VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-013-0158-6
Math. Slovaca **63** (2013), No. 5, 1101–1112

THE ARC LENGTH VARIATIONAL FORMULA ON THE EXPONENTIAL MANIFOLD

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(*Communicated by Július Korbaš*)

ABSTRACT. In this paper, we mainly consider the first and second arc length variational problems on the exponential statistical manifold, and give the variational formulae.

1. Introduction

Information geometry introduced in 1980s has proposed new applications of differential geometry. It began as a study of natural structures possessed by families of probability distributions. A family of probability distributions such as normal distributions naturally has a Riemannian metric. Moreover, it has a dualistic structure of affine connections, and this duality is useful in statistics. Recently some authors ([7]–[11]) have investigated the structures of the state space of thermodynamic parameters based upon the differential geometric approach to parametric statistics developed by Chentsov ([4]), Efron ([5]), Amari etc. ([1,2]), Shima and Yagi ([10]) and Yilmaz ([11]) discussed geometric structures of Hessian manifolds.

It is desirable to study the geometric structure of the statistical manifolds, about which some interesting results have already been obtained. For example,

2010 Mathematics Subject Classification: Primary 53A30; Secondary 53B25.
Keywords: arc length variation, exponential manifold, Fisher metric.
Supported by NSF of China (10671181).
Research of the forth author was supported by the National Natural Science Foundation of China (10871218).

Springer

Matsuzoe ([7]) has given an equiaffine structure on the submanifolds of the exponential manifold, and Furuhata ([6]) studied the characters of hypersurfaces of statistical manifolds. Since the induced connection of pull-back bundle has not been defined exactly in information geometry, we consider if the arc length variation can be expressed by dual connection. Fortunately, the answer is definite.

In this paper, we give the first and second arc length variational formulae on the exponential statistical manifold. This paper includes four sections. In Section 1, we introduce three main theorems, and in Section 2 the fundamental structure of information geometry is introduced. Finally, in Section 3 and 4, we give the proofs of the main theorems and some propositions implied by the main theorems.

Theorem 1.1. *Let $\gamma\colon [a,b] \to S$ be a smooth curve in the exponential manifold S, and the ratio of t and arc length parameter s be constant. If $\Phi\colon [a,b]\times(-\varepsilon,\varepsilon) \to S$ is a variation of γ, denoting $L(u) = L(\gamma_u)$ the arc length of γ_u, $u \in (-\varepsilon,\varepsilon)$, then we have*

$$L'(0) = \frac{b-a}{l}\left(\langle U, \gamma'(t)\rangle|_a^b - \int_a^b \langle \nabla_{\frac{\partial}{\partial t}}\gamma'(t), U\rangle \mathrm{d}t\right), \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the Fisher metric of the exponential manifold S, and ∇ is the Riemannian connection on $\langle\cdot,\cdot\rangle$.

Theorem 1.2. *Let $\gamma\colon [a,b] \to S$ be a geodesic on 1-connection in the exponential manifold S. For any variation $\Phi\colon [a,b]\times(-\varepsilon,\varepsilon) \to S$ of γ, we have*

$$\begin{aligned} L''(0) = {}& \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \gamma'(t)\rangle|_a^b + \frac{b-a}{2l}\frac{\partial}{\partial t}\langle U, U\rangle|_a^b \\ & - \frac{b-a}{2l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle \mathrm{d}t - \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), U\rangle|_a^b \\ & + \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}} U\rangle \mathrm{d}t \\ & - \frac{(b-a)^3}{4l^3}\int_a^b \left(2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle - T(U, \gamma'(t), \gamma'(t))\right)^2 \mathrm{d}t, \end{aligned} \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the Fisher metric of the exponential manifold S.

