RIEMANNIAN HOLONOMY GROUPS OF STATISTICAL MANIFOLDS

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ABSTRACT. Normal distribution manifolds play essential roles in the theory of information geometry, so do holonomy groups in classification of Riemannian manifolds. After some necessary preliminaries on information geometry and holonomy groups, it is presented that the corresponding Riemannian holonomy group of the d-dimensional normal distribution is $SO\left(\frac{d(d+3)}{2}\right)$, $d = 1, 2, 3$. As a generalization on exponential family, a list of holonomy groups follows.

1. INTRODUCTION

Statistical manifolds, which consist of probability distribution functions, are the main objects in information geometry. In order to describe their geometric structures, a series of concepts such as α-connections, dual connections, and particularly Fisher metric[?], which makes statistical manifold Riemannian manifold, are introduced and studied. As a special example, normal distribution manifolds, defined in Definition5.1, are of great importance. When S. Amari initiated the theory of information geometry[?, ?], he found that the sectional curvature of the monistic normal distribution manifold is $-\frac{1}{2}$, which is recalled in the proof of Lemma 5.7, implying its isometry to a hyperbolic space. Rather than an amazing result, it is also the trigger for Amari to develop information geometry. Some basic defintions and results on information geometry are presented in Section 2.

Around 1926, É. Carten introduced holonomy groups in order to study and classify symmetric spaces. Indeed, he has classified irreducible symmetric spaces by considering holonomy groups. As part of the generalization of parallel tranpotations, holonomy could be defined on any vector boudle with connections[?]. Hence, as S. S. Chern believed, it plays an important role in the theory of connections. However there are seldom brilliant results except for Ambrose-Singer holonomy theorem [?, ?]. When coming to Riemannian holonomy groups, the classification for irreducible cases was solved in 1955 by M. Berger[?] and J. Simons[?]. After introducing defintions and propsitions on holonomy groups in Section 3, several useful results on classification are presented in terms of theorems and corollaries in Section 4 and also in terms of two tables in Appendix. Although, holonomy group fails to classify non-isometric manifolds due to its small number of classes, it is still essential and is applied to many fields including string theory.

In this paper, we concentrate on the Riemannian holonomy groups of statistical manifolds, especially the normal distribution manifolds. After calculating the holonomy groups of normal distribution manifolds in Theorem 5.3, Theorem 6.3 follows as a generalization on exponential family.

2. INFORMATION GEOMETRY ON STATISTICAL MANIFOLDS

We call
$$S := \{p(x; \theta) \,|\, \theta \in \Theta\}$$
a statistical manifold if x is a random variable in sample space X and $p(x; \theta)$ is the probability density function, which satisfies certain regular conditions. Here, $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$ is an

n-dimensional vector in some open subset $\Theta \subset \mathbb{R}^n$, and θ can be viewed as the coordinates on manifold S.

Definition 2.1. An n-dimensional parametric statistical model $\Theta = \{p_\theta | \theta \in \Theta\}$ is called an exponential family or of exponential type, if the probability density function can be expressed in terms of functions $C, F_1, \ldots, F_n$ and a convex function ϕ on Θ of the form ([?])

$$p(x; \theta) = \exp\{C(x) + \Sigma_i \theta_i F_i(x) - \phi(\theta)\}.$$

In addition, we call $\{\theta_i\}$ the natural parameters and ϕ the potential function.

Definition 2.2. The Riemannian metric on statistical manifolds is defined by the Fisher information matrix[?]:

$$g_{ij}(\theta) := E[(\partial_i l)(\partial_j l)] = \int (\partial_i l)(\partial_j l) p(x; \theta) dx, \quad i, j = 1, 2, \ldots, n,$$

where E denotes the expectation, $\partial_i := \frac{\partial}{\partial \theta_i}$, and $l := l(x; \theta) = \log p(x; \theta)$.

Proposition 2.3. *Suppose S is an exponential family with coordinates θ_i and potential function ϕ. Then the Fisher metric is given by*

$$g_{ij} = \partial_i \partial_j \phi.$$

Definition 2.4. A family of connections $\nabla^{(\alpha)}$ defined by Amari as follows

$$< \nabla_A^{(\alpha)} B, C > := E[(ABl)(Cl)] + \frac{1-\alpha}{2} E[(Al)(Bl)(Cl)]$$

are called α-connections, where $A, B, C \in \mathfrak{X}(S)$, $ABl = A(Bl)$, and $\alpha \in \mathbb{R}$ is the parameter.

Proposition 2.5. *The connection $\nabla^{(\alpha)}$ is torsion free for all α. The only Riemannian connection, with respect to Fisher metric, is $\nabla = \nabla^{(0)}$.*

Remark 2.6. In this paper, we focus on the Riemannian case.

Theorem 2.7. *If the Riemannian connection coefficients and α-connection coefficients are denoted by Γ_{ijk} and $\Gamma_{ijk}^{(\alpha)}$, respectively, then*

$$\Gamma_{ijk}^{(\alpha)} = \Gamma_{ijk} - \frac{\alpha}{2} T_{ijk},$$

where $T_{ijk} := E[(\partial_i l)(\partial_j l)(\partial_k l)]$. Note that $\Gamma_{ijk}^{(0)} = \Gamma_{ijk}$, which coincides with Proposition 2.4.

Proposition 2.8. *Suppose S is an exponential family with coordinates θ_i and potential function ϕ. Then the Riemannian connection coefficients are given by*

$$\Gamma_{ijk} = \partial_i \partial_j \partial_k \phi.$$

Definition 2.9. The Riemannian curvature tensor of α-connections is defined by

$$R_{ijkl}^{(\alpha)} = (\partial_j \Gamma_{ik}^{(\alpha)s} - \partial_i \Gamma_{jk}^{(\alpha)s}) + (\Gamma_{jtl}^{(\alpha)} \Gamma_{ik}^{(\alpha)t} - \Gamma_{itl}^{(\alpha)} \Gamma_{jk}^{(\alpha)t}),$$

where $\Gamma_{jk}^{(\alpha)s} = \Gamma_{jki}^{(\alpha)} g^{is}$ and (g^{is}) is the inverse matrix of the metric matrix (g_{mn}). Einstein notation is also used here.

