The Probabilistic and Combinatorial Interpretations of the Bernoulli Symbol $\mathcal{B}$

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Outlines

Definition

The Bernoulli numbers $(B_n)_{n=0}^{\infty}$ and Bernoulli polynomials $(B_n(x))_{n=0}^{\infty}$ can be defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \text{ and } \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

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Namely

$$\frac{t}{e^t - 1}\bullet = \mathbb{E}[\bullet]$$

Probabilistic Interpretation

For <u>independent</u> random variables X and Y, if $\mathbb{E}\left[e^{tX}\right] = F(x)$ and $\mathbb{E}\left[e^{tY}\right] = G(x)$, then

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$$B_n(x) = \mathbb{E}\left[(\mathcal{B}+x)^n\right] = \frac{[t^n]\, e^{\mathcal{B}t} e^{xt}}{n!} = \frac{[t^n]\, \frac{te^{xt}}{e^t-1}}{n!}.$$

Generalization

- Bernoulli:

$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \Leftrightarrow B_n(x) = (x + \mathcal{B})^n$$

- Nörlund:

$$\left(\frac{t}{e^t - 1} \right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} \Leftrightarrow B_n^{(p)}(x) = (x + \mathcal{B}_1 + \cdots + \mathcal{B}_p)^n$$

- Bernoulli-Barnes: for $\mathbf{a} = (a_1, \ldots, a_k)$.

$$e^{tx} \prod_{i=1}^{k} \frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x) \frac{t^n}{n!}$$

$$\Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|} \left(x + \mathbf{a} \cdot \vec{\mathcal{B}} \right)^n, \text{ where } \begin{cases} \mathbf{a} = (a_1, \ldots, a_k) \\ \vec{\mathcal{B}} = (\mathcal{B}, \ldots, \mathcal{B}_k) \\ \mathbf{a} \cdot \vec{\mathcal{B}} = \sum_{l=1}^{k} a_l \mathcal{B}_l \\ |\mathbf{a}| = \prod_{l=1}^{k} a_l \end{cases}$$

Several Results

Bernoulli-Barnes

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n\left(\mathbf{a};x\right)\frac{t^n}{n!}\Leftrightarrow B_n\left(\mathbf{a};x\right)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\vec{\mathcal{B}}\right)^n$$

Theorem[L. Jiu, V. Moll and C. Vignat]

$$f\left(x-\mathbf{a}\cdot\vec{\mathcal{B}}\right)=\sum_{\ell=0}^{n}\sum_{|L|=\ell}|\mathbf{a}|_{L^*}\,f^{(n-\ell)}\left(x+\left(\mathbf{a}\cdot\vec{\mathcal{B}}\right)_L\right).$$

The multiple zeta function

$$\zeta_r\left(n_1,\dots,n_r\right)=\sum_{0<k_1<\cdots<k_r}\frac{1}{k_1^{n_1}\cdots k_r^{n_r}}=\sum_{k_1,\dots,k_r=1}^{\infty}\frac{1}{k_1^{n_1}\left(k_1+k_2\right)^{n_2}\cdots\left(k_1+\cdots+k_r\right)^{n_r}}$$

Theorem[L. Jiu, V. Moll and C. Vignat]

$$\zeta_r\left(-n_1,\dots,-n_r\right)=\prod_{k=1}^{r}(-1)^{n_k}\,\mathcal{C}_{1,\dots,k}^{n_k+1},$$

where

$$\mathcal{C}_1^n=\frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n=\frac{\left(\mathcal{C}_1+\mathcal{B}_2\right)^n}{n},\dots,\mathcal{C}_{1,\dots,k+1}^n=\frac{\left(\mathcal{C}_{1,\dots,k}+\mathcal{B}_{k+1}\right)^n}{n}$$

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Proof.

