MATH-1030
Matrix Theory & Linear Algebra I

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May 7th, 2019

1. 1st assignment of today: read the syllabus posted on Brightspace.
   - Lin Jiu, Chase 307, Lin.Jiu@dal.ca Office hours: Mondays and Tuesdays 12:00--1:30 PM or by appointment
     Office hours THIS WEEK: Wednesday (May 8th) and Thursday (9th) 12:00–1:30 PM
   - Learning Centre: Chase 119. Tutors are available Monday–Friday 1–5pm
   - Homework (10%), Classwork (10%), Midterm I (20%), Midterm II (20%), Final Exam (40%)
     - You need to pass the final (50% or higher) to pass the course.
     - Midterm I: May 21, Midterm II: June 6th, 6:00-7:30 PM, Dunn 117
     - Final Exam: June 20, 6:00-9:00 PM, Dunn 117
     - Homework on BrightSpace (WeBWork) (five problem sets with 2 points for each)
     - Classwork: on each lecture days (not including today and days with exams, i.e., 10 times), there will be one problem in the middle of the lecture. You can form a group of 2 or 3 to turn your classwork in. Each problem is worth 1 point
2. 2nd assignment of the today: find one or two partners.
3. For midterms and final, you will expect questions of
   - True of False without explanations; multiple choice; ⟵NO PARTIAL CREDITS
   - Tu-re of False WITH explanations, shorted or detailed answers: ⟵PARTIAL CREDITS
4. A missed midterm cannot be written at another time. If you miss the midterm without prior permission, then it will count as a 0. Exceptions are made in two cases: (1) if you obtain the instructor's prior permission to miss a midterm, or (2) if you have an officially valid excuse such as a medical doctor's note. In these cases, the weight of the missed midterm will be shifted to the final exam (e.g., the final exam will then count 60% instead of 40%). There is no make-up option for the final exam except in cases of an officially valid excuse such as a medical doctor's note.

1. LINEAR:
   - Example of a linear equation:
   $$2x = 8$$
   - Example of a non-linear equation:
   $$2x^{2} = 8$$
   - Example of another linear equation:
   $$2x + 3y + 10z - 5w = 13 \qquad (*)$$
2. SOLUTION(S) of $(*)$
   - Solution 1: $x = 2,\ y = 3,\ z = w = 0.$
   $$2 \cdot 2 + 3 \cdot 3 = 4 + 9 = 13.$$
   - Solution 2: $x = 0,\ y = 1,\ z = 1,\ w = 0.$
   $$3 \cdot 1 + 10 \cdot 1 = 13.$$
   - Solution 3: $x = \frac{13}{2},\ y = z = w = 0.$
   $$2 \cdot \frac{13}{2} = 13.$$

Fact: There are infinitely many solutions of $(*)$.

Tuples

$$2x + 3y + 10z - 5w = 13 \qquad (*)$$

Notation

We shall use the ***tuple*** notation for the solutions.

$$(x, y, z, w) = (2, 3, 0, 0)$$

Tuple:

$$\begin{array}{rcl} (x, y) & \text{-pair} & \text{2-tuple} \\ (x, y, z) & \text{-triple} & \text{3-tuple} \\ & \cdots & \text{etc.} \end{array}$$

Remark

Note that tuples are *ordered*, and therefore different from sets. For example,

$$(2, 3) \neq (3, 2) \quad \text{while} \quad \{2, 3\} = \{3, 2\}.$$

The set of solutions of $(*)$ can be written(/listed) as

$$\left\{(2, 3, 0, 0), (0, 1, 1, 0), \left(\frac{13}{2}, 0, 0, 0\right), \ldots\right\}$$

Example

$$\begin{aligned} 2x + 3y + 10z - 5w &= 13 \quad (*) \\ x + 2y + 3z + 4w &= 5 \\ y - z + 2w &= 0 \end{aligned}$$

Definition

A *solution* of a system of equations is an assigned of numbers to the variables, making all the equation true.

Example

$(2, 3, 0, 0)$ is not a solution, since $1 \cdot 2 + 2 \cdot 3 = 8 \neq 5$.

$(0, 1, 1, 0)$ is a solution: $\begin{array}{l} 2 + 3 = 5 \\ 1 - 1 = 0 \end{array}$

Definition

A *linear equation* is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b. \tag{1}$$

Here,

$$\begin{aligned} x_1, \ldots, x_n &\quad - \quad \text{variables/unknowns} \\ a_1, \ldots, a_n &\quad - \quad \text{coefficients} \\ b &\quad - \quad \text{constant term/right-hand side (RHS)} \end{aligned}$$

If an equation is written in the form of (1), it is in *standard form*.

Example

$$2x_1 + 3x_2 + 0x_3 = 5$$

is a linear equation in standard form.

Example

$$2x - 3y + 5 = 7z + 2y$$

It is a linear equation $(x_1, x_2, x_3) = (x, y, z)$, but not in standard form. We can rewrite it to standard form by algebra:

$$2x + (-5)y + (-7)z = -5,$$

or

$$2x - 5y - 7z = -5.$$

Definition

A *system of linear equations* is a list of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{2}$$

Similarly, this is the *standard form*, and

$$x_1, \ldots, x_n - \text{variables/unknowns}; \quad a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn} - \text{coefficients}; \quad b_1, \ldots, , b_m - \text{constant term}$$

A system of equations is *inconsistent* if it has no solution. It is called *consistent* if it has one or more solutions.

Example

$$\begin{aligned} 2x - 7w &= 5 \\ 2y + 3x &= 15 \\ x + w &= y + 5 \end{aligned} \quad \text{a system of linear equations, but not in standard form} \Rightarrow \begin{aligned} 2x + 0y - 7w &= 5 \\ 3x + 2y + 0w &= 15 \\ x - y + w &= 5 \end{aligned}$$

Solution

$$(x, y, w) = \left(\frac{185}{39}, \frac{5}{13}, \frac{25}{39}\right). \quad \text{consistent}$$

Example

Are the following systems consistent or inconsistent?

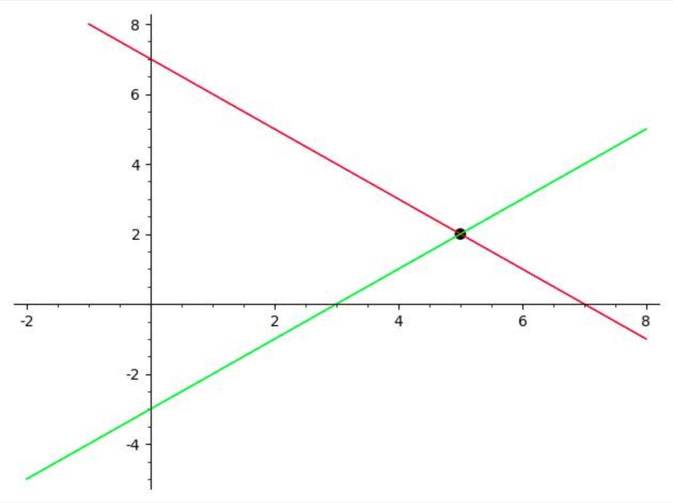
$$(1) \begin{cases} x + y &= 5 \\ 2x + 2y &= 11 \end{cases} \quad (2) \begin{cases} x &= 0 \\ x &= 1 \end{cases} \quad (3) 0x = 1 \quad (4) \begin{cases} x + y &= 7 \\ x - y &= 3 \\ x + 2y &= 1 \end{cases} \boxed{\text{All are inconsistent.}}$$

When there are 2 or 3 variables, we can visualize the system.

Example

$$\begin{cases} x + y = 7 \\ x - y = 3 \end{cases}$$

Each equation represents a line.



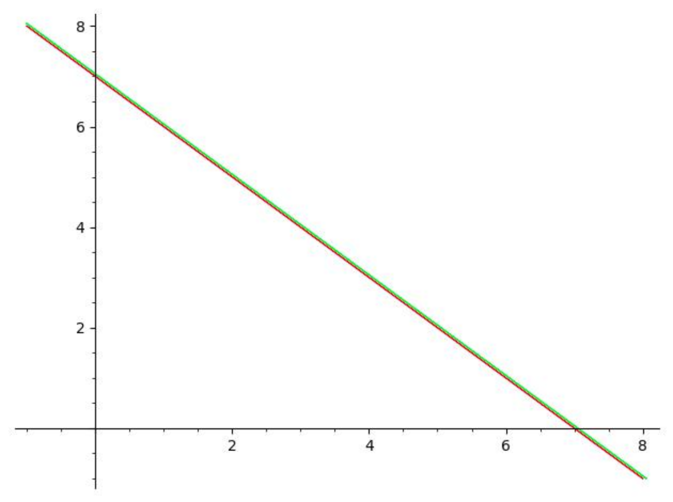
This system has a unique solution

$$(x, y) = (5, 2).$$

Geometric interpretation

Example

$$\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$$

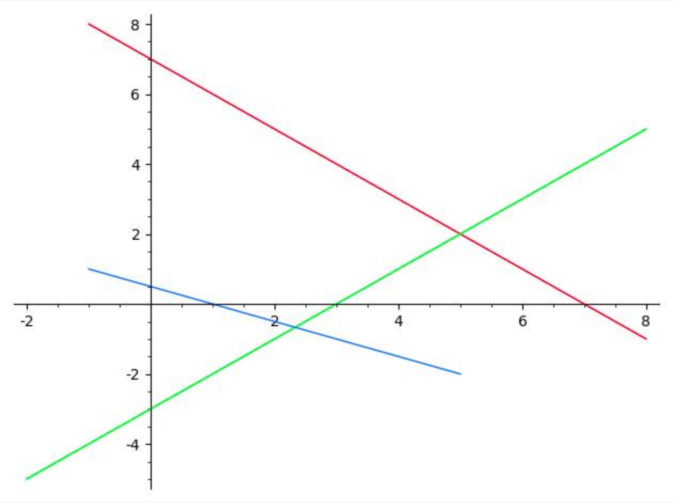


This system has infinity many solutions

$$\{(5,0),(4,1),(0,5),\ldots\}$$

Example

$$\begin{cases} x+y=7 \\ x-y=3 \\ x+2y=1 \end{cases}$$



This system has no solution (inconsistent).

Example

$$\begin{cases} x+y+z=1 \\ 2x+y-z=5 \\ x-y+2z=7 \end{cases}$$

It has a unique solution (SageMath). This particular example involves three planes intersection in a point. There are many other possibilities for how 3 planes intersect (e.g., two or all three planes could be parallel).

Elementary operations:

- Interchange (swap) two equations.
- Multiply an equation by a <u>**non-zero**</u> number. For example,

$$x + 2y = 5 \sim 2x + 4y = 10.$$

- Add a multiple of one equation to another. For example,

$$\begin{aligned} E1 &: 2x + y = 5 \\ E2 &: x - 2y = 0 \end{aligned} \quad \overset{E1 \leftarrow E1 - 2E2}{\sim} \quad \begin{aligned} E1 &: 0x + 5y = 5 \\ E2 &: x - 2y = 0 \end{aligned}$$

  More precisely,

$$\begin{aligned} LHS &: (2x + y) - 2(x - 2y) = 5y \\ RHS &: 5 - 2 \cdot 0 = 5. \end{aligned}$$

  From this, we can easily see $y = 1$ and then $x = 2$.

Performing an elementary operation does NOT change the set of solutions of the system of equations, but (hopefully) it will make the system *simpler*.

Remark

The symbol "$\sim$" means that tow systems of equations are *equivalent*, i.e., they have the SAME solutions.

Example

$$\begin{aligned} E1 &: x+2y+z=8\\ E2 &: 2x+y+z=7\\ E3 &: x-y-z=-4 \end{aligned}$$

$$\boxed{\overset{E2\leftarrow E2-2E1}{\sim}}\quad \begin{aligned} E1 &: x+2y+z=8\\ E2 &: \quad -3y-z=-9\\ E3 &: x-y-z=-4 \end{aligned}$$

$$\boxed{\overset{E3\leftarrow E3-E1}{\sim}}\quad \begin{aligned} E1 &: x+2y+z=8\\ E2 &: \quad -3y-z=-9\\ E3 &: \quad -3y-2z=-12 \end{aligned}$$

$$\boxed{\overset{E3\leftarrow E3-E2}{\sim}}\quad \begin{aligned} E1 &: x+2y+z=8\\ E2 &: -3y-z=-9\\ E3 &: \qquad -z=-3 \end{aligned}$$

$$\boxed{\overset{E3\leftarrow -E3}{\sim}}\quad \begin{aligned} E1 &: x+2y+z=8\\ E2 &: -3y-z=-9\\ E3 &: \qquad z=3 \end{aligned}$$

Back Substitution:

$$z=3 \overset{E2}{\Longrightarrow} -3y-3=-9 \Longrightarrow -3y=-6 \Rightarrow y=2$$

$$(y,z)=(2,3) \overset{E1}{\Longrightarrow} x+4+3=8 \Longrightarrow x=1$$

This system has the unique solution

$$(x,y,z)=(1,2,3)$$

It is consistent

Matrix Notation

Definition

The *augmented matrix* of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

Example

$$\begin{aligned} E1 : x + 2y + z &= 8 \\ E2 : 2x + y + z &= 7 \Rightarrow \\ E3 : x - y - z &= -4 \end{aligned} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right]$$

Notation

The following shorter notation of elementary operations will apply:

- Swap rows R_i and R_j: $R_i \longleftrightarrow R_j$.
- Multiply R_i by a nonzero number a: $R_i \longleftarrow aR_i$.
- Add a times R_i to R_j: $R_j \longleftarrow R_j + aR_i$.

Example

$$
\begin{aligned}
E1 &: x + 2y + z = 8 \\
E2 &: 2x + y + z = 7 \\
E3 &: x - y - z = -4
\end{aligned}
\Longrightarrow
\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right]
\qquad R_2 \leftarrow R_2 - 2R_1
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 1 & -1 & -1 & -4 \end{array}\right]
\qquad R_3 \leftarrow R_3 - R_1
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & -3 & -2 & -12 \end{array}\right]
\qquad R_3 \leftarrow R_3 - R_2
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & -1 & -3 \end{array}\right]
\qquad R_3 \leftarrow -R_3
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

Substitution as before. Or, we can continue as

$$
\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right]
\qquad R_2 \leftarrow R_2 + R_3
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right]
\qquad R_2 \leftarrow -\frac{R_2}{3}
$$

$$
\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]
\qquad R_1 \leftarrow R_1 - 2R_2 - R_3 \sim
\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]
\Rightarrow (x, y, z) = (1, 2, 3).
$$

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Matrix Theory & Linear Algebra I

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Definition

The *augmented matrix* of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

is

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right].$$

Notation

The following shorter notation of elementary operations will apply:

- Swap rows R_i and R_j: $R_i \longleftrightarrow R_j$.
- Multiply R_i by a nonzero number a: $R_i \longleftarrow aR_i$.
- Add a times R_i to R_j: $R_j \longleftarrow R_j + aR_i$.

Example

$$\begin{aligned} E1 &: x+2y+z=8 \\ E2 &: 2x+y+z=7 \Rightarrow \\ E3 &: x-y-z=-4 \end{aligned} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right] \quad R_2 \leftarrow R_2 - 2R_1$$

$$\begin{array}{rccc|c} & 2 & 1 & 1 & 7 \\ -2\cdot(& 1 & 2 & 1 & 8\ ) \\ \hline & 2 & 1 & 1 & 7 \\ - & 2 & 4 & 2 & 16 \\ \hline & 0 & -3 & -1 & -9 \end{array}$$

$$\begin{aligned} E1 &: x+2y+z=8 \\ E2 &: 2x+y+z=7 \Rightarrow \\ E3 &: x-y-z=-4 \end{aligned} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 1 & -1 & -1 & -4 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 1 & -1 & -1 & -4 \end{array}\right] \quad \boxed{R_3 \leftarrow R_3 - R_1} \quad \begin{array}{rccc|c} & 1 & -1 & -1 & -4 \\ - & 1 & 2 & 1 & 8 \\ \hline & 0 & -3 & -2 & -12 \end{array}$$

$$\begin{aligned} E1 &: x+2y+z=8 \\ E2 &: 2x+y+z=7 \Rightarrow \\ E3 &: x-y-z=-4 \end{aligned} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 1 & -1 & -1 & -4 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_1}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & -3 & -2 & -12 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & -3 & -2 & -12 \end{array}\right] \quad \boxed{R_3 \leftarrow R_3 - R_2} \quad \boxed{\begin{array}{rccc|c} & 0 & -3 & -1 & -9 \\ - & 0 & -3 & -2 & -12 \\ \hline & 0 & 0 & 1 & 3 \end{array}} \overset{R_3 \leftarrow R_3 - R_2}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} E1 &: x+2y+z=8 \\ E2 &: 2x+y+z=7 \\ E3 &: x-y-z=-4 \end{aligned} \Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right] \sim\sim\sim\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{aligned} E1 &: x+2y+z=8 \\ E2 &: -3y-z=-9 \\ E3 &: z=3 \end{aligned}$$

Back Substitution:

$$z=3 \overset{E2}{\Longrightarrow} -3y-3=-9 \Longrightarrow y=2$$
$$(y,z)=(2,3) \overset{E1}{\Longrightarrow} x+2\cdot 2+3=8 \Longrightarrow x=1$$

Or, we can continue as follows

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \boxed{R_2 \leftarrow R_2 + R_3} \quad \boxed{\begin{array}{rrrr|r} & 0 & -3 & -1 & -9 \\ + & 0 & 0 & 1 & 3 \\ \hline & 0 & -3 & 0 & -6 \end{array}} \quad \overset{R_2 \leftarrow R_2 + R_3}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 1 & 3 \end{array}\right] \overset{R_2 \leftarrow -\frac{R_2}{3}}{\sim} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \boxed{R_2 \leftarrow R_1 - R_3} \quad \boxed{\begin{array}{rrrr|r} & 1 & 2 & 1 & 8 \\ - & 0 & 0 & 1 & 3 \\ \hline & 1 & 2 & 0 & 5 \end{array}} \quad \sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \boxed{R_1 \leftarrow R_1 - 2R_2} \quad \boxed{\begin{array}{rrrr|r} & 1 & 2 & 0 & 5 \\ -2\cdot(& 0 & 1 & 0 & 2) \\ \hline & 1 & 0 & 0 & 1 \end{array}} \quad \sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} E1 &: x+2y+z=8 \\ E2 &: 2x+y+z=7 \\ E3 &: x-y-z=-4 \end{aligned} \Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 1 & 7 \\ 1 & -1 & -1 & -4 \end{array}\right] \sim\sim\sim\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{aligned} E1 &: x=1 \\ E2 &: y=2 \\ E3 &: z=3 \end{aligned}$$

Example

$$\begin{array}{r} E1: x+2y+z=7 \\ E2: 2x+2y-3z=0 \\ E3: 3x+4y-2z=6 \end{array} \Longrightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ \color{red}{2} & 2 & -3 & 0 \\ 3 & 4 & -2 & 6 \end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ \color{red}{3} & 4 & -2 & 6 \end{array}\right] \qquad R_3 \leftarrow R_3 - 3R_1$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ 0 & \color{red}{-2} & -5 & -15 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

This an inconsistent system. The last indicates an equation

$$0x + 0y + 0z = 0 = -1.$$

If the augmented matrix contains a row of all 0's on the LHS (coefficients) and a non-zero value on the RHS (constant term), as

$$\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & b(\neq 0) \end{array}\right],$$

then the system is inconsistent.
[Question] Any other possible cases? (unique solution, no solution, more solutions (infinitely many))

Example

$$\begin{aligned} E1 &: x+2y+z=7 \\ E2 &: 2x+2y-3z=0 \\ E3 &: 3x+4y-2z=7 \end{aligned} \Longrightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ {\color{red}2} & 2 & -3 & 0 \\ 3 & 4 & -2 & 7 \end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ {\color{red}3} & 4 & -2 & 7 \end{array}\right] \qquad R_3 \leftarrow R_3 - 3R_1$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ 0 & {\color{red}-2} & -5 & -14 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_2 \sim \boxed{\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & -2 & -5 & -14 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

We were expecting an equation involving only z. However, there is no such equation, which means z can be *any number*. We let

$$\boxed{z=s},$$

where s is called a *parameter*(free variable), i.e., s is any number. We substitute back:

$$R_2: \quad -2y-5z=-14 \Rightarrow y = 7-\frac{5}{2}z = 7-\frac{5}{2}s$$

$$R_1: \quad x+2y+z=7 \Rightarrow x = 7-2y-z = 7-2\left(7-\frac{5}{2}s\right)-z = -7+4s.$$

General solution (parametrized solution):

$$(x,y,z) = \left(4s-7, 7-\frac{5}{2}s, s\right) \quad \text{for any number } s$$

Special solutions:

s	0	1	2	3
(x,y,z)	$(-7,7,0)$	$(-3,\frac{9}{2},1)$	$(1,2,2)$	$(5,-\frac{1}{2},3)$

Example

$$\begin{aligned} E1 &: x+2y+3z=6 \\ E2 &: 2x+4y+6z=12 \\ E3 &: -x-2y-3z=-6 \end{aligned} \quad (E2=2E1,\ E3=-E1) \implies \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 4 & 6 & 12 \\ -1 & -2 & -3 & -6 \end{array}\right] \qquad R_2 \leftarrow R_2 - 2R_1$$

$$\sim \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -3 & -6 \end{array}\right] \qquad R_3 \leftarrow R_3 + R_1$$

$$\sim \boxed{\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}$$

The last two rows are trivial, which indicates that both y and z can be any number. Let

$$z = s, \ y = t,$$

for parameters s and t.

$$R_1 : x + 2y + 3z = 6 \Rightarrow x = 6 - 2y - 3z = 6 - 2t - 3s.$$

The (2-parameter) general solution is

$$(x, y, z) = (6 - 2t - 3s, t, s).$$

Special solutions are

$t \backslash s$	0	1	2	3
0	$(6,0,0)$	$(3,0,1)$	$(0,0,2)$	$(-3,0,3)$
1	$(4,1,0)$	$(1,1,1)$	$(-2,1,2)$	$(-5,1,3)$
2	$(2,2,0)$	$(-1,2,1)$	$(-4,2,2)$	$(-7,2,3)$
3	$(0,3,0)$	$(-3,3,1)$	$(-6,3,2)$	$(-9,3,3)$

Echelon Form

Definition

An entry of an augmented matrix is called a *leading entry* or *pivot entry* if it is the leftmost non-zero entry of a row. An augmented matrix is in *echelon form* (also called *row echelon form*) if

1. All rows of zeros are below all non-zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of any row above it.

A column containing a pivot entry is also called a *pivot column*.

Example

$$\left[\begin{array}{cccccc|c} \color{red}{*} & * & * & * & * & * & * \\ 0 & \color{red}{*} & * & * & * & * & * \\ 0 & 0 & 0 & \color{red}{*} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} *- & \text{nonzero entry;} \\ \color{red}{*}- & \text{leading/pivot entry.} \end{cases}$$

Echelon forms:

$$\left[\begin{array}{ccccc|c} 0 & \color{red}{5} & 2 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & \color{red}{1} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccccc|c} \color{red}{1} & 4 & 0 & 0 & 5 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 2 & 0 \\ 0 & 0 & 0 & \color{red}{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \color{red}{1} \end{array}\right], \quad \left[\begin{array}{ccc|c} \color{red}{3} & 0 & 6 & 2 \\ 0 & \color{red}{1} & 4 & 0 \\ 0 & 0 & \color{red}{2} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Not in echelon form

$$\left[\begin{array}{ccc|c} \color{blue}{0} & \color{blue}{0} & \color{blue}{0} & \color{blue}{0} \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ \color{blue}{2} & 4 & -6 \\ \color{blue}{4} & 0 & 7 \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & \color{blue}{2} & 3 & 3 \\ \color{blue}{1} & 5 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Goal: Use elementary row operations to transform the system of equations to echelon form.
Input: Augmented matrix
Output: Augmented matrix in echelon form *row equivalent* to the input.

Algorithm: Gaussian Elimination

This algorithm provides a method for using row operations to take a matrix to its echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first non-zero column. This is the first pivot column, and the position at the top of this column will be the position of the first pivot entry. Switch rows if necessary to place a non-zero number in the first pivot position.
2. Use row operations to make the entries below the first pivot entry (in the first pivot column) equal to zero.
3. Ignoring the row containing the first pivot entry, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more non-zero rows left.

Example

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{array}\right] \quad R_1 \leftrightarrow R_2 \sim \left[\begin{array}{ccccc|c} 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{array}\right] R_4 \leftarrow R_4 - \frac{1}{2}$$

$$\sim \left[\begin{array}{ccccc|c} 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & \frac{3}{2} & 3 & \frac{7}{2} & 3 \end{array}\right] = \left[\begin{array}{ccccc|c} \multicolumn{6}{c}{\text{[redacted]}} \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & \frac{3}{2} & 3 & \frac{7}{2} & 3 \end{array}\right] \quad R_2 \leftrightarrow R_4 \left[\begin{array}{ccccc|c} 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & \frac{3}{2} & 3 & \frac{7}{2} & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 & 5 \end{array}\right] R_4 \leftarrow R_4 - 2R_3$$

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & w & v \end{array} \\ \sim \left[\begin{array}{ccccc|c} 0 & 2 & 1 & 0 & 1 & 4 \\ 0 & 0 & \frac{3}{2} & 3 & \frac{7}{2} & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right] \end{array}$$

$$\begin{array}{lr} E1 & 2y + z + v = 4 \\ E2 & \frac{3}{2}z + 3w + \frac{7}{2}v = 3 \\ E3 & w + 2v = 3 \\ E4 & -v = -1 \end{array}$$

Terminology

1. Any column containing a pivot entry is called a *pivot column*.
2. The corresponding variables are called *pivot variables*.
3. Any variable that is not a pivot variable is called a *free variable*.

In the previous example, x is a free variable and y, z, w, v are pivot variables.

Back substitution

- Every free variable becomes a parameter.
- Every pivot/leading variable has an equation, which we can use to find its value.

Example

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & w & v \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We have

$$\begin{aligned}
v &: \quad \text{free variable} \quad v = t \\
w &: \quad \text{from } R_2\text{, pivot variable} \quad w - 2v = 4 \Longrightarrow w = 2v + 4 = 2t + 4 \\
z &: \quad \text{free variable} \quad z = s \\
y &: \quad \text{from } R_1\text{, pivot variable} \quad y + 2z + 3v = 5 \Longrightarrow y = 5 - 2z - 3v = 5 - 2s - 3t \\
x &: \quad \text{free variable} \quad x = r
\end{aligned}$$

General solution

$$(x, y, z, w, v) = (r, 5 - 2s - 3t, s, 2t + 4, t).$$

Rank

Remark

1. If a system is *inconsistent*, then the RHS is a pivot column.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & \boxed{7} \end{array}\right] \longrightarrow \text{inconsistent, no solution.} \quad \left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & b(\neq 0) \end{array}\right]$$

2. If the system is consistent, the number of free variables determines the number of solutions:
   - no free variables$\Longrightarrow$no parameters$\Longrightarrow$unique solution;
   - at least one free variable $\Longrightarrow$ infinitely many solution.

Definition

The *rank* of a matrix is the number of pivot entries of its echelon form.

Example

$$A = \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$rank(A) = 2 \qquad rank(B) = 3$$

Remark

If a system of m equations in n variables has rank r and is consistent, then there are $n-r$ free variables, i.e., $n-r$ parameters in the general solution.

- $n=r \Rightarrow n-r=0 \Rightarrow$unique solution
- $n>r \Rightarrow n-r>0 \Rightarrow$infinitely many solutions.

Example

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & w & v \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

A system of 4 equations in 5 variables. rank$=2$. $m=4$, $n=5$ and $r=2$.

$$5-2=3 \text{ free variables: } (x,z,v)=(r,s,t)$$

$$\begin{aligned} v: &= t \\ w: &= 2t+4 \\ z: &= s \\ y: &= 5-2s-3t \\ x: &= r \end{aligned}$$

General solution

$$(x,y,z,w,v)=(t,5-2s-3t,s,2t+4,t).$$

Consider system

$$\begin{aligned} x + 2y + z &= 5 \\ x + 3y + 2z &= 9 \\ 2x + 5y + 3z &= 14 \end{aligned}$$

and its corresponding matrix form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 1 & 3 & 2 & 9 \\ 2 & 5 & 3 & 14 \end{array}\right]$$

Find its rank.

Lecture resumes at 7:45.

Example

Solve the linear system

$$\begin{aligned} x+2y+z&=5\\ x+3y+2z&=9\\ 2x+5y+3z&=14 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1&2&1&5\\ 1&3&2&9\\ 2&5&3&14 \end{array}\right] \quad R_2\leftarrow R_2-R_1 \quad \sim \left[\begin{array}{ccc|c} 1&2&1&5\\ 0&1&1&4\\ 2&5&3&14 \end{array}\right] R_3\leftarrow R_3-2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1&2&1&5\\ 0&1&1&4\\ 0&1&1&4 \end{array}\right] \quad R_3-R_2 \quad \sim \left[\begin{array}{ccc|c} 1&2&1&5\\ 0&1&1&4\\ 0&0&0&0 \end{array}\right] \quad \begin{aligned} x+2y+z&=5\\ y+z&=4 \end{aligned}$$

pivot variables $\underline{x \text{ and } y}$; free variable $\underline{z}$

$$\begin{aligned} z=t \quad &\overset{R_2}{\Longrightarrow} \quad y=4-z=4-t\\ &\overset{R_1}{\Longrightarrow} \quad x=5-2y-z=5-2(4-t))-t=t-3 \end{aligned}$$

General solution:

$$(x,y,z)=(-3+t,4-t,t).$$

It is also written as column form:

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} -3+t\\ 4-t\\ t \end{bmatrix} = \begin{bmatrix} -3\\ 4\\ 0 \end{bmatrix} + t\begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix}.$$

We shall come back to this form in Chapter 2.

It is similar to Gaussian elimination, but with

- more elementary row operators;
- and less back substitution.

Definition

A matrix B is in *reduced echelon form* if it is in echelon form and additionally

- all pivot entries are equal to 1;
- all entries above pivot entries are equal to 0.