3. HOLONOMY GROUPS

3.1. **Holonomy of a Connection in a Vector Bundle.**

Definition 3.1. Let E be a rank r vector bundle over a smooth manifold M and ∇ be a connection on E. Given a piecewise smooth loop $\gamma : [0, 1] \to M$ based at $x \in M$, we let P_γ denote the parallel transportation of ∇ and E_x denote the fiber over x. Then, $P_\gamma : E_x \to E_x$ is an invertible linear transformation, hence an element of $GL(E_x) \cong GL(r, \mathbb{R})$. The holonomy group of ∇ based at x is defined by

$$H_x(\nabla) := \{P_\gamma \in GL(E_x) | \gamma \text{ is a loop based at } x\}.$$

The restricted holonomy group based at x is its subgroup defined by

$$H_x^0(\nabla) := \{P_\gamma \in GL(E_x) | \gamma \text{ is a contratible loop based at } x\}.$$

Proposition 3.2. *If M is connected(hence path connected), then the holonomy groups on different base points are conjugate of one another in $GL(r, \mathbb{R})$. In concrete, if $x, y \in M$, and γ is a path from x to y, then*

$$H_y(\nabla) = P_\gamma H_x(\nabla) P_\gamma^{-1}.$$

As a result, we shall always omit the base point and denote the group by $H(\nabla)$. While only considering one connection ∇, we could further reduce the notation for the group by H. Now, here are several properties for the holonomy groups.

Proposition 3.3. *Let E be a rank r vector bundle over a connected manifold M, and ∇ be a connection on E, then*
  (1) H^0 is a conneted, Lie-subgroup of $GL(r, \mathbb{R})$;
  (2) H^0 is the identity component of H, hence the determinant of every matrix element is positive;
  (3) if, in addition, M is simply connected, then $H^0 = H$;
  (4) ∇ is flat iff $H^0(\nabla) = 0$.

3.2. Riemannian Holonomy.

Definition 3.4. The Riemannian holonomy group of a Riemannian manifold (M, g) is just the holonomy group of the Levi-Civita connection ∇ on the tangent bundle TM.

In another word, the Riemannian holonomy is a special case.

Proposition 3.5. *Let M be an n-dimensional Riemannian manifold and H denote its Riemannian holonomy group, then*
  (1) H is a (compact) (closed) Lie-subgroup of $O(n)$(n-dimensional orthogonal group);
  (2) if M is orientable, then H is a subgroup of the special orthogonal group $SO(n)$.

4. Classification of Riemannian Holonomy Groups

Theorem 4.1. *Every locally symmetric Riemannian manifold is locally isometric to a symmetric space.*

Hence, we only need to consider symmetric spaces. We begin with the de Rham decomposition theorem.

Theorem 4.2. *(de Rham) Suppose that M is a complete, simply connected Riemannian manifold, then it is isometric to $\mathbb{R}^k \times M^1 \times \cdots \times M^m$, where $k \geqslant 0$ and each M^i is an irreducible, complete, and simply connected Riemannian manifold. Moreover, the dimension k and manifolds $M^1, \ldots, M^m$ are uniquely(up to the order) determined by M.*

Corollary 4.3. *Let $M = \mathbb{R}^k \times M^1 \times \cdots \times M^m$ be the de Rham decomposition, H be the holonomy group of M, and H_i be the holonomy of M^i, then $H \cong H_1 \times \cdots \times H_m$.*

By de Rham decomposition, simply connected irreducible symmetric spaces are essential. The holonomy of a symmetric space can be derived by the holonomy of its factors. Therefore, we only need to find all simply connected irreducible symmetric spaces and their holonomy groups.

In fact, all simply connected irreducible symmetric spaces M are of the form $M \cong G/K$, where G is the group of isometric transformations on M and K is its isotropy subgroup. There are three types of such spaces(where κ denotes the curvature of M):
(1) Euclidean type: $\kappa = 0$ and M is isometric to a Euclidean space;
(2) compact type: $\kappa \geqslant 0$(not identically 0);
(3) non-compact type: $\kappa \leq 0$(not identically 0).
In all cases there are two classes:
Class A: G is a real simple Lie group;

Class B: G is either the product of a compact simple Lie group with itself (compact type), or a complexification of such a Lie group (non-compact type).

All these types are completly classified by É. Cartan. Please also see [?] for details. We only give one of the four tables of symmetric spaces.

Theorem 4.4. *(É. Cartan) The seven infinite series and twelve exceptional Riemannian symmetric spaces in Table 1(in Appendix) give all Riemannian symmetric spaces of class A and non-compact type(called type II).*

Based on Theorems 4.1-4.4, the holonomy groups of a locally symmetric Riemannian manifold are completely classified. The remaining problem is to classify all non-locally symmetric Riemannian manifolds with an irreducible holonomy group. This problem is solved by M. Berger([?]) in 1955 and J. Simons([?]) in 1962 in terms of the following theorem.

Theorem 4.5. *(M. Berger) The complete classification of possible holonomy groups for simply connected Riemannian manifolds which are irreducible and nonsymmetric is in Table 2 (in Appendix).*

From Berger's list, several direct corollaries follow.

Corollary 4.6. *Let $n = dim(M)$ and H be the holonomy group of M. Then*
(1) if n is odd and $n \neq 7$, $H = SO(n)$;
(2) if $n = 7$, $H = SO(7)$ or G_2;
(3) if M is not an Einstein manifold and n is even, $H = SO(n)$ or $U(\frac{n}{2})$.

Actually, all the cases on Berger's list occur, which means for every group H on the list, there exists a manifold that admits H as its holonomy group.

Remark 4.7. The content about holonomy groups and symmetric spaces mentioned in Section 3 and 4 is discussed in [?]-[?].

