# PRINCIPAL BUNDLES AND HOLONOMY GROUPS ON STATISTICAL MANIFOLDS

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Abstract. We study the principle bundles and holonomy groups over statistical manifolds. For principle bundles, after obtaining the $\alpha$-structure of frame bundles with respect to $\alpha$-connections, a concrete example of normal distribution manifold follows. For holonomy groups, we prove that the Riemannian holonomy group of the $d$-dimensional normal distribution is $\mathrm{SO}\left(d\left(d+3\right)/2\right)$, for $d = 1, 2, 3$. A list of holonomy groups on exponential family appears in the end.

## 1. Introduction

Statistical manifolds, which consist of probability distribution functions, are the main objects in information geometry. In order to describe their geometric structures, a series of concepts such as $\alpha$-connections, dual connections, and particularly Fisher metric [1], which makes statistical manifold Riemannian manifold, are introduced and studied. As a special example, normal distribution manifolds, defined in Definition 4.1, are of great importance. When Amari initiated the theory of information geometry [2, 3], he found that the sectional curvature of the univariate normal distribution manifold is $-\frac{1}{2}$, which is recalled in the proof of Lemma 4.7, implying its isometry to a hyperbolic space. Rather than an amazing result, it is also the trigger for Amari to develop information geometry. Some basic definitions and results on information geometry are presented in Section 2.

The recognition of fibre bundles took place in the period 1935–1940. After the first definition by Whitney, the theory of fibre bundles has been developed by many mathematicians such as Hopf, Stiefel and Steenrod [8]. Nowadays, the theory of fibre bundles, especially (differentiable) principal bundles, plays an important role in many fields such as differential geometry, algebraic topology, etc. As an extraordinary example, the proof, by Chern through a global approach, of Gauss-Bonnet-Chern formula [4], which lays the foundation of global differential geometry, involves principal bundles and connections in the key step. In particular, since the concept of connections is of great importance in differential geometry, connections on principal bundles attract much attention. From then on, increasing concerns have been focused on the theory of fibre bundles and connections on principal bundles. Subsection 2.2 introduces basic results on principal bundles, in order to obtain results in Section 3, where we give the $\alpha$-structure on frame bundles, which are certainly principal bundles, over statistical manifolds, in terms of Theorem 3.8 and Corollary 3.10. It turns out that the $\alpha$-structures on frame bundles of statistical manifolds are always easier to handle due to the linear structure on the matrix Lie group $\mathrm{GL}(n, \mathbb{R})$ . As an example, Subsection 3.2 discusses the $\alpha$-structures over manifold of normal distributions as both an application on concrete case and a verification of results in Subsection 3.1.

Around 1926, Carten introduced holonomy groups in order to study and classify symmetric spaces. Indeed, he has classified irreducible symmetric spaces by considering holonomy groups. As part of the generalization of parallel transportations, holonomy could be defined on any vector bundle with connections [7]. As Chern believed, it plays an important role in the theory of connections. However there are seldom brilliant results except for Ambrose-Singer holonomy theorem [9, 10]. When coming to Riemannian holonomy groups, the classification for irreducible cases was

2000 *Mathematics Subject Classification.* 53C05 53C29 62A01.

*Key words and phrases.* Normal Distribution, Exponential Family, Principal Bundles, Holonomy Group.

This subject is supported by the National Natural Science Foundations of China (No. 61179031, No. 10932002.)

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solved in 1955 by Berger [11] and Simons [12]. After introducing definitions and propositions on holonomy groups in Subsection 2.3, several useful results on classification are presented in terms of theorems and corollaries also in terms of two tables in Appendix. Although, holonomy group fails to classify non-isometric manifolds due to its small number of classes, it is still essential and is applied to many fields including string theory. Here, we obtain that the Riemannian holonomy groups of the normal distribution manifolds is provided in Theorem 4.3, while Theorem 4.22 follows as a generalization on exponential family.

## 2. Preliminaries

### 2.1. Information geometry on statistical manifolds.

**Definition 2.1.** We call $S \coloneqq \{p\,(x;\theta)\,|\theta \in \Theta\}$ a *statistical manifold* if $x$ is a random variable in sample space $X$ and $p\,(x;\theta)$ is the probability density function (pdf), satisfying certain regular conditions. $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$ is an $n$-dimensional vector in some open subset $\Theta \subset \mathbb{R}^n$, which can also be viewed as the coordinates on $S$.

**Definition 2.2.** An $n$-dimensional parametric statistical model $\Theta = \{p_\theta | \theta \in \Theta\}$ is called an *exponential family* or *of exponential type*, if the pdf can be expressed in terms of functions $C, F_1, \ldots, F_n$ and a convex function $\phi$ on $\Theta$ of the form [39]

$$p(x;\theta) = \exp\left\{C(x) + \sum_i \theta_i F_i(x) - \phi(\theta)\right\}.$$

In addition, we call $\{\theta_i\}$ the *natural parameters* and $\phi$ the *potential function*.

**Definition 2.3.** The Riemannian metric on statistical manifolds is defined by the *Fisher information matrix* [1]:

$$g_{ij}(\theta) \coloneqq E[(\partial_i l)(\partial_j l)] = \int (\partial_i l)(\partial_j l) p(x;\theta) \mathrm{d}x, \quad i,j = 1,2,\ldots,n,$$

where $E$ denotes the expectation, $\partial_i \coloneqq \partial/\partial\theta_i$, and $l = l(x;\theta) \coloneqq \log p(x;\theta)$.

**Proposition 2.4.** *Suppose $S$ is an exponential family with coordinates $\theta_i$ and potential function $\phi$. Then the Fisher metric is given by $g_{ij} = \partial_i\partial_j\phi$, and the Riemannian connection coefficients are given by*

$$\Gamma_{ijk} = \partial_i\partial_j\partial_k\phi.$$

**Definition 2.5.** A family of connections $\nabla^{(\alpha)}$, defined by Amari,

$$< \nabla^{(\alpha)}_A B, C > \coloneqq E[(ABl)(Cl)] + \frac{1-\alpha}{2} E[(Al)(Bl)(Cl)]$$

are called *$\alpha$-connections*, where $A, B, C \in \mathfrak{X}(S)$, $ABl = A(Bl)$, and $\alpha \in \mathbb{R}$ is the parameter.

**Proposition 2.6.** *The connection $\nabla^{(\alpha)}$ is torsion free for all $\alpha$. The only Riemannian connection, with respect to Fisher metric, is $\nabla = \nabla^{(0)}$.*

**Theorem 2.7.** *If the Riemannian connection coefficients and $\alpha$-connection coefficients are denoted by $\Gamma_{ijk}$ and $\Gamma^{(\alpha)}_{ijk}$, respectively, then*

$$\Gamma^{(\alpha)}_{ijk} = \Gamma_{ijk} - \frac{\alpha}{2} T_{ijk},$$

*where $T_{ijk} \coloneqq E[(\partial_i l)(\partial_j l)(\partial_k l)]$. Note that $\Gamma^{(0)}_{ijk} = \Gamma_{ijk}$, which coincides with Proposition 2.6.*

**Definition 2.8.** The *Riemannian curvature tensor of $\alpha$-connections* is defined by

$$R^{(\alpha)}_{ijkl} = (\partial_j \Gamma^{(\alpha)s}_{ik} - \partial_i \Gamma^{(\alpha)s}_{jk}) + (\Gamma^{(\alpha)}_{jtl}\Gamma^{(\alpha)t}_{ik} - \Gamma^{(\alpha)}_{itl}\Gamma^{(\alpha)t}_{jk}),$$

where $\Gamma^{(\alpha)s}_{jk} = \Gamma^{(\alpha)}_{jki} g^{is}$ and $\left(g^{is}\right)$ is the inverse matrix of the metric matrix $(g_{mn})$. Einstein notation is also used here.

**Definition 2.9.** We call the statistical manifold $S$ *$\alpha$-flat* if $R^{(\alpha)}_{ijkl} = 0$ holds, and the coordinates $\theta$ *$\alpha$-affine* if $\Gamma^{(\alpha)}_{ijk} = 0$, respectively in some open set.

**Fact 2.10.** *$\alpha$-flatness of $S$ is equivalent to existence of $\alpha$-affine coordinates on it.*

**Definition 2.11.** A (piecewise) smooth curve $\gamma\colon [0,1] \to S$ on $S$ is called an *$\alpha$-geodesic* if

$$\nabla^{(\alpha)}_{\gamma'(t)}\gamma'(t) = 0.$$

2.2. **Principal bundles.**

**Definition 2.12.** Suppose that $P$, $M$, and $G$ are all smooth manifolds, where $G$ is also a (right) Lie transformation group on $P$ and $\pi\colon P \to M$ is a smooth surjection. $(P, \pi, M, G)$ is called a *principal (differentiable) (fibre) bundle* if the following are true.

(1) The action of $G$ on $P$ is free, i.e., if $ug = u$, $\forall u \in P$, then $g$ is the identity in $G$;
(2) $\pi^{-1}(\pi(p)) = pG \coloneqq \{pg | g \in G\}$, $\forall p \in P$;
(3) $\forall x \in M$, there exist $U \in \mathcal{N}(x) \coloneqq \{U | x \in U,\ U \text{ is an open set in } M\}$ and a diffeomorphism $\Phi_U\colon \pi^{-1}(U) \to U \times G$, where $\Phi_U$ has two components, i.e., $\Phi_U = (\pi, \phi_U)$, s.t. $\phi_U\colon \pi^{-1}(U) \to G$ satisfies

$$\phi_U(pg) = \phi_U(p)g,\ p \in P,\ g \in G.$$

$G$ is called the *structure group* of principal bundle $P$ and the pair $(\pi^{-1}(U),\ \Phi_U)$ is called the *local trivialization.*

**Definition 2.13.** Suppose that $(P, \pi, M, G)$ is a principal bundle, $(\pi^{-1}(U),\ \Phi_U)$ and $(\pi^{-1}(V),\ \Phi_V)$ are two local trivializations.

$$\begin{aligned} g_{UV}\colon U \cap V &\to G, \\ x &\mapsto \phi_U(p)(\phi_V(p))^{-1},\, p \in \pi^{-1}(x) \end{aligned}$$

is called the *transition function* between $(\pi^{-1}(U),\ \Phi_U)$ and $(\pi^{-1}(V),\ \Phi_V)$.

**Definition 2.14.** $(F(E), \widetilde{\pi}, M, \mathrm{GL}(r;\mathbb{R}))$ is called the *frame bundle associated with vector bundle* $(E, \pi, M, \mathbb{R}^r, \mathrm{GL}(r;\mathbb{R}))$. In particular, when $E = TM$, the tangent bundle of manifold $M$, $F(M) \coloneqq F(TM)$ is called the *frame bundle of manifold $M$*.

Frame bundle is one of the most important types of principal bundles because of its various useful structures. Some results hold only on frame bundles rather than on general principal bundles. Amazingly, the transition functions of frame bundle are quite nature: the Jacobi matrix, as stated in the next theorem.

**Theorem 2.15.** *$(F(E), \widetilde{\pi}, M, \mathrm{GL}(r;\mathbb{R}))$ and $(E, \pi, M, \mathbb{R}^r, \mathrm{GL}(r;\mathbb{R}))$ have the same family of transition functions. In particular, the common transition functions of $(F(M), \widetilde{\pi}, M, \mathrm{GL}(n;\mathbb{R}))$ and $(TM, \pi, M, \mathbb{R}^n, \mathrm{GL}(n;\mathbb{R}))$ are the Jacobian matrix of coordinates: $(g_{\alpha\beta}(x))_{ij} = (\partial x^i_\alpha / \partial x^j_\beta)$.*

**Definition 2.16.** Let $(P, \pi, M, G)$ be a principal bundle.

$$V_p \coloneqq \ker \pi_* = \{X \in T_pP | \pi_*(X) = 0\}$$

is called the *vertical subspace* of $T_pP$.

