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AN EXTENSION OF THE METHOD OF BRACKETS

IVAN GONZALEZ, KAREN KOHL, LIN JIU, AND VICTOR H. MOLL

Abstract. The method of brackets is an efficient method for the evaluation of a large class of definite integrals on the half-line. It is based on a small collection of rules, some of which are heuristic. The extension discussed here is based on the concepts of null and divergent series. These are formal representations of functions, whose coefficients a_n have meromorphic representations for $n \in \mathbb{C}$, but might vanish or blow up when $n \in \mathbb{N}$. These ideas are illustrated with the evaluation of a variety of entries from the classical table of integrals by Gradshteyn and Ryzhik.

1. Introduction

The evaluation of definite integrals have a long history dating from the work of Eudoxus of Cnidus (408-355 BC) with the creation of the method of exhaustion. The history of this problem is reported in [17]. A large variety of methods developed for the evaluations of integrals may be found in older Calculus textbooks, such as the collection by J. Edwards [4, 5]. As the number of examples grew, they began to be collected in *tables of integrals*. The table compiled by I. S. Gradshteyn and I. M. Ryzhik [16] is the most widely used one, now in its 8^{th}-edition.

The interest of the last author in this topic began with entry 3.248.5 in [14]

$$I = \int_0^\infty (1+x^2)^{-3/2} \left[\varphi(x) + \sqrt{\varphi(x)} \right]^{-1/2} \, dx \tag{1.1}$$

where $\varphi(x) = 1 + \frac{4}{3}x^2(1+x^2)^{-2}$. The value $\pi/2\sqrt{6}$ given in the table is incorrect, as a direct numerical evaluation will confirm. Since an evaluation of the integral still elude us, the editors of the table found an ingenious temporary solution to this problem: it does not appear in [15] nor in the latest edition [16]. This motivated an effort to present proofs of all entries in Gradshteyn-Ryzhik. It began with [19] and has continued with several short papers. These have appeared in *Revista Scientia*, the latest one being [1].

The work presented here deals with the *method of brackets*. This is a new method for integration developed in [11, 12, 13] in the context of integrals arising from Feynman diagrams. It consists of a small number of rules that converts the integrand into a collection of series. The success of the method depends on the ability to give closed-form expressions for these series. Some of these heuristic rules are currently being placed on solid ground [2]. The reader will find in [8, 9, 10] a large collection of examples that illustrate the power and flexibility of this method.

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The operational rules are described in Section 2. The method applies to functions that can be expanded in a formal power series

$$f(x) = \sum_{n=0}^{\infty} a(n) x^{\alpha n + \beta - 1}, \tag{1.2}$$

where α, $\beta \in \mathbb{C}$ and the coefficients $a(n) \in \mathbb{C}$. (The extra -1 in the exponent is for a convenient formulation of the operational rules). The adjective *formal* refers to the fact that the expansion is used to integrate over $[0, \infty)$, even though it might be valid only on a proper subset of the half-line.

Note 1.1. There is no precise description of the complete class of functions f for which the method can be applied. At the moment, it is a working assumption, that the coefficients $a(n)$ in (1.2) are expressions that admit a unique meromorphic continuation to $n \in \mathbb{C}$. This is required, since the method involves the evaluation of $a(n)$ for n not a natural number, hence an extension is needed. For example, the Bessel function

$$I_0(x) = \sum_{n=0}^{\infty} \frac{1}{n!^2} \left(\frac{x}{2} \right)^{2n} \tag{1.3}$$

has $\alpha = 2$, $\beta = 1$ and $a(n) = 1/2^{2n} n!^2$ can be written as $a(n) = 1/2^{2n}\Gamma^2(n+1)$ and now the evaluation, say at $n = \frac{1}{2}$, is possible. The same observation holds for the Bessel function J_0

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{x}{2} \right)^{2n}. \tag{1.4}$$

The goal of the present work is to produce *non-classical series representations* for functions f, which do not have expansions like (1.2). These representations are formally of the type (1.2) but the coefficients $a(n)$ might all (or some) vanish, or be divergent expressions. The examples show how to use these representations in conjunction with the method of brackets to evaluate definite integrals. The examples presented here come from the table [16]. This process is, up to now, completely heuristic. These non-classical series are classified according to the following types:

1) `Totally (partially) divergent series`. Each term (some of the terms) in the series is a divergent value. For example,

$$\sum_{n=0}^{\infty} \Gamma(-n) x^n \text{ and } \sum_{n=0}^{\infty} \frac{\Gamma(n-3)}{n!} x^n. \tag{1.5}$$

2) `Totally (partially) null series`. Each term (some of the terms) in the series vanishes. For example,

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(-n)} x^n \text{ and } \sum_{n=0}^{\infty} \frac{1}{\Gamma(3-n)} x^n. \tag{1.6}$$

This type includes series where all but finitely many terms vanish. These are polynomials in the corresponding variable.

3) `Formally divergent series`. This is a classical divergent series: the terms are finite but the sum of the series diverges. For example,

$$\sum_{n=0}^{\infty} \frac{n!^2}{(n+1)\,(2n)!} 5^n. \tag{1.7}$$

In spite of the divergence of these series, they will be used in combination with the method of brackets to evaluate a variety of definite integrals. Examples of these type of series are given next.

Some examples of functions that admit non-classical representations are given next.

• The *exponential integral* with the partially divergent series

$$\text{Ei}(-x) = -\int_1^{\infty} t^{-1} e^{-xt}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n\, n!}. \tag{1.8}$$

• The *Bessel K_0-function*

$$K_0(x) = \int_0^{\infty} \frac{\cos xt\, dt}{(t^2+1)^{1/2}} \tag{1.9}$$

with totally null representation

$$K_0(x) = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma^2(n+\frac{1}{2})}{n!\,\Gamma(-n)} \left(\frac{4}{x^2} \right)^n \tag{1.10}$$

and the totally divergent one

$$K_0(x) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-n)}{n!} \left(\frac{x^2}{4} \right)^n. \tag{1.11}$$

Section 2 presents the rules of the method of brackets. Section 3 shows that the bracket series associated to an integral is independent of the presentation of the integrand. The remaining sections use the method of brackets and non-classical series to evaluate definite integrals. Section 4 contains the exponential integral $\text{Ei}(-x)$ in the integrand, Section 5 has the Tricomi function $U(a,b;x)$ (as an example of the confluent hypergeometric function), Section 6 is dedicated to integrals with the Airy function $\text{Ai}(x)$ and then Section 7 has the Bessel function $K_\nu(x)$, with special emphasis on $K_0(x)$. Section 8 gives examples of definite integral whose value contains the Bessel function $K_\nu(x)$. The final section has a new approach to the evaluation of bracket series, based on a differential equation involving parameters.

2. The method of brackets

The method of brackets evaluates integrals over the half line $[0, \infty)$. It is based on a small number of rules reviewed in this section.

Definition 2.1. For $a \in \mathbb{C}$, the symbol

$$\langle a \rangle = \int_0^{\infty} x^{a-1}\, dx \tag{2.1}$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)} \tag{2.2}$$

is called the *indicator* associated to the index n. The notation $\phi_{n_1 n_2 \cdots n_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{n_1}\phi_{n_2}\cdots\phi_{n_r}$.

Note 2.1. The indicator ϕ_n will be used in the series expressions used in the method of brackets. For instance (1.8) is written as

$$\mathrm{Ei}(-x) = \sum_n \phi_n \frac{x^n}{n} \tag{2.3}$$

and (1.11) as

$$K_0(x) = \frac{1}{2}\sum_n \phi_n \Gamma(-n)\left(\frac{x^2}{4}\right)^n. \tag{2.4}$$

In the process of implementing the method of brackets, these series will be evaluated for $n \in \mathbb{C}$, not necessarily positive integers. Thus the notation for the indices does not include its range of values.

Rules for the production of bracket series

The first part of the method is to associate to the integral

$$I(f) = \int_0^\infty f(x)\,dx \tag{2.5}$$

a bracket series. This is done following two rules:

Rule $\mathbf{P_1}$. Assume f has the expansion

$$f(x) = \sum_{n=0}^\infty \phi_n a(n) x^{\alpha n+\beta-1}. \tag{2.6}$$

Then $I(f)$ is assigned the *bracket series*

$$I(f) = \sum_n \phi_n a(n)\langle \alpha n + \beta\rangle. \tag{2.7}$$

Note 2.2. The series including the indicator ϕ_n have indices *without* limits, since its evaluation requires to take n outside $\mathbb{N}$.

Rule $\mathbf{P_2}$. For $\alpha \in \mathbb{C}$, the multinomial power $(u_1+u_2+\cdots+u_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1,n_2,\ldots,n_r} \phi_{n_1\,n_2\cdots n_r} u_1^{n_1}\cdots u_r^{n_r}\frac{\langle -\alpha+n_1+\cdots+n_r\rangle}{\Gamma(-\alpha)}. \tag{2.8}$$

The integer r is called *the dimension of the bracket series.*

Rules for the evaluation of a bracket series

The next set of rules associates a complex number to a bracket series.

Rule $\mathbf{E_1}$. The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n a(n)\langle \alpha n + b\rangle = \frac{1}{|\alpha|}a(n^*)\Gamma(-n^*), \tag{2.9}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an+b=0$.

Note 2.3. The rule E_1 is a version of the *Ramanujan's Master Theorem.* This theorem requires an extension of the coefficients $a(n)$ from $n \in \mathbb{N}$ to $n \in \mathbb{C}$. The assumptions imposed on the function f is precisely for the application of this result. A complete justification of this rule is provided in [2]. Making the remaining rules rigorous is the subject of current work.

The next rule provides a value for multi-dimensional bracket series where the number of sums is equal to the number of brackets.

Rule $\mathbf{E_2}$. Assume the matrix $B = (b_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1, n_2, \cdots, n_r} \phi_{n_1 \cdots n_r} a(n_1, \cdots, n_r) \langle b_{11}n_1 + \cdots + b_{1r}n_r + c_1 \rangle \cdots \langle b_{r1}n_1 + \cdots + b_{rr}n_r + c_r \rangle$$

$$= \frac{1}{|\det(B)|} a(n_1^*, \cdots n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if B is singular.

Rule $\mathbf{E_3}$. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$\text{index} = \text{number of sums} - \text{number of brackets}. \tag{2.10}$$

In the case of a multi-dimensional bracket series of positive index, then the system generated by the vanishing of the coefficients has a number of free parameters. The solution is obtained by computing all the contributions of maximal rank in the system by selecting these free parameters. Series expressed in the same variable and converging in a common region are added. Divergent series are discarded.

Example 2.2. A generic bracket series of index 1 has the form

$$\sum_{n_1,\, n_2} \phi_{n_1, n_2} C(n_1, n_2) A^{n_1} B^{n_2} \langle a_{11}n_1 + a_{12}n_2 + c_1 \rangle, \tag{2.11}$$

where a_{11}, a_{12}, c_1 are fixed coefficients, A, B are parameters and $C(n_1, n_2)$ is a function of the indices.

The Rule E_3 is used to generate two series by leaving first n_1 and then n_2 as free parameters. The Rule E_1 is used to assign a value to the corresponding series: n_1 as a free parameter produces

$$T_1 = \frac{B^{-c_1/a_{12}}}{|a_{12}|} \sum_{n_1=0}^{\infty} \phi_{n_1} \Gamma\left(\frac{a_{11}n_1 + c_1}{a_{12}} \right) C\left(n_1, -\frac{a_{11}n_1 + c_1}{a_{12}} \right) \left(AB^{-a_{11}/a_{12}} \right)^{n_1};$$

n_2 as a free parameter produces

$$T_2 = \frac{A^{-c_1/a_{11}}}{|a_{11}|} \sum_{n_2=0}^{\infty} \phi_{n_2} \Gamma\left(\frac{a_{12}n_2 + c_1}{a_{11}} \right) C\left(-\frac{a_{12}n_2 + c_1}{a_{11}}, n_2 \right) \left(BA^{-a_{12}/a_{11}} \right)^{n_2}.$$

The series T_1 and T_2 are expansions of the solution in terms of different parameters

$$x_1 = AB^{-a_{11}/a_{12}} \text{ and } x_2 = BA^{-a_{12}/a_{11}}. \tag{2.12}$$

Observe that $x_2 = x_1^{a_{12}/a_{11}}$. Therefore the bracket series is assigned the value T_1 *or* T_2. If one of the series is a null-series or divergent, it is discarded. If *both* series are discarded, the method of brackets does not produce a value for the integral that generates the bracket series.