5. Holonomy Groups of Normal Distribution Manifolds

Definition 5.1. Let $PD(d, \mathbb{R})$ be the set of all real d-ordered positive definite symmetric matrices. The d-dimensional normal distribution manifold is defined by

$$N^d := \left\{ p(x, \mu, \Sigma) = \frac{\exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}}{(2\pi)^{\frac{d}{2}}(\det \Sigma)^{\frac{1}{2}}} \middle| \mu \in \mathbb{R}^d, \Sigma \in PD(d, \mathbb{R}) \right\}.$$

Remark 5.2.
(1)Here the dimension d is the dimension of normal distributions. As a manifold, the dimension is not hard to compute by

$$\dim \left(N^d \right) = \dim \left(\mathbb{R}^d \times PD \left(d, \mathbb{R} \right) \right) = d + \frac{d\,(d+1)}{2} = \frac{d\,(d+3)}{2}.$$

(2)Also, S. Amari proved that lower dimensional normal distribution manifold can be embedded into higher dimensional ones, i.e. if $d_1 < d_2$, we could have $N^{d_1} \subset N^{d_2}$. This is bacause lower distributions could be treated as higher distributions with restrictions.

Obviously, N^d is also an exponential family as in Definition 2.1. Our main result is the following theorem.

Theorem 5.3. *Let N^d be the d-dimensional normal distribution manifold, g be the Fisher metric and $\nabla = \nabla^{(0)}$ be the corresponding Levi-Civita connection. Suppose H_d is the Riemannian holonomy group and H_d^0 is the restricted Riemannian holonomy group, then*

$$H_d = H_d^0 = SO(\frac{d(d+3)}{2}), \ d = 1, 2, 3.$$

Remark 5.4. Together with Remark 5.2, we recognize the result as the first column in Table 2. It shows not only the orientability of the manifold, but also that the Fisher metric is a generic Riemannian metric.

Since some preparation is needed to prove the theorem, we first start with several lemmas.

Lemma 5.5. N^d *is simply connected for all* $d \in \mathbb{N}$.

Proof. As a topological space, N^d is homeomorphic to the parameter space $\mathbb{R}^d \times PD(d, \mathbb{R}) \subset \mathbb{R}^{\frac{d(d+3)}{2}}$, as we stated in Remark 5.2. The $\mathbb{R}^d$ part is contractible showing $\pi_1(\mathbb{R}^d) = 0$. By the theory of linear algebra, we see that if $A, B \in PD(d, \mathbb{R})$, then $(1-t)A + tB \in PD(d, \mathbb{R}), \forall t \in [0, 1]$. Therefore, the space $PD(d, \mathbb{R})$ is convex, hence also contractible. In particular, $\pi_1(PD(d, \mathbb{R})) = 0$. By the theory of algebraic topology, we have

$$\pi_1(N^d) \cong \pi_1(\mathbb{R}^d \times PD(d, \mathbb{R})) \cong \pi_1(\mathbb{R}^d) \times \pi_1(PD(d, \mathbb{R})) = 0$$

as desired. $\square$

Corollary 5.6. $H_d = H_d^0$.

Proof. This result is straightforward by applying Lemma 5.5 and part (3) of Proposition 3.3. $\square$

Lemma 5.7. N^1 *is isometric to the 2-dimensional hyperbolic space, denoted by* $H(2)$.

Proof. By definition, we have

$$N^1 = \left\{ p(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \middle| \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+ \right\}.$$

Let $\theta_1 = \frac{\mu}{\sigma^2}$ and $\theta_2 = -\frac{1}{2\sigma^2}$ be the natural coordinates of the exponential family. Then the Fisher metric matrix is given by

$$[g_{ij}] = \begin{bmatrix} \sigma^2 & 2\mu\sigma^2 \\ 2\mu\sigma^2 & 2\sigma^2(2\mu^2 + \sigma^2) \end{bmatrix},$$

and the curvature tensor follows $R_{1212} = \frac{1}{\sigma^6}$.

As a result, the sectional curvature(also the Gaussian curvature) is $\kappa = -\frac{1}{2}$, which is a negative constant. Thus, N^1 is a complete simply connected manifold with constant sectional curvature $-\frac{1}{2}$, hence is the space form and isometric to 2-dimensional hyperbolic space $H(2)$ with

$$\dim N^1 = \frac{1(1+3)}{2} = 2.$$

$\square$

Lemma 5.8. *The n-dimensional hyperbolic space* $H(n)$ *is a symmetric space for all* $n \in \mathbb{Z}_+$.

Proof. Consider $s = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix} \in M(n+1, \mathbb{R})$ where I_n is the $n \times n$ identity matrix and

$$O(n, 1) := \{A \in GL(n+1, \mathbb{R}) | A^T s A = s\}.$$

Here, $O(n, 1)$ is called the Lorentz group consisting of all linear transformation on $\mathbb{R}^{n+1}$ maintaining the invariance of the Lorentz inner product, which is defined by

$$\langle X, Y \rangle_L := \sum_{i=1}^{n} x^i y^i - x^{n+1} y^{n+1} = X^T s Y,$$

$X = (x^1, \ldots, x^{n+1}), Y = (y^1, \ldots, y^{n+1}) \in \mathbb{R}^{n+1}$. Note that $O(n, 1)$ has 4 components and the one containing I is

$$G = \{A = (a_{ij}) \in O(n, 1) | \det A = 1, a_{(n+1)(n+1)} \geqslant 1\},$$

which is a connected Lie group and acts on the Lorentz space $\mathbb{R}_L^{n+1} = (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_L)$ keeping $H(n) = \{ X = (x^1, \ldots, x^{n+1})^T \in \mathbb{R}^{n+1} | \langle X, X \rangle_L = -1,\ x^{n+1} > 0 \}$ invariant. The Lorentz inner product induces a Riemannian metric g on $H(n)$. Also,

$$\sigma : G \longrightarrow G$$
$$A \longmapsto sAs$$

is an involution automorphism on G. Note that the fixed point subgroup

$$\begin{aligned} K_\sigma &= \{ A \in G | \sigma(A) = A \} = G \cap O(n+1) \\ &= \left\{ A = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \middle| B \in SO(n) \right\} \cong SO(n). \end{aligned}$$

Thus, K_σ is a compact connected Lie group, which means that (G, K_σ, σ) is a Riemannian symmetric pair, and $H(n) = G/K_\sigma$ is a Riemannian symmetric space.