Theorem 1.3. *Let $\gamma\colon [a,b] \to S$ be a geodesic on -1-connection in the exponential manifold S. For any variation $\Phi\colon [a,b]\times(-\varepsilon,\varepsilon) \to S$ of γ, we have*

$$\begin{aligned} L''(0) = {} & \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \gamma'(t)\rangle|_a^b + \frac{b-a}{2l}\frac{\partial}{\partial t}\langle U, U\rangle|_a^b \\ & - \frac{b-a}{2l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle \mathrm{d}t - \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), U\rangle|_a^b \\ & + \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}} U\rangle \mathrm{d}t - \frac{(b-a)^3}{4l^3}\int_a^b (2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle \\ & - 2\langle \nabla^{(1)}_{\frac{\partial}{\partial t}}\gamma'(t), U\rangle - T(U, \gamma'(t), \gamma'(t)))^2 \mathrm{d}t. \end{aligned} \tag{1.3}$$

2. Preliminaries

Definition 2.1. An n-dimensional set of probability density functions $S = \{p_\theta \mid \theta \in \Theta \subset R^n\}$ for random variable $x \in \Omega \subseteq R$ is said to be an exponential family when the density functions can be expressed in terms of functions $\{C, F_1, \ldots, F_n\}$ on R and a function φ on Θ as

$$p_\theta(x) = \mathrm{e}^{\{C(x)+\sum\limits_i \theta^i F_i(x) - \varphi(\theta)\}}. \tag{2.1}$$

Then we call $\{\theta^i\}$ its natural coordinates, and φ its potential function.

From the normalization condition $\int\limits_\Omega p_\theta(x)\delta x = 1$ we obtain

$$\varphi(\theta) = \log \int\limits_\Omega \mathrm{e}^{\{C(x)+\sum\limits_i \theta^i F_i(x)\}} \mathrm{d}x. \tag{2.2}$$

From the definition of an exponential family, and letting $\partial_i = \frac{\partial}{\partial \theta^i}$, we use the log-likelihood function $l(\theta, x) = \log(p_\theta(x))$ to obtain

$$\partial_i l(\theta, x) = F_i(x) - \partial_i \varphi(\theta)$$

and

$$\partial_i \partial_j l(\theta, x) = -\partial_i \partial_j \varphi(\theta). \tag{2.3}$$

The Fisher information metric g on the n-dimensional space of parameters $\Theta \subset R^n$, equivalently on the set $S = \{p_\theta \mid \theta \in \Theta \subset R^n\}$, has the coefficients

$$g_{ij} = -\int_\Omega \partial_i \partial_j l(\theta, x) p(\theta, x) \mathrm{d}x = \partial_i \partial_j \varphi(\theta) = \varphi_{ij}(\theta). \tag{2.4}$$

Then, (S, g) is a Riemannian manifold with the Levi-Civita connection given by

$$\begin{aligned} \langle \nabla_{\partial_i} \partial_j, \partial_h \rangle g^{hk} &= \Gamma_{ij}^k(\theta) \\ &= \frac{1}{2} \sum g^{kh} (\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}) \\ &= \sum \frac{1}{2} g^{kh} \partial_i \partial_j \partial_h \varphi(\theta). \end{aligned} \tag{2.5}$$

Next we obtain a family of symmetric connections that included the Levi-Civita case and has significance in mathematical statistics. For any $\alpha \in R$ the function $\Gamma_{ij,k}^{(\alpha)}$ of each $\theta \in \Theta$ is defined by

$$\begin{aligned} \langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle &= \Gamma_{ij,k}^{(\alpha)}(\theta) \\ &= \int_\Omega \left(\partial_i \partial_j l(\theta, x) + \frac{1-\alpha}{2} \partial_i l(\theta, x) \partial_j l(\theta, x) \right) \partial_k l(\theta, x) p_\theta \mathrm{d}x \\ &= \frac{1-\alpha}{2} \partial_i \partial_j \partial_k \varphi(\theta). \end{aligned} \tag{2.6}$$

If $\alpha = 0$ in (2.6), then $\nabla^{(0)}$ is the Riemannian connection defined in (2.5); if $\alpha = 1$, then $\nabla^{(1)}$ is an affine with respect to natural coordinates $\{\theta^i\}$, that is $\nabla_{\partial_i}^{(1)} \partial_j = 0$.

