q-Analogues on Hankel Determinants: the q-Euler Numbers and the q-Binomial Transform

Lin Jiu

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Part of the work was sponsored by the Kunshan Municipal Government research funding.

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The nth *Hankel determinants* of a given sequence $\mathrm{a} = (a_0, a_1, \ldots,)$ is the determinant of the nth *Hankel matrix*

$$H_n(\mathrm{a}) = H_n(a_k) = \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix}$$

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Theorem

$H_n(\mathrm{C}) = 1$ *for all* $n = 0, 1, \ldots$

Bernoulli and Euler Polynomials

Definition

The *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$ are given by their exponential generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

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Theorem (Al-Salam and Carlitz)

$$H_n(B_k) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{(k!)^6}{(2k)!(2k+1)!} \quad \textit{and} \quad H_n(E_k) = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} (k!)^2.$$

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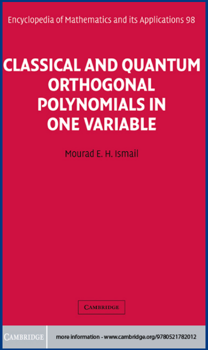

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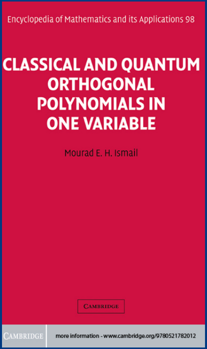

- $\mathrm{c} = (c_0, c_1, \ldots, c_n, \ldots)$
- Orthogonal polynomials P_n, w. r. t. c:

$$P_n(y) y^r \Big|_{y^k = c_k} = 0, 0 \leq r \leq n-1.$$

- $P_n(y) =$

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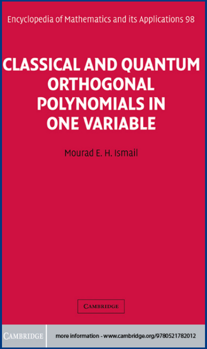

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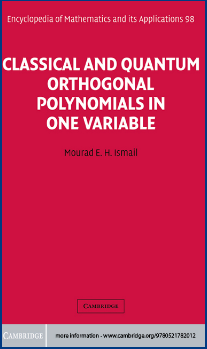

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$$P_{n+1} = (y+s_n)P_n(y) - t_n P_{n-1}(y) \Rightarrow \begin{cases} \sum_{n=0}^{\infty} c_n z^n = \cfrac{c_0}{1+s_0 z - \cfrac{t_1 z^2}{1+s_1 z - \cfrac{t_2 z^2}{\ddots}}} \\ H_n(\mathrm{c}) = c_0^{n+1} t_1^n t_2^{n-1} \cdots t_n \end{cases}$$

Early Work with Karl Dilcher

K. Dilcher and L. Jiu

- Hankel determinants of shifted sequences of Bernoulli and Euler numbers, Contrib. Discrete Math. 18 (2023), 146–175.
- Hankel Determinants of sequences related to Bernoulli and Euler Polynomials, Int. J. Number Theory 18 (2022), 331–359.
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We computed the Hankel determinants of the following sequences:

$$B_{2n+1}\left(\tfrac{x+1}{2}\right) \qquad E_{2k}\left(\tfrac{x+1}{2}\right)$$
$$E_{2k+1}\left(\tfrac{x+1}{2}\right) \qquad E_{2k+2}\left(\tfrac{x+1}{2}\right)$$
$$B_k\left(\frac{x+r}{q}\right) \pm B_k\left(\frac{x+s}{q}\right) \qquad kE_{k-1}(x)$$
$$B_{k,\chi_q} \qquad (q=3,4,6)$$
$$\frac{B_{k,\chi_{2q,\ell}}}{k+1} \qquad (q=3,4;\ell=1,2)$$
$$E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right) \qquad (2k+1)E_{2k}$$
$$(2^{2k+2}-1)B_{2k+2} \qquad (2k+1)B_{2k}(\tfrac{1}{2})$$
$$(2k+3)B_{2k} \qquad (2k+2)E_{2k+1}(1)$$