**Definition 2.17.** For principal bundle $(P, \pi, M, G)$, $H \subset TP$ is called a *connection* on $P$ if

(1) $T_pP = V_p \oplus H_p,\ p \in P$;
(2) $(R_g)_{*p}(H_p) = H_{pg},\ p \in P, g \in G$;
(3) and $\forall X \in \mathfrak{X}(P)$, its projections to $V$ and $H$: $v(X)$ and $h(X)$, are both smooth.

In other words, a connection $H$ is a smooth decomposition of tangent spaces on $P$: vertical subspace $V$ and horizontal subspace $H$, where the latter is right-invariant.

**Definition 2.18.** Let $(P, \pi, M, G)$ be a principal bundle, and $\mathfrak{g}$ be the Lie algebra of structure group $G$.

$$\begin{aligned} \tau\colon \mathfrak{g} &\to \mathfrak{X}(P), \\ A &\mapsto \tau(A),\ \tau(A)(p) := (R_p)_{*e}(A) \end{aligned}$$

is called the *fundamental vector field* induced by $A$, where $R_p\colon G \to \pi^{-1}(\pi(p))$, and $R_p(g) := R(p, g) = p \cdot g \in \pi^{-1}(\pi(p))$.

Obviously, the set of all fundamental vector fields is a Lie algebra isomorphic to $\mathfrak{g}$.

**Definition 2.19.** Let $(P, \pi, M, G)$ be a principal bundle, and $\mathfrak{g}$ be the Lie algebra $G$. $\theta\colon \mathfrak{X}(G) \to \mathfrak{g}$, defined by

$$\theta(X_g) := L_{g*}^{-1}(X_g),\ \ X_g \in TgG$$

is called the *canonical* 1-*form on* $G$. Furthermore, let $g_{\alpha\beta}$ be the transition functions, $\theta_{\alpha\beta}\colon \mathfrak{X}(U_\alpha \cap U_\beta) \to \mathfrak{g}$ is given by $\theta_{\alpha\beta} := g_{\alpha\beta}^*\theta$, that is, $\theta_{\alpha\beta}$ is a $\mathfrak{g}$-valued-1-form on $M$, defined as the pull-back $\mathfrak{g}$-valued-1-form of $\theta$ on $G$ by $g_{\alpha\beta}$.

**Theorem 2.20.** *Suppose that $(P, \pi, M, G)$ is a principal bundle, and $\mathfrak{g}$ is the Lie algebra $G$. The following definitions of connections are equivalent.*

DEF. 1 *A connection on $P$ is a smooth $M$-distribution $H \subset TP$ s.t.*
- (1) $T_pP = V_p \oplus H_p,\ \ p \in P,$
- (2) $(R_g)_*(H_p) = H_{pg},\ \ p \in P, g \in G.$

DEF. 2 *A connection on $P$ is a smooth $\mathfrak{g}$-valued-1-form field $\omega$ on $P$ s.t.*
- (3) $\omega(\tau(A)) = A,\ \ A \in \mathfrak{g},$
- (4) $R_g^*(\omega(X)) = Ad_{g^{-1}}(\omega(X)),\ \ g \in G, X \in TP.$

DEF. 3 *A connection on $P$ is a family of smooth $\mathfrak{g}$-valued-1-form fields $\omega_\alpha$ on $U_\alpha$ s.t.*
- (5) $\omega_\beta(p) = Ad(g_{\alpha\beta}^{-1}(p)) \circ \omega_\alpha(p) + \theta_{\alpha\beta}(p),\ \ p \in U_\alpha \cap U_\beta.$

**Definition 2.21.** Assume that $(P, \pi, M, G)$ is a principal bundle, $\mathfrak{g}$ is the Lie algebra of $G$, and $H$ is a connection on $P$. $\omega\colon \mathfrak{X}(P) \to \mathfrak{g}$, defined by

$$\omega(X) := \sigma_{u*}^{-1}(v(X)),\ \ X \in T_uP$$

is called the *connection form of* $(P, H)$. Here $\sigma_u\colon G \to uG$ is the left action of $G$ on $P$.

It is easy to check that $\omega$ is vertical: $\omega(H) = 0$. In fact, if we have a $\mathfrak{g}$-valued-1-form $\omega$ satisfying conditions DEF. 2 in Theorem 2.20, then $H := \ker(\omega)$ is a connection on $P$ with $\omega$ as its connection form, which is also right-covariant.

**Corollary 2.22.** *Let $(E, \pi, M, \mathbb{R}^r, \mathrm{GL}(r; \mathbb{R}))$ and $(F(E), \widetilde{\pi}, M, \mathrm{GL}(r; \mathbb{R}))$ be a vector bundle and its associated frame bundle, respectively. Then there exists a* 1-1 *correspondence between the connections on $E$ and the connections on $F(E)$.*

**Corollary 2.23.** *There exists a* 1-1 *correspondence between the connections on $M$ and $F(M)$.*

**Definition 2.24.** Let $\pi_{*b}\colon H_b \to T_{\pi(b)}M$. For any $X \in \mathfrak{X}(M)$, there exists a unique $\widetilde{X} = \pi_*^{-1}(X) \in \mathfrak{X}(P)$, called the *horizontal lift of* $X$, s.t. $\pi_*(\widetilde{X}) = X$.

**Theorem 2.25.** *A vector field on $P$ is right-invariant if and only if it is the horizontal lift of some vector filed on $M$.*

**Definition 2.26.** Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a smooth curve on $M$. $\widetilde{\gamma}\colon (-\epsilon, \epsilon) \to P$ is called the *horizontal lift of* $\gamma$ if

$$\pi(\widetilde{\gamma}(t)) = \gamma(t),\ \ \widetilde{\gamma}'(t) \in H_{\widetilde{\gamma}(t)},\ \ t \in (-\epsilon, \epsilon).$$

**Theorem 2.27.** *Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a smooth curve on $M$ with $\gamma(0) = p$. Then,*

(1) *for any $b \in \pi^{-1}(p)$, there exists a unique horizontal lift $\widetilde{\gamma}$ s.t. $\widetilde{\gamma}(0) = b$.*

(2) *Let* $\widetilde{\gamma}_1$ *be another smooth curve with* $\widetilde{\gamma}_1(0) = bg$, $g \in G$, *then* $\widetilde{\gamma}_1$ *is also a horizontal lift of* $\gamma$ *if and only if* $\widetilde{\gamma}_1(t) = \widetilde{\gamma}(t)g$, $t \in (-\epsilon, \epsilon)$.

As a result, horizontal lift curve is unique when initial point is fixed. Furthermore, all other horizontal lifts are just formed by right-translations.

**Definition 2.28.** Denote by $(P, \pi, M, G, H, \omega)$ a principal bundle with connection $H$ and connection form $\omega$.

$$\Omega := \mathrm{d}\omega + \frac{1}{2}\omega \wedge \omega$$

is called the *curvature form*, where $\Omega$ is a $\mathfrak{g}$-valued-2-form on $P$.

**Proposition 2.29.** *The second structure equation holds that*

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\ \omega, \omega\ ] = \mathrm{d}\omega \circ h.$$

**Definition 2.30.** Let $(F(M), \pi, M, \mathrm{GL}(n;\mathbb{R}))$ be a frame bundle over $M$. $\theta\colon T(F(M)) \to \mathbb{R}^n$, defined by

$$\theta(Y_u) := u^{-1}(\pi_* Y_u),\ Y_u \in T_u F(M)$$

is called the *canonical* 1-*form on* $F(M)$, where

$$u\colon \mathbb{R}^n \to T_{\pi(u)}M,\ u(\xi) := u\xi, \xi \in \mathbb{R}^n.$$

In fact, the canonical 1-from $\theta$ can only be defined on frame bundles.

**Definition 2.31.** For any $\xi \in \mathbb{R}^n$, $H(\xi)\colon F(M) \to H$ s.t. $H(\xi)_u := \pi_*^{-1}(u\xi)$ is called the *fundamental horizontal vector field*, where $\pi_*\colon H_u \to T_{\pi(u)}M$ is linear isomorphism.

Fundamental horizontal vector fields and fundamental vertical vector fields are horizontal and vertical, respectively, therefore "orthogonal" to each other. In addition, they form a basis of $T(F(M))$, which implies that $T(F(M))$ is a trivial bundle, or parallelizable.

**Definition 2.32.** Let $(F(M), \pi, M, \mathrm{GL}(n;\mathbb{R}), H, \omega)$ be a frame bundle with connection $H$ and connection form $\omega$. $\Theta := \mathrm{d}\theta \circ h$ is called the *torsion form on* $F(M)$, where $\Theta$ is a $\mathbb{R}^n$-valued-2-form on $F(M)$.

**Proposition 2.33.** *The first structure equation holds that* $\Theta = \mathrm{d}\theta + \omega \wedge \theta$.

In fact, the first and the second structure equations are similar to the structure equations on a smooth manifold with a connection.

**Theorem 2.34.** *Suppose that* $(F(M), \pi, M, \mathrm{GL}(n;\mathbb{R}), H, \omega)$ *is a frame bundle with connection* $H$ *and connection form* $\omega$. *Then the torsion form* $\Theta$ *and the curvature form* $\Omega$ *satisfy the following equations*

$$\mathrm{d}\Theta = \Omega \wedge \theta - \omega \wedge \Theta,\ \textit{and}\ \mathrm{d}\Omega = \Omega \wedge \omega,$$

*and which are called the first and the second Bianchi idendities, respectively.*

**Theorem 2.35.** *Denote by* $(F(M), \pi, M, \mathrm{GL}(n;\mathbb{R}), H, \Omega, \Theta)$ *a principal bundle with connection* $H$, *connection form* $\omega$, *curvature form* $\Omega$ *and torsion form* $\Theta$. *For any* $X, Y, Z \in T_pM$, $W \in \mathfrak{X}(M)$, $u \in \pi^{-1}(p)$, *we have*

$$\nabla_X W := u\widetilde{X}(\theta(\widetilde{W})),$$

$$T(X, Y) := u(\Theta(\widetilde{X}, \widetilde{Y})),$$

*and*

$$R(X, Y)Z := u(\Omega(\widetilde{X}, \widetilde{Y})u^{-1}(Z)),$$

*where* $\widetilde{X}$, $\widetilde{Y}$ *and* $\widetilde{W}$ *are the horizontal lifts of the vector fields* $X$, $Y$ *and* $W$, *respectively.*

The right-hand sides of all formulas involving geometric structures on frame bundle $F(M)$ are irrelevant to the base manifold $M$, which means that geometric structures on base manifold can be calculated on bundles. The importance lies in that geometric structures on frame bundle are often easier to handle.