In fact, $H(n)$ is just of the type BDI in Table 1 with $p = 1$ and $q = n - 1$. $\qquad\square$

Proposition 5.9. $H_1 = SO(2)$.

Proof. It follows from Lemma 5.7, Lemma 5.8 and Corollary 4.3. $\qquad\square$

Lemma 5.10. *Let (M^n, g) be a Riemannian manifold, if M is isometric to $M_1^{n_1} \times M_2^{n_2}$, where $n_1 + n_2 = n$. Suppose matrix $K_{n \times n}$ is the sectional curvature matrix of M, then K must be block diagonal. All geometric structures are presented in appendix*

Remark 5.11. Lemma 5.10 is elementary in theory of submanifold, see also [44].

Lemma 5.12. N^d *is irreducible for $d = 1, 2, 3$.*

Proof. Lemma 5.7 implies that N^1 is irreducible. When d=2,3, a direct computation of sectional curvature shows that the sectional curvature matrix for N^d is not block diagonal, hence irreducible (see appendix). $\qquad\square$

Remark 5.13. In general, the covariance matrix Σ is not necessarily diagonal, hence N^d is very likely to be irreducible, for all $d \in \mathbb{N}$.

Lemma 5.14. N^d *is not symmetric for all $d \geq 2$.*

Proof. When $d = 2, 3$, appendix shows that κ is neither negative nor positive(of course not identically 0 either). We conclude that N^2 and N^3 are not of Euclidean type, compact type or non-compact type as well, hence not a symmetric space.
$\qquad\square$

Corollary 5.15. H^d *must lie in the following Berger's list for $d = 2, 3$.*

$SO\left(\frac{d(d+3)}{2}\right)$	$U\left(\frac{d(d+3)}{4}\right)$	$SU\left(\frac{d(d+3)}{4}\right)$	$Sp\left(\frac{d(d+3)}{8}\right) \cdot Sp(1)$	$Sp\left(\frac{d(d+3)}{8}\right)$	G_2	$Spin(7)$

Proof. H^d is simply connected by Lemma 5.5, irreducible by Lemma 5.12 and also nonsymmetric by Lemma 5.14. Hence Theorem 4.5(M. Berger) applies. $\qquad\square$

Lemma 5.16. H^d *is neither G_2 nor $Spin(7)$.*

Proof. $dim(N^d) = \frac{d(d+3)}{2} = 2, 5, 9, 14, 20, \ldots$. However, Berger's list indicates that every manifold with G_2 or $Spin(7)$ as its holonomy group must be of dimension 7 or 8, respectively. As a result, H^d is neither G_2 nor $Spin(7)$. $\qquad\square$

Lemma 5.17. H^d *is not equal to any of the following groups*

$$SU(\frac{d(d+3)}{4}),\ Sp(\frac{d(d+3)}{8}) \cdot Sp(1),\ Sp(\frac{d(d+3)}{8}).$$

Proof. By the comments in Berger's list we see that a manifold possesses holonomy groups $SU(\frac{d(d+3)}{4})$, $Sp(\frac{d(d+3)}{8}) \cdot Sp(1)$ or $Sp(\frac{d(d+3)}{8})$ must be an Einstein manifold, namely, there exists a constant k, s.t.,

$$Ric = kg.$$

However, it is obvious that N^2 and N^3 are not Einstein manifolds (see appendix), which implies that H^d is not equal to any of the following groups:

$$SU(\frac{d(d+3)}{4}),\ Sp(\frac{d(d+3)}{8}) \cdot Sp(1),\ Sp(\frac{d(d+3)}{8}).$$

$\square$

Lemma 5.18. *N^d is not Kählerian for all $d \in \mathbb{N}$.*

Proof. Takano([?]-[?]) has proved that $(N^d, \nabla^{(\alpha)})$ admits an almost complex structure $J^{(\alpha)}$ that is parallel to the α-connection $\nabla^{(\alpha)}$ only if $\alpha = \pm 1$. Recall Proposition 2.4 that $\nabla^{(\alpha)}$ is the Levi-Civita connection if and only if $\alpha = 0$. Hence N^d does not admit a Kähler metric. $\square$

A direct corollary follows.

Corollary 5.19. *$H^d \neq U(\frac{d(d+3)}{4})$.*

Based on all preparations, we could show the proof of theorem 5.3.

Proof. When $d = 1$, Proposition 5.9 proves the case.

When $d \geqslant 2$ we get a possible list in corollary 5.15. Furthermore, Lemma 5.16, Lemma 5.17 and Corollary 5.19 rule out all possible groups except for $SO(\frac{d(d+3)}{2})$, as desired.

Thus, to sum up, we could conclude that

$$H^d = H_0^d = SO(\frac{d(d+3)}{2}),\ d = 1, 2, 3.$$

$\square$

In fact, part of our results about the normal distribution manifolds can be generalized to the exponential family.

6. HOLONOMY GROUP OF EXPONENTIAL FAMILY

Let S be an exponential family with dimension n and H be its holonomy group.

Lemma 6.1. *S is not Kählerian.*

Proof. Similarly as that in lemma 5.18, S admits an almost complex structure $J^{(\alpha)}$ that is parallel to the α-connection $\nabla^{(\alpha)}$ only if $\alpha = \pm 1$, which is not the Levi-Civita connection([?]-[?]). $\square$

Corollary 6.2. *H is not equal to any of following groups*

$$U(\frac{n}{2}),\ SU(\frac{n}{2}),\ Sp(\frac{n}{4}).$$

Proof. Lemma 6.1 implies that H is not a subgroup of $U(\frac{n}{2})$. Noting that

$$Sp(\frac{n}{4}) < SU(\frac{n}{2}) < U(\frac{n}{2}),$$

hence H cannot be any of them. $\square$

Now, after ruling out several cases, the following theorem holds

Theorem 6.3. *If S is a simply connected nonsymmetric n-dimensional exponential family with irreducible holonomy group H, then H must be one of the following groups*

