Quick Review of Long Cal I

△ Function: $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ Domain: $\mathbb{R}$ $(-\infty, \infty)$

$g(x) = \sqrt{x+1}$ $x \geq -1$; $h(x) = \ln x$ $x > 0$

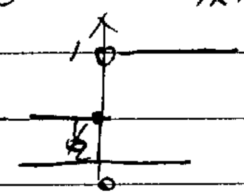
- Odd, Even, Increasing / Decreasing / Monotone
- Composition $f \circ g(x) = |\sqrt{x+1}| = \sqrt{x+1}$
- 1-1, onto, Inverse Function
- Trigonometric: sin cos tan cot sec csc
  Exponential / Logarithmic: e^x a^x $\ln x$ $\log_a x$

△ Limit: $f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 5 & x = 3 \end{cases} \Rightarrow \begin{cases} f(3) = 5 \\ \lim_{x\to 3} f(x) = 6 \end{cases}$

- Factorization
- Conjugate $\lim_{x\to 1} \frac{x^2+x-2}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(x-1)(x+2)(\sqrt{x}+1)}{(x-1)} = 6$
- L'Hospital, $\lim_{t\to 0} \frac{\sin t}{t} = 1$, $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$

Squeeze THM.

- Left / Right Limit:



$\lim_{x\to 0^-} f(x) = \frac{1}{2}$

$\lim_{x\to 0^+} f(x) = 1$

△ Derivatives: $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

- slope of tangent line at a
- position function$'$ = velocity function
  velocity function$'$ = acceleration function
- Notation $y = f(x)$, y', $f'(x)$, $\frac{dy}{dx} = \frac{d}{dx}(y)$
- $(c)' = 0$, $(e^x)' = e^x$, $(\sin x)' = \cos x$
  $(x^n)' = nx^{n-1}$, $(\ln x)' = \frac{1}{x}$

$\sin x \xrightarrow{\,'\,} \cos x \xrightarrow{\,'\,} -\sin x \xrightarrow{\,'\,} -\cos x \xrightarrow{\,'\,} \sin x$

° Technique.

- Product Rule: $(fg)' = f'g + fg'$ $\left((f+g)' = f' + g'\right)$
- Quotient Rule. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Chain Rule: $(f \circ g(x))' = f'(g(x)) \cdot g'(x)$

  Ex: $(\sin(\sin(\sin(x))))' = ?$

  $(e^{x^2})' = e^{x^2} \cdot 2x = 2xe^{x^2}$
- Implicit.

  $x^2 + y^2 = 25 \quad \frac{dy}{dx} = ?$

  $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) = 0$

  $\parallel$

  $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

△ Trigonometric, P214. $(\arctan x)' = \frac{1}{1+x^2}$

3.8 Exponential Growth / Decay.

Key. "Rate of change is proportional to the size"

$\frac{dy}{dt} = ky$ $\begin{cases} k>0 & \text{law of natural growth} \\ k<0 & \text{decay} \end{cases}$

THM. $y(t) = y(0)e^{kt}$

△ Population Growth : $P(t) = y(t)$

World Population	Year	
2560 M	~~2560~~ 1950	$t=0$
3040 M	1960	$t=10$

[Q]. $k = ?$ Year 2014 $(t=64)$ $P=?$

Solution. $P(t) = P(0)e^{kt} = 2560e^{kt}$

$3040 = P(10) = 2560\, e^{k10} \Rightarrow k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017185$

$P(64) = 7689.45$

△ Radioactive Decay.

° Notation. mass of remaining radioactive substances $m(t)$

$m(t) = m_0 e^{kt}$ $(k<0)$

° Half-life: The time require for half of any given quantity to decay.

$m \xrightarrow[\text{half-life}]{} \frac{1}{2}m$

Ex. The half-life of radium-226 is 1590 years

~~(a)~~ A sample has 100 mg radium-226. Find the formula of the sample that remains after t years.

(=) Find the mass after 1000 years.

Solution.

$$m(t) = m_0 e^{kt} = 100\, e^{kt}$$

(一) $m(1590) = \frac{1}{2}(100) = 50 = 100\, e^{k1590} \Rightarrow \frac{1}{2} = e^{k1590}$ $\left| \frac{1}{2}m = m e^{k1590} \right|$

$$\Rightarrow \boxed{k = -\frac{\ln 2}{1590}} \qquad k = -\frac{\ln 2}{\text{half-life}}$$

$$\therefore m(t) = 100\, e^{-(\ln 2)t/1590}$$

(二) $m(1000) = 100\, e^{-(\ln 2)1000/1590} = 65 \text{ mg}$

▷ Newton's law of Cooling.

T_s : temperature of the surroundings.

$$\boxed{\frac{dy}{dt} = ky} \iff \frac{dT}{dt} = k(T - T_s)$$

Define $y(t) = T(t) - T_s$ $\qquad \frac{dy}{dt} = ky$

$$y(t) = \underbrace{(T(0) - T_s)}_{y(0)}\, e^{kt}$$

$$T(t) = (T(0) - T_s)\, e^{kt} + T_s$$

Ex. object: a bottle of soda (Coke)

Temperatures: $\begin{cases} \text{room} & 72°F \\ \text{frige} & 44°F \end{cases}$

After 30 mins, the soda becomes 61°F

[Q]: What is the temperature after ANOTHER 30 mins?

Solution. $61 = T(30) = (72 - 44)e^{k\cdot 30} + 44$

$$\Rightarrow k = \ln\left(\tfrac{17}{28}\right)/30 \approx -0.01663$$

$$T(60) = (72 - 44)e^{k\cdot 60} + 44 = 54.3$$

§3.9 Related Rates

Ex. Air is being pumped into a spherical balloon so that the volume increases at a rate of 100 cm³/s. How fast is the radius increasing when the diameter is 50cm?

Solution Notation: $\begin{cases} \text{Volume: } V \\ \text{Time: } t \\ \text{Radius: } r \\ \text{Diameter: } d = 2r \end{cases}$

Formula: $V(r) = \frac{4}{3}\pi r^3$

Given: $\frac{dV}{dt} = 100 \text{ cm}^3/s$

Unknown: $\frac{dr}{dt}$ when $r = 25 \Leftrightarrow d = 50$

Chain Rule: $f(g(x))' = f'(g(x)) \cdot g'(x)$

If let $u = g(x)$, $[f(g(x))]' = f'(u) \cdot g'(x) = f'(u) \cdot u'(x)$

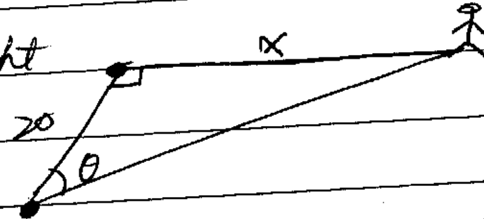
$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}$$

$$\boxed{\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}} = 4\pi r^3 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot 100$$

When $r = 25$, $\frac{dr}{dt} = \frac{1}{25\pi}$

Ex. A man walks along a straight path at speed of 4 ft/s. A searchlight is located 20 ft from the path and is kept focused on him. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Notation. Distance x Given ~~x~~ $\frac{dx}{dt} = 4$

Angle θ

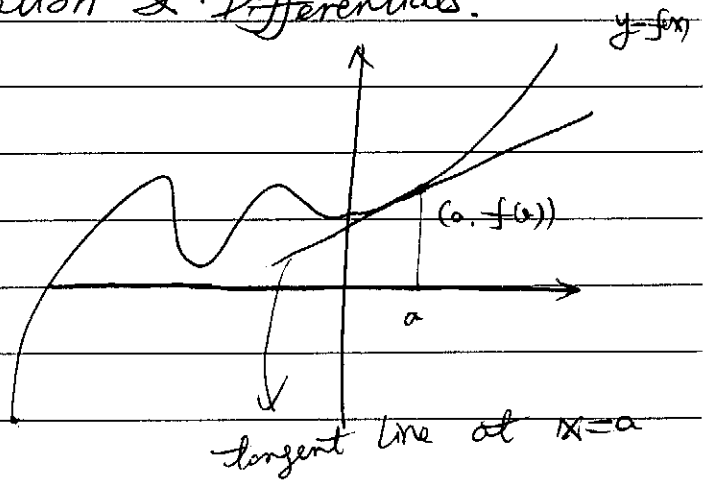
Time t.   Unknown $\frac{d\theta}{dt}$ when $x = 15$.

Formula. $\tan\theta = \frac{x}{20}$ $\cancel{x = 20\tan\theta}$

Chain Rule. $\underbrace{\frac{dx}{dt}}_{4} = \underbrace{\frac{dx}{d\theta}}_{20\sec^2\theta} \frac{d\theta}{dt}$ $\Rightarrow \frac{d\theta}{dt} = \frac{1}{5}\cos^2\theta$

$x = 15, \Rightarrow \cos\theta = \frac{4}{5} \Rightarrow \frac{d\theta}{dt} = \frac{16}{125}$

§3.10 Linear Approximation & Differentials.