Denoting $\eta^i = \partial_i \varphi(\theta)$, then $\{\eta_i\}$ are the dual coordinates of $\{\theta^i\}$, and $\partial^i = \frac{\partial}{\partial \eta_i}$. From [1], we see that $\nabla^{(-1)}$ is flat, that is $\nabla_{\partial^i}^{(-1)} \partial^j = 0$.

The cubic tensor of the exponential manifold is defined as follows:

$$T_{ijk}(\theta) = \int \partial_i l(\theta, x) \partial_j l(\theta, x) \partial_k l(\theta, x) p_\theta \mathrm{d}x = \partial_i \partial_j \partial_k \varphi(\theta). \tag{2.7}$$

Definition 2.2. Let $\gamma\colon [a, b] \to S$ be a smooth curve in the exponential manifold S. If there exists an $\varepsilon > 0$, and let smooth map $\Phi\colon [a, b] \times (-\varepsilon, \varepsilon) \to S$ be such that

$$\Phi(t, 0) = \gamma(t) = p(\theta(t, 0), x), \qquad \text{for all} \quad t \in [a, b],$$

then Φ is a variation of γ, and if

$$\Phi(a, u) = \gamma(a), \quad \Phi(b, u) = \gamma(b), \qquad \text{for all} \quad u \in (-\varepsilon, \varepsilon),$$

then Φ is a variation with endpoints of γ.

Now suppose that Φ is a variation of γ, for a fixed $u \in (-\varepsilon, \varepsilon)$, let $\gamma_u : [a, b] \to M$ such that

$$\gamma_u(t) = \Phi(t, u), \qquad \text{for all} \quad t \in [a, b];$$

then, $\{\gamma_u\}$ are variational curves of Φ.

On the other hand, for any $t \in [a, b]$, there exists a smooth curve

$$\sigma_t \colon (-\varepsilon, \varepsilon) \to M, \qquad \text{for all} \quad u \in (-\varepsilon, \varepsilon);$$

then, $\{\sigma_t\}$ are sectional curves of Φ.

Next, we give the definition of a variational vector field of Φ as follows:

$$\widetilde{T} = \Phi_{*(t,u)} \left(\frac{\partial}{\partial t} \right), \qquad \widetilde{U} = \Phi_{*(t,u)} \left(\frac{\partial}{\partial u} \right), \tag{2.8}$$

where $\widetilde{T}$ and $\widetilde{U}$ are the variational vector fields of curve families $\{\gamma_u\}$ and $\{\sigma_t\}$.

Denote

$$\begin{aligned} \widetilde{T} l(\theta, x) &= \Phi_{*(t,u)} \left(\frac{\partial}{\partial t} \right) l(\theta, x) = \frac{\partial}{\partial t} l(\theta(t, u), x), \\ \widetilde{U} l(\theta, x) &= \Phi_{*(t,u)} \left(\frac{\partial}{\partial u} \right) l(\theta, x) = \frac{\partial}{\partial u} l(\theta(t, u), x). \end{aligned} \tag{2.9}$$

Especially, it is easy to check that

$$\widetilde{T} l(\theta, x)|_{u=0} = \gamma'(t)(l(\theta(t, 0), x)), \tag{2.10}$$

and

$$\widetilde{U} l(\theta, x)|_{u=0} = \sigma'(u)(l(\theta(t, u), x))|_{u=0}, \tag{2.11}$$

where we denote $\sigma'(u) = U$.

Using the same method as [3], we can get the following result.

Proposition 2.3. *Let $\gamma \colon [a, b] \to S$ be a smooth curve in the exponential manifold S. Then U_t a smooth vector field along γ, where $U_t \in T_{\gamma(t)} S$, then there exists a variation Φ of γ with U as its variational vector field.*

To prove the main theorems, we need the following facts.