Holonomy	Dimension
$SO(m)$	$n = m$
$Sp(m) \cdot SP(1)$	$n = 4m$
G_2	$n = 7$
$Spin(7)$	$n = 8$

Corollary 6.4. *Suppose S is a simply connected nonsymmetric n-dimensional exponential family with irreducible holonomy group H. We have*

(1) if $n \neq 7, 8$, then H is either $SO(n)$ or $Sp(m) \cdot Sp(1)$, where $n = 4m$;

(2) if $n = 7$, then H is either $SO(7)$ or G_2;

(3) if $n \neq 7$ and n is odd, then $H = SO(n)$;

(4) if $n = 2(2m+1)$, then $H = SO(n)$;

(5) if $n = 8$, then $H = SO(8)$, or $Sp(2) \cdot Sp(1)$ or $Spin(7)$;

(6) if $n = 4m$ where $m \neq 2$, then H is either $SO(n)$ or $Sp(m) \cdot Sp(1)$;

(7) if S is not an Einstein manifold, then $H = SO(n)$.

Proof. (1)-(6) directly follow from Theorem 6.3, hence the remaining is to prove (7). If H is a subgroup of G_2 or $Spin(7)$, then the Ricci curvature must be identically $0([?])$, implying S is an Einstein manifold, which is a contradiction. $\square$

Since almost all common examples of exponential families are not Einstein, the holonomy groups of almost all exponential families are $SO(n)$. There is only one exception, the monistic normal distribution manifold N^1, which is Einstein. However, $H^1 = SO(2) = SO\left(\dim N^1\right)$ (Proposition 5.9) which coincides with our results.

7. CONCLUSION

After some preliminaries about information geometry and holonomy groups, two main results, Theorem 5.3 and Theorem 6.3 are proved. Theorem 5.3 shows that the holonomy groups of normal distribution manifolds are special orthogonal groups for all dimensions. In addition, a list of possible holonomy groups for general exponential families is presented in Theorem 6.3.

8. APPENDIX

We presents two tables mentioned in Section 4 on classification of Riemannian holonomy groups and the geometric structure of N^d used in Section 5 here.

$$N^2 = \{p(x, y, \mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}) = \frac{1}{2\pi\sqrt{\sigma_1\sigma_2 - \sigma_{12}^2}} \exp\{-AB\} | \mu_1, \mu_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in \mathbb{R}_+, \sigma_{12} = cov(X, Y)\},$$

where $A = \frac{1}{2(\sigma_2\sigma_2 - \sigma_{12}^2)}$, $B = \sigma_2(x - \mu_1)^2 - 2\sigma_{12}(x - \mu_1)(y - \mu_2) + \sigma_1(y - \mu_2)^2$. This is a 5-dimensional manifold. It is obvious that

$$p(x, y) = \log\left(\frac{1}{2\pi\sqrt{\Delta}} e^{-\frac{1}{2\Delta}(\sigma_2(x-\mu_1)^2 - 2\sigma_{12}(x-\mu_1)(y-\mu_2) + \sigma_1(y-\mu_2)^2)}\right)$$

$$= \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\Delta} x + \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\Delta} y + \frac{-\sigma_2}{2\Delta} x^2 + \frac{\sigma_{12}}{2\Delta} xy + \frac{-\sigma_1}{2\Delta} y^2$$

$$- (\log(2\pi\sqrt{\Delta}) + \frac{\mu_2{}^2\sigma_1 + \mu_1{}^2\sigma_2 - 2\mu_1\mu_2\sigma_{12}}{2\Delta}),$$

where $\Delta = \sigma_1\sigma_2 - \sigma_{12}^2$. Hence, the coordinates for exponential family are

$$\theta_1 = \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\Delta}, \theta_2 = \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\Delta}, \theta_3 = -\frac{\sigma_2}{2\Delta}, \theta_4 = -\frac{\sigma_{12}}{\Delta}, \theta_5 = -\frac{\sigma_1}{2\Delta},$$

and $\Delta = \frac{1}{4\theta_3\theta_5 - \theta_4{}^2}$. While the potential function is

$$\phi(\theta) = \log(2\pi\sqrt{\Delta}) + \frac{\mu_2{}^2\sigma_1 + \mu_1{}^2\sigma_2 - 2\mu_1\mu_2\sigma_{12}}{2\Delta}$$

$$= \log(2\pi\sqrt{\Delta}) - \Delta(\theta_2{}^2\theta_3 - \theta_1\theta_2\theta_4 + \theta_1{}^2\theta_5).$$

The components g_{ij}, $i, j = 1, 2, 3, 4, 5$ of the Fisher metric matrix are given by