DEF.

$$y - f(a) = f'(a)(x-a)$$

$$y = \underbrace{f'(a)(x-a) + f(a)}$$

$$L(x) = f(a) + f'(a)(x-a)$$

Linear Approximation

Tangent Line Approximation

Ex. $f(x) = \sqrt{x+3}$ $\quad a = 1$ $\quad f(a) = f(1) = 2$

$$f'(x) = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(a) = f'(1) = \frac{1}{4}$$

$$L(x) = f(a) + f'(a)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{x}{4} + \frac{7}{4}$$

(一) $\sqrt{3.98} = f(0.98) = 1.99499373\ldots$ $\quad$] Error $= 0.0000062$

$L(0.98) = \frac{0.98}{4} + \frac{7}{4} = 1.995$

(二) $\sqrt{4.05} = f(1.05) = 2.01246117\ldots$ $\quad$] Error $= 0.0003\ldots$

$L(1.05) = 2.0125$

△ Differentials. dy, dx

$$dy = f'(x)dx \quad \text{if} \quad y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$$\begin{cases} \Delta y = f(x+\Delta x) - f(x) \\ dy = f'(x)dx \end{cases}$$

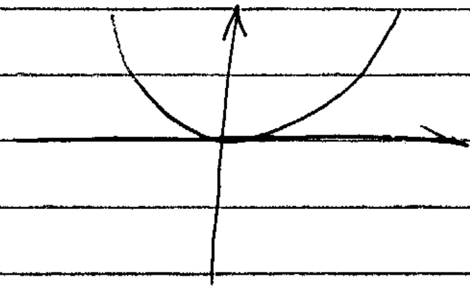
Ex. $f(x) = x^3$ $\quad x = 2.$ $\quad \overset{dx}{\overset{\|}{\Delta x}} = 0.01$, $\quad f'(x) = 3x^2$

$\Delta y = f(2.01) - f(2) = 2.01^3 - 2^3 =$

$dy = f'(2) \cdot 0.01 = 0.12$

§4.3 Derivatives Affect the Shape of a Graph

△ f' Affects f.



Ex. $f(x) = x^2$

$$f'(x) = 2x \begin{cases} \geq 0 & x \geq 0 \\ < 0 & x < 0 \end{cases}$$

THM. [Increasing/Decreasing Test]

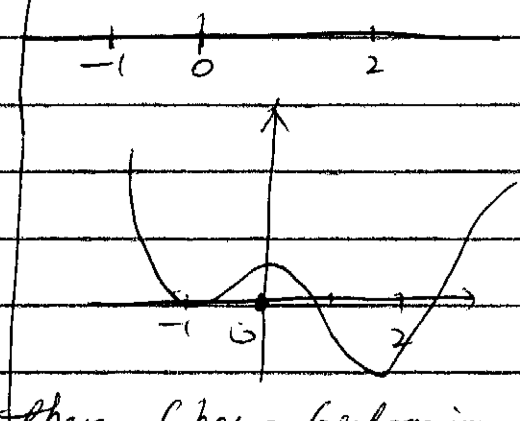
(a) If $f'(x) > 0$ on an interval, then f is increasing on it

(b) ——— < 0 ——————————— decreasing ———

Ex. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12(x+1)x(x-2)$$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, +\infty)$
f'	$-$	$+$	$-$	$+$
f	↘	↗	↘	↗



THM. [The 1st Derivative Test]

Suppose c is a critical number of a continuous function f

(a) If f' changes from $+$ to $-$, then f has a local maximum at c

(b) ——————— $-$ to $+$ ——————— minimum

(c) If f' does not change sign at c, NOTHING

Ex. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$, $f'(x) = 12(x+1)x(x-2)$

critical numbers. $-1, 0, 2$

Ex. $f(x) = x^3$, $f'(x) = x^2$, critical number $x = 0$, $f'(x) \geq 0$

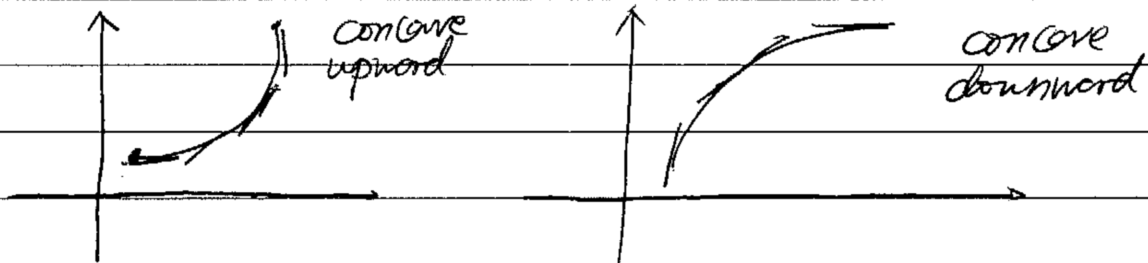
△ f'' Affects f

DEF.

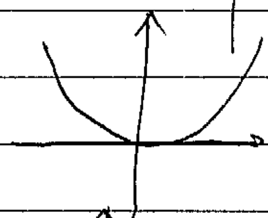
(a) If the graph of f lies above all its tangents on an interval I, then it is called concave upward

(b) ——— below ———
——— downward

Ex.

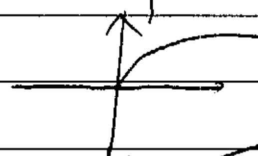


Ex. (一) $f(x) = x^2$



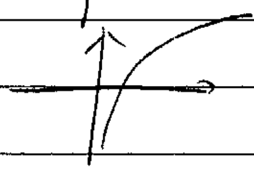
concave upward

(二) $f(x) = \sqrt{x}$



concave downward

(三) $f(x) = \ln x$



concave downward

THM. [Concavity Test]

(a) If $f''(x) > 0$ for all x in I, then it is concave upward on I

(b ——— < 0 ——— downward —

Ex. (一) $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2 > 0$

(二) $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}}$

$$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-\frac{3}{2}} = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$= \frac{-1}{4\sqrt{x^3}} < 0$$

(三) $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2} < 0$

<u>Recall</u>.

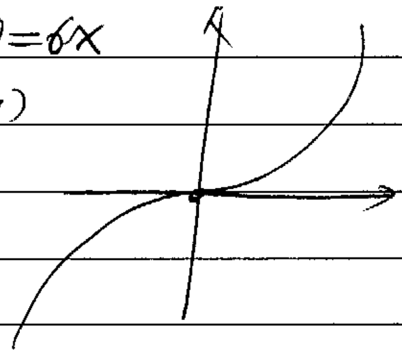
f'	$+$	$-$	0	change signs at c
	↗	↘	critical number	local maximum/minimum
f''	$+$	$-$		change signs at c
	CU	CD		$\boxed{\text{inflection point}}$

<u>DEF.</u> [Inflection Point].
Points where f is continuous at and also changes concavity at.

<u>Ex.</u> $f(x) = x^3$, $f'(x) = 3x^2$ $f''(x) = 6x$

Interval	$(-\infty, 0)$	$(0, +\infty)$
f''	$-$	$+$

0 is an inflection point



<u>THM.</u> [The 2^{nd} Derivative Test]
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minium at c.
(b) ——— < 0 ——— maximum ——.

<u>Ex</u>. (一) $f(x) = x^2$, $\begin{cases} f'(x) = 2x \\ f''(x) = 2 \end{cases}$, $f'(0) = 0$, $f''(0) = 2$

(二) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
$f'(x) = 12x(x+1)(x-2) = 12x^3 - 12x^2 - 24x$
$f'(x) = 36x^2 - 24x - 24$

$$\begin{cases} f'(-1) = 0 & f''(-1) = 36 > 0 \\ f'(0) = 0 & f''(0) = -24 < 0 \\ f'(2) = 0 & f''(2) = 72 > 0 \end{cases}$$

§4.5 Summery of Curve Sketching.

STEPS.

A. Domain

B. Intercepts, $\begin{cases} f(0) & y\text{-intercepts} \\ f(x)=0 & x\text{-intercepts} \end{cases}$

C. Symmetry, $\begin{cases} \text{Even.} & f(-x)=f(x) & x^2 \\ \text{Odd} & f(-x)=-f(x) & x^3 \\ \text{Periodic} & f(x+p)=f(x) & \sin x,\ p=2\pi \end{cases}$

D. Asymptotes.

(一) Horizontal, $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$ $\tan^{-1}x$

(二) Vertical: $\lim_{x\to a^{\pm}} f(x)=\pm\infty$ $\frac{1}{x}$

(三) Slant* $\lim_{x\to\infty}[f(x)-(mx+b)]=0$

$\to y=mx+b$ slant asymptote

E. Intervals of Increase/Decrease f'

F. Local Maximum/Minimum

G. Concavity and Inflection Points f''

H. Sketch

Ex. $f(x)=x+\frac{1}{x}$

A. Domain $=\{x\neq 0\}=(-\infty,0)\cup(0,\infty)$

B. $\begin{cases} 0 \text{ is not in domain} \Rightarrow \text{no } y\text{-intercept} \\ f(x)=x+\frac{1}{x}=0 \Rightarrow x^2+1=0,\ \text{no solution} \Rightarrow \text{no } x\text{-intercept} \end{cases}$