Some special cases will clarify the rules to follow in the use of the series T_1 and T_2. Suppose $a_{12} = -a_{11}$, then

$$T_1 = \frac{B^{-c_1/a_{11}}}{|a_{11}|} \sum_{n_1=0}^{\infty} \phi_{n_1} \Gamma\left(n_1 + \frac{c_1}{a_{11}}\right) C\left(n_1, -n_1 - \frac{c_1}{a_{11}}\right) (AB)^{n_1} \tag{2.13}$$

and

$$T_2 = \frac{A^{-c_1/a_{11}}}{|a_{11}|} \sum_{n_2=0}^{\infty} \phi_{n_2} \Gamma\left(n_2 + \frac{c_1}{a_{11}}\right) C\left(-n_2 - \frac{c_1}{a_{11}}, n_2\right) (AB)^{n_2} \tag{2.14}$$

and since both series are expansions in the same parameter (AB), *their values must be added* to compute the value associated to the bracket series. On the other hand, if $a_{12} = -2a_{11}$, then

$$T_1 = \frac{B^{c_1/2a_{11}}}{2|a_{11}|} \sum_{n_1=0}^{\infty} \phi_{n_1} \Gamma\left(-\frac{1}{2}n_1 - \frac{c_1}{2a_{11}}\right) C\left(n_1, \frac{1}{2}n_1 + \frac{c_1}{2a_{11}}\right) \left(AB^{1/2}\right)^{n_1}$$

and

$$T_2 = \frac{A^{-c_1/a_{11}}}{|a_{11}|} \sum_{n_2=0}^{\infty} \phi_{n_2} \Gamma\left(-2n_2 + \frac{c_1}{a_{11}}\right) C\left(2n_2 - \frac{c_1}{a_{11}}, n_2\right) \left(A^2B\right)^{n_2}.$$

Splitting the sum in T_1 according to the parity of the indices produces a power series in A^2B when $n_1 = 2n_3$ is even and for n_1 odd a second power series in the same argument A^2B times an extra factor $AB^{1/2}$. Since these are expansions in the same argument, they have to be added to count their contribution to the bracket series.

Note 2.4. It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

Note 2.5. The extension presented in this work shows how to use these divergent series in the evaluation of definite integrals. Example 9.3 illustrates this procedure.

Rule $\mathbf{E_4}$. In the evaluation of a bracket series, repeated series are counted only once. For instance, a convergent series appearing repeated in the same region of convergence should be counted only once.

An example of Rule E_4 appears in Section 4.

Note 2.6. A systematic procedure in the simplification of the series has been used throughout the literature: express factorials in terms of the gamma function and the transform quotients of gamma terms into Pochhammer symbols, defined by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{2.15}$$

Any presence of a Pochhammer with a negative index k is transformed by the rule

$$(a)_{-k} = \frac{(-1)^k}{(1-a)_k}, \quad \text{for } k \in \mathbb{N}. \tag{2.16}$$

In the special case when a *is also* a negative integer, the rule

$$(-km)_{-m} = \frac{k}{k+1} \cdot \frac{(-1)^m (km)!}{((k+1)m)!} \tag{2.17}$$

holds. This value is justified in [7]. The duplication formula

$$(a)_{2n} = 2^{2n} \left(\frac{a}{2} \right)_n \left(\frac{a+1}{2} \right)_n \tag{2.18}$$

is also used in the simplifications.

Many of the evaluations are given as values of the hypergeometric functions

$${}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{2.19}$$

with $(a)_n$ as in (2.15). It is often that the value of ${}_2F_1$ at $z = 1$ is required. This is given by the classical formula of Gauss:

$${}_2F_1 \left(\begin{matrix} a \;\; b \\ c \end{matrix} \middle| 1 \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{2.20}$$

Note 2.7. The extension considered here is to use the method of brackets to functions that do not admit a series representation as described in Rule P_1. For example, the Bessel function $K_0(x)$ has a singular expansion of the form

$$K_0(x) = -\left(\gamma - \ln 2 + \ln x\right) I_0(x) + \sum_{j=0}^{\infty} \frac{H_j}{j!^2} \frac{x^{2j}}{2^{2j}} \tag{2.21}$$

(see [21, 10.31.2]). Here $I_0(x)$ is the Bessel function given in (1.3), $H_j = \displaystyle\sum_{k=1}^{j} \frac{1}{k}$ is the harmonic number and $\gamma = \displaystyle\lim_{j \to \infty} (H_j - \ln j)$ is Euler's constant. The presence of the logarithm term in (2.21) does not permit a direct application of the method of brackets. An alternative is presented in Section 7.

3. Independence of the factorization

The evaluation of a definite integral by the method of brackets begins with the association of a bracket series to the integral. It is common that the integrand contains several factors from which the bracket series is generated. This representation is not unique. For example, the integral

$$I = \int_0^{\infty} e^{-ax} J_0(x)\, dx \tag{3.1}$$

is associated the bracket series

$$\sum_{n_1, n_2} \phi_{n_1,n_2} \frac{a^{n_1}}{2^{2n_2}\Gamma(n_2+1)} \langle n_1 + 2n_2 + 1 \rangle, \tag{3.2}$$

and rewriting (3.1) as

$$I = \int_0^{\infty} e^{-ax/2} e^{-ax/2} J_0(x)\, dx, \tag{3.3}$$

provides the second bracket series

$$\sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{n_1+n_2}}{2^{n_1+n_2+2n_3}\Gamma(n_3+1)} \langle n_1+n_2+2n_3+1 \rangle \tag{3.4}$$

associated to (3.1). It is shown next that all such bracket series representations of an integral produce the same the value.

Theorem 3.1. *Assume $f(x) = g(x)h(x)$, where f, g and h have expansions as in (1.2). Then, the method of brackets assigns the same value to the integrals*

$$I_1 = \int_0^\infty f(x)\, dx \text{ and } I_2 = \int_0^\infty g(x)h(x)\, dx. \tag{3.5}$$

Proof. Suppose that

$$\begin{aligned} f(x) &= \sum_n \phi_n a(n) x^{\alpha n+\beta} \\ g(x) &= \sum_{n_1} \phi_{n_1} b\left(n_1\right) x^{\alpha n_1+\beta_1} \\ h(x) &= \sum_{n_2} \phi_{n_2} c\left(n_2\right) x^{\alpha n_2+\beta_2}. \end{aligned}$$

Then

$$I_1 = \int_0^\infty f(x)dx = \sum_n \phi_n a\left(n\right) \langle \alpha n+\beta+1 \rangle = \frac{1}{|\alpha|} a(-s)\Gamma(s), \tag{3.6}$$

with $s = (1+\beta)/\alpha$.

To evaluate the second integral, observe that

$$\begin{aligned} g(x)h(x) &= x^{\beta_1+\beta_2} \left(\sum_{n_1=0}^\infty \frac{(-1)^{n_1}}{n_1!} b(n_1) x^{\alpha n_1} \right) \left(\sum_{n_2=0}^\infty \frac{(-1)^{n_2}}{n_2!} c(n_2) x^{\alpha n_2} \right) \\ &= x^{\beta_1+\beta_2} \sum_{n=0}^\infty F(n) x^{\alpha n}, \end{aligned}$$

with

$$\begin{aligned} F(n) &= \sum_{k=0}^n \frac{(-1)^k}{k!} b(k) \frac{(-1)^{n-k}}{(n-k)!} c(n-k) \\ &= \frac{(-1)^n}{n!} \sum_{k=0}^n \binom{n}{k} b(k) c(n-k). \end{aligned} \tag{3.7}$$

This yields

$$f(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left[\sum_{k=0}^n \binom{n}{k} b(k)c(n-k) \right] x^{\alpha n+\beta_1+\beta_2} \tag{3.8}$$

and matching this with (3.6) gives $\beta = \beta_1+\beta_2$ and

$$a(n) = \sum_{k=0}^n \binom{n}{k} b(k)c(n-k) = \sum_{k=0}^\infty \frac{(-1)^k}{k!} (-n)_k b(k) c(n-k). \tag{3.9}$$

Now, the method of brackets gives

$$I_2 = \int_0^\infty g(x)h(x)\,dx = \sum_{n_1,n_2} \phi_{n_1,n_2} b(n_1)c(n_2)\langle \alpha n_1 + \alpha n_2 + \beta + 1\rangle \tag{3.10}$$

and it yields two series as solutions

$$\begin{aligned} T_1 &= \frac{1}{|\alpha|}\sum_n \phi_n \Gamma\left(n+s\right) b(n)c(-n-s) \\ T_2 &= \frac{1}{|\alpha|}\sum_n \phi_n \Gamma\left(n+s\right) b(-n-s)c(n), \end{aligned} \tag{3.11}$$

with $s = (\beta+1)/\alpha$. Comparing with (3.6) shows that $I_1 = I_2$ is equivalent to

$$\Gamma(s)a(-s) = \sum_n \phi_n \Gamma(n+s) b(n)c(-s-n), \tag{3.12}$$

that is,

$$a(-s) = \sum_n \phi_n (s)_n b(n)c(-s-n). \tag{3.13}$$

The identity (3.13) is the extension of (3.9) from $n \in \mathbb{N}$ to $s \in \mathbb{C}$. This extension is part of the requirements on the functions f explained in Note 1.1. The proof is complete. □

It is direct to extend the result to the case of a finite number of factors.

Theorem 3.2. *Assume f admits a representation of the form $f(x) = \prod_{i=1}^{r} f_i(x)$. Then the value of the integral, obtained by method of brackets, is the same for both series representations.*

4. The exponential integral

The *exponential integral function* is defined by the integral formula

$$\text{Ei}(-x) = -\int_1^\infty \frac{\exp(-xt)}{t}\,dt, \text{ for } x > 0. \tag{4.1}$$

(See [16, 8.211.1]). The method of brackets is now used to produce a non-classical series for this function. Start by replacing the exponential function by its power series to obtain

$$\text{Ei}(-x) = -\sum_{n_1} \phi_{n_1} x^{n_1} \int_1^\infty t^{n_1-1}\,dt \tag{4.2}$$

and then use the method of brackets to produce

$$\int_1^\infty t^{n_1-1}\,dt = \int_0^\infty (y+1)^{n_1-1}\,dy = \sum_{n_2,\,n_3} \phi_{n_2 n_3} \frac{\langle -n_1+1+n_2+n_3\rangle\,\langle n_2+1\rangle}{\Gamma(-n_1+1)}.$$

Replace this in (4.2) to obtain

$$\text{Ei}(-x) = -\sum_{n_1,n_2,n_3} \phi_{n_1 n_2 n_3} x^{n_1} \frac{\langle -n_1+1+n_2+n_3\rangle\;\langle n_2+1\rangle}{\Gamma(-n_1+1)}. \tag{4.3}$$

The evaluation of this series by the method of brackets generates two identical terms for $\text{Ei}(-x)$:

$$\text{Ei}(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,\Gamma(n+1)} x^n. \tag{4.4}$$

Only one of them is kept, according to Rule E_4. This is a partially divergent series (from the value at $n=0$), written as

$$\text{Ei}(-x) = \sum_n \phi_n \frac{x^n}{n}. \tag{4.5}$$

The next example illustrates how to use this partially divergent series in the evaluation of an integral.

Example 4.1. Entry 6.223 of [16] gives the Mellin transform of the exponential integral as

$$\int_0^\infty x^{\mu-1}\text{Ei}(-bx)\,dx = -\frac{b^{-\mu}}{\mu}\Gamma(\mu). \tag{4.6}$$

To verify this, use the partially divergent series (4.5) and the method of brackets to obtain

$$\begin{aligned} \int_0^\infty x^{\mu-1}\text{Ei}(-bx)\,dx &= \sum_n \phi_n \frac{b^n}{n} \int_0^\infty x^{\mu+n-1}\,dx \\ &= \sum_n \phi_n \frac{b^n}{n} \langle \mu+n \rangle \\ &= -\frac{b^{-\mu}}{\mu}\Gamma(\mu), \end{aligned} \tag{4.7}$$

as claimed.