Example

$$\begin{aligned} x+2y+z&=5\\ x+3y+2z&=9\\ 2x+5y+3z&=14 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5\\ 1 & 3 & 2 & 9\\ 2 & 5 & 3 & 14 \end{array}\right] \sim \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5\\ 0 & 1 & 1 & 4\\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Echelon Form}} \quad R_1 \leftarrow R_1 - 2R_2 \quad \sim \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3\\ 0 & 1 & 1 & 4\\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

pivot variables x and y; free variable z

$$z=t \ \text{ Equations: } \begin{cases} x-z=-3\\ y+z=4 \end{cases} \Longrightarrow \begin{cases} y=4-z &= 4-t\\ x=-3+z &= -3+t \end{cases} \Longrightarrow (x,y,z)=(-3+t,4-t,t)$$

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} -3\\ 4\\ 0 \end{bmatrix} + t\begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix} = \begin{bmatrix} -3\\ 4\\ 0 \end{bmatrix} + \begin{bmatrix} t\\ -t\\ t \end{bmatrix} = \begin{bmatrix} -3+t\\ 4-t\\ t \end{bmatrix}.$$

$$\begin{aligned} 2x+y+w&=8\\ x+y+z&=6\\ 3x+2y+z+v&=-2 \end{aligned} \rightarrow \left[\begin{array}{ccccc|c} 2&1&0&1&0&8\\ 1&1&1&0&0&6\\ 3&2&1&0&1&-2 \end{array}\right] R_1 \leftrightarrow R_2 \quad \sim \left[\begin{array}{ccccc|c} 1&1&1&0&0&6\\ 2&1&0&1&0&8\\ 3&2&1&0&1&-2 \end{array}\right] \begin{array}{l} R_2 \leftarrow R_2-2R_1\\ R_3 \leftarrow R_3-3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1&1&1&0&0&6\\ 0&-1&-2&1&0&-4\\ 0&-1&-2&0&1&-20 \end{array}\right] R_2 \leftarrow -R_2 \quad \sim \left[\begin{array}{ccccc|c} 1&1&1&0&0&6\\ 0&1&2&-1&0&4\\ 0&-1&-2&0&1&-20 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1-R_2\\ R_3 \leftarrow R_3+R_2 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1&0&-1&1&0&2\\ 0&1&2&-1&0&4\\ 0&0&0&-1&1&-16 \end{array}\right] R_3 \leftarrow -R_3 \quad \sim \left[\begin{array}{ccccc|c} 1&0&-1&1&0&2\\ 0&1&2&-1&0&4\\ 0&0&0&1&-1&16 \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1-R_3\\ R_2 \leftarrow R_2+R_3 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1&0&-1&0&1&-14\\ 0&1&2&0&-1&20\\ 0&0&0&1&-1&16 \end{array}\right]$$

free variables $z=s$ and $v=t$

$$\begin{aligned} R_1 &: x=-14+z-v=-14+s-t\\ R_2 &: y=20-2z+v=20-2s+t\\ R_3 &: w=16+v=16+t \end{aligned}$$

General solution

$$\begin{bmatrix} x\\ y\\ z\\ w\\ v \end{bmatrix} = \begin{bmatrix} -14\\ 20\\ 0\\ 16\\ 0 \end{bmatrix} + t\begin{bmatrix} -1\\ 1\\ 0\\ 1\\ 1 \end{bmatrix} + s\begin{bmatrix} 1\\ -2\\ 1\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} -14\\ 20\\ 0\\ 16\\ 0 \end{bmatrix} + \begin{bmatrix} -t\\ t\\ 0\\ t\\ t \end{bmatrix} + \begin{bmatrix} s\\ -2s\\ s\\ 0\\ 0 \end{bmatrix} = \begin{bmatrix} -14-t+s\\ 20+t-2s\\ s\\ 16+t\\ t \end{bmatrix}$$

Uniqueness

Theorem

Reduced echelon forms are unique.

Definition

Two matrices are called *row equivalent*, if one can be obtained from the other by elementary row operations.

Example

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right], \quad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right]$$

$$\begin{aligned} B &\overset{R_1 \leftarrow \frac{1}{2} R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right] \\ &\overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 6 & 5 & 7 \end{array}\right] \\ &\overset{R_3 \leftarrow R_3 - 5R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{array}\right] \\ &\overset{R_2 \leftarrow 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right] = A \end{aligned}$$

A and B are row equivalent.

Example

Theorem

Two matrices are row equivalent if and only if (iff) they have the same reduced echelon form.

Example

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right], \quad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right]$$

$$A \underset{\sim}{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right] \underset{\sim}{R_2 \leftarrow \frac{1}{3} R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$B \underset{\sim}{R_1 \leftarrow \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right] \underset{\sim}{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 6 & 5 & 7 \end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, we confirm that A and B are row equivalent.

MATH-1030
Matrix Theory & Linear Algebra I

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Homogeneous systems of equations

Definition

A system of equations is called homogeneous if each of the constant terms is equal to 0. A homogeneous system therefore has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

a_{ij}—coefficients

x_j—variables

Example

$$\begin{aligned} 2x + 3y + 4z &= 7 \\ x - 2y &= 0 \\ x + y + z &= 3 \end{aligned}$$

Non-homogenous

$$\begin{aligned} 2x + 3y + 4z &= 0 \\ x - 2y &= 0 \\ x + y + z &= 0 \end{aligned}$$

Homogenous

Example

Definition

A system of equations is called homogeneous if each of the constant terms is equal to 0. A homogeneous system therefore has the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0
\end{aligned}
$$

Remark

Homogeneous systems are always consistent. They always have the following (trivial) solution $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.
Therefore, we want to know, for a given homogeneous system, does it have non-trivial solutions? Infinitely many solutions.

Example

$$\begin{aligned} 2x+3y+4z&=0\\ x-2y&=0\\ x+y+z&=0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0\\ 1 & -2 & 0 & 0\\ 1 & 1 & 1 & 0 \end{array}\right] \overset{R_1\leftrightarrow R_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 2 & 3 & 4 & 0\\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\overset{\substack{R_2\leftarrow R_2-2R_1\\ R_3\leftarrow R_3-R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 0 & 7 & 4 & 0\\ 0 & 3 & 1 & 0 \end{array}\right] \overset{R_2\leftarrow R_2-2R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0\\ 0 & 1 & 2 & 0\\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$\overset{\substack{R_1\leftarrow R_1+2R_2\\ R_3\leftarrow R_3-3R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & -5 & 0 \end{array}\right] \overset{R_3\leftarrow \frac{-R_3}{5}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\overset{\substack{R_1\leftarrow R_1-4R_3\\ R_2\leftarrow R_2-2R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow (x,y,z)=(0,0,0)$$

Example

$$\begin{aligned} 2x + 3y + 4z &= 0 \\ x - 2y &= 0 \\ x + y + z &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} \color{red}{2} & 3 & 4 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim\sim\sim\sim \left[\begin{array}{ccc|c} \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 \end{array}\right]$$

Thus, it only has the trivial solution

$$(x, y, z) = (0, 0, 0)$$

[Question] What is the rank of the augmented matrix (or its reduced echelon form)? (Elementary row operations do not change the rank)

$$rank = 3 = \# \text{ of unknowns.}$$

Fact

If a homogeneous system of n variables has rank n, then it only has the trivial solution.

$$\# \text{ of free variables/parameters} = \# \text{ of unknowns} - rank$$

Example

$$\begin{aligned} x+3y-2z&=0\\ 2x+4y-2z&=0\\ y-z&=0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0\\ 2 & 4 & -2 & 0\\ 0 & 1 & -1 & 0 \end{array}\right] \overset{R_2\leftarrow R_2-2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0\\ 0 & -2 & 2 & 0\\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0\\ 0 & 1 & -1 & 0\\ 0 & -2 & 2 & 0 \end{array}\right] \overset{\substack{R_1\leftarrow R_1-3R_2\\ R_3\leftarrow R_3+2R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

3 variables with rank 2:

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array}\\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Longrightarrow \begin{cases} \text{pivot variables} & x, y\\ \text{free variable} & z = t \end{cases}$$

Example

$$\begin{aligned} x + 3y - 2z &= 0 \\ 2x + 4y - 2z &= 0 \\ y - z &= 0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim\sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} \color{red}{1} & 0 & 1 & 0 \\ 0 & \color{red}{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$z = t \Rightarrow \begin{cases} R_1: & x + z = 0 \\ R_2: & y - z = 0 \end{cases} \Rightarrow \begin{cases} x = -t \\ y = t \end{cases}$$

$$(x, y, z) = (-t, t, t)$$

General solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{basic solution}(t = 1) - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Theorem

Theorem

Given a homogeneous system with n variables and of rank r, the system is always consistent:

1. *if $r = n$, it only has the unique, trivial solution; (all zeros)*
2. *if $r < n$, there are non-trivial solutions. In this case, the number of parameters is $n - r$ and there are $n - r$ basic solutions.*

Example

$$\begin{aligned} x + 4y + 3z &= 0 \\ 3x + 12y + 9z &= 0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \overset{R_2 \leftarrow R_2 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{parameters: } z = s, \; y = t \Longrightarrow x = 0 - 4y - 3z = -4t - 3s$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4t - 3s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

basic solutions: $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Theorem

Theorem

If the general solution has the form

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = s_1 \begin{bmatrix} \color{red}{\bullet} \\ \color{red}{\vdots} \\ \color{red}{\bullet} \end{bmatrix} + \cdots + s_r \begin{bmatrix} \color{blue}{\bullet} \\ \color{blue}{\vdots} \\ \color{blue}{\bullet} \end{bmatrix}$$

basic solutions: $\begin{bmatrix} \color{red}{\bullet} \\ \color{red}{\vdots} \\ \color{red}{\bullet} \end{bmatrix}, \ldots, \begin{bmatrix} \color{blue}{\bullet} \\ \color{blue}{\vdots} \\ \color{blue}{\bullet} \end{bmatrix}$

Theorem

Theorem

If the general solution has the form

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = s_1 \begin{bmatrix} \color{red}\bullet \\ \color{red}\vdots \\ \color{red}\bullet \end{bmatrix} + \cdots + s_r \begin{bmatrix} \color{blue}\bullet \\ \color{blue}\vdots \\ \color{blue}\bullet \end{bmatrix}$$

basic solutions:

$$\begin{bmatrix} \color{red}\bullet \\ \color{red}\vdots \\ \color{red}\bullet \end{bmatrix}, \ldots, \begin{bmatrix} \color{blue}\bullet \\ \color{blue}\vdots \\ \color{blue}\bullet \end{bmatrix}$$

Definition

Let A be a system of linear equations. The *associated homogeneous system* is the system B, which has the same left-hand side as A, but all 0's on the right-hand side.

Example

$$\begin{aligned} x + 4y + 3z &= 2 \\ 3x + 12y + 9z &= 6 \end{aligned} \xrightarrow{\text{associated homogeneous system}} \begin{aligned} x + 4y + 3z &= \color{blue}0 \\ 3x + 12y + 9z &= \color{blue}0 \end{aligned}$$

Comparison

$$\begin{array}{r} x+4y+3z=2 \\ 3x+12y+9z=6 \end{array} (A) \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 3 & 12 & 9 & 6 \end{array}\right] \overset{R_2 \leftarrow R_2 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variables: $z = s,\ y = t \Longrightarrow x = 2 - 4y - 3z = 2 - 4t - 3s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 4t - 3s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{r} x+4y+3z=0 \\ 3x+12y+9z=0 \end{array} (B) \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \overset{R_2 \leftarrow R_2 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4t - 3s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{array}{c} (x,y,z) = (2,0,0) \\ \text{is a solution to } A \end{array}}$$

Comparison

Theorem

Let A be a system of equations, and let B be the associated homogeneous system. Then

$$\text{the general solution of } A(GA) = \text{a particular solution of } A(PA) + \text{the general solution of } B(GB).$$

Example

$$\begin{aligned} x + 4y + 3z &= 2 \\ 3x + 12y + 9z &= 6 \end{aligned} (A) \longrightarrow \begin{aligned} x + 4y + 3z &= 0 \\ 3x + 12y + 9z &= 0 \end{aligned} (B) \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{GA} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{PA} + \underbrace{t \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{GB}$$

Summary of Chpt. 1

- Given a system (or just one equation), tell whether it is linear:

$$\underbrace{2x+4y=0}_{\text{linear}} \qquad \underbrace{\begin{cases} 3x+5y=3 \\ 2xy=4 \end{cases}}_{\text{NOT linear}}$$

- Given a linear system, write down the augmented matrix:

$$\begin{aligned} x+2y+z&=0 \\ x+3y+2z&=3 \\ 2x+5y+3z&=3 \end{aligned} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 3 \\ 2 & 5 & 3 & 3 \end{array}\right]$$

and vice versa.

Summary of Chpt. 1

- The ***most important*** eliminations:
  Gaussian elimination and Gauss-Jordan elimination

Gaussian Elimination	Gauss-Jordan Elimination
$\downarrow$	$\downarrow$
echelon form	redued echelon form
$\downarrow$	$\downarrow$
$\left[\begin{array}{ccc\|c} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$

through THREE elementary row operations

$$R_i \longleftrightarrow R_j \qquad R_j \longleftarrow aR_j \qquad R_j \longleftarrow R_j + aR_i \quad (i \neq j, a \neq 0)$$

Summary of Chpt. 1

- (reduced) echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 3 \\ 2 & 5 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z & = 0 \\ y + z & = 3 \end{cases}$$

1. Consistent or inconsistent? For the example, it is consistent.

   Inconsistent example: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & \boxed{7} \end{array}\right]$
2. Pivot variable(s): x and y;
3. Free variable(s): $z = t$
4. rank=# of pivot entries$= 2$

- (reduced) echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 3 \\ 2 & 5 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x + 2y + z & = 0 \\ y + z & = 3 \end{cases}$$

5. **Solution** $z = t$ free, x, y pivot
   - echelon form $\Longrightarrow$ back substitution $y = 3 - z = 3 - t$
     $x = -z - 2y = -6 + t$
   - reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} + t \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ 0 \end{bmatrix} + t \begin{bmatrix} \bullet \\ \bullet \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 + t \\ 3 - t \\ t \end{bmatrix}$$

Summary of Chpt. 1

6. Determine whether two systems have the same solutions. Compare the *reduced* echelon forms of the two systems

$$\begin{cases} x + 2y + z &= 3 \\ 3y + 3z &= 3 \\ x + y &= 2 \end{cases} \qquad \begin{cases} 2x + 4y + 2z &= 6 \\ 2x + 5y + 3z &= 7 \\ x + 6y + 5z &= 7 \end{cases}$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right], \qquad B = \left[\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ 2 & 5 & 3 & 7 \\ 1 & 6 & 5 & 7 \end{array}\right]$$

So they have the same solutions. (Two matrices are row equivalent.)

Summary of Chpt. 1

- Some important facts:
  1. A system of linear equations can have 0, 1 and infinitely many solutions.
     0 solution—inconsistent   1 or infinitely many—consistent
  2. A consistent system of m linear equations, n variables and rank r.   # of free variables/parameters $= n - r$
     $n = r$ unique solution
     $n > r$ infinitely many

Hard & Tricky Problem: What happens if $r > n$? For example, a system of 3 equations with 2 variables and rank 3.
The answer is that the system if ***inconsistent***.

$$\begin{array}{c}\#\text{ of free variables}\\ \| \\ 2-3=-1\end{array} \qquad \left[\begin{array}{cc|c} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array}\right] \sim \left[\begin{array}{cc|c} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{array}\right]$$

Classwork

Solve the homogeneous system

$$\begin{aligned} x + 2y + z &= 0 \\ x + 3y + 2z &= 0 \\ 2x + 5y + 3z &= 0 \end{aligned}$$

Lecture resumes at 7:45.

Solution

$$\begin{aligned} x + 2y + z &= 0 \\ x + 3y + 2z &= 0 \\ 2x + 5y + 3z &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 5 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

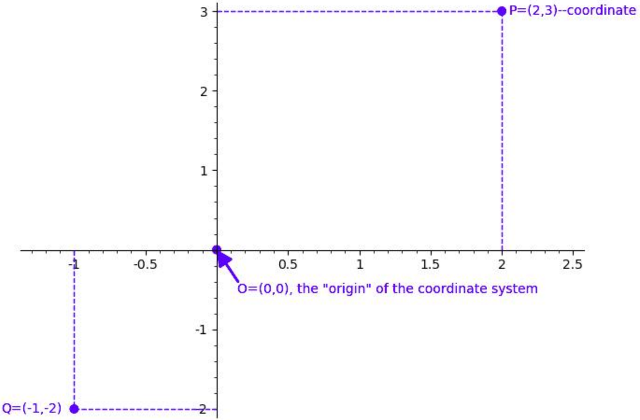
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

z is free: $z = t$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Coordinate system

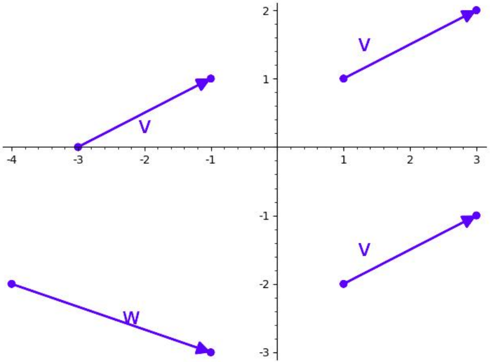
On the plane ($\mathbb{R}^2$):



The coordinate system allows us to describe every point in the plane by a pair of real numbers, called the *coordinates* of the point.

Vectors

Unlike a point, which describes a location in a coordinate system, a vector describes an offset or a distance and direction.



$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{2 and 1 of } v \text{ are called components.} \qquad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Space

Zero vector

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

More generally,

- a *point* in an n-dimensional space is given by n coordinates and written as

$$P = (x_1, \ldots, x_n)$$

- a *vector* in an n-dimensional space is given by n components and written as

$$\mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Space

Definition

The set of n-dimensional column vectors is denoted by

$$\mathbb{R}^n := \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_1, \ldots, x_n \in \mathbb{R} \right\}$$

and is called <u>*n-dimensional Euclidean space*</u>.

Example

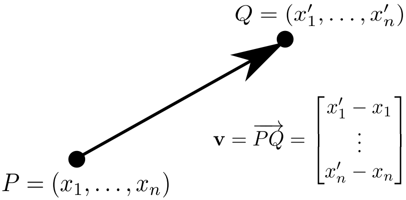
$\mathbb{R}^2$: Euclidean plane

$\mathbb{R}^3$: vectors in 3-dimensional Euclidean space

$\mathbb{R}^n$: vectors in n-dimensional Euclidean space

Points VS Vectors

1. Given two points P and Q, we can define a vector $\mathbf{v} = \overrightarrow{PQ}$

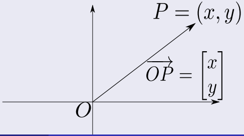


2. In n-dimensional space, the *coordinate vector* of a point $P = (x_1, \ldots, x_n)$ is the vector

$$\overrightarrow{OP} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$O = (0, 0, \ldots, 0)$ is the origin.

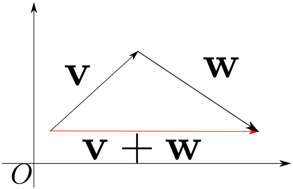
Example ($n = 2$)



Addition of vectors

If

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\color{red}n} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{\color{red}n} \end{bmatrix} \Longrightarrow v + w = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$



Slide **w** so that the tail of **w** is on the tip of **v**. Then draw the arrow which goes from the tail of **v** to the tip of **w** . This arrow represents the vector $\mathbf{v} + \mathbf{w}$.

Addition of vectors

Example

$$\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}1\\7\end{bmatrix}\qquad \begin{bmatrix}1\\2\\3\\-4\\5\end{bmatrix}+\begin{bmatrix}1\\1\\0\\-1\\1\end{bmatrix}=\begin{bmatrix}2\\3\\3\\-5\\6\end{bmatrix}$$

Negation

$$\mathbf{v}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\Longrightarrow -\mathbf{v}=\begin{bmatrix}-x_1\\-x_2\\\vdots\\-x_n\end{bmatrix}\qquad \text{Rotate } \mathbf{v} \text{ by } 180^\circ$$

Properties of Addition

Properties

(A1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(A2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(A3) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(A4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Example

$$
\begin{array}{rcl}
 & & \mathbf{u} + \mathbf{v} + \mathbf{w} + (-\mathbf{u}) + \mathbf{w} + (-\mathbf{w}) \\
(A2) & = & \mathbf{u} + \mathbf{v} + \mathbf{w} + (-\mathbf{u}) + (\mathbf{w} + (-\mathbf{w})) \\
(A4) & = & \mathbf{u} + \mathbf{v} + \mathbf{w} + (-\mathbf{u}) + \mathbf{0} \\
(A3) & = & \mathbf{u} + \mathbf{v} + \mathbf{w} + (-\mathbf{u}) \\
(A1) & = & \mathbf{u} + (-\mathbf{u}) + \mathbf{v} + \mathbf{w} \\
(A4) & = & \mathbf{0} + \mathbf{v} + \mathbf{w} \\
(A3) & = & \mathbf{v} + \mathbf{w}.
\end{array}
$$

Properties of Addition

Properties

(A1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

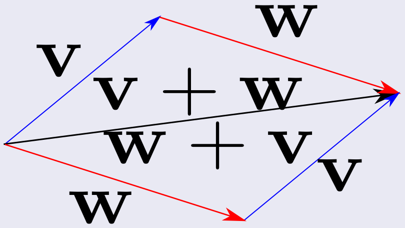
(A2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{v} + (\mathbf{u} + \mathbf{w})$

(A3) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(A4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Remark

(A1)

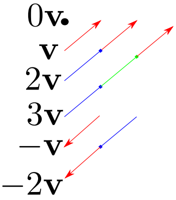


Scalar Multiplication

$$a \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

Example

$$3 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$



Properties

$$\begin{aligned}
&\textbf{(SM1)} \quad k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v} \\
&\textbf{(SM2)} \quad (k+\ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u} \\
&\textbf{(SM3)} \quad k(\ell\mathbf{u}) = (k\ell)\mathbf{u} \\
&\textbf{(SM4)} \quad 1 \cdot \mathbf{u} = \mathbf{u}
\end{aligned}$$

Example

$$\begin{aligned}
& && \mathbf{u} + 2\mathbf{v} - 3\mathbf{u} - 5\mathbf{v} + 6\mathbf{u} \\
\text{(A1)} &= && (\mathbf{u} - 3\mathbf{u} + 6\mathbf{u}) + (2\mathbf{v} - 5\mathbf{v}) \\
\text{(SM2)} &= && (1 - 3 + 6)\mathbf{u} + (2 - 5)\mathbf{v} \\
&= && 4\mathbf{u} - 3\mathbf{v}
\end{aligned}$$

Example

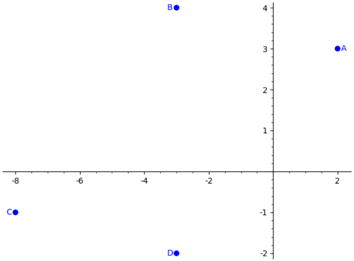
$$\begin{aligned} 2x + y + w &= 8 \\ x + y + z &= 6 \\ 3x + 2y + z + v &= -2 \end{aligned} \rightarrow \left[\begin{array}{ccccc|c} 2 & 1 & 0 & 1 & 0 & 8 \\ 1 & 1 & 1 & 0 & 0 & 6 \\ 3 & 2 & 1 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & -14 \\ 0 & 1 & 2 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 & -1 & 16 \end{array}\right]$$

free variables $z = s$ and $v = t$
General solution:

$$\begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} -14 \\ 20 \\ 0 \\ 16 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 \\ 20 \\ 0 \\ 16 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ t \\ 0 \\ t \\ t \end{bmatrix} + \begin{bmatrix} s \\ -2s \\ s \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 - t + s \\ 20 + t - 2s \\ s \\ 16 + t \\ t \end{bmatrix}$$

Example

Consider four points $A(2,3)$, $B(-3,4)$, $C(-8,-1)$ and $D(-3,-2)$

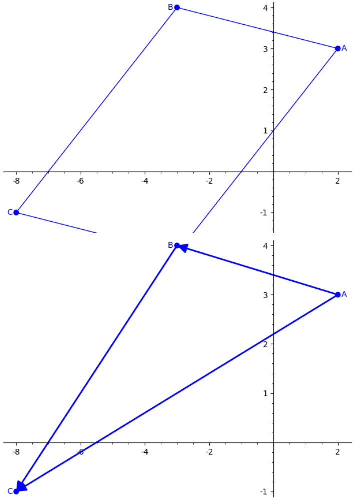


$$\overrightarrow{AB} = \begin{bmatrix} -3-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \quad \overrightarrow{DC} = \begin{bmatrix} -8-(-3) \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \overrightarrow{AB}$$

$$\overrightarrow{AD} = \begin{bmatrix} -3-2 \\ -2-3 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \quad \overrightarrow{CB} = \begin{bmatrix} -3-(-8) \\ 4-(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = -\overrightarrow{AD} = \overrightarrow{DA}$$

A, B, C, D form a parallelogram

$A(2,3)$, $B(-3,4)$, $C\,(-8,-1)$ $D\,(-3,-2)$



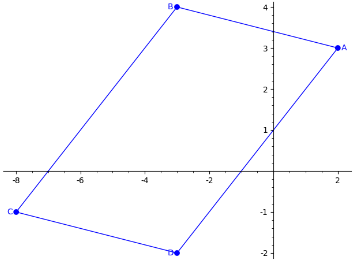
$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\overrightarrow{AB}+\overrightarrow{BC}=\begin{bmatrix}-5\\1\end{bmatrix}+\begin{bmatrix}-5\\-5\end{bmatrix}=\begin{bmatrix}-10\\-4\end{bmatrix}$$

$$\overrightarrow{AC}=\begin{bmatrix}-8-2\\-1-3\end{bmatrix}=\begin{bmatrix}-10\\-4\end{bmatrix}$$

$\overrightarrow{AB}-\overrightarrow{BC}=?$ in terms of $\overrightarrow{XY}$, for X and Y from A, B, C and D.

$A(2,3)$, $B(-3,4)$, $C\,(-8,-1)$ $D\,(-3,-2)$



$$\overrightarrow{AB}-\overrightarrow{BC}=\overrightarrow{AB}+\overrightarrow{CB}=\overrightarrow{DC}+\overrightarrow{CB}=\overrightarrow{DB}$$

$$\begin{aligned}\overrightarrow{AB}-\overrightarrow{BC} &= \begin{bmatrix}-5\\1\end{bmatrix}-\begin{bmatrix}-5\\-5\end{bmatrix}=\begin{bmatrix}-5\\1\end{bmatrix} \\ &= \begin{bmatrix}0\\6\end{bmatrix}\end{aligned}$$

$$\overrightarrow{DB}=\begin{bmatrix}-3-(-3)\\4-(-2)\end{bmatrix}=\begin{bmatrix}0\\6\end{bmatrix}$$

MATH-1030
Matrix Theory & Linear Algebra I

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May 16th, 2019

Recall

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \Longrightarrow \mathbf{v} + \mathbf{w} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}. \qquad a \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ -5 \\ 6 \end{bmatrix} \qquad 3 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

Linear Combination

Definition

A *linear combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an expression of the form

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \left(= \sum_{k=1}^{n} a_k \mathbf{v}_k \right).$$

Example

Is the vector $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$? By definition, we need to determine whether there exist scalars a_1 and a_2 such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$$

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \iff \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 + 3a_2 \\ 2a_1 + 5a_2 \\ a_1 + 2a_2 \end{bmatrix} \iff \begin{cases} a_1 + 3a_2 &= 2 \\ 2a_1 + 5a_2 &= 3 \\ a_1 + 2a_2 &= 1 \end{cases}$$

$$\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \quad \mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \Longleftrightarrow \begin{cases} a_1 + 3a_2 & = 2 \\ 2a_1 + 5a_2 & = 3 \\ a_1 + 2a_2 & = 1 \end{cases}$$

A system of linear equations. Of course $E3 - E1$ gives a_2 but we would like to review the Gauss-Jordan elimination.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{array}\right] \underset{\sim}{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{array}\right] \underset{\sim}{\substack{R_3 \leftarrow R_3 - R_2 \\ R_2 \leftarrow -R_2}} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\underset{\sim}{R_1 \leftarrow R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{w} = -\mathbf{v}_1 + \mathbf{v}_2$$

So the answer is yes, $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Example

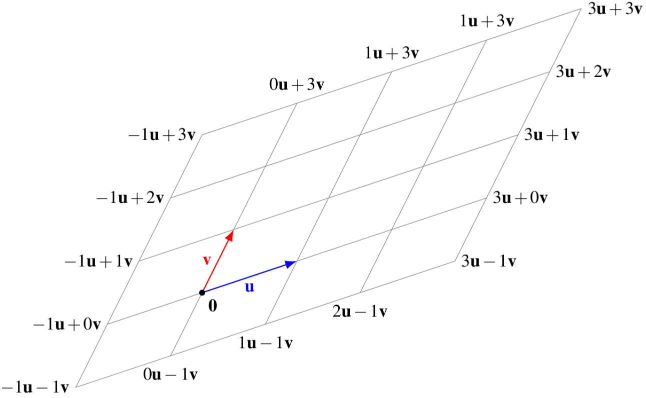
Is the vector $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$?

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \Leftrightarrow \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 + 3a_2 \\ 2a_1 + 5a_2 \\ a_1 + 2a_2 \end{bmatrix} \Leftrightarrow \begin{cases} a_1 + 3a_2 & = 2 \\ 2a_1 + 5a_2 & = 3 \\ a_1 + 2a_2 & = 2 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 2 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_2}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent. Such a_1 and a_2 do not exist. $\mathbf{u}$ is NOT a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Geometrically....



The set of all linear combinations of **u** and **v** is the set of (position vectors of) points in the <u>*plane spanned*</u> by **u** and **v**. Here, **u** and **v** are not parallel, i.e., $\mathbf{v} \neq k\mathbf{u}$ for some k. (Otherwise $a\mathbf{u} + b\mathbf{v} = a\mathbf{u} + bk\mathbf{u} = (a + bk)\,\mathbf{u}$ —a line)

Length/Norm

Definition

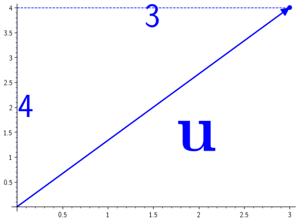
The *norm* of a vector $\mathbf{u} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is given by

$$\|\mathbf{u}\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

Example

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{By definition,} \quad \|\mathbf{u}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix} \Longrightarrow \|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2 + (-2)^2} = \sqrt{1 + 4 + 9 + 4} = \sqrt{18} = 3\sqrt{2}$$



By Pythagorean's Theorem, let the length of **u** be c, then

$$c^2 = 3^2 + 4^2 \Rightarrow c = \sqrt{3^2 + 4^2} = 5.$$

Fact

Geometrically, $\|\mathbf{u}\|$ *is the* <u>*length*</u> *of the vector* **u**.

Remark

We do not distinguish the norm of a vector and the length of a vector.

Unit vector

Definition

A vector is called a *unit vector* if it has length 1.

Example

The following vectors are unit vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is NOT a unit vector. $\|\mathbf{u}\| = 5 \neq 1$.

Unit vector

Definition

If $\mathbf{u}$ is any non-zero vector, then $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector, which is called the result of *normalizing* the vector $\mathbf{u}$.

Remark

1. If $\mathbf{u} \neq \mathbf{0}$ (non-zero), $\|\mathbf{u}\|$ is a positive number (length) , so is $\frac{1}{\|\mathbf{u}\|}$.

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} \text{ is the } \textit{scalar multiplication} \text{ of } \frac{1}{\|\mathbf{u}\|} \text{ and } \mathbf{u}$$

2. If $\mathbf{u} = \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $\|\mathbf{0}\| = \sqrt{0^2 + \cdots + 0^2} = 0$, so that $\frac{1}{\|\mathbf{0}\|}$ does not exist.

3. Sometimes we write

$$\frac{\mathbf{u}}{\|\mathbf{u}\|}$$

Unit vector

Example

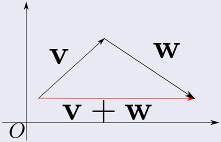
Normalize the vector $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Since $\|\mathbf{u}\| = \sqrt{3^2 + 4^2} = 5$,

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \cdot 3 \\ \frac{1}{5} \cdot 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Let $\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$: $\quad \|\mathbf{v}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1 \quad$ $\mathbf{v}$ is a unit vector.