C. $f(-x)=-x+\frac{1}{-x}=-f(x)$ odd

D. $\lim_{x\to\infty} f(x)=\infty$, $\lim_{x\to-\infty} f(x)=-\infty$ no-horizontal-asymptote

$\lim_{x\to 0^+} f(x)=+\infty$, $\lim_{x\to 0^-} f(x)=-\infty$

*. $\lim_{x\to+\infty}[f(x)-x]=0=\lim_{x\to-\infty}[f(x)-x]$

E. $f'(x) = 1 - \frac{1}{x^2}$. $\quad f'(x) = 0 \Rightarrow x = \pm 1$

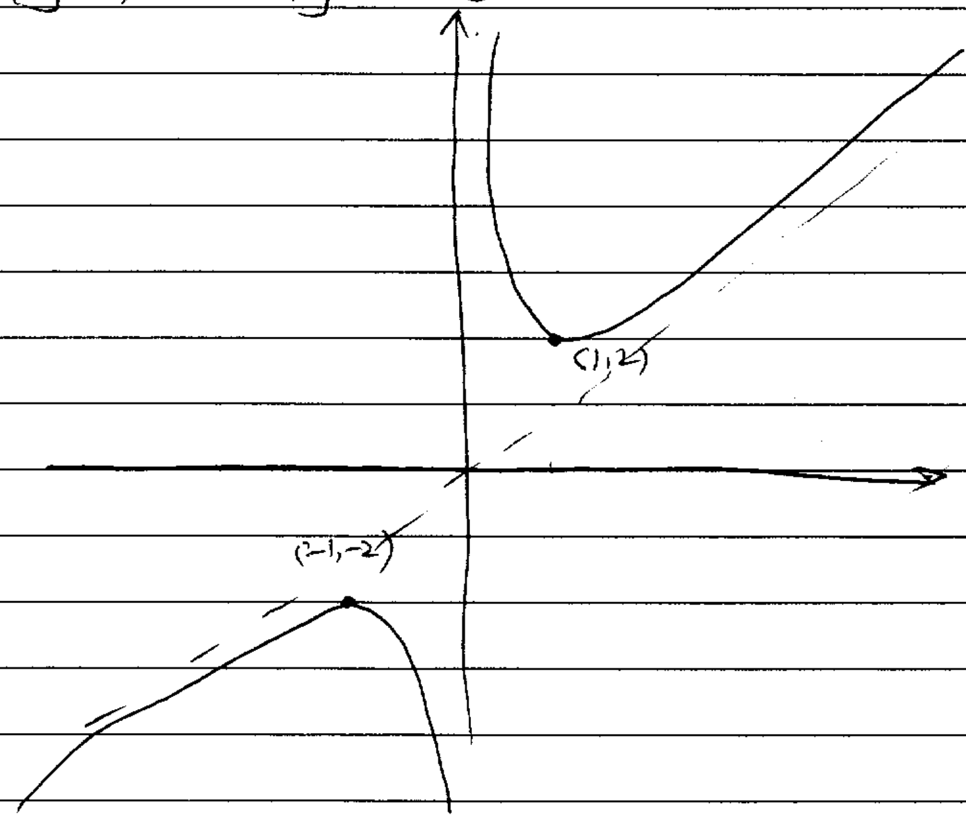
	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
f'	$+$	$-$	$-$	$+$
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

F. $f''(x) = \frac{2}{x^3}$ $\quad \begin{cases} f''(1) = 2 > 0 & \text{local min} \\ f''(-1) = -2 < 0 & \text{local max} \end{cases}$

G. $f''(x) = 0$ has no solution

$\begin{cases} f''(x) > 0 & \text{if } x > 0 \\ f''(x) < 0 & \text{if } x < 0 \end{cases}$

H.



Ex. $f(x) = \frac{1}{2}(e^{x} + e^{-x})$

A. Domain $= (-\infty, +\infty)$

B. $\begin{cases} f(0) = \frac{1}{2}(1+1) = 1 \Rightarrow y\text{-intercept is } (0,1) \\ f(x) > 0 \Rightarrow f(x) = 0 \text{ has no solution} \Rightarrow \text{no } x\text{-intercept} \end{cases}$

C. $f(-x) = \frac{1}{2}(e^{-x} + e^{x}) = f(x) \Rightarrow$ Even.

D. Every point x, $f(x)$ is well defined, so no vertical asymptote exists. Also

$$\lim_{x\to+\infty} f(x) = +\infty = \lim_{x\to-\infty} f(x)$$

shows no horizontal asymptote exists.

E. $f'(x) = \frac{1}{2}(e^{x} + e^{-x}\cdot(-1)) = \frac{1}{2}(e^{x} - e^{-x})$

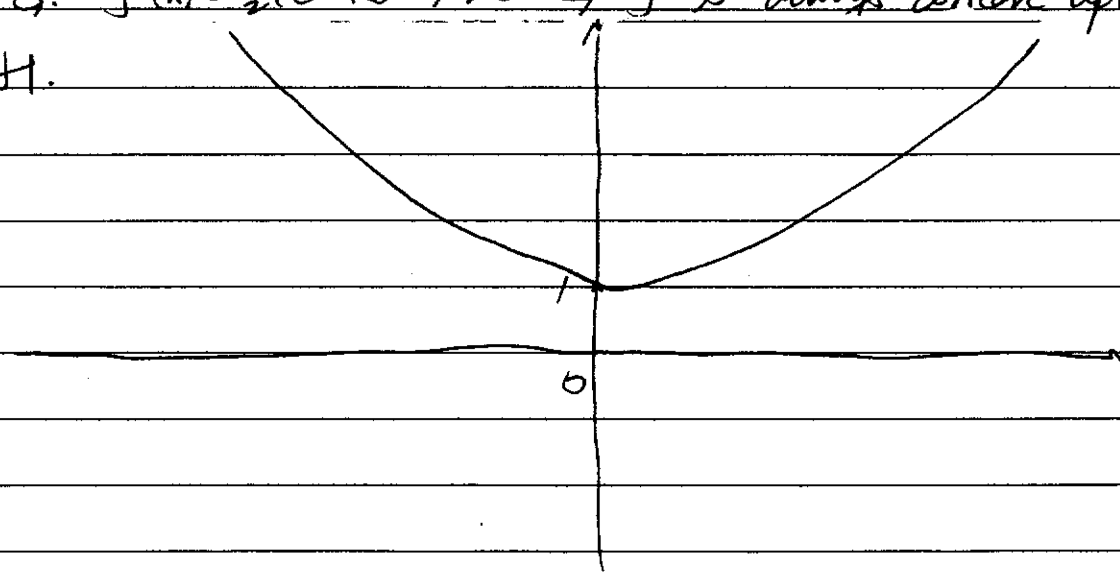
$f'(x) = 0 \Leftrightarrow e^{x} - e^{-x} = 0 \Leftrightarrow e^{2x} - 1 = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$.

	$(-\infty, 0)$	$(0, +\infty)$
f'	<0	>0
f	$\searrow$	$\nearrow$

F. Since the only critical number is 0, we only need to see this point. From E, f' changes from $-$ to $+$, then $f(0) = 1$ is a local min.

G. $f''(x) = \frac{1}{2}(e^{x} + e^{-x}) > 0 \Rightarrow f$ is always concave upward.

H.



Ex. $y = f(x) = x^3 - 12x^2 + 36x$.

A. Domain $= (-\infty, +\infty)$

B. $f(0) = 0 \Rightarrow$ y-intercept (0.0)

$f(x) = 0 \Leftrightarrow x(x-6)^2 = 0 \Leftrightarrow x = 0.6 \Rightarrow$ x-intercepts $(0.0), (6.0)$

C. $f(-x) = (-x)^3 - 12(-x)^2 + 36(-x) = -x^3 - 12x^2 - 36x \neq \begin{cases} f(x) \\ -f(x) \end{cases}$

No symmetry.

D. No vertical asymptote exists since $f(x)$ is defined on $(-\infty, +\infty)$

$\lim_{x\to+\infty} f(x) = +\infty$, $\lim_{x\to-\infty} f(x) = -\infty \Rightarrow$ No Horizontal Asymptote.

E. $f'(x) = 3x^2 - 24x + 36 = 3(x^2 - 8x + 12) = 3(x-2)(x-6)$

$f'(x) = 0 \Leftrightarrow x = 2, 6 \Rightarrow$ critical numbers are $2, 6$

	$(-\infty, 2)$	$(2, 6)$	$(6, +\infty)$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

F. f' changes from $+$ to $-$ at $2 \Rightarrow$ local max $f(2) = 32$

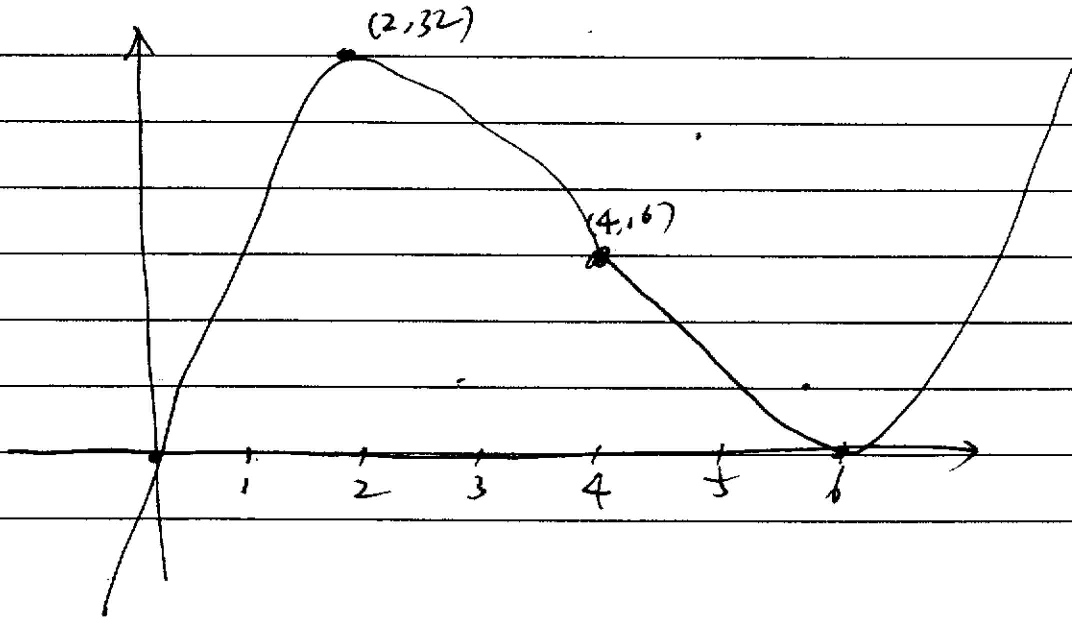
f' changes from $-$ to $+$ at $6 \Rightarrow$ local min $f(6) = 0$

G. $f''(x) = 6x - 24 = 6(x-4)$

$f''(x) = 0 \Leftrightarrow x = 4$ inflection point $(4, f(4)) = (4, 16)$

	$(-\infty, 4)$	$(4, +\infty)$
f''	<0	>0
f	downward	upward

H.



