3.8 EXPONENTIAL GROWTH AND DECAY

KEY: "Rate of change is proportional to the size"

$$\frac{dy}{dx} = ky, \text{ where } \begin{cases} k > 0 & \text{law of natural growth} \\ k < 0 & \text{law of natural decay} \end{cases}$$

Theorem. *The solution to* $\frac{dy}{dx} = ky$ *is*

$$y(x) = y(0)\, e^{kt}.$$

Example. [Population Growth] Consider that

Year	World Population
1950	2560 Million
1960	3040 M

[Question]: What is the population in the year 2020?

Solution. Define the function of population by $P(t)$, then

$$\frac{dP}{dt} = kt,$$

for some $k > 0$. Thus

$$P(t) = P(0)\, e^{kt}.$$

Intepretate the table to get that

$$t = 0 \leftrightarrow \text{Year } 1950 \leftrightarrow P(0) = 2560 \Rightarrow P(t) = 2560e^{kt}.$$

and

$$t = 10 \leftrightarrow \text{Year } 1960 \leftrightarrow P(10) = 3040.$$

So,

$$3040 = P(10) = 2560e^{k10} \Rightarrow 10k = \ln\frac{3040}{2560} \Rightarrow k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017.$$

Now,

$$P(70) = 2560e^{70k} \approx 8524 \text{ Million}.$$

Example. [Radioactive decay]

$m(t)$= mass remaining at time t; $m_0 = m(0)$;

Half-life. Time required for any quantity decay to 1/2, denoted by t_{HL}

$$m(t) = m_0 e^{kt},\ t < 0.$$

Then

$$m(t_{HL}) = \frac{1}{2}m_0 = m_0 e^{kt_{HL}} \Leftrightarrow \frac{1}{2}e^{kt_{HL}} \Leftrightarrow t_{HL} = \frac{1}{k}\ln\frac{1}{2} = \frac{1}{k}\ln\left(2^{-1}\right) = -\frac{\ln 2}{k}.$$

Radium-226 has $t_{HL} = 1590$ years and we have a sample of 100 mg.

[Q]What is the mass after 1000 years? How long does it take to decay to 30 mg?

Solution. As we computed

$$1590 = t_{HL} = -\frac{\ln 2}{k} \Rightarrow k = -\frac{\ln 2}{1590}.$$

Then

$$m(t) = m_0 e^{kt} = 100e^{-\frac{\ln 2}{1590}t}.$$

(1)$m(1000) = 100e^{-\frac{1590}{\ln 2}\cdot 1000} \approx 65$ mg

(2)We need to solve

$$30 = m(t) = 100e^{-\frac{\ln 2}{1590}t} \Leftrightarrow \frac{3}{10} = e^{-\frac{\ln 2}{1590}t} \Leftrightarrow -\frac{\ln 2}{1590}t = \ln\frac{3}{10} \Rightarrow t \approx 2762.$$

Example. [Newton's Law of Cooling]

$T(t)$= temperature at time t; T_s=surrounding temperature. Then

$$\frac{dT(t)}{dt} = k(T(t) - T_s) \Rightarrow T(t) = T_s + (T(0) - T_s)e^{kt},\ k < 0.$$

Suppose we have a bottle of soda at 72° F and placed in a refrigerator of temperatue 44°F. Then, we know that

$$T(0) = 72 \text{ and } T_s = 44.$$

Suppose we also know that after 30 minutes, the bottle becomes 61°F. Thus

$$61 = T(30) = 44 + (72 - 44)e^{k30} \Rightarrow k = \frac{1}{30}\ln\left(\frac{17}{28}\right) \approx -0.01663.$$

[Question]What will happen after an hour?

Solution.

$$T(60) = 4 + (72 - 44)e^{k60} \approx 56.3.$$

[Question]How long does it take to drop the tempurature to 50° F?

Solution.

$$50 = T(t) = T(60) = 4 + (72 - 44)e^{kt} \Rightarrow t \approx 92.6$$

Example. [Contiuously Compounded Interest]

\$1000 is invested at 6%, compounded annually, then

$$\begin{cases} f(1) = 1000 \times 1.06 = 1060 & \text{after 1 year} \\ f(2) = 1000 \times 1.06^2 = 1123.60 & \text{after 2 years} \\ f(t) = 1000 \times (1.06)^t & \text{in general} \end{cases}$$

Thus, we have the formula

$$A_0(1 + r)^t.$$

However, if the interest is compounded n times a year, it becomes

$$A_0\left(1 + \frac{r}{n}\right)^{nt}.$$

For instance if it is daily compounded, then

$$f(t) = 1000 \times \left(1 + \frac{0.06}{365}\right)^{365t}.$$

Now, if let $n \to \infty$, then

$$A(t) = \lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}.$$

3.9 RELATED RATES

KEY: Two things are related by simple geometric/physical formula. Then use chain rule (implicit differentiation) to find the "related rates".

Example. Air is being pumped into a spherical balloon so that its volume increases at a rate $100\text{cm}^3/\text{s}$. How fast is the radius of balloon increasing when the diameter is 50 cm?

Solution. Two things: volume: V and radius r (diameter) with

$$V = \frac{4}{3}\pi r^3.$$

It is given that

$$\frac{dV}{dt} = 100.$$

We want to compute

$$\frac{dr}{dt} =? \text{ when } r = 25.$$

Now from

$$V = \frac{4}{3}\pi r^3.$$

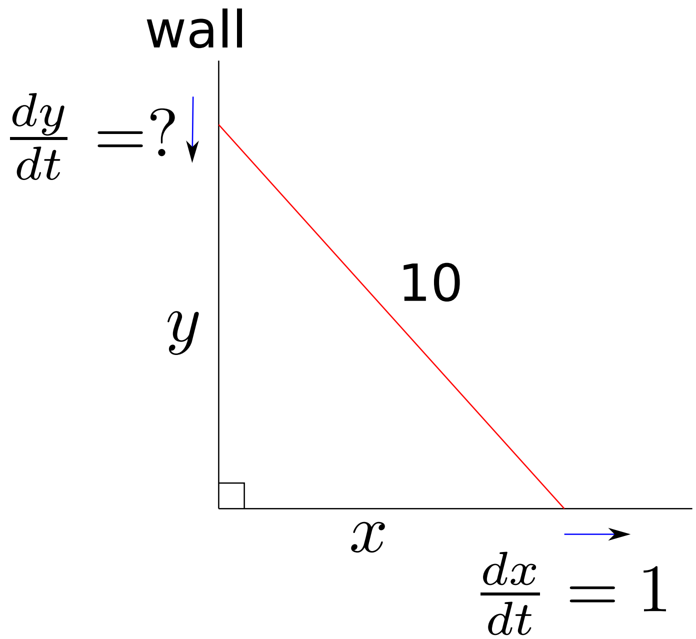
Taking derivatives WITH RESPECT TO time t gives

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}.$$

Thus, when $r = 25$

$$\frac{dr}{dt} = \frac{100}{4\pi (25)^2} = \frac{1}{25\pi}\,(\text{cm/s}).$$

Example. A 10-ft long ladder rests against a vertical wall. It the bottom slides away from the way at rate 1ft/s, how fast is the top sliding down when the bottom is 6 ft from the wall?



Solution. As we known

$$x^2 + y^2 = 10^2 = 100.$$

Differentiate both sides WITH RESPECT TO time t to get