$b_k, k \geq 1$	b_0	Prop.	$b_k, k \geq 1$	b_0	Prop.
B_{k-1}	0	3.1	$E_{k+3}(1)$	$(\frac{-1}{4})$	5.2
B_{2k}	(1)	6.1	$E_{2k-1}(1)$	0	3.3
$(2k+1)B_{2k}$	(1)	6.2	$E_{2k+5}(1)$	$(\frac{1}{2})$	5.1
$(2^{2k}-1)B_{2k}$	(0)	3.4	$E_k(1)/k!$	(1)	3.6
$(2k+1)E_{2k}$	0	3.5	$E_{2k-1}(1)/(2k-1)!$	0	6.3
E_{2k-2}	0	7.3	$E_{2k-2}(\frac{x+1}{2})$	0	7.2
$E_{k-1}(1)$	0	3.2	$E_{k-1}^{(p)}$	$\alpha \in \mathbb{R}$	8.1

TABLE 2. Summary of results.

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Proposition

V_n is invertible iff $n < r$.

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Dr. Shane Chern

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The q-Bernoulli numbers were introduced by Carlitz as

$$\beta_n := \frac{1}{(1-q)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k+1}{[k+1]_q},$$

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Conjecture (L. J)

$$H_n(\beta_k) = (-1)^{\binom{n+1}{2}} q^{\frac{(n-1)n(n+1)}{6}} \frac{\prod_{k=1}^{n} [k]_q^{6(n+1-k)}}{\prod_{k=1}^{2n+1} [k]_q^{2n+2-k}}.$$

q-Bernoulli

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RE: q-Bernoulli
To: Karl Dilcher, Shane Chern

February 18, 2023 at 11:12 PM

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See More from Karl Dilcher

THE STARTING POINT

1. The Carlitz β

Carlitz [1, 2] generated the Bernoulli numbers to the sequence β_n, by the recurrence:

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k q^{k+1}-\beta_m=\begin{cases}1, & m=1;\\ 0, & m>1,\end{cases}$$

with also the value $\beta_0=1$.

Definition 1. The q-bracket is defined by

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for all $x\in\mathbb{R}$ and $q>0$. The q-factorial is then defined by

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Theorem (F. Chapton and J. Zeng, 2017)

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1. The Carlitz β

Carlitz [1, 2] generated the Bernoulli numbers to the sequence β_n, by the recurrence:

$$\sum_{k=0}^{n} \binom{n}{k} \beta_k q^{k+1} - \beta_m = \begin{cases} 1, & m = 1; \\ 0, & m > 1, \end{cases}$$

with also the value $\beta_0 = 1$.

Definition 1. The q-bracket is defined by

$$[x]_q := \frac{1 - q^x}{1 - q},$$

for all $x \in \mathbb{R}$ and $q > 0$. The q-factorial is then defined by

$$[k]_q! := [k]_q [k-1]_q \cdots [1]_q.$$

Conjecture 2.

$$H_n(\beta_k) = (-1)^{\binom{n+1}{2}} q^{\frac{(n-1)n(n+1)}{6}} \frac{\prod_{k=1}^{n} [k]_q^{6(n+1-k)}}{\prod_{k=1}^{2n+1} [k]_q^{2n+2-k}}.$$

Theorem (F. Chapton and J. Zeng, 2017)

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- F. Chapoton and J. Zeng, “Nombres de q-Bernoulli-Carlitzet fractions continues”,J. Théor. Nombres Bordeaux 29 (2017), no. 2, pp. 347-368.

q-Euler

Definition

The q-Euler numbers were introduced by Carlitz as

$$\epsilon_n := \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1+q}{1+q^{k+1}}.$$

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Theorem (S. Chern and L. J.)

$$\det_{0\le i,j\le n} \left(\epsilon_{i+j}\right) = \frac{(-1)^{\binom{n+1}{2}} q^{\frac{1}{4}\binom{2n+2}{3}}}{(1-q)^{n(n+1)}} \prod_{k=1}^{n} \frac{\left(q^2,q^2;q^2\right)_k}{\left(-q,-q^2,-q^2,-q^3;q^2\right)_k}$$

$$\det_{0\le i,j\le n} \left(\epsilon_{i+j+1}\right) = \frac{(-1)^{\binom{n+2}{2}} q^{\frac{1}{4}\binom{2n+4}{3}}}{(1-q)^{n(n+1)}\left(1+q^2\right)^{n+1}} \prod_{k=1}^{n} \frac{\left(q^2,q^4;q^2\right)_k}{\left(-q^2,-q^3,-q^3,-q^4;q^2\right)_k}$$

$$\det_{0\le i,j\le n} \left(\epsilon_{i+j+2}\right) = \frac{(-1)_2^{(n+2)} q^{\frac{1}{4}\binom{2n+4}{3}} (1+q)^n \left(1-(-1)^n q^{(n+2)^2}\right)}{(1-q)^{n(n+1)}\left(1+q^2\right)^{2(n+1)}\left(1+q^3\right)^{n+1}}$$

$$\times \prod_{k=1}^{n} \frac{\left(q^4,q^4;q^2\right)_k}{\left(-q^3,-q^4,-q^4,-q^5;q^2\right)_k}$$