Example 4.2. Entry 6.228.2 in [16] is

$$G(\nu,\mu,\beta) = \int_0^\infty x^{\nu-1}e^{-\mu x}\text{Ei}(-\beta x)\,dx = -\frac{\Gamma(\nu)}{\nu(\beta+\nu)^\nu}\,{}_2F_1\left(\begin{matrix} 1 \;\; \nu \\ \nu+1 \end{matrix}\,\middle|\, \frac{\mu}{\beta+\mu}\right). \tag{4.8}$$

The partially divergent series (4.5) is now used to establish this formula. First form the bracket series

$$G(\nu,\mu,\beta) = \sum_{n_1,n_2} \phi_{n_1,n_2} \frac{\beta^{n_1}\mu^{n_2}}{n_1} \langle n_1+n_2+\nu \rangle. \tag{4.9}$$

Rule E_1 yields two cases from the equation $n_1+n_2+\nu=0$:
Case 1: $n_2 = -n_1-\nu$ produces

$$T_1 = \mu^{-\nu} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{\Gamma(n_1+\nu)}{n_1} \left(\frac{\beta}{\mu}\right)^{n_1}, \tag{4.10}$$

which is discarded since it is partially divergent (due to the term $n_1=0$).
Case 2: $n_1 = -n_2-\nu$ gives

$$T_2 = -\beta^{-\nu} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2!} \left(\frac{\mu}{\beta}\right)^{n_2} \frac{\Gamma(n_2+\nu)}{n_2+\nu}, \tag{4.11}$$

and using

$$\Gamma(n_2+\nu) = (\nu)_{n_2}\Gamma(\nu) \text{ and } n_2+\nu = \frac{\Gamma(n_2+\nu+1)}{\Gamma(n_2+\nu)} = \frac{(\nu+1)_{n_2}\Gamma(\nu+1)}{(\nu)_{n_2}\Gamma(\nu)} \tag{4.12}$$

equation (4.11) becomes

$$\begin{aligned} T_2 &= -\frac{\Gamma(\nu)}{\nu\,\beta^\nu}\sum_{n_2=0}^{\infty}\frac{(\nu)_{n_2}(\nu)_{n_2}}{n_2!\,(\nu+1)_{n_2}}\left(-\frac{\mu}{\beta}\right)^{n_2} \\ &= -\frac{\Gamma(\nu)}{\nu\,\beta^\nu}\,{}_2F_1\left(\left.\begin{matrix}\nu \quad \nu\\ \nu+1\end{matrix}\right| -\frac{\mu}{\beta}\right). \end{aligned} \tag{4.13}$$

The condition $|\mu| < |\beta|$ is imposed to guarantee the convergence of the series. Finally, the transformation rule (see entry 9.131.1 in [16])

$${}_2F_1\left(\left.\begin{matrix}\alpha \quad \beta\\ \gamma\end{matrix}\right| z\right) = (1-z)^{-\alpha}\,{}_2F_1\left(\left.\begin{matrix}\alpha \quad \gamma-\beta\\ \gamma\end{matrix}\right| \frac{z}{z-1}\right) \tag{4.14}$$

with $\alpha=\beta=\nu$, $\gamma=\nu+1$ and $z=-\mu/\beta$ yields (4.8).

Example 4.3. The next evaluation is entry 6.232.2 in [16]:

$$G(a,b) = \int_0^\infty \mathrm{Ei}(-ax)\cos bx\,dx = -\frac{1}{b}\tan^{-1}\left(\frac{b}{a}\right). \tag{4.15}$$

A direct application of the method of brackets using

$$\cos x = {}_0F_1\left(\left.\begin{matrix}-\\ \frac{1}{2}\end{matrix}\right| -\frac{x^2}{4}\right) \tag{4.16}$$

gives

$$G(a,b) = \sqrt{\pi}\sum_{n_1,n_2}\phi_{n_1,n_2}\frac{b^{2n_1}a^{n_2}}{2^{2n_1}\Gamma(n_1+\frac{1}{2})\,n_2}\langle 2n_1+n_2+1\rangle. \tag{4.17}$$

This produces two series for $G(a,b)$:

$$T_1 = \frac{\sqrt{\pi}}{b}\sum_{n_2=0}^{\infty}\frac{(-1)^{n_2}}{n_2!}\frac{\Gamma(\frac{1}{2}(n_2+1))}{n_2\,\Gamma(-\frac{1}{2}n_2)}\left(\frac{2a}{b}\right)^{n_2}, \tag{4.18}$$

and

$$T_2 = -\frac{\sqrt{\pi}}{a}\sum_{n_1=0}^{\infty}\frac{(-1)^{n_1}}{n_1!}\frac{\Gamma(2n_1+1)}{(2n_1+1)\Gamma(n_1+\frac{1}{2})}\left(\frac{b^2}{4a^2}\right)^{n_1}. \tag{4.19}$$

The analysis begins with a simplification of T_2. Use the duplication formula for the gamma function

$$\frac{\Gamma(2u)}{\Gamma(u)} = \frac{2^{2u-1}}{\sqrt{\pi}}\Gamma(u+\tfrac{1}{2}) \tag{4.20}$$

and write

$$\frac{1}{2n_1+1} = \frac{(1)_{n_1}\left(\frac{1}{2}\right)_{n_1}}{n_1!\,\left(\frac{3}{2}\right)_{n_1}} \tag{4.21}$$

to obtain

$$T_2 = -\frac{1}{a}\,{}_2F_1\left(\left.\begin{matrix}1 \quad \frac{1}{2}\\ \frac{3}{2}\end{matrix}\right| -\frac{b^2}{a^2}\right), \tag{4.22}$$

provided $|b| < |a|$ to guarantee convergence. The form (4.15) comes from the identity

$$ {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} & 1 \\ \frac{3}{2} & \end{matrix} \right| -z^2 \right) = \frac{\tan^{-1} z}{z} \tag{4.23} $$

(see 9.121.27 in [16]).

The next step is the evaluation of T_1. Separating the sum (4.18) into even and odd indices yields

$$ \begin{aligned} T_1 &= \frac{\sqrt{\pi}}{2b} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{\Gamma\left(n+\frac{1}{2}\right)}{n\Gamma(-n)} \left(\frac{4a^2}{b^2} \right)^n \\ &\quad - \frac{\sqrt{\pi}}{b} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \frac{\Gamma(n+1)}{(2n+1)\Gamma\left(-n-\frac{1}{2}\right)} \left(\frac{2a}{b} \right)^{2n+1}, \end{aligned} \tag{4.24} $$

and in hypergeometric form

$$ \begin{aligned} T_1 &= -\frac{\pi}{2b} \, {}_2F_1\left(\left. \begin{matrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \end{matrix} \right| -\frac{a^2}{b^2} \right) + \frac{a}{b^2} \, {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} & 1 \\ \frac{3}{2} & \end{matrix} \right| -\frac{a^2}{b^2} \right) \\ &= -\frac{\pi}{2b} + \frac{1}{b} \tan^{-1}\left(\frac{a}{b} \right). \end{aligned} \tag{4.25} $$

and this is the same as (4.15).

The evaluation of entry 6.232.1 in [16]

$$ \int_0^{\infty} \operatorname{Ei}(-ax) \sin bx \, dx = -\frac{1}{2b} \ln\left(1 + \frac{b^2}{a^2} \right) \tag{4.26} $$

is obtained in a similar form.

Example 4.4. Entry 6.782.1 in [16] is

$$ B(z) = \int_0^{\infty} \operatorname{Ei}(-x) J_0(2\sqrt{zx}) \, dx = \frac{e^{-z}-1}{z}. \tag{4.27} $$

Here

$$ J_0(x) = {}_0F_1\left(\left. \begin{matrix} - \\ 1 \end{matrix} \right| -\frac{x^2}{4} \right) \tag{4.28} $$

is the classical Bessel function defined in (1.4). Therefore

$$ J_0(2\sqrt{zx}) = \sum_{n_2} \phi_{n_2} \frac{z^{n_2}}{\Gamma(n_2+1)} x^{n_2}. \tag{4.29} $$

The standard procedure using the partially divergent series (4.4) now gives

$$ B(z) = \sum_{n_1,n_2} \phi_{n_1,n_2} \frac{1}{n_1} \frac{z^{n_2}}{\Gamma(n_2+1)} \langle n_1 + n_2 + 1 \rangle, \tag{4.30} $$

which gives the convergent series

$$ T_1 = -\sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1!} \frac{(1)_{n_1}}{(2)_{n_1}} z^{n_1} = -{}_1F_1\left(\left. \begin{matrix} 1 \\ 2 \end{matrix} \right| -z \right) = \frac{e^{-z}-1}{z}, \tag{4.31} $$

and the series

$$T_2 = -\frac{1}{z} \sum_{n_2=0}^{\infty} \frac{(-z)^{-n_2}}{\Gamma(1-n_2)}. \tag{4.32}$$

Observe that the expression T_2 contains a single non-vanishing term, so it is of the partially null type. An alternative form of T_2 is to write

$$\begin{aligned} T_2 &= -\frac{1}{z} \sum_{n_2=0}^{\infty} \frac{(-z^{-1})^{n_2}}{\Gamma(1-n_2)} \\ &= -\frac{1}{z} \sum_{n_2=0}^{\infty} \frac{(-z^{-1})^{n_2}}{\Gamma(1)\,(1)_{-n_2}} \\ &= -\frac{1}{z} \sum_{n_2=0}^{\infty} (z^{-1})^{n_2} (0)_{n_2}\ (1)_{n_2}\ \frac{(z^{-1})^{n_2}}{n_2!} \\ &= -\frac{1}{z} {}_2F_0\left(\left. \begin{matrix} 0 \ 1 \\ - \end{matrix} \right| \frac{1}{z} \right). \end{aligned} \tag{4.33}$$

The series ${}_2F_0\left(\left. \begin{matrix} a \ b \\ - \end{matrix} \right| z \right)$ diverges, unless one of the parameters a or b is a non-positive integer, in which case the series terminates and it reduces to a polynomial. This is precisely what happens here: only the term for $n_2 = 0$ is non-vanishing and T_2 reduces to

$$T_2 = -\frac{1}{z}. \tag{4.34}$$

This gives the asymptotic behavior $B(z) \sim -1/z$, consistent with the value of T_1 for large z. This phenomena occurs every time one obtains a series of the form ${}_pF_q(z)$ with $p \geq q+2$ when the series diverges. The truncation represents an asymptotic approximation of the solution.

5. The Tricomi function

The confluent hypergeometric function, denoted by ${}_1F_1\left(\left. \begin{matrix} a \\ c \end{matrix} \right| z \right)$, defined in (2.19), arises when two of the regular singular points of the differential equation for the Gauss hypergeometric function ${}_2F_1\left(\left. \begin{matrix} a \ b \\ c \end{matrix} \right| z \right)$, given by

$$z(1-z)y'' + (c - (a+b+1)z)y' - aby = 0, \tag{5.1}$$

are allowed to merge into one singular point. More specifically, if we replace z by z/b in ${}_2F_1\left(\left. \begin{matrix} a \ b \\ c \end{matrix} \right| z \right)$, then the corresponding differential equation has singular points at 0, b and ∞. Now let $b \to \infty$ so as to have infinity as a confluence of two singularities. This results in the function ${}_1F_1\left(\left. \begin{matrix} a \\ c \end{matrix} \right| z \right)$ so that

$${}_1F_1\left(\left. \begin{matrix} a \\ c \end{matrix} \right| z \right) = \lim_{b \to \infty} {}_2F_1\left(\left. \begin{matrix} a \ \ b \\ c \end{matrix} \right| z \right), \tag{5.2}$$

and the corresponding differential equation

$$zy'' + (c-z)y' - ay = 0, \tag{5.3}$$

known as the confluent hypergeometric equation. Evaluation of integrals connected to this equation are provided in [3].

The equation (5.3) has two linearly independent solutions:

$$M(a,b;x) = {}_1F_1\left(\left. \begin{matrix} a \\ b \end{matrix} \right| x \right), \tag{5.4}$$

known as the Kummer function and the *Tricomi function* with integral representation

$$U(a,b;x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \exp(-xt)(1+t)^{b-a-1}\, dt, \tag{5.5}$$

and hypergeometric form

$$U(a,b;x) = \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1\left(\left. \begin{matrix} 1+a-b \\ 2-b \end{matrix} \right| x \right) + \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1\left(\left. \begin{matrix} a \\ b \end{matrix} \right| x \right). \tag{5.6}$$

A direct application of the method of brackets gives

$$\begin{aligned} U(a,b;x) &= \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} \left(\sum_{n_1} \phi_{n_1} x^{n_1} t^{n_1} \right) \left(\sum_{n_2,n_3} \phi_{n_2,n_3} t^{n_3} \frac{\langle 1+a-b+n_2+n_3 \rangle}{\Gamma(1+a-b)} \right) dt \\ &= \frac{1}{\Gamma(a)} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} x^{n_1} \frac{\langle 1+a-b+n_2+n_3 \rangle}{\Gamma(1+a-b)} \langle a+n_1+n_3 \rangle. \end{aligned}$$

This is a bracket series of index 1 and its evaluation produces three terms:

$$\begin{aligned} U_1(a,b;x) &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1\left(\left. \begin{matrix} a \\ b \end{matrix} \right| x \right), \\ U_2(a,b;x) &= \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1\left(\left. \begin{matrix} 1+a-b \\ 2-b \end{matrix} \right| x \right), \\ U_3(a,b;x) &= x^{-a} {}_2F_0\left(\left. \begin{matrix} a \quad 1+a-b \\ - \end{matrix} \right| -\frac{1}{x} \right). \end{aligned}$$

The first two are convergent in the region $|x| < 1$ and their sum yields (5.6). The series U_3 is formally divergent, the terms are finite but the series is divergent.