$g_{11} = \sigma_1$,

$g_{12} = \sigma_{12}$,

$g_{13} = 2\mu_1\sigma_1$,

$g_{14} = \sigma_1\mu_2 + \sigma_{12}\mu_1$,

Label	G	K	Dimension	Rank	Geometric interpretation
AI	$SL(n;\mathbb{R})$	$SO(n)$	$\frac{(n-1)(n+2)}{2}$	n-1	Set of $\mathbb{R}P_{hyp}^n$'s in $\mathbb{C}P_{hyp}^n$
AII	$SL(n;\mathbb{H})$	$Sp(n)$	$(n-1)(2n+1)$	n-1	Set of $\mathbb{H}P_{hyp}^{n-1}$'s in $\mathbb{C}P_{hyp}^{2n-1}$
AIII	$SU(p,q)$	$S(U(p)\times U(q))$	$2pq$	$\min(p,q)$	$G_p(p,q;\mathbb{C})$
BDI	$SO_0(p,q)$	$SO(p)\times SO(q)$	pq	$\min(p,q)$	$G_p(p,q;\mathbb{R})$
DIII	$SO(n;\mathbb{H})$	$U(n)$	$n(n-1)$	$\left[\frac{n}{2}\right]$	Set of $\mathbb{C}P_{hyp}^{n-1}$'s in $\mathbb{R}P_{hyp}^{2n-1}$
CI	$Sp(n;\mathbb{R})$	$U(n)$	$n(n+1)$	n	Set of $\mathbb{C}P_{hyp}^n$'s in $\mathbb{H}P_{hyp}^n$
CII	$Sp(p,q)$	$Sp(p)\times Sp(q)$	$4pq$	$\min(p,q)$	$G_p(p,q;\mathbb{H})$
EI	E_6^6	$Sp(4)$	42	6	Antichains of $(\mathbb{C}\otimes\mathbb{O})P_{hyp}^2$
EII	E_6^2	$SU(6)\times SU(2)$	40	4	Set of the $(\mathbb{C}\otimes\mathbb{H})P_{hyp}^2$'s in $(\mathbb{C}\otimes\mathbb{O})P_{hyp}^2$
EIII	E_6^{-14}	$SO(10)\times SO(2)$	32	2	Rosenfeld's hyperbolic projective plane $(\mathbb{C}\otimes\mathbb{O})P_{hyp}^2$
EIV	E_6^{-26}	F_4	26	2	Set of $\mathbb{O}P^2$'s the in $(\mathbb{C}\otimes\mathbb{O})P_{hyp}^2$
EV	E_7^7	$SU(8)$	70	7	Antichains of $(\mathbb{H}\otimes\mathbb{O})P_{hyp}^2$
EVI	E_7^{-5}	$SO(12)\times SU(2)$	64	4	Rosenfeld hyperbolic projective plane $(\mathbb{H}\otimes\mathbb{O})P_{hyp}^2$
EVII	E_7^{-25}	$E_6\times SO(2)$	54	3	Set of the $(\mathbb{C}\otimes\mathbb{O})P_{hyp}^2$'s in $(\mathbb{H}\otimes\mathbb{O})P_{hyp}^2$
EVIII	E_8^8	$SO(16)$	128	8	Rosenfeld projective plane $(\mathbb{O}\otimes\mathbb{O})P_{hyp}^2$
EIX	E_8^{-24}	$E_7\times SU(2)$	112	4	Set of the $(\mathbb{H}\otimes\mathbb{O})P_{hyp}^2$'s in $(\mathbb{O}\otimes\mathbb{O})P_{hyp}^2$
FI	F_4^4	$Sp(3)\times SU(2)$	28	4	Set of the $\mathbb{H}P_{hyp}^2$'s in $\mathbb{O}P_{hyp}^2$
FII	F_4^{-20}	$SO(9)$	16	1	Hyperbolic Cayley projective plane $\mathbb{O}P_{hyp}^2$
G	G_2^2	$SU(2)\times SU(2)$	8	2	Set of non-division $\mathbb{H}$ subalgebras of the non-division $\mathbb{O}$

Table 1: One of four lists of Riemannian symmetric spaces

H	Dimension	Type of manifold	Comments
$SO(n)$	n	Oriented manifold	Generic Metric
$U(n)$	$2n$	Kähler manifold	Kähler
$SU(n)$	$2n$	Calabi-Yau manifold	Ricci-flat, Kähler
$Sp(n)\cdot Sp(1)$	$4n$	Quaternion-Kähler manifold	Eistein
$Sp(n)$	$4n$	Hyperkähler manifold	Ricci-flat, Kähler
G_2	7	G_2 manifold	Ricci-flat
$Spin(7)$	8	$Spin(7)$ manifold	Ricci-flat

Table 2: List of Riemannian holonomy groups

$g_{15} = 2\sigma_{12}\mu_2,$

$g_{22} = \sigma_{22},$

$g_{23} = 2\sigma_{12}\mu_1,$

$g_{24} = \sigma_2\mu_1 + \sigma_{12}\mu_2,$

$g_{25} = 2\sigma_2\mu_2,$

$g_{33} = 2\sigma_1\left(\sigma_1 + 2\mu_1{}^2\right),$

$g_{34} = 2\sigma_{12}\sigma_{11} + 2\mu_1\sigma_1\mu_2 + 2\mu_1{}^2\sigma_{12},$

$g_{35} = 2\sigma_{12}\left(\sigma_{12} + 2\mu_1\mu_2\right),$

$g_{44} = \sigma_1\sigma_2 + \sigma_1\mu_2{}^2 + \mu_1{}^2\sigma_2 + 2\mu_1\sigma_{12}\mu_2 + \sigma_{12}{}^2,$

$g_{45} = 2\sigma_2\sigma_{12} + 2\sigma_{22}\mu_1\mu_2 + 2\sigma_{12}\mu_2{}^2,$

$g_{55} = 2\sigma_2\left(\sigma_2 + 2\mu_2{}^2\right).$

The components K_{ij}, $i,j = 1,2,3,4,5$ of the sectional curvature are given by

$K_{ii} = 0$, $i = 1,2,3,4,5,$

$K_{12} = 1/4 - 1/4\alpha^2,$

$K_{13} = -1/2 + 1/2\alpha^2,$

$K_{14} = -1/4\dfrac{\left(-3\sigma_2{}^2\sigma_1 - \sigma_1{}^2\sigma_{22} - \sigma_2{}^2\mu_1{}^2 + \sigma_1\mu_1{}^2\sigma_{22}\right)\alpha^2}{\sigma_{11}{}^2\sigma_{22} + \sigma_1\mu_1{}^2\sigma_{22} + \sigma_2{}^2\sigma_1 - \sigma_2{}^2\mu_1{}^2} - 1/4\dfrac{\sigma_1{}^2\sigma_{22} + 3\sigma_2{}^2\sigma_1 - \sigma_1\mu_1{}^2\sigma_{22} + \sigma_2{}^2\mu_1{}^2}{\sigma_1{}^2\sigma_{22} + \sigma_1\mu_1{}^2\sigma_{22} + \sigma_2{}^2\sigma_1 - \sigma_2{}^2\mu_1{}^2},$