§4.4 Indeterminate Forms & L'Hospital's Rule.

Ex. $\lim\limits_{x\to 3} \frac{x^2-9}{x-3} = \lim\limits_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim\limits_{x\to 3}(x+3) = 6$

Fact. $\lim\limits_{x\to 3}(x^2-9) = 0$, $\lim\limits_{x\to 3}(x-3) = 0$ "$\frac{0}{0}$" Type.

Ex. $\lim\limits_{x\to \infty} \frac{x+2}{2x-1} = \lim\limits_{x\to 0} \frac{1+\frac{2}{x}}{2-\frac{1}{x}} = \frac{1}{2}$

Fact $\lim\limits_{x\to\infty}(x+2) = \infty = \lim\limits_{x\to\infty}(2x-1)$ "$\frac{\infty}{\infty}$" Type

THM. [L'Hospital's Rule]
Suppose f and g are differentiable on an open interval I that contains a. When computing $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ reaches either "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" type, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

if the LHS (right hand side) exists, or is $\pm\infty$.

Ex. $\lim\limits_{x\to 3}\frac{x^2-9}{x-3} = \lim\limits_{x\to 3}\frac{2x}{1} = 6$

$\lim\limits_{x\to\infty}\frac{x+2}{2x-1} = \lim\limits_{x\to\infty}\frac{1}{2} = \frac{1}{2}$

Ex. $\lim\limits_{x\to 1}\frac{\ln x}{x-1}$.

$\lim\limits_{x\to 1}\ln x = 0 = \lim\limits_{x\to 1}(x-1)$. $\lim\limits_{x\to 1}\frac{\ln x}{x-1} \overset{L'}{=} \lim\limits_{x\to 1}\frac{(\frac{1}{x})}{1} = 1$

Ex. $\lim\limits_{x\to 0}\frac{\sin x}{x} \overset{L'}{=} \lim\limits_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0}\frac{\cos x}{1} = 1$

Ex. $\lim\limits_{x\to\infty}\frac{e^x}{x^2+x} \overset{L'}{=} \lim\limits_{x\to 0}\frac{e^x}{2x+1} \overset{L'}{=} \lim\limits_{x\to 0}\frac{e^x}{2} = \infty$

Ex. $\lim\limits_{x\to 1} \dfrac{x^2-x-2}{x-1}$

If use L'Hospital, $\lim\limits_{x\to 1} \dfrac{x^2-x-2}{x-1} = \lim\limits_{x\to 1} \dfrac{2x-1}{1} = 1$, which is wrong!

$\lim\limits_{x\to 1}(x-1) = 0 \neq -2 = \lim\limits_{x\to 1}(x^2-x-2)$. This is not a "$\frac{0}{0}$" type.

$\lim\limits_{x\to 1} \dfrac{x^2-x-2}{x-1} = \dfrac{-2}{0} = -\infty$

Remark. Check the type before applying L'Hospital's Rule

• Indeterminate Product: $\boxed{\text{"}0\cdot\pm\infty\text{" Type}}$

Ex.

$\lim\limits_{x\to 0^+} x\ln x$, $\quad \lim\limits_{x\to 0^+} x = 0$. $\lim\limits_{x\to 0^+} \ln x = -\infty$

Method I $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \underbrace{\dfrac{\ln x}{\left(\frac{1}{x}\right)}}_{\text{"}\frac{\infty}{\infty}\text{"}} \overset{L'}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$

Method II? $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \underbrace{\dfrac{x}{\left(\frac{1}{\ln x}\right)}}_{\text{"}\frac{0}{0}\text{"}} \overset{L'}{=} \lim\limits_{x\to 0^+} \dfrac{1}{-\frac{1}{x\ln^2 x}}$

WRONG approach $= \lim\limits_{x\to 0^+} -x\ln^2 x$ WORSE than before!

• Indeterminate Difference.

Ex. $\lim\limits_{x\to \frac{\pi}{2}^-} (\sec x - \tan x)$ "$\infty - \infty$"

$\lim\limits_{x\to \frac{\pi}{2}^-} \sec x = \infty = \lim\limits_{x\to \frac{\pi}{2}^+} \tan x$

$\lim\limits_{x\to \frac{\pi}{2}^-} (\sec x - \tan x) = \lim\limits_{x\to \frac{\pi}{2}} \underbrace{\dfrac{1-\sin x}{\cos x}}_{\text{"}\frac{0}{0}\text{"}} \overset{L'}{=} \lim\limits_{x\to \frac{\pi}{2}} \dfrac{-\cos x}{-\sin x} = 0$

• Indeterminate Power. $\lim_{x\to a}[f(x)]^{g(x)}$

"0^0" "∞^0", "1^∞"

KEY: Natural logarithmic: $\ln[f(x)]^{g(x)} = g(x)\cdot\ln[f(x)]$

$$\begin{cases} \ln "0^0" = "0\cdot\ln 0" = 0\cdot(-\infty) \\ \ln\infty^0 = "0\cdot\ln\infty" = 0\cdot\infty \\ \ln 1^\infty = \infty\cdot\ln 1 = \infty\cdot 0 \end{cases}$$

Ex. $\lim_{x\to 0}\underset{\substack{\uparrow\\ 0^0}}{x^x} = ?$

$y = x^x \Rightarrow \ln y = x\ln x$

$\Rightarrow \lim_{x\to 0}\ln y = \lim_{x\to 0} x\ln x = 0$

$\Rightarrow \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0}\ln y} = e^0 = 1$

Ex. $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$

"1^∞", $y = (1+\sin 4x)^{\cot x} \Rightarrow \ln y = \underset{"\infty\cdot 0"}{\cot x\,[\ln(1+\sin 4x)]}$

$= \dfrac{\ln(1+\sin 4x)}{\tan x}$

$\lim_{x\to 0^+}\ln y \overset{L'}{=} \lim_{x\to 0^+}\dfrac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$

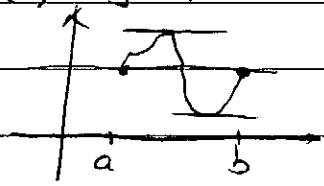
$\lim_{x\to 0^+} y = e^{\lim_{x\to 0}\ln y} = e^4$

§4.2 The Mean Value Theorem.

THM. [Rolle's]

If (一) f is continuous on $[a,b]$, (二) f' exists on (a,b) (differentiable), (三) $f(a)=f(b)$, then there exists a number c in (a,b) such that $f'(c)=0$

Remark



Ex. (一) $f(x)=2$ constant function $a=0.\ b=3 \Rightarrow$ Any c

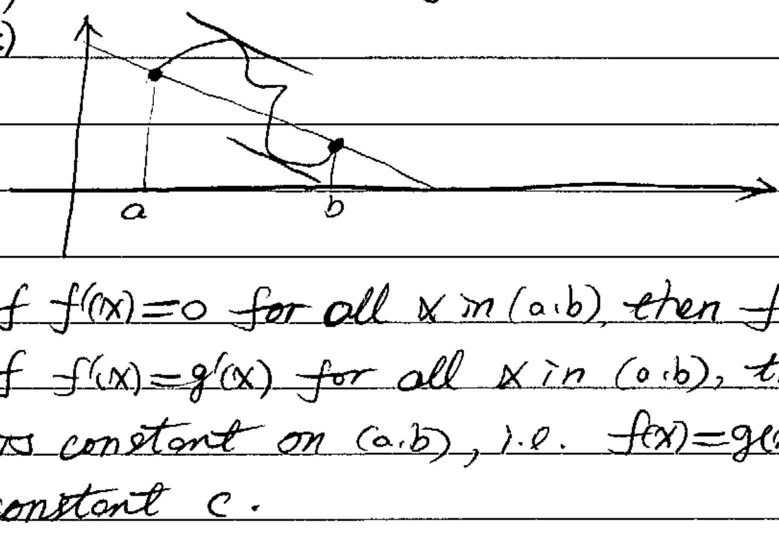
(二) $f(x)=x^2$ $a=-2,\ b=2$ $c=0$

THM. [Mean Value]

If (一) f is continuous on $[a,b]$, (二) f' exists on (a,b), then there exists a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Remark (一) When $f(a)=f(b) \Rightarrow f'(c)=0 \Rightarrow$ Rolle's

(二)



THM. If $f'(x)=0$ for all x in (a,b), then $f(x)$ is constant on (a,b)

Corol. If $f'(x)=g'(x)$ for all x in (a,b), then $f(x)-g(x)$ is constant on (a,b), i.e. $f(x)=g(x)+C$ for some constant C.