Remarks

1.

$$(A;q)_n := \prod_{k=1}^{n}(1 - Aq^{k-1}), \quad (A_1, A_2, \ldots, A_n, q) := \prod_{j=1}^{n}(A_j;q)_n$$

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2.
$$\lim_{q\to 1}\det_{0\le i,j\le n}(\epsilon_{i+j}) = \left(-\frac{1}{16}\right)^{\binom{n+1}{2}}\prod_{k=1}^{n}((2k)!!)^2 = \left(-\frac{1}{4}\right)^{\binom{n+1}{2}}\prod_{k=1}^{n}(k!)^2.$$

3. We also use the big q-Jacobi polynomials
$$\mathscr{J}_{\ell,n}(z) := {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{n+\ell+1}, z \\ q^{\ell+1}, 0 \end{matrix} ; q, q\right)$$

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Recall

$B_{2n+1}\left(\frac{x+1}{2}\right)$	$E_{2k}\left(\frac{x+1}{2}\right)$
$E_{2k+1}\left(\frac{x+1}{2}\right)$	$E_{2k+2}\left(\frac{x+1}{2}\right)$
$B_k\left(\frac{x+r}{q}\right) \pm B_k\left(\frac{x+s}{q}\right)$	$kE_{k-1}(x)$
B_{k,χ_q}	$(q = 3, 4, 6)$
$\frac{B_{k,\chi_{2q,\ell}}}{k+1}$	$(q = 3, 4; \ell = 1, 2)$
$E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right)$	$(2k+1)E_{2k}$
$(2^{2k+2}-1)B_{2k+2}$	$(2k+1)B_{2k}(\frac{1}{2})$
$(2k+3)B_{2k}$	$(2k+2)E_{2k+1}(1)$

$b_k, k \geq 1$	b_0	Prop.	$b_k, k \geq 1$	b_0	Prop.
B_{k-1}	0	3.1	$E_{k+3}(1)$	$(\frac{-1}{4})$	5.2
B_{2k}	(1)	6.1	$E_{2k-1}(1)$	0	3.3
$(2k+1)B_{2k}$	(1)	6.2	$E_{2k+5}(1)$	$(\frac{1}{2})$	5.1
$(2^{2k}-1)B_{2k}$	(0)	3.4	$E_k(1)/k!$	(1)	3.6
$(2k+1)E_{2k}$	0	3.5	$E_{2k-1}(1)/(2k-1)!$	0	6.3
E_{2k-2}	0	7.3	$E_{2k-2}(\frac{x+1}{2})$	0	7.2
$E_{k-1}(1)$	0	3.2	$E_{k-1}^{(p)}$	$\alpha \in \mathbb{R}$	8.1

TABLE 2. Summary of results.

Binomial Transform

Theorem

Given a sequence $\mathrm{c} = (c_0, c_1, \ldots)$ *and defined the sequence of polynomials*

$$c_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell x^{k-\ell},$$

then

$$H_n(c_k) = H_n(c_k(x)).$$

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then

$$H_n(c_k) = H_n(c_k(x)).$$

$$H_n(B_k(x)) = H_n(B_k) \quad H_n(E_n(x)) = 2^{n(n+1)} H_n(E_k).$$

$B_{2n}\left(\frac{x+1}{2}\right)$

$H_1\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{12}x^2+\frac{1}{45}$
$H_2\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$-\frac{1}{540}x^6+\frac{97}{18900}x^4-\frac{11}{4725}x^2+\frac{16}{55125}$
$H_3\left(B_{2k}\left(\frac{1+x}{2}\right)\right)$	$\frac{1}{42000}x^{12}-\frac{121}{441000}x^{10}+\frac{153}{154000}x^8-\frac{17441}{12262250}x^6+\frac{8369}{11036025}x^4-\frac{1632}{9634625}x^2+\frac{256}{18883865}$

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Theorem (S. Chern, L. J., S. Li, and L. Wang)