Fact 2.4. *If γ is a geodesic with respect to Levi-Civita connection in exponential manifold, we have $\bigtriangledown_{\frac{\partial}{\partial t}} \gamma'(t) = 0$.*

Fact 2.5. *If γ is a geodesic with respect to the 1-connection in the exponential manifold S, and $\{\theta^i\}$ is natural coordinate system on S, then $\overset{(1)}{\bigtriangledown}_{\frac{\partial}{\partial t}} \gamma'(t) = 0$, where $\gamma'(t) = \mathrm{d}\theta^i t \frac{\partial}{\partial i}$.*

Fact 2.6. *If γ is a geodesic with respect to the -1-connection in the exponential manifold S, and $\{\eta^i\}$ is dual coordinate system of $\{\theta^i\}$ on S, then $\bigtriangledown^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)=0$, where $\gamma'(t) = \mathrm{d}\eta^i t\frac{\partial}{\partial^i}$.*

3. The proof of Theorem 1.1

P r o o f. Let $\widetilde{T}$ and $\widetilde{U}$ be the vector fields defined by (2.9). The arc length formula is as follows:

$$L(u) = \int_a^b \langle \widetilde{T}, \widetilde{T}\rangle^{\frac{1}{2}} \mathrm{d}t, \tag{3.1}$$

Taking the derivative of (3.1) with respect to u, we obtain

$$L'(u) = \int_a^b \frac{\frac{\partial}{\partial u}\langle \widetilde{T}, \widetilde{T}\rangle}{2\langle \widetilde{T}, \widetilde{T}\rangle^{\frac{1}{2}}} \mathrm{d}t, \tag{3.2}$$

and

$$\begin{aligned}\frac{\partial}{\partial u}\langle \widetilde{T}, \widetilde{T}\rangle = 2\int \frac{\partial \widetilde{T}l(\theta,x)}{\partial u}\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x \\ + \int \widetilde{T}l(\theta,x)^2\frac{\partial l(\theta,x)}{\partial u}p(\theta,x)\mathrm{d}x,\end{aligned} \tag{3.3}$$

where $\theta = \theta(t, u)$.

From the right hand side (3.3), we get

$$\begin{aligned}\int \frac{\partial \widetilde{T}l(\theta,x)}{\partial u}\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x = \frac{\partial}{\partial t}\int \widetilde{U}l(\theta,x)\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x \\ - \int \widetilde{U}l(\theta,x)\frac{\partial \widetilde{T}l(\theta,x)}{\partial t}p(\theta,x)\mathrm{d}x \\ - \int \widetilde{U}l(\theta,x)\widetilde{T}l(\theta,x)^2 p(\theta,x)\mathrm{d}x,\end{aligned} \tag{3.4}$$

and (3.3) is equivalent to the following identity,

$$\frac{\partial}{\partial u}\langle \widetilde{T}, \widetilde{T}\rangle = 2\frac{\partial}{\partial t}\langle \widetilde{U}, \widetilde{T}\rangle - 2\int \widetilde{U}l(\theta,x)\frac{\partial \widetilde{T}l(\theta,x)}{\partial t}p(\theta,x)\mathrm{d}x - T(\widetilde{U}, \widetilde{T}, \widetilde{T}). \tag{3.5}$$

By the definition of α-connection, it is easy to check that

$$\langle \bigtriangledown^{(\alpha)}_{\frac{\partial}{\partial t}}\widetilde{T}, \widetilde{U}\rangle = \int \frac{\partial \widetilde{T}l(\theta,x)}{\partial t}\widetilde{U}l(\theta,x)p(\theta,x)\mathrm{d}x + \frac{1-\alpha}{2}T(\widetilde{T}, \widetilde{T}, \widetilde{U}). \tag{3.6}$$