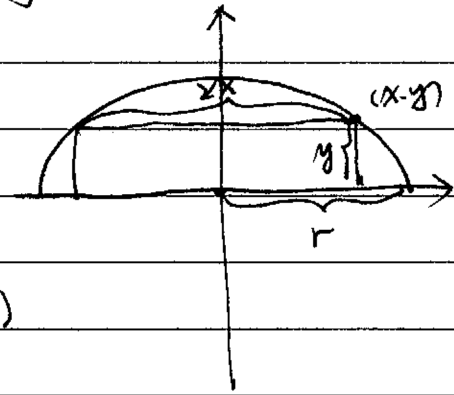
Ex. $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$.

Let $f(x)=\tan^{-1}x+\cot^{-1}x$. Then $f'(x)=\frac{1}{1+x^2}-\frac{1}{1+x^2}=0 \Rightarrow f(x)=C$

$C=f(1)=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$

§4.7 Optimization Problems.

Ex. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



$A = 2xy$

$x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2}$

$\Rightarrow A(x) = 2x\sqrt{r^2-x^2}$

[Q] Find the max of $A(x)$ on $[0, r)$

$$A'(x) = 2\sqrt{r^2-x^2} + 2x\cdot\frac{-2x}{2\sqrt{r^2-x^2}}$$

$$= 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}}$$

$$= \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}}$$

$A'(x)=0 \Leftrightarrow r^2=2x^2 \Leftrightarrow x=\frac{r}{\sqrt{2}}$ ← This is the only critical number, must be max.

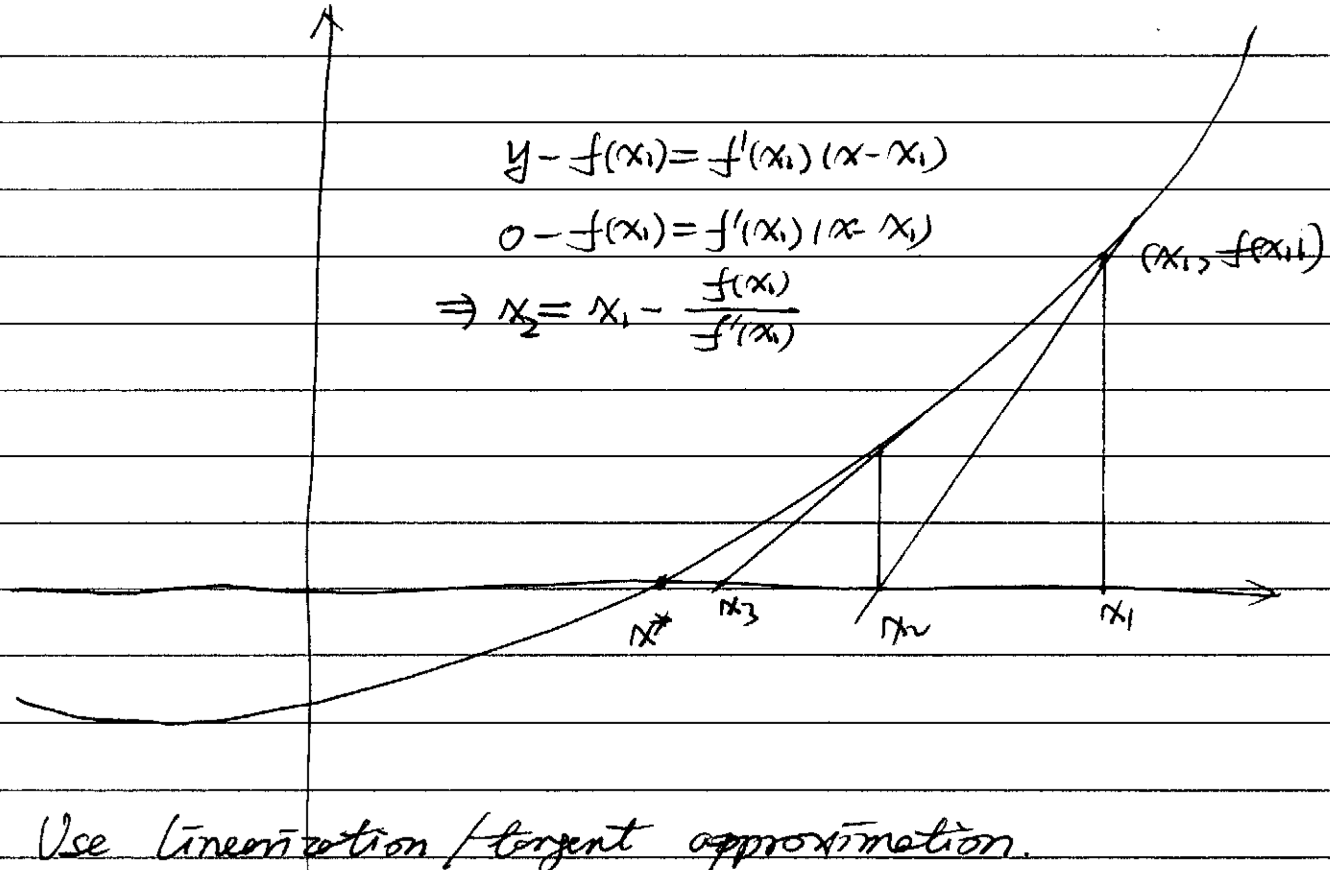
$A''(x) = \frac{-6r^2x+4x^3}{(r^2-x^2)^{3/2}}$, $A''\left(\frac{r}{\sqrt{2}}\right) = -\frac{8r}{\sqrt{r^3}} < 0$

$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2-\frac{r^2}{2}} = r^2$

§4.8 Newton's Method

<u>Aim</u>: Find the zero/root of a function $f(x)$.
i.e. solve $f(x)=0$.

<u>Ex</u>. $48x(1+x)^{60} - (1+x)^{60} + 1 = 0$



$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$0 - f(x_1) = f'(x_1)(x - x_1)$$
$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Use linearization / tangent approximation.

$$\underline{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Pick x_1, recurrence $\Rightarrow$ $x^* = \lim_{n\to\infty} x_n$

<u>Ex</u>. $\sqrt[6]{2} = 2^{\frac{1}{6}} = ?$

$x^6 = 2$, $\underbrace{x^6 - 2}_{f(x)} = 0$, $f'(x) = 6x^5$ $\Rightarrow x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$

$x_1 = 1$, $x_2 = 1 - \frac{1-2}{6} = \frac{7}{6} = 1.166667$

$x_3 = \frac{7}{6} - \frac{(\frac{7}{6})^6 - 2}{6(\frac{7}{6})^5} = 1.12644318$

...

$x \approx 1.2246205$ —

§4.9 Antiderivatives.

DEF. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Ex. $f(x) = 3x^2$

$F_1(x) = x^3$, $F_2(x) = x^3 + 1$, $F_3(x) = x^3 + 2014$

THM. If F is an antiderivative of f on an interval I, so is $F(x) + C$, where C is an arbitrary constant.

Terminology. Most General Antiderivative

$$F(x) + C$$

Ex. (一) $f(x) = \sin x$

$(\cos x)' = -\sin x \Rightarrow (-\cos x)' = \sin x$

$F(x) = -\cos x + C$.

(二) $f(x) = \frac{1}{x}$

$(\ln x)' = \frac{1}{x}$ $\quad x > 0$ $\qquad (\ln|x|)' = \frac{1}{x}$

$(\ln(-x))' = \frac{-1}{-x} = \frac{1}{x}$ $\quad x < 0$

$F(x) = \ln|x| + C$

P345. Table

Ex. Find all function g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

$(x^n)' = nx^{n-1}$

$f(x) = x^n \quad n \neq -1$

$F(x) = \frac{x^{n+1}}{n+1} + C$

$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$

$\sin x \to -\cos x$, $\quad 4\sin x \to -4\cos x$

$x^4 \to \frac{x^5}{5} \qquad 2x^4 \to \frac{2x^5}{5}$

$x^{-\frac{1}{2}} \to \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$

$g(x) = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$.

DEF. [Indefinite Integrals], $\int f(x)\,dx = F(x) + C$

$\int x^2 dx = \frac{x^3}{3} + c$ $\int \frac{1}{x} dx = \ln|x| + c$

$\int \sin x \, dx = -\cos x \cancel{+} c$ $\int e^x dx = e^x + c$

Ex. [1st Order Differential Equation].