Laws of norm/length

- $\|\mathbf{u}\| \geq 0$. $\|\mathbf{u}\| = 0$ if and only if (iff) $\mathbf{u} = \mathbf{0}$
- $\|k\mathbf{u}\| = |k| \, \|\mathbf{u}\|$
- Triangle inequality $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$



Unit vector

Laws of norm/length

1. $\|\mathbf{u}\| \geq 0$. $\|\mathbf{u}\| = 0$ if and only if (iff) $\mathbf{u} = \mathbf{0}$
2. $\|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|$
3. Triangle inequality $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

Example

Let $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ $\quad \|\mathbf{u}\| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4 + 9 + 36} = 7$

$$-2\mathbf{u} = \begin{bmatrix} -4 \\ -6 \\ 12 \end{bmatrix} \Longrightarrow \|2\mathbf{u}\| = \sqrt{(-4)^2 + (-6)^2 + 12^2} = 14 = |-2|\,\|\mathbf{u}\|$$

Remark

If $\mathbf{u}$ is non-zero $\left\| \frac{1}{\|\mathbf{u}\|}\mathbf{u} \right\| = \left| \frac{1}{\|\mathbf{u}\|} \right| \cdot \|\mathbf{u}\| \stackrel{1}{=} \frac{1}{\|\mathbf{u}\|} \cdot \|\mathbf{u}\| = 1 \Rightarrow \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is a unit vector

Dot Product

Definition

Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

The *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \bullet \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \left(= \sum_{k=1}^{n} u_k v_k \right).$$

Example

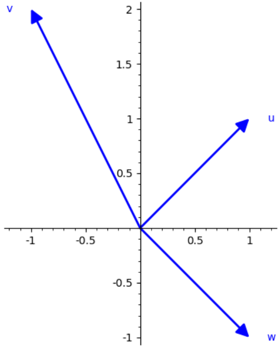
$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot 2 + (-1) \cdot 0 + 0 \cdot 0 = 2 + 0 + 0 = 2.$$

Examples

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 1 \cdot (-1) + 1 \cdot 2 = -1 + 2 = 1 \\ \mathbf{v} \cdot \mathbf{w} &= (-1) \cdot 1 + 2 \cdot (-1) = -3 \\ \mathbf{u} \cdot \mathbf{w} &= 1 \cdot 1 + 1 \cdot (-1) = 0 \end{aligned}$$

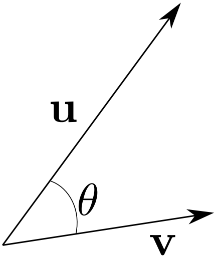
$$\mathbf{u} \cdot \mathbf{w} = 0, \quad \mathbf{u} \cdot \mathbf{v} > 0, \quad \mathbf{v} \cdot \mathbf{w} < 0$$



We can observe from the picture that:

- $\mathbf{u}$ and $\mathbf{w}$ is perpendicular ($= 90^\circ$);
- the (smaller) angle between $\mathbf{u}$ and $\mathbf{v}$ is acute ($< 90^\circ$);
- the (smaller) angle between $\mathbf{v}$ and $\mathbf{w}$ is obtuse ($> 90^\circ$);

Geometric interpretation



- We only consider the smaller angle (the one $\leq 180^\circ$) between two vectors.
- $$\boxed{\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cos\theta}$$
  Recall that if both **u** and **v** are non-zero,
  $$\|\mathbf{u}\| > 0,\ \|\mathbf{v}\| > 0 \text{ and } \cos\theta \begin{cases} > 0, & \theta < 90^\circ \\ = 0, & \theta = 90^\circ \\ < 0, & \theta > 90^\circ \end{cases}$$
- If the angle θ is acute ($\theta < 90^\circ$), then $\mathbf{u}\cdot\mathbf{v} > 0$;
- If the angle θ is obtuse ($\theta > 90^\circ$), then $\mathbf{u}\cdot\mathbf{v} < 0$;
- If two vectors are perpendicular ($\theta = 90^\circ$), then $\mathbf{u}\cdot\mathbf{v} = 0$;
- Vice versa: $\mathbf{u}\cdot\mathbf{v}$ gives certain information for the angle between **u** and **v**

Example

Find the angle between

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Directly,

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot 2 + (-1) \cdot 0 + 0 \cdot 0 = 2.$$

In addition,

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \qquad \|\mathbf{v}\| = \sqrt{2^2 + 0^2 + 0^2} = 2.$$

Thus, let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{2} \cdot 2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ.$$

How about $\sin\theta$? $\sin 45^\circ = \frac{1}{\sqrt{2}}$ $\sin^2\theta + \cos^2\theta = 1 \Longrightarrow \sin^2\theta = 1 - \cos^2\theta = \frac{1}{2}$

$$0^\circ \leq \theta \leq 180^\circ \Longrightarrow \sin\theta \geq 0 \Longrightarrow \sin\theta = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Laws

- $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ (Be careful, it is not associative! $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\neq\mathbf{u}(\mathbf{v}\cdot\mathbf{w})$)
  The dot product gives a scalar(number). We can NOT compute $\mathbf{u}\cdot\mathbf{v}\cdot\mathbf{w}$! In addition,

$$\mathbf{u}=\begin{bmatrix}1\\0\end{bmatrix},\ \mathbf{v}=\begin{bmatrix}1\\1\end{bmatrix},\ \mathbf{w}=\begin{bmatrix}0\\1\end{bmatrix}\Longrightarrow\begin{cases}\mathbf{u}\cdot\mathbf{v} & =1\\ \mathbf{v}\cdot\mathbf{w} & =1\end{cases}\Longrightarrow\begin{cases}(\mathbf{u}\cdot\mathbf{v})\mathbf{w} & =\mathbf{w}\\ (\mathbf{v}\cdot\mathbf{w})\mathbf{u} & =\mathbf{u}\end{cases}$$

- $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2\geq 0,$ $\quad$ $\mathbf{u}\cdot\mathbf{u}=0$ if and only if (iff) $\mathbf{u}=\mathbf{0}$

$$\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|\|\mathbf{u}\|\cos 0^\circ=\|\mathbf{u}\|^2$$

- $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w},$ $\quad$ $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$
- $(k\mathbf{u})\cdot\mathbf{v}=k(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(k\mathbf{v})$
- $|\mathbf{u}\cdot\mathbf{v}|\leq\|\mathbf{u}\|\|\mathbf{v}\|$

$$|\mathbf{u}\cdot\mathbf{v}|=|\|\mathbf{u}\|\cdot\|\mathbf{v}\|\cos\theta|=\|\mathbf{u}\|\cdot\|\mathbf{v}\|\,|\cos\theta|\leq\|\mathbf{u}\|\cdot\|\mathbf{v}\|$$

Example

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 \geq 0,$ $\qquad \mathbf{u} \cdot \mathbf{u} = 0$ if and only if (iff) $\mathbf{u} = \mathbf{0}$
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w},$ $\qquad \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
5. $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$

Problem. Using the properties of the dot product, prove the parallelogram identity:

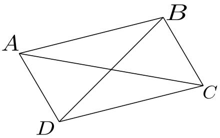
$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$$

Proof.

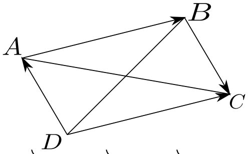
$$\begin{aligned}
& \|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 \\
\overset{2}{=} \; & (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
\overset{3}{=} \; & \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} + (\mathbf{a}) \cdot (-\mathbf{b}) + (-\mathbf{b}) \cdot \mathbf{a} + (-\mathbf{b}) \cdot (-\mathbf{b}) \\
\overset{4}{=} \; & \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + (-1)^2 \mathbf{b} \cdot \mathbf{b} \\
\overset{1}{=} \; & \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
= \; & 2\mathbf{a} \cdot \mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} \\
\overset{2}{=} \; & 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2
\end{aligned}$$

Example

$$\|\mathbf{a}+\mathbf{b}\|^2+\|\mathbf{a}-\mathbf{b}\|^2=2\|\mathbf{a}\|^2+2\|\mathbf{b}\|^2$$



Given a parallelogram $ABCD$, if we denote the length of the line segment AC by $|AC|$ and similarly for other segments, we have $|AC|^2+|BD|^2=|AB|^2+|CD|^2+|BC|^2+|AD|^2=2|AB|^2+2|BC|^2$



$\overrightarrow{AB}=\overrightarrow{DC}=\mathbf{a}$
$\overrightarrow{BC}=\mathbf{b}$
$\overrightarrow{DA}=-\mathbf{b}$

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{a}+\mathbf{b}$$
$$\overrightarrow{DB}=\overrightarrow{DA}+\overrightarrow{AB}=-\mathbf{b}+\mathbf{a}=\mathbf{a}-\mathbf{b}$$
$$\|\overrightarrow{AB}\|=\|\overrightarrow{DC}\|=\|\mathbf{a}\|$$
$$\|\overrightarrow{AD}\|=\|\overrightarrow{BC}\|=\|\mathbf{b}\|$$

Definition ((This will not appear in the tests))

Read section 2.6.5 for

$$\mathrm{comp}_{\mathbf{v}}(\mathbf{u}):=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\ \text{(a scalar)}\quad\text{and}\quad \mathrm{proj}_{\mathbf{v}}(\mathbf{u})=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\ \text{(a vector)}$$

Classwork

Let θ be the angle between vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Find $\cos\theta$ and $\sin\theta$

Lecture resumes at 7:45.

Solution

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\0\end{bmatrix} = 1\cdot 1 + 0\cdot(-2) + 2\cdot 0 = 1$$

Meanwhile

$$\|\mathbf{u}\| = \sqrt{1^2+0^2+2^2} = \sqrt{5}, \qquad \|\mathbf{v}\| = \sqrt{1^2+(-2)^2+0^2} = \sqrt{5}$$

Thus,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{5}\cdot\sqrt{5}} = \frac{1}{5}$$

From $\cos^2\theta + \sin^2\theta = 1$, we have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{25} = \frac{24}{25}$$

Notice that $\theta \leq 180^\circ$, which indicates that $\sin\theta \geq 0$.

$$\sin\theta = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}.$$

Cross Product ONLY in $\mathbb{R}^3$

Definition

Given

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

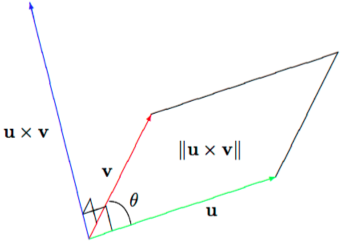
the *cross product* of **u** and **v** is defined by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

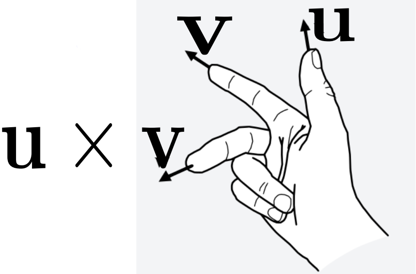
Example

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 4 \cdot 1 - 1 \cdot 0 \\ 1 \cdot 2 - 3 \cdot 1 \\ 3 \cdot 0 - 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$$

Geometric Meaning



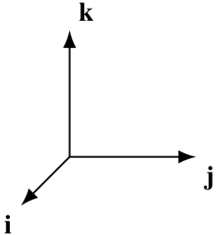
- $\mathbf{u} \times \mathbf{v}$ is orthogonal(perpendicular) to both $\mathbf{u}$ and $\mathbf{v}$.
- $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin\theta$ is the <u>*area*</u> of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a right-hand system.



Use your <u>**right hand**</u>

- thumb $\mathbf{u}$
- index finger $\mathbf{v}$
- then the middle finger gives your the direction of $\mathbf{u} \times \mathbf{v}$.

Right-hand coordinate system



$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Any vector in $\mathbb{R}^3$ can be uniquely written as a linear combination of **i**, **j**, and **k**:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Table

$$\begin{array}{ccc} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array}$$

Laws

$$\begin{array}{ccc} \mathbf{i}\times\mathbf{i}=\mathbf{0} & \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{j}\times\mathbf{j}=\mathbf{0} & \mathbf{j}\times\mathbf{k}=\mathbf{i} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \end{array}$$

1. $\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}$
2. $\mathbf{u}\times\mathbf{u}=\mathbf{0} \qquad (\sin 0^\circ = 0)$
3. $\mathbf{u}\times(\mathbf{v}+\mathbf{w})=\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$,
4. $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=\mathbf{u}\times\mathbf{w}+\mathbf{v}\times\mathbf{w}$,
5. $(k\mathbf{u})\times\mathbf{v}=k(\mathbf{u}\times\mathbf{v})=\mathbf{u}\times(k\mathbf{v})$

Example

$$\mathbf{u}=\begin{bmatrix}3\\4\\1\end{bmatrix} \qquad \mathbf{v}=\begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= (3\mathbf{i}+4\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+\mathbf{k}) \\ &\overset{3,4,5}{=} 6\mathbf{i}\times\mathbf{i}+3\mathbf{i}\times\mathbf{k}+8\mathbf{j}\times\mathbf{i}+4\mathbf{j}\times\mathbf{k}+2\mathbf{k}\times\mathbf{i}+\mathbf{k}\times\mathbf{k} \\ \text{Table} &= 6\mathbf{0}-3\mathbf{j}-8\mathbf{k}+4\mathbf{i}+2\mathbf{j}+\mathbf{0} \\ &= 4\mathbf{i}-\mathbf{j}-8\mathbf{k} \\ &= \boxed{\begin{bmatrix}4\\-1\\-8\end{bmatrix}} = \begin{bmatrix}4\cdot1-1\cdot0\\1\cdot2-3\cdot1\\3\cdot0-4\cdot2\end{bmatrix} \end{aligned}$$

Area

1. Find the area of the parallelogram spanned by

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

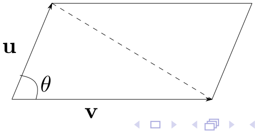
Answer is

$$\|\mathbf{u} \times \mathbf{v}\| = \left\| \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix} \right\| = \sqrt{4^2 + (-1)^2 + (-8)^2} = \sqrt{16 + 1 + 64} = \sqrt{81} = 9.$$

2. Find the area of the triangle that has

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

as two of its sides.
Answer is



$$\frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{9}{2}$$

Box product

- Addition— $\mathbf{u}+\mathbf{v}$ vector+vector=vector;
- Scalar Multiplication— $k\mathbf{u}$ scalar·vector=vector;
- Dot product— $\mathbf{u}\cdot\mathbf{v}$ vector·vector=scalar/number;
- Cross product— $\mathbf{u}\times\mathbf{v}$ vector×vector=vector;
- Norm/Length— $\|\mathbf{u}\|$ ‖vector‖ =scalar/number;
- Normalization— $\frac{1}{\|\mathbf{u}\|}\mathbf{u}$ =unit vector;

Definition

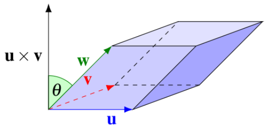
The *box product* of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$. It is equal to the *signed volume* of the *parallelepiped* spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The *sign* is

- positive, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right-handed system;
- negative, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a left-handed system;

The actual, unsigned volume is

$$\text{volume} = |(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|.$$

Box product



$$\text{Volume} = \text{Base Area} \times \text{Height}$$
$$\text{Base Area} = \|\mathbf{u} \times \mathbf{v}\|$$
$$\text{Height} = \|\mathbf{w}\| \cos\theta$$
$$\text{Volume} = \|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos\theta = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

Fact

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ *See Proposition 2.51 of the textbook.*

Example

Find the volume of the parallelepiped determined by the vectors

$$\begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}, \text{ and } \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

Box product

$$\begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}, \text{ and } \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} (-7)(-6) - (-5)(-2) \\ (-5)1 - 1(-6) \\ 1(-2) - (-7)1 \end{bmatrix} = \begin{bmatrix} 32 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 32 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = 32 \cdot (-3) + 1 \cdot 2 + 5 \cdot 3 = -79$$

$$\text{Volume} = |-79| = 79.$$

Summary of Chpt. 2

1. Points and ***Vectors***
2. Operations:
   - Addition—$\mathbf{u}+\mathbf{v}$
   - Scalar Multiplication—$k\mathbf{u}$
   - Dot product—$\mathbf{u}\cdot\mathbf{v}$
   - Cross product—$\mathbf{u}\times\mathbf{v}$
   - Norm/Length—$\|\mathbf{u}\|$;
   - Normorlization—$\frac{1}{\|\mathbf{u}\|}\mathbf{u}$
   - Box Product $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

   - Definition (Calculations)
   - Laws/Properties
   - Geometric Interpretations
3. Linear combinations $\Longleftrightarrow$ Linear systems

MATH-1030
Matrix Theory & Linear Algebra I

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May 21st, 2019

A system of linear equations

- Standard form
$$\begin{aligned} 3x + 2y + 4z &= 7 \\ x - y + z &= 2 \\ 2y + 3z &= 5 \end{aligned}$$
- Vector form $$x \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3x + 2y + 4z \\ x - y + z \\ 0x + 2y + 3z \end{bmatrix}$$
- *Goal of the chapter*: Matrix form

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix} \longleftarrow \text{GOAL}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & 4 & 7 \\ 1 & -1 & 1 & 2 \\ 0 & 2 & 3 & 5 \end{array}\right]$$

Definition

Definition

An *$m \times n$-matrix* is an array of scalars with m rows and n columns.

$$m \text{ rows} \left\{ \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \right.$$

- The dimension of the matrix is $m \times n$.
- The scalars in the matrix are called *entries*. An entry at the ith row and jth column is an (i, j)-entry.

Example

If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, 2 is the $(1, 1)$-entry; both $(2, 1)$ and $(2, 2)$ entries are 0. The dimension of A is 2×3.

Equality, Addition

Two matrices are *equal* if they have the same dimension (i.e., the same number of rows and columns) and the same entries.

Example

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition

We can *add* two $m \times n$ matrices (same dimension) and get an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

where

$$c_{ij} = a_{ij} + b_{ij}, \qquad 1 \leq i \leq m, \ 1 \leq j \leq n$$

Example

1. $$\begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2+2 & 3+2 \\ -1+1 & 0+1 \\ 5+0 & 0+(-1) \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 0 & 1 \\ 5 & -1 \end{bmatrix}$$

2. $$\begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \qquad \text{not defined}$$

Remark

Recall the addition of vectors:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Scalar multiplication

Definition

$$k \cdot \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{bmatrix}$$

Example

$$2 \cdot \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 6 \end{bmatrix}$$

Remark

Recall the scalar multiplication of vectors:

$$a \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} au_1 \\ \vdots \\ au_n \end{bmatrix}$$

Negation, Zero Matrix

Definition

1. If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, then $-A := (-1) \cdot A = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} & \cdots & -a_{mn} \end{bmatrix}$

$$-\begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ -5 & 0 \end{bmatrix}$$

2. The zero matrix of dimension $m \times n$ is the $m \times n$ matrix with all entries 0:

$$0 = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}\right\} m \text{ rows}$$

Remark

It depends on the context that whether 0 is the scalar or the zero matrix of a certain dimension.

Laws

Let A, B, C, and 0 have the same dimension.

- Addition
  1. $(A+B)+C = A+(B+C)$
  2. $A+B = B+A$
  3. $A+0 = A = 0+A$
  4. $A+(-A) = 0$
- Scalar Multiplication
  1. $(k+\ell)A = kA+\ell A$
  2. $k(A+B) = kA+kB$
  3. $k(\ell A) = (k\ell)A$
  4. $1A = A, \quad (-1)A = -A, \quad 0A = 0$

For vectors:

(A1) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(A2) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(A3) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(A4) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(SM1) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v}$

(SM2) $(k+\ell)\mathbf{u} = k\mathbf{u}+\ell\mathbf{u}$

(SM3) $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$

(SM4) $1\cdot\mathbf{u} = \mathbf{u}$

Fact

A column vector in $\mathbb{R}^n$ is a special $n\times 1$ matrix.

For example, $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}$ *is a 2×1 matrix.*

Multiplication

Fact (Row and Column Vectors)

1. *A* <u>*column vector*</u> $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ *is an* $n \times 1$ *matrix.*
2. *A* <u>*row vector*</u> $\begin{bmatrix} x_1, \cdots, x_n \end{bmatrix}$ *is a* $1 \times n$ *matrix.*
3. *A* <u>*scalar*</u> x *is, sometimes, usefully, considered as a* 1×1 *matrix* $\begin{bmatrix} x \end{bmatrix}$. *If the matrix contains only* 1 *entry, the square brackets "[]" can be considered as redundant.*

Definition (Matrix Multiplication)

We can multiply an $m \times n$ matrix by a $k \times \ell$ matrix <u>***ONLY if***</u> $n = k$. In this case, the product is an $m \times \ell$ matrix. In general

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{n\ell} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{m\ell} \end{bmatrix},$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

Multiplication

Remember this: c_{ij} is determined by the *i-th row of the first matrix* and the *j-th column of the second matrix*. More precisely,

- the i-th row of the first matrix is the row vector:

$$\begin{bmatrix} a_{i1}, a_{i2}, \dots, a_{in} \end{bmatrix} \xRightarrow{\text{column vector}} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} \in \mathbb{R}^n$$

- the j-th column of the second matrix is $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} \in \mathbb{R}^n$

- compute the dot product

$$\begin{bmatrix} a_{i1} \\ \vdots \\ a_{in} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = c_{ij}.$$

Example

Example

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 4 \\ 2 & 6 \end{bmatrix} = ? \qquad \text{dimensions: } 2\times 3 \text{ and } 4 \times 2 \quad 3 \neq 4 \Longrightarrow \text{not defined.}$$

Remark

This is why we have dot and cross product for vectors but not regular "multiplication" .

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Both have dimension $n \times 1$.

$$\# \text{ of colums in } \mathbf{u} = 1 \neq n = \# \text{ of rows in } \mathbf{v}$$

Example

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = ?$$

First of all, the dimensions are 2×4 and 4×3. The result is a matrix of dimension 2×3. Now, we write down the two rows of the first matrix:

$$\begin{aligned} R_1 &: \begin{bmatrix} 2 & 1 & 1 & -1 \end{bmatrix} \\ R_2 &: \begin{bmatrix} 0 & 2 & 1 & 4 \end{bmatrix} \end{aligned} \xRightarrow{\text{column vectors}} \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

and the three column vectors from the second matrix

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 4 \end{bmatrix} \quad \Bigg| \quad \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\begin{aligned}
a_{11} &= \mathbf{v}_1 \cdot \mathbf{w}_1 = 2 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 + (-1) \cdot 1 = 2 \\
a_{12} &= \mathbf{v}_1 \cdot \mathbf{w}_2 = 2 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 + (-1) \cdot 1 = 0 \\
a_{13} &= \mathbf{v}_1 \cdot \mathbf{w}_3 = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 + (-1) \cdot 0 = 4 \\
a_{21} &= \mathbf{v}_2 \cdot \mathbf{w}_1 = 0 \cdot 1 + 2 \cdot 0 + 1 \cdot 1 + 4 \cdot 1 = 5 \\
a_{22} &= \mathbf{v}_2 \cdot \mathbf{w}_2 = 0 \cdot 0 + 2 \cdot 1 + 1 \cdot 0 + 4 \cdot 1 = 6 \\
a_{23} &= \mathbf{v}_2 \cdot \mathbf{w}_3 = 0 \cdot 1 + 2 \cdot 2 + 1 \cdot 0 + 4 \cdot 0 = 4
\end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \mathbf{v}_1 \cdot \mathbf{w}_3 \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \mathbf{v}_2 \cdot \mathbf{w}_3 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 5 & 6 & 4 \end{bmatrix}$$

Alternative calculation

column-method: Let A be an $m \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$ is a vector, which can be treated as an $n \times 1$ matrix. $A\mathbf{v}$ is an $m \times 1$ matrix as well as a vector in $\mathbb{R}^m$.

row-method: Let A be an $m \times n$ matrix and $\mathbf{u}$ a $1 \times m$ matrix. Namely $\mathbf{u} = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}$ is a row vector. $\mathbf{u}A$ is a $1 \times n$ matrix as well as a *row* vector in $\mathbb{R}^n$.

Example

$$\begin{aligned}
\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} &= \left[\begin{array}{c|c|c|c} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{array}\right] \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\
&= 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 5 \end{bmatrix}
\end{aligned}$$

Alternative calculation

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix}}_{A} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{array} \end{array} = \begin{bmatrix} A\mathbf{w}_1 & A\mathbf{w}_2 & A\mathbf{w}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 6 & 4 \end{bmatrix}$$

Row and column methods

<u>*Row-method*</u> and <u>*column-method*</u>: Read Proposition 4.23 and 4.29. Let $A_{m\times n}$ and $B_{n\times\ell}$ be matrices such that

$$A=\begin{bmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_m\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_\ell\end{bmatrix}, \quad \text{for} \begin{cases}\text{row vectors } \mathbf{a}_1,\dots,\mathbf{a}_n\\ \text{column vectors } \mathbf{b}_1,\dots,\mathbf{b}_\ell\end{cases} \quad \text{in } \mathbb{R}^n$$

$$AB=\overbrace{\begin{bmatrix}A\mathbf{b}_1 & \cdots & A\mathbf{b}_\ell\end{bmatrix}}^{\text{colume method}}=\overbrace{\begin{bmatrix}\mathbf{a}_1B\\ \vdots\\ \mathbf{a}_mB\end{bmatrix}}^{\text{row method}}=\begin{bmatrix}\mathbf{a}_1\mathbf{b}_1 & \cdots & \mathbf{a}_1\mathbf{b}_\ell\\ \vdots & \ddots & \vdots\\ \mathbf{a}_m\mathbf{b}_1 & \cdots & \mathbf{a}_m\mathbf{b}_\ell\end{bmatrix}$$

Example (Column method)

$$\underbrace{\begin{bmatrix}2 & 1 & 1 & -1\\ 0 & 2 & 1 & 4\end{bmatrix}}_{A}\begin{array}{c}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\\ \begin{array}{ccc}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{array}\end{array}=[A\mathbf{w}_1\ A\mathbf{w}_2\ A\mathbf{w}_3]$$

$$A\mathbf{w}_1=1\begin{bmatrix}2\\0\end{bmatrix}+0\begin{bmatrix}1\\2\end{bmatrix}+1\begin{bmatrix}1\\1\end{bmatrix}+1\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$$

Example (row method)

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 6 & 4 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} &= 2\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 0 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} &= 0\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 4 \end{bmatrix} \end{aligned}$$

MATH-1030
Matrix Theory & Linear Algebra I

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Multiplication

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1\ell} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{n\ell} \end{bmatrix}_{k \times \ell}$$

AB exists ***ONLY if*** $n = k$.

$$AB = \begin{bmatrix} c_{11} & \cdots & c_{1\ell} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{m\ell} \end{bmatrix}_{m \times \ell}, \quad c_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

formula, row method, column method

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} \Rightarrow AB = \begin{bmatrix} \mathbf{a}_1 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_\ell \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \cdots & \mathbf{a}_1\mathbf{b}_\ell \\ \vdots & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \cdots & \mathbf{a}_m\mathbf{b}_\ell \end{bmatrix}$$

$$B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_\ell \end{bmatrix}$$

Example $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} = ?$

First of all the dimensions are 2×2 and 2×2. $\Rightarrow$ Result: a 2×2-matrix.

$$A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \Rightarrow \begin{cases}\mathbf{a}_1 = [1\ 2]\\ \mathbf{a}_2 = [3\ 4]\end{cases} \quad \Bigg| \quad B = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} \Rightarrow \mathbf{b}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

(I) Formula:

$$\begin{aligned}
\mathbf{a}_1\mathbf{b}_1 &= \begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 1 \cdot 1 + 2 \cdot 1 = 3\\
\mathbf{a}_1\mathbf{b}_2 &= \begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}0\\1\end{bmatrix} = 1 \cdot 0 + 2 \cdot 1 = 2\\
\mathbf{a}_2\mathbf{b}_1 &= \begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 3 \cdot 1 + 4 \cdot 1 = 7\\
\mathbf{a}_2\mathbf{b}_2 &= \begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}0\\1\end{bmatrix} = 3 \cdot 0 + 4 \cdot 1 = 4
\end{aligned}$$

Thus,

$$AB = \begin{bmatrix}\mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2\\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2\end{bmatrix} = \begin{bmatrix}3 & 2\\7 & 4\end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{a}_1 = [1\ 2] \\ \mathbf{a}_2 = [3\ 4] \end{cases} \quad \Bigg| \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(II) Column method: $AB = A\,[\mathbf{b}_1\ \mathbf{b}_2] = [A\mathbf{b}_1\ A\mathbf{b}_2]$

$$\begin{aligned} A\mathbf{b}_1 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ A\mathbf{b}_2 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{aligned} \quad \Rightarrow AB = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$$

(III) Row method: $AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \end{bmatrix}$

$$\begin{aligned} \mathbf{a}_1 B &= [1\ 2] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1\,[1\ 0] + 2\,[1\ 1] = [1\ 0] + [2\ 2] = [3\ 2] \\ \mathbf{a}_2 B &= [3\ 4] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 3\,[1\ 0] + 4\,[1\ 1] = [3\ 0] + [4\ 4] = [7\ 4] \end{aligned}$$

$$\Rightarrow AB = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$$

Remarks

1. Let
$$\mathbf{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n \Longrightarrow \mathbf{v} \cdot \mathbf{w} = a_1 b_1 + \cdots + a_n b_n$$
$$\underbrace{\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}}_{1 \times n} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{n \times 1} = [\mathbf{v} \cdot \mathbf{w}]$$
We identify a 1×1 matrix and a scalar.

2. Recall our goal:
$$\begin{aligned} 3x + 2y + 4z &= 7 \\ x - y + z &= 2 \\ 0x + 2y + 3z &= 5 \end{aligned} \Leftrightarrow \begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \begin{pmatrix} \text{column} \\ \text{method} \end{pmatrix} = \begin{bmatrix} 3x + 2y + 4z \\ x - y + z \\ 0x + 2y + 3z \end{bmatrix}$$

Identity matrix

Definition

The n-th *identity* matrix is a square matrix of dimension $n \times n$, given by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$$

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Laws

1. $(AB)C = A(BC)$
2. Most of the times, $\boxed{AB \neq BA}$
3. $A(B+C) = AB + AC$
4. $(A+B)C = AC + BC$
5. $k(AB) = (kA)B = A(kB)$
6. If A has dimension $m \times n$, $AI_n = A = I_mA$.

ALWAYS remember that since matrix multiplication is not commutative, we need to specify multiplying a matrix to the LEFT or to the RIGHT.

Example

1. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

2. $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + bd \end{bmatrix}$

$\begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} ac & bc \\ ad & bd \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Not defined

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

Remark

Please REMEMBER: matrix multiplication is totally different from the multiplication of numbers/scalars.