Combining (3.6) with (3.5), we obtain

$$\frac{\partial}{\partial u}\langle\widetilde{T},\widetilde{T}\rangle = 2\frac{\partial}{\partial t}\langle\widetilde{U},\widetilde{T}\rangle - 2\langle\nabla^{(\alpha)}_{\frac{\partial}{\partial t}}\widetilde{T},\widetilde{U}\rangle - \alpha T(\widetilde{U},\widetilde{T},\widetilde{T}). \tag{3.7}$$

On the other hand, by the assumption of the theorem, the ratio of t and arc length parameter s is constant, we have $|\gamma'(t)| =$ constant, and

$$l = L(\gamma) = \int_a^b \langle\gamma'(t),\gamma'(t)\rangle^{\frac{1}{2}}\mathrm{d}t = |\gamma'(t)|(b-a),$$

that is

$$|\gamma'(t)| = \frac{l}{b-a}.$$

When $u = 0$, $\widetilde{T}$ is tangent to $\gamma(t)$, and the first arc length variation formula is given as follows:

$$\begin{aligned}
L'(0) &= \frac{b-a}{l}\int_a^b \left(2\frac{\partial}{\partial t}\langle U,\gamma'(t)\rangle - 2\langle\nabla^{(\alpha)}_{\frac{\partial}{\partial t}}\gamma'(t),U\rangle - \alpha T(U,\gamma'(t),\gamma'(t))\right)\mathrm{d}t \\
&= \frac{b-a}{l}\langle U,\gamma'(t)\rangle|_a^b - \frac{b-a}{l}\int_a^b \left(\langle\nabla^{(\alpha)}_{\frac{\partial}{\partial t}}\gamma'(t),U\rangle + \frac{1}{2}\alpha T(U,\gamma'(t),\gamma'(t))\right)\mathrm{d}t \\
&= \frac{b-a}{l}\left(\langle U,\gamma'(t)\rangle|_a^b - \int_a^b \langle\nabla_{\frac{\partial}{\partial t}}\gamma'(t),U\rangle\mathrm{d}t\right),
\end{aligned} \tag{3.8}$$

where ∇ is the Riemannian connection on Fisher metric g. □

Proposition 3.1. *If Φ is a variation with fixed endpoints, namely, $U(a) = U(b) = 0$, then we have*

$$L'(0) = -\frac{b-a}{l}\int_a^b \langle\nabla_{\frac{\partial}{\partial t}}\gamma'(t),U\rangle\mathrm{d}t. \tag{3.9}$$

Particularly, if $\gamma(t)$ is a geodesic with respect to Riemannian connection in the exponential manifold S, using Fact 2.4., we can see that γ is the critical point of the arc length function.

4. The proofs of Theorem 1.2 and 1.3

To prove Theorem 1.2, we need to prove three lemmas first.

Lemma 4.1.

$$\frac{\partial}{\partial u}\langle \widetilde{U}, \widetilde{T}\rangle = \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{U}, \widetilde{T}\rangle + \frac{1}{2} T(\widetilde{U}, \widetilde{U}, \widetilde{T}) + \frac{1}{2}\frac{\partial}{\partial t}\langle \widetilde{U}, \widetilde{U}\rangle. \tag{4.1}$$