Find $f(x)$ such that $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$

$$f(x) = \int f'(x) dx$$
$$= \int \left[e^x + \frac{20}{1+x^2}\right] dx$$

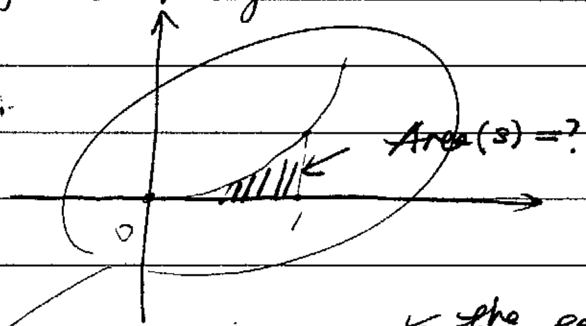
$$= e^x + 20 \arctan x + c \quad \Rightarrow f'(x) = e^x + \frac{20}{1+x^2}$$

$f(0) = 1 + 20 \cdot 0 + c = 2 \Rightarrow c = -3$

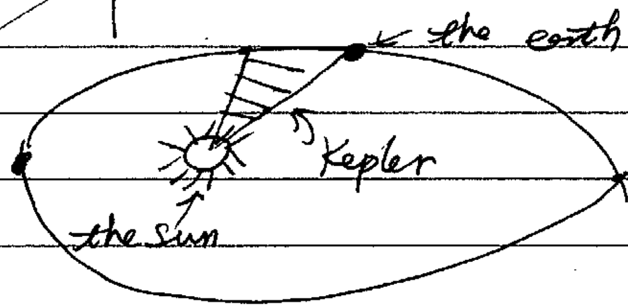
$f(x) = e^x + 20 \arctan x - 3.$

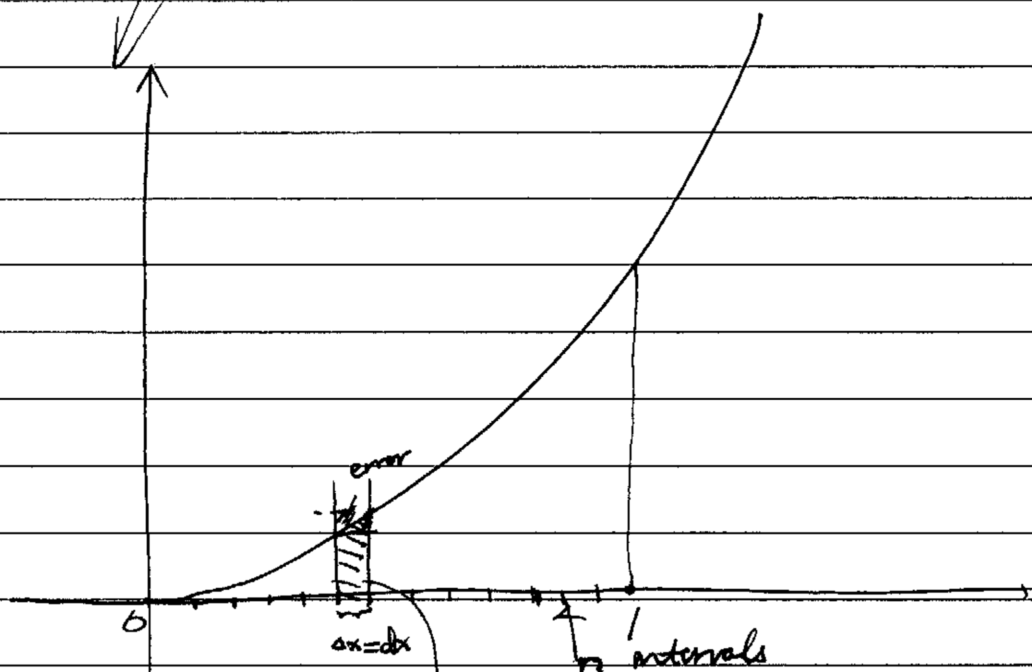
§5.1 ~ 5.2. Definite Integrals.

[Q]. $f(x) = x^2$



ASIDE.

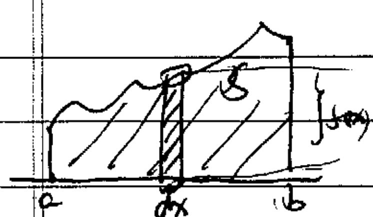




$$\text{Area}(S) = \sum \text{Area}(S_i) \approx \sum A(\text{Rectangle})$$

$$= \lim_{|dx| \to 0} \sum A(\text{Rectangle}) = \lim_{n \to \infty} \sum A(\text{Rectangle})$$

Notation

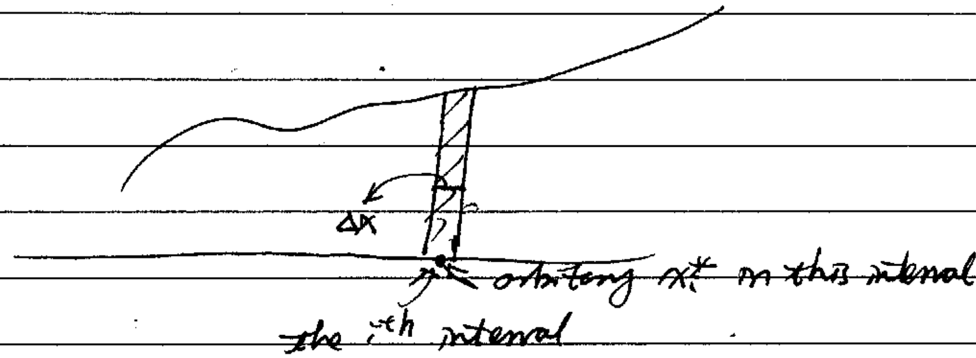


$$\lim \text{sum} = \lim \sum_i f(x)\,dx$$

$$= \int_a^b f(x)\,dx = \text{Area}(S)$$

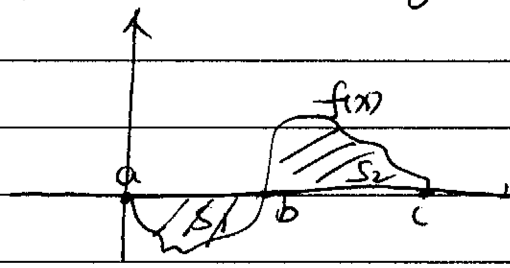
DEF: [Definite Integral]

$$\text{Area}(S)=\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$



THM. If f is continuous on $[a,b]$ (or has only a finite number of jump discontinuities), then f is integrable on $[a,b]$
i.e., $\int_a^b f(x)\,dx$ exists.

Remark. IMPORTANT:
This area is a signed ~~arg~~ area.

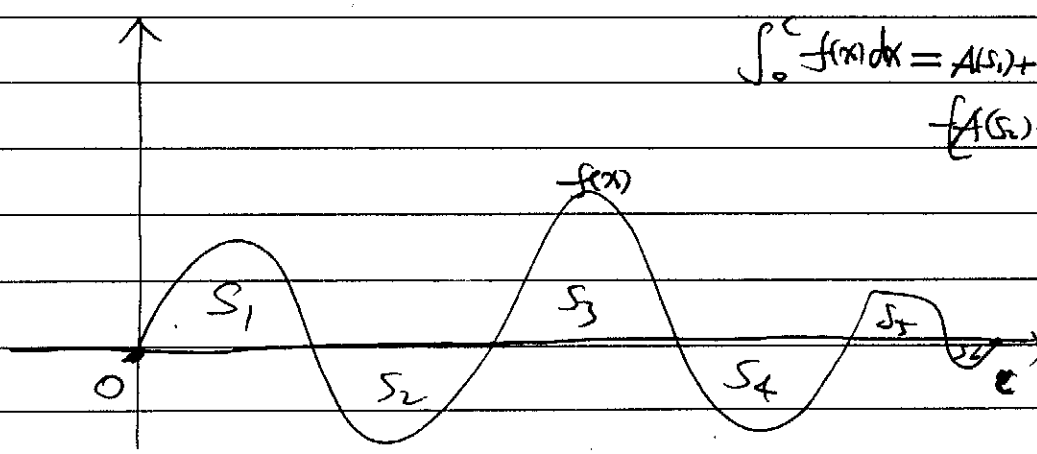


$$\int_b^c f(x)\,dx = \text{Area}(S_2) > 0$$

$$\int_a^b f(x)\,dx = -\text{Area}(S_1) < 0$$

$$\int_a^c f(x)\,dx = \text{Area}(S_2) - \text{Area}(S_1)$$

$$\int_0^c f(x)\,dx = A(S_1)+A(S_3)+A(S_5) - [A(S_2)+A(S_4)+A(S_6)]$$

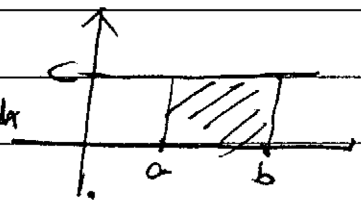


Properties:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

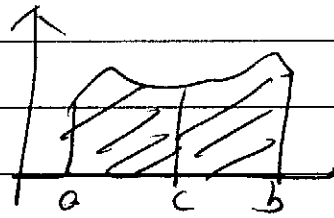
   When $b=a$ $\int_a^a f(x)\,dx = -\int_a^a f(x)\,dx \Rightarrow \int_a^a f(x)\,dx = 0$

2. $\int_a^b C\,dx = \underbrace{C}_{\text{constant}}(b-a)$



3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\int_a^b C f(x)\,dx = C\int_a^b f(x)\,dx$



5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

6. If $f(x) \geq 0$ ~~for~~ $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

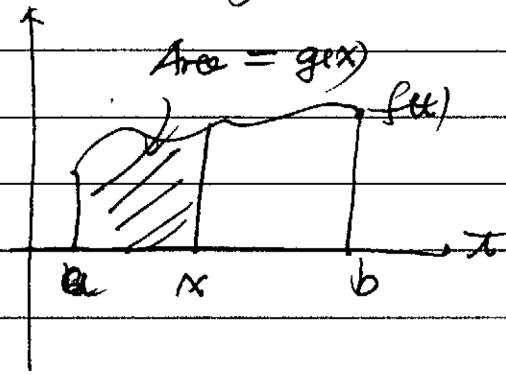
   $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

§5-3. Fundamental Theorem of Calculus.

THM. Suppose f is continuous on $[a,b]$

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
   Antiderivative

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative.