Example 5.1. The Mellin transform of the Tricomi function is given by

$$I(a,b;\beta) = \int_0^\infty x^{\beta-1} U(a,b,x)\, dx. \tag{5.7}$$

Entry 7.612.1 of [16]

$$\int_0^\infty x^{\beta-1} {}_1F_1\left(\left. \begin{matrix} a \\ b \end{matrix} \right| -x \right) dx = \frac{\Gamma(\beta)\Gamma(a-\beta)\Gamma(b)}{\Gamma(b-\beta)\Gamma(a)} \tag{5.8}$$

is used in the evaluation of $I(a,b,\beta)$. A proof of (5.8) appears in [3].

The first evaluation of (5.7) uses the hypergeometric representation (5.6) and the formula (5.8). This is a traditional computation. Direct substitution gives

$$\begin{aligned} I(a,b,\beta) &= \frac{\Gamma(b-1)}{\Gamma(a)} \int_0^\infty x^{\beta-b} {}_1F_1 \left(\left. \begin{matrix} 1+a-b \\ 2-b \end{matrix} \right| x \right) dx + \\ & \quad \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \int_0^\infty x^{\beta-1} {}_1F_1 \left(\left. \begin{matrix} a \\ b \end{matrix} \right| x \right) dx \\ &= -(-1)^{-\beta+b} \frac{\Gamma(b-1)}{\Gamma(a)} \frac{\Gamma(\beta-b+1)\Gamma(a-\beta)\Gamma(2-b)}{\Gamma(1+a-b)\Gamma(1-\beta)} \\ & \quad +(-1)^{-\beta} \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \frac{\Gamma(\beta)\Gamma(a-\beta)\Gamma(b)}{\Gamma(b-\beta)\Gamma(a)}. \end{aligned}$$

The result

$$\int_0^\infty x^{\beta-1} U(a,b,x)\, dx = \frac{\Gamma(a-\beta)\Gamma(\beta-b+1)\Gamma(\beta)}{\Gamma(a)\Gamma(a-b+1)} \tag{5.9}$$

follows from simplification of the previous expression.

The second evaluation of (5.7) uses the method of brackets and the divergent series U_3. It produces the result directly. Start with

$$\begin{aligned} I(a,b,\beta) &= \int_0^\infty x^{\beta-1} U(a,b,x)\, dx \\ &= \int_0^\infty x^{\beta-a-1} {}_2F_0 \left(\left. \begin{matrix} a \;\; 1+a-b \\ - \end{matrix} \right| -\frac{1}{x} \right) dx \\ &= \sum_n \phi_n (a)_n (1+a-b)_n \langle \beta-a-n \rangle. \end{aligned}$$

A standard evaluation by the method of brackets now reproduces (5.9).

Example 5.2. The evaluation of

$$J(a,b;\mu) = \int_0^\infty e^{-\mu x} U(a,b,x)\, dx \tag{5.10}$$

is given next. Start with the expansions

$$\exp(-\mu x) = \sum_{n_1} \phi_{n_1} \mu^{n_1} x^{n_1} \tag{5.11}$$

and

$$\begin{aligned} U(a,b,x) &= x^{-a}\, {}_2F_0 \left(\left. \begin{matrix} a \;\; 1+a-b \\ - \end{matrix} \right| -\frac{1}{x} \right) \\ &= \frac{x^{-a}}{\Gamma(a)\Gamma(1+a-b)} \sum_{n_2} \phi_{n_2} \Gamma(a+n_2)\Gamma(1+a-b+n_2) x^{-n_2}, \end{aligned}$$

to write

$$J(a,b;\mu) = \frac{1}{\Gamma(a)\Gamma(1+a-b)} \sum_{n_1,n_2} \phi_{n_1,n_2} \mu^{n_1} \Gamma(a+n_2)\Gamma(1+a-b+n_2) \langle n_1-a-n_2+1 \rangle.$$

This yields the two series

$$\begin{aligned} J_1(a,b;\mu) &= \frac{1}{\Gamma(a)\Gamma(1+a-b)} \sum_n \phi_n \Gamma(a-1-n)\Gamma(n+1)\Gamma(2-b+n)\mu^n \\ &= \frac{\Gamma(2-b)}{(a-1)\Gamma(1+a-b)} {}_2F_1\left(\begin{matrix} 1 \ \ 2-b \\ 2-a \end{matrix} \middle| \mu \right), \end{aligned}$$

and

$$\begin{aligned} J_2(a,b;\mu) &= \frac{\mu^{a-1}}{\Gamma(a)\Gamma(1+a-b)} \sum_n \phi_n \Gamma(-a+1-n)\Gamma(a+n)\Gamma(1+a-b+n)\mu^n \\ &= \mu^{a-1}\Gamma(1-a) {}_1F_0\left(\begin{matrix} 1+a-b \\ - \end{matrix} \middle| \mu \right) \\ &= \frac{\mu^{a-1}\,\Gamma(1-a)}{(1-\mu)^{1+a-b}}. \end{aligned}$$

In the case $|\mu| < 1$, both J_1 and J_2 are convergent. Therefore

$$\int_0^\infty \exp(-\mu x) U(a,b,x)\,dx = \frac{\Gamma(2-b)}{(a-1)\Gamma(1+a-b)} {}_2F_1\left(\begin{matrix} 1 \ \ 2-b \\ 2-a \end{matrix} \middle| \mu \right) + \frac{\mu^{a-1}\,\Gamma(1-a)}{(1-\mu)^{1+a-b}}.$$

In the case $\mu = 1$, the series J_2 diverges, so it is discarded. This produces

$$\int_0^\infty e^{-x} U(a,b,x)\,dx = \frac{\Gamma(2-b)}{(a-1)\Gamma(1+a-b)} {}_2F_1\left(\begin{matrix} 1 \ \ 2-b \\ 2-a \end{matrix} \middle| 1 \right). \tag{5.12}$$

Gauss' value (2.20) gives

$$\int_0^\infty e^{-x} U(a,b,x)\,dx = \frac{\Gamma(2-b)}{\Gamma(2-b+a)}. \tag{5.13}$$

In particular, if a is a positive integer, say $a = k$, then

$$\int_0^\infty e^{-x} U(k,b,x)\,dx = \frac{1}{(b-2)_k}. \tag{5.14}$$

This result is summarized next.

Proposition 5.3. *Let*

$$J(a,b;\mu) = \int_0^\infty e^{-\mu x} U(a,b,x)\,dx. \tag{5.15}$$

Then, for $|\mu| < 1$,

$$J(a,b,\mu) = \frac{\Gamma(2-b)}{(a-1)\Gamma(1+a-b)} {}_2F_1\left(\begin{matrix} 1 \ \ 2-b \\ 2-a \end{matrix} \middle| \mu \right) + \frac{\mu^{a-1}\,\Gamma(1-a)}{(1-\mu)^{1+a-b}}, \tag{5.16}$$

and for $\mu = 1$,

$$J(a,b;1) = \frac{\Gamma(2-b)}{\Gamma(2-b+a)}. \tag{5.17}$$

In the special case $a = k \in \mathbb{N}$,

$$J(k,b;1) = \frac{1}{(b-2)_k}. \tag{5.18}$$

6. The Airy function

The Airy function, defined by the integral representation

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt \right) dt \tag{6.1}$$

satisfies the equation

$$\frac{d^2y}{dx^2} - xy = 0, \tag{6.2}$$

and the condition $y \to 0$ as $x \to \infty$. A second linearly independent solution of (6.2) is usually taken to be

$$\mathrm{Bi}(x) = \frac{1}{\pi} \int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt \right) + \sin\left(\frac{t^3}{3} + xt \right) \right] dt. \tag{6.3}$$

Using (4.16) produces

$$\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{\pi} \sum_{n_1} \phi_n \frac{1}{\left(\frac{1}{2}\right)_{n_1} 2^{2n_1}} \int_0^\infty \left(\frac{t^3}{3} + xt \right)^{2n_1} dt \\
&= \frac{1}{\pi} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{x^{n_2} \langle -2n_1 + n_2 + n_3 \rangle}{\left(\frac{1}{2}\right)_{n_1} 2^{2n_1} \Gamma(-2n_1) 3^{n_3}} \int_0^\infty t^{3n_3+n_2} \, dt \\
&= \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{x^{n_2}}{\sqrt{\pi}\Gamma(-2n_1)\Gamma\left(\frac{1}{2}+n_1\right) 2^{2n_1} 3^{n_3}} \langle -2n_1 + n_2 + n_3 \rangle \langle 3n_3 + n_2 + 1 \rangle.
\end{aligned}$$

The usual resolution of this bracket series gives three cases:

$$T_1 = \frac{1}{2} \sqrt{\frac{3}{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(-\frac{1}{2} - 3n)}{\Gamma(-2n)} \left(\frac{3}{4} \right)^n x^{3n+1/2} \tag{6.4}$$

a totally null series,

$$T_2 = \frac{1}{6^{2/3}\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(\frac{1}{6} - \frac{n}{3})}{\Gamma(\frac{1}{3} - \frac{2n}{3})} \left(\frac{3}{4} \right)^{n/3} x^n \tag{6.5}$$

a partially divergent series (at the index $n = 18$), and

$$T_3 = \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(3n+1)\Gamma(n+\frac{1}{2})}{\Gamma(-n)\Gamma(2n+1)} \left(\frac{4}{3} \right)^n x^{-3n-1} \tag{6.6}$$

a totally null series, as T_1 was.

Example 6.1. The series for $\mathrm{Ai}(x)$ are now used to evaluate the Mellin transform

$$I(s) = \int_0^\infty x^{s-1} \mathrm{Ai}(x) \, dx. \tag{6.7}$$

This integral is now computed using the three series T_j given above. Using first the value of T_1 and the formulas

$$\Gamma(2u) = \frac{2^{2u-1}}{\sqrt{\pi}} \Gamma(u)\Gamma(u+\tfrac{1}{2}) \text{ and } \Gamma(3u) = \frac{3^{3u-\frac{1}{2}}}{2\pi} \Gamma(u)\Gamma(u+\tfrac{1}{3})\Gamma(u+\tfrac{2}{3}) \tag{6.8}$$

(these appear as 8.335.1 and 8.335.2 in [16], respectively), give

$$\begin{aligned} I(s) &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \sum_n \phi_n \left(\frac{3}{4}\right)^n \frac{\Gamma(-\frac{1}{2}-3n)}{\Gamma(-2n)} \langle s+3n+\tfrac{1}{2}\rangle \\ &= \frac{1}{6}\sqrt{\frac{3}{\pi}} \left(\frac{3}{4}\right)^{-s/3-1/6} \frac{\Gamma\left(\frac{2s+1}{6}\right)\Gamma(s)}{\Gamma\left(\frac{2s+1}{3}\right)}. \\ &= 3^{-(s+2)/3} \frac{\Gamma(s)}{\Gamma(\frac{s+2}{3})} \\ &= \frac{3^{(4s-7)/6}}{2\pi} \Gamma\left(\frac{s+1}{3}\right)\Gamma\left(\frac{s}{3}\right). \end{aligned} \tag{6.9}$$

Similar calculations, using T_2 or T_3, give the same result. This result is stated next.