$$(-1)^{n+1} B_{2n} \sim \frac{2\,(2n)!}{(2\pi)^{2n}}.$$

$\square$

Lemma

Uniqueness is equivalent to existence of constants C and D, such that

$$\left|\bar{B}_n\right| \leq C D^n n!.$$

Cumulants

$$K(t) := \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \log \mathbb{E}\left[e^{tX}\right] = \log\left(\sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!} t^n\right).$$

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Theorem

[Faà di Bruno's formula] For moments $(m_n)_{n=0}^{\infty}$ and cumulants $(\kappa_n)_{n=1}^{\infty}$ it holds that

$$m_n = Y_n(\kappa_1, \ldots, \kappa_n) \text{ and } \kappa_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! Y_{n,k}(m_1, \ldots, m_{n-k+1}),$$

where, the partial or incomplete exponential Bell polynomial is given by

$$Y_{n,k}\left(x_1, \ldots, x_{n-k+1}\right) := \sum_{\substack{j_1+\cdots+j_{n-k+1}=k \\ j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n}} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

and the n^{th} complete exponential Bell polynomial is given by the sum

$$Y_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} Y_{n,k}\left(x_1, \ldots, x_{n-k+1}\right) = \sum_{k=\left(\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}\right) \vdash n} \frac{n!}{k_1! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n}.$$

Cumulants

Theorem

$$B_n\left(\frac{1}{2}\right) = Y_n\left(0, -\frac{B_2}{2}, -\frac{B_3}{3}, \ldots, -\frac{B_n}{n}\right),$$

and

$$B_n = -n \sum_{k=1}^{n} (-1)^{k-1} (k-1)!\, Y_{n,k}\left(B_0\left(\frac{1}{2}\right), \ldots, B_{n-k+1}\left(\frac{1}{2}\right)\right).$$

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[M. Hoffman]

$$Y_k\left(\frac{B_2 \cdot 1!}{2 \cdot 2!}, \frac{B_4 \cdot 2!}{4 \cdot 4!}, \dots, \frac{B_{2k} \cdot k!}{(2k) \cdot (2k)!}\right) = \frac{k!}{2^{2k}(2k+1)!}.$$

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Remark

The first one is a special result of B. Robinstein: https://arxiv.org/abs/0911.3069.

Cumulants

Consider different moment generating function

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \frac{\sinh\frac{t}{2}}{\frac{t}{2}}$$

Theorem

It also holds that

$$B_n = n\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\,Y_{n,k}\left(0, \frac{1}{4\cdot 3}, 0, \ldots, \frac{1+(-1)^{n-k+2}}{2^{n-k+2}(n-k+2)}\right).$$

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$$f\left(x\right):=\sum_{k=0}^{\infty}\frac{B_{2k}k!}{(2k)\,(2k)!}=\log\left(\frac{e^x-1}{x}\right)-\frac{x}{2}$$

and

$$\sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}\,(2k+1)!}=\frac{\sinh\left(\frac{x}{2}\right)}{\frac{x}{2}}=e^{f(x)}.$$

Continued Fractions & Orthogonal Polynomials

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$$(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n d\mu(x) \quad \overset{?}{\Rightarrow} \quad (P_n(x))_{n=1}^{\infty} \sim \int_{\mathbb{R}} P_n(x) P_m(x) \, \mathrm{d}\mu(x) = C_n \delta_{m,n}$$

Continued Fractions & Orthogonal Polynomials

$$
\begin{aligned}
(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n d\mu(x) \quad &\overset{?}{\Rightarrow} \quad (P_n(x))_{n=1}^{\infty} \sim \int_{\mathbb{R}} P_n(x) P_m(x)\, \mathrm{d}\mu(x) = C_n \delta_{m,n} \\
&\Rightarrow \quad P_{n+1}(x) = (x + s_n) P_n(x) - t_n P_{n-1}(x) \\
&\Rightarrow \quad \sum_{n=0}^{\infty} m_n x^n = \cfrac{m_0}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - \cdots}}}
\end{aligned}
$$