**Theorem 2.36.** *Suppose that $(F(M), H, \Omega, \Theta)$ is a frame bundle over $(M, \nabla, T, R)$, where $\nabla$ is the connection of $M$ induced by $H$. Also let $T$ and $R$ denote the torsion tensor and curvature tensor on $M$, respectively. For any $X, Y, Z \in \mathfrak{X}(M)$, we have*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

*and*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

**Corollary 2.37.** *Let $(F(M), H, \Omega, \Theta)$ be the frame bundle over $(M, \nabla, T, R)$. Then*

$$\begin{aligned} \gamma\colon (-\epsilon,\epsilon) \to M \text{ is a geodesic} &\iff \nabla_{\gamma'}\gamma' = 0 \iff \tilde{\gamma}'(\theta(\tilde{\gamma}')) = 0; \\ (M,\nabla) \text{ is torsion free} &\iff \quad T = 0 \quad \iff \quad \Theta = 0; \\ (M,\nabla) \text{ is flat} &\iff \quad R = 0 \quad \iff \quad \Omega = 0. \end{aligned}$$

Based on these results, there are simple approaches to determine whether a curve $\gamma$ on $(M, \nabla)$ is a geodesic, and whether $(M, \nabla)$ is torsion free, flat or not.

## 2.3. **Holonomy groups.**

### 2.3.1. *Holonomy of a connection in a vector bundle.*

**Definition 2.38.** Let $E$ be a rank $r$ vector bundle over a smooth manifold $M$ and $\nabla$ be a connection on $E$. Given a piecewise smooth loop $\gamma\colon [0,1] \to M$ based at $x \in M$, we let $P_\gamma$ denote the parallel transportation of $\nabla$ and $E_x$ denote the fiber over $x$. Then, $P_\gamma\colon E_x \to E_x$ is an invertible linear transformation, so an element of $\mathrm{GL}(E_x) \cong \mathrm{GL}(r, \mathbb{R})$. The *holonomy group of* $\nabla$ based at $x$ is defined by

$$H_x(\nabla) := \{P_\gamma \in \mathrm{GL}(E_x) | \gamma \text{ is a loop based at } x\}.$$

The *restricted holonomy group based at* $x$ is its subgroup defined by

$$H_x^0(\nabla) := \{P_\gamma \in \mathrm{GL}(E_x) | \gamma \text{ is a contratible loop based at } x\}.$$

**Proposition 2.39.** *If $M$ is connected (path connected), then the holonomy groups on different base points are conjugate of one another in $\mathrm{GL}(r, \mathbb{R})$. In concrete, if $x$, $y \in M$, and $\gamma$ is a path from $x$ to $y$, then*

$$H_y(\nabla) = P_\gamma H_x(\nabla) P_\gamma^{-1}.$$

As a result, we shall always omit the base point and denote the group by $H(\nabla)$. While only considering one connection $\nabla$, we could further reduce the notation for the group by $H$. Now, here are several properties for the holonomy groups.

**Proposition 2.40.** *Let $E$ be a rank $r$ vector bundle over a connected manifold $M$, and $\nabla$ be a connection on $E$, then*

1. *$H^0$ is a conneted, Lie-subgroup of $\mathrm{GL}(r, \mathbb{R})$;*
2. *$H^0$ is the identity component of $H$, therefore the determinant of every matrix element is positive;*
3. *if, in addition, $M$ is simply connected, then $H^0 = H$;*
4. *$\nabla$ is flat iff $H^0(\nabla) = 0$.*

2.3.2. *Riemannian Holonomy.*

**Definition 2.41.** The *Riemannian holonomy group* of a Riemannian manifold $(M, g)$ is just the holonomy group of the Levi-Civita connection $\nabla$ on the tangent bundle $TM$.

**Proposition 2.42.** *Let $M$ be an $n$-dimensional Riemannian manifold and $H$ denote its Riemannian holonomy group, then*

*(1) $H$ is a (compact) (closed) Lie-subgroup of* $\mathrm{O}(n)$ *($n$-dimensional orthogonal group);*
*(2) if $M$ is orientable, then $H$ is a subgroup of the special orthogonal group* $\mathrm{SO}(n)$.

2.3.3. *Classification of Riemannian holonomy groups.*

**Theorem 2.43.** *Every locally symmetric Riemannian manifold is locally isometric to a symmetric space.*

As a result, we only need to consider symmetric spaces. We begin with the de Rham decomposition theorem.

**Theorem 2.44** (de Rham)**.** *Suppose that $M$ is a complete, simply connected Riemannian manifold, then it is isometric to $\mathbb{R}^k \times M^1 \times \cdots \times M^m$, where $k \geqslant 0$ and each $M^i$ is an irreducible, complete, and simply connected Riemannian manifold. Moreover, the dimension $k$ and manifolds $M^1, \ldots, M^m$ are uniquely (up to the order) determined by $M$.*

**Corollary 2.45.** *Let $M = \mathbb{R}^k \times M^1 \times \cdots \times M^m$ be the de Rham decomposition, $H$ be the holonomy group of $M$, and $H_i$ be the holonomy of $M^i$, then $H \cong H_1 \times \cdots \times H_m$.*

By de Rham decomposition, simply connected irreducible symmetric spaces are essential. The holonomy of a symmetric space can be derived by the holonomy of its factors. Therefore, we only need to find all simply connected irreducible symmetric spaces and their holonomy groups.

In fact, all simply connected irreducible symmetric spaces $M$ are of the form $M \cong G/K$, where $G$ is the group of isometric transformations on $M$ and $K$ is its isotropy subgroup. There are three types of such spaces (where $\kappa$ denotes the curvature of $M$):

(1) Euclidean type: $\kappa = 0$ and $M$ is isometric to a Euclidean space;
(2) compact type: $\kappa \geqslant 0$ (not identically 0);
(3) non-compact type: $\kappa \leq 0$ (not identically 0).

In all cases there are two classes:

Class A $G$ is a real simple Lie group;
Class B $G$ is either the product of a compact simple Lie group with itself (compact type), or a complexification of such a Lie group (non-compact type).

All these types are completly classified by Cartan [27] in details. We only give one of the four tables of symmetric spaces.

**Theorem 2.46** (Cartan)**.** *The seven infinite series and twelve exceptional Riemannian symmetric spaces in Table 1 (in Appendix) give all Riemannian symmetric spaces of class A and non-compact type(called type II).*

Based on Theorems 2.43–2.46, the holonomy groups of a locally symmetric Riemannian manifold are completely classified. The remaining problem is to classify all non-locally symmetric Riemannian manifolds with an irreducible holonomy group. This problem is solved by Berger [11] in 1955 and Simons [12] in 1962 in terms of the following theorem.

**Theorem 2.47** (Berger)**.** *The complete classification of possible holonomy groups for simply connected Riemannian manifolds which are irreducible and nonsymmetric is in Table 2 (in Appendix).*

From Berger's list, several direct corollaries follow.

**Corollary 2.48.** *Let $n = dim(M)$ and $H$ be the holonomy group of $M$. Then*

(1) *if $n$ is odd and $n \neq 7$, $H = \mathrm{SO}(n)$;*

(2) *if* $n = 7$, $H = \mathrm{SO}(7)$ *or* $G_2$*;*
(3) *if* $M$ *is not an Einstein manifold and* $n$ *is even,* $H = \mathrm{SO}(n)$ *or* $\mathrm{U}(n/2)$.

Actually, all the cases on Berger's list occur, which means for every group $H$ on the list, there exists a manifold that admits $H$ as its holonomy group.

*Remark* 2.49. The content about holonomy groups and symmetric spaces mentioned above is discussed in [5]-[38].

## 3. $\alpha$-structure on Frame Bundles over Statistical Manifolds

3.1. **General results.** Throughout this section, we let $S = \{p(x;\theta|\theta \in \Theta)\}$ be an $n$-dimensional statistical manifold with coordinates charts $\{(U_\beta, x^i_\beta)|\beta \in J\}$. Moreoever, Define $e^\beta_i := \partial/\partial x^i_\beta$ and $\omega^i_\beta := \mathrm{d}x^i_\beta$ , which is the dual 1-form of $e^\beta_i$ on $U_\beta$, $\forall 1 \le i \le n$. Then, let $(\omega_\beta)^k_j := (\Gamma_\beta)^k_{ji}\omega^i_\beta$ denote the connection form of the Riemannian connection $\nabla$.

**Definition 3.1.** The *$\alpha$-connection form* is defined by $(\omega^{(\alpha)}_\beta)^k_j := (\Gamma^{(\alpha)}_\beta)^k_{ji}\omega^i_\beta$, which is a $\mathrm{GL}(n;\mathbb{R})$-valued-1-form on $U_\beta$.

*Remark* 3.2. The indices here are of different meanings. The super index $\alpha$ with parentheses is the same index with respect to $\alpha$-connection $\nabla^{(\alpha)}$, while the lower index $\beta$ follows from the index of coordinates $\left\{\left(U_\beta, x^i_\beta\right)\right\}$.

**Definition 3.3.** Let $F(S)$ be the frame bundle over $S$ with local trivialization $\{(U_\beta, \phi_\beta, \Phi_\beta)|\beta \in J\}$. Define

$$\widetilde{\omega}^{(\alpha)}_\beta(u) := Ad(\phi^{-1}_\beta) \circ \pi^*\omega^{(\alpha)}_\beta(u) + \phi^*_\beta\theta(u),\ u \in \pi^{-1}(U_\beta).$$

Then, by Theorem 2.20, $\widetilde{\omega}^{(\alpha)} := (\widetilde{\omega}^{(\alpha)}_\beta)$ is a well defined $\mathrm{GL}(n;\mathbb{R})$-valued-1-form globally on $F(S)$. So there exists a unique connection on $F(S)$ with $\widetilde{\omega}^{(\alpha)}$ as its connection form, which is denoted by $H^{(\alpha)}$. Now, $(H^{(\alpha)}, \widetilde{\omega}^{(\alpha)})$ is a family of connections on the principal bundle $F(S)$.

With such connections on $F(S)$, geometric structures can be defined as that in Subsection 2.2.

**Definition 3.4.** Let $(F(s), H^{(\alpha)}, \widetilde{\omega}^{(\alpha)})$ be the frame bundle with respect to $\alpha$-connection over $S$. $\Theta^{(\alpha)} := \mathrm{d}\theta \circ h^{(\alpha)}$ and $\Omega^{(\alpha)} := \mathrm{d}\widetilde{\omega}^{(\alpha)} \circ h^{(\alpha)}$ are called the *$\alpha$-torsion form* and *$\alpha$-curvature form* on $F(S)$, respectively.

**Definition 3.5.** $\forall \xi \in \mathbb{R}^n$, a vector field $H^{(\alpha)}(\xi) : F(S) \to H^{(\alpha)}$ defined by $H^{(\alpha)}(\xi)_u := \pi^{-1}_*(u\xi)$ is called the *fundamental $\alpha$-horizontal vector field*, where $\pi_* \colon H^{(\alpha)}_u \to T_{\pi(u)}M$ is linear isomorphism.

This definition is an analog to Definition 2.31, corresponding to different connections on frame bundles.

Properties of $\mathrm{GL}\,(n;\mathbb{R})$ and direct computation give the following lemma

**Lemma 3.6.** *Denote by* $(F(S), H^{(\alpha)}, \widetilde{\omega}^{(\alpha)}, \Theta^{(\alpha)}, \Omega^{(\alpha)})$ *the frame bundle with $\alpha$-connection, also* $(H^{(\alpha)}, \widetilde{\omega}^{(\alpha)})$*, $\alpha$-torsion form* $\Theta^{(\alpha)}$ *and $\alpha$-curvature form* $\Omega^{(\alpha)}$*. Then we have*

(1) $\theta(H^{(\alpha)}(\xi)) = \xi,\ \xi \in \mathbb{R}^n$;
(2) $R_{g*}(H^{(\alpha)}(\xi)_u) = H^{(\alpha)}(g^{-1}\xi)_{ug},\ g \in \mathrm{GL}(n;\mathbb{R}^n),\ \xi \in \mathbb{R}^n$;
(3) $[\tau(A), H^{(\alpha)}(\xi)] = H^{(\alpha)}(A\xi),\ A \in gl(n;\mathbb{R}^n),\ \xi \in \mathbb{R}^n$.