Ex. $f(t) = t^2$. $g(x) = \int_0^x t^2\,dt \Rightarrow g'(x) = f(x) = x^2$

$$g(x) = F(x) - F(0) = \left(\frac{x^3}{3} + c\right) - \left(\frac{0^3}{3} + c\right) = \frac{x^3}{3}$$

Ex. (-) $g(x) = \int_0^x \sqrt{1+t^2}\,dt$ $\qquad f(t) = \sqrt{1+t^2}$

$$g'(x) = f(x) = \sqrt{1+x^2}$$

(=) $\frac{d}{dx}\left(\int_1^{x^4} \sec t\,dt\right)$

$g(x) = \int_1^{x^4} \sec t\,dt \qquad g'(x) = ?$

$u(x) = x^4$. $\quad g(u) = \int_1^u \sec t\,dt$

$\frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx}$ ← Chain Rule

$= \sec u \cdot 4x^3$

$= 4x^3 \sec(x^4)$

Chain Rule

$\Rightarrow g(x) = \int_a^{u(x)} f(t)\,dt$

$g'(x) = f(u(x)) \cdot u'(x)$

(三)* $\frac{d}{dx}\left(\int_x^{x^2} \cos t\, dt\right)$

$$g(x) = \int_x^{x^2} \cos t\, dt = \int_x^0 \cos t\, dt + \int_0^{x^2} \cos t\, dt$$

$$= -\int_0^x \cos t\, dt + \int_0^{x^2} \cos t\, dt$$

$$g'(x) = \frac{d}{dx}\left(-\int_0^x \cos t\, dt\right) + \frac{d}{dx}\left(\int_0^{x^2} \cos t\, dt\right)$$

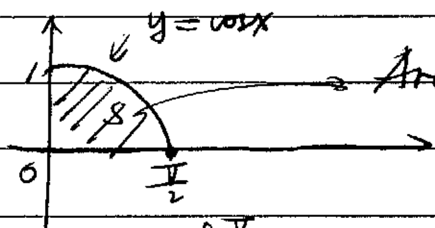
$$= -\cos x + \cos x^2 \cdot 2x = 2x\cos(x^2) - \cos x$$

Remark. $g(x) = \int_x^{x^2} \cos t\, dt$ $\qquad \int \cos t\, dt = \underline{\sin t} + C$

$$= \sin(x^2) - \sin(x)$$

$$g'(x) = \cos(x^2) \cdot 2x - \cos x$$

Ex.



Area(S) = ? $\qquad \int \cos x\, dx = \underline{\sin x} + C$

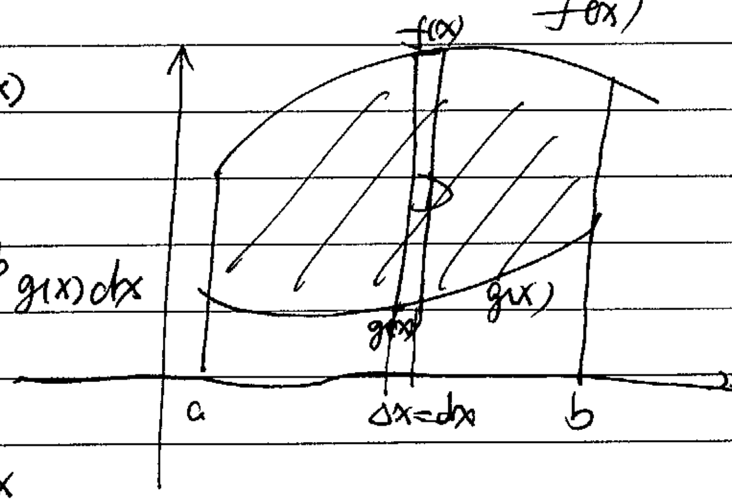
$$\text{Area}(S) = \int_0^{\frac{\pi}{2}} \cos x\, dx = \sin x \Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0 = 1$$

Ex. (一) $\int_1^3 e^x\, dx = e^x \Big|_{x=1}^{x=3} = e^3 - e^1$ $\qquad \int e^x\, dx = \underline{e^x} + C$

$$= e^3 - e$$

§6.1 Areas Between Curves.



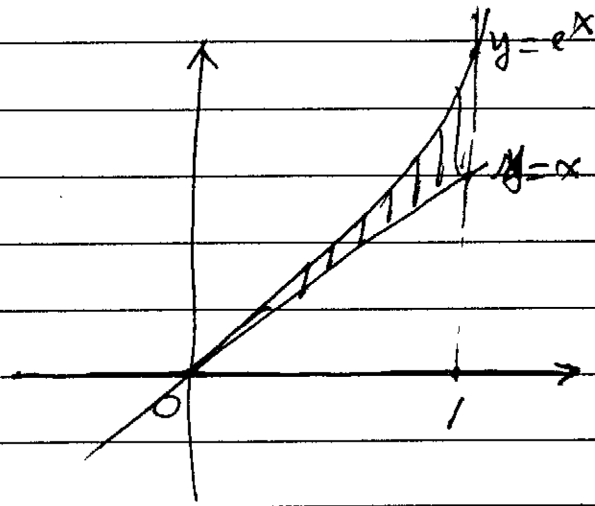
Easy Case. $f(x) \geq g(x)$

$a \leq x \leq b$

$$Area(D) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$Area(D) = \int_a^b \underbrace{[f(x)-g(x)]}_{\text{one side}}\,dx$$

Ex. Find the area of the region bounded by
$y=e^x$, $y=x$ $x=0$, $x=1$



$$Area = \int_0^1 (e^x - x)\,dx$$

$$= \int_0^1 e^x\,dx - \int_0^1 x\,dx$$

$$= e^x \Big|_{x=0}^{x=1} - \frac{x^2}{2}\Big|_{x=0}^{x=1}$$

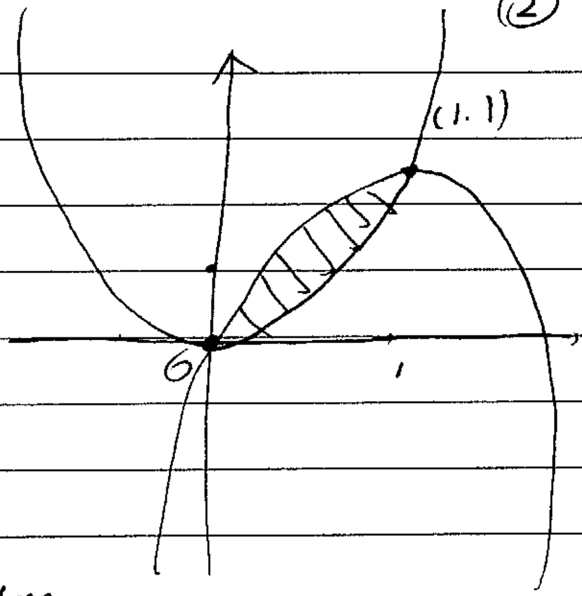
$$= e - 1 - \frac{1}{2} = e - 1.5$$

Ex. Find the area of the region bounded by $y=x^2$ & $y=2x-x^2$

IMPORTANT STEP.

Find the intersection points.

$$\begin{cases} y = x^2 \\ y = 2x - x^2 \end{cases} \Rightarrow x^2 = 2x - x^2 \Rightarrow x^2 = x \quad x(x-1)=0 \quad \begin{cases} (0,0) \\ (1,1) \end{cases}$$



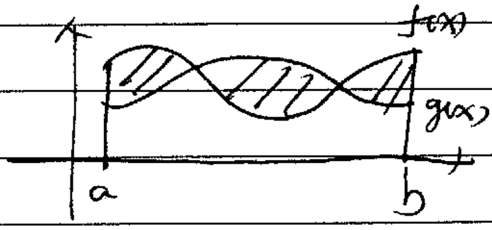
$$Area = \int_0^1 \left[(2x-x^2)-x^2\right] dx$$

$$= \int_0^1 2(x-x^2)\, dx$$

$$= 2\cdot\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3}$$

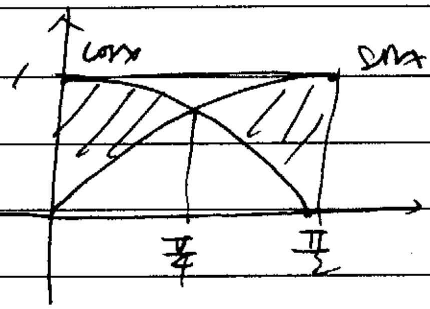
Normal Case.