P r o o f. By the definition of Fisher metric, we have

$$\begin{aligned}
\frac{\partial}{\partial u}\langle \widetilde{U}, \widetilde{T}\rangle &= \frac{\partial}{\partial u}\int \widetilde{U}l(\theta,x)\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x \\
&= \int \frac{\partial}{\partial u}(\widetilde{U}l(\theta,x))\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x \\
&\quad + \int \widetilde{U}l(\theta,x)\frac{\partial}{\partial u}(\widetilde{T}l(\theta,x))p(\theta,x)\mathrm{d}x \\
&\quad + \int \widetilde{U}l(\theta,x)\widetilde{T}l(\theta,x)\widetilde{T}l(\theta,x)p(\theta,x)\mathrm{d}x \\
&= \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{U}, \widetilde{T}\rangle + \frac{1}{2}\frac{\partial}{\partial t}\int \widetilde{U}l(\theta,x)\widetilde{U}l(\theta,x)p(\theta,x)\mathrm{d}x \qquad (4.2)\\
&\quad - \frac{1}{2}T(\widetilde{U}, \widetilde{U}, \widetilde{T}) + T(\widetilde{U}, \widetilde{U}, \widetilde{T}) \\
&= \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{U}, \widetilde{T}\rangle + \frac{1}{2} T(\widetilde{U}, \widetilde{U}, \widetilde{T}) + \frac{1}{2}\frac{\partial}{\partial t}\langle \widetilde{U}, \widetilde{U}\rangle.
\end{aligned}$$

□

Lemma 4.2.

$$\frac{\partial}{\partial u}\langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial t}} \widetilde{T}, \widetilde{U}\rangle = \frac{\partial}{\partial t}\langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{T}, \widetilde{U}\rangle - \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{T}, \bigtriangledown^{(-1)}_{\frac{\partial}{\partial t}} \widetilde{U}\rangle + \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial t}} \widetilde{T}, \bigtriangledown^{(-1)}_{\frac{\partial}{\partial u}} \widetilde{U}\rangle. \tag{4.3}$$

P r o o f. It is similar to the proof of Lemma 4.1, so we omit it here. □

Lemma 4.3.

$$\frac{\partial}{\partial t}T(\widetilde{U}, \widetilde{U}, \widetilde{T}) - \frac{\partial}{\partial u}T(\widetilde{U}, \widetilde{T}, \widetilde{T}) = \langle \bigtriangledown^{(1)}_{\frac{\partial}{\partial t}} \widetilde{T}, \bigtriangledown^{(-1)}_{\frac{\partial}{\partial u}} \widetilde{U}\rangle - \langle \bigtriangledown^{(-1)}_{\frac{\partial}{\partial t}} \widetilde{T}, \bigtriangledown^{(1)}_{\frac{\partial}{\partial u}} \widetilde{U}\rangle. \tag{4.4}$$

P r o o f.

$$\begin{aligned}
&\frac{\partial}{\partial t}T(\widetilde{U}, \widetilde{U}, \widetilde{T}) - \frac{\partial}{\partial u}T(\widetilde{U}, \widetilde{T}, \widetilde{T}) \\
&= 2\int \frac{\partial}{\partial t}(\widetilde{U}l(\theta,x))(\widetilde{U}l(\theta,x))(\widetilde{T}l(\theta,x))p(\theta,x)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int(\widetilde{U}l(\theta,x))^2\frac{\partial}{\partial t}(\widetilde{T}l(\theta,x))p(\theta,x)\mathrm{d}x\\
&-2\int\frac{\partial}{\partial u}(\widetilde{T}l(\theta,x))(\widetilde{U}l(\theta,x))(\widetilde{T}l(\theta,x))p(\theta,x)\mathrm{d}x\\
&-\int(\widetilde{T}l(\theta,x))^2\frac{\partial}{\partial u}(\widetilde{U}l(\theta,x))p(\theta,x)\mathrm{d}x\\
=&\int(\widetilde{U}l(\theta,x))^2\frac{\partial}{\partial t}(\widetilde{T}l(\theta,x))p(\theta,x)\mathrm{d}x-\int(\widetilde{T}l(\theta,x))^2\frac{\partial}{\partial u}(\widetilde{U}l(\theta,x))p(\theta,x)\mathrm{d}x\\
=&\langle\nabla^{(-1)}_{\frac{\partial}{\partial u}}\widetilde{U},\nabla^{(1)}_{\frac{\partial}{\partial t}}\widetilde{T}\rangle-\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}\widetilde{U},\nabla^{(-1)}_{\frac{\partial}{\partial t}}\widetilde{T}\rangle. \qquad (4.5)
\end{aligned}
$$