- $3 \cdot 4 = 12 = 4 \cdot 3$ but most of the times $AB \neq BA$.
- $x^2 = 0 \Rightarrow x = 0$, but $A^2 = AA = 0$ does not mean $A = 0$.
  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ This actually shows we can have $A \neq 0 \neq B$, but $AB = 0$
  Recall that, for numbers, if $xy = 0$ then either $x = 0$ or $y = 0$.
- $ab = ac \Rightarrow b = c$ if $a \neq 0$ but $AB = AC$ does not imply that $B = C$.
  $AB = AC \Rightarrow AB - AC = 0 \Rightarrow A(B - C) = 0$
  Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow B - C = A$
  and $A(B - C) = AA = A^2 = 0$.
- Exercise 4.4.12–Exercise 4.4.18

Simplification

$$(AB)C = A(BC) \qquad A(B+C) = AB + AC \qquad k(AB) = (kA)B = A(kB)$$
$$AB \neq BA \qquad (A+B)C = AC + BC \qquad AI_n = A = I_mA$$

Example

Let X be 2×2, satisfying $3X \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} = X \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 3 & -1 \end{bmatrix}$

$\boxed{X = ?}$

- $3X \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = X \left(3 \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \right) = X \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix}$
- $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 4 \\ 3 \cdot 1 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 2 \cdot 3 & 1 \cdot 5 + 2 \cdot (-1) \\ 3 \cdot (-1) + 4 \cdot 3 & 3 \cdot 5 + 4 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 9 & 11 \end{bmatrix}$

$$X \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix} = X \begin{bmatrix} 5 & 3 \\ 9 & 11 \end{bmatrix} \Rightarrow X \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix} - X \begin{bmatrix} 5 & 3 \\ 9 & 11 \end{bmatrix} = - \begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -2 & -8 \\ -3 & -12 \end{bmatrix} = X \left(\begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ 9 & 11 \end{bmatrix} \right) = X \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = X.$$

Further question

$$3X\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix} + \begin{bmatrix}2\\ 3\end{bmatrix}\begin{bmatrix}1 & 4\end{bmatrix} = X\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & 5\\ 3 & -1\end{bmatrix} \Rightarrow X = -\begin{bmatrix}2 & 8\\ 3 & 12\end{bmatrix}$$

Problem

How about $3X\begin{bmatrix}2 & 0\\ 3 & 4\end{bmatrix} + \begin{bmatrix}2\\ 3\end{bmatrix}\begin{bmatrix}1 & 4\end{bmatrix} = X\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & 5\\ 3 & -1\end{bmatrix}$?

$$3X\begin{bmatrix}2 & 0\\ 3 & 4\end{bmatrix} - X\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & 5\\ 3 & -1\end{bmatrix} = X\begin{bmatrix}6 & 0\\ 9 & 12\end{bmatrix} - X\begin{bmatrix}5 & 3\\ 9 & 11\end{bmatrix} = X\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix}$$

Namely, it holds that $X\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix} = -\begin{bmatrix}2\\ 3\end{bmatrix}\begin{bmatrix}1 & 4\end{bmatrix} = -\begin{bmatrix}2 & 8\\ 3 & 12\end{bmatrix}$

Of course, we can set $X = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and observe that

$$-\begin{bmatrix}2 & 8\\ 3 & 12\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix} = \begin{bmatrix}a & -3a+b\\ c & -3c+d\end{bmatrix}$$

Use the FOUR equations to solve for a, b, c, and d. Or, we can introduce the *inverse matrix*.

Matrix Inverses

Definition

1. Let A be a matrix. We say that A is *left invertible* if there exists a matrix B such that

$$BA = I \qquad \text{(Here we do not specify the dimension)}$$

Then B is called a *left inverse* of A. $A_{m\times n} \Rightarrow B_{n\times m}$
2. A is *right invertible* if there exists a matrix B such that

$$AB = I.$$

Then B is called a *right inverse* of A.
3. A is *invertible* if it has both *left and right inverses*. In this case, A is a square matrix (Theorem 4.48) and both left and right inverses coincide. Namely, there exist a (square) matrix B (with the same dimension as A) such that

$$AB = BA = I. \qquad \text{We write } B = A^{-1}.$$

Fact

Not every matrix is invertible. Not every square matrix is invertible.

Examples

1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. We can compute that

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

   and

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$$

   B is a right inverse of A, but not a left inverse.

2. Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 - 3 \cdot 0 & 1 \cdot 3 - 3 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = BA.$$

Example

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = I \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$3X\begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \end{bmatrix} = X\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 3 & -1 \end{bmatrix} \Rightarrow X\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}$$

Now, multiplying both sides by $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, *to the right*

$$X\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

- The left-hand side is $X\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = XI = X$
- The right-hand side is

$$-\begin{bmatrix} 2 & 8 \\ 3 & 12 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 2\cdot 1 + 8\cdot 0 & 2\cdot 3 + 8\cdot 1 \\ 3\cdot 1 + 12\cdot 0 & 3\cdot 3 + 12\cdot 1 \end{bmatrix} = -\begin{bmatrix} 2 & 14 \\ 3 & 21 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & -14 \\ -3 & -21 \end{bmatrix}$$

Find the inverses

Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 3 \end{bmatrix}$, check if A is invertible. If so, find the inverse.

Method: Make an augmented system $\left[A \mid I\right]$. Attempt row reductions to get $\left[I \mid B\right]$. If this succeeds, then A is invertible and $B = A^{-1}$.

$$\left[A \mid I\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \overset{R_3 \leftarrow R_3 - 2R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 - 2R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\overset{\substack{R_1 \leftarrow R_1 + R_3 \\ R_2 \leftarrow R_2 - R_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

A is invertible with $A^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 2 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix}$ Exercise: check $AA^{-1} = A^{-1}A = I$

Find the inverses

Given $A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, check if A is invertible. If so, find the inverse.

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \overset{R_1 \leftarrow R_1 + 3R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Remark

After Midterm II, we will study an alternative method to compute the inverse, which sometimes is simpler for 2×2 matrices.

Example

Let $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 0 & 2 \\ 2 & -2 & 4 \end{bmatrix}$. Find A^{-1} if it exists.

Example

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & -2 & 4 & 0 & 0 & 1 \end{array}\right] \underset{R_3 \leftarrow R_3 - 2R_1}{\overset{R_2 \leftarrow R_2 - R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

From here, we see $\operatorname{rank}(A) = 2 < 3$. So the (unique) reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is no way to obtain I on the left-hand side of this augmented matrix. Hence, there is no way to complete the algorithm, and the inverse of A does not exist. Namely, A is NOT invertible.

Remark (A being $n \times n$)

If $\operatorname{rank}(A) = n$, $[A \mid I] \sim [I \mid B]$ succeeds to get $B = A^{-1}$. (full rank)
If $\operatorname{rank}(A) < n$, then A is not invertible.

Laws & Properties

1. Inverse is unique. If both B and C are inverses of A, then $B = C$.

Proof.

$C = IC = (BA)C = B(AC) = BI = B.$ □

2. $I^{-1} = I$.
3. If both A and B are invertible, $(AB)^{-1} = B^{-1}A^{-1}$. $(B^{-1}A^{-1}AB = I.)$ $(1/(xy) = (1/x)(1/y))$
4. If A is invertible, so is A^{-1}. Moreover, $(A^{-1})^{-1} = A$.
5. If A is invertible and $k \neq 0$. $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$.

Theorem

1. *If A is invertible, then A is a square matrix.*
2. *If A is a square matrix, then the following are equivalent:*
    - *A is invertible;*
    - *A has a left inverse;*
    - *A has a right inverse;*
3. *Let A be an $m \times n$ matrix.*
    - *If A is left invertible, then $m \geq n$.*
    - *If A is right invertible, then $m \leq n$.*
    - *If A is invertible, then $m = n$.*

- *$\mathrm{rank}(A) \leq m$, $\mathrm{rank}(A) \leq n$*
- *left invertible iff $\mathrm{rank}(A) = n$*
- *right invertible iff $\mathrm{rank}(A) = m$*

Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Determine whether A is invertible. If A is invertible, find A^{-1}.

Lecture resumes at 7:50

Solution

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -2 & 1 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow -\frac{1}{2} R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 - R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}$$

Solve system of equations

Consider the system $\begin{aligned} x+z&=1\\ x-y+z&=3.\\ x+y-z&=2 \end{aligned}$ We can write it as $A\mathbf{a}=\mathbf{b}$ where

$$A=\begin{bmatrix}1 & 0 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}, \quad \mathbf{a}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \quad \mathbf{b}=\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}.$$

From the previous slide, we have

$$A^{-1}=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & -1 & 0\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$$

Multiplying both sides of $A\mathbf{a}=\mathbf{b}$ by A^{-1} to the LEFT:

$$A^{-1}A\mathbf{a}=A^{-1}\mathbf{b} \Longleftrightarrow \mathbf{a}=A^{-1}\mathbf{b}$$

$$\mathbf{a}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=A^{-1}\mathbf{b}=\begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & -1 & 0\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}0\cdot 1+\frac{1}{2}\cdot 3+\frac{1}{2}\cdot 2\\ 1\cdot 1-1\cdot 3+0\cdot 2\\ 1\cdot 1-\frac{1}{2}\cdot 3-\frac{1}{2}\cdot 2\end{bmatrix}=\begin{bmatrix}\frac{5}{2}\\ -2\\ -\frac{3}{2}\end{bmatrix}.$$

Solve system of equations

How about another system $\begin{aligned} x+z&=0\\ x-y+z&=1\\ x+y-z&=3 \end{aligned}$? $\begin{bmatrix}0\\1\\3\end{bmatrix}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = A^{-1}\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & -1 & 0\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\1\\3\end{bmatrix} = \begin{bmatrix}2\\-1\\-2\end{bmatrix}$$

This illustrates that for a system $A\mathbf{x}=\mathbf{b}$ where A^{-1} exists, it is easy to find the solution when the vector $\mathbf{b}$ is changed.

$$\begin{aligned} x+z&=0\\ x-y+z&=0 \qquad \text{(homogeneous)}\\ x+y-z&=0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = A^{-1}\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & -1 & 0\\ 1 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\cdot 0+\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0\\ 1\cdot 0-1\cdot 0+0\cdot 0\\ 1\cdot 0-\frac{1}{2}\cdot 0-\frac{1}{2}\cdot 0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Application (Solve matrix equation)

Find a matrix X such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$AX = C \Longrightarrow A^{-1}AX = A^{-1}C = IX = X.$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{R_1 \leftarrow R_1 - R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

Thus,

$$X = A^{-1}C = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 - 1\cdot 3 & 2\cdot 2 - 1\cdot 4 \\ -1\cdot 1 + 1\cdot 3 & -1\cdot 2 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 2 \end{bmatrix}.$$

Application (Solve matrix equation)

Find a matrix X such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Recall the column method:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a + c \\ a + 2c \end{bmatrix} \Rightarrow c = 2, a = -1$$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b + d \\ b + 2d \end{bmatrix} \Rightarrow d = 2, b = 0$$

Brief Explanation

Consider the example of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 4 & 3 \end{bmatrix}$. Let $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that

$$AB = A \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the column method,

$$A \begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

it is equivalent to three systems (with the same LHS A): Recall the reduced echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 3 & 0 \end{array}\right] \underset{\sim\sim\sim\sim\sim\sim}{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & g \end{array}\right]$$

Brief Explanation

Similarly

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 3 & 0 \end{array}\right] \underset{\sim\sim\sim\sim\sim\sim\sim}{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & h \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 4 & 3 & 1 \end{array}\right] \underset{\sim\sim\sim\sim\sim\sim\sim}{\text{Row operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & c \\ 0 & 1 & 0 & f \\ 0 & 0 & 1 & i \end{array}\right]$$

KEY: The row operations for the three systems are the same, completely determined by A (the coefficients)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \underset{\sim\sim\sim\sim\sim\sim\sim}{\text{Row operations}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Symbolic matrix equation

Assume all matrices A, B, C and X are $n \times n$ and also assume all matrices are invertible as needed. If X satisfies $AX(X+C)^{-1} = B$, solve for X.

Solution. Multiplying both sides by $X + C$ to the right

$$AX(X+C)^{-1}(X+C) = B(X+C) \qquad (*)$$

$$AX = BX + BC \Longrightarrow AX - BX = BC \Longrightarrow (A-B)X = BC$$

Multiply both sides by $(A-B)^{-1}$ to the left

$$(A-B)^{-1}(A-B)X = (A-B)^{-1}BC = X. \qquad (**)$$

$$X = (A-B)^{-1}BC$$

If you multiply matrices to the wrong side, you will have either in $(*)$

$$(X+C)AX(X+C)^{-1} = (X+C)B$$

or in $(**)$

$$(A-B)X(A-B)^{-1} = BC(A-B)^{-1}$$

Transpose

Definition

The *transpose* of an $m \times n$ matrix A, denoted by A^T, is an $n \times m$ matrix, whose (i, j)-entry is just the (j, i)-entry of A. Namely, it is obtained from A by switching rows and columns, , i.e., the i-th column of A^T is the i-th row of A. More precisely,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Transpose

Example

- $$B = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 0 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$$
- $$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} \Longrightarrow \mathbf{v}^T = \begin{bmatrix} 1 & 0 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \mathbf{v}$$

Remark

Now we see that the transpose of a row (resp. column) vector is a column (resp. row) vector.

Dot product

Recall that for $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

$$\underline{\mathbf{v} \cdot \mathbf{w}} = v_1 w_1 + \cdots + v_n w_n = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \underline{\mathbf{v}^T \mathbf{w}}.$$

Remark

To save space, we sometimes use the *transpose of a row vector* to express the *corresponding column vector*. Namely,

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T.$$

Laws of transposes

$$(A+B)^T = A^T + B^T \qquad (rA)^T = rA^T$$
$$(A^T)^T = A \qquad (AB)^T = B^T A^T$$
$$0^T = 0 \qquad I^T = I \qquad (A^{-1})^T = (A^T)^{-1}$$

Example

Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \qquad \Longrightarrow C = AB = \begin{bmatrix} 10 & -1 \\ 3 & -1 \\ -1 & -2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} \qquad \Longrightarrow B^T A^T = \begin{bmatrix} 10 & 3 & -1 \\ -1 & -1 & -2 \end{bmatrix} = C^T$$

Namely, Let $\mathbf{v} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\mathbf{w} = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$.

$$\mathbf{v} \cdot \mathbf{w} = 3 = \mathbf{w} \cdot \mathbf{v} \qquad \Leftrightarrow \qquad \mathbf{v}^T \mathbf{w} = 3 = \mathbf{w}^T \mathbf{v}.$$
$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = 3 = [3] = [3]^T = \left(\mathbf{v}^T \mathbf{w}\right)^T = \mathbf{w}^T \left(\mathbf{v}^T\right)^T = \mathbf{w}^T \mathbf{v} = \mathbf{w} \cdot \mathbf{v}$$

(Anti-)Symmetric matrices

Definition

A matrix A is called *symmetric* if $A^T = A$. It is said to be *antisymmetric* (sometimes also called *skew symmetric*) if $A^T = -A$.

Example

- $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 3 & 10 & 15 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{are symmetric.}$$
- $$B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 10 \\ -3 & -10 & 0 \end{bmatrix} \; C = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} \; 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{are antisymmetric.}$$
- $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is neither.

Trace (*)

Definition

Let A be an $n \times n$-matrix (square matrix). The *trace*, denoted by $tr(A)$ is the sum of its diagonal entries.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \Longrightarrow tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 3 & 10 & 15 \end{bmatrix} \Longrightarrow tr(A) = 1 + 4 + 15 = 20.$$

Summary of Chpt. 3

The only object is matrix.

1. dimension (check before calculation), entry, row vectors & column vectors.
2. addition & scalar multiplication similar to those for vectors
3. **multiplication**:
   - requirement for dimensions
   - calculation formula, row & column methods
4. **inverse**
   - definition and laws/properties
   - find the inverse
   - applications: solving linear systems matrix equations (including symbolic equations)
5. transpose

MATH-1030
Matrix Theory & Linear Algebra I

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Definition

Given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the **set of all their linear combinations**. In symbols,

$$\mathit{span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k \mid a_1, \ldots, a_k \in \mathbb{R}\}\,.$$

Example

Which of the following vectors are in the span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$?

(a) $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$ (b)$\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ (c) $\mathbf{t} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ What do we want?

Determine whether there exist a_1 and a_2 such that

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \qquad \mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \qquad \mathbf{t} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$$

Solution

$$\mathbf{v}_1 = [1\ 2\ 2]^T,\ \mathbf{v}_2 = [2\ 0\ 2]^T,\ \mathbf{u} = [5\ 2\ 4]^T,\ \mathbf{w} = [3\ 2\ 3]^T,\ \mathbf{t} = [1\ -2\ 0]^T.$$

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\begin{bmatrix}1\\2\\2\end{bmatrix} + a_2\begin{bmatrix}2\\0\\2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 0\\2 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$(a)\left[\begin{array}{cc|c}1 & 2 & 5\\2 & 0 & 2\\2 & 2 & 4\end{array}\right] \quad (b)\left[\begin{array}{cc|c}1 & 2 & 3\\2 & 0 & 2\\2 & 2 & 3\end{array}\right] \quad (c)\left[\begin{array}{cc|c}1 & 2 & 1\\2 & 0 & -2\\2 & 2 & 0\end{array}\right]$$

$$\left[\begin{array}{cc|ccc}1 & 2 & 5 & 3 & 1\\2 & 0 & 2 & 2 & -2\\2 & 2 & 4 & 3 & 0\end{array}\right] \overset{\substack{R_2\leftarrow R_2-2R_1\\R_3\leftarrow R_3-2R_1}}{\sim} \left[\begin{array}{cc|ccc}1 & 2 & 5 & 3 & 1\\0 & -4 & -8 & -4 & -4\\0 & -2 & -6 & -3 & -2\end{array}\right]$$

$$\overset{\substack{R_2\leftarrow -\frac{1}{4}R_2\\R_3\leftarrow -\frac{1}{2}R_3}}{\sim} \left[\begin{array}{cc|ccc}1 & 2 & 5 & 3 & 1\\0 & 1 & 2 & 1 & 1\\0 & 1 & 3 & \frac{3}{2} & 1\end{array}\right]$$

$$\overset{R_3\leftarrow R_3-R_2}{\sim} \left[\begin{array}{cc|ccc}1 & 2 & 5 & 3 & 1\\0 & 1 & 2 & 1 & 1\\0 & 0 & 1 & \frac{1}{2} & 0\end{array}\right]$$

Solution

$$\mathbf{v}_1 = [1\ 2\ 2]^T,\ \mathbf{v}_2 = [2\ 0\ 2]^T,\ \mathbf{u} = [5\ 2\ 4]^T,\ \mathbf{w} = [3\ 2\ 3]^T,\ \mathbf{t} = [1\ \ -2\ 0]^T.$$

$$\left[\begin{array}{cc|ccc} 1 & 2 & 5 & 3 & 1 \\ 2 & 0 & 2 & 2 & -2 \\ 2 & 2 & 4 & 3 & 0 \end{array}\right] \sim\sim\sim \left[\begin{array}{cc|ccc} 1 & 2 & 5 & 3 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

Only the system of (c) is consistent. Therefore, only $\mathbf{t}$ is the in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. $\mathbf{t} \in span\{\mathbf{v}_1, \mathbf{v}_2\}$

What is the solution? $(a_1, a_2) = (-1, 1)$

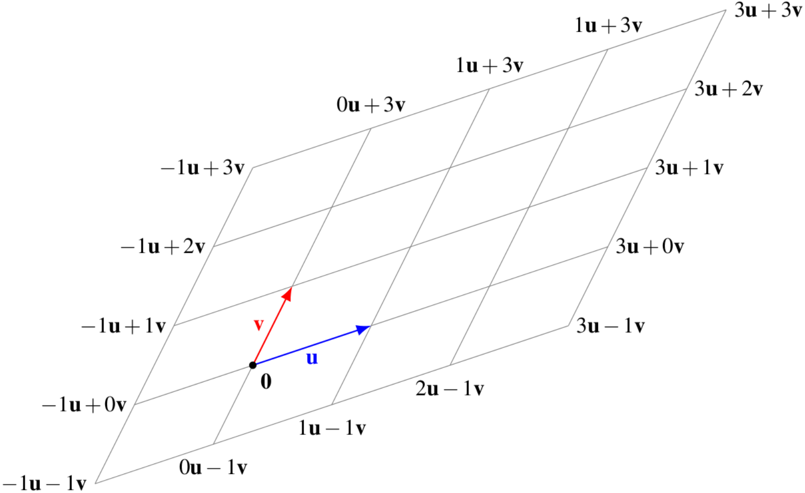
$$\mathbf{v}_2 - \mathbf{v}_1 = [2\quad 0\quad 2]^T - [1\quad 2\quad 2]^T = \left([1\quad 2\quad 2] - [2\quad 0\quad 2]\right)^T = [1\quad -2\quad 0]^T = \mathbf{t}$$

Is this solution important?

No, the key is to know whether the system is consistent.

Geometric Interpretation of Span

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be non-parallel vectors. Then, *span* $\{\mathbf{v}_1, \mathbf{v}_2\}$, the set of all linear combinations, is a *plane* through the origin.



Geometric Interpretation of Span

Remark

- The span of 1 vector is usually a line. ($span\{\mathbf{0}\}$ is just the origin.)
- The span of 2 vectors is usually a plane. (It could be a line if both are parallel) (If $\mathbf{v} = k\mathbf{u}$ for some k (parallel), then $a\mathbf{u} + b\mathbf{v} = a\mathbf{u} + bk\mathbf{u}$ $= (a + bk)\,\mathbf{u}$ — a line)
- The span of 3 vectors is usually a 3-dimensional space. (It could be a plane or a line)
- Higher-dimensional spaces:

Example (in $\mathbb{R}^6$)

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}^T, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}^T,$$
$$\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 & 3 & 7 & 4 \end{bmatrix}^T, \quad \mathbf{v}_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & 0 \end{bmatrix}^T.$$

$$span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

Redundancy

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

They are not mutually parallel. However, it is not hard to see the fact:

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \mathbf{v}_3$$

Thus, $\mathbf{v}_3$ is "redundant", i.e., $span\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right\} = span\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$.

- $span\left\{\mathbf{v}_1, \mathbf{v}_2\right\} \subseteq span\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right\}$: $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0\mathbf{v}_3$
- $span\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right\} \subseteq span\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$:
  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\left(\mathbf{v}_1 + \mathbf{v}_2\right) = \left(a_1 + a_3\right)\mathbf{v}_1 + \left(a_2 + a_3\right)\mathbf{v}_2$

Remark

This is an example that a span of 3 vectors is a plane.

Redundancy

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors and consider $span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. We say that the vector $\mathbf{v}_j$ is *redundant* if $\mathbf{v}_j$ is a linear combination of the vectors ***preceding*** it, i.e., $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$. In this case, $\mathbf{v}_j$ contributes nothing to the span, i.e., $span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = span\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots \mathbf{v}_k\}$.

Example

Find the redundant vectors in the sequence

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $\mathbf{v}_1$ is not redundant. There is no $\mathbf{v}_0$. In fact, $\mathbf{v}_1$ is never redundant unless $\mathbf{v}_1 = \mathbf{0}$. $span\emptyset = \{\mathbf{0}\}$
- $\mathbf{v}_2 = 2\mathbf{v}_1$, so it is redundant.
- $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so it is not redundant. why?

Example

$$\mathbf{v}_1=\begin{bmatrix}1\\1\\0\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}2\\2\\0\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad \mathbf{v}_4=\begin{bmatrix}-1\\-2\\1\end{bmatrix},\quad \mathbf{v}_5=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\mathbf{v}_1$ is $\underline{\text{not redundant}}$. $\mathbf{v}_2$ is $\underline{\text{redundant}}$. $\mathbf{v}_3$ is $\underline{\text{not redundant}}$. 3***rd component***

- $\mathbf{v}_4$?

$$\left[\begin{array}{ccc|c}1&2&1&-1\\1&2&0&-2\\0&0&1&1\end{array}\right]\overset{R_2\leftarrow R_2-R_1}{\sim}\left[\begin{array}{ccc|c}1&2&1&-1\\0&0&-1&-1\\0&0&1&1\end{array}\right]\overset{R_3\leftarrow R_3+R_2}{\sim}\left[\begin{array}{ccc|c}1&2&1&-1\\0&0&-1&-1\\0&0&0&0\end{array}\right]$$

consistent$\Longrightarrow$ $\underline{\text{redundant}}$.

- $\mathbf{v}_5$?

$$\left[\begin{array}{cccc|c}1&2&1&-1&1\\1&2&0&-2&1\\0&0&1&1&1\end{array}\right]\overset{R_2\leftarrow R_2-R_1}{\sim}\left[\begin{array}{cccc|c}1&2&1&-1&1\\0&0&-1&-1&0\\0&0&1&1&1\end{array}\right]\overset{R_3\leftarrow R_3+R_2}{\sim}\left[\begin{array}{cccc|c}1&2&1&-1&1\\0&0&-1&-1&0\\0&0&0&0&1\end{array}\right]$$

inconsistent$\Longrightarrow$$\underline{\text{not redundant}}$.

$$span\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\}=span\{\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_5\}$$

Casting-out algorithm

Input: A finite sequence of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$

Output: Redundant vectors

Steps: (1) Form an $n \times k$ matrix whose columns are the k given vectors.
(2) Reduce it to an echelon form.
(3) Non-pivot columns correspond to redundant vectors.

Example

Find the redundant vectors in the sequence

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \overset{R_3 \leftarrow R_3 + R_2}{\sim} \begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Non-pivot vectors are $\mathbf{v}_2$ and $\mathbf{v}_4$, which are redundant.

$span\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\right\} = span\left\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\right\}$

Example

Find the redundant vectors in the sequence

$$\mathbf{v}_1 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\2\\2\\0\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix}1\\3\\3\\1\end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&1&1&1\\0&1&2&1&3&2\\0&1&2&1&3&3\\0&0&0&1&1&4\end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix}0&\color{red}{1}&2&1&3&2\\0&0&0&1&1&1\\0&1&2&1&3&3\\0&0&0&1&1&4\end{bmatrix}$$

$$\underset{\sim}{R_3 \leftarrow R_3 - R_1} \begin{bmatrix}0&\color{red}{1}&2&2&3&2\\0&0&0&\color{red}{1}&1&1\\0&0&0&0&0&1\\0&0&0&1&1&4\end{bmatrix}$$

$$\underset{\sim}{R_4 \leftarrow R_4 - R_2} \begin{bmatrix}0&\color{red}{1}&2&2&3&2\\0&0&0&\color{red}{1}&1&1\\0&0&0&0&0&\color{red}{1}\\0&0&0&0&0&3\end{bmatrix}$$

Example

Find the redundant vectors in the sequence

$$\mathbf{v}_1 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\2\\2\\0\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix}1\\3\\3\\1\end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$\begin{bmatrix}0&0&0&1&1&1\\0&1&2&1&3&2\\0&1&2&1&3&3\\0&0&0&1&1&4\end{bmatrix} \sim\sim\sim \begin{bmatrix}0&\color{red}{1}&2&2&3&2\\0&0&0&\color{red}{1}&1&1\\0&0&0&0&0&\color{red}{1}\\0&0&0&0&0&3\end{bmatrix} \overset{R_4 \leftarrow R_4 - 3R_3}{\sim} \begin{bmatrix}0&\color{red}{1}&2&2&3&2\\0&0&0&\color{red}{1}&1&1\\0&0&0&0&0&\color{red}{1}\\0&0&0&0&0&0\end{bmatrix}$$

Non-pivot vectors are $\mathbf{v}_1$, $\mathbf{v}_3$ and $\mathbf{v}_5$, which are redundant.

$$span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} = span\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$$

Remark

$$\begin{aligned}
span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} &= \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \mid a_1, a_2, a_3 \in \mathbb{R}\} \\
span\{\mathbf{v}_1, \mathbf{v}_2\} &= \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \mid a_1, a_2 \in \mathbb{R}\} \\
span\{\mathbf{v}_1\} &= \{a_1\mathbf{v}_1 \mid a_1 \in \mathbb{R}\} \\
span\emptyset &= \{\mathbf{0}\} \text{ the set containing only the zero vector.}
\end{aligned}$$

Linear (in)dependence

Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called

- *linearly independent* if there are no redundant vectors;
- *linearly dependent* if it is not linearly independent, i.e., if there are redundant vectors.

Example

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linear (in)dependence

Definition

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is called

- *linearly independent* if there are no redundant vectors;
- *linearly dependent* if it is not linearly independent, i.e., if there are redundant vectors.

Example

Are the following vectors linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \overset{R_3 \leftarrow R_3 - R_2}{\sim} \begin{bmatrix} \color{red}{1} & 0 & 1 \\ 0 & \color{red}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\mathbf{v}_3$ is redundant, we see they are linearly dependent.

Extended casting-out algorithm

- The ordinary casting-out algorithm.
- Except we go to the ***reduced echelon form***.
- The entries of the non-pivot columns determine the coefficients for writing each redundant vector as a linear combination of preceding non-redundant vectors.

Example

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent? $\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_2$
(b) If not, show how each redundant vector can be written as a linear combination of preceding vectors.

Example

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\2\\2\\2\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}3\\1\\1\\1\end{bmatrix}$$

Solution

(a) By the casting-out algorithm,

$$\begin{bmatrix}1&1&0&3\\0&1&2&1\\0&1&2&1\\0&1&2&1\end{bmatrix} \overset{\substack{R_3\leftarrow R_3-R_2\\R_4\leftarrow R_4-R_2}}{\sim} \begin{bmatrix}1&1&0&3\\0&1&2&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \Rightarrow$$

$\mathbf{v}_3$ and $\mathbf{v}_4$ are redundant.

(b) Proceed to the reduced echelon form:

$$\begin{bmatrix}1&1&0&3\\0&1&2&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \overset{R_1\leftarrow R_1-R_2}{\sim} \begin{bmatrix}1&0&-2&2\\0&1&2&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \Rightarrow \begin{cases}\mathbf{v}_3 = (-2)\mathbf{v}_1 + 2\mathbf{v}_2\\ \mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_2\end{cases}$$

Remark

You might (or might not) get confused, by comparing with an early example

$$\begin{aligned} 2x+y+w&=8\\ x+y+z&=6\\ 3x+2y+z+v&=-2 \end{aligned} \rightarrow \left[\begin{array}{ccccc|c} 2&1&0&1&0&8\\ 1&1&1&0&0&6\\ 3&2&1&0&1&-2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1&0&-1&0&1&-14\\ 0&1&2&0&-1&20\\ 0&0&0&1&-1&16 \end{array}\right]$$

$$z=s \text{ and } v=t \Longrightarrow \quad \begin{bmatrix} x\\ y\\ z\\ w\\ v \end{bmatrix} = \begin{bmatrix} -14\\ 20\\ 0\\ 16\\ 0 \end{bmatrix} + t\begin{bmatrix} -1\\ 1\\ 0\\ 1\\ 1 \end{bmatrix} + s\begin{bmatrix} 1\\ -2\\ 1\\ 0\\ 0 \end{bmatrix}$$

We have to switch the signs for the entries, except for those on the right-hand side.

$$\begin{bmatrix} 1&1&0&3\\ 0&1&2&1\\ 0&1&2&1\\ 0&1&2&1 \end{bmatrix} \sim \begin{bmatrix} 1&0&-2&2\\ 0&1&2&1\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{v}_3=(-2)\mathbf{v}_1+2\mathbf{v}_2\\ \mathbf{v}_4=2\mathbf{v}_1+\mathbf{v}_2 \end{cases}$$

Remark

The reason is: when expressing $\mathbf{v}_3$ and $\mathbf{v}_4$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, they ARE on the right-hand side.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \underset{\sim}{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - R_2}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{\sim}{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \mathbf{v}_3 \\ \| \\ (-2)\mathbf{v}_1 \\ + \\ 2\mathbf{v}_2 \end{array}$$

Classwork

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ linearly independent?

(b) If not, show how each redundant vector can be written as a linear combination of preceding vectors.

Lecture resumes at 7:45.

Solution

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{bmatrix}
&\overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{bmatrix} \\
&\overset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & 0 & 1 \end{bmatrix} \\
&\overset{R_2 \leftarrow -R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 0 & 1 \end{bmatrix} \\
&\overset{\substack{R_3 \leftarrow R_3 + R_2 \\ R_1 \leftarrow R_1 - R_2}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}
\end{aligned}$$

Solution

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\overset{R_3 \leftarrow -R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\overset{\substack{R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 - R_3}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2 = -\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_5 = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4$$

Alternative characterization

Theorem

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent iff (if and only if) the vector equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

*has **only** the trivial solution. Namely it only has the solution $a_1 = \cdots = a_k = 0$.*

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$$

The equation above ($k = 3$) becomes

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0} \Longleftrightarrow \begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -2 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Homogeneous}$$

Check the rank of the coefficient matrix. Find its (reduced) echelon form:

Example

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & 2 & -2 \\ 0 & 3 & 3 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \overset{R_3 \leftarrow R_3 - 3R_1}{\sim} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 \leftarrow R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank is $2 < 3$, so the vector equation has non-trivial solutions

- The three vectors, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly **dependent**.
- In particular,

$$\mathbf{v}_3 = -4\mathbf{v}_1 + \mathbf{v}_2 \iff 4\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0} \iff (a_1, a_2, a_3) = (4, -1, 1)$$

This is the basic solution. General solution: $t\,(4, -1, 1)$

$$t\,(4\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) = 4t\mathbf{v}_1 - t\mathbf{v}_2 + t\mathbf{v}_3 = t\mathbf{0} = \mathbf{0}$$

Proof

Theorem

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent iff the vector equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

*has **only** the trivial solution.*

Proof.

