+1) \\
& - (664078540666702251488371015680000 + \cdots + 5976795675008958464n^{27})r_{4,5}(n+2) \\
& + (36337840931616555318702833664000 + \cdots + 1591499100740648896n^{27})r_{4,5}(n+3) \\
& + (1737772868400007324872130560000 + \cdots + 3900964176134144n^{27})r_{4,5}(n+4) \\
& - (36446102109669030849285120000 + \cdots + 51561082388480n^{27})r_{4,5}(n+5) \\
& - (154404486709237819219968000 + \cdots + 138110042112n^{27})r_{4,5}(n+6).
\end{aligned}$$

Accordingly, $R_{4,5}(0,z)$ satisfies a differential equation of order 33 and degree 6.

Corollary

As $n \to \infty$, $r_{4,5}(n) \sim C_{4,5} \cdot 80^n n^{-\frac{5}{2}}$, where the constant $C_{4,5} \approx 0.0353$.

Theorem

The sequence $r_{4,5}(n)$ satisfies a recurrence of order 6:

$$\begin{aligned}
0 = &(236482206192589127006772264960000 + \cdots + 312808771118086225920n^{27})r_{4,5}(n) \\
&+ (880540948213763261498004602880000 + \cdots + 228813823317859368 96n^{27})r_{4,5}(n+1) \\
&- (664078540666702251488371015680000 + \cdots + 597679567500895846 4n^{27})r_{4,5}(n+2) \\
&+ (363378409316165553187028336640000 + \cdots + 1591499100740648 96n^{27})r_{4,5}(n+3) \\
&+ (173777286840000732487213056000 0 + \cdots + 3900964176134144n^{27})r_{4,5}(n+4) \\
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The Pólya number is approximately $\boldsymbol{P}_{4,5}(0) \approx 0.01560$.

$R_{3,5}(0, z)$

Fact

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On May 8, 2024, at 9:10 PM, Christoph Koutschan <christoph.koutschan@ricam.oeaw.ac.at> wrote:

Hi Lin,

your observation is correct: I have not implemented any procedures to automatically parallelize the computations in HolonomicFunctions.

What can be done (and what I did in the past), but it is a bit cumbersome, is to parallelize the computation of FindCreativeTelescoping by using evaluation-interpolation, and chinese remaindering. For this purpose, the option Mode -> Modular can be used. This way you can let the different primes and evaluation points be computed in parallel, but you have to put together the results "manually". A necessary condition is that the computation is feasible for special values and modulo prime. This you can see when you follow the verbose output (Printlevel = 5).

Let me know in case you need further information.
Best wishes,
Christoph

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$$\left\{\widetilde{r}_{3,5}(n)\right\}_{n=0}^{54} = \{1, 80, 71280, \ldots,$$

$$\underbrace{20198893220533155882232776\ldots33034974414173089331 2000}_{\text{200 digits!}}$$

Final Remarks

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Problem

Let $N \geq 1$ be fixed. As sequences in M, are $\mathrm{ord}(M, N)$ and $\deg(M, N)$ unimodal? If so, do the peak positions take place around a half of N?