□

P r o o f o f T h e o r e m 1.2. Taking the second derivative in (3.1), we have

$$
\begin{aligned}
L''(u)&=\mathrm{d}^2u^2L(\gamma(u))\\
&=\int_a^b\frac{\frac{\partial^2}{\partial u^2}\langle\widetilde{T},\widetilde{T}\rangle}{2\langle\widetilde{T},\widetilde{T}\rangle^{\frac{1}{2}}}\mathrm{d}t-\int_a^b\frac{(\frac{\partial}{\partial u}\langle\widetilde{T},\widetilde{T}\rangle)^2}{4\langle\widetilde{T},\widetilde{T}\rangle^{\frac{3}{2}}}\mathrm{d}t.
\end{aligned} \qquad (4.6)
$$

Let $\alpha=1$ in (3.7), and taking the derivative with respect to u, we have

$$
\begin{aligned}
&\frac{\partial^2}{\partial u^2}\langle\widetilde{T},\widetilde{T}\rangle\\
&=2\frac{\partial^2}{\partial u\partial t}\langle\widetilde{T},\widetilde{U}\rangle-2\frac{\partial}{\partial u}\langle\nabla^{(1)}_{\frac{\partial}{\partial t}}\widetilde{T},\widetilde{U}\rangle-\frac{\partial}{\partial u}T(\widetilde{T},\widetilde{T},\widetilde{U}).
\end{aligned} \qquad (4.7)
$$

Substituting (4.1), (4.3) and (4.4) into (4.7), we obtain

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2}\langle\widetilde{T},\widetilde{T}\rangle=&\,2\frac{\partial}{\partial t}\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}\widetilde{U},\widetilde{T}\rangle+\frac{\partial^2}{\partial t^2}\langle\widetilde{U},\widetilde{U}\rangle\\
&-\langle\nabla^{(-1)}_{\frac{\partial}{\partial u}}\widetilde{U},\nabla^{(1)}_{\frac{\partial}{\partial t}}\widetilde{T}\rangle-\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}\widetilde{U},\nabla^{(-1)}_{\frac{\partial}{\partial t}}\widetilde{T}\rangle\\
&-2\frac{\partial}{\partial t}\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}\widetilde{T},\widetilde{U}\rangle+2\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}\widetilde{T},\nabla^{(-1)}_{\frac{\partial}{\partial t}}\widetilde{U}\rangle.
\end{aligned} \qquad (4.8)
$$

If $u=0$, and γ is a geodesic with respect to the 1-connection, we obtain

$$
\begin{aligned}
L''(0)=&\,\frac{b-a}{l}\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}U,\gamma'(t)\rangle|_a^b+\frac{b-a}{2l}\frac{\partial}{\partial t}\langle U,U\rangle|_a^b\\
&-\frac{b-a}{2l}\int_a^b\langle\nabla^{(1)}_{\frac{\partial}{\partial u}}U,\nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), U\rangle|_a^b + \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}}U\rangle \mathrm{d}t \\
&- \frac{(b-a)^3}{4l^3}\int_a^b \left(2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle - T(U,\gamma'(t),\gamma'(t))\right)^2 \mathrm{d}t. \qquad (4.9)
\end{aligned}
$$

On the other hand, If $u = 0$, and γ is a geodesic with respect to the -1-connection, then

$$
\begin{aligned}
&L''(0) = \\
&= \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}U, \gamma'(t)\rangle|_a^b + \frac{b-a}{2l}\frac{\partial}{\partial t}\langle U, U\rangle|_a^b - \frac{b-a}{2l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}U, \nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle \mathrm{d}t \\
&- \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), U\rangle|_a^b + \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}}U\rangle \mathrm{d}t \\
&- \frac{(b-a)^3}{4l^3}\int_a^b \left(2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle - 2\langle \nabla^{(1)}_{\frac{\partial}{\partial t}}\gamma'(t), U\rangle - T(U,\gamma'(t),\gamma'(t))\right)^2 \mathrm{d}t \qquad (4.10)
\end{aligned}
$$