We need to prove two directions.

- "if part" $\boxed{\text{If } a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0} \text{ has only the trivial solution, then } \mathbf{v}_1, \ldots, \mathbf{v}_k \text{ are linearly independent.}}$

  By contradiction. Suppose the statement is false, i.e., $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent. WLOG (=without loss of generality), we let $\mathbf{v}_j$ be redundant. This means, there are $a_1, \ldots, a_{j-1}$ such that
  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{j-1}\mathbf{v}_{j-1} = \mathbf{v}_j \Longrightarrow a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{j-1}\mathbf{v}_{j-1} - \mathbf{v}_j = \mathbf{0}.$

  $$(a_1, \ldots, a_{j-1}, -1, 0, \ldots, 0) \text{ is a non-trivial solution.}$$

  This is a contradiction. Therefore, the statement is true, i.e., $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Proof

Theorem

The vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent iff the vector equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

has ***only*** *the trivial solution.*

Proof (Cont'd).

If $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent, then $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ only has the trivial solution.

- "only if part"

  Again, to prove it by contradiction, we assume

  $$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

  has a non-trivial solution, say $(a_1, \dots, a_k) \neq (0, \dots, 0)$. Let j be the largest index such that $a_j \neq 0$, i.e.,

  $$(a_1, \dots, a_k) = (a_1, \dots, a_j, 0, \dots, 0) \Longrightarrow a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_j\mathbf{v}_j = \mathbf{0}$$

  Thus,
  $$\mathbf{v}_j = -\frac{a_1}{a_j}\mathbf{v}_1 - \frac{a_2}{a_j}\mathbf{v}_2 - \cdots - \frac{a_1}{a_j}\mathbf{v}_{j-1}.$$

  This contradicts that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent.

Properties of linear independence

1. Reordering

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, then any reordering of this sequence of vectors is also linearly independent.

Example

If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, then $\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

2.Subset

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, so are $\mathbf{v}_1, \ldots, \mathbf{v}_j$ for any $j < k$.

Remark

This seems to work only for the first $j(< k)$ vector. Combining with the "Reordering" property, we see any j vectors chosen from $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Properties of linear independence

3. Dimension

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be linearly independent in $\mathbb{R}^n$. Then, $k \leq n$.

Proof.

The casting-out algorithm involves an $n \times k$ matrix. If $k > n$,

$$\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n & \mathbf{v}_{n+1} & \cdots & \mathbf{v}_k \end{bmatrix}$$

i.e., there are more columns than rows, there must be at least one non-pivot column. This implies $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must be linearly dependent if $k > n$. □

Remark

There will be a general version of this property for subspaces.

Example

Assume $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent in $\mathbb{R}^n$.
[Question] Determine whether the following vectors are linearly independent:

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \qquad \mathbf{w}_2 = \mathbf{v}_2 + 2\mathbf{v}_3, \qquad \mathbf{w}_3 = 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$$

We use the alternative characterization of linear independence.
Consider the vector equation

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = \mathbf{0}$$

and ask whether this equation has a non-trivial solution.

- If "yes", then $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly dependent.
- If "no", then $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent.

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1(\mathbf{v}_1 + \mathbf{v}_2) + a_2(\mathbf{v}_2 + 2\mathbf{v}_3) + a_3(2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3)$$

Simplify

$$(a_1 + 2a_3)\mathbf{v}_1 + (a_1 + a_2 + a_3)\mathbf{v}_2 + (2a_2 - a_3)\mathbf{v}_3 = \mathbf{0}$$

Because $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, we know that

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{0} \Longrightarrow (b_1, b_2, b_3) = (0,0,0) \Longrightarrow \begin{aligned} a_1 + 2a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \\ 2a_2 - a_3 &= 0 \end{aligned}$$

Example

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = \mathbf{0} \Leftrightarrow \begin{aligned} a_1 + 2a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \\ 2a_2 - a_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

This system only has the trivial solution $(a_1, a_2, a_3) = (0, 0, 0)$.
$\mathbf{w}_1$,$\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent.

Remark (Linear transformation)

$$\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3\end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$$

After Midterm II, we shall learn an alternative method.

Uniqueness of linear combinations

Theorem

*Assume $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent and $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then, $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in a **unique** way.*

Proof.

Assume there are two ways:

$$\begin{aligned}
\mathbf{w} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k \\
\mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k
\end{aligned}$$

Subtraction yields

$$\mathbf{0} = (b_1 - c_1)\,\mathbf{v}_1 + (b_2 - c_2)\,\mathbf{v}_2 + \cdots + (b_k - c_k)\,\mathbf{v}_k$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, the only solution to the equation is

$$b_1 - c_1 = b_2 - c_2 = \cdots = b_k - c_k = 0 \Longleftrightarrow b_1 = c_1,\; b_2 = c_2,\; \ldots, b_k = c_k.$$

They are the same linear combination. □

Subspaces in $\mathbb{R}^n$

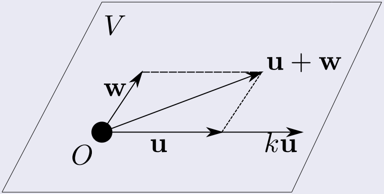
Definition

A subset V of $\mathbb{R}^n$ is called a *subspace* of $\mathbb{R}^n$ if the following 3 conditions hold:

- $\mathbf{0} \in V$;
- "V is closed under addition" : Whenever $\mathbf{u}, \mathbf{w} \in V$, then $\mathbf{u} + \mathbf{w} \in V$.
- "V is closed under scalar multiplication" : Whenever $\mathbf{u} \in V$ and $k \in \mathbb{R}$, then $k\mathbf{u} \in V$.

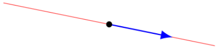
Example

- Let V be a plane through the origin in 3-dimensional space. Then, V is a subspace of $\mathbb{R}^3$.

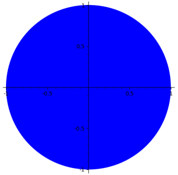


Examples

- Let W be a line through the origin. Then, it is a subspace.



- Let A be the closed unit disc. $A = \left\{ \begin{bmatrix} x & y \end{bmatrix}^T \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \right\}$.



This is not a subspace:

- $\mathbf{0} \in A$
- $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in A$ but $\mathbf{u} + \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin A$ $\left(1^2 + 1^2 = 2 > 1\right)$;

  $2\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin A$

Solutions to a homogeneous system

Example

The solution space of a system of homogeneous equations. Let A be an $m \times n$ matrix and V be the set of solutions of the homogeneous system

$$A\mathbf{u} = \mathbf{0} \qquad \text{i.e., } V = \{\mathbf{u} \in \mathbb{R}^n \mid A\mathbf{u} = \mathbf{0}\}.$$

Then, V is a subspace of $\mathbb{R}^n$.

1. $\mathbf{0} \in V$, $\Longleftarrow A\mathbf{0} = \mathbf{0}$ the trivial solution;
2. for any $\mathbf{u}, \mathbf{w} \in V$, $\Longrightarrow A\mathbf{u} = \mathbf{0} = A\mathbf{w} \Longrightarrow A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ $\Longrightarrow \mathbf{u} + \mathbf{w} \in V$;
3. for any $\mathbf{u} \in V$, $k \in \mathbb{R}$, $\Longrightarrow A\mathbf{u} = \mathbf{0} \Longrightarrow A(k\mathbf{u}) = k(A\mathbf{u}) = k\mathbf{0} = \mathbf{0}$ $\Longrightarrow k\mathbf{u} \in V$

Remark

The solution space of a homogeneous system $=$ *span* {basic solutions}

Span

Example

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then, *span* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subspace.

1. $\mathbf{0} \in \mathit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \Longleftarrow \mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k$;
2. Addition: Assume $\mathbf{u}, \mathbf{w} \in \mathit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$

$$\begin{aligned} \mathbf{u} &= a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k \\ \mathbf{w} &= b_1\mathbf{v}_1 + \cdots + b_k\mathbf{v}_k \end{aligned} \Longrightarrow \mathbf{u} + \mathbf{w} = (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_k + b_k)\mathbf{v}_k$$

$$\Longrightarrow \mathbf{u} + \mathbf{w} \in \mathit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

3. Scalar multiplication:

$$\mathbf{u} \in \mathit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \Longrightarrow \mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$$

$$\Longrightarrow r\mathbf{u} = (ra_1)\mathbf{v}_1 + \cdots + (ra_k)\mathbf{v}_k \in \mathit{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

Span

Theorem

Every subspace V of $\mathbb{R}^n$ is equal to a span. Moreover, we can write $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, where $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

Proof.

- If $V = \{\mathbf{0}\}$, then $V = span\emptyset$ and we are done.
- If $V \neq \{\mathbf{0}\}$, then V contains some non-zero vector. Pick any non-zero vector $\mathbf{u}_1 \in V$. If $V = span\{\mathbf{u}_1\}$, we are done.
- Otherwise, pick $\mathbf{u}_2$ in V but not in $span\{\mathbf{u}_1\}$. If $V = span\{\mathbf{u}_1, \mathbf{u}_2\}$, we are done.
- Otherwise, we pick $\mathbf{u_3} \in V \backslash span\{\mathbf{u}_1, \mathbf{u}_2\}$ ($A \backslash B = \{x \in A \mid x \notin B\}$) and check $V \stackrel{?}{=} span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ etc.

Note that:

1. After j steps, vectors $\mathbf{u}_1, \ldots, \mathbf{u}_j$ are linearly independent.
2. Since there are at most n linearly independent vectors in $\mathbb{R}^n$, this process must stop after finitely many steps.

□

MATH-1030
Matrix Theory & Linear Algebra I

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Recall

1. Linearly (in)dependent
   - (extended) casting-out algorithm
   - homogeneous system
2. Span of a set of vectors
   - Redundant vectors can be eliminated.
   - For example, a span of three vectors, may not be a space.
3. Subspaces of $\mathbb{R}^n$
   - lines, planes containing the origin
   - the solution space of homogeneous system
   - **Theorem**. Every subspace V of $\mathbb{R}^n$ is equal to a span. Moreover, we can write $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, where $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

Bases

Definition

Let V be a subspace of $\mathbb{R}^n$. $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a *basis* of V if the following two conditions hold:

1. $V = span\,\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$
2. $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

Remark

The space V is "exactly" the span of the basis. None of the vectors in a basis is redundant.

Example

1. A basis of $\mathbb{R}^3$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is called the **standard basis** of $\mathbb{R}^3$. Check the right-hand system: **i**, **j**, **k** in Chpt. 2.

Examples

- $\mathbb{R}^3 = span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \Longleftarrow$ every vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$: $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$
- It is obvious that they are linearly independent. I_3 has rank 3.

Proposition

Let $\mathbf{e}_i$ be the vector in $\mathbb{R}^n$ whose i-th component is 1 and all of whose other components are 0. In other words, $\mathbf{e}_i$ is the i-th column of the identity matrix.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. It is called the standard basis of $\mathbb{R}^n$.

$$rank(I_n) = n.$$

Examples

For the standard basis of $\mathbb{R}^n$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

This is also an *orthonormal* basis (orthogonal+normal)

- Orthogonal: For $i \neq j$, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$
- Normal: $\|\mathbf{e}_i\| = 1$

You will study more in MATH-2040 (§11)

Examples

2. A non-standard basis of $\mathbb{R}^3$: $\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\mathbf{u}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

- $\begin{bmatrix}x\\y\\z\end{bmatrix} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \to \left[\begin{array}{ccc|c}1&1&1&x\\1&1&0&y\\1&0&0&z\end{array}\right]$

  $rank = 3 \Rightarrow$ consistent & unique solution $\left[\begin{array}{ccc|c}1&1&1&x\\1&1&0&y\\1&0&0&z\end{array}\right]$

  $$\overset{\substack{R_2\leftarrow R_2-R_1\\R_3\leftarrow R_3-R_1}}{\sim} \left[\begin{array}{ccc|c}1&1&1&x\\0&0&-1&y-x\\0&-1&-1&z-x\end{array}\right] \overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c}1&1&1&x\\0&-1&-1&z-x\\0&0&-1&y-x\end{array}\right]$$

  $\Longrightarrow \mathbb{R}^3 = span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$

- $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent : $x = y = z = 0$

  $$\left[\begin{array}{ccc|c}1&1&1&0\\1&1&0&0\\1&0&0&0\end{array}\right] \overset{\substack{R_2\leftarrow R_2-R_1\\R_3\leftarrow R_3-R_1}}{\sim} \left[\begin{array}{ccc|c}1&1&1&0\\0&0&-1&0\\0&-1&0&0\end{array}\right] \overset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c}1&1&1&0\\0&-1&0&0\\0&0&-1&0\end{array}\right]$$

  rank$= 3 \Longrightarrow$unique trivial solution $\Longrightarrow$linearly independent.

Examples

Remark

Recall Theorem 1.26 Consider a system of m equations in n variables, and assume that the coefficient matrix has rank r . Assume further that the system is ***consistent***.

- If $r = n$, then the system has a unique solution.
- If $r < n$, then the system has infinitely many solutions, with $n - r$ parameters.

Now, suppose we ***do not assume the system is consistent***, but let $m = n = r$. The system is consistent, since there are no zero rows for the coefficient matrix in its echelon form $(m = n = r)$. Moreover, this system has a unique solution.

$$\#\text{ of equations} = \#\text{ of variables} = rank \Longrightarrow \text{unique solution}$$

Example

3. The columns of an invertible matrix. Let A be an ***invertible*** $n \times n$ matrix and $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be the columns of A.

$$A = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \Rightarrow \left[\begin{array}{c|c} A & I \end{array}\right] \sim \left[\begin{array}{c|c} I & A^{-1} \end{array}\right] \Longrightarrow rank(A) = n$$

Examples

$$A\mathbf{x} = \mathbf{v} \iff \left[\begin{array}{ccc|c} \mathbf{u}_1 & \cdots & \mathbf{u}_n & \mathbf{v} \end{array}\right] \text{ has a unique solution.}$$

- $\mathbf{v} = \begin{bmatrix} x_1 \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$: $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \implies \mathbb{R}^n = span\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$
- $\mathbf{v} = \mathbf{0} \implies \mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent.

The columns of an $n \times n$ invertible matrix form a basis of $\mathbb{R}^n$.
4. Find a basis for the space

$$V = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

It is obvious that the second vector is redundant. $\mathbf{u}_2 = 2\mathbf{u}_1$
It suffices to cast out the redundant vectors.

- The result will only contain linearly independent vectors.
- And this does not change the span (still V).

Examples

4. Find a basis for the space $V = span\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\4\\2\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\}$.

Solution. Use the casting-out algorithm:

$$\begin{bmatrix}1&2&1&4&3\\2&4&0&4&4\\0&0&1&2&1\end{bmatrix} \overset{R_2\leftarrow R_2-2R_1}{\sim} \begin{bmatrix}1&2&1&4&3\\0&0&-2&-4&-2\\0&0&1&2&1\end{bmatrix}$$

$$\overset{R_2\leftarrow -\frac{1}{2}R_2}{\sim} \begin{bmatrix}1&2&1&4&3\\0&0&1&2&1\\0&0&1&2&1\end{bmatrix}$$

$$\overset{R_3\leftarrow R_3-R_2}{\sim} \begin{bmatrix}1&2&1&4&3\\0&0&1&2&1\\0&0&0&0&0\end{bmatrix}$$

This means $\mathbf{u}_2$, $\mathbf{u}_4$ and $\mathbf{u}_5$ are redundant. Therefore,

$$V = span\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\4\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\4\\2\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\} = V = span\left\{\overbrace{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}}^{\text{basis}}\right\}.$$

Examples

5. Find a basis of the **solution space** of the system of homogeneous equations

$$\begin{bmatrix}1&2&1&4&3\\2&4&0&4&4\\0&0&1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Leftrightarrow\left[\begin{array}{ccccc|c}1&2&1&4&3&0\\2&4&0&4&4&0\\0&0&1&2&1&0\end{array}\right]$$

Solution.

$$\left[\begin{array}{ccccc|c}1&2&1&4&3&0\\2&4&0&4&4&0\\0&0&1&2&1&0\end{array}\right]\underset{\sim}{R_2\leftarrow R_2-2R_1}\underset{\sim}{R_2\leftarrow -\tfrac{1}{2}R_2}\left[\begin{array}{ccccc|c}1&2&1&4&3&0\\0&0&1&2&1&0\\0&0&1&2&1&0\end{array}\right]$$

$$\underset{\sim}{R_3\leftarrow R_3-R_2}\underset{\sim}{R_1\leftarrow R_1-R_2}\left[\begin{array}{ccccc|c}1&2&0&2&2&0\\0&0&1&2&1&0\\0&0&0&0&0&0\end{array}\right]$$

Examples

$$\begin{bmatrix}1&2&1&4&3\\2&4&0&4&4\\0&0&1&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\sim\left[\begin{array}{ccccc|c}1&2&0&2&2&0\\0&0&1&2&1&0\\0&0&0&0&0&0\end{array}\right]$$

x and z are pivot while y, w and v are free.

$$\begin{matrix}y=t\\w=s\\v=r\end{matrix}\Rightarrow\begin{matrix}x=-2y-2w-2v=-2t-2s-2r\\z=\quad -2w-v\quad=-2s-r\end{matrix}\Longrightarrow\begin{bmatrix}x\\y\\z\\w\\v\end{bmatrix}=\begin{bmatrix}-2t-2s-2r\\t\\-2s-r\\s\\r\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\\w\\v\end{bmatrix}=\begin{bmatrix}-2t-2s-2r\\t\\-2s-r\\s\\r\end{bmatrix}=t\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}-2\\0\\-2\\1\\0\end{bmatrix}+r\begin{bmatrix}-2\\0\\-1\\0\\1\end{bmatrix}$$

Examples

The solution space is *span* $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent due to the 1's in red. Basis:

$$\left\{ [-2\ 1\ 0\ 0\ 0]^T, [-2\ 0\ -2\ 1\ 0]^T, [-2\ 0\ -1\ 0\ 1]^T \right\}$$

Fact

The solution space of a homogeneous system is the span of basic solution(s). Basic solutions are linearly independent. (Parameters)

6. Consider the subspace V of $\mathbb{R}^3$, consisting of all vectors $\begin{bmatrix} x & y & z \end{bmatrix}^T$ with $x = z$. In symbols,

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x = z \right\}.$$

Find a basis for V.

Solution. Solve the homogeneous system $x - z = 0$. V is the solution space of the equation $x = z$, the homogeneous system $x - z = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} z = t \Rightarrow x = z = t \\ y = s \end{array} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x = z \right\} = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Coordinate

Definition

Let $B = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be a basis of some subspace V. Then, every vector $\mathbf{w} \in V$ can be **uniquely** written as a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_k$:

$$\mathbf{w} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k.$$

We say $a_1, \dots, a_k$ are the *coordinates of* $\mathbf{w}$ *with respect to the basis* B. We shall use the notation

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}^T.$$

Example

1. Let $V = \mathbb{R}^3$.
   - With respect to the standard basis $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

   every vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are the coordinates.

Examples

- Recall a non-standard basis of $\mathbb{R}^3$: $B = \left\{ \mathbf{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & 1 & 0 & y \\ 1 & 0 & 0 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & -1 & -1 & z-x \\ 0 & 0 & -1 & y-x \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & z \\ 0 & 1 & 0 & y-z \\ 0 & 0 & 1 & x-y \end{array}\right]$$

Every vector $\mathbf{v} = \begin{bmatrix}x\\y\\z\end{bmatrix} = z\mathbf{u}_1 + (y-z)\,\mathbf{u}_2 + (x-y)\,\mathbf{u}_3$

$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix}z\\y-z\\x-y\end{bmatrix} = A\begin{bmatrix}x\\y\\z\end{bmatrix} \Longrightarrow A = \begin{bmatrix}0 & 0 & 1\\0 & 1 & -1\\1 & -1 & 0\end{bmatrix}$ Exercise:

$$A^{-1} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{v} = z\mathbf{u}_1 + (y-z)\,\mathbf{u}_2 + (x-y)\,\mathbf{u}_3 = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}z\\y-z\\x-y\end{bmatrix}$$

Example

2. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a basis for a certain 3-dimensional subspace of $\mathbb{R}^4$. Let $\mathbf{w} = \begin{bmatrix} 3 & 2 & 6 & 1 \end{bmatrix}^T$. Find the coordinates of $\mathbf{w}$ with respect to (w. r. t.) B, i.e, $[\mathbf{w}]_B$.

Solution. We need to solve $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_2\mathbf{u}_3 = \mathbf{w}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\underset{\sim}{R_4 \leftarrow R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{w} = \mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3.$$

Example

3. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Find the vector $\mathbf{v}$ whose coordinates w. r. t. B are $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$.

Solution.

$$\mathbf{v} = 2\mathbf{u}_1 + (-1)\,\mathbf{u}_2 + 1\mathbf{u}_3 = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Fact

vector$\Leftrightarrow$coordinates One direction is easier than the other, which is due to that "vector$\Rightarrow$coordinates" needs to calculate the (left) inverse.

Classwork

Let

$$S=\left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\;\middle|\;\begin{matrix}x+y=z+w\\x+z=y+w\end{matrix}\right\}.$$

(a) Find a basis B for S.

(b) Obviously, $\mathbf{v}=[3\ 3\ 3\ 3]^T\in S$. Find $[\mathbf{v}]_B$

Lecture resumes at 7:55.

Solution

$$S=\left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\;\middle|\;\begin{matrix}x+y=z+w\\x+z=y+w\end{matrix}\right\}\Rightarrow\begin{matrix}x+y-z-w=0\\x-y+z-w=0\end{matrix}\Rightarrow\begin{bmatrix}1&1&-1&-1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

(a)

$$\left[\begin{array}{cccc|c}1&1&-1&-1&0\\1&-1&1&-1&0\end{array}\right]\overset{R_2\leftarrow R_2-R_1}{\sim}\left[\begin{array}{cccc|c}1&1&-1&-1&0\\0&-2&2&0&0\end{array}\right]$$

$$\overset{R_2\leftarrow -\frac{1}{2}R_2}{\sim}\left[\begin{array}{cccc|c}1&1&-1&-1&0\\0&1&-1&0&0\end{array}\right]$$

$$\overset{R_1\leftarrow R_1-R_2}{\sim}\left[\begin{array}{cccc|c}1&0&0&-1&0\\0&1&-1&0&0\end{array}\right]$$

$$\begin{matrix}z=t\\w=s\end{matrix}\Rightarrow\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=t\begin{bmatrix}0\\1\\1\\0\end{bmatrix}+s\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

Solution

$$S=\left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix}\;\middle|\;\begin{matrix}x+y=z+w\\x+z=y+w\end{matrix}\right\}=span\left\{\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}\Rightarrow B=\left\{\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$$

(b) Easily by observation, $[\mathbf{v}]_B=\begin{bmatrix}3\\3\end{bmatrix}$. Or, we can try to solve

$$\begin{bmatrix}3\\3\\3\\3\end{bmatrix}=\mathbf{v}=a\begin{bmatrix}0\\1\\1\\0\end{bmatrix}+b\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$\left[\begin{array}{cc|c}0&1&3\\1&0&3\\1&0&3\\0&1&3\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&3\\0&1&3\\1&0&3\\0&1&3\end{array}\right]\sim\left[\begin{array}{cc|c}1&0&3\\0&1&3\\0&0&0\\0&0&0\end{array}\right]$$

Properties of Bases

Theorem

Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are linearly independent elements of span $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$. Then, $r \leq s$.

Proof.

For $j = 1, \ldots, r$, since $\mathbf{u}_j \in span\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$, $\mathbf{u}_j = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \cdots + a_{sj}\mathbf{v}_s$.
Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{sr} \end{bmatrix} \quad \text{an } s \times r \text{ matrix.}$$

Assume $r > s$. The system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x} \neq \mathbf{0}$ for the system

$$\begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{sr} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

for all $j = 1, \ldots, s$

$$x_1 a_{j1} + \cdots + x_r a_{jr} = 0$$

Properties of Bases

$$
\begin{aligned}
x_1\mathbf{u}_1+\cdots+x_r\mathbf{u}_r = & \quad x_1\left(a_{11}\mathbf{v}_1+a_{21}\mathbf{v}_2+\cdots+a_{s1}\mathbf{v}_s\right)+\cdots \\
+ & \quad x_r\left(a_{1r}\mathbf{v}_1+a_{2r}\mathbf{v}_2+\cdots+a_{sr}\mathbf{v}_s\right) \\
= & \quad \left(x_1a_{11}+\cdots+x_ra_{1r}\right)\mathbf{v}_1+\cdots \\
+ & \quad \left(x_1a_{r1}+\cdots+x_ra_{rr}\right)\mathbf{v}_r \\
= & \quad 0\mathbf{v}_1+\cdots+0\mathbf{v}_r=\mathbf{0}
\end{aligned}
$$

We have a non-trivial linear combination of $\{\mathbf{u}_1,\ldots\mathbf{u}_r\}$ giving the zero vector $\mathbf{0}$. This violates that $\mathbf{u}_1,\ldots,\mathbf{u}_r$ are linearlly independent $\Rightarrow r\leq s$.

Theorem

*Let V be a subspace of $\mathbb{R}^n$. Then, **any two bases** of V have **the same number of elements**.*

Proof.

Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ and $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ be two bases of V.

- $V=span\{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ and $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ are linearly independent. $\Longrightarrow r\leq s$.
- $V=span\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ and $\{\mathbf{u}_1,\ldots,\mathbf{u}_s\}$ are linearly independent. $\Longrightarrow s\leq r$.

Therefore, $r=s$. $\square$

Dimension

Definition

Let V be a subspace of $\mathbb{R}^n$ with basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then we say V has dimension k. $\dim V = k$.

Properties

1. Every subspace V of $\mathbb{R}^n$ has a basis and therefore a dimension.
2. Every linearly independent set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r \in V$ can be **extended** to a basis.
3. Every spanning set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of V can be **shrunk** to a basis of V.
4. If V and W are subspaces of $\mathbb{R}^n$ with $V \subseteq W$ (V is a subset of W), then $\dim V \leq \dim W$.
5. Let V be a k-dimensional space. Then,
   - every linearly independent set has $\leq k$ vectors;
   - every spanning set has $\geq k$ vectors.
6. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in a k-dimensional space V.

   $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ if and only if $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

   Namely, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a basis iff they are linearly independent.

Proofs

1. Every subspace V of $\mathbb{R}^n$ has a basis and therefore a dimension. (See the slides on Tuesday that $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$)
2. Every linearly independent set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r \in V$ can be **extended** to a basis.
   Proof. By (1), V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_s$. Apply the casting-out algorithm to $\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s$.
   - The output form a basis for $span\{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\} = span\{\mathbf{v}_1, \ldots, \mathbf{v}_s\} = V$;
   - $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are linearly independent, so they "survived".
3. Every spanning set $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of V can be **shrunk** to a basis of V.
   Proof. Casting-out algorithm removes redundant vectors.
4. If V and W are subspaces of $\mathbb{R}^n$ with $V \subseteq W$ then $\dim V \leq \dim W$.
   Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis of V. (2) allows us to **extend** it to a basis of W. Thus, $\dim V \leq \dim W$.
5. Let V be a k-dimensional space. Then,
   - every linearly independent set has $\leq k$ vectors;
   - every spanning set has $\geq k$ vectors.

     Proof.
     - Apply the first property of bases: Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are linearly independent elements of $span\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then, $r \leq k$.
     - If $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ with $r < k$, then $\dim V \leq r < k$, which is a contradiction.

Proofs

6. Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be vectors in a k-dimensional space V.

$$V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \text{ if and only if } \mathbf{u}_1, \ldots, \mathbf{u}_k \text{ are linearly independent.}$$

Namely, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a basis iff they are linearly independent.

Proof.

We show both directions.

- If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, by (2) we **extend** it to a basis of V. $\dim V = k = \#$ of elements in the basis, $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is A basis. $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.
- If $V = span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, (3) allows us to **shrunk** to a basis of V, containing k elements. . No one is redundant. $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

$\square$

Example

Extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis of $\mathbb{R}^4$, where $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$.

Solution. Recall the standard basis of $\mathbb{R}^4$ that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. We apply the casing-out algorithm to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 - 2R_1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{R_3 \leftarrow R_3 + R_2 \\ R_4 \leftarrow R_4 - 2R_2}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

Basis: $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_2, \mathbf{e}_3\}$

$$\overset{R_4 \leftarrow R_4 + 2R_3}{\sim} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Example

Let $V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \, x + 2y + z - w = 0 \right\}$. Note $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in V$.
Extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to a basis of V.
Solution. First of all, solve for V. $[1\ 2\ 1\ -1 \mid 0]$ $y = t$, $z = s$, $w = r$
$x = -2t - s + r$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} + r \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_3}$$

Casting-out algorithm is applied to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

$$\begin{bmatrix} 1 & -2 & -2 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis: $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$

Column, row and null spaces of a matrix

Definition

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be an $m \times n$-matrix.

- The *column space* of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

$$col(A) = span \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}.$$

- The *row space* of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

$$row(A) = span \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \end{bmatrix}^T, \begin{bmatrix} a_{21} & \cdots & a_{2n} \end{bmatrix}^T, \cdots, \begin{bmatrix} a_{m1} & \cdots & a_{mn} \end{bmatrix}^T \right\}.$$

- The *null space* of A is the subspace of $\mathbb{R}^n$ given by the solutions to the homogeneous system $A\mathbf{v} = \mathbf{0}$. $null(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}\}$.

Column, row and null spaces of a matrix

Theorem

If A is an $m \times n$ matrix of rank r, then, $dim(col(A)) = dim(row(A)) = r$, and $dim(null(A)) = n - r$.

Definition

The dimension of the null space is also called the *nullity* of A, i.e., $nullity(A) = dim(null(A))$.

Theorem (Rank-Nullity Theorem)

For an $m \times n$ matrix A: $rank(A) + nullity(A) = n$.

Example

Find a basis and the dimension of each of (a) the column space, (b) the row space (c) and the null space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 3 & 2 & 5 & 4 & 7 \\ 5 & 0 & 5 & 5 & 10 \end{bmatrix}$$

Example

Find the reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 3 & 2 & 5 & 4 & 7 \\ 5 & 0 & 5 & 5 & 10 \end{bmatrix} \overset{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 5R_1}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 0 & -4 & -4 & -2 & -2 \\ 0 & -10 & -10 & -5 & -5 \end{bmatrix}$$

$$\overset{\substack{R_2 \leftarrow -\frac{1}{4} R_2 \\ R_3 \leftarrow -\frac{1}{10} R_3}}{\sim} \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\overset{\substack{R_3 \leftarrow R_3 - R_2 \\ R_1 \leftarrow R_1 - 2R_2}}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Column space: pivot vectors are the first two columns.

$$\text{Basis} : \left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\} \qquad dim(col(A)) = rank(A) = 2.$$

2. Row space: pivot vectors are the first two rows.

$$\text{Basis} : \left\{ \begin{bmatrix} 1 & 2 & 3 & 2 & 3 \end{bmatrix}^T, \begin{bmatrix} 3 & 2 & 5 & 4 & 7 \end{bmatrix}^T \right\} \qquad dim(row(A)) = 2.$$

Example

3. Null space: $A\mathbf{v} = \mathbf{0}$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 3 & 0 \\ 3 & 2 & 5 & 4 & 7 & 0 \\ 5 & 0 & 5 & 5 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} x_3 = s \\ x_4 = t \implies \\ x_5 = r \end{array} \qquad \begin{array}{l} x_1 = -x_3 - x_4 - 2x_5 = -s - t - 2r \\ x_2 = -x_3 - \frac{1}{2}x_4 - \frac{1}{2}x_5 = -s - \frac{1}{2}t - \frac{1}{2}r \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad dim(null(A)) = 3.$

Remark

Be careful when interchanging rows.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for the row space: $\{R_1, R_3, R_4\}$.