Continued Fractions & Orthogonal Polynomials

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\end{aligned}$$

Theorem [J. Touchard]

The polynomial sequence (ϕ_n), define by

$$\phi_{n+1}(z) = \left(z + \frac{1}{2}\right) \phi_n(z) + \omega_n \phi_{n-1}(z)$$

satisfies for any $0 \leq r < n$, $\underline{\mathcal{B}^r \phi_n(\mathcal{B}) = 0}$, where

$$\omega_n = \frac{n^4}{4(2n+1)(2n-1)}.$$

How

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$$\psi_1(z) := \psi'(z) := (\log(\Gamma(z)))''$$

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$$\log(\mathcal{B} + z) = \psi\left(\left|z - \frac{1}{2}\right| + \frac{1}{2}\right).$$

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Theorem

$$\varphi_{n+1}(z,x) := \left(z + \frac{1}{2} - x\right)\varphi_n(z,x) + \omega_n \varphi_{n-1}(z,x)$$

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(Generalized) Motzkin Numbers

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$

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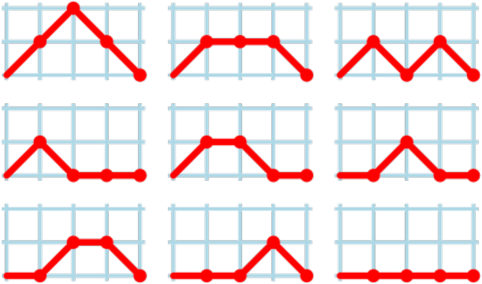

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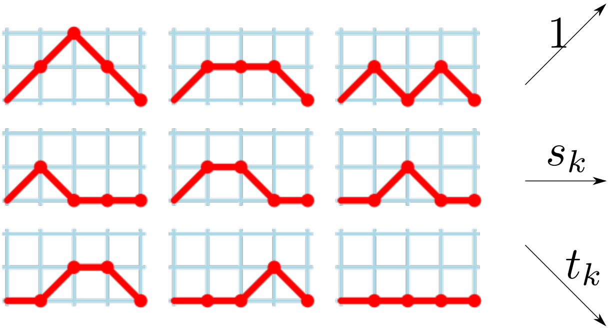

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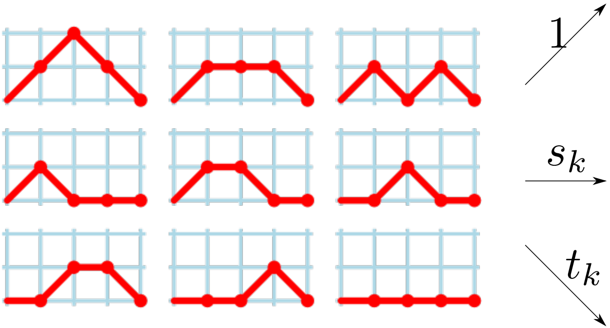

$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}$$

Combinatorial Interpretation

Theorem

Define $\left(M_{n,k}^{\mathbf{x},\omega}\right)_{n,k=0}^{\infty}$, *by* $M_{0,0}^{\mathbf{x},\omega} = 1$, $M_{n,k}^{\mathbf{x},\omega} = 0 \text{ if } k > n$, *and the recurrence*

$$M_{n+1,k}^{\mathbf{x},\omega} = M_{n,k-1}^{\mathbf{x},\omega} + x_k M_{n,k}^{\mathbf{x},\omega} - \omega_{k+1} M_{n,k+1}^{\mathbf{x},\omega},$$

where $\mathbf{x} = (x_n)_{n=0}^{\infty}$ *is given by* $x_n = x - \frac{1}{2}$, *and* $\omega = (\omega_n)_{n=1}^{\infty}$ *by* $\omega_n = \frac{n^4}{4(2n+1)(2n-1)}$. *Then,* $M_{n,0}^{\mathbf{x},\omega} = B_n(x)$.