$K_{15} = -1/2\dfrac{\left(\sigma_1\mu_2{}^2\sigma_{22} - \sigma_{22}\sigma_2{}^2 - \sigma_2{}^2\mu_2{}^2\right)\alpha^2}{\sigma_{22}{}^2\sigma_1 + 2\sigma_1\mu_2{}^2\sigma_{22} - 2\sigma_2{}^2\mu_2{}^2} - 1/2\dfrac{-\sigma_1\mu_2{}^2\sigma_{22} + \sigma_{22}\sigma_2{}^2 + \sigma_2{}^2\mu_2{}^2}{\sigma_{22}{}^2\sigma_1 + 2\sigma_1\mu_2{}^2\sigma_{22} - 2\sigma_2{}^2\mu_2{}^2},$

$K_{23} = -1/2\dfrac{\left(-\sigma_2{}^2\sigma_1 - \sigma_{12}{}^2\mu_1{}^2 + \sigma_1\mu_1{}^2\sigma_{22}\right)\alpha^2}{\sigma_1{}^2\sigma_{22} + 2\sigma_1\mu_1{}^2\sigma_{22} - 2\sigma_2{}^2\mu_1{}^2} - 1/2\dfrac{-\sigma_1\mu_1{}^2\sigma_{22} + \sigma_2{}^2\mu_1{}^2 + \sigma_2{}^2\sigma_1}{\sigma_1{}^2\sigma_{22} + 2\sigma_1\mu_1{}^2\sigma_{22} - 2\sigma_2{}^2\mu_1{}^2},$

$K_{24} = -1/4\dfrac{\left(-\sigma_{22}{}^2\sigma_1 + \sigma_1\mu_2{}^2\sigma_{22} - 3\sigma_{22}\sigma_2{}^2 - \sigma_2{}^2\mu_2{}^2\right)\alpha^2}{\sigma_{22}{}^2\sigma_1 + \sigma_1\mu_2{}^2\sigma_{22} + \sigma_{22}\sigma_2{}^2 - \sigma_2{}^2\mu_2{}^2} - 1/4\dfrac{\sigma_{22}{}^2\sigma_1 - \sigma_1\mu_2{}^2\sigma_{22} + 3\sigma_{22}\sigma_2{}^2 + \sigma_2{}^2\mu_2{}^2}{\sigma_{22}{}^2\sigma_1 + \sigma_1\mu_2{}^2\sigma_{22} + \sigma_{22}\sigma_{12}{}^2 - \sigma_2{}^2\mu_2{}^2},$

$K_{25} = -1/2 + 1/2\alpha^2,$

$K_{34} = -1/2\dfrac{\left(-\sigma_1{}^3\sigma_{22} - \sigma_{11}{}^3\mu_2{}^2 - 3\sigma_1{}^2\mu_1{}^2\sigma_{22} + \sigma_2{}^2\sigma_1{}^2 + 2\sigma_1{}^2\mu_1\sigma_2\mu_2 + \sigma_1\sigma_{22}\mu_1{}^4 + 2\sigma_1\sigma_2{}^2\mu_1{}^2 - \sigma_2{}^2\mu_1{}^4\right)\alpha^2}{\sigma_1{}^3\sigma_{22} + \sigma_1{}^3\mu_2{}^2 + 3\sigma_1{}^2\mu_1{}^2\sigma_{22} - 2\sigma_1{}^2\mu_1\sigma_2\mu_2 - \sigma_2{}^2\sigma_1{}^2 + 2\sigma_1\sigma_{22}\mu_1{}^4 - 2\sigma_1\sigma_2{}^2\mu_1{}^2 - 2\sigma_2{}^2\mu_1{}^4}$
$\qquad - 1/2\dfrac{\sigma_1{}^3\sigma_{22} + \sigma_2{}^2\mu_1{}^4 - \sigma_2{}^2\sigma_1{}^2 + \sigma_1{}^3\mu_2{}^2 + 3\sigma_1{}^2\mu_1{}^2\sigma_{22} - 2\sigma_1{}^2\mu_1\sigma_2\mu_2 - \sigma_1\sigma_{22}\mu_1{}^4 - 2\sigma_1\sigma_2{}^2\mu_1{}^2}{\sigma_1{}^3\sigma_{22} + \sigma_1{}^3\mu_2{}^2 + 3\sigma_1{}^2\mu_1{}^2\sigma_{22} - 2\sigma_1{}^2\mu_1\sigma_2\mu_2 - \sigma_2{}^2\sigma_1{}^2 + 2\sigma_1\sigma_{22}\mu_1{}^4 - 2\sigma_1\sigma_2{}^2\mu_1{}^2 - 2\sigma_2{}^2\mu_1{}^4},$

$K_{35} = -\dfrac{\left(-\sigma_1\sigma_{22}\sigma_2{}^2 - 2\sigma_1\sigma_{22}\mu_1\sigma_2\mu_2 + \sigma_1\sigma_{22}\mu_1{}^2\mu_2{}^2 - \sigma_1\sigma_2{}^2\mu_2{}^2 - \sigma_2{}^2\mu_1{}^2\sigma_{22} - \mu_1{}^2\sigma_2{}^2\mu_2{}^2 + \sigma_2{}^4 + 4\sigma_2{}^3\mu_1\mu_2\right)\alpha^2}{\sigma_1{}^2\sigma_{22}{}^2 + 2\sigma_{11}{}^2\mu_2{}^2\sigma_{22} + 2\sigma_1\sigma_{22}{}^2\mu_1{}^2 + 4\sigma_1\sigma_{22}\mu_1{}^2\mu_2{}^2 - 4\sigma_2{}^3\mu_1\mu_2 - 4\mu_1{}^2\sigma_2{}^2\mu_2{}^2 - \sigma_2{}^4}$
$\qquad - \dfrac{\sigma_1\sigma_{22}\sigma_2{}^2 + 2\sigma_{11}\sigma_{22}\mu_1\sigma_2\mu_2 - \sigma_1\sigma_{22}\mu_1{}^2\mu_2{}^2 + \sigma_1\sigma_2{}^2\mu_2{}^2 + \sigma_2{}^2\mu_1{}^2\sigma_{22} + \mu_1{}^2\sigma_2{}^2\mu_2{}^2 - \sigma_2{}^4 - 4\sigma_2{}^3\mu_1\mu_2}{\sigma_1{}^2\sigma_{22}{}^2 + 2\sigma_1{}^2\mu_2{}^2\sigma_{22} + 2\sigma_1\sigma_{22}{}^2\mu_1{}^2 + 4\sigma_{11}\sigma_{22}\mu_1{}^2\mu_2{}^2 - 4\sigma_2{}^3\mu_1\mu_2 - 4\mu_1{}^2\sigma_2{}^2\mu_2{}^2 - \sigma_2{}^4},$