Midterm II

- There will be 20 problems.
- The True/False problems will NOT require explanations.
- True/False, Multiple choice, Short answer

Remark

Please be aware of that: in the final exam, short answer problems will be replaced by detailed answer problems. Namely, to obtain full credits, you need to show all the steps, i.e., only providing the final answer is NOT sufficient.

Summary of Chpt. 5

$\boxed{\text{span}}$ $\boxed{\text{linearly (in)dependent vectors}}$ $\boxed{\text{basis}}$

1. span: set of linear combinations *span* $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$
   Natural question: Do we need all the $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$?
   To answer: casting-out algorithm By casting out the *redundant* vectors, the remaining vectors are linearly independent.
   Every subspace is a span of a set of vectors.
2. linearly (in-)dependent vectors: Besides (extended) casting-out algorithm, we have another criterion to determine whether a set of vectors. The corresponding homogeneous system:

$$a_1\mathbf{u}_1 + \cdots + a_\ell\mathbf{u}_\ell = \mathbf{0}.$$

   check whether it ONLY has the trivial solution.
3. basis: a set of linearly independent vectors, whose span gives the subspace
   - The number of vectors in a basis is fixed. dimension of a subspace
   - ***Shrink*** and ***extend*** to obtain a basis.
   - Bases for the row, column and null spaces of a matrix

Summary of Chpt. 5

There are lots of examples in this chapter. The key is the elementary row operation.

- Read the slides/textbook carefully, by focusing on the examples.
- Make sure, for different type of problems, you understand the steps & reasons.

MATH-1030
Matrix Theory & Linear Algebra I

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June 4th, 2019

Vector functions

Example (Function $y = f(x)$)

$$f(x) = x^2 \Longrightarrow f(0) = 0,\ f(1) = 1,\ f(2) = 4, \dots$$

Example (Vector function $\mathbf{w} = f(\mathbf{v})$, with both input and output vectors)

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + z \\ y^2 - xz \end{bmatrix} \Longrightarrow f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 + 0 \\ 0^2 - 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 + 0 \\ 1^2 - 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 + 1 \\ 1^2 - 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Examples

Remark

If f inputs an n-dimensional vector and outputs an m-dimensional vector, we write

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

Example

(a) The example above is $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ y^2 - xz \end{bmatrix}$$

(b)

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ 2y \end{bmatrix}$$

is an example $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Linear Transformation

Recall that at the very beginning of this class, we quickly introduced linear equations and non-linear equations. We shall also classify the vector functions.

Definition

A vector function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called *linear* or a *linear transformation* if

1. $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$;
2. $T(k\mathbf{v}) = kT(\mathbf{v})$.

Namely, T is linear iff T preserves the addition and scalar multiplication of the vectors. Note that the addition and scalar multiplication on both sides of the two identities are different, since the vectors on both sides usually do not have the same dimension. $\quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \quad T(\mathbf{v}), T(\mathbf{w}) \in \mathbb{R}^m$

Example

Which of the following vector functions are linear?

$$(a)\ T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2z \\ y - z \end{bmatrix},\ (b)\ T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ x - z \end{bmatrix},\ (c)\ T_3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ yz \end{bmatrix}$$

Solution

(c) $T_3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ yz \end{bmatrix}$ is obviously not linear since it involves x^2 and yz.

$$T_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \\ 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ T_3\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.\ \left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

However, $2T_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 0 \end{bmatrix} = T_3\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

It violates the second condition. Namely, the scalar multiplication is not preserved.

(a) $T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2z \\ y - z \end{bmatrix}$ is linear. Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{w} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \in \mathbb{R}^3$ and $k \in \mathbb{R}$.

We need to check

- $T_1(\mathbf{v} + \mathbf{w}) \overset{???}{=} T_1(\mathbf{v}) + T_1(\mathbf{w})$
- $T_1(k\mathbf{v}) \overset{???}{=} kT_1(\mathbf{v})$

Example

$$(a)\ T_1\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+2z\\y-z\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad \mathbf{w}=\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$$

- Check $T_1(\mathbf{v}+\mathbf{w}) \overset{???}{=} T_1(\mathbf{v})+T_1(\mathbf{w})$;

$$T_1(\mathbf{v})=T_1\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}x+2z\\y-z\end{bmatrix} \qquad T_1(\mathbf{w})=\begin{bmatrix}x'+2z'\\y'-z'\end{bmatrix}$$

$$\Longrightarrow T_1(\mathbf{v})+T_1(\mathbf{w})=\begin{bmatrix}x+x'+2z+2z'\\y'+y'-z-z'\end{bmatrix}$$

On the other hand,

$$\mathbf{v}+\mathbf{w}=\begin{bmatrix}x+x'\\y+y'\\z+z'\end{bmatrix}\Rightarrow T_1(\mathbf{v}+\mathbf{w})=\begin{bmatrix}x+x'+2(z+z')\\y+y'-(z+z')\end{bmatrix}=\begin{bmatrix}x+x'+2z+2z'\\y'+y'-z-z'\end{bmatrix}$$

$$\Longrightarrow T_1(\mathbf{v})+T_1(\mathbf{w})=T_1(\mathbf{v})+T_1(\mathbf{w})$$

Example

$$(a)\ T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2z \\ y-z \end{bmatrix},\ \ \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ \ \mathbf{w} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

- Check $T_1(k\mathbf{v}) \overset{???}{=} kT_1(\mathbf{v})$;

$$T_1(\mathbf{v}) = \begin{bmatrix} x+2z \\ y-z \end{bmatrix} \qquad kT_1(\mathbf{v}) = \begin{bmatrix} k(x+2z) \\ k(y-z) \end{bmatrix} = \begin{bmatrix} kx+2kz \\ ky-kz \end{bmatrix}$$

Meanwhile,

$$k\mathbf{v} = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix} \Longrightarrow T_1(k\mathbf{v}) = \begin{bmatrix} kx+2kz \\ ky-kz \end{bmatrix} = kT_1(\mathbf{v})$$

Remark

It is not surprising that both components of $T_1(x+2z, y-z)$ are linear in x, y, and z. If a vector function is linear, all its components are linear functions.

Example

$$(b)\ T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ x - z \end{bmatrix}$$

Apparently, both components of T_2 are linear functions. However, T_2 is not linear, due to the constant 2.

$$T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad T_2\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow 2T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \neq T_2\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

In general, due to this 2, $T_2(\mathbf{v}) = \begin{bmatrix} 2 \\ * \end{bmatrix}$ and $T_2(k\mathbf{v}) = \begin{bmatrix} 2 \\ ** \end{bmatrix}$ so that

$T_2(\mathbf{v}) \neq T_2(k\mathbf{v})$.

Remark

For a vector function, even if all the components are linear functions, this does guarantee the function to be linear. (KEY: Without constant terms)

Linear Transformation

Lemma

Every linear transformation T satisfies $T(\mathbf{0}) = \mathbf{0}$.

Example

Recall $T_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ x - z \end{bmatrix}$. $T_2(\mathbf{0}) = T_2\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \mathbf{0}$.

Proof.

Let $\mathbf{v}$ be any vector in $\mathbb{R}^n$. Then, $T(\mathbf{v}) \in \mathbb{R}^m$

$$\mathbf{0} = 0 \cdot T(\mathbf{v}) \overset{(2)}{=} T(0 \cdot \mathbf{v}) = T(\mathbf{0})$$

□

Remark

$\boxed{\text{This is only necessary, but not sufficient.}}$ Recall $T_3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ yz \end{bmatrix} \Rightarrow T_3\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ but NOT linear

Matrix transformation

Theorem

Let A be an $m \times n$ matrix. The function

$$\begin{aligned} T : \mathbb{R}^n &\rightarrow \mathbb{R}^m \\ \mathbf{v} &\mapsto A\mathbf{v} \qquad (T(\mathbf{v}) = A\mathbf{v}) \end{aligned}$$

is always linear. Such a function is called a matrix transformation.

Proof.

Just check the two conditions.

1. $T(\mathbf{v}+\mathbf{w}) = A(\mathbf{v}+\mathbf{w}) = A\mathbf{v} + A\mathbf{w} = T(\mathbf{v}) + T(\mathbf{w})$
2. $T(k\mathbf{v}) = A(k\mathbf{v}) = kA\mathbf{v} = kT(\mathbf{v})$

$\square$

Remark

The dimensions are compatible.

- $\mathbf{v} \in \mathbb{R}^n$ — an $n \times 1$ matrix;
- $A_{m\times n}$ —$A\mathbf{v}$ is well defined and has dimension $m \times 1$ — a vector in $\mathbb{R}^m$

Example

Recall the previous example

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2z \\ y-z \end{bmatrix}$$

It is actually a matrix transformation. $T_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$T_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2z \\ y-z \end{bmatrix}_{2\times 1} = \begin{bmatrix} \quad \end{bmatrix}_{{\color{blue}2}\times{\color{red}3}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition

Every linear transformation is a matrix transformation. (Proof comes later)

Remark

Given a vector function, if all the components are linear without the constant term, we can write it as a matrix transformation. Therefore, it is linear.

Example

Let $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x-2w+z \\ x-y \\ 2y+z \end{bmatrix}$. *Is* T *linear? If yes, write it as a matrix transformation.*

Solution. All the three components are linear without the constant term. So, T is linear.

Since $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$, we should find a 3×4 matrix A such that $T(\mathbf{v}) = A\mathbf{v}$.

$$\begin{bmatrix} x-2w+z \\ x-y \\ 2y+z \end{bmatrix} = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -y \\ 2y \end{bmatrix} + \begin{bmatrix} z \\ 0 \\ z \end{bmatrix} + \begin{bmatrix} -2w \\ 0 \\ 0 \end{bmatrix}$$

$$= x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Linear combination

Remark

If T is a linear transformation, then T preserves not only addition and scalar multiplication, but all linear combinations:

$$\begin{aligned} T\left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k\right) &\overset{(1)}{=} T\left(a_1\mathbf{v}_1\right) + T\left(a_2\mathbf{v}_2\right) + \cdots + T\left(a_k\mathbf{v}_k\right) \\ &\overset{(2)}{=} a_1 T\left(\mathbf{v}_1\right) + a_2 T\left(\mathbf{v}_2\right) + \cdots + a_k T\left(\mathbf{v}_k\right) \end{aligned}$$

Example

Assume $T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T(\mathbf{u}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $T(\mathbf{u}_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $T\left(4\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3\right)$.

Solution.

$$\begin{aligned} T\left(4\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3\right) &= 4T\left(\mathbf{u}_1\right) + 2T\left(\mathbf{u}_2\right) - T\left(\mathbf{u}_3\right) \\ &= 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{aligned}$$

Example

Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be linear. If

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

find $T\left(\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}\right)$.

Solution. For simplicity, we let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. We first need to express $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

Example

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 2 & 3 & 1 & 1 \\ 1 & 3 & 1 & 4 \end{array}\right] \overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & -7 & -5 \\ 1 & 3 & 1 & 4 \end{array}\right] \overset{R_3 \leftarrow R_3 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & -7 & -5 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & -7 & -5 \\ 0 & 0 & 11 & 11 \end{array}\right] \overset{R_3 \leftarrow \frac{1}{11} R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & -7 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 + 7R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \overset{R_1 \leftarrow R_1 - 4R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 - R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \mathbf{v} = -3\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$$

Example

$$\mathbf{v} = -3\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3$$

$$T(\mathbf{u}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad T(\mathbf{u}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{aligned} T(\mathbf{v}) &= T(-3\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3) \\ &= -3T(\mathbf{u}_1) + 2T(\mathbf{u}_2) + T(\mathbf{u}_3) \\ &= -3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \end{aligned}$$

What will happen if $\mathbf{v}$ is NOT a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$?
[Answer]: Then, we do not know $T(\mathbf{v})$.

Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$$

and a linear transformation T satisfy

$$T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Find $T(\mathbf{u})$.

Lecture resumes at 7:45.

Solution

$$\mathbf{v}_1 = \begin{bmatrix}2\\2\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\3\\3\end{bmatrix}, \mathbf{u} = \begin{bmatrix}3\\1\\7\end{bmatrix}, T(\mathbf{v}_1) = \begin{bmatrix}1\\1\end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix}0\\2\end{bmatrix}, T(\mathbf{v}_3) = \begin{bmatrix}2\\2\end{bmatrix}$$

First of all, we try to write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Namely, find a_1, a_2, a_3 such that $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$.

$$\begin{bmatrix}3\\1\\7\end{bmatrix} = a_1\begin{bmatrix}2\\2\\0\end{bmatrix} + a_2\begin{bmatrix}1\\0\\1\end{bmatrix} + a_3\begin{bmatrix}1\\3\\3\end{bmatrix} = \begin{bmatrix}2&1&1\\2&0&3\\0&1&3\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}2&1&1&3\\2&0&3&1\\0&1&3&7\end{array}\right] \underset{\sim}{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c}2&1&1&3\\0&-1&2&-2\\0&1&3&7\end{array}\right] \underset{\sim}{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|c}2&1&1&3\\0&-1&2&-2\\0&0&5&5\end{array}\right]$$

$$\underset{\sim}{R_3 \leftarrow \tfrac{1}{5}R_3} \left[\begin{array}{ccc|c}2&1&1&3\\0&-1&2&-2\\0&0&1&1\end{array}\right] \underset{\sim}{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|c}2&1&1&3\\0&-1&0&-4\\0&0&1&1\end{array}\right] \underset{\sim}{R_2 \leftarrow -R_2} \left[\begin{array}{ccc|c}2&1&1&3\\0&1&0&4\\0&0&1&1\end{array}\right]$$

$$\underset{\sim}{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c}2&0&1&-1\\0&1&0&4\\0&0&1&1\end{array}\right] \underset{\sim}{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|c}2&0&0&-2\\0&1&0&4\\0&0&1&1\end{array}\right] \underset{\sim}{R_1 \leftarrow \tfrac{1}{2}R_1} \left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&4\\0&0&1&1\end{array}\right]$$

Solution

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}, T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{u} = -\mathbf{v}_1 + 4\mathbf{v}_2 + \mathbf{v}_3 \Rightarrow T(\mathbf{u}) &= T(-\mathbf{v}_1 + 4\mathbf{v}_2 + \mathbf{v}_3) \\ &= -T(\mathbf{v}_1) + 4T(\mathbf{v}_2) + T(\mathbf{v}_3) \\ &= -\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 9 \end{bmatrix} \end{aligned}$$

Next, we study several properties.

Basis

Theorem

A linear transformation T is ***completely*** *determined by its action on a basis.*

Example

An unknown linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}.$$

Find a formula for T. (formula=matrix transformation form)
Solution.

$$\begin{aligned} T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) &= T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = xT\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + zT\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \\ &= x\begin{bmatrix}2\\3\end{bmatrix} + y\begin{bmatrix}1\\2\end{bmatrix} + z\begin{bmatrix}-1\\1\end{bmatrix} = \boxed{\begin{bmatrix}2 & 1 & -1\\3 & 2 & 1\end{bmatrix}}\begin{bmatrix}x\\y\\z\end{bmatrix} \end{aligned}$$

Let $A = \begin{bmatrix}2 & 1 & -1\\3 & 2 & 1\end{bmatrix}$. Then, $T(\mathbf{v}) = A\mathbf{v}$.

Matrix transformation

Theorem

Every linear transformation is a matrix transformation.

Proof.

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation. Recall the **standard basis** of $\mathbb{R}^n$, i.e, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Now, denote

$$\mathbf{w}_1 = T(\mathbf{e}_1), \ \mathbf{w}_2 = T(\mathbf{e}_2), \ \ldots, \mathbf{w}_n = T(\mathbf{e}_n).$$

Form $A = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}$, i.e., A is the matrix with $\mathbf{w}_1, \ldots, \mathbf{w}_n$ being its columns and dimension $m \times n$ ($\mathbf{w}_k \in \mathbb{R}^m$). We shall show that $T(\mathbf{v}) = A\mathbf{v}$.

For all $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow T(\mathbf{v}) = T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1\mathbf{w}_1 + \cdots + x_n\mathbf{w}_n$

This implies

$$T(\mathbf{v}) = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{v}.$$

□

Example

Revisit the class work

$$\mathbf{v}_1=\begin{bmatrix}2\\2\\0\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}1\\0\\1\end{bmatrix},\mathbf{v}_3=\begin{bmatrix}1\\3\\3\end{bmatrix},\mathbf{u}=\begin{bmatrix}3\\1\\7\end{bmatrix},T(\mathbf{v}_1)=\begin{bmatrix}1\\1\end{bmatrix},T(\mathbf{v}_2)=\begin{bmatrix}0\\2\end{bmatrix},T(\mathbf{v}_3)=\begin{bmatrix}2\\2\end{bmatrix}$$

1. Is $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ a basis for $\mathbb{R}^3$? Check whether they are linearly independent:
$$\begin{bmatrix}2&1&1\\2&0&3\\0&1&3\end{bmatrix}\sim\sim\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left(\left[\begin{array}{ccc|c}2&1&1&3\\2&0&3&1\\0&1&3&7\end{array}\right]\sim\left[\begin{array}{ccc|c}1&0&0&-1\\0&1&0&4\\0&0&1&1\end{array}\right]\right)$$
They are linearly independent and form a basis for $\mathbb{R}^3$. $\dim\mathbb{R}^3=3$.
2. Can we find a formula for T? To apply the same method, we need to know $T(\mathbf{e}_1),T(\mathbf{e}_2),T(\mathbf{e}_3)$. Since $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis, we can express $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ as linear combinations of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$. Find a, b, c such that

$$\mathbf{e}_1=a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_3\Leftrightarrow\begin{bmatrix}2&1&1\\2&0&3\\0&1&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}\Leftrightarrow\left[\begin{array}{ccc|c}2&1&1&1\\2&0&3&0\\0&1&3&0\end{array}\right]$$

Example

Similarly, find d, e, f such that

$$\mathbf{e}_2 = d\mathbf{v}_1 + e\mathbf{v}_2 + f\mathbf{v}_3 \Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 2 & 0 & 3 & 1 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

and g, h, i such that

$$\mathbf{e}_3 = g\mathbf{v}_1 + h\mathbf{v}_2 + i\mathbf{v}_3 \Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

Observe that (or, trying to solve three systems together)

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = I_3$$

Find

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}^{-1}$$

Example

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \overset{R_2 \leftarrow R_2 - R_1}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 + R_2}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 5 & -1 & 1 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow \frac{R_3}{5}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$\overset{\substack{R_2 \leftarrow R_2 - 2R_3 \\ R_1 \leftarrow R_1 - R_3}}{\sim} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -\frac{3}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 + R_2 - R_3}{\sim} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & -1 & 0 & -\frac{3}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$\overset{\substack{R_2 \leftarrow -R_2 \\ R_1 \leftarrow \frac{1}{2} R_1}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{10} & \frac{1}{5} & -\frac{3}{10} \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

Example

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{5} & -\frac{3}{10} \\ \frac{3}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{aligned} \mathbf{e}_1 &= \frac{3}{10}\mathbf{v}_1 + \frac{3}{5}\mathbf{v}_2 - \frac{1}{5}\mathbf{v}_3 \\ \mathbf{e}_2 &= \frac{1}{5}\mathbf{v}_1 - \frac{3}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_3 \\ \mathbf{e}_3 &= -\frac{3}{10}\mathbf{v}_1 + \frac{2}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_3 \end{aligned} \Longrightarrow \begin{aligned} T(\mathbf{e}_1) &= \frac{3}{10}T(\mathbf{v}_1) + \frac{3}{5}T(\mathbf{v}_2) - \frac{1}{5}T(\mathbf{v}_3) \\ T(\mathbf{e}_2) &= \frac{1}{5}T(\mathbf{v}_1) - \frac{3}{5}T(\mathbf{v}_2) + \frac{1}{5}T(\mathbf{v}_3) \\ T(\mathbf{e}_3) &= -\frac{3}{10}T(\mathbf{v}_1) + \frac{2}{5}T(\mathbf{v}_2) + \frac{1}{5}T(\mathbf{v}_3) \end{aligned}$$

$$\Longrightarrow \begin{aligned} T(\mathbf{e}_1) &= \frac{3}{10}\begin{bmatrix}1\\1\end{bmatrix} + \frac{3}{5}\begin{bmatrix}0\\2\end{bmatrix} - \frac{1}{5}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{10}\\ \frac{11}{10}\end{bmatrix} \\ T(\mathbf{e}_2) &= \frac{1}{5}\begin{bmatrix}1\\1\end{bmatrix} - \frac{3}{5}\begin{bmatrix}0\\2\end{bmatrix} + \frac{1}{5}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\ -\frac{3}{5}\end{bmatrix} \\ T(\mathbf{e}_3) &= -\frac{3}{10}\begin{bmatrix}1\\1\end{bmatrix} + \frac{2}{5}\begin{bmatrix}0\\2\end{bmatrix} + \frac{1}{5}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}\frac{1}{10}\\ \frac{9}{10}\end{bmatrix} \end{aligned}$$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_1) = \begin{bmatrix} -\frac{1}{10} \\ \frac{11}{10} \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} \frac{1}{10} \\ \frac{9}{10} \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3) = x\begin{bmatrix} -\frac{1}{10} \\ \frac{11}{10} \end{bmatrix} + y\begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix} + z\begin{bmatrix} \frac{1}{10} \\ \frac{9}{10} \end{bmatrix}$$

$$S = \begin{bmatrix} -\frac{1}{10} & \frac{3}{5} & \frac{1}{10} \\ \frac{11}{10} & -\frac{3}{5} & \frac{9}{10} \end{bmatrix} \Rightarrow T(\mathbf{v}) = S\mathbf{v}$$

Example

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_1) = \begin{bmatrix} -\frac{1}{10} \\ \frac{11}{10} \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} \frac{1}{10} \\ \frac{9}{10} \end{bmatrix}, \quad S = \begin{bmatrix} -\frac{1}{10} & \frac{3}{5} & \frac{1}{10} \\ \frac{11}{10} & -\frac{3}{5} & \frac{9}{10} \end{bmatrix}$$

Recall the steps $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{5} & -\frac{3}{10} \\ \frac{3}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$

$$T(\mathbf{e}_1) = \frac{3}{10}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_2) = \frac{1}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \quad S = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} A^{-1}$$

$$T(\mathbf{e}_3) = -\frac{3}{10}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

by the column method.

Theorem

Theorem (Exercise 6.2.2)

Assume that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis of $\mathbb{R}^n$. For all $i = 1, \ldots, n$, let $\mathbf{v}_i = T(\mathbf{u}_i)$. Let A be the matrix that has $\mathbf{u}_1, \ldots, \mathbf{u}_n$ as its columns, and let B be the matrix that has $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as its columns. Show that A is invertible and the matrix of T is BA^{-1}.

Example

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, T(\mathbf{u}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{u}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, T(\mathbf{u}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \text{is invertible since} \\ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \text{ is a basis} \end{matrix} \Rightarrow \text{linearly independent} \Rightarrow rank(A) = 3$$

$$A^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{5} & -\frac{3}{10} \\ \frac{3}{5} & -\frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}, S = BA^{-1} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{5} & \frac{1}{10} \\ \frac{11}{10} & -\frac{3}{5} & \frac{9}{10} \end{bmatrix}$$

Composition

For functions, we can compose two functions: $f(x) = x^2$, $g(x) = x + 1$

$$g \circ f(x) = g(f(x)) = g\left(x^2\right) = x^2 + 1$$

Similarly, consider two linear transformations:

$$\begin{array}{rccc} T: & \mathbb{R}^m & \to & \mathbb{R}^n \\ & \mathbf{v} & \mapsto & A\mathbf{v} \end{array} \qquad \begin{array}{rc} S: & \mathbb{R}^k \to \mathbb{R}^m \\ & \mathbf{w} \mapsto B\mathbf{w} \end{array}$$

We can compose T and S and get that

$$\begin{array}{rcl} T \circ S : \mathbb{R}^k & \to & \mathbb{R}^n \\ \mathbf{w} & \mapsto & T(S(\mathbf{w})) = T(B\mathbf{w}) = A(B\mathbf{w}) = (AB)\mathbf{w} \end{array}$$

So the **matrix multiplication** originally was to solve the question:
How can we compose two linear transformations?

Inverse

Definition

Let $T, S : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Suppose that for each $\mathbf{v} \in \mathbb{R}^n$,

$$(S \circ T)(\mathbf{v}) = \mathbf{v}$$

and

$$(T \circ S)(\mathbf{v}) = \mathbf{v}.$$

Then S is called the inverse of T, and we write $S = T^{-1}$.

Theorem (Theorem 6.22: Matrix of the inverse transformation)

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the corresponding $n \times n$-matrix. Then T has an inverse if and only if the matrix A is invertible. In this case, the matrix of T^{-1} is A^{-1}.

Example

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, T(\mathbf{v}_3) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$S = \begin{bmatrix} -\frac{1}{10} & \frac{3}{5} & \frac{1}{10} \\ \frac{11}{10} & -\frac{3}{5} & \frac{9}{10} \end{bmatrix} = BA^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}^{-1}$$

We define

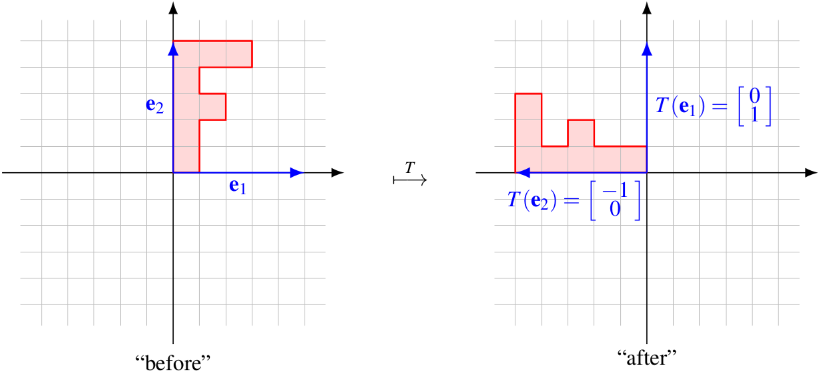
$$\begin{array}{rccc} T_1 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3, \\ & \mathbf{e}_1 & \mapsto & \mathbf{v}_1 \\ & \mathbf{e}_2 & \mapsto & \mathbf{v}_2 \\ & \mathbf{e}_3 & \mapsto & \mathbf{v}_3 \end{array} \qquad \begin{array}{rccc} T_2 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & \mathbf{e}_1 & \mapsto & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ & \mathbf{e}_2 & \mapsto & \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ & \mathbf{e}_3 & \mapsto & \begin{bmatrix} 2 \\ 2 \end{bmatrix} \end{array}$$

$$T_1(\mathbf{u}) = A\mathbf{u} \qquad T_2(\mathbf{w}) = B\mathbf{w}.$$

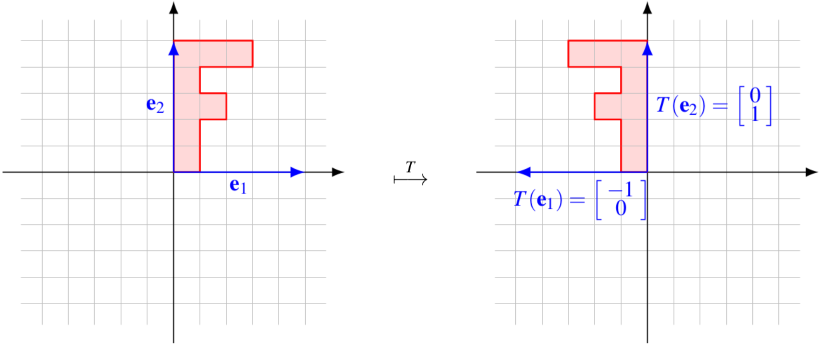
$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is another basis for $\mathbb{R}^3$, A is invertible (i.e., T_1 has an inverse map T_1^{-1} with matrix transformation of A^{-1})

$$T = T_2 \circ T_1^{-1} \Leftrightarrow S = BA^{-1}$$

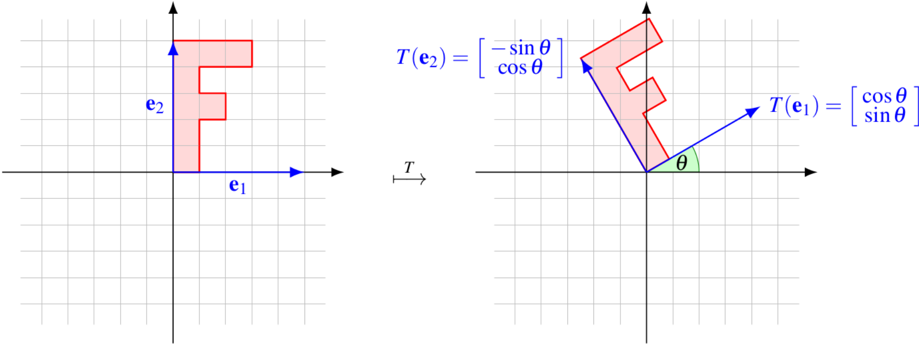
1. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation by 90°.



2. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflection about the y-axis



3. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ counterclockwise rotation by θ.



Summary of Chpt. 6

Linear Transformations:

1. definition: Trick: All components are linear without the constant term.
2. property: $T(\mathbf{0}) = \mathbf{0}$ preserves addition and scalar multiplication $\Leftrightarrow$ preserves linear combination
3. linear transformations are all matrix transformations: $T(\mathbf{v}) = A\mathbf{v}$
   - Given certain values, e.g., $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $T(\mathbf{v}_3)$, find $T(\mathbf{u})$ for $\mathbf{u} \in \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
   - Given $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$, $T(\mathbf{v}_3)$ on a basis, find the matrix of the transformation.
4. composition$\leftrightarrow$matrix multiplication
5. inverse

MATH-1030
Matrix Theory & Linear Algebra I

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Determinant

Goal

To an $n \times n$-matrix A, we will define a scalar $\det(A)$, called the *determinant* of A. In particular, we will use the determinant to determine some properties of a square matrix.

- invertible?
- rank?

Recall: An $n \times n$ matrix is invertible iff $\operatorname{rank}(A) = n$.

Claim

A is invertible iff $\det(A) \neq 0$.

We shall begin with the case $n = 2$.

Definition

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is defined by $\det(A) := ad - bc$

Examples

Example

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \Longrightarrow \det(A) = 2 \cdot 3 - 1 \cdot 0 = 6$$

$$B = \begin{bmatrix} -3 & 7 \\ 0 & 3 \end{bmatrix} \Longrightarrow \det(B) = (-3) \cdot 3 - 7 \cdot 0 = -9$$

$$C = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} \Longrightarrow \det(C) = 3 \cdot 8 - 6 \cdot 4 = 24 - 24 = 0.$$

By the elementary row operations, we see $C \overset{R_1 \leftarrow \frac{1}{3} R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \overset{R_2 \leftarrow R_2 - 4R_1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Remark

These three examples above shows that the determinant can be positive, negative and zero. Also, both A and B are in echelon forms, so $rank(A) = rank(B) = 2$, while $rank(C) = 1 < 2$. Please recall for dot product: $\mathbf{u} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{u}$ and $\mathbf{v}$ are orthogonal/perpendicular. Later on we shall see $\det(A) = 0 \Leftrightarrow rank(A) < n$ for an $n \times n$ matrix A.