Also, following Proposition 2.33 and necessary computations, it is not hard to obtain the next proposition.

**Proposition 3.7.** *Let* $(F(S), H^{(\alpha)}, \widetilde{\omega}^{(\alpha)}, \Theta^{(\alpha)}, \Omega^{(\alpha)})$ *be the frame bundle over $S$. Then we have*

$$\Theta^{(\alpha)} = \mathrm{d}\theta + \widetilde{\omega}^{(\alpha)} \wedge \theta \text{ and } \Omega^{(\alpha)} = \mathrm{d}\widetilde{\omega}^{(\alpha)} + \widetilde{\omega}^{(\alpha)} \wedge \widetilde{\omega}^{(\alpha)}.$$

**Theorem 3.8.** *$\forall X, Y, Z \in \mathfrak{X}(M)$, we have*

$$\nabla_X^{(\alpha)} Y = u(\widetilde{X}^{(\alpha)}(\theta(\widetilde{Y}^{(\alpha)}))), \tag{3.1}$$

$$T^{(\alpha)}(X,Y) = u(\Theta^{(\alpha)}(\widetilde{X},\widetilde{Y})) = \nabla_X^{(\alpha)} Y - \nabla_Y^{(\alpha)} X - [X,Y], \tag{3.2}$$

*and*

$$R^{(\alpha)}(X,Y)Z = u(\Omega^{(\alpha)}(\widetilde{X},\widetilde{Y})u^{-1}(Z)) = \nabla_X^{(\alpha)}\nabla_Y^{(\alpha)} Z - \nabla_Y^{(\alpha)}\nabla_X^{(\alpha)} Z - \nabla_{[X,Y]}^{(\alpha)} Z, \tag{3.3}$$

*where $\widetilde{X}^{(\alpha)}$ is the horizontal lift of $X$ corresponding to connection $H^{(\alpha)}$.*

*Proof.* Apparently, (3.1) follows from Corollary 2.22 and Theorem 2.35. Direct computation also verifies (3.2) and (3.3). □

*Remark* 3.9. Theorem 3.8 shows that the geometric structure on the base manifold can be represented by that on the principle bundle, w. r. t. the $\alpha$-connections.

As a consequence of the Theorem 3.8 and analog of Corollary 2.37, the following corollary holds.

**Corollary 3.10.** *If $(F(S), H^{(\alpha)}, \Theta^{(\alpha)}, \Omega^{(\alpha)})$ is the frame bundle over $(S, \nabla^{(\alpha)}, T^{(\alpha)}, R^{(\alpha)})$.*

$$\begin{aligned}
\gamma\colon (-\epsilon,\epsilon)\to M \text{ is an } \alpha\text{-geodesic} &\iff \nabla_{\gamma'}^{(\alpha)}\gamma' = 0 \iff \widetilde{\gamma}'(\theta(\widetilde{\gamma}'^{(\alpha)})) = 0;\\
(M,\nabla^{(\alpha)}) \text{ is torsion free} &\iff T^{(\alpha)} = 0 \iff \Theta^{(\alpha)} = 0;\\
(M,\nabla^{(\alpha)}) \text{ is flat} &\iff R^{(\alpha)} = 0 \iff \Omega^{(\alpha)} = 0.
\end{aligned}$$

3.2. **Principle boundles over normal distribution manifold.** Recall the manifold of one-dimensional normal distributions, defined by

$$S := \{p(x;\theta^1,\theta^2) | (\theta^1,\theta^2) \in \mathbb{R}\times\mathbb{R}_+\},$$

where $\theta = (\theta^1,\theta^2) = (\mu,\sigma)$ are coordinates and

$$p(x;\theta^1,\theta^2) = p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

is the pdf of a one-dimensional normal distribution with expectation $\mu$ and standard derivation $\sigma$. For the geometric structures on it, directly we could compute that

$$l(x,\theta) = \log p(x,\theta) = -\frac{(x-\mu)^2}{2\sigma^2} - \log(\sqrt{2\pi}\sigma),$$

Consequently, the Riemannian metric and its inverse are given by

$$g = (g_{ij}) = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix},\ g^{-1} = (g^{ij}) = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{\sigma^2}{2} \end{bmatrix}.$$

Furthermore, one can compute

$$\begin{cases} T_{111} = T_{122} = T_{212} = T_{221} = 0, \\ T_{112} = T_{121} = T_{211} = \frac{2}{\sigma^3}, \\ T_{222} = \frac{8}{\sigma^3}. \end{cases}$$

and

$$\begin{cases} \Gamma_{111}^{(\alpha)} = \Gamma_{122}^{(\alpha)} = \Gamma_{212}^{(\alpha)} = \Gamma_{221}^{(\alpha)} = 0, \\ \Gamma_{112}^{(\alpha)} = \frac{1-\alpha}{\sigma^3}, \\ \Gamma_{121}^{(\alpha)} = \Gamma_{211}^{(\alpha)} = -\frac{1+\alpha}{\sigma^3}, \\ \Gamma_{222}^{(\alpha)} = -\frac{2+4\alpha}{\sigma^3}. \end{cases}$$

to obtain the curvature tensor

$$R_{1212}^{(\alpha)} = \frac{1-\alpha^2}{\sigma^4}.$$

After computing the main $\alpha$-structures on $S$ above, we switch to the study of the same structures but in the "bundle version", which means all of them can be calculated on the principal bundle (the frame bundle $F(S)$ over $S$). Before the calculation, we make some necessary preparations.

Therefore, coefficients of $\alpha$-connections are

$$\begin{cases}(\Gamma^{(\alpha)})^1_{11} = (\Gamma^{(\alpha)})^2_{12} = (\Gamma^{(\alpha)})^2_{21} = (\Gamma^{(\alpha)})^1_{22} = 0,\\ (\Gamma^{(\alpha)})^2_{11} = \frac{1-\alpha}{2\sigma},\\ (\Gamma^{(\alpha)})^1_{12} = (\Gamma^{(\alpha)})^1_{21} = -\frac{1+\alpha}{\sigma},\\ (\Gamma^{(\alpha)})^2_{22} = -\frac{1+2\alpha}{\sigma}.\end{cases}$$

and then follows the connection form on $S$

$$\omega^{(\alpha)} = ((\omega^{(\alpha)})^i_j) = \begin{bmatrix} -\frac{1+\alpha}{\sigma}d\theta^2 & -\frac{1+\alpha}{\sigma}d\theta^1 \\ \frac{1-\alpha}{2\sigma}d\theta^1 & -\frac{1+2\alpha}{\sigma}d\theta^2\end{bmatrix}.$$

Now consider the frame bundle $F(S)$ over $S$. Since $S$ has a global coordinates neighbourhood, the bundle is trivial, i.e., $F(S) = S \times \mathrm{GL}(2;\mathbb{R})$. For any $u \in F(S)$, $u$ represents a basis $(e_1, e_2)^T$ of $T_{\pi(u)}S$. If $(e_1, e_2)^T = A(\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2})^T$, the coordinates of $u$ is $u = (u^1, \dots, u^6)$, where

$$u^3 = A_{11}, u^4 = A_{12}, u^5 = A_{21}, u^6 = A_{22}.$$

Furthermore, $\pi(u) = (\theta^1, \theta^2) = (u^1, u^2)$, so we have

$$\pi_*\left(\frac{\partial}{\partial u^1}\bigg|_u\right) = \frac{\partial}{\partial\theta^1}\bigg|_{\pi(u)},\ \pi_*\left(\frac{\partial}{\partial u^2}\right)\bigg|_u = \frac{\partial}{\partial\theta^2}\bigg|_{\pi(u)}.$$

And the local trivialization is

$$\begin{aligned}\Phi\colon F(S) &\to S \times \mathrm{GL}(2;\mathbb{R}),\\ u &\mapsto (u^1, u^2, A) = (\theta^1, \theta^2, A).\end{aligned}$$

By Definition 3.3,

$$\widetilde{\omega}^{(\alpha)}(u) = \mathrm{Ad}(\phi^{-1}(u)) \circ \pi^*\omega^{(\alpha)} + \phi^*\theta(u).$$

Since $u = (u^1, u^2, \dots, u^6)$ and $X_u = X^i \frac{\partial}{\partial u^i}\big|_u$, we have

$$\begin{aligned}\widetilde{\omega}^{(\alpha)}(X_u) &= \mathrm{Ad}(\phi^{-1}(u)) \circ \pi^*\omega^{(\alpha)}(X_u) + \phi^*\theta(X_u)\\ &= \mathrm{Ad}(\phi^{-1}(u)) \circ \omega^{(\alpha)}(\pi_*(X_u)) + \theta(\phi_*(X_u))\\ &= A\omega^{(\alpha)}\left(X^1\frac{\partial}{\partial\theta^1}\bigg|_{(\theta^1,\theta^2)} + X^2\frac{\partial}{\partial\theta^2}\bigg|_{(\theta^1,\theta^2)}\right)A^{-1} + A^{-1}B\\ &= A\begin{bmatrix} -\frac{1+\alpha}{\sigma}X^2 & -\frac{1+\alpha}{\sigma}X^1 \\ \frac{1-\alpha}{2\sigma}X^1 & -\frac{1+2\alpha}{\sigma}X^2\end{bmatrix}A^{-1} + A^{-1}B,\end{aligned}$$

where

$$A = \begin{bmatrix} u^3 & u^4 \\ u^5 & u^6\end{bmatrix},\ B = \begin{bmatrix} X^3 & X^4 \\ X^5 & X^6\end{bmatrix}.$$

So the horizontal space is

$$H_u^{(\alpha)} = \ker(\widetilde{\omega}^{(\alpha)}) = \left\{X \in T_uF(S)\middle| A\begin{bmatrix} -\frac{1+\alpha}{\sigma}X^2 & -\frac{1+\alpha}{\sigma}X^1 \\ \frac{1-\alpha}{2\sigma}X^1 & -\frac{1+2\alpha}{\sigma}X^2\end{bmatrix}A^{-1} + A^{-1}B = 0\right\}.$$

In particutlar, let $(e_1, e_2) = (\frac{\partial}{\partial\theta^1}\big|_{\pi(u)}, \frac{\partial}{\partial\theta^2}\big|_{\pi(u)})$, then we obtain $u = (u^1, u^2, 1, 0, 0, 1)$ and $\phi(u) = (\theta^1, \theta^2, I_{2\times2})$, where $I_{2\times2}$ is the $2\times2$ identity matrix. Furthermore, $\forall X \in T_uF(S)$, $X = X^i\frac{\partial}{\partial u^i}\big|_u$, we have

$$\pi_*(X) = X^1\frac{\partial}{\partial\theta^1}\bigg|_{(\theta^1,\theta^2)} + X^2\frac{\partial}{\partial\theta^2}\bigg|_{(\theta^1,\theta^2)}.$$

So, the corresponding horizontal space becomes

$$H_u^{(\alpha)} = \left\{X_u \in T_uF(S)\middle|\begin{bmatrix} -\frac{1+\alpha}{\sigma}X^2 & -\frac{1+\alpha}{\sigma}X^1 \\ \frac{1-\alpha}{2\sigma}X^1 & -\frac{1+2\alpha}{\sigma}X^2\end{bmatrix} + \begin{bmatrix} X^3 & X^4 \\ X^5 & X^6\end{bmatrix} = 0\right\}.$$