$K_{45} = 1/2\dfrac{\left(\sigma_1\sigma_{22}{}^3 + 3\sigma_{11}\sigma_{22}{}^2\mu_2{}^2 - \sigma_1\sigma_{22}\mu_2{}^4 + \sigma_{22}{}^3\mu_1{}^2 - \sigma_{22}{}^2\sigma_2{}^2 - 2\sigma_{22}{}^2\mu_1\sigma_2\mu_2 - 2\sigma_{22}\sigma_2{}^2\mu_2{}^2 + \sigma_2{}^2\mu_2{}^4\right)\alpha^2}{\sigma_1\sigma_{22}{}^3 + 3\sigma_1\sigma_{22}{}^2\mu_2{}^2 + 2\sigma_1\sigma_{22}\mu_2{}^4 + \sigma_{22}{}^3\mu_1{}^2 - 2\sigma_{22}{}^2\mu_1\sigma_2\mu_2 - \sigma_{22}{}^2\sigma_2{}^2 - 2\sigma_{22}\sigma_2{}^2\mu_2{}^2 - 2\sigma_2{}^2\mu_2{}^4}$
$\qquad + 1/2\dfrac{-\sigma_1\sigma_{22}{}^3 - 3\sigma_1\sigma_{22}{}^2\mu_2{}^2 + \sigma_1\sigma_{22}\mu_2{}^4 - \sigma_{22}{}^3\mu_1{}^2 + \sigma_{22}{}^2\sigma_{12}{}^2 + 2\sigma_{22}{}^2\mu_1\sigma_2\mu_2 + 2\sigma_{22}\sigma_2{}^2\mu_2{}^2 - \sigma_2{}^2\mu_2{}^4}{\sigma_1\sigma_{22}{}^3 + 3\sigma_1\sigma_{22}{}^2\mu_2{}^2 + 2\sigma_1\sigma_{22}\mu_2{}^4 + \sigma_{22}{}^3\mu_1{}^2 - 2\sigma_{22}{}^2\mu_1\sigma_2\mu_2 - \sigma_{22}{}^2\sigma_2{}^2 - 2\sigma_{22}\sigma_2{}^2\mu_2{}^2 - 2\sigma_2{}^2\mu_2{}^4}.$

The components Ric_{ij}, $i,j = 1,2,3,4,5$ of the Ricci tensor are given by

$Ric_{11} = 1/2\sigma_1\alpha^2 - 1/2\sigma_1,$

$Ric_{12} = 1/2\sigma_2\alpha^2 - 1/2\sigma_2,$

$Ric_{13} = \mu_1\sigma_1\alpha^2 - \mu_1\sigma_1,$

$Ric_{14} = \left(1/2\sigma_1\mu_2 + 1/2\sigma_2\mu_1\right)\alpha^2 - 1/2\sigma_1\mu_2 - 1/2\sigma_2\mu_1,$

$Ric_{15} = -\sigma_2\mu_2 + \alpha^2\sigma_2\mu_2,$

$Ric_{22} = 1/2\sigma_{22}\alpha^2 - 1/2\sigma_{22},$

$Ric_{23} = -\sigma_2\mu_1 + \alpha^2\sigma_2\mu_1,$

$Ric_{24} = \left(1/2\sigma_{22}\mu_1 + 1/2\sigma_2\mu_2\right)\alpha^2 - 1/2\sigma_{22}\mu_1 - 1/2\sigma_2\mu_2,$

$Ric_{25} = -\sigma_{22}\mu_2 + \sigma_{22}\alpha^2\mu_2,$

$Ric_{33} = 2\sigma_1\left(\sigma_1 + \mu_1{}^2\right)\alpha^2 - 2\sigma_1\left(\sigma_1 + \mu_1{}^2\right),$

$Ric_{34} = \left(2\sigma_2\sigma_1 + \mu_1\sigma_1\mu_2 + \mu_1{}^2\sigma_2\right)\alpha^2 - 2\sigma_2\sigma_1 - \mu_1\sigma_1\mu_2 - \mu_1{}^2\sigma_2,$

$Ric_{35} = \left(-\sigma_1\sigma_{22} + 3\sigma_2{}^2 + 2\mu_1\sigma_2\mu_2\right)\alpha^2 + \sigma_1\sigma_{22} - 3\sigma_2{}^2 - 2\mu_1\sigma_2\mu_2,$

$Ric_{44} = \left(3/2\sigma_1\sigma_{22} + 1/2\sigma_1\mu_2{}^2 + 1/2\mu_1{}^2\sigma_{22} + 1/2\sigma_2{}^2 + \mu_1\sigma_2\mu_2\right)\alpha^2 - 3/2\sigma_{11}\sigma_{22} - 1/2\sigma_1\mu_2{}^2 - 1/2\mu_1{}^2\sigma_{22}$
$\qquad - 1/2\sigma_2{}^2 - \mu_1\sigma_2\mu_2,$

$Ric_{45} = \left(2\sigma_{22}\sigma_2 + \sigma_{22}\mu_1\mu_2 + \sigma_2\mu_2{}^2\right)\alpha^2 - 2\sigma_{22}\sigma_2 - \sigma_{22}\mu_1\mu_2 - \sigma_2\mu_2{}^2,$

$Ric_{55} = 2\sigma_{22}\left(\sigma_{22} + \mu_2{}^2\right)\alpha^2 - 2\sigma_{22}\left(\sigma_{22} + \mu_2{}^2\right).$

Remark 8.1. The computations of N^3, even N^d for all $d \in \mathbb{N}$ are similar to that of N^2 (in fact, they are calculated by a common program), but the results are too large to be presented here.

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