Examples

Remark

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \Longrightarrow \det(A) = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 2 \cdot 3 - 1 \cdot 0 = 6$$

$$B = \begin{bmatrix} -3 & 7 \\ 0 & 3 \end{bmatrix} \Longrightarrow \det(B) = \begin{vmatrix} -3 & 7 \\ 0 & 3 \end{vmatrix} = (-3) \cdot 3 - 7 \cdot 0 = -9$$

$$C = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} \Longrightarrow \det(C) = \begin{vmatrix} 3 & 6 \\ 4 & 8 \end{vmatrix} = 3 \cdot 8 - 6 \cdot 4 = 24 - 24 = 0.$$

3×3

Definition

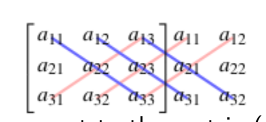
For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant is defined by

$$\det(A) := a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

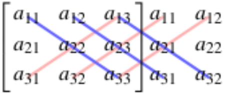
Of course, it is almost impossible to memorize.

Remark

Here is a picture for help.



1. Copy the first two columns next to the matrix (right-hand side)
2. The blue lines correspond to the positive terms of the determinant: $a_{11}a_{22}a_{33}$, $a_{12}a_{23}a_{31}$, and $a_{13}a_{21}a_{32}$.
3. The red lines correspond to the negative terms: $a_{31}a_{22}a_{13}$, $a_{32}a_{23}a_{11}$, and $a_{33}a_{21}a_{12}$



$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{matrix} 0 & 1 \\ 3 & 1 \\ 1 & 1 \end{matrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} \\ &= 0 \cdot 1 \cdot (-1) + 2 \cdot 0 \cdot 1 + 2 \cdot 3 \cdot 1 - (2 \cdot 1 \cdot 1 + 0 \cdot 0 \cdot 1 + 1 \cdot 3 \cdot (-1)) \\ &= 6 - (-1) \\ &= 7 \end{aligned}$$

Cross product

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}.$$

Recall the right-hand side system $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Theorem

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & v_1 & \mathbf{i} \\ u_2 & v_2 & \mathbf{j} \\ u_3 & v_3 & \mathbf{k} \end{vmatrix}.$$

Namely, $\mathbf{u} \times \mathbf{v} = \det(A)$ *for* $A = \begin{bmatrix} u_1 & v_1 & \mathbf{i} \\ u_2 & v_2 & \mathbf{j} \\ u_3 & v_3 & \mathbf{k} \end{bmatrix}$

Cross product

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \quad A = \begin{bmatrix} u_1 & v_1 & \mathbf{i} \\ u_2 & v_2 & \mathbf{j} \\ u_3 & v_3 & \mathbf{k} \end{bmatrix} \quad \left| \begin{array}{ccc|cc} u_1 & v_1 & \mathbf{i} & u_1 & v_1 \\ u_2 & v_2 & \mathbf{j} & u_2 & v_2 \\ u_3 & v_3 & \mathbf{k} & u_3 & v_3 \end{array} \right.$$

$$\begin{aligned} \det(A) &= u_1 v_2 \mathbf{k} + v_1 \mathbf{j} u_3 + \mathbf{i} u_2 v_3 - \mathbf{i} v_2 u_3 - u_1 \mathbf{j} v_3 - v_1 u_2 \mathbf{k} \\ &= (u_2 v_3 - v_2 u_3)\mathbf{i} + (v_1 u_3 - u_1 v_3)\mathbf{j} + (u_1 v_2 - v_1 u_2)\mathbf{k} \\ &= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \end{aligned}$$

Triangular matrices

Definition

A square matrix $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ is *upper triangular* if $a_{ij} = 0$ whenever $i > j$.

In other words, a matrix is upper triangular if entries below the main diagonal are 0.

Similarly, a square matrix is *lower triangular* if all entries above the main diagonal are 0.

Let $*$ refers to any non-zero numbers:

$$\underbrace{\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}}_{\text{upper triangular}} \qquad \underbrace{\begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}}_{\text{lower triangular}}$$

Triangular matrices

Theorem

For an $n \times n$ triangular matrix A, the determinant is equal to the product of the entries on the main diagonal. Namely,

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 16 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 3 & 33 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Longrightarrow \det(A) = \begin{vmatrix} 1 & 2 & 3 & 16 \\ 0 & 2 & 6 & -7 \\ 0 & 0 & 3 & 33 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdot (-1) = -6.$$

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 4 & 1 & -2 & 0 \\ 10000000 & 8000 & 9 & 2 \end{vmatrix} = (-1) \cdot 4 \cdot (-2) \cdot 2 = 16$$

Row operations

Remark

Be careful that matrices of the following forms are not triangular matrices:

$$\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

Recall the three elementary row operations:

$$R_i \longleftrightarrow R_j, \; R_j \longleftarrow aR_j, \; R_j \longleftarrow R_j + aR_i \quad (i \neq j, a \neq 0)$$

Theorem

Let A be an $n \times n$-matrix.

1. *If B is obtained from A by switching two rows, then* $\det(B) = -\det(A)$;
2. *If B is obtained from A by multiplying one row by a non-zero scalar k, then* $\det(B) = k\det(A)$;
3. *If B is obtained from A by adding a multiple of one row to another row, then* $\det(B) = \det(A)$;

Example

$$\begin{cases} A \overset{R_i \longleftrightarrow R_j}{\sim} B & \Rightarrow \det(B) = -\det(A) \\ A \overset{R_j \longleftarrow kR_j}{\sim} B & \Rightarrow \det(B) = k\det(A) \\ A \overset{R_j \longleftarrow R_j + aR_i}{\sim} B & \Rightarrow \det(B) = \det(A) \end{cases}$$

Let $A = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix}$, find $\det(A)$.

1. We see

$$\begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 8 & 1 \\ 2 & 4 & -8 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\sim} \begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & -4 & -3 \end{bmatrix} \overset{R_3 \leftarrow R_3 - \frac{1}{3}R_2}{\sim} \begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\det(A) = {\color{red}-} \begin{vmatrix} 1 & 8 & 1 \\ 2 & 4 & -8 \\ 1 & 4 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 1 & 4 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & -4 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & 0 & \frac{1}{3} \end{vmatrix}$$

Example

$$\begin{cases} A \overset{R_i \longleftrightarrow R_j}{\sim} B & \Rightarrow \det(B) = -\det(A) \\ A \overset{R_j \longleftarrow kR_j}{\sim} B & \Rightarrow \det(B) = k\det(A) \\ A \overset{R_j \longleftarrow R_j + aR_i}{\sim} B & \Rightarrow \det(B) = \det(A) \end{cases} \qquad A = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\det(A) = -\begin{vmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = -1 \cdot (-12) \cdot \frac{1}{3} = 4$$

2. Alternatively,

$$\begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_2 \leftarrow \frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

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$$\begin{vmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \frac{1}{2}\det(A). \Rightarrow \det(A) = 2\begin{vmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \cdots = 2\begin{vmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 0 & 0 & -\frac{1}{3} \end{vmatrix} = 4$$

MATH-1030
Matrix Theory & Linear Algebra I

Lin Jiu

Dalhousie University

June 11th, 2019

Recall

The determinant can be considered as a function: $\det : \{\text{square matrices}\} \longrightarrow \mathbb{R}$.

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
  $-(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$
- Triangular: If A is upper/lower triangular, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Remark

Be careful that matrices of the following forms are not triangular matrices:

$$\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 0 \\ 3 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

MAIN DIAGONAL

Row operations

Recall the three elementary row operations:

$$R_i \longleftrightarrow R_j,\ R_j \longleftarrow aR_j,\ R_j \longleftarrow R_j + aR_i \quad (i \neq j, a \neq 0)$$

Theorem

Let A be an $n \times n$-matrix.

1. *If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$;*
2. *If B is obtained from A by multiplying one row by a non-zero scalar k, then $\det(B) = k\det(A)$;*
3. *If B is obtained from A by adding a multiple of one row to another row, then $\det(B) = \det(A)$;*

Example

Let $A = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix}$, find $\det(A)$.

$$\begin{cases} A \overset{R_i \longleftrightarrow R_j}{\sim} B & \Rightarrow \det(B) = -\det(A) \\ A \overset{R_j \longleftarrow kR_j}{\sim} B & \Rightarrow \det(B) = k\det(A) \\ A \overset{R_j \longleftarrow R_j + aR_i}{\sim} B & \Rightarrow \det(B) = \det(A) \end{cases}$$

1. We see

$$\underbrace{\begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix}}_{A} \overset{R_1 \leftrightarrow R_2}{\sim} \underbrace{\begin{bmatrix} 1 & 8 & 1 \\ 2 & 4 & -8 \\ 1 & 4 & -2 \end{bmatrix}}_{A_1} \overset{R_2 \leftarrow R_2 - 2R_1}{\sim} \underbrace{\begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 1 & 4 & -2 \end{bmatrix}}_{A_2}$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\sim} \underbrace{\begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & -4 & -3 \end{bmatrix}}_{A_3} \overset{R_3 \leftarrow R_3 - \frac{1}{3}R_2}{\sim} \underbrace{\begin{bmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{A_4}$$

$\det(A_4) = 1(-12)\frac{1}{3} = -4$ By the properties:
$\det(A_4) = \det(A_3) = \det(A_2) = \det(A_1) = -\det(A) \Longrightarrow \det(A) = 4$

Example

$$\begin{cases} A \overset{R_i \longleftrightarrow R_j}{\sim} B & \Rightarrow \det(B) = -\det(A) \\ A \overset{R_j \longleftarrow kR_j}{\sim} B & \Rightarrow \det(B) = k\det(A) \\ A \overset{R_j \longleftarrow R_j + aR_i}{\sim} B & \Rightarrow \det(B) = \det(A) \end{cases} \qquad A = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\det(A) = -\begin{vmatrix} 1 & 8 & 1 \\ 0 & -12 & -10 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = -1 \cdot (-12) \cdot \frac{1}{3} = 4$$

2. Alternatively,

$$\begin{bmatrix} 2 & 4 & -8 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_2 \leftarrow \frac{1}{2} R_2}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 0 & 2 & 2 \end{bmatrix} \overset{R_3 \leftarrow R_3 - \frac{1}{3} R_2}{\sim} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \frac{1}{2}\det(A). \Rightarrow \det(A) = 2\begin{vmatrix} 1 & 2 & -4 \\ 1 & 8 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \cdots = 2\begin{vmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = 4$$

Recall

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \overset{R_2 \leftarrow R_2 - \frac{a_{21}}{a_{11}} R_1}{\sim} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\overset{R_3 \leftarrow R_3 - \frac{a_{31}}{a_{11}} R_1}{\sim} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} \\ 0 & a_{32} - \frac{a_{12}a_{31}}{a_{11}} & a_{33} - \frac{a_{12}a_{31}}{a_{11}} \end{bmatrix}$$

$$\overset{R_3 \leftarrow R_3 - \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}} R_2}{\sim} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} \\ 0 & 0 & b \end{bmatrix},$$

where $b = a_{33} - \frac{a_{12}a_{31}}{a_{11}} - \frac{a_{11}a_{32} - a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}\left(a_{23} - \frac{a_{13}a_{21}}{a_{11}}\right)$ Eventually, by simplification,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} \\ 0 & 0 & b \end{vmatrix} = b\,(a_{11}a_{22} - a_{12}a_{21})$$

Minors

Goal

We shall compute the determinant of an $n \times n$, through the **cofactor expansion**. Basically, this expansion express an $n \times n$ determinant as a linear combination of $(n-1) \times (n-1)$ determinants.

Definition

Let A be an $n \times n$ matrix. The ijth <u>*minor*</u> of A is the determinant of the $(n-1) \times (n-1)$ matrix obtained from A , by deleting the ith row and jth column. We write M_{ij} for the ijth minor.

Example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} \cancel{1} & \cancel{-1} & \cancel{2} \\ \cancel{-3} & 5 & 1 \\ \cancel{0} & 2 & 3 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 1 \cdot 2 = 13$$

Example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix} \Longrightarrow M_{11} = \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = 13$$

$$M_{21} = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = (-1) \cdot 3 - 2 \cdot 2 = -7$$

$$M_{31} = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix} = (-1) \cdot 1 - 2 \cdot 5 = -11$$

$$M_{12} == \begin{vmatrix} -3 & 1 \\ 0 & 3 \end{vmatrix} = -9$$

$$M_{13} = \begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix} = -6 \qquad M_{22} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3 \qquad M_{23} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2$$

$$M_{32} = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = 7 \qquad M_{33} = \begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} = 2$$

Cofactor

Definition

The ijth cofactor of A is $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$.

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

Example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{matrix} M_{11} = 13 & M_{12} = -9 & M_{13} = -6 \\ M_{21} = -7 & M_{22} = 3 & M_{23} = 2 \\ M_{31} = -11 & M_{32} = 7 & M_{33} = 2 \end{matrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \Longrightarrow \begin{matrix} C_{11} = 13 & C_{12} = 9 & C_{13} = -6 \\ C_{21} = 7 & C_{22} = 3 & C_{23} = -2 \\ C_{31} = -11 & C_{32} = -7 & C_{33} = 2 \end{matrix}$$

Determinant expansion

Theorem

*Let A be an $n \times n$ matrix. Then $\det(A)$ is calculated by picking a **row** (or **column**) and taking the product of each entry in that row (column) with its cofactor and adding these products together. This process is known as expanding along the ith row (or column).*

$$\det(A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- *along the 1st row:* $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
- *along the last column:* $\det(A) = a_{1n}C_{1n} + a_{2n}C_{2n} + \cdots + a_{nn}C_{nn}$
- *along the ith row:* $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$
- *along the jth column:* $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

Remark

Remember that: we only pick ONE row or column.

Example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{array}{ccc} C_{11} = 13 & C_{12} = 9 & C_{13} = -6 \\ C_{21} = 7 & C_{22} = 3 & C_{23} = -2 \\ C_{31} = -11 & C_{32} = -7 & C_{33} = 2 \end{array}$$

- along the 1st row:

$$\det(A) = 1 \cdot 13 + (-1) \cdot 9 + 2 \cdot (-6) = 13 - 9 - 12 = -8$$

- along the 2nd row:

$$\det(A) = (-3) \cdot 7 + 5 \cdot 3 + 1 \cdot (-2) = -21 + 15 - 2 = -8$$

- along the 3rd row:

$$\det(A) = 0 \cdot (-11) + 2 \cdot (-7) + 3 \cdot 2 = 0 - 14 + 6 = -8$$

- along the 1st column:

$$\det(A) = 1 \cdot 13 + (-3) \cdot 7 + 0 \cdot (-11) = 13 - 21 + 0 = -8$$

- along the 2nd column:

$$\det(A) = (-1) \cdot 9 + 5 \cdot 3 + 2 \cdot (-7) = -9 + 15 - 14 = -8$$

- along the 3rd column:

$$\det(A) = 2 \cdot (-6) + 1 \cdot (-2) + 3 \cdot 2 = -12 - 2 + 6 = -8$$

Example

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{matrix} C_{11} = 13 & C_{12} = 9 & C_{13} = -6 \\ C_{21} = 7 & C_{22} = 3 & C_{23} = -2 \\ C_{31} = -11 & C_{32} = -7 & C_{33} = 2 \end{matrix} \quad \left(\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \right)$$

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{vmatrix} &= 0 \cdot \begin{vmatrix} -1 & 2 \\ 5 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} \\ &= -2\,(1 \cdot 1 - 2 \cdot (-3)) + 3\,(1 \cdot 5 - (-1) \cdot (-3)) = -8 \end{aligned}$$

For a triangular matrix

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} &= a_{11} \cdot (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \\ &= a_{11} \cdot a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} \\ &= \cdots = a_{11} a_{22} \cdots a_{nn} \end{aligned}$$

Example

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix},\ \mathbf{v}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\Rightarrow \mathbf{u}\times\mathbf{v}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}=\begin{vmatrix}u_1 & v_1 & \mathbf{i}\\u_2 & v_2 & \mathbf{j}\\u_3 & v_3 & \mathbf{k}\end{vmatrix}.$$

Expand along the last column:

$$\begin{aligned}\begin{vmatrix}u_1 & v_1 & \mathbf{i}\\u_2 & v_2 & \mathbf{j}\\u_3 & v_3 & \mathbf{k}\end{vmatrix} &= \mathbf{i}\begin{vmatrix}u_2 & v_2\\u_3 & v_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}u_1 & v_1\\u_3 & v_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}u_1 & v_1\\u_2 & v_2\end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix}u_2 & v_2\\u_3 & v_3\end{vmatrix}+\mathbf{j}\begin{vmatrix}u_3 & v_3\\u_1 & v_1\end{vmatrix}+\mathbf{k}\begin{vmatrix}u_1 & v_1\\u_2 & v_2\end{vmatrix}\\ &= (u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}\\ &= \begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}\end{aligned}$$

Example

Compute $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & -3 & 5 & 1 \\ 0 & 0 & 2 & 3 \end{vmatrix}$ $\qquad \left(\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}\right)$

Solution. The key is to pick a row/column containing as many 0 as possible.
last row:

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & -3 & 5 & 1 \\ 0 & 0 & 2 & 3 \end{vmatrix} &= -0 \cdot || + 0 \cdot || - 2 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & -3 & 5 \end{vmatrix} \\
\left(\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}\right) &= -2 \underbrace{\left(1 \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \right)}_{\text{1st row}} + 3 \underbrace{\left(1 \begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} \right)}_{\text{1st column}} \\
&= -2\left((1+6) - 2(-2)\right) + 3\left((5-3) + (-5)\right) \\
&= -2(7+4) + 3(2-5) = -22 - 9 = -31
\end{aligned}
$$

Example

$$
\begin{aligned}
\begin{vmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 & 2 \end{vmatrix}
&= \begin{vmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \color{red}{4} & 0 \\ 1 & 0 & 0 & 5 & 2 \end{vmatrix} \\
&= 4 \cdot (-1)^{4+4} \begin{vmatrix} 1 & 0 & 1 & 3 \\ 0 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} \\
&= 4 \left((-1) \cdot (-1)^{2+2} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} \right) = -4 \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} \\
&= -4 \left(1 \cdot (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} + 2 \cdot (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right) \\
&= -4 \left((-3) + 2 \cdot (1 - 1) \right) = 12
\end{aligned}
$$

Classwork

Evaluate the determinant:

$$\begin{vmatrix} 1 & 1 & 0 & -1 \\ 2 & 3 & 1 & 2 \\ 4 & 4 & 3 & -2 \\ 3 & 4 & 1 & 3 \end{vmatrix}$$

Lecture resumes at 7:45.

Solution

Expand along the first row seems to be (a little) easier:

$$\begin{vmatrix} 1 & 1 & 0 & -1 \\ 2 & 3 & 1 & 2 \\ 4 & 4 & 3 & -2 \\ 3 & 4 & 1 & 3 \end{vmatrix} = 1 \begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 4 & 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 1 & 2 \\ 4 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 3 & 2 \\ 4 & 4 & -2 \\ 3 & 4 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 3 & 1 \\ 4 & 4 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 4 & 1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 2 \\ 4 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 4 & 4 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

- $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 4 & 1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 4 & -2 \\ 4 & 3 \end{vmatrix} + 2 \begin{vmatrix} 4 & 3 \\ 4 & 1 \end{vmatrix} = 3\,(9+2) - (12+8) + 2\,(4-12) = -3$
- $\begin{vmatrix} 2 & 1 & 2 \\ 4 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix} = 2 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 4 & -2 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = 2\,(9+2) - (12+6) + 2\,(4-9) = -6$

Solution

$$\begin{vmatrix} 1 & 1 & 0 & -1 \\ 2 & 3 & 1 & 2 \\ 4 & 4 & 3 & -2 \\ 3 & 4 & 1 & 3 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 4 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 2 \\ 4 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 2 \\ 4 & 4 & -2 \\ 3 & 4 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 & 1 \\ 4 & 4 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 4 & 1 & 3 \end{vmatrix}}_{-3} - \underbrace{\begin{vmatrix} 2 & 1 & 2 \\ 4 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix}}_{-6} + \begin{vmatrix} 2 & 3 & 1 \\ 4 & 4 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

- $\begin{vmatrix} 2 & 3 & 1 \\ 4 & 4 & 3 \\ 3 & 4 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 4 \\ 3 & 4 \end{vmatrix} = 2(4-12) - 3(4-9) + (16-12) = 3$

Thus,

$$\begin{vmatrix} 1 & 1 & 0 & -1 \\ 2 & 3 & 1 & 2 \\ 4 & 4 & 3 & -2 \\ 3 & 4 & 1 & 3 \end{vmatrix} = (-3) - (-6) + 3 = 6.$$

Solution

Alternatively, we can also use the row operations:

$$\underbrace{\begin{bmatrix}1 & 1 & 0 & -1\\ 2 & 3 & 1 & 2\\ 4 & 4 & 3 & -2\\ 3 & 4 & 1 & 3\end{bmatrix}}_{A} \underset{\sim}{\begin{matrix}R_2 \leftarrow R_2 - 2R_1\\ R_3 \leftarrow R_3 - 4R_1\\ R_4 \leftarrow R_4 - 3R_1\end{matrix}} \underbrace{\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 1 & 1 & 4\\ 0 & 0 & 3 & 2\\ 0 & 1 & 1 & 6\end{bmatrix}}_{A_1}$$

$$\underset{\sim}{R_4 \leftarrow R_4 - R_2} \underbrace{\begin{bmatrix}1 & 1 & 0 & -1\\ 0 & 1 & 1 & 4\\ 0 & 0 & 3 & 2\\ 0 & 0 & 0 & 2\end{bmatrix}}_{A_2}$$

$$\begin{vmatrix}1 & 1 & 0 & -1\\ 2 & 3 & 1 & 2\\ 4 & 4 & 3 & -2\\ 3 & 4 & 1 & 3\end{vmatrix} = \det(A) = \det(A_1) = \det(A_2) = 1 \cdot 1 \cdot 3 \cdot 2 = 6.$$

Properties of determinants

- $\det(I) = 1$
- $\det(AB) = \det(A)\det(B)$ This is important
- A is invertible iff $\det(A) \neq 0$, and in this case $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right)$$

- $\det\left(A^T\right) = \det(A)$
- $\det(kA) = k^n \det(A)$ (Recall that $A \overset{R_j \longleftarrow kR_j}{\sim} B \Rightarrow \det(B) = k\det(A)$)
- If A has a row of zeros or a column of zeros, then $\det(A) = 0$.
- If A has two equal rows or columns, then $\det(A) = 0$. $(\mathrm{rank}(A) < n)$

Full rank

For an $n \times n$ square matrix A, it is said to have full rank if $rank(A) = n$. Let A be a full rank $n \times n$-matrix.

- For $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_n]^T$ and $\mathbf{b} = [b_1 \ b_2 \ \ldots \ b_n]^T \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ is consistent and has a unique solution.
- Write the column vectors of A as $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \ldots \ \mathbf{v}_n]$. The casting-out algorithm will result in an echelon form of rank n, so that every column is pivot. $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent $\Rightarrow$ they form a basis of $\mathbb{R}^n$. Alternatively, you can let $\mathbf{b} = \mathbf{0}$ in the system above. The unique solution is the trivial solution, indicating linear independence.
- The reduced echelon form of A is I_n. $\det(A) = C\det(I_n)$ for some constant $C \neq 0$. Since $\det(I_n) = 1$, $\det(A) = C \neq 0$.
- $[A|I] \sim \left[I|A^{-1}\right]$ so that A is invertible.
- $rank(A^T) = n$

Example

1. How many solutions: $\begin{cases} 5x + 2y & = -9 \\ 7x + 3y & = -13 \end{cases}$? (Midterm I, Problem 11)

Solution. $\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 5 \cdot 3 - 2 \cdot 7 \ = 1 \neq 0$

2. Which of the following is a basis of $\mathbb{R}^3$? (Midterm II, Problem 14)

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ (B) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix} \right\}$ (D) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

Solution. A basis of $\mathbb{R}^3$ should contain 3 vectors.

For (A), $\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 0 \cdot 1 = 0$ —not linearly independent.

Example

3. Which two vectors would extend the set $\left\{\begin{bmatrix}2\\-6\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\-1\\0\end{bmatrix}\right\}$ to a basis of $\mathbb{R}^4$? $\left(\begin{matrix}\text{Midterm II}\\ \text{Problem 16}\end{matrix}\right)$

(A) $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ (B) $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ (C) $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ (D) $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

Solution. We can compute determinants one by one:

$$\text{(A)} \begin{vmatrix}2 & 1 & 0 & 0\\ -6 & -2 & 1 & 0\\ -2 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = 1\begin{vmatrix}2 & 1 & 0\\ -6 & -2 & 1\\ -2 & -1 & 0\end{vmatrix} = 1\cdot(-1)\begin{vmatrix}2 & 1\\ -2 & -1\end{vmatrix} = 0$$

$$\text{Similarly, (B)} \begin{vmatrix}2 & 1 & 1 & 1\\ -6 & -2 & 0 & 0\\ -2 & -1 & 0 & 1\\ 0 & 0 & 0 & 0\end{vmatrix} = 0,\ \text{(D)} \begin{vmatrix}2 & 1 & 1 & 0\\ -6 & -2 & 0 & 1\\ -2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{vmatrix} = 0$$

Inverse

Definition

Let A be an $n \times n$ matrix.

- The *cofactor matrix* of A is $cof\,(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$.
- The *adjugate* of A is $adj\,(A) = (cof\,(A))^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$.

Theorem

$$A^{-1} = \frac{1}{\det(A)} adj\,(A)$$

Example: $\left(A^{-1} = \frac{1}{\det(A)} adj\,(A)\right)$

Recall an early example:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 5 & 1 \\ 0 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{matrix} C_{11} = 13 & C_{12} = 9 & C_{13} = -6 \\ C_{21} = 7 & C_{22} = 3 & C_{23} = -2 \\ C_{31} = -11 & C_{32} = -7 & C_{33} = 2 \end{matrix} \Longrightarrow \det(A) = -8$$

$$cof\,(A) = \begin{bmatrix} 13 & 9 & -6 \\ 7 & 3 & -2 \\ -11 & -7 & 2 \end{bmatrix} \qquad adj(A) = \begin{bmatrix} 13 & 7 & -11 \\ 9 & 3 & -7 \\ -6 & -2 & 2 \end{bmatrix}$$

$$\Longrightarrow A^{-1} = \frac{1}{-8} \begin{bmatrix} 13 & 7 & -11 \\ 9 & 3 & -7 \\ -6 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{13}{8} & -\frac{7}{8} & \frac{11}{8} \\ -\frac{9}{8} & -\frac{3}{8} & \frac{7}{8} \\ \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Check $AA^{-1} = I$ as an exercise

Example 2×2

Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \implies \det(A) = 1 \cdot 4 - 2(-3) = 10 \neq 0.$

$$\begin{matrix} M_{11} = 4 & M_{12} = -3 \\ M_{21} = 2 & M_{22} = 1 \end{matrix} \implies \begin{matrix} C_{11} = 4 & C_{12} = 3 \\ C_{21} = -2 & C_{22} = 1 \end{matrix}$$

$$cof(A) = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} \implies adj(A) = cof(A)^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

Theorem

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$\square$

Cramer's rule (*)

Theorem

Suppose A is an invertible $n \times n$-matrix and we wish to solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_n]^T$ and $\mathbf{b} = [b_1 \ b_2 \ \ldots \ b_n]^T \in \mathbb{R}^n$.
Then x_i can be computed by the rule $x_i = \det(A_i)/\det(A)$ where A_i is the matrix obtained by replacing the ith column of A with $\mathbf{b}$.

Example

$$\begin{cases} 5x + 2y &= -9 \\ 7x + 3y &= -13 \end{cases} \Rightarrow A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \Rightarrow \det(A) = 1 \neq 0. \ \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -9 \\ -13 \end{bmatrix}$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} -9 & 2 \\ -13 & 3 \end{vmatrix}}{1} = (-9) \cdot 3 - 2(-13) = -1$$

$$\boxed{\begin{aligned} 5(-1) + 2(-2) &= -5 - 4 = -9 \\ 7(-1) + 3(-2) &= -7 - 6 = -13 \end{aligned}}$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 5 & -9 \\ 7 & -13 \end{vmatrix}}{1} = 5 \cdot (-13) - (-9) \cdot 7 = -2$$

Summary of Chpt. 7

Determinant: ONLY square matrices have determinants.

1. Calculation
   - 2×2 and triangular;
   - Row operations; $\begin{cases} A \overset{R_i \longleftrightarrow R_j}{\sim} B & \Rightarrow \det(B) = -\det(A) \\ A \overset{R_j \longleftarrow kR_j}{\sim} B & \Rightarrow \det(B) = k\det(A) \\ A \overset{R_j \longleftarrow R_j + aR_i}{\sim} B & \Rightarrow \det(B) = \det(A) \end{cases}$
   - minor, cofactor, expansion
   - You should be able to compute a 4×4 or 5×5 determinant.
2. Property:
   - $\det(A) \neq 0$
   - $rank(A) = n$
   - row/column vectors of A are linearly independent $\Rightarrow$ form a basis of $\mathbb{R}^n$
   - A is invertible with $\det(A^{-1}) \neq 0$ $A^{-1} = \frac{1}{\det(A)} adj(A)$
   - the system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\Rightarrow$if $\mathbf{b} = \mathbf{0}$, the homogeneous system has only the trivial solution

MATH-1030
Matrix Theory & Linear Algebra I

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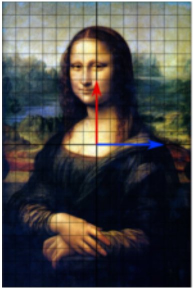

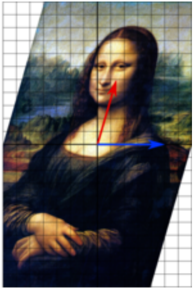

The red arrow changes direction but the blue arrow does not. The blue arrow is an eigenvector of this mapping because it does not change direction, and since its length is unchanged, its eigenvalue is 1.

Definition

Let A be an $n \times n$ matrix, $\mathbf{v}$ be a **non-zero** vector, and λ be a scalar such that $A\mathbf{v} = \lambda\mathbf{v}$. Then,

- $\mathbf{v}$ is called an *eigenvector* of A,
- and λ is called the corresponding *eigenvalue*.

Example

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Which of the following are eigenvectors of A?

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $A\mathbf{u}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{u}_1$ YES eigenvalue 2
- $A\mathbf{u}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{u}_2$ YES eigenvalue 1
- $A\mathbf{u}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq k\mathbf{u}_3$ NO

2. Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Which are the eigenvalues of A? Find the corresponding eigenvalues too.

Solution.

- $A\mathbf{u}_1 = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 + (-3) \cdot 3 \\ 6 \cdot 1 + (-5) \cdot 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -9 \end{bmatrix} \neq \lambda \mathbf{u}_1$ NO
- $A\mathbf{u}_2 = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -2\mathbf{u}_2$ YES $\boxed{\begin{array}{r} \text{Eigenvalue} \\ \lambda = -2 \end{array}}$
- $A\mathbf{u}_3 = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{u}_3$ YES $\boxed{\begin{array}{r} \text{Eigenvalue} \\ \lambda = 1 \end{array}}$
- $A\mathbf{u}_4 = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{u}_4$ NO An eigenvector must be non-zero.