Under such circumstance, the horizontal projection of $X = X^i \frac{\partial}{\partial u^i} \in T_u F(S)$ is

$$h^{(\alpha)}(X) = X^1 \frac{\partial}{\partial u^1} + X^2 \frac{\partial}{\partial u^2} + \frac{1+\alpha}{\sigma} X^2 \frac{\partial}{\partial u^3} + \frac{1+\alpha}{\sigma} X^1 \frac{\partial}{\partial u^4} - \frac{1-\alpha}{2\sigma} X^1 \frac{\partial}{\partial u^5} + \frac{1+2\alpha}{\sigma} X^2 \frac{\partial}{\partial u^6}.$$

Then the horizontal lift of $X = X^i \frac{\partial}{\partial \theta^i} \in T_{(\theta^1,\theta^2)} S$ can also be expressed as

$$\widetilde{X}_u = X^1 \frac{\partial}{\partial u^1} + X^2 \frac{\partial}{\partial u^2} + \frac{1+\alpha}{\sigma} X^2 \frac{\partial}{\partial u^3} + \frac{1+\alpha}{\sigma} X^1 \frac{\partial}{\partial u^4} - \frac{1-\alpha}{2\sigma} X^1 \frac{\partial}{\partial u^5} + \frac{1+2\alpha}{\sigma} X^2 \frac{\partial}{\partial u^6}.$$

In particular, letting $X = \frac{\partial}{\partial \theta^1}$ and $Y = \frac{\partial}{\partial \theta^2}$, we get

$$\widetilde{X} = \frac{\partial}{\partial u^1} + \frac{1+\alpha}{\sigma} \frac{\partial}{\partial u^4} - \frac{1-\alpha}{2\sigma} \frac{\partial}{\partial u^5},$$

and

$$\widetilde{Y} = \frac{\partial}{\partial u^2} + \frac{1+\alpha}{\sigma} \frac{\partial}{\partial u^3} + \frac{1+2\alpha}{\sigma} \frac{\partial}{\partial u^6}.$$

Thus,

$$\nabla^{(\alpha)}_X X = \nabla^{(\alpha)}_{\frac{\partial}{\partial \theta^1}} \frac{\partial}{\partial \theta^1} = (\Gamma^{(\alpha)})^1_{11} \frac{\partial}{\partial \theta^1} + (\Gamma^{(\alpha)})^2_{11} \frac{\partial}{\partial \theta^2} = \frac{1-\alpha}{2\sigma} \frac{\partial}{\partial \theta^2}, \tag{3.4}$$

$$\nabla^{(\alpha)}_X Y = \nabla^{(\alpha)}_{\frac{\partial}{\partial \theta^1}} \frac{\partial}{\partial \theta^2} = (\Gamma^{(\alpha)})^1_{12} \frac{\partial}{\partial \theta^1} + (\Gamma^{(\alpha)})^2_{12} \frac{\partial}{\partial \theta^2} = -\frac{1+\alpha}{\sigma} \frac{\partial}{\partial \theta^1}, \tag{3.5}$$

and

$$\nabla^{(\alpha)}_Y Y = \nabla^{(\alpha)}_{\frac{\partial}{\partial \theta^2}} \frac{\partial}{\partial \theta^2} = (\Gamma^{(\alpha)})^1_{22} \frac{\partial}{\partial \theta^1} + (\Gamma^{(\alpha)})^2_{22} \frac{\partial}{\partial \theta^2} = -\frac{1+2\alpha}{\sigma} \frac{\partial}{\partial \theta^2}. \tag{3.6}$$

On the other hand, consider

$$\gamma_1(t) := (u^1 + t, u^2, 1, 0 + \frac{1+\alpha}{\sigma} t, 0 - \frac{1-\alpha}{2\sigma} t, 1)$$

and

$$\gamma_2(t) := (u^1, u^2 + t, 1 + \frac{1+\alpha}{\sigma} t, 0, 0, 1 + \frac{1+2\alpha}{\sigma} t).$$

It is obvious that

$$\gamma_1(0) = \gamma_2(0) = u, \gamma_1'(0) = \widetilde{X}, \gamma_2'(0) = \widetilde{Y}.$$

Thus,

$$\begin{aligned} u(\widetilde{X}_u(\theta(\widetilde{X}))) &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(\widetilde{X}) \circ \gamma_1 \right) \\ &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left( \frac{1}{1 + \frac{1-\alpha^2}{2\sigma^2} t^2} \begin{bmatrix} 1 & -\frac{1+\alpha}{\sigma} t \\ \frac{1-\alpha}{2\sigma} t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) \\ &= \frac{1-\alpha}{2\sigma} \frac{\partial}{\partial \theta^2} = \nabla^{(\alpha)}_X X, \end{aligned} \tag{3.7}$$

$$\begin{aligned} u(\widetilde{X}_u(\theta(\widetilde{Y}))) &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(\widetilde{Y}) \circ \gamma_1 \right) \\ &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left( \frac{1}{1 + \frac{1-\alpha^2}{2\sigma^2} t^2} \begin{bmatrix} 1 & -\frac{1+\alpha}{\sigma} t \\ \frac{1-\alpha}{2\sigma} t & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right) \\ &= -\frac{1+\alpha}{\sigma} \frac{\partial}{\partial \theta^1} = \nabla^{(\alpha)}_X Y, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} u(\widetilde{Y}_u(\theta(\widetilde{Y}))) &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(\widetilde{Y}) \circ \gamma_2 \right) \\ &= u\left( \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \begin{bmatrix} \frac{1}{1+\frac{1+\alpha}{\sigma} t} & 0 \\ 0 & \frac{1}{1+\frac{1+2\alpha}{\sigma} t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= -\frac{1+2\alpha}{\sigma} \frac{\partial}{\partial \theta^2} = \nabla^{(\alpha)}_Y Y. \end{aligned} \tag{3.9}$$

Formulas from (3.4) to (3.9) verify (3.1) of Theorem 3.8 directly. Also, since both torsion $T^{(\alpha)}$ and curvature $R^{(\alpha)}$ can be derived by connection $\nabla^{(\alpha)}$, similarly one could compute to check (3.2) and (3.3). In fact, we have

$$g(u(\Omega^{(\alpha)}(\widetilde{X},\widetilde{Y})u^{-1}(Y)),Y)=\frac{1-\alpha^2}{\sigma^4}=R^{(\alpha)}_{1212}$$

as desired.

These results show how our bundle approach simplifies the calculation because all operations on the structure group $\mathrm{GL}(2,\mathbb{R})$ are easier to handle. Since $\mathrm{GL}(n,\mathbb{R})$ is a matrix Lie group, the right translation and tangent mapping are actually products between matrices, which are linear and therefore convenient to calculate.

## 4. Holonomy groups on statistical manifold

### 4.1. **Normal distribution manifold.**

**Definition 4.1.** Let $\mathrm{SPD}(d,\mathbb{R})$ be the set of all real $d$-ordered positive definite symmetric matrices. The $d$-dimensional normal distribution manifold is defined by

$$N^d:=\left\{p(x,\mu,\Sigma)=\frac{\exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}}{(2\pi)^{\frac{d}{2}}(\det\Sigma)^{\frac{1}{2}}}\middle|\mu\in\mathbb{R}^d,\Sigma\in\mathrm{SPD}(d,\mathbb{R})\right\}.$$

*Remark* 4.2.

(1) Here the dimension $d$ is the dimension of normal distributions. As a manifold, the dimension is not hard to compute by

$$\dim\left(N^d\right)=\dim\left(\mathbb{R}^d\times\mathrm{SPD}\left(d,\mathbb{R}\right)\right)=d+\frac{d\left(d+1\right)}{2}=\frac{d\left(d+3\right)}{2}.$$

(2) Also, Amari proved that lower dimensional normal distribution manifold can be embedded into higher dimensional ones, i.e. if $d_1<d_2$, we could have $N^{d_1}\subset N^{d_2}$. The intuition is the distribution in lower dimension could be treated as a distribution in higher dimension with restrictions on some of the components.

Obviously, $N^d$ is also an exponential family as in Definition 2.2.

**Theorem 4.3.** *Let $N^d$ be the $d$-dimensional normal distribution manifold, $g$ be the Fisher metric and $\nabla=\nabla^{(0)}$ be the corresponding Levi-Civita connection. Suppose $H^d$ is the Riemannian holonomy group and $H^d_0$ is the restricted Riemannian holonomy group, then*

$$H^d=H^d_0=\mathrm{SO}\left(\frac{d(d+3)}{2}\right),\ d=1,2,3.$$

*Remark* 4.4. Together with Remark 4.2, we recognize the result as the first column in Table 2. It shows not only the orientability of the manifold, but also that the Fisher metric is a generic Riemannian metric.

Since some preparation is needed to prove the theorem, we first start with several lemmas.

**Lemma 4.5.** *$N^d$ is simply connected for all $d\in\mathbb{N}$.*

*Proof.* As a topological space, $N^d$ is homeomorphic to the parameter space $\mathbb{R}^d\times\mathrm{SPD}(d,\mathbb{R})\subset\mathbb{R}^{\frac{d(d+3)}{2}}$, as we stated in Remark 4.2. The $\mathbb{R}^d$ part is contractible showing $\pi_1(\mathbb{R}^d)=0$. By the theory of linear algebra, we see that if $A,B\in\mathrm{SPD}(d,\mathbb{R})$, then $(1-t)A+tB\in\mathrm{SPD}(d,\mathbb{R})$, $\forall t\in[0,1]$. Therefore, the space $\mathrm{SPD}(d,\mathbb{R})$ is convex, therefore also contractible. In particular, $\pi_1(\mathrm{SPD}(d,\mathbb{R}))=0$. By the theory of algebraic topology, we have

$$\pi_1(N^d)\cong\pi_1(\mathbb{R}^d\times\mathrm{SPD}(d,\mathbb{R}))\cong\pi_1(\mathbb{R}^d)\times\pi_1(\mathrm{SPD}(d,\mathbb{R}))=0.$$

□

**Corollary 4.6.** $H^d=H^d_0$.

*Proof.* This result is straightforward by applying Lemma 4.5 and part (3) of Proposition 2.40. □

**Lemma 4.7.** *$N^1$ is isometric to the 2-dimensional hyperbolic space, denoted by $H(2)$.*

*Proof.* By definition, we have

$$N^1 = \left\{ p(x,\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \middle| \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+ \right\}.$$

Let $\theta_1 = \frac{\mu}{\sigma^2}$ and $\theta_2 = -\frac{1}{2\sigma^2}$ be the natural coordinates of the exponential family. Then the Fisher metric matrix is given by

$$[g_{ij}] = \begin{bmatrix} \sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & 2\sigma^2(2\mu^2+\sigma^2) \end{bmatrix},$$

and the curvature tensor follows $R_{1212} = \frac{1}{\sigma^6}$.