Hard Case:



$$Area = \int_a^b |f(x)-g(x)|\, dx$$

Ex. $y=\sin x$, $y=\cos x$, $x=0$, $x=\frac{\pi}{2}$



$$A = \int_0^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$$

$$= \int_0^{\frac{\pi}{4}} |\sin x - \cos x|\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} |\sin x - \cos x|\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx$$

$$= (\sin x + \cos x)\Big|_{x=0}^{x=\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}}$$

$$= 2\sqrt{2} - 2$$

Special Case: $A = \int_c^d |f(y)-g(y)|\, dy$

Ex. $y = x-1$ $\quad y^2 = 2x+6$

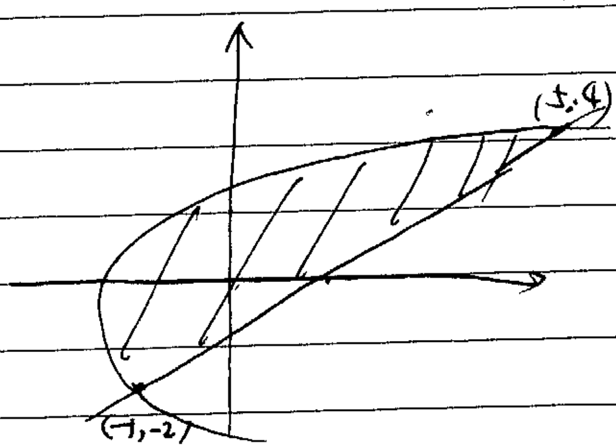
$x = y+1$ $\quad x = \frac{1}{2}(y^2-6)$

$$\begin{cases} y = x-1 \\ y^2 = 2x+6 \end{cases}$$

$\cancel{(x-1)^2 = 2x+6}$

$\cancel{x^2-4x-5=0}$

$\cancel{(x-5)(x+1)=0}$



$\Rightarrow (-1,-2),\ (5,4)$

$$A = \int_{-2}^{4} \left[(y+1) - \frac{1}{2}(y^2-6) \right] dy$$

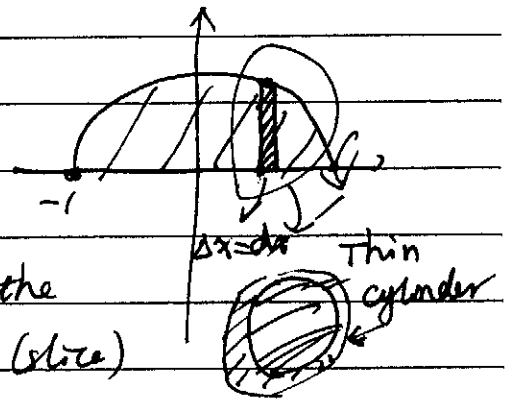
$$= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy$$

$$= \left(-\frac{1}{6}y^3 + \frac{y^2}{2} + 4y\right)\Big|_{y=-2}^{y=4} = 18$$

§6.2 Volumes.

Question: Find the volume of the solid obtained by rotating ______ R ______ about ______ L ______.

Ex. R: The region bounded by $y=\sqrt{1-x^2}$ and x-axis
L: x-axis.



(i) $V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi$

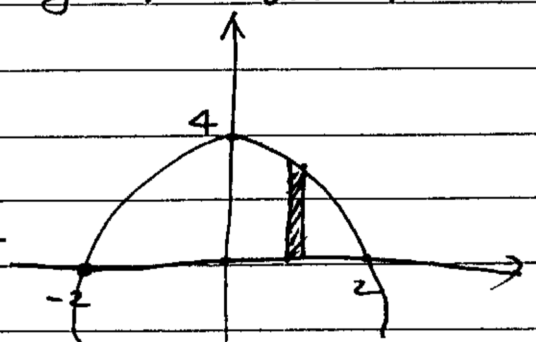
(ii) Thickness $=\Delta x=dx$
Base: disk, Radius = Distance from the outmost point (slice) to L

Radius $r=y=\sqrt{1-x^2}$
Area of Base $=\pi r^2=\pi(1-x^2)$

Volume of small cylinder $=\pi(1-x^2)\cdot dx$
Sum + limit $=\int$

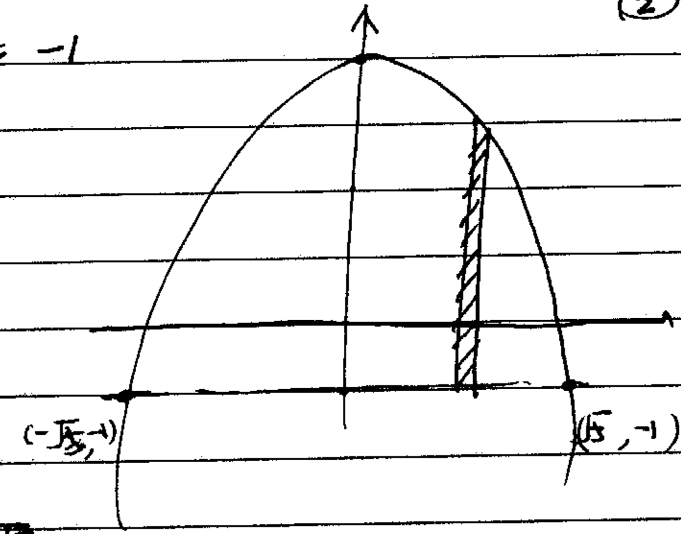
$$V=\int_{-1}^{1}\pi(1-x^2)\,dx=\pi\left(x-\frac{x^3}{3}\right)\Big|_{x=-1}^{x=1}=\frac{4}{3}\pi$$

Ex. R: The region bounded by $y=4-x^2$ and x-axis
L: x-axis



Thickness $=dx$
Radius $=y=4-x^2$
$V=\int_{-2}^{2}\pi y^2\,dx=\int_{-2}^{2}\pi(4-x^2)^2\,dx=\frac{512\pi}{15}$

Ex. R: $y = 4 - x^2$, $\cancel{y}= -1$
L: $y = -1$



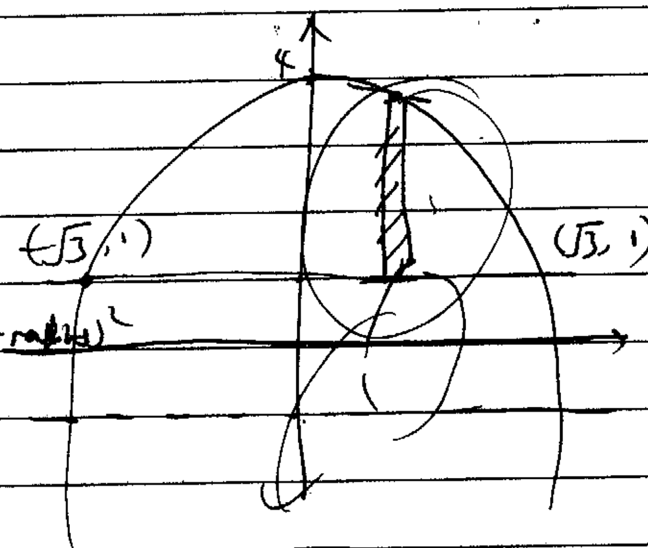
$$V = \int_{-\sqrt{5}}^{\sqrt{5}} A \cdot dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \pi r^2 \, dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \pi \cancel{(y+1)^2 dx}$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \pi (4 - x^2 + 1)^2 dx = \frac{80\sqrt{5}}{3}\pi$$

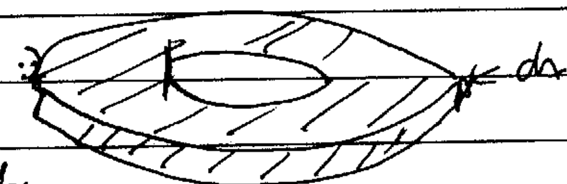
Ex. R: $y = 4 - x^2$ $y = 1$
L: $y = -1$



Thickness $= dx$

$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

$$= \pi(y+1)^2 - \pi(2)^2$$



$$= \pi(y^2 + 2y - 3)$$

$$V = \int_{-\sqrt{3}}^{\sqrt{3}} A\,dx = \int_{-\sqrt{3}}^{\sqrt{3}} \pi(y^2+2y-3)\,dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \pi\left[(4-x^2)^2 + 2(4-x^2) - 3\right] dx$$

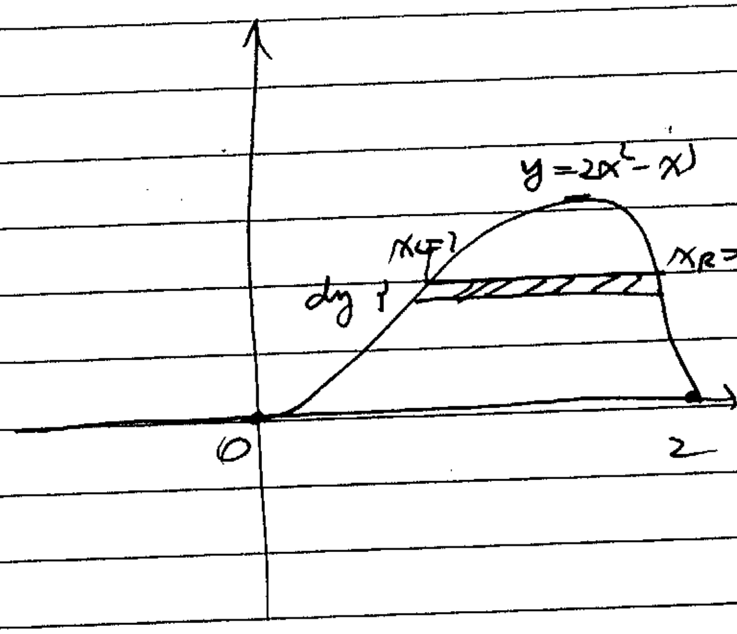
$$= \cancel{\frac{128\sqrt{3}}{5}\pi} \quad \frac{128\sqrt{3}}{5}\pi$$

§6.3 Volumes by Cylindrical shells

Ex. R: $y = 2x^2 - x^3$ x-axis

L: y-axis.

$$V = \int (\pi x_R^2 - \pi x_L^2)\,dy$$



$$V = \underline{\lim \sum} \,[\text{circumference}][\text{height}][\text{thickness}]$$

$$= \int_0^2 2\pi x \cdot y \quad dx$$

$$= \int_0^2 2\pi x\,(2x^2 - x^3)\,dx$$

$$= \frac{16}{5}\pi$$