Lemma 6.2. *The Mellin transform of the Airy function is given by*

$$\int_0^\infty x^{s-1}\mathrm{Ai}(x)\,dx = \frac{1}{2\pi} 3^{(4s-7)/6} \Gamma\left(\frac{s+1}{3}\right)\Gamma\left(\frac{s}{3}\right). \tag{6.10}$$

7. The Bessel function K_ν

This section presents series representations for the Bessel function $K_\nu(x)$ defined by the integral representation

$$K_\nu(x) = \frac{2^\nu \Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} x^\nu \int_0^\infty \frac{\cos t\,dt}{(x^2+t^2)^{\nu+\frac{1}{2}}}, \tag{7.1}$$

given as entry 8.432.5 in [16]. Using the representation (4.16) of $\cos t$ as ${}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix}\middle| -\frac{t^2}{4}\right)$ and using Rule P_2 in Section 2 to expand the binomial in the integrand as a bracket series gives

$$K_\nu(x) = 2^\nu \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{x^{2n_3+\nu}}{2^{2n_1}\Gamma(n_1+\frac{1}{2})} \langle \nu+\tfrac{1}{2}+n_2+n_3\rangle\langle 2n_1+2n_2+1\rangle. \tag{7.2}$$

The usual procedure to evaluate this bracket series gives three expressions:

$$\begin{aligned} T_1 &= 2^{\nu-1}x^{-\nu} \sum_n \phi_n \Gamma(\nu-n)\left(\frac{x^2}{4}\right)^n, \\ T_2 &= 2^{-1-\nu}x^{\nu} \sum_n \phi_n \Gamma(-\nu-n)\left(\frac{x^2}{4}\right)^n, \\ T_3 &= 2^\nu \sum_n \phi_n \frac{2^{2n}}{\Gamma(-n)} \Gamma(n+\nu+\tfrac{1}{2})\Gamma(n+\tfrac{1}{2})x^{-2n-\nu-1}. \end{aligned} \tag{7.3}$$

The series T_3 is a totally null series for K_ν. In the case $\nu \notin \mathbb{N}$, the series T_1 and T_2 are finite and $K_\nu(x) = T_1 + T_2$ gives the usual expression in terms of the Bessel I_ν function

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin\pi\nu}, \tag{7.4}$$

as given in entry 8.485 in [16].

In the case $\nu = k \in \mathbb{N}$, the series T_1 is partially divergent (the terms $n = 0, 1, \ldots, k$ have divergent coefficients) and the series T_2 is totally divergent (every coefficient is divergent). In the case $\nu = 0$, both the series T_1 and T_2 become

$$\text{Totally divergent series for } K_0(x) = \frac{1}{2}\sum_n \phi_n \Gamma(-n)\left(\frac{x^2}{4}\right)^n, \tag{7.5}$$

using Rule E_4 to keep a single copy of the divergent series. This complements the

$$\text{Totally null series for } K_0(x) = \sum_n \phi_n \frac{2^{2n}}{\Gamma(-n)}\Gamma^2(n+\tfrac{1}{2})x^{-2n-1}. \tag{7.6}$$

The examples presented below illustrate the use of these divergent series in the computation of definite integrals with the Bessel function K_0 in the integrand. Entries in [16] with K_0 as the result of an integral have been discussed in [6].

Example 7.1. Entry 6.511.12 of [16] states that

$$\int_0^\infty K_0(x)\,dx = \frac{\pi}{2}. \tag{7.7}$$

To verify this result, use the totally null representation (7.6) to obtain

$$\begin{aligned} \int_0^\infty K_0(x)\,dx &= \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} 4^n \int_0^\infty x^{-2n-1}\,dx \\ &= \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} 4^n \langle -2n \rangle. \end{aligned} \tag{7.8}$$

The value of the bracket series is

$$\begin{aligned} \int_0^\infty K_0(x)\,dx &= \frac{1}{2}\Gamma\left(n+\tfrac{1}{2}\right)^2 4^n \Big|_{n=0} \\ &= \frac{\pi}{2}. \end{aligned} \tag{7.9}$$

Example 7.2. The Mellin transform

$$G(\beta, s) = \int_0^\infty x^{s-1} K_0(\beta x)\,dx \tag{7.10}$$

is evaluated next. The totally divergent series (7.5) yields

$$G(\beta, s) = \frac{1}{2}\sum_n \phi_n \Gamma(-n) \frac{\beta^{2n}}{2^{2n}} \langle 2n + s \rangle \tag{7.11}$$

and a direct evaluation of the brackets series using Rule E_1 gives

$$G(\beta, s) = \frac{2^{s-2}}{\beta^s}\Gamma^2\left(\frac{s}{2}\right). \tag{7.12}$$

Now using the totally null representation (7.6) gives the bracket series

$$G(\beta, s) = \sum_n \phi_n \frac{2^{2n}\Gamma^2(n+\frac{1}{2})}{\beta^{2n+1}\,\Gamma(-n)} \langle s - 1 - 2n \rangle. \tag{7.13}$$

One more application of Rule E_1 gives (7.12) again.

Example 7.3. Entry 6.611.9 of [16] is

$$\int_0^\infty e^{-ax} K_0(bx)\,dx = \frac{1}{\sqrt{b^2-a^2}} \cos^{-1}\left(\frac{a}{b}\right), \tag{7.14}$$

for $\text{Re}\,(a+b)>0$. The totally divergent representation (7.5) and the series for the exponential function (5.11) give the bracket series

$$\int_0^\infty e^{-ax} K_0(bx)\,dx = \frac{1}{2}\sum_{n_1,n_2} \phi_{n_1 n_2} \Gamma(-n_2) \frac{a^{n_1} b^{2n_2}}{2^{2n_2}} \langle n_1 + 2n_2 + 1\rangle. \tag{7.15}$$

The usual procedure gives two expressions:

$$T_1 = \frac{1}{2a}\sum_n \phi_n \Gamma(2n+1)\Gamma(-n)\left(\frac{b^2}{4a^2}\right)^n, \tag{7.16}$$

which is discarded since it is divergent and

$$T_2 = \frac{1}{2b}\sum_{n=0}^\infty \frac{\Gamma\left(\frac{n+1}{2}\right)^2}{n!}\left(-\frac{2a}{b}\right)^n. \tag{7.17}$$

Separating the series according to the parity of the index n yields

$$T_2 = \frac{1}{2b}\left[\pi \sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n}{n!}\left(\frac{a^2}{b^2}\right)^n - \frac{2a}{b}\sum_{n=0}^\infty \frac{(1)_n^2}{n!\,\left(\frac{3}{2}\right)_n}\left(\frac{a^2}{b^2}\right)^n\right]. \tag{7.18}$$

The identity [16, 9.121.1]

$${}_2F_1\left(\left.\begin{matrix}-n, b\\ b\end{matrix}\right| -z\right) = (1+z)^n, \tag{7.19}$$

with $n = -\frac{1}{2}$ gives

$$\frac{\pi}{2b}\sum_{n=0}^\infty \frac{\left(\frac{1}{2}\right)_n}{n!}\left(\frac{a^2}{b^2}\right)^n = \frac{\pi}{2}\frac{1}{\sqrt{b^2-a^2}}. \tag{7.20}$$

The identity

$$-\frac{a}{b^2}\sum_{n=0}^\infty \frac{(1)_n^2}{n!\,\left(\frac{3}{2}\right)_n}\left(\frac{a}{b}\right)^{2n} = -\frac{1}{\sqrt{b^2-a^2}}\sin^{-1}\left(\frac{a}{b}\right) \tag{7.21}$$

comes from the Taylor series

$$\frac{2x\sin^{-1}x}{\sqrt{1-x^2}} = \sum_{n=1}^\infty \frac{2^{2n}x^{2n}}{n\binom{2n}{n}}. \tag{7.22}$$

(See Theorem 7.6.2 in [20] for a proof). The usual argument now gives

$$T_2 = \int_0^\infty e^{-ax}K_0(bx)\,dx = \frac{1}{\sqrt{b^2-a^2}}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{a}{b}\right)\right], \tag{7.23}$$

an equivalent form of (7.14).

Example 7.4. The next example,

$$\int_0^\infty x\sin(bx)K_0(ax)\,dx = \frac{\pi b}{2}(a^2+b^2)^{-3/2}, \tag{7.24}$$

appears as entry 6.691 in [16]. The factor $\sin bx$ in integrand is expressed as a series:

$$\sin(bx) = b\,x\,{}_0F_1\left(\left. \begin{matrix} - \\ \frac{3}{2} \end{matrix} \right| -\frac{b^2x^2}{4} \right) = b\Gamma\left(\tfrac{3}{2}\right) \sum_{n_2} \phi_{n_2} \frac{\left(\frac{b^2}{4}\right)^{n_2}}{\Gamma\left(n_2+\frac{1}{2}\right)} x^{2n_2+1} \tag{7.25}$$

and the Bessel factor is replaced by its totally-null representation (7.6)

$$K_0(ax) = \frac{1}{a} \sum_{n_1} \phi_{n_1} \frac{\Gamma\left(n_1+\frac{1}{2}\right)^2}{\Gamma(-n_1)} \left(\frac{4}{a^2}\right)^{n_1} x^{-2n_1-1}. \tag{7.26}$$

This yields

$$\int_0^\infty x\sin(bx)K_0(ax)\,dx = \Gamma\left(\frac{3}{2}\right) \sum_{n_1,n_2} \phi_{n_1,n_2} \frac{\Gamma\left(n_1+\frac{1}{2}\right)^2}{\Gamma\left(n_2+\frac{3}{2}\right)\Gamma(-n_1)} \frac{4^{n_1-n_2}b^{2n_2+1}}{a^{2n_1+1}} \langle 2+2n_2-2n_1\rangle. \tag{7.27}$$

These representation produces two solutions S_1 and S_2, one per free index, that *are identical*. The method of brackets rules states that one only should be taken. This is:

$$S_1 = \frac{\sqrt{\pi}\,b}{a^3} \sum_{k=0}^\infty \frac{\Gamma\left(k+\frac{3}{2}\right)(-1)^k b^{2k}}{k!\,a^{2k}}. \tag{7.28}$$

The result now follows from the identity

$$\sum_{k=0}^\infty \frac{\left(\frac{3}{2}\right)_k}{k!} \left(-\frac{b}{a}\right)^k = {}_1F_0\left(\left. \begin{matrix} \frac{3}{2} \\ - \end{matrix} \right| -\frac{b}{a} \right) \tag{7.29}$$

and the binomial theorem obtaining

$${}_1F_0\left(\left. \begin{matrix} \frac{3}{2} \\ - \end{matrix} \right| x \right) = \frac{1}{(1-x)^{3/2}}. \tag{7.30}$$

Example 7.5. The next example in this section evaluates

$$G(a,b) = \int_0^\infty J_0(ax)K_0(bx)\,dx. \tag{7.31}$$

From the representation

$$J_0(ax) = \sum_{n_1} \phi_{n_1} \frac{a^{2n_1}x^{2n_1}}{2^{2n_1}\Gamma(n_1+1)} \tag{7.32}$$

and the null-series (1.10) it follows that

$$G(a,b) = \sum_{n_1,n_2} \phi_{n_1,n_2} \frac{a^{2n_1}2^{2(n_2-n_1)}\Gamma^2(n_2+\frac{1}{2})}{\Gamma(n_1+1)\Gamma(-n_2)b^{2n_2+1}} \langle 2n_1-2n_2\rangle. \tag{7.33}$$

This bracket series generates two identical series, so only one is kept to produce

$$G(a,b) = \frac{1}{2b}\sum_n \phi_n \frac{\Gamma^2(n+\frac{1}{2})}{\Gamma(n+1)}\left(\frac{a^2}{b^2}\right)^n = \frac{\pi}{2b}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\middle| -\frac{a^2}{b^2}\right) = \frac{1}{b}\mathbf{K}\left(\frac{ia}{b}\right). \tag{7.34}$$

Here $\mathbf{K}(z)$ is the elliptic integral of the first kind. Using the identity

$$\mathbf{K}(iz) = \frac{1}{\sqrt{z^2+1}}\mathbf{K}\left(\frac{z}{\sqrt{z^2+1}}\right) \tag{7.35}$$

yields

$$G(a,b) = \frac{1}{\sqrt{a^2+b^2}}\mathbf{K}\left(\frac{a}{\sqrt{a^2+b^2}}\right). \tag{7.36}$$

Example 7.6. The next example evaluates

$$H(a) = \int_0^\infty K_0^2(ax)\,dx. \tag{7.37}$$

Naturally $H(a) = H(1)/a$, but it is convenient to keep a as a parameter. The problem is generalized to

$$H_1(a,b) = \int_0^\infty K_0(ax)K_0(bx)\,dx, \tag{7.38}$$

and $H(a) = H_1(a,a)$. The evaluation uses the totally divergent series (7.5)

$$K_0(ax) = \sum_{n_1} \phi_{n_1} \frac{a^{2n_1}\Gamma(-n_1)}{2^{2n_1+1}} x^{2n_1} \tag{7.39}$$

as well as the integral representation (see 8.432.6 [16]) and the corresponding bracket series

$$K_0(bx) = \frac{1}{2}\int_0^\infty \exp\left(-t - \frac{b^2x^2}{4t}\right)\frac{dt}{t} = \sum_{n_2,n_3} \phi_{n_2,n_3} \frac{b^{2n_3}x^{2n_3}}{2^{2n_3+1}} \langle n_2 - n_3\rangle. \tag{7.40}$$

Then

$$H_1(a,b) = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_1}b^{2n_3}\Gamma(-n_1)}{2^{2n_1+2n_3+2}} \langle n_2 - n_3\rangle \langle 2n_1 + 2n_3 + 1\rangle. \tag{7.41}$$

The evaluation of this bracket series requires an extra parameter ε and to consider

$$H_2(a,b,\varepsilon) = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_1}b^{2n_3}\Gamma(-n_1)}{2^{2n_1+2n_3+2}} \langle n_2 - n_3 + \varepsilon\rangle \langle 2n_1 + 2n_3 + 1\rangle. \tag{7.42}$$

Evaluating this brackets series produces three values, one divergent, which is discarded, and two others:

$$
\begin{aligned}
T_2 &= \frac{1}{4a} c^{\varepsilon} \sum_n \phi_n \Gamma(-n-\varepsilon)\Gamma^2(\varepsilon+n+\tfrac{1}{2})c^n \\
T_3 &= \frac{1}{4a} \sum_n \phi_n \Gamma(-n+\varepsilon)\Gamma^2(n+\tfrac{1}{2})c^n,
\end{aligned}
\tag{7.43}
$$

with $c = b^2/a^2$. Converting the Γ-factors into Pochhammer symbols produces

$$
\begin{aligned}
T_2 &= \frac{1}{4a} c^{\varepsilon} \Gamma(-\varepsilon)\Gamma^2\left(\tfrac{1}{2}+\varepsilon\right) {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}+\varepsilon \;\; \frac{1}{2}+\varepsilon \\ 1+\varepsilon \end{matrix} \right| c \right) \\
T_3 &= \frac{\pi}{4a} \Gamma(\varepsilon) \; {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} \;\; \frac{1}{2} \\ 1-\varepsilon \end{matrix} \right| c \right).
\end{aligned}
\tag{7.44}
$$

This yields

$$
H_2(a,b,\varepsilon) = \frac{\pi}{4a} \left[\Gamma(\varepsilon) {}_2F_1\left(\left. \begin{matrix} \frac{1}{2} \;\; \frac{1}{2} \\ 1-\varepsilon \end{matrix} \right| c \right) - c^{\varepsilon} \frac{\Gamma^2(\frac{1}{2}+\varepsilon)}{\varepsilon\,\Gamma(\varepsilon)\sin\pi\varepsilon} {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}+\varepsilon \;\; \frac{1}{2}+\varepsilon \\ 1+\varepsilon \end{matrix} \right| c \right) \right].
$$

Let $c \to 1$ ($b \to a$) and use Gauss' formula (2.20) to obtain

$$
{}_2F_1\left(\left. \begin{matrix} \frac{1}{2} \;\; \frac{1}{2} \\ 1-\varepsilon \end{matrix} \right| 1 \right) = \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma^2\left(\frac{1}{2}-\varepsilon\right)} \quad \text{and} \quad {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}+\varepsilon \;\; \frac{1}{2}+\varepsilon \\ 1+\varepsilon \end{matrix} \right| 1 \right) = \frac{\Gamma(1+\varepsilon)\Gamma(-\varepsilon)}{\Gamma^2\left(\frac{1}{2}\right)},
$$

and this produces

$$
\begin{aligned}
H_2(a,a,\varepsilon) &= \frac{\Gamma(-\varepsilon)^2\Gamma^2\left(\varepsilon+\frac{1}{2}\right)\Gamma(\varepsilon+1)}{4\pi a} + \frac{\pi\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{4a\,\Gamma^2\left(\frac{1}{2}-\varepsilon\right)} \\
&= \frac{\pi}{4a} \left[\frac{\Gamma^2(-\varepsilon)\Gamma(\varepsilon+1)\Gamma^2(\varepsilon+\frac{1}{2})}{\pi^2} + \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\Gamma(\varepsilon)}{\Gamma^2(\frac{1}{2}-\varepsilon)} \right].
\end{aligned}
$$

Expanding $H_2(a,a,\varepsilon)$ in powers of ε gives

$$
H(a,a,\varepsilon) = \frac{\pi^2}{4a} - \frac{\pi^2}{4a}(\gamma + 4\ln 2)\varepsilon + o(\varepsilon). \tag{7.45}
$$

Letting $\varepsilon \to 0$ gives

$$
\int_0^\infty K_0^2(ax)\,dx = \frac{\pi^2}{4a}. \tag{7.46}
$$

Example 7.7. The final example in this section is the general integral

$$
I(a,b;\nu,\lambda;\rho) = \int_0^\infty x^{\rho-1} K_\nu(ax) K_\lambda(bx)\,dx. \tag{7.47}
$$

The case $a = b$ appears in [18].

The evaluation uses the integral representation

$$
K_\nu(ax) = \frac{(ax)^\nu}{2^{\nu+1}} \int_0^\infty \exp\left(-t - \frac{a^2x^2}{4t}\right) \frac{dt}{t^{\nu+1}} \tag{7.48}
$$

appearing in [16, 8.432.6]. This produces the bracket series representation

$$
K_\nu(ax) = \frac{1}{2^{\nu+1}} \sum_{n_1,n_2} \phi_{n_1,n_2} \frac{a^{2n_2+\nu}}{2^{2n_2}} x^{2n_2+\nu} \langle n_1 - n_2 - \nu \rangle. \tag{7.49}
$$

The second factor uses the totally null representation (1.10)

$$K_\lambda(bx) = 2^\lambda \sum_{n_3} \phi_{n_3} \frac{2^{2n_3}\Gamma(n_3+\lambda+\frac{1}{2})\Gamma(n_3+\frac{1}{2})}{\Gamma(-n_3)b^{2n_3+\lambda+1}} \frac{1}{x^{2n_3+\lambda+1}}. \tag{7.50}$$

Replacing in (7.47) produces the bracket series

(7.51)

$$I(a,b;\nu,\lambda;\rho) = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_2+\nu}2^{\lambda-\nu-1+2n_3-2n_2}\Gamma(n_3+\lambda+\frac{1}{2})\Gamma(n_3+\frac{1}{2})}{b^{2n_3+\lambda+1}\,\Gamma(-n_3)} \times \langle n_1-n_2-\nu\rangle\langle \rho+\nu-\lambda+2n_2-2n_3-1\rangle.$$

The vanishing of the brackets gives the system of equations

$$\begin{aligned} n_1-n_2 &= \nu \\ 2n_2-2n_3 &= -\rho-\nu+\lambda+1. \end{aligned} \tag{7.52}$$

The matrix of coefficients is of rank 2, so it produces three series as candidates for values of the integral, one per free index.

Case 1: n_1 free. Then $n_2 = n_1-\nu$ and $n_3 = \frac{\rho-\nu-\lambda-1}{2}+n_1$. This gives

$$T_1 = 2^{\rho-3}\frac{b^{\nu-\rho}}{a^\nu}\Gamma\left(\frac{\rho-\nu+\lambda}{2}\right)\Gamma\left(\frac{\rho-\nu-\lambda}{2}\right)\Gamma(\nu)\,{}_2F_1\left(\left.\begin{matrix}\frac{\rho-\nu+\lambda}{2} \quad \frac{\rho-\nu-\lambda}{2}\\ 1-\nu\end{matrix}\right| \frac{a^2}{b^2}\right).$$

Case 2: n_2 free. Then $n_1 = n_2+\nu$ and $n_3 = \frac{\rho+\nu-\lambda-1}{2}+n_2$. This gives

$$T_2 = 2^{\rho-3}\frac{a^\nu}{b^{\nu+\rho}}\Gamma(-\nu)\Gamma\left(\frac{\rho+\nu+\lambda}{2}\right)\Gamma\left(\frac{\rho+\nu-\lambda}{2}\right)\Gamma(\nu)\,{}_2F_1\left(\left.\begin{matrix}\frac{\rho+\nu+\lambda}{2} \quad \frac{\rho+\nu-\lambda}{2}\\ 1+\nu\end{matrix}\right| \frac{a^2}{b^2}\right).$$

Case 3: n_3 free. Then $n_2 = n_3+\frac{\lambda-\rho-\nu+1}{2}$ and $n_1 = n_3+\frac{\lambda-\rho+\nu+1}{2}$. This produces

$$T_3 = 2^{\rho-3}\frac{a^{-\rho+\lambda+1}}{b^{\lambda+1}}\sum_n \frac{\phi_n}{\Gamma(-n)}\Gamma\left(\frac{\rho+\nu-\lambda-1}{2}-n\right)\Gamma\left(\frac{\rho-\nu-\lambda-1}{2}-n\right) \Gamma\left(n+\lambda+\tfrac{1}{2}\right)\Gamma(n+\tfrac{1}{2})\left(\frac{a^2}{b^2}\right)^n.$$

This series has the value zero. This proves the next statement:

Proposition 7.8. *The integral*

$$I(a,b;\nu,\lambda;\rho) = \int_0^\infty x^{\rho-1}K_\nu(ax)K_\lambda(bx)\,dx \tag{7.53}$$

is given by

$$\begin{aligned} I(a,b;\nu,\lambda;\rho) = {} & 2^{\rho-3}\frac{b^{\nu-\rho}}{a^\nu}\Gamma(\nu)\Gamma\left(\frac{\rho-\nu+\lambda}{2}\right)\Gamma\left(\frac{\rho-\nu-\lambda}{2}\right)\,{}_2F_1\left(\left.\begin{matrix}\frac{\rho-\nu+\lambda}{2} \quad \frac{\rho-\nu-\lambda}{2}\\ 1-\nu\end{matrix}\right| \frac{a^2}{b^2}\right) \\ & + 2^{\rho-3}\frac{a^\nu}{b^{\nu+\rho}}\Gamma(-\nu)\Gamma\left(\frac{\rho+\nu+\lambda}{2}\right)\Gamma\left(\frac{\rho+\nu-\lambda}{2}\right)\,{}_2F_1\left(\left.\begin{matrix}\frac{\rho+\nu+\lambda}{2} \quad \frac{\rho+\nu-\lambda}{2}\\ 1+\nu\end{matrix}\right| \frac{a^2}{b^2}\right). \end{aligned}$$

Some special cases of this evaluation are interesting in their own right. Consider first the case $a = b$. Using Gauss' theorem (2.20) it follows that

$$T_1 = \frac{2^{\rho-3}\,\Gamma(\nu)\Gamma\left(\frac{\rho+\lambda-\nu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\nu}{2}\right)\Gamma(1-\nu)\Gamma(1-\rho)}{a^{\rho}\,\Gamma\left(1-\frac{\rho+\nu+\lambda}{2}\right)\Gamma\left(1-\frac{\rho+\nu-\lambda}{2}\right)} \tag{7.54}$$

and

$$T_2 = \frac{2^{\rho-3}\,\Gamma(-\nu)\Gamma\left(\frac{\rho+\lambda+\nu}{2}\right)\Gamma\left(\frac{\nu+\rho-\lambda}{2}\right)\Gamma(\nu+1)\Gamma(1-\rho)}{a^{\rho}\,\Gamma\left(1-\frac{\rho-\nu-\lambda}{2}\right)\Gamma\left(1-\frac{\rho-\nu+\lambda}{2}\right)}. \tag{7.55}$$

Proposition 7.9. *The integral*

$$J(a;\nu,\lambda;\rho) = \int_0^\infty x^{\rho-1}K_\nu(ax)K_\lambda(ax)\,dx \tag{7.56}$$

is given by

$$J(a;\nu,\lambda;\rho) = \frac{2^{\rho-3}\,\Gamma(\nu)\Gamma\left(\frac{\rho+\lambda-\nu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\nu}{2}\right)\Gamma(1-\nu)\Gamma(1-\rho)}{a^{\rho}\,\Gamma\left(1-\frac{\rho+\nu+\lambda}{2}\right)\Gamma\left(1-\frac{\rho+\nu-\lambda}{2}\right)} + \frac{2^{\rho-3}\,\Gamma(-\nu)\Gamma\left(\frac{\rho+\lambda+\nu}{2}\right)\Gamma\left(\frac{\nu+\rho-\lambda}{2}\right)\Gamma(\nu+1)\Gamma(1-\rho)}{a^{\rho}\,\Gamma\left(1-\frac{\rho-\nu-\lambda}{2}\right)\Gamma\left(1-\frac{\rho-\nu+\lambda}{2}\right)}.$$

The next special case is to take $a = b$ and $\lambda = \nu$. Then

$$T_1 = \frac{2^{\rho-3}}{a^{\rho}}\frac{\Gamma(\nu)\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho}{2}-\nu\right)\Gamma(1-\nu)\Gamma(1-\rho)}{\Gamma\left(1-\frac{\rho}{2}-\nu\right)\Gamma\left(1-\frac{\rho}{2}\right)} \tag{7.57}$$

and

$$T_2 = \frac{2^{\rho-3}}{a^{\rho}}\frac{\Gamma(-\nu)\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho}{2}+\nu\right)\Gamma(\nu+1)\Gamma(1-\rho)}{\Gamma\left(1-\frac{\rho}{2}+\nu\right)\Gamma\left(1-\frac{\rho}{2}\right)}. \tag{7.58}$$

This proves the next result:

Proposition 7.10. *The integral*

$$L(a;\nu,\rho) = \int_0^\infty x^{\rho-1}K_\nu^2(ax)\,dx \tag{7.59}$$

is given by

$$L(a;\nu,\rho) = \frac{2^{\rho-3}}{a^{\rho}}\left[\frac{\Gamma(\nu)\Gamma(1-\nu)\Gamma\left(\frac{\rho}{2}-\nu\right)}{\Gamma\left(1-\frac{\rho}{2}-\nu\right)} + \frac{\Gamma(-\nu)\Gamma(1+\nu)\Gamma\left(\frac{\rho}{2}+\nu\right)}{\Gamma\left(1-\frac{\rho}{2}+\nu\right)}\right].$$

The last special case is $\rho = 1$; that is, the integral

$$M(a,b;\nu,\lambda) = \int_0^\infty K_\nu(ax)K_\lambda(bx)\,dx. \tag{7.60}$$

It is shown that the usual application of the method of brackets yield only divergent series, so a new approach is required.

The argument begins with converting the brackets series in (7.51) to

(7.61)
$$M(a,b;\nu,\lambda) = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_2+\nu} 2^{\lambda-\nu-1+2n_3-2n_2} \Gamma(n_3+\lambda+\frac{1}{2})\Gamma(n_3+\frac{1}{2})}{b^{2n_3+\lambda+1}\,\Gamma(-n_3)} \times \langle n_1-n_2-\nu\rangle\langle \nu-\lambda+2n_2-2n_3\rangle.$$

A routine application of the method of brackets gives three series

$$\begin{aligned} T_1 &= \frac{b^{\nu-1}}{4a^{\nu}}\Gamma\left(\frac{1-\nu+\lambda}{2}\right)\Gamma\left(\frac{1-\nu-\lambda}{2}\right)\Gamma(\nu)\, {}_2F_1\left(\left. \begin{matrix} \frac{1-\nu+\lambda}{2} & \frac{1-\nu-\lambda}{2} \\ & 1-\nu \end{matrix} \right| \frac{a^2}{b^2} \right) \\ T_2 &= \frac{a^{\nu}}{4b^{\nu+1}}\Gamma\left(\frac{1+\nu+\lambda}{2}\right)\Gamma\left(\frac{1+\nu-\lambda}{2}\right)\Gamma(-\nu)\, {}_2F_1\left(\left. \begin{matrix} \frac{1+\nu+\lambda}{2} & \frac{\nu-\lambda+1}{2} \\ & 1+\nu \end{matrix} \right| \frac{a^2}{b^2} \right) \end{aligned}$$

and a totally null series T_3. Gauss' value (2.20) shows that T_1 and T_2 diverge when $a \to b$. Therefore (7.61) is replaced by

(7.62)
$$M(a,b;\nu,\lambda) = \lim_{\varepsilon\to 0} \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_2+\nu} 2^{\lambda-\nu-1+2n_3-2n_2} \Gamma(n_3+\lambda+\frac{1}{2})\Gamma(n_3+\frac{1}{2})}{b^{2n_3+\lambda+1}\,\Gamma(-n_3)} \times \langle n_1-n_2-\nu+\varepsilon\rangle\langle \nu-\lambda+2n_2-2n_3\rangle.$$

Proceeding as before produces a null series that is discarded and also

$$\begin{aligned} T_1 &= \frac{a^{-\nu+2\varepsilon}}{4b^{1-\nu+2\varepsilon}}\Gamma(\nu-\varepsilon)\Gamma\left(\frac{1+\lambda-\nu}{2}+\varepsilon\right)\Gamma\left(\frac{1-\lambda-\nu}{2}+\varepsilon\right) \\ &\quad \times {}_2F_1\left(\left. \begin{matrix} \frac{1-\nu+\lambda}{2}+\varepsilon & \frac{1-\nu-\lambda}{2}+\varepsilon \\ & 1-\nu+\varepsilon \end{matrix} \right| \frac{a^2}{b^2} \right) \\ T_2 &= \frac{a^{\nu}}{4b^{1+\nu}}\Gamma(-\nu+\varepsilon)\Gamma\left(\frac{1+\lambda+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\nu}{2}\right) \\ &\quad \times {}_2F_1\left(\left. \begin{matrix} \frac{1+\nu-\lambda}{2} & \frac{1+\nu+\lambda}{2} \\ & 1+\nu-\varepsilon \end{matrix} \right| \frac{a^2}{b^2} \right). \end{aligned}$$

In the limit as $b \to a$, these become

$$\begin{aligned} T_1 &= \frac{\Gamma(\nu-\varepsilon)\Gamma\left(\frac{1+\lambda-\nu}{2}+\varepsilon\right)\Gamma\left(\frac{1-\lambda-\nu}{2}+\varepsilon\right)\Gamma(1-\nu+\varepsilon)\Gamma(-\varepsilon)}{4a\,\Gamma\left(\frac{1-\nu-\lambda}{2}\right)\Gamma\left(\frac{1-\nu+\lambda}{2}\right)} \\ T_2 &= \frac{\Gamma(-\nu+\varepsilon)\Gamma\left(\frac{1+\lambda+\nu}{2}\right)\Gamma\left(\frac{1-\lambda+\nu}{2}\right)\Gamma(1+\nu-\varepsilon)\Gamma(-\varepsilon)}{4a\,\Gamma\left(\frac{1+\nu+\lambda}{2}-\varepsilon\right)\Gamma\left(\frac{1+\nu-\lambda}{2}-\varepsilon\right)}. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ gives

(7.63)
$$\int_0^\infty K_\nu(ax)K_\lambda(ax)\,dx = \frac{\pi^2}{4a\sin\pi\nu}\left[\tan\left(\frac{\pi}{2}(\lambda+\nu)\right) - \tan\left(\frac{\pi}{2}(\lambda-\nu)\right)\right].$$

In the special case $\lambda = \nu$, it follows that

(7.64)
$$\int_0^\infty K_\nu^2(ax)\,dx = \frac{\pi^2}{4a\cos\pi\nu}.$$

This value generalizes (7.46).

8. An example with an integral producing the Bessel function

The evaluation of integrals in Section 7 contain the Bessel function K_ν in the integrand. This section uses the method developed in the current work to evaluate some entries in [16] where the answer involves K_0.

Example 8.1. The first example is entry 6.532.4 in [16]

$$\int_0^\infty \frac{xJ_0(ax)}{x^2+b^2}\,dx = K_0(ab). \tag{8.1}$$

The analysis begins with the series

$$\begin{aligned} J_0(ax) &= \sum_{n=0}^{\infty} \frac{1}{n!^2}\left(-\frac{a^2x^2}{4}\right)^n \\ &= \sum_{n_1=0}^{\infty} \phi_{n_1} \frac{a^{2n_1}}{2^{2n_1}\,\Gamma(n_1+1)} x^{2n_1} \end{aligned} \tag{8.2}$$

Rule P_2 gives

$$\frac{1}{x^2+b^2} = \sum_{n_2,n_3} \phi_{n_2,n_3} x^{2n_2} b^{2n_3} \langle 1+n_2+n_3 \rangle. \tag{8.3}$$

Therefore

$$\int_0^\infty \frac{xJ_0(ax)}{x^2+b^2}\,dx = \sum_{n_1,n_2,n_3} \phi_{n_1,n_2,n_3} \frac{a^{2n_1}b^{2n_3}}{2^{2n_1}\,\Gamma(n_1+1)} \langle 1+n_2+n_3\rangle\langle 2+2n_1+2n_2\rangle. \tag{8.4}$$

The method of brackets produces three series as candidates for solutions, one per free index n_1, n_2, n_3:

$$\begin{aligned} T_1 &= \frac{1}{2}\sum_{n=0}^{\infty} \phi_n \Gamma(-n)\left(\frac{a^2b^2}{4}\right)^n \\ T_2 &= \frac{2}{a^2b^2}\sum_{n=0}^{\infty} \phi_n \frac{\Gamma^2(n)}{\Gamma(-n)}\left(\frac{4}{a^2b^2}\right)^n \\ T_3 &= \frac{1}{2}\sum_{n=0}^{\infty} \phi_n \Gamma(-n)\left(\frac{a^2b^2}{4}\right)^n. \end{aligned} \tag{8.5}$$

The fact that $T_1 = T_3$ and using Rule E_4 shows that only one of these series has to be counted. Since T_1 and T_2 are non-classical series of distinct variables, the integral is assigned either value. Observe that T_2 is the totally null representation of $K_0(ab)$ given in (1.11). This confirms (8.1). The fact that T_3 is also a value for the integral gives another totally divergent representation for K_0:

$$K_0(x) = \frac{2}{x^2}\sum_{n=0}^{\infty} \phi_n \frac{\Gamma^2(n+1)}{\Gamma(-n)}\left(\frac{4}{x^2}\right)^n. \tag{8.6}$$

To test its validity, the integral in Example 7.1 is evaluated again, this time using (8.6):

$$\begin{aligned}\int_0^\infty K_0(x)dx &= \int_0^\infty \frac{2}{x^2}\sum_n \phi_n \frac{\Gamma^2(n+1)}{\Gamma(-n)} 2^{2n}x^{-2n} \\ &= \sum_n \phi_n 2^{2n+1}\frac{\Gamma^2(n+1)}{\Gamma(-n)}\int_0^\infty x^{-2n-2}\,dx \\ &= \sum_n \phi_n 2^{2n+1}\frac{\Gamma^2(n+1)}{\Gamma(-n)}\langle -2n-1\rangle.\end{aligned} \tag{8.7}$$

The bracket series is evaluated using Rule E_1 to confirm (7.7).

Example 8.2. Entry 6.226.2 in [16] is

$$\int_0^\infty \text{Ei}\left(-\frac{a^2}{4x}\right)e^{-\mu x}\,dx = -\frac{2}{\mu}K_0(a\sqrt{\mu}). \tag{8.8}$$

The evaluation starts with the partially divergent series (4.5)

$$\text{Ei}\left(-\frac{a^2}{4x}\right) = \sum_{n_1=0}^\infty \phi_{n_1}\frac{a^{2n_1}}{n_1 2^{2n_1}}\frac{1}{x^{n_1}} \tag{8.9}$$

and this yields

$$\int_0^\infty \text{Ei}\left(-\frac{a^2}{4x}\right)e^{-\mu x}\,dx = \sum_{n_1,n_2}\phi_{n_1 n_2}\frac{a^{2n_1}\mu^{n_2}}{n_1 2^{2n_1}}\langle n_2-n_1+1\rangle. \tag{8.10}$$

The method of brackets gives two series. The first one

$$\begin{aligned}T_1 &= \frac{1}{\mu}\sum_{n_1}\phi_{n_1}\frac{\Gamma(1-n_1)}{n_1 2^{2n_1}}(a^2\mu)^{n_1} \\ &= -\frac{1}{\mu}\sum_{n_1}\phi_{n_1}\frac{\Gamma(-n_1)}{2^{2n_1}}(a^2\mu)^{n_1} \\ &= -\frac{2}{\mu}K_0(a\sqrt{\mu}),\end{aligned} \tag{8.11}$$

using (1.11). The second series is

$$T_2 = \sum_{n_2}\phi_{n_2}\frac{a^{2n_2+2}\mu^{n_2}}{(n_2+1)2^{2(n_2+1)}}\Gamma(-n_2-1). \tag{8.12}$$

Now shift the index by $m = n_2+1$ to obtain

$$\begin{aligned}T_2 &= \sum_m \phi_{m-1}\frac{a^{2m}\mu^{m-1}}{m2^{2m}}\Gamma(-m). \\ &= -\frac{1}{\mu}\sum_m \phi_m\Gamma(-m)\frac{a^{2m}\mu^m}{2^{2m}}.\end{aligned}$$

This is the same sum as T_1 in the second line of (8.11). Recall that the summation indices are placed after the conversion of the indicator ϕ_{n_2} to its expression in terms of the gamma function. According to Rule E_4, the sum T_2 is discarded. This establishes (8.8).

9. A new use of the method of brackets

This section introduces a procedure to evaluate integrals of the form

$$I(a_1, a_2) = \int_0^\infty f_1(a_1x)f_2(a_2x)\,dx. \tag{9.1}$$

Differentiating with respect to the parameters leads to

$$a_1\frac{\partial I(a_1,a_2)}{\partial a_1} + a_2\frac{\partial I(a_1,a_2)}{\partial a_2} = \int_0^\infty x\frac{d}{dx}\left[f_1(a_1x)f_2(a_2x)\right]\,dx. \tag{9.2}$$

Integration by parts produces

$$I(a_1,a_2) = xf_1(a_1x)f_2(a_2x)\Big|_0^\infty - \left(a_1\frac{\partial I(a_1,a_2)}{\partial a_1} + a_2\frac{\partial I(a_1,a_2)}{\partial a_2}\right). \tag{9.3}$$

A direct extension to many parameters leads to the following result.

Theorem 9.1. *Let*

$$I(a_1,\cdots,a_n) = \int_0^\infty \prod_{j=1}^n f(a_jx)\,dx. \tag{9.4}$$

Then

$$I(a_1,\cdots,a_n) = x\prod_{j=1}^n f_j(a_jx)\Big|_0^\infty - \sum_{j=1}^n a_j\frac{\partial I(a_1,\cdots,a_n)}{\partial a_j}. \tag{9.5}$$

Example 9.2. The integral

$$I(a,b) = \int_0^\infty e^{-ax}J_0(bx)\,dx \tag{9.6}$$

is evaluated first by a direct application of the method of brackets and then using Theorem 9.1.

The bracket series for $I(a,b)$

$$I(a,b) = \sum_{n_1,n_2}\phi_{n_1,n_2}\frac{a^{n_1}b^{2n_2}}{2^{2n_2}\Gamma(n_2+1)}\langle n_1+2n_2+1\rangle \tag{9.7}$$

is obtained directly from (5.11)

$$e^{-ax} = \sum_{n_1}\phi_{n_1}a^{n_1}x^{n_1} \tag{9.8}$$

and

$$J_0(bx) = {}_0F_1\left(\begin{matrix}-\\1\end{matrix}\middle|-\frac{(bx)^2}{4}\right) = \sum_{n_2}\phi_{n_2}\frac{b^{2n_2}}{\Gamma(n_2+1)2^{2n_2}}x^{2n_2}. \tag{9.9}$$

Solving for n_1 in the equation coming from the vanishing of the bracket gives $n_1 = -2n_2 - 1$, which yields

$$T_1 = \sum_{n_2=0}^\infty \frac{(-1)^{n_2}}{n_2!}\frac{a^{-2n_2-1}b^{2n_2}}{2^{2n_2}}\frac{\Gamma(2n_2+1)}{\Gamma(n_2+1)}. \tag{9.10}$$

To simplify this sum transform the gamma factors via (2.15) and use the duplication formula (2.18) to produce

$$T_1 = \frac{1}{a} \sum_{n_2=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{n_2}}{n_2!} \left(-\frac{b^2}{a^2}\right)^n = \frac{1}{a} {}_1F_0 \left(\left. \begin{matrix} \frac{1}{2} \\ - \end{matrix} \right| -\frac{b^2}{a^2} \right). \tag{9.11}$$

The identity ${}_1F_0 \left(\left. \begin{matrix} c \\ - \end{matrix} \right| z \right) = (1-z)^{-c}$ gives $T_1 = \dfrac{1}{\sqrt{a^2+b^2}}$. A direct calculation shows that the series obtained from solving for n_2 yields the same solution, so it discarded. Therefore

$$\int_0^{\infty} e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{a^2+b^2}}. \tag{9.12}$$

The evaluation of this integral using Theorem 9.1 begins with checking that the boundary terms vanish. This comes from the asymptotic behavior $J_0(x) \sim 1$ as $x \to 0$ and $J_0(x) \sim \sqrt{\dfrac{2}{\pi x}} \cos x$ as $x \to \infty$. The term

$$a \frac{\partial I(a,b)}{\partial a} = \sum_{n_1,n_2} \phi_{n_1 n_2} \frac{n_1 a^{n_1} b^{2n_2}}{2^{2n_2} \Gamma(n_2+1)} \langle n_1 + 2n_2 + 1 \rangle. \tag{9.13}$$

This generates two series

$$T_1 = \frac{1}{b} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{n \Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)} \left(\frac{2a}{b}\right)^n \tag{9.14}$$

and

$$T_2 = -\frac{1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(2n+2)}{\Gamma(n+1)} \left(\frac{b^2}{4a^2}\right)^n. \tag{9.15}$$

These are both convergent series, but in distinct regions: T_1 converges when $|4a^2| < |b^2|$ and T_2 in $|4a^2| > |b^2|$. Therefore they both give the value of the integral. The series T_2 is easier to evaluate and this leads to

$$a\frac{\partial I}{\partial a} = -\frac{a^2}{(a^2+b^2)^{3/2}} \quad \text{and} \quad b\frac{\partial I}{\partial b} = -\frac{b^2}{(a^2+b^2)^{3/2}}. \tag{9.16}$$

Replacing in (9.3) gives

$$I(a,b) = \frac{a^2}{(a^2+b^2)^{3/2}} + \frac{b^2}{(a^2+b^2)^{3/2}} = \frac{1}{\sqrt{a^2+b^2}}, \tag{9.17}$$

and confirms (9.12).

Example 9.3. Entry 6.222 in [16] is

$$\begin{aligned} I(a_1,a_2) &= \int_0^{\infty} \text{Ei}(-a_1 x)\text{Ei}(-a_2 x)\, dx \\ &= \left(\frac{1}{a_1} + \frac{1}{a_2}\right) \ln(a_1+a_2) - \frac{\ln a_1}{a_2} - \frac{\ln a_2}{a_1}. \end{aligned} \tag{9.18}$$

In particular

$$\int_0^{\infty} \text{Ei}^2(-ax)\, dx = \frac{2\ln 2}{a}. \tag{9.19}$$

The evaluation of this integral by the method of brackets begins with the partially divergent series for $\text{Ei}(-x)$ which yields (using (2.3) = (4.5)):

$$I(a_1, a_2) = \sum_{n_1, n_2} \phi_{n_1, n_2} \frac{a_1^{n_1} a_2^{n_2}}{n_1 n_2} \langle n_1 + n_2 + 1 \rangle. \tag{9.20}$$

The usual procedure requires the relation $n_1 + n_2 + 1 = 0$ and taking n_1 as the free parameter gives

$$I_1(a_1, a_2) = -\frac{1}{a_2} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}}{n_1(n_1+1)} \left(\frac{a_1}{a_2} \right)^{n_1}, \tag{9.21}$$

and when n_2 as free parameter one obtains the series

$$I_2(a_1, a_2) = -\frac{1}{a_1} \sum_{n_2=0}^{\infty} \frac{(-1)^{n_2}}{n_2(n_2+1)} \left(\frac{a_2}{a_1} \right)^{n_2}. \tag{9.22}$$

These two series correspond to different expansions: the first one in $x = a_1/a_2$ and the second one in $x^{-1} = a_2/a_1$. Both series are partially divergent, so the Rule E_3 states that these sums must be discarded. The usual method of brackets fails for this problem.

The solution using Theorem 9.1 is described next. An elementary argument shows that $x\text{Ei}(-x) \to 0$ as $x \to 0$ or ∞. Then (9.3) becomes

(9.23)

$$\begin{aligned} I(a_1, a_2) &= -a_1 \frac{\partial I}{\partial a_1} - a_2 \frac{\partial I}{\partial a_2} \\ &= -\sum_{n_1, n_2} \phi_{n_1, n_2} \frac{a_1^{n_1} a_2^{n_2}}{n_2} \langle n_1 + n_2 + 1 \rangle - \sum_{n_1, n_2} \phi_{n_1, n_2} \frac{a_1^{n_1} a_2^{n_2}}{n_1} \langle n_1 + n_2 + 1 \rangle, \\ &\equiv S_1 + S_2, \end{aligned}$$

using (9.20) to compute the partial derivatives. The method of brackets gives two series for each of the sums S_1 and S_2:

$$T_{1,1} = \frac{1}{a_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{a_1}{a_2} \right)^n \tag{9.24}$$

$$T_{1,2} = -\frac{1}{a_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \left(\frac{a_2}{a_1} \right)^n \tag{9.25}$$

$$T_{2,1} = -\frac{1}{a_2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \left(\frac{a_1}{a_2} \right)^n \tag{9.26}$$

$$T_{2,2} = \frac{1}{a_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{a_2}{a_1} \right)^n, \tag{9.27}$$

the series $T_{1,1}$ and $T_{1,2}$ come from the first sum S_1 and $T_{2,1}$, $T_{2,2}$ from S_2. Rule E_3 indicates that the value of the integral is either

$$I(a_1, a_2) = T_{1,1} + T_{2,1} \quad \text{or} \quad I(a_1, a_2) = T_{1,2} + T_{2,2}; \tag{9.28}$$

the first form is an expression in a_1/a_2 and the second one in a_2/a_1.

The series $T_{1,1}$ is convergent when $|a_1| < |a_2|$ and it produces the function

$$f(a_1, a_2) = \frac{1}{a_1} \log\left(1 + \frac{a_1}{a_2}\right) \tag{9.29}$$

and $T_{2,2}$ is also convergent and is gives

$$g(a_1, a_2) = \frac{1}{a_2} \log\left(1 + \frac{a_2}{a_1}\right). \tag{9.30}$$

Observe that, according to (9.28) to complete the evaluation of $I(a_1, a_2)$, some of the series required are partially divergent series. The question is how to make sense of these divergent series. The solution proposed here is, for instance, to interpret $T_{2,1}$ as a partially divergent series attached to the function $g(a_1, a_2)$. Therefore, the sum in (9.28), the term $T_{2,1}$ is replaced by $g(a_1, a_2)$ to produce

$$\begin{aligned} I(a_1, a_2) &= f(a_1, a_2) + g(a_1, a_2) \\ &= \frac{1}{a_1} \log\left(1 + \frac{a_1}{a_2}\right) + \frac{1}{a_2} \log\left(1 + \frac{a_2}{a_1}\right), \end{aligned} \tag{9.31}$$

and this confirms (9.18). A similar interpretation of $T_{1,2} + T_{2,2}$ gives the same result.

10. Conclusions

The method of brackets consists of a small number of heuristic rules used for the evaluation of definite integrals on $[0, +\infty)$. The original formulation of the method applied to functions that admit an expansion of the form $\sum_{n=0}^{\infty} a(n) x^{\alpha n + \beta - 1}$. The results presented here extend this method to functions, like the Bessel function K_ν and the exponential integral Ei, where the expansions have expansions of the form $\sum_{n=0}^{\infty} \Gamma(-n) x^n$ (where all the coefficients are divergent) or $\sum_{n=0}^{\infty} \frac{1}{\Gamma(-n)} x^n$ (where all the coefficients vanish). A variety of examples illustrate the validity of this formal procedure.

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Instituto de Física y Astronomía, Universidad de Valparaiso, Valparaiso, Chile
E-mail address: ivan.gonzalez@uv.cl

Department of Mathematics, University of Southern Mississippi, Long Beach, MS 39560
E-mail address: karen.kohl@usm.edu

Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria
E-mail address: ljiu@risc.uni-linz.ac.at

Department of Mathematics, Tulane University, New Orleans, LA 70118
E-mail address: vhm@math.tulane.edu