Bernoulli symbol on multiple zeta values at negative integers

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RICAM, Austrian Science of Academy

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JCT LEV ACADEMIC CENTER

Acknowledgement

Joint Work with

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Prof. Christophe Vignat

Outline

1 Bernoulli Numbers, Polynomials, Symbol
- Bernoulli numbers and Bernoulli polynomials
- Bernoulli Symbol $\mathcal{B}$

2 Multiple Zeta Values
- Definitions and analytic continuation
- Generalized Bernoulli symbol :$\mathcal{C}$

3 An Interesting Result

Bernoulli Numbers & Bernoulli Polynomials

Definition

Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are given by their exponential generating functions: $(B_{2n+1} = 0)$

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Examples

$$1^n + 2^n + \cdots + N^n = \frac{1}{n+1} \sum_{i=1}^{n+1} \binom{n+1}{i} B_{n+1-i} N^i = \frac{B_{n+1}(N+1) - B_{n+1}}{n+1}.$$

Riemann-zeta: for $n \in \mathbb{Z}_+$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \zeta(-n) = -\frac{B_{n+1}}{n+1} = (-1)^n \frac{B_{n+1}}{n+1}.$$

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Key Idea:

$$\mathcal{B}^n \mapsto B_n : \text{ i.e., super index} \leftrightarrow \text{lower index.}$$

Why?

Simplification

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k = (\mathcal{B} + x)^n.$$

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Visualization

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow \left[(\mathcal{B}+x)^n\right]' = n(\mathcal{B}+x)^{n-1}.$$

New Aspect (Probabilitistic Interpretation)

$\exists p(t)$ on $\mathbb{R}$ s. t. (moment)

$$\mathcal{B}^n = B_n = \int_{\mathbb{R}} t^n p(t)\, dt.$$

Theorem[Density of $\mathcal{B}$](A. Dixit, V. H. Moll, and C. Vignat)

$\mathcal{B} \sim \imath L_B - \frac{1}{2}$, where

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- $$B_n(x) = (\mathcal{B} + x)^n = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + \iota t - \frac{1}{2}\right)^n \operatorname{sech}^2(\pi t)\, dt.\ \left(\frac{t}{e^t - 1} \mid e^{tx}\right)$$

- Norlünd:

$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} \Leftrightarrow B_n^{(p)} = \left(\underbrace{\mathcal{B}_1 + \cdots + \mathcal{B}_p}_{\text{i. i. d.}} + x\right)^n$$

- Bernoulli-Barnes: for $\mathbf{a} = (a_1, \ldots, a_p)$, $|\mathbf{a}| = \prod_{l=1}^{p} a_l \neq 0$

$$e^{tx}\prod_{i=1}^{p}\frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x)\frac{t^n}{n!} \Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|}\left(x + \mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

MZV:Defintion

Recall

Riemann-zeta: for $n \in \mathbb{Z}_+$, the AC $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$.

Definition

$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2} \cdots (k_1+\cdots+k_r)^{n_r}}$$

B. Sadaoui's analytic continuation is based on Raabe's identity by linking

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1,\infty)^r} \frac{d\mathbf{x}}{(x_1+a_1)\cdots(x_1+a_1+\cdots+x_r+a_r)^{n_r}}$$

to the multiple zeta function

$$Z(\mathbf{n}, \mathbf{z}) = \sum_{k_1,\ldots,k_r>0} \frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

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MZV: Analytic Continuation

Theorem(Sadaoui)

$$\zeta_r(-n_1,\ldots,-n_r) = (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

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Example

$$\zeta(-n)=(-1)^n\,\mathcal{C}^{n+1}=(-1)^n\,\frac{B_{n+1}}{n+1}.$$

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Results

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$$\zeta_r\left(n_1,\ldots,n_r;z_1,\ldots,z_r\right):=\sum_{k_1,\ldots,k_r>0}\frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

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Theorem(L. Jiu, V. H. Moll and C. Vignat)

- Recurrence:

$$\zeta_r(-\mathbf{n};\mathbf{z}) = \frac{(-1)^{n_r}}{n_r+1}\sum_{l=0}^{n_r+1}\binom{n_r+1}{l}(-1)^l\,\zeta_{r-1}(-\mathbf{n}_{r-2},-n_{r-1}-l;\mathbf{z}_{r-1})\,B_{n_1+1-l}(z_r);$$

- Contiguity: for $\mathcal{Z}_r^l = \zeta_r(-\mathbf{n}_{r-1},-n_r-l;\mathbf{z})$:

$$\zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r+1) = \zeta_r(-\mathbf{n};\mathbf{z}_{r-1},z_r) + (-1)^{n_r}(z_r-\mathcal{Z}_{r-1})^{n_r};$$

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Theorem(L. Jiu, V. H. Moll and C. Vignat)

- Generating Function

$$F_r(w_1,\ldots,w_r) := \sum_{n_1,\ldots,n_r\ge 0} \frac{w_1^{n_1}\cdots w_r^{n_r}}{n_1!\cdots n_r!}\zeta_r(-n_1,\ldots,-n_r)$$
$$= \left(F_1(w_r,-\partial_{r-1})\cdots F_1(w_2,-\partial_1)\right)\bullet F_1(w_1,0),$$

where $\partial_i = \partial/\partial w_i$ and

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MZV: Another Approach

S. Akiyama and Y. Tanigawa uses Euler-Maclaurin summation formula to derive (the recurrence)

$$\begin{aligned}\zeta_r\left(-n_1,\ldots,-n_r\right) \quad = \quad & -\frac{\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r-1\right)}{1+n_r} \\ & -\frac{\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r\right)}{2} \\ & +\sum_{q=1}^{n_r}\left(-n_r\right)_q a_q\zeta_{r-1}\left(-n_1,\ldots,-n_{r-2},-n_{r-1}-n_r+q\right),\end{aligned}$$

where $a_q = B_{q+1}/\left(q+1\right)!$.

Remark

$$B_1 = -\frac{1}{2} \text{ and } \left(-n\right)_{-1} = -\frac{1}{n+1}$$

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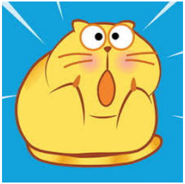

This shows the two appoaches, by Raabe's identity and Euler-Maclaurin summation formula, lead to analytic continuations of MZVs, which coincide on negative integer values.

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Well,.... I do not know.....

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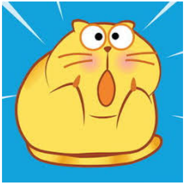

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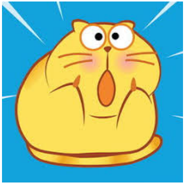

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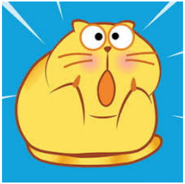

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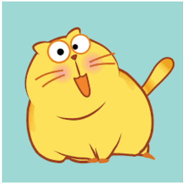

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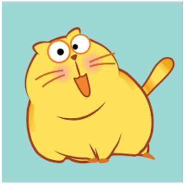

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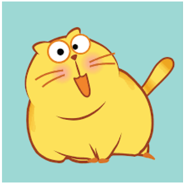

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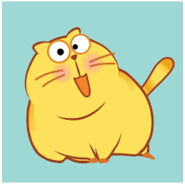

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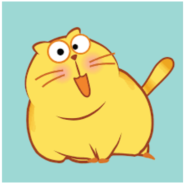

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