As a result, the sectional curvature(also the Gaussian curvature) is $\kappa = -\frac{1}{2}$, which is a negative constant. Thus, $N^1$ is a complete simply connected manifold with constant sectional curvature $-\frac{1}{2}$, which is the space form and isometric to 2-dimensional hyperbolic space $H(2)$ with

$$\dim N^1 = \frac{1\,(1+3)}{2} = 2.$$

□

**Lemma 4.8.** *The $n$-dimensional hyperbolic space $H(n)$ is a symmetric space for all $n \in \mathbb{Z}_+$.*

*Proof.* Consider $s = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix} \in M(n+1,\mathbb{R})$ where $I_n$ is the $n \times n$ identity matrix and

$$\mathrm{O}(n,1) := \{A \in \mathrm{GL}(n+1,\mathbb{R}) | A^T s A = s\}.$$

Here, $\mathrm{O}(n,1)$ is called the Lorentz group consisting of all linear transformation on $\mathbb{R}^{n+1}$ maintaining the invariance of the Lorentz inner product, which is defined by

$$\langle X, Y \rangle_L := \sum_{i=1}^{n} x^i y^i - x^{n+1} y^{n+1} = X^T s Y,$$

$X = (x^1,\ldots,x^{n+1}), Y = (y^1,\ldots,y^{n+1}) \in \mathbb{R}^{n+1}$. Note that $\mathrm{O}(n,1)$ has 4 components and the one containing $I$ is

$$G = \{A = (a_{ij}) \in \mathrm{O}(n,1) | \det A = 1, a_{(n+1)(n+1)} \geqslant 1\},$$

which is a connected Lie group and acts on the Lorentz space $\mathbb{R}^{n+1}_L = (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_L)$ keeping $H(n) = \{X = (x^1,\ldots,x^{n+1})^T \in \mathbb{R}^{n+1} | \langle X, X\rangle_L = -1,\ x^{n+1} > 0\}$ invariant. The Lorentz inner product induces a Riemannian metric $g$ on $H(n)$. Also,

$$\begin{aligned} \sigma\colon G &\longrightarrow G \\ A &\longmapsto sAs \end{aligned}$$

is an involution automorphism on $G$. Note that the fixed point subgroup

$$\begin{aligned} K_\sigma &= \{A \in G | \sigma(A) = A\} = G \cap \mathrm{O}(n+1) \\ &= \left\{ A = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \middle| B \in \mathrm{SO}(n) \right\} \cong \mathrm{SO}(n). \end{aligned}$$

Thus, $K_\sigma$ is a compact connected Lie group, which means that $(G, K_\sigma, \sigma)$ is a Riemannian symmetric pair, and $H(n) = G/K_\sigma$ is a Riemannian symmetric space.

In fact, $H(n)$ is just of the type BDI in Table 1 with $p = 1$ and $q = n-1$. □

**Proposition 4.9.** $H^1 = \mathrm{SO}(2)$.

*Proof.* It follows from Lemma 4.7, Lemma 4.8 and Corollary 2.45. □

**Lemma 4.10.** *Let $(M^n, g)$ be a Riemannian manifold, if $M$ is isometric to $M_1^{n_1} \times M_2^{n_2}$, where $n_1 + n_2 = n$. Suppose matrix $K_{n\times n}$ is the sectional curvature matrix of $M$, then $K$ must be block diagonal. All geometric structures are presented in Appendix.*

*Remark* 4.11. Lemma 4.10 is elementary in theory of submanifold. See also [44].

**Lemma 4.12.** *$N^d$ is irreducible for $d = 1, 2, 3$.*

*Proof.* Lemma 4.7 implies that $N^1$ is irreducible. When $d = 2$ and $d = 3$, a direct computation of sectional curvature shows that the sectional curvature matrix for $N^d$ is not block diagonal, therefore irreducible (see Appendix). □

*Remark* 4.13. In general, the covariance matrix $\Sigma$ is not necessarily diagonal, so $N^d$ is very likely to be irreducible, for all $d \in \mathbb{N}$.

**Lemma 4.14.** *$N^d$ is not symmetric for all $d \geq 2$.*

*Proof.* When $d = 2$ and 3, Appendix shows that $\kappa$ is neither negative nor positive(of course not identically 0 either). We conclude that $N^2$ and $N^3$ are not of Euclidean type, compact type or non-compact type as well, therefore not a symmetric space. □

**Corollary 4.15.** *$H^d$must lie in the following Berger's list for $d = 2$, 3.*

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathrm{SO}\left(\frac{d(d+3)}{2}\right)$ | $\mathrm{U}\left(\frac{d(d+3)}{4}\right)$ | $\mathrm{SU}\left(\frac{d(d+3)}{4}\right)$ | $\mathrm{Sp}\left(\frac{d(d+3)}{8}\right)\cdot\mathrm{Sp}\,(1)$ | $\mathrm{Sp}\left(\frac{d(d+3)}{8}\right)$ | $G_2$ | $\mathrm{Spin}\,(7)$ |

.

*Proof.* For $d = 1, 2, 3$, $H^d$ is simply connected by Lemma 4.5, irreducible by Lemma 4.12 and also nonsymmetric by Lemma 4.14. Then theorem 2.47 applies. □

**Lemma 4.16.** *$H^d$ is neither $G_2$ nor* Spin(7)*.*

*Proof.* $dim(N^d) = \frac{d(d+3)}{2} = 2, 5, 9, 14, 20, \ldots$. However, Berger's list indicates that every manifold with $G_2$ or Spin(7) as its holonomy group must be of dimension 7 or 8, respectively. As a result, $H^d$ is neither $G_2$ nor Spin(7). □

**Lemma 4.17.** *$H^d$ is not equal to any of the following groups*

$$\mathrm{SU}\left(\frac{d(d+3)}{4}\right),\ \mathrm{Sp}\left(\frac{d(d+3)}{8}\right)\cdot\mathrm{Sp}(1),\ \mathrm{Sp}\left(\frac{d(d+3)}{8}\right).$$

*Proof.* By the comments in Berger's list, a manifold possesses holonomy groups $\mathrm{SU}\left(\frac{d(d+3)}{4}\right)$, $\mathrm{Sp}\left(\frac{d(d+3)}{8}\right)\cdot\mathrm{Sp}(1)$ or $\mathrm{Sp}\left(\frac{d(d+3)}{8}\right)$ must be an Einstein manifold, namely, there exists a constant $k$, s.t., $Ric = kg$. However, it is obvious that $N^2$ and $N^3$ are not Einstein manifolds (see Appendix), which implies that $H^d$ is not equal to any of the following groups:

$$\mathrm{SU}\left(\frac{d(d+3)}{4}\right),\ \mathrm{Sp}\left(\frac{d(d+3)}{8}\right)\cdot\mathrm{Sp}(1),\ \mathrm{Sp}\left(\frac{d(d+3)}{8}\right).$$

□

**Lemma 4.18.** *$N^d$ is not Kählerian for all $d \in \mathbb{N}$.*

*Proof.* Takano [40]–[44] has proved that $(N^d, \nabla^{(\alpha)})$ admits an almost complex structure $J^{(\alpha)}$ that is parallel to the $\alpha$-connection $\nabla^{(\alpha)}$ only if $\alpha = \pm 1$. Recall Proposition 2.6 that $\nabla^{(\alpha)}$ is the Levi-Civita connection if and only if $\alpha = 0$. This implies that $N^d$ does not admit a Kähler metric. □

A direct corollary follows.

**Corollary 4.19.** $H^d \neq \mathrm{U}\left(\frac{d(d+3)}{4}\right)$.

Based on all preparations, we could show the proof of Theorem 4.3.

*Proof of Theorem 4.3.*
When $d = 1$, Proposition 4.9 proves the case.
When $d = 2$, 3, we get a possible list in Corollary 4.15. Furthermore, Lemma 4.17, Lemma 4.18 and Corollary 4.19 rule out all possible groups except for $\mathrm{SO}\left(\frac{d(d+3)}{2}\right)$, as desired.

Thus, to sum up, we could conclude that

$$H^d = H_0^d = SO(\frac{d(d+3)}{2}),\ d = 1, 2, 3.$$

□

In fact, part of our results about the normal distribution manifolds can be generalized to the exponential family.

4.2. **Exponential family.** Let $S$ be an exponential family with dimension $n$ and $H$ be its holonomy group.

**Lemma 4.20.** *$S$ is not Kählerian.*

*Proof.* Similarly as that in lemma 4.18, $S$ admits an almost complex structure $J^{(\alpha)}$ that is parallel to the $\alpha$-connection $\nabla^{(\alpha)}$ only if $\alpha = \pm 1$, which is not the Levi-Civita connection [40]–[44]. □

**Corollary 4.21.** *$H$ is not equal to any of following groups*

$$\mathrm{U}\left(\frac{n}{2}\right),\ \mathrm{SU}\left(\frac{n}{2}\right),\ \mathrm{Sp}\left(\frac{n}{4}\right).$$

*Proof.* Lemma 4.20 implies that $H$ is not a subgroup of $\mathrm{U}\left(\frac{n}{2}\right)$. Noting that

$$\mathrm{Sp}\left(\frac{n}{4}\right) < \mathrm{SU}\left(\frac{n}{2}\right) < \mathrm{U}\left(\frac{n}{2}\right),$$

so $H$ cannot be any of them. □

Now, after ruling out several cases, the following theorem holds

**Theorem 4.22.** *If $S$ is a simply connected nonsymmetric $n$-dimensional exponential family with irreducible holonomy group $H$, then $H$ must be one of the following groups*

| *Holonomy* | *Dimension* |
|---|---|
| $\mathrm{SO}(m)$ | $n = m$ |
| $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$ | $n = 4m$ |
| $G_2$ | $n = 7$ |
| $\mathrm{Spin}(7)$ | $n = 8$ |

.

**Corollary 4.23.** *Suppose $S$ is a simply connected nonsymmetric $n$-dimensional exponential family with irreducible holonomy group $H$. We have*

1. *if $n \neq 7, 8$, then $H$ is either $\mathrm{SO}(n)$ or $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$, where $n = 4m$;*
2. *if $n = 7$, then $H$ is either $\mathrm{SO}(7)$ or $G_2$;*
3. *if $n \neq 7$ and $n$ is odd, then $H = \mathrm{SO}(n)$;*
4. *if $n = 2(2m+1)$, then $H = \mathrm{SO}(n)$;*
5. *if $n = 8$, then $H = \mathrm{SO}(8)$, or $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ or $\mathrm{Spin}(7)$;*
6. *if $n = 4m$ where $m \neq 2$, then $H$ is either $\mathrm{SO}(n)$ or $\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$;*
7. *if $S$ is not an Einstein manifold, then $H = \mathrm{SO}(n)$.*

*Proof.* (1)-(6) directly follow from Theorem 4.22, so the remaining is to prove (7). If $H$ is a subgroup of $G_2$ or $\mathrm{Spin}(7)$, then the Ricci curvature must be identically 0 [13], implying $S$ is an Einstein manifold, which is a contradiction. □

Since almost all common examples of exponential families are not Einstein, the holonomy groups of almost all exponential families are $\mathrm{SO}(n)$. There is only one exception, the monistic normal distribution manifold $N^1$, which is Einstein. However, $H^1 = \mathrm{SO}(2) = \mathrm{SO}\left(\dim N^1\right)$ (Proposition 4.9) which coincides with our results.

## 5. Conclusion

After some preliminaries about information geometry, fibre bundles and holonomy groups, two main results, Theorem 3.8 and Theorem 4.3 are proved. Theorem 3.8 shows that the holonomy groups of normal distribution manifolds are special orthogonal groups for some dimensions. In addition, a list of possible holonomy groups for general exponential families is presented in Theorem 4.3. In addition, Prof. R. Byrant proposed the following conjecture.