Example

3. Let $A = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix}$. The eigenvalues are 2, 3 and -3. Find the corresponding eigenvector.

Solution. (1) We need to solve $\mathbf{v}_1$ from the equation: $A\mathbf{v}_1 = 2\mathbf{v}_1 \iff A\mathbf{v}_1 = 2I\mathbf{v}_1$ $\iff \mathbf{0} = A\mathbf{v}_1 - 2I\mathbf{v}_1 = (A - 2I)\,\mathbf{v}_1$ $\boxed{(A-2I)\,\mathbf{v}_1 = \mathbf{0}}$ –Homogeneous system

$$A - 2I = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ -6 & -3 & 2 \\ 3 & -6 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ -6 & -3 & 2 & 0 \\ 3 & -6 & 4 & 0 \end{array}\right] \overset{R_1 \leftarrow -\frac{1}{5}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -6 & -3 & 2 & 0 \\ 3 & -6 & 4 & 0 \end{array}\right] \overset{\substack{R_2 \leftarrow R_2 + 6R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & -6 & 4 & 0 \end{array}\right] \overset{R_3 \leftarrow R_3 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2 \leftarrow -\frac{1}{3}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{2}{3}t \\ t \end{bmatrix} \overset{t=3}{\Longrightarrow} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ (to avoid fractions) <u>Exercise check: $A\mathbf{v}_1 = 2\mathbf{v}_1$</u>

(2) $A\mathbf{v}_2 = 3\mathbf{v}_2 \iff A\mathbf{v}_2 = 3I\mathbf{v}_2 \iff (A-3I)\,\mathbf{v}_2 = \mathbf{0}$ — $\boxed{(A-3I)\,\mathbf{v}_2 = \mathbf{0}}$

$$A-3I = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 0 \\ -6 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix} \overset{R_1 \leftarrow -\frac{1}{6}R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -6 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix}$$

$$\overset{\substack{R_2 \leftarrow R_2 + 6R_1 \\ R_3 \leftarrow R_3 - 3R_1}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 2 \\ 0 & -6 & 3 \end{bmatrix} \overset{R_2 \leftarrow -\frac{R_2}{4}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -6 & 3 \end{bmatrix} \overset{R_3 \leftarrow R_3 + 6R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix} \xRightarrow{t=2} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Double-check:

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3\cdot 0 + 0\cdot 1 + 0\cdot 2 \\ -6\cdot 0 - 1\cdot 1 + 2\cdot 2 \\ 3\cdot 0 - 6\cdot 1 + 6\cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = 3\mathbf{v}_2$$

Example

(3) $A\mathbf{v}_3 = -3\mathbf{v}_3 \iff (A+3I)\,\mathbf{v}_3 = \mathbf{0}$ — $\boxed{(A+3I)\,\mathbf{v}_3 = \mathbf{0}}$

$$A+3I = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -6 & 2 & 2 \\ 3 & -6 & 9 \end{bmatrix} \overset{\substack{R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3}}{\sim} \begin{bmatrix} -6 & 2 & 2 \\ 3 & -6 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 \leftarrow -\frac{R_6}{6}}{\sim} \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 3 & -6 & 9 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_2 \leftarrow R_2 - 3R_1}{\sim} \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_2 \leftarrow \frac{-R_2}{5}}{\sim} \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 \leftarrow R_1 + \frac{R_2}{3}}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} \overset{t=1}{\Longrightarrow} \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Double-check:

$$A\mathbf{v}_3 = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -1 \\ -6 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix} = -3\mathbf{v}_3$$

Remark

From the two examples above, we see, for an square matrix A:

1. Given an eigenvector $\mathbf{v}$, we can compute $A\mathbf{v} = \lambda\mathbf{v}$ to find the corresponding eigenvalue.
2. Given an eigenvalue λ, since $A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda I)\mathbf{v} = \mathbf{0}$, we can proceed to find the solution space of the homogeneous system.
3. It is obvious that for an eigenvalue λ, there are more than one eigenvectors:
   - If $A\mathbf{v} = \lambda\mathbf{v}$, for any non-zero scalar k: $A(k\mathbf{v}) = k(A\mathbf{v}) = k\lambda\mathbf{v} = \lambda(k\mathbf{v})$;
   - If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A with respect to the SAME eigenvalue λ: $A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = \lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 = \lambda(\mathbf{v}_1 + \mathbf{v}_2)$
   - This is natural since for an eigenvalue λ, the solution space of the homogeneous system: $(A - \lambda I)\mathbf{v} = 0$ contains all the eigenvectors and the zero vector, which is a subspace. Naturally, it is closed under addition and scalar multiplication.
4. Just given the matrix A, can we find the eigenvalues and eigenvectors?

Finding eigenvalues

Without given any information, we can first find the eigenvalues and then the corresponding eigenvectors.

$$\text{a squre matrix} \longrightarrow \text{eigenvalue(s)} \longrightarrow \text{eigenvectors.}$$

In general, λ is an eigenvalue if and only if there exists a non-zero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \iff A\mathbf{v} = \lambda I\mathbf{v} \iff A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0} \iff (A - \lambda I)\,\mathbf{v} = \mathbf{0}$$

$\iff$ The homogeneous system $(A - \lambda I)\,\mathbf{v} = \mathbf{0}$ has non-trivial solutions.

$\iff \det(A - \lambda I) = 0 \longleftarrow$ This only involves λ not $\mathbf{v}$.

Therefore, we can take the following two steps:

1. Solve the equation $\det(A - \lambda I) = 0$. Solutions $\lambda_1, \ldots, \lambda_n$ are eigenvalues.
2. For each λ_i, to find the corresponding eigenvectors, we solve the homogeneous system $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$. In particular, we choose the **basic solutions** as the representatives of the eigenvectors.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Solution. $\underline{\text{1. Solve } \lambda \text{ from the equation } \det(A - \lambda I) = 0.}$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1{-\lambda} & 1 \\ -2 & 4{-\lambda} \end{bmatrix}$$

$\boxed{A - \lambda I \text{ is obtained by adding } -\lambda \text{ to all the main diagonal entries.}}$

$$\det(A - \lambda I) = (1 - \lambda)(4 - \lambda) - 1(-2) = 4 - \lambda - 4\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6$$

$$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$$

$$\boxed{ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)}$$

$\Longrightarrow \det(A - \lambda I) = 0$ has two solutions: $\lambda_1 = 2$ and $\lambda_2 = 3$, which are the eigenvalues of A.

2. For each eigenvalue (λ_1, λ_2 here), find the corresponding eigenvector, by solving the homogeneous system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

Example

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \Longrightarrow \lambda_1 = 2, \quad \lambda_3 = 3$$

- $\underline{\lambda_1 = 2}$: $A - 2I = \begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ Row operations:

  $$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \overset{R_1 \leftarrow -R_1}{\sim} \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \overset{R_2 \leftarrow R_2 + 2R_1}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

  Suppose the two unknows are x and y. Then, $y = t$ is free and $x = y = t$. General solution: $\begin{bmatrix} t \\ t \end{bmatrix}$ and we pick the basic solution $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, as the representative of the eigenvectors with respect to $\lambda_1 = 2$.

  Check:

  $$A\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ (-2) \cdot 1 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \mathbf{v}_1.$$

Example

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \Longrightarrow \lambda_1 = 2, \quad \lambda_3 = 3$$

- $\underline{\lambda_2 = 3}$: $A - 3I = \begin{bmatrix} 1-3 & 1 \\ -2 & 4-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$ Row operations:
  $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \overset{R_2 \leftarrow R_2 - R_1}{\sim} \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \overset{R_1 \leftarrow -\frac{1}{2} R_1}{\sim} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$
  Suppose the two unknows are x and y. Then, $y = t$ is free and $x = y/2 = t/2$.
  General solution: $\begin{bmatrix} t/2 \\ t \end{bmatrix}$ and we pick the basic solution $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, as the representative of the eigenvectors with respect to $\lambda_2 = 3$.
  Check:$A\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \lambda_2 \mathbf{v}_2$.

Example

Therefore, for matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

It has two eigenvalues: $\lambda_1 = 2, \quad \lambda_3 = 3$

- For $\lambda_1 = 2$, the basis of corresponding eigenvectors is

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

- For $\lambda_2 = 3$, the basis of corresponding eigenvectors is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

How about a 3×3 matrix?

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix}$

Example

$A = \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix}$ $\boxed{\begin{array}{c}\text{Solve the equation}\\ \det(A-\lambda I)=0\end{array}}$: $A - \lambda I = \begin{bmatrix} -2-\lambda & -3 & 6 \\ 0 & 1-\lambda & 0 \\ -3 & -3 & 7-\lambda \end{bmatrix}$

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} -2-\lambda & -3 & 6 \\ 0 & 1-\lambda & 0 \\ -3 & -3 & 7-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -2-\lambda & 6 \\ -3 & 7-\lambda \end{vmatrix} \\
&= (1-\lambda)\left[(-2-\lambda)(7-\lambda) - 6(-3)\right] \\
&= (1-\lambda)\left[-14+2\lambda-7\lambda+\lambda^2+18\right] \\
&= (1-\lambda)\left(\lambda^2-5\lambda+4\right) \\
&= (1-\lambda)(\lambda-1)(\lambda-4) \\
&= -(\lambda-1)^2(\lambda-4)
\end{aligned}$$

The roots of $\det(A-\lambda I) = -(\lambda-1)^2(\lambda-4) = 0$ are the eigenvalues of A: $\lambda_1 = 1$ and $\lambda_2 = 4$

- For $\lambda_1 = 1$: $A - I = \begin{bmatrix} -3 & -3 & 6 \\ 0 & 0 & 0 \\ -3 & -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\begin{array}{l} y = t \\ z = s \end{array} \Rightarrow x = -t + 2s}$

  General solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ basic solutions: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Example

$$A = \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix} \Rightarrow A - 4I = \begin{bmatrix} -6 & -3 & 6 \\ 0 & -3 & 0 \\ -3 & -3 & 3 \end{bmatrix} \underset{\sim}{\overset{R_1 \leftarrow R_1/(-6)}{\overset{R_2 \leftarrow R_2/(-3)}{R_3 \leftarrow R_3/(-3)}}} \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 1 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} \underset{\sim}{\overset{R_1 \leftarrow R_1 - R_2/2}{R_3 \leftarrow R_3 - R_2/2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y = 0,\ z = t \Longrightarrow x = z = t$

General solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow$ Basic solution: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

A has two eigenvalues: $\lambda_1 = 1,\ \lambda_2 = 4$

- Eigenvalue $\lambda = 1$: Multiplicity 2 ; Basis of eigenvectors : $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$
- Eigenvalue $\lambda = 4$: Multiplicity 1 ; Basis of eigenvectors : $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Classwork

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A

Lecture resumes at 7:55.

Solution

$$A=\begin{bmatrix}1&3&3\\0&4&3\\0&0&7\end{bmatrix}\implies A-\lambda I=\begin{bmatrix}1-\lambda&3&3\\0&4-\lambda&3\\0&0&7-\lambda\end{bmatrix}$$

$$\implies \det(A-\lambda I)=(1-\lambda)(4-\lambda)(7-\lambda)$$

$$\implies \det(A-\lambda I)=0 \text{ has three roots: } \lambda_1=1,\ \lambda_2=4,\ \lambda_3=7$$

- $\lambda_1=1$: $A-I=\begin{bmatrix}0&3&3\\0&3&3\\0&0&6\end{bmatrix}\sim\begin{bmatrix}0&1&1\\0&1&1\\0&0&1\end{bmatrix}\sim\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

  $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}t\\0\\0\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$. Basic solution: $\mathbf{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$.

- $\lambda_2=4$: $A-4I=\begin{bmatrix}-3&3&3\\0&0&3\\0&0&3\end{bmatrix}\sim\begin{bmatrix}1&-1&-1\\0&0&1\\0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}$

  $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}t\\t\\0\end{bmatrix}=t\begin{bmatrix}1\\1\\0\end{bmatrix}$. Basic solution: $\mathbf{v}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Solution

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix} \implies \det(A - \lambda I) = (1-\lambda)(4-\lambda)(7-\lambda)$$

- $\lambda_3 = 7$: $A - 7I = \begin{bmatrix} -6 & 3 & 3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Basic solution: $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

A has three eigenvalues:

- Eigenvalue $\lambda = 1$: Basis of eigenvectors : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- Eigenvalue $\lambda = 4$: Basis of eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
- Eigenvalue $\lambda = 7$: Basis of eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Characteristic polynomial

Definition

If A is a square matrix , the expression $p(\lambda) = \det(A - \lambda I)$ is always a polynomial in λ. It is called the *characteristic polynomial* of A. Moreover, the eigenvalues of A are the **roots** of the characteristic polynomial.

Example

1. $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$ The characteristic polynomial is :

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & 4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(7-\lambda).$$

2. $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 1(-2) = (\lambda-3)(\lambda-2)$$

Properties

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 7 \end{bmatrix} \Longrightarrow p_1(\lambda) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & 4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(7-\lambda).$$

- If A is an $n \times n$ matrix, the degree of $p(\lambda)$ is n. There are n eigenvalues counting multiplicity.

  $$B = \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix} \Longrightarrow p_2(\lambda) = (4-\lambda)(\lambda-1)^2 \Rightarrow \begin{cases} \lambda_1 = \lambda_2 = 1 \\ \lambda_3 = 4 \end{cases}$$

- The constant term of the characteristic polynomial $= p(0) = \det(A)$
  $p_1(0) = 1 \cdot 4 \cdot 7 = \det(A)$ Exercise: check, $p_2(0) = 4 = \det(B)$
- (*) The $n-1$ term

  $$p(\lambda) = (-1)^n \lambda^n + \underline{(-1)^{n-1}\, tr(A) \lambda^{n-1}} + \cdots + \det(A).$$

- For triangluar matrics: $p(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$ Eigenvalues are the entries on the main diagonal.

Multiplicity

Recall the example $B = \begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix} \Longrightarrow p_2(\lambda) = (4-\lambda)(\lambda-1)^2$

B has two eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$

- Eigenvalue $\lambda_1 = 1$: Multiplicity 2 ; Basis of eigenvectors : $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$
- Eigenvalue $\lambda_2 = 4$: Multiplicity 1 ; Basis of eigenvectors : $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Focus on $\lambda_1 = 1$, the solution space of the homogeneous system $(A - I)\mathbf{v} = \mathbf{0}$.

has dimension 2, since the basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ has two vectors.

Multiplicity

Consider another example $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1. Find the characteristic polynomial: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$
$$= (2-\lambda)^2 (1-\lambda)$$
2. Find the roots (eigenvalues):
   - $\lambda_1 = 1$ with multiplicity 1
   - and $\lambda_2 = 2$ with multiplicity 2.
3. Find the eigenvectors:
   - $\underline{\lambda_1 = 1}$: $A - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
   - $\underline{\lambda_2 = 2}$: $A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Multiplicity

We see for $\lambda_1 = 2$, it has multiplicity 2 in the equation $p(\lambda) = \det(A - \lambda I) = 0$ and the dimension of the solution space of $(A - 2I)\mathbf{v} = \mathbf{0}$ is 1.

Definition

Let A be a square matrix.

1. If λ is a root of the characteristic polynomial of multiplicity k, we say that λ is an eigenvalue of *algebraic multiplicity* k.
2. For an eigenvalue λ, the solution space $\{\mathbf{v} | (A - \lambda I)\mathbf{v} = \mathbf{0}\} = \{\mathbf{v} \mid A\mathbf{v} = \lambda \mathbf{v}\}$, denoted by E_λ, is called an *eigenspace* of A, with respect to λ.
3. If λ has an m-dimensional eigenspace , i.e., $\dim E_\lambda = m$, we say λ has *geometric multiplicity* m.

Theorem

$$1 \leq \textit{geometric multiplicity} \leq \textit{algebraic multiplicity}.$$

Examples

As we computed before:

- Matrix $\begin{bmatrix} -2 & -3 & 6 \\ 0 & 1 & 0 \\ -3 & -3 & 7 \end{bmatrix}$ has two eigenvalues:
  - Eigenvalue $\lambda_1 = 1$: algebraic multiplicity $=$ geometric multiplicity $= 2$
  - Eigenvalue $\lambda_2 = 4$: algebraic multiplicity $=$ geometric multiplicity $= 1$
- Matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has two eigenvalues:
  - Eigenvalue $\lambda_1 = 1$: algebraic multiplicity $=$ geometric multiplicity $= 1$
  - Eigenvalue $\lambda_2 = 2$: algebraic multiplicity $= 2$, while geometric multiplicity $= 1$

Preview

Next time (the last lecture) we shall see that:
An $n \times n$-matrix is *diagonalizable* iff for all its eigenvalues, the algebraic multiplicity$=$ geometric multiplicity.

MATH-1030
Matrix Theory & Linear Algebra I

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Recall

Definition

Let A be an $n \times n$ matrix, $\mathbf{v}$ be a **non-zero** vector, and λ be a scalar such that $A\mathbf{v} = \lambda\mathbf{v}$. Then,

- $\mathbf{v}$ is called an *eigenvector* of A,
- and λ is called the corresponding *eigenvalue*.

Remark

An eigenvector cannot be a zero vector, but an eigenvalue can be 0.

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To find eigenvalues and eigenvectors, we need to :

1. Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$
2. Find the eigenvalues, which are the roots of $p(\lambda) = 0$.
3. For each λ_i, solve the homogeneous system $(A - \lambda I)\,\mathbf{v} = \mathbf{0}$, by providing the basic solutions.

Recall

Definition

Let A be a square matrix.

1. If λ is a root of the characteristic polynomial of multiplicity k, we say that λ is an eigenvalue of *algebraic multiplicity* k.
2. For an eigenvalue λ, the solution space $\{\mathbf{v}|(A-\lambda I)\mathbf{v}=\mathbf{0}\}=\{\mathbf{v}\mid A\mathbf{v}=\lambda\mathbf{v}\}$, denoted by E_λ, is called an *eigenspace* of A, with respect to λ.
3. If λ has an m-dimensional eigenspace, i.e., $\dim E_\lambda = m$, we say λ has *geometric multiplicity* m.

Theorem

$$1 \leq \textit{geometric multiplicity} \leq \textit{algebraic multiplicity}.$$

Diagonal matrices ($n \times n$)

Definition

A diagonal matrix is of the form $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$.
Namely, only the entries on the main diagonal can be non-zero.

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark

It does not require $d_1, \ldots, d_n$ to be nonzero.

Diagonal matrices

Adding, subtracting, multiplying diagonal matrices are **much easier** and results are also diagonal matrices.

- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$
- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a \cdot c + 0 \cdot 0 & a \cdot 0 + 0 \cdot c \\ 0 \cdot c + b \cdot 0 & 0 \cdot 0 + b \cdot d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$

  In particular, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} \Longrightarrow \left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right)^{100} = \begin{bmatrix} a^{100} & 0 \\ 0 & b^{100} \end{bmatrix}$.

In general: all operations are computed componentwise on the diagonal.

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} + \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} c_1+d_1 & 0 & \cdots & 0 \\ 0 & c_2+d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n+d_n \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} c_1 d_1 & 0 & \cdots & 0 \\ 0 & c_2 d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n d_n \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$. Find a matrix B such that $B^2 = A$.

Solution. From the computation above, we see $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ satisfies

$$B^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} = A$$

Of course there are other solutions

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Recall an early example $A = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix}$

$$p(A) = -(\lambda - 1)^2 (\lambda - 4)$$

- $\lambda = 1$ basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$
- $\lambda = 4$ basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\Longrightarrow \boxed{\begin{array}{l} A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ A \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \\ A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \end{array}}$$

Let $P = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow AP = \begin{bmatrix} A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} & A \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} & A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

$$P \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} = \begin{bmatrix} P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & P \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 2 & P \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \\ 1 & 0 & \\ 0 & 1 & \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix}.$$

We see $\boxed{AP = PD} \iff A = PDP^{-1}$.

Diagonalization

Theorem

*Let A be an $n \times n$ matrix, with n **linearly independent** eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$ $k = 1, \ldots, n$ (allowing repetitions of $\lambda_1, \ldots, \lambda_n$) Then, there exists an invertible matrix P and a diagonal matrix D such that*

$$P^{-1}AP = D \iff A = PDP^{-1}$$

In the case, we say that A is diagonalizable. In particular,

- *P is the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.*
- *D is the matrix whose main diagonal entries are $\lambda_1, \ldots, \lambda_n$.*

Remark

The key is that A has n linearly independent eigenvectors.

Example

Diagonalize the matrix $A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$

Solution. 1. Characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -2 \\ 1 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - (-2)1 = 10 - 7\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 7\lambda + 12 = (\lambda - 3)(\lambda - 4)$$

2. Eigenvalues: $\det(A - \lambda I) = 0 \Longrightarrow \lambda_1 = 3 \quad \lambda_2 = 4$
3. Eigenvectors
   - $A - 3I = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Longrightarrow x = -2y \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
   - $A - 4I = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Longrightarrow x = -y \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
4. Diagonalize: $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \frac{adj(P)}{\det(P)}$

$M_{11} = 1,\ M_{12} = 1,$
$M_{21} = -1,\ M_{22} = -2$
$\Rightarrow cof(P) = \begin{bmatrix} 1 & -1 \\ -(-1) & -2 \end{bmatrix} \Rightarrow adj(P) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

Example

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \Rightarrow \lambda_1 = 3, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \lambda_2 = 4, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \frac{adj(P)}{\det(P)} = \frac{1}{-1} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$P^{-1}AP = D \Longleftrightarrow A = PDP^{-1}$$

5. Check:

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-1)\cdot 2 + (-1)\cdot 1 & (-1)\cdot(-2) + (-1)\cdot 5 \\ 1\cdot 2 + 2\cdot 1 & 1\cdot(-2) + 2\cdot 5 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -3 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-3)\cdot(-2) + (-3)\cdot 1 & (-3)\cdot(-1) + (-3)\cdot 1 \\ 4\cdot(-2) + 8\cdot 1 & 4\cdot(-1) + 8\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = D \end{aligned}$$

Example

Diagonalize the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$

Solution. 1. Characteristic polynomial

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 4-\lambda & -1 \\ -2 & -4 & 4-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix} \\
&= (2-\lambda)\left[(4-\lambda)(4-\lambda) - (-4)(-1)\right] \\
&= (2-\lambda)\left[\lambda^2 - 8\lambda + 16 - 4\right] = (2-\lambda)\left(\lambda^2 - 8\lambda + 12\right) \\
&= (2-\lambda)(\lambda-2)(\lambda-6) = -(\lambda-2)^2(\lambda-6)
\end{aligned}$$

2. Eigenvalues: $\det(A-\lambda I) = 0 \Longrightarrow \lambda_1 = \lambda_2 = 2, \quad \lambda_3 = 6$

3. Eigenvectors

- For $\lambda = 2$, $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & -1 \\ -2 & -4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = -2y + z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t+s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \Longrightarrow \begin{array}{r} \lambda_1 = \lambda_2 = 2 \\ \lambda_3 = 6 \end{array}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- For $\lambda = 6$, $A - 6I = \begin{bmatrix} -4 & 0 & 0 \\ 1 & -2 & -1 \\ -2 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = 0 \\ y = -\frac{z}{2} \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -r/2 \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \Longrightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \Longrightarrow PDP^{-1} = A$$

Remark

- The order is not unique. Since both $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors with respect to $\lambda = 2$, we can let

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

Or, we can consider

$$D_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1/2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Remark

The key is to make sure that eigenvalues and eigenvectors are located at the same columns.

- We do not really require the basic solution(s)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \Longrightarrow D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1/2 \\ 0 & 1 & 1 \end{bmatrix}$$

- $$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

- $$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad P_4 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

During the break, please consider this question.
Diagonalize

$$A = \begin{bmatrix} 8 & 0 & 10 \\ -6 & -3 & -6 \\ -5 & 0 & -7 \end{bmatrix}.$$

Namely, find a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}$$

Lecture resumes at 7:30.

Solution

$$A = \begin{bmatrix} 8 & 0 & 10 \\ -6 & -3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

1. Characteristic polynomial: $\det(A - \lambda I) = \begin{vmatrix} 8-\lambda & 0 & 10 \\ -6 & -3-\lambda & -6 \\ -5 & 0 & -7-\lambda \end{vmatrix}$

$$= -(\lambda+3)\begin{vmatrix} 8-\lambda & 10 \\ -5 & -7-\lambda \end{vmatrix}$$
$$= -(\lambda+3)\left[(8-\lambda)(-7-\lambda) - (-5)\cdot 10\right] = -(\lambda+3)\left(\lambda^2-\lambda-6\right)$$
$$= -(\lambda+3)(\lambda+2)(\lambda-3)$$

2. Eigenvalues: $\det(A-\lambda I) = 0 \Longrightarrow \lambda_1 = -3, \quad \lambda_2 = -2, \quad \lambda_3 = 3$
3. Eigenvectors

- For $\lambda_1 = -3$, $A + 3I = \begin{bmatrix} 11 & 0 & 10 \\ -6 & 0 & -6 \\ -5 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{10}{11} \\ 1 & 0 & 1 \\ 1 & 0 & \frac{4}{5} \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & \frac{10}{11} \\ 1 & 0 & \frac{4}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -\frac{1}{11} \\ 0 & 0 & -\frac{1}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 8 & 0 & 10 \\ -6 & -3 & -6 \\ -5 & 0 & -7 \end{bmatrix} \Rightarrow \lambda_1 = -3, \quad \lambda_2 = -2, \quad \lambda_3 = 3, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- For $\lambda_2 = -2$, $A + 2I = \begin{bmatrix} 10 & 0 & 10 \\ -6 & -1 & -6 \\ -5 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 6 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- For $\lambda_3 = 3$, $A - 3I = \begin{bmatrix} 5 & 0 & 10 \\ -6 & -6 & -6 \\ -5 & 0 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 8 & 0 & 10 \\ -6 & -3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

$$\lambda_1 = -3, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_2 = -2, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \lambda_3 = 3, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus,

$$P = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

such that

$$A = PDP^{-1}$$

Application

Example

Let $A = \begin{bmatrix} -3 & 0 & 0 \\ -6 & -1 & 2 \\ 3 & -6 & 6 \end{bmatrix}$. Find A^{100}

Solution

$P = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Longrightarrow P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \Longrightarrow$

$$P^{-1}AP = D \Leftrightarrow A = PDP^{-1}.$$

$$A^{100} = \underbrace{\left(PDP^{-1}\right)\left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right)\left(PDP^{-1}\right)}_{\text{100 copies}}$$

$= PD\cancel{P^{-1}P}D\cancel{P^{-1}P}D\cdots\cancel{P^{-1}P}DP^{-1} = PD^{100}P^{-1}$. Thus,

$$A^{100} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4^{100} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \underline{\begin{bmatrix} -b+2 & -b+1 & 2b-2 \\ 0 & 1 & 0 \\ -b+1 & -b+1 & 2b-1 \end{bmatrix}}_{b=4^{100}}$$

Application

Example

Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$. Find a square root of A. Namely, find B such that $B^2 = A$.

Solution

$P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \implies P^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$
$\implies P^{-1}AP = D \Leftrightarrow A = PDP^{-1}$. We just need to find a square root of D. Let

$$E = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2 \end{bmatrix} \implies E^2 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \cdot \sqrt{3} & 0 \\ 0 & 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = D$$

Now, consider $B := PEP^{-1}$.

$$B^2 = \left(PEP^{-1}\right)\left(PEP^{-1}\right) = PEP^{-1}PEP^{-1} = PE^2P^{-1} = PDP^{-1} = A.$$

$$B := PEP^{-1} = \begin{bmatrix} -2 + 2\sqrt{3} & -4 + 2\sqrt{3} \\ 2 - \sqrt{3} & 4 - \sqrt{3} \end{bmatrix}$$

Remark

1. Since $A = PDP^{-1}$,
$$A^n = PD^nP^{-1} \qquad \text{and} \qquad A^{\frac{1}{n}} = PD^{\frac{1}{n}}P^{-1}$$
where (for $m = n$ or $m = 1/n$).
$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}^m = \begin{bmatrix} d_1^m & 0 & 0 & \cdots & 0 \\ 0 & d_2^m & 0 & \cdots & 0 \\ 0 & 0 & d_3^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^m \end{bmatrix}$$
However, to get the square root, we need $d_1, \ldots, d_n \geq 0$.

2. Recall another example $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ which has:
   - eigenvalue $\lambda = 1$, multiplicity 1, Basis of eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$;
   - eigenvalue $\lambda = 2$ multiplicity 2, Basis of eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$;

In this case, A is NOT diagonalizable. **(*n* linearly independent eigenvectors.)**

Remark

Theorem

Eigenvectors with respect to distinct eigenvalues are linearly independent. Let A be an $n \times n$ matrix and $\lambda_1, \dots, \lambda_k$ be its distinct eigenvalues. For each λ_j, pick ANY eigenvector $\mathbf{v}_j$ with respect to λ_j. Then, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are linearly independent.

Proof. (* You do not need to remember the proof).

Suppose $\mathbf{v}_j = a_1\mathbf{v}_1 + \cdots + a_{j-1}\mathbf{v}_{j-1}$, for $a_1, \dots, a_{j-1}$ are not all zero.
$\lambda_j\mathbf{v}_j = A\mathbf{v}_j = A\left(a_1\mathbf{v}_1 + \cdots + a_{j-1}\mathbf{v}_{j-1}\right) = a_1\lambda_1\mathbf{v}_1 + \cdots + a_{j-1}\lambda_{j-1}\mathbf{v}_{j-1}$
$\Longrightarrow \mathbf{0} = a_1\left(\lambda_1 - \lambda_j\right)\mathbf{v}_1 + \cdots + a_{j-1}\left(\lambda_{j-1} - \lambda_j\right)\mathbf{v}_{j-1}$.
From here, one can prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are linearly independent by induction on j and contradiction. $\square$

Remark

As we know, the solution space of a homogeneous system $=$ *span* {basic solutions}. Namely, the basic solutions of the *eigenspace* for a given eigenvalue are linearly independent.

Remark

Theorem

Let A be an $n \times n$ matrix and $\lambda_1, \ldots, \lambda_k$ $(k \leq n)$ be its distinct eigenvalues. A is diagonalizable iff for each λ_j, $j = 1, \ldots, k$, its algebraic multiplicity equals its geometric multiplicity. In particular, if A has n distinct eigenvalues i.e., $k = n$, A is diagonalizable.

Example

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A is of "the closest" form to a diagonal matrix.
Jordan canonical form (*)

Summary of Chpt. 8

- Eigenvalues and eigenvectors for square matrix A
  - For nonzero vector $\mathbf{v}$: $A\mathbf{v} = \lambda\mathbf{v}$
- Find the eigenvalues and eigenvectors:
  1. Find the characteristic polynomials $p(\lambda) = \det(A - \lambda I)$
  2. Find the roots of $p(\lambda)$, i.e., eigenvalues $p(\lambda) = 0$ $\lambda_1, \ldots, \lambda_n$
  3. For each λ_j, find the eigenvectors by solving the homogeneous system $(A - \lambda_j I)\mathbf{v} = \mathbf{0}$.
- Diagonalize a matrix A is diagonalizable iff A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with respect to eigenvalues $\lambda_1, \ldots, \lambda_n$ (allowing repetitions). Then,

$$P = [\mathbf{v}_1, \ldots, \mathbf{v}_n], \ D = diag\{\lambda_1, \ldots, \lambda_n\}$$

such that

$$AP = PD \Longleftrightarrow A = PDP^{-1}$$

  - Compute A^n
  - Find B such that $B^m = A$