holds. This completes the proof of Theorem 1.2. $\square$

Particularly, if Φ is a variation with fixed endpoints, and γ is a geodesic with respect to the 1-connection, then thanks to fact 2.3, the following equality holds:

$$
\begin{aligned}
&L''(0) = \\
&= \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}}U, \gamma'(t)\rangle|_a^b - \frac{b-a}{2l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}U, \nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle \mathrm{d}t \\
&+ \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}}U\rangle \mathrm{d}t \qquad (4.11) \\
&- \frac{(b-a)^3}{4l^3}\int_a^b \left(2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle - T(U,\gamma'(t),\gamma'(t))\right)^2 \mathrm{d}t.
\end{aligned}
$$

In the case γ is a geodesic with respect to the -1-connection, using fact 2.4, we obtain

$$\begin{aligned} L''(0) = \frac{b-a}{l}\langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \gamma'(t)\rangle\big|_a^b - \frac{b-a}{2l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}} U, \nabla^{(-1)}_{\frac{\partial}{\partial t}}\gamma'(t)\rangle \mathrm{d}t \\ + \frac{b-a}{l}\int_a^b \langle \nabla^{(1)}_{\frac{\partial}{\partial u}}\gamma'(t), \nabla^{(-1)}_{\frac{\partial}{\partial t}} U\rangle \mathrm{d}t \qquad (4.12) \\ - \frac{(b-a)^3}{4l^3}\int_a^b \left(2\frac{\partial}{\partial t}\langle U, \gamma'(t)\rangle - 2\langle \nabla^{(1)}_{\frac{\partial}{\partial t}}\gamma'(t), \widetilde{U}\rangle - T(U,\gamma'(t),\gamma'(t))\right)^2 \mathrm{d}t. \end{aligned}$$

Acknowledgement. The second author would like to express his thanks to professors D. Krupka and O. Krupkova for their help.

REFERENCES

[1] AMARI, S.: *Differential Geometric Methods in Statistics*. Lecture Notes in Statist. 28, Springer-Verlag, Berlin, 1985.

[2] AMARI, S.—NAGAOKA, H.: *Methods of Information Geometry*, Amer. Math. Soc./Oxford University Press, Providence, RI/Oxford, 2000.

[3] BOOTHBY, W. M.: *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, Inc., New York, 1986.

[4] CHENTSOV, N. N.: *Statistical Decision Rules and Optimal Inference*, Nauka Press, Moscow, 1976.

[5] EFRON, B.: *Defining the curvature of a statistical problem (with applications to second order efficiency)*, Ann. Statist. **3** (1975), 1189–1242.

[6] FURUHATA, H.: *Hypersurfaces in statistical manifolds*, Differential Geom. Appl. **27** (2009), 420–429.

[7] MATSUZOE, H.—TAKEUCHI, J.—AMARI, S.: *Equiaffine structures on statistical manifolds and Bayesian statistics*, Differential Geom. Appl. **24** (2006), 567–578.

[8] NOMIZU, K.—SASAKI, T.: *Affine Differential Geometry: Geometry of Affine Immersions*, Cambridge University Press, Cambridge, 1994.

[9] PORTESI, M.—PLASTINO, A.—PENNINI, F.: *Information geometry and phase transition*, Internat. J. Modern Phys. B **20** (2006), 5250–5253.

[10] SHIMA, H.—YAGI, K.: *Geometry of Hessian manifolds*, Differential Geom. Appl. **7** (1997), 277–290.

[11] YILMAZ, M. Y.—BEKTAS, M.: *A survey on curvature of Hessian manifolds*, Chaos Solitons Fractals **38** (2008), 620–630.

Received 21. 4. 2011
Accepted 24. 5. 2011

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