**Conjecture 5.1** (R. Bryant)**.** *The affine holonomy group of normal distribution manifold is the special linear group.*

## 6. Acknowledgement

The first author would like to thank Prof. R. Bryant for his valuable discussions on the Theorem 4.3 and Conjecture 5.1.

The last author was supported by Austrian Science Fund (FWF) grant FWF-Projekt 29467.

## 7. Appendix

7.1. **Tables of classification of Riemannian holonomy groups.** We first presents two tables mentioned in Subsection 2.3.

Table 1: One of four lists of Riemannian symmetric spaces

| Label | G | K | Dimension | Rank | Geometric interpretation |
|---|---|---|---|---|---|
| AI | $\mathrm{SL}(n;\mathbb{R})$ | $\mathrm{SO}(n)$ | $\frac{(n-1)(n+2)}{2}$ | $n-1$ | Set of $\mathbb{R}P^n_{hyp}$'s in $\mathbb{C}P^n_{hyp}$ |
| AII | $\mathrm{SL}(n;\mathbb{H})$ | $\mathrm{Sp}(n)$ | $2n^2-n-1$ | $n-1$ | Set of $\mathbb{H}P^{n-1}_{hyp}$'s in $\mathbb{C}P^{2n-1}_{hyp}$ |
| AIII | $\mathrm{SU}(p,q)$ | $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ | $2pq$ | $\min(p,q)$ | $G_p(p,q;\mathbb{C})$ |
| BDI | $\mathrm{SO}_0(p,q)$ | $\mathrm{SO}(p)\times\mathrm{SO}(q)$ | $pq$ | $\min(p,q)$ | $G_p(p,q;\mathbb{R})$ |
| DIII | $\mathrm{SO}(n;\mathbb{H})$ | $\mathrm{U}(n)$ | $n(n-1)$ | $[\frac{n}{2}]$ | Set of $\mathbb{C}P^{n-1}_{hyp}$'s in $\mathbb{R}P^{2n-1}_{hyp}$ |
| CI | $\mathrm{Sp}(n;\mathbb{R})$ | $\mathrm{U}(n)$ | $n(n+1)$ | n | Set of $\mathbb{C}P^n_{hyp}$'s in $\mathbb{H}P^n_{hyp}$ |
| CII | $\mathrm{Sp}(p,q)$ | $\mathrm{Sp}(p)\times\mathrm{Sp}(q)$ | $4pq$ | $\min(p,q)$ | $G_p(p,q;\mathbb{H})$ |
| EI | $E_6^6$ | $Sp(4)$ | 42 | 6 | Antichains of $(\mathbb{C}\otimes\mathbb{O})P^2_{hyp}$ |
| EII | $E_6^2$ | $\mathrm{SU}(6)\times\mathrm{SU}(2)$ | 40 | 4 | Set of the $(\mathbb{C}\otimes\mathbb{H})P^2_{hyp}$'s in $(\mathbb{C}\otimes\mathbb{O})P^2_{hyp}$ |
| EIII | $E_6^{-14}$ | $\mathrm{SO}(10)\times\mathrm{SO}(2)$ | 32 | 2 | Rosenfeld's hyperbolic projective plane $(\mathbb{C}\otimes\mathbb{O})P^2_{hyp}$ |
| EIV | $E_6^{-26}$ | $F_4$ | 26 | 2 | Set of $\mathbb{O}P^2$'s the in $(\mathbb{C}\otimes\mathbb{O})P^2_{hyp}$ |
| EV | $E_7^7$ | $\mathrm{SU}(8)$ | 70 | 7 | Antichains of $(\mathbb{H}\otimes\mathbb{O})P^2_{hyp}$ |
| EVI | $E_7^{-5}$ | $\mathrm{SO}(12)\times\mathrm{SU}(2)$ | 64 | 4 | Rosenfeld hyperbolic projective plane $(\mathbb{H}\otimes\mathbb{O})P^2_{hyp}$ |
| EVII | $E_7^{-25}$ | $E_6\times\mathrm{SO}(2)$ | 54 | 3 | Set of the $(\mathbb{C}\otimes\mathbb{O})P^2_{hyp}$'s in $(\mathbb{H}\otimes\mathbb{O})P^2_{hyp}$ |
| EVIII | $E_8^8$ | $\mathrm{SO}(16)$ | 128 | 8 | Rosenfeld projective plane $(\mathbb{O}\otimes\mathbb{O})P^2_{hyp}$ |
| EIX | $E_8^{-24}$ | $E_7\times\mathrm{SU}(2)$ | 112 | 4 | Set of the $(\mathbb{H}\otimes\mathbb{O})P^2_{hyp}$'s in $(\mathbb{O}\otimes\mathbb{O})P^2_{hyp}$ |
| FI | $F_4^4$ | $\mathrm{Sp}(3)\times\mathrm{SU}(2)$ | 28 | 4 | Set of the $\mathbb{H}P^2_{hyp}$'s in $\mathbb{O}P^2_{hyp}$ |
| FII | $F_4^{-20}$ | $\mathrm{SO}(9)$ | 16 | 1 | Hyperbolic Cayley projective plane $\mathbb{O}P^2_{hyp}$ |
| G | $G_2^2$ | $\mathrm{SU}(2)\times\mathrm{SU}(2)$ | 8 | 2 | Set of non-division $\mathbb{H}$ subalgebras of the non-division $\mathbb{O}$ |

Table 2: List of Riemannian holonomy groups

| $H$ | Dimension | Type of manifold | Comments |
|---|---|---|---|
| SO($n$) | $n$ | Oriented manifold | Generic Metric |
| U($n$) | $2n$ | Kähler manifold | Kähler |
| SU($n$) | $2n$ | Calabi-Yau manifold | Ricci-flat, Kähler |
| Sp($n$) · Sp(1) | $4n$ | Quaternion-Kähler manifold | Eistein |
| Sp($n$) | $4n$ | Hyperkähler manifold | Ricci-flat, Kähler |
| $G_2$ | 7 | $G_2$ manifold | Ricci-flat |
| Spin(7) | 8 | Spin(7) manifold | Ricci-flat |

7.2. **Computation of $N^2$ and $N^3$.** Next, we present the computation of $N^d$ for $d = 2$ and 3, used in Subsection 4.1.

$$N^2 = \{p(x, y, \mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}) = \frac{\exp\{-AB\}}{2\pi\sqrt{\sigma_1\sigma_2 - \sigma_{12}^2}} | \mu_1, \mu_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in \mathbb{R}_+, \sigma_{12} = cov(X, Y)\},$$

where $A = \frac{1}{2(\sigma_2\sigma_2 - \sigma_{12}^2)}$, $B = \sigma_2(x-\mu_1)^2 - 2\sigma_{12}(x-\mu_1)(y-\mu_2)+\sigma_1(y-\mu_2)^2$. This is a 5-dimensional manifold. It is obvious that

$$\begin{aligned} p(x, y) &= \log(\frac{1}{2\pi\sqrt{\Delta}} e^{-\frac{1}{2\Delta}(\sigma_2(x-\mu_1)^2 - 2\sigma_{12}(x-\mu_1)(y-\mu_2)+\sigma_1(y-\mu_2)^2)}) \\ &= \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\Delta} x + \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\Delta} y + \frac{-\sigma_2}{2\Delta} x^2 + \frac{\sigma_{12}}{2\Delta} xy + \frac{-\sigma_1}{2\Delta} y^2 \\ &\quad - (\log(2\pi\sqrt{\Delta}) + \frac{{\mu_2}^2\sigma_1 + {\mu_1}^2\sigma_2 - 2\mu_1\mu_2\sigma_{12}}{2\Delta}), \end{aligned}$$

where $\Delta = \sigma_1\sigma_2 - \sigma_{12}^2$. Therefore, the coordinates for exponential family are

$$\theta_1 = \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\Delta}, \theta_2 = \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\Delta}, \theta_3 = -\frac{\sigma_2}{2\Delta}, \theta_4 = -\frac{\sigma_{12}}{\Delta}, \theta_5 = -\frac{\sigma_1}{2\Delta},$$

and $\Delta = \frac{1}{4\theta_3\theta_5 - {\theta_4}^2}$. While the potential function is

$$\begin{aligned} \phi(\theta) &= \log(2\pi\sqrt{\Delta}) + \frac{{\mu_2}^2\sigma_1 + {\mu_1}^2\sigma_2 - 2\mu_1\mu_2\sigma_{12}}{2\Delta} \\ &= \log(2\pi\sqrt{\Delta}) - \Delta({\theta_2}^2\theta_3 - \theta_1\theta_2\theta_4 + {\theta_1}^2\theta_5). \end{aligned}$$

The the Fisher metric matrix $(g_{ij})_{5\times5}$ is given, where the lower triangular entries are omitted due to symmetry, by

$$\begin{pmatrix} \sigma_1 & \sigma_{12} & 2\mu_1\sigma_1 & 2\sigma_{12}\mu_2 & \sigma_1\mu_2 + \sigma_{12}\mu_1 \\ & \sigma_2 & 2\sigma_{12}\mu_1 & 2\sigma_2\mu_2 & \sigma_2\mu_1 + \sigma_{12}\mu_2 \\ & & 2\sigma_1\left(\sigma_1 + 2\mu_1^2\right) & 2\sigma_{12}\left(\sigma_{12} + 2\mu_1\mu_2\right) & 2\left(\sigma_{12}\sigma_1 + \mu_1\sigma_1\mu_2 + \mu_1^2\sigma_{12}\right) \\ & & & 2\sigma_2(\sigma_2 + 2\mu_2^2) & 2\left(\sigma_2\sigma_{12} + \sigma_2\mu_1\mu_2 + \sigma_{12}\mu_2^2\right) \\ & & & & \sigma_1\sigma_2 + \sigma_1\mu_2^2 + \sigma_2\mu_1^2 + 2\mu_1\mu_2\sigma_{12} + \sigma_{12}^2 \end{pmatrix}.$$

For the components $K_{ij}$, $i, j = 1, 2, 3, 4, 5$ of the sectional curvature, we list them as follows.