For every $n \geq 0$, $H_n(B_{2k}\left(\frac{x+1}{2}\right))$ is a polynomial in x of degree $n(n+1)$ with leading coefficient

$$\left[x^{n(n+1)}\right] H_n\left(B_{2k}\left(\frac{1+x}{2}\right)\right) = (-1)^{\binom{n+1}{2}} \prod_{j=1}^{n} \frac{(j!)^6}{(2j)!(2j+1)!}.$$

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$$\det\begin{pmatrix} 1 & x^2 & \cdots & x^{2n} \\ x^2 & x^4 & \cdots & x^{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{2n} & x^{2n+2} & \cdots & x^{4n} \end{pmatrix}$$

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q-Binomial

Recall that

$$H_n(c_k) = H_n(c_k(x)), \quad \text{where } c_k(x) := \sum_{\ell=0}^{k} \binom{k}{\ell} c_\ell x^{k-\ell}.$$

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Problem

What if

$$\binom{k}{\ell} \mapsto \begin{bmatrix} k \\ \ell \end{bmatrix}_q := \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-\ell+1})}{(1-q)(1-q^2)\cdots(1-q^\ell)}?$$

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Theorem (The q-binomial theorem)

$$\prod_{k=0}^{n-1} (1+q^k t) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k.$$

Two problems

Problem

What are $H_n(\alpha_k(x))$ and $H_n(\tilde{\alpha}_k(x))$, where

$$\alpha_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{k-\ell} x^{\ell} \quad \text{and} \quad \widetilde{\alpha}_k(x) := \sum_{\ell=0}^{k} q^{\binom{\ell}{2}} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \alpha_{\ell} x^{k-\ell} ?$$

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Remark

For $\tilde{\alpha}_k(x)$, we can define $\beta_\ell = q^{\binom{\ell}{2}} \alpha_\ell$.

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Theorem (S. Chern, L. J., S. Li, and L. Wang)

1. *For every $n \geq 0$, $H_n(\alpha_k(x))$ is a polynomial in x of degree $n(n+1)$ with leading coefficient*

$$\left[x^{n(n+1)}\right] H_n\left(\alpha_k(x)\right) = \alpha_0^{n+1} (-1)^{\binom{n+1}{2}} q^{3\binom{n+1}{3}} \prod_{j=1}^{n} \left(1 - q^j\right)^{n+1-j}.$$

2. *For every $n \geq 0$, $H_n(\tilde{\alpha}_k(x))$ is a polynomial in x of degree $n(n+1)/2$ with leading coefficient*

$$\left[x^{\frac{n(n+1)}{2}}\right] H_n\left(\widetilde{\alpha}_k(x)\right) = \alpha_0 \alpha_1 \cdots \alpha_n (-1)^{\binom{n+1}{2}} q^{2\binom{n+1}{3}} \prod_{j=1}^{n} \left(1 - q^j\right)^{n+1-j}.$$

TABLE 2. Coefficients in $H_2(\alpha_k(x))$

x^6	$-q^3(1-q)^3(1+q)\alpha_0^3$
x^5	$-q^2(1-q)^3(1+q)(1+2q)\alpha_0^2\alpha_1$
x^4	$-q(1-q)^2(1+q)\left[(1+2q+q^2-q^3)\alpha_0\alpha_1^2-(1+2q^2)\alpha_0^2\alpha_2\right]$
x^3	$(1-q)^2(1+q)\left[(2+q+q^2+2q^3)\alpha_0\alpha_1\alpha_2-(1-q)\alpha_0^2\alpha_3-(1+2q+2q^2+q^3)\alpha_1^3\right]$
x^2	$(1-q)\left[(1+3q+q^2-q^3)\alpha_0\alpha_1\alpha_3+(1-q-q^2-q^3-q^4)\alpha_0\alpha_2^2-\alpha_0^2\alpha_4-(1+2q-2q^3-q^4)\alpha_1^2\alpha_2\right]$
x^1	$-(1-q)\left[\alpha_0\alpha_1\alpha_4-(1-q^2)\alpha_0\alpha_2\alpha_3+(1+2q)\alpha_1\alpha_2^2-(1+2q+q^2)\alpha_1^2\alpha_3\right]$
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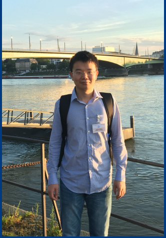
Shuhan Li



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