Combinatorial Interpretation

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Define $\left(M^{\mathbf{x},\omega}_{n,k}\right)_{n,k=0}^{\infty}$, *by* $M^{\mathbf{x},\omega}_{0,0} = 1$, $M^{\mathbf{x},\omega}_{n,k} = 0 \text{ if } k > n$, *and the recurrence*

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where $\mathbf{x} = (x_n)_{n=0}^{\infty}$ *is given by* $x_n = x - \frac{1}{2}$, *and* $\omega = (\omega_n)_{n=1}^{\infty}$ *by* $\omega_n = \frac{n^4}{4(2n+1)(2n-1)}$. *Then,* $M^{\mathbf{x},\omega}_{n,0} = B_n(x)$. *In addition, the lattice path interpretation allows us to define the infinite-dimensional matrix*

$$R_{\mathbf{x},\omega} := \begin{pmatrix} x-\frac{1}{2} & -\omega_1 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{1}{2} & -\omega_2 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\omega_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{1}{2} & -\omega_{n+1} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Matrix Computation

Direct computations shows

$$R_{\mathbf{x},\omega,4} = \begin{pmatrix} x - 1/2 & -\frac{1}{12} & 0 & 0 \\ 1 & x - 1/2 & -\frac{4}{15} & 0 \\ 0 & 1 & x - 1/2 & -\frac{81}{140} \\ 0 & 0 & 1 & x - 1/2 \end{pmatrix}$$

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$$\left(R^4_{\mathbf{x},\omega,4}\right) = \begin{pmatrix} x^4-2x^3+x^2-\frac{1}{30} & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix},$$

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where noting

$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}.$$

Euler Analogue

Definition

Euler numbers $(E_n)_{n=0}^{\infty}$ and Euler polynomials $(E_n(x))_{n=0}^{\infty}$

$$\operatorname{sech}(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

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$$E_n(x) = \int_{\mathbb{R}} \left(x - \frac{1}{2} + \imath t\right)^n \operatorname{sech}(\pi t)\, \mathrm{d}t.$$

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$$\mathcal{E}^n := \mathbb{E}\left[\mathcal{E}^n\right] = E_n$$

Conversely, it holds that $\mathbb{E}\left[L_E^n\right] = \left(\frac{\imath}{2}\right)^n E_n$ and $\mathbb{E}\left[e^{tL_E}\right] = \sec\left(\frac{t}{2}\right)$.

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$$\begin{cases} E_{2n} = 1 - \sum_{k=1}^{n} \binom{2n}{2k-1} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k} \\ B_{2n} = \frac{2n}{2^{2n}(2^{2n}-1)} \sum_{k=0}^{n-1} \binom{2n-1}{2k} E_{2k} \end{cases}$$

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$$2\beta\left(\frac{s+1}{2}\right) \sim \sum_{j=1}^{\infty} \frac{E_j}{s^{j+1}}$$

Possible Extension to Nörlund Polynomials

$$\left(\frac{t}{e^t - 1}\right)^p e^{xt} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} \Leftrightarrow B_n^{(p)}(x) = (\mathcal{B}_1 + \cdots + \mathcal{B}_p + x)^n .$$

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$$\frac{\Gamma(z+x)}{\Gamma(z+x+1-p)\,z^p} \sim \sum_{n=0}^{\infty} \frac{(p-n)_n}{n!} B_n^{(p)}(x)\,\frac{1}{z^{n+1}}$$

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$$\log(\mathcal{B}_1 + \cdots + \mathcal{B}_p + z) = -H_{p-1} + \frac{\mathrm{d}^{p-1}}{\mathrm{d}z^{p-1}}\left[\binom{z-1}{p-1}\psi\left(z - \lfloor\frac{p}{2}\rfloor\right)\right]$$

where $H_n := 1 + 1/2 + \cdots + 1/n$, is the n-th harmonic number and $\lfloor\rfloor$ is the floor function.

End

Thank you