$K_{ii} = 0$, $i = 1, 2, 3, 4, 5$,
$K_{12} = 1/4 - 1/4\alpha^2$,
$K_{13} = -1/2 + 1/2\alpha^2$,
$$K_{14} = -\frac{\left(-3\sigma_2^2\sigma_1 - \sigma_1^2\sigma_2 - \sigma_2^2\mu_1^2 + \sigma_1\mu_1^2\sigma_2\right)\alpha^2}{4\left(\sigma_1^2\sigma_2 + \sigma_1\mu_1^2\sigma_2 + \sigma_2^2\sigma_1 - \sigma_2^2\mu_1^2\right)} - \frac{\sigma_1^2\sigma_2 + 3\sigma_2^2\sigma_1 - \sigma_1\mu_1^2\sigma_2 + \sigma_2^2\mu_1^2}{4\left(\sigma_1^2\sigma_2 + \sigma_1\mu_1^2\sigma_2 + \sigma_2^2\sigma_1 - \sigma_2^2\mu_1^2\right)},$$
$$K_{15} = -\frac{\left(\sigma_1\mu_2^2\sigma_2 - \sigma_2\sigma_2^2 - \sigma_2^2\mu_2^2\right)\left(1 - \alpha^2\right)}{2\left(2\sigma_2^2\sigma_1 + 2\sigma_1\mu_2^2\sigma_2 - 2\sigma_2^2\mu_2^2\right)},$$
$$K_{23} = -\frac{\left(-\sigma_2^2\sigma_1 - \sigma_2^2\mu_1^2 + \sigma_1\mu_1^2\sigma_2\right)\alpha^2 - \left(\sigma_1\mu_1^2\sigma_2 + \sigma_2^2\mu_1^2 + \sigma_2^2\sigma_1\right)}{2\left(\sigma_1^2\sigma_2 + 2\sigma_1\mu_1^2\sigma_2 - 2\sigma_2^2\mu_1^2\right)},$$

$$
\begin{aligned}
K_{24} &= -\frac{\left(-\sigma_2^2\sigma_1 + \sigma_1\mu_2^2\sigma_2 - 3\sigma_2\sigma_2^2 - \sigma_2^2\mu_2^2\right)\left(1-\alpha^2\right)}{4\left(\sigma_2^2\sigma_1 + \sigma_1\mu_2^2\sigma_2 + \sigma_2\sigma_2^2 - \sigma_2^2\mu_2^2\right)},\\
K_{25} &= -1/2 + 1/2\alpha^2,\\
K_{34} &= -\frac{\left(-\sigma_1^3\sigma_2 - \sigma_1^3\mu_2^2 - 3\sigma_1^2\mu_1^2\sigma_2 + \sigma_2^2\sigma_1^2 + 2\sigma_1^2\mu_1\sigma_2\mu_2\right)\alpha^2}{2\left(\sigma_1^3\sigma_2 + \sigma_1^3\mu_2^2 + 3\sigma_1^2\mu_1^2\sigma_2 - 2\sigma_1^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_1^2 + 2\sigma_1\sigma_2\mu_1^4 - 2\sigma_1\sigma_2^2\mu_1^2 - 2\sigma_2^2\mu_1^4\right)}\\
&\quad - \frac{\left(\sigma_1\sigma_2\mu_1^4 + 2\sigma_1\sigma_2^2\mu_1^2 - \sigma_2^2\mu_1^4\right)\alpha^2}{2\left(\sigma_1^3\sigma_2 + \sigma_1^3\mu_2^2 + 3\sigma_1^2\mu_1^2\sigma_2 - 2\sigma_1^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_1^2 + 2\sigma_1\sigma_2\mu_1^4 - 2\sigma_1\sigma_2^2\mu_1^2 - 2\sigma_2^2\mu_1^4\right)}\\
&\quad - \frac{\sigma_1^3\sigma_2 + \sigma_2^2\mu_1^4 - \sigma_2^2\sigma_1^2 + \sigma_1^3\mu_2^2 + 3\sigma_1^2\mu_1^2\sigma_2}{2\left(\sigma_1^3\sigma_2 + \sigma_1^3\mu_2^2 + 3\sigma_1^2\mu_1^2\sigma_2 - 2\sigma_1^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_1^2 + 2\sigma_1\sigma_2\mu_1^4 - 2\sigma_1\sigma_2^2\mu_1^2 - 2\sigma_2^2\mu_1^4\right)},\\
&\quad + \frac{2\sigma_1^2\mu_1\sigma_2\mu_2 + \sigma_1\sigma_2\mu_1^4 + 2\sigma_1\sigma_2^2\mu_1^2}{2\left(\sigma_1^3\sigma_2 + \sigma_1^3\mu_2^2 + 3\sigma_1^2\mu_1^2\sigma_2 - 2\sigma_1^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_1^2 + 2\sigma_1\sigma_2\mu_1^4 - 2\sigma_1\sigma_2^2\mu_1^2 - 2\sigma_2^2\mu_1^4\right)}\\
K_{35} &= \frac{\left(\sigma_1\sigma_2\sigma_2^2 + 2\sigma_1\sigma_2\mu_1\sigma_2\mu_2 + \sigma_1\sigma_2^2\mu_2^2 + \sigma_2^2\mu_1^2\sigma_2 + \mu_1^2\sigma_2^2\mu_2^2\right)\alpha^2}{\sigma_1^2\sigma_2^2 + 2\sigma_1^2\mu_2^2\sigma_2 + 2\sigma_1\sigma_2^2\mu_1^2 + 4\sigma_1\sigma_2\mu_1^2\mu_2^2 - 4\sigma_2^3\mu_1\mu_2 - 4\mu_1^2\sigma_2^2\mu_2^2 - \sigma_2^4}\\
&\quad - \frac{\left(\sigma_1\sigma_2\mu_1^2\mu_2^2 + \sigma_2^4 + 4\sigma_2^3\mu_1\mu_2\right)\alpha^2}{\sigma_1^2\sigma_2^2 + 2\sigma_1^2\mu_2^2\sigma_2 + 2\sigma_1\sigma_2^2\mu_1^2 + 4\sigma_1\sigma_2\mu_1^2\mu_2^2 - 4\sigma_2^3\mu_1\mu_2 - 4\mu_1^2\sigma_2^2\mu_2^2 - \sigma_2^4},\\
&\quad + \frac{\sigma_1\sigma_2\mu_1^2\mu_2^2 + \sigma_2^4 + 4\sigma_2^3\mu_1\mu_2}{\sigma_1^2\sigma_2^2 + 2\sigma_1^2\mu_2^2\sigma_2 + 2\sigma_1\sigma_2^2\mu_1^2 + 4\sigma_1\sigma_2\mu_1^2\mu_2^2 - 4\sigma_2^3\mu_1\mu_2 - 4\mu_1^2\sigma_2^2\mu_2^2 - \sigma_2^4}\\
&\quad - \frac{\sigma_1\sigma_2\sigma_2^2 + 2\sigma_1\sigma_2\mu_1\sigma_2\mu_2 + \sigma_1\sigma_2^2\mu_2^2 + \sigma_2^2\mu_1^2\sigma_2 + \mu_1^2\sigma_2^2\mu_2^2}{\sigma_1^2\sigma_2^2 + 2\sigma_1^2\mu_2^2\sigma_2 + 2\sigma_1\sigma_2^2\mu_1^2 + 4\sigma_1\sigma_2\mu_1^2\mu_2^2 - 4\sigma_2^3\mu_1\mu_2 - 4\mu_1^2\sigma_2^2\mu_2^2 - \sigma_2^4}\\
K_{45} &= \frac{\left(\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + \sigma_2^3\mu_1^2 + \sigma_2^2\mu_2^4\right)\alpha^2}{2\left(\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + 2\sigma_1\sigma_2\mu_2^4 + \sigma_2^3\mu_1^2 - 2\sigma_2^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_2^2 - 2\sigma_2\sigma_2^2\mu_2^2 - 2\sigma_2^2\mu_2^4\right)}\\
&\quad - \frac{\left(\sigma_1\sigma_2\mu_2^4 + \sigma_2^2\sigma_2^2 + 2\sigma_2^2\mu_1\sigma_2\mu_2 + 2\sigma_2\sigma_2^2\mu_2^2\right)\alpha^2}{2\left(\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + 2\sigma_1\sigma_2\mu_2^4 + \sigma_2^3\mu_1^2 - 2\sigma_2^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_2^2 - 2\sigma_2\sigma_2^2\mu_2^2 - 2\sigma_2^2\mu_2^4\right)}.\\
&\quad + \frac{\sigma_1\sigma_2\mu_2^4 + \sigma_2^2\sigma_{12}^2 + 2\sigma_2^2\mu_1\sigma_2\mu_2 + 2\sigma_2\sigma_2^2\mu_2^2}{2\left(\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + 2\sigma_1\sigma_2\mu_2^4 + \sigma_2^3\mu_1^2 - 2\sigma_2^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_2^2 - 2\sigma_2\sigma_2^2\mu_2^2 - 2\sigma_2^2\mu_2^4\right)}\\
&\quad - \frac{\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + \sigma_2^3\mu_1^2 + \sigma_2^2\mu_2^4}{2\left(\sigma_1\sigma_2^3 + 3\sigma_1\sigma_2^2\mu_2^2 + 2\sigma_1\sigma_2\mu_2^4 + \sigma_2^3\mu_1^2 - 2\sigma_2^2\mu_1\sigma_2\mu_2 - \sigma_2^2\sigma_2^2 - 2\sigma_2\sigma_2^2\mu_2^2 - 2\sigma_2^2\mu_2^4\right)}
\end{aligned}
$$

The components $Ric_{ij}$, $i, j = 1, 2, 3, 4, 5$ of the Ricci tensor are given by

$$
\begin{aligned}
Ric_{12} &= 1/2\sigma_2\alpha^2 - 1/2\sigma_2,\\
Ric_{13} &= \mu_1\sigma_1\alpha^2 - \mu_1\sigma_1,\\
Ric_{14} &= \left(1/2\sigma_1\mu_2 + 1/2\sigma_2\mu_1\right)\alpha^2 - 1/2\sigma_1\mu_2 - 1/2\sigma_2\mu_1,\\
Ric_{15} &= -\sigma_2\mu_2 + \alpha^2\sigma_2\mu_2,\\
Ric_{22} &= 1/2\sigma_{22}\alpha^2 - 1/2\sigma_{22},\\
Ric_{23} &= -\sigma_2\mu_1 + \alpha^2\sigma_2\mu_1,\\
Ric_{24} &= \left(1/2\sigma_{22}\mu_1 + 1/2\sigma_2\mu_2\right)\alpha^2 - 1/2\sigma_{22}\mu_1 - 1/2\sigma_2\mu_2,\\
Ric_{25} &= -\sigma_{22}\mu_2 + \sigma_{22}\alpha^2\mu_2,\\
Ric_{33} &= 2\sigma_1\left(\sigma_1 + {\mu_1}^2\right)\alpha^2 - 2\sigma_1\left(\sigma_1 + {\mu_1}^2\right),\\
Ric_{34} &= \left(2\sigma_2\sigma_1 + \mu_1\sigma_1\mu_2 + {\mu_1}^2\sigma_2\right)\alpha^2 - 2\sigma_2\sigma_1 - \mu_1\sigma_1\mu_2 - {\mu_1}^2\sigma_2,\\
Ric_{35} &= \left(-\sigma_1\sigma_{22} + 3{\sigma_2}^2 + 2\mu_1\sigma_2\mu_2\right)\alpha^2 + \sigma_1\sigma_{22} - 3{\sigma_2}^2 - 2\mu_1\sigma_2\mu_2,\\
Ric_{44} &= \left(3/2\sigma_1\sigma_{22} + 1/2\sigma_1{\mu_2}^2 + 1/2{\mu_1}^2\sigma_{22} + 1/2{\sigma_2}^2 + \mu_1\sigma_2\mu_2\right)\alpha^2 - 3/2\sigma_{11}\sigma_{22} - 1/2\sigma_1{\mu_2}^2\\
&\qquad - 1/2{\mu_1}^2\sigma_{22} - 1/2{\sigma_2}^2 - \mu_1\sigma_2\mu_2,\\
Ric_{45} &= \left(2\sigma_{22}\sigma_2 + \sigma_{22}\mu_1\mu_2 + \sigma_2{\mu_2}^2\right)\alpha^2 - 2\sigma_{22}\sigma_2 - \sigma_{22}\mu_1\mu_2 - \sigma_2{\mu_2}^2,\\
Ric_{55} &= 2\sigma_{22}\left(\sigma_{22} + {\mu_2}^2\right)\alpha^2 - 2\sigma_{22}\left(\sigma_{22} + {\mu_2}^2\right).
\end{aligned}
$$

*Remark* 7.1. The computations of $N^3$, even $N^d$ for all $d \in \mathbb{N}$ are similar to that of $N^2$ (in fact, they are calculated by a common program), but the results are too large to be presented here.

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