"Random Walks" for Harmonic Sums

Lin Jiu

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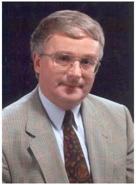
Acknowledgment


Dr. J. Ablinger


Prof. J. Blümlein


Prof. P. Paule


Prof. C. Schneider


Prof. C. Vignat

Outlines

Beginning-Partition

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Partition zeta functions

Robert Schneider

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Abstract

We exploit transformations relating generalized q-series, infinite products, sums over integer partitions, and continued fractions, to find partition-theoretic formulas to compute the values of constants such as π, and to connect sums over partitions to the Riemann zeta function, multiple zeta values, and other number-theoretic objects.

Keywords: Partitions, q-series, Zeta functions

1 Introduction, notations and central theorem

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function $\zeta(s)$ emerged nearly fully-formed from Euler's pioneering work [3, 8]. Euler discovered the magical-seeming generating function for the partition function $p(n)$

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n, \tag{1}$$

in which the q-Pochhammer symbol is defined as $(z;q)_0 := 1$, $(z;q)_n := \prod_{k=0}^{n-1}(1-zq^k)$ for $n \geq 1$, and $(z;q)_\infty = \lim_{n\to\infty}(z;q)_n$ if the product converges, where we take $z \in \mathbb{C}$ and $q := e^{i2\pi\tau}$ with $\tau \in \mathbb{H}$ (the upper half-plane). He also discovered the beautiful product formula relating the zeta function $\zeta(s)$ to the set $\mathbb{P}$ of primes

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to solve the so-called Basel problem, finding the exact value of $\zeta(2)$; he went on to find an exact formula for $\zeta(2k)$ for every $k \in \mathbb{Z}^+$ [8]. Euler's approach to these problems, interweaving infinite products, infinite sums and special functions, permeates number theory.

Very much in the spirit of Euler, here we consider certain series of the form $\sum_{\lambda\in\mathcal{P}}\phi(\lambda)$, where the sum is taken over the set $\mathcal{P}$ of integer partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, as well as the "empty partition" $\emptyset$, and where $\phi : \mathcal{P} \to \mathbb{C}$. We might sum

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Define the *partition-theoretic generaliztion of Riemann-zeta function* as

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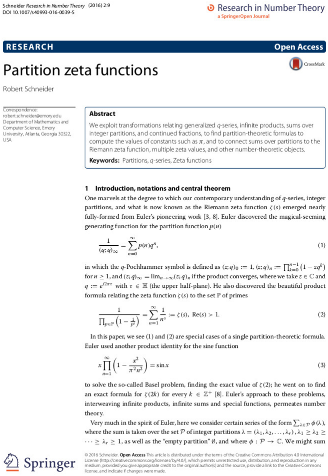

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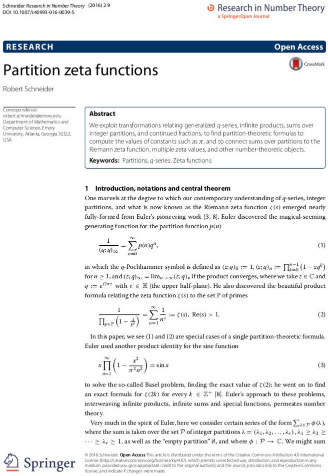

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Harmonic Sums

DEF: harmonic sum

$$S_{a_1,\ldots,a_k}(N)=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}}\cdots\frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}.$$

$k=1,\ a_1>0,\ N=\infty$

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In particular, if $a = m \in \mathbb{N}$ and $m \geq 2$, by considering $\xi_m := \exp\left(\frac{2\pi\iota}{m}\right)$ (M. Chamberland and A. Straub)

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$$B(N,1+t)=\frac{1}{N}\sum_{k=0}^{\infty}(-t)^k S_{1_k}(N).$$

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Multiple Beta Function

$$B\left(\alpha_1,\ldots,\alpha_m\right) := \frac{\Gamma\left(\alpha_1\right)\cdots\Gamma\left(\alpha_m\right)}{\Gamma\left(\alpha_1+\cdots+\alpha_m\right)} = \int_{\Omega_m}\prod_{i=1}^{m} x_i^{\alpha_i-1}d\boldsymbol{x},$$

where $\Omega_m = \left\{\left(x_1,\ldots,x_m\right)\in\mathbb{R}_+^m : x_1+\cdots+x_{m-1}<1,\ x_1+\cdots+x_m=1\right\}$.

Prop.

$$S_{m_k}\left(\infty\right) = \frac{\left(-1\right)^{mk}\left(m-1\right)!}{\left(mk\right)!}\int_{\Omega_m}\ln^{mk}\left(\prod_{j=0}^{m-1}x_{j+1}^{\xi_m^j}\right)d\boldsymbol{x},\ \xi_m = \exp\left(\frac{2\pi\iota}{m}\right)$$

$$\zeta\left(m\right) = \frac{\left(-1\right)^m}{m}\int_0^1\int_0^{1-x_1}\cdots\int_0^{1-x_1-\cdots-x_{m-2}} \ln^m\left(x_1^{\xi_m^0}\cdots x_{m-1}^{\xi_m^{m-2}}\left(1-x_1-\cdots-x_{m-1}\right)^{\xi_m^{m-1}}\right)dx_{m-1}\cdots dx_1.$$

Integral Representation

Multiple Beta Function

$$B(\alpha_1,\ldots,\alpha_m) := \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m)} = \int_{\Omega_m}\prod_{i=1}^{m} x_i^{\alpha_i-1}d\boldsymbol{x},$$

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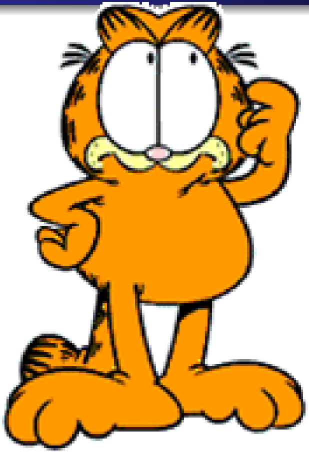
BREAK



$S_{1_k}(N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}$

Label N sites as follows:

$$\underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \cdots \quad \underset{N-1}{\bullet} \quad \underset{N}{\bullet}$$

We start a random walk at site "N", with the rules: (as a pawn)

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namely:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

For example, suppose we are at site "6":

$$\underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \underset{4}{\bullet} \quad \underset{5}{\bullet} \quad \overset{\text{here}}{\underset{6}{\bullet}} \quad \underset{7}{\bullet} \quad \underset{8}{\bullet} \quad \cdots \quad \underset{N}{\bullet}$$

Then, the next step only allows to walk to sites $\{1,2,3,4,5,6\}$, with probabilities:

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$S_{1_k}(N)$

On the other hand, the transition matrix of sites $\{1,\dots,N\}$ is:

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

i.e.,

$$P_{N|1} = \left(p_{i,j}^{(1)}\right) \text{ with } p_{ij}^{(1)} = \mathbb{P}\left(i \to j\right) = \frac{1}{i}.$$

Therefore, after $k+1$ steps, entries of $P_{N|1}^{k+1}$ give the transition probabilities among sites. In particular,

$$\left(P_{N|1}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N} S_{1_k}\left(N\right),$$

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Matrix Representation

$$S_{1_k}(N) \to S_{m_k}(N) \to S_{a_k}(N) \to S_{a_1,\ldots,a_k}(N)$$

Recall

$$S_{a_1,\ldots,a_k}(N) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}$$

For $l = 1, \ldots, k$

$$P_{N|a_l} = \begin{pmatrix} \operatorname{sign}(a_l) & 0 & \cdots & 0 \\ \frac{1}{2^{|a_l|}} & \frac{1}{2^{|a_l|}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \cdots & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} \end{pmatrix}.$$

THM.

Denote $a_0 = 1$, then

$$S_{a_1,\ldots,a_k}(N) = N \cdot \left(P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k}\right)_{N,1} = N \cdot \left(\prod_{l=0}^{k} P_{N|a_l}\right)_{N,1}.$$

Matrix Representation

$$S_{\mathbf{1}_k}(N) \to S_{\mathbf{m}_k}(N) \to S_{\mathbf{a}_k}(N) \to S_{a_1,\ldots,a_k}(N)$$

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$S_{a_k}(N)$ with $a > 1$

$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^a} & \frac{1}{2^a} & \cdots & 0 & 1-\frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^a} & \frac{1}{N^a} & \cdots & \frac{1}{N^a} & 1-\frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} P_{N|a} & \overrightarrow{\left(1-\frac{1}{i^{a-1}}\right)} \\ \hline \underbrace{(0,\ldots,0)}_{N} & 1 \end{array}\right)$$

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N & \mathfrak{N} \end{array}$$

with

$$\mathbb{P}(\mathfrak{N} \to \mathfrak{N}) = 1 \text{ and } \mathbb{P}(i \to \mathfrak{N}) = 1 - \frac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1} = 1) = \frac{1}{N^a} S_{a_k}(N).$$

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$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \cdots & \bullet & & \bullet \\ 1 & 2 & 3 & & N & & \color{red}{\mathfrak{N}} \end{array}$$

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BREAK

Diagonalization

$a = 1$

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \text{ and } \left(P_{N|1}^{k+1}\right)_{N,1} = \frac{1}{N} S_{1_k}(N)$$

$$P_{N|1} = Q_{N|1} \operatorname{diag}\left\{1, \ldots, \frac{1}{N}\right\} Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag}\left\{1, \ldots, \frac{1}{N^{k+1}}\right\} Q_{N|1}^{-1}.$$

$$Q_{N|1} = \left(\frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}}\right) \text{ and } Q_{N|1}^{-1} = \left((-1)^{i+j}\binom{N-1}{i-1}\binom{i-1}{j-1}\right)$$

$$\frac{1}{N} S_{1_k}(N) = \left(P_{N|1}^{k+1}\right)_{N,1} = \sum_{l=1}^{N} \frac{1}{l^{k+1}} (-1)^{l-1} \binom{N-1}{l-1},$$

K. Dilcher:

$$\sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = S_{1_k}(N).$$

Diagonalization

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Diagonalization

$a > 0$

$$P_{N|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2^a} & \frac{1}{2^a} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N^a} & \frac{1}{N^a} & \cdots & \frac{1}{N^a} \end{pmatrix} \text{ and } \left(P_{N|a}^{k+1}\right)_{N,a} = \frac{1}{N^a} S_{a_k}(N).$$

Diagonalization implies:

$$S_{a_k}(N) = \sum_{l=1}^{N} \left(\prod_{\substack{n=1 \\ n \neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}.$$

Recall

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k}.$$

When $a = 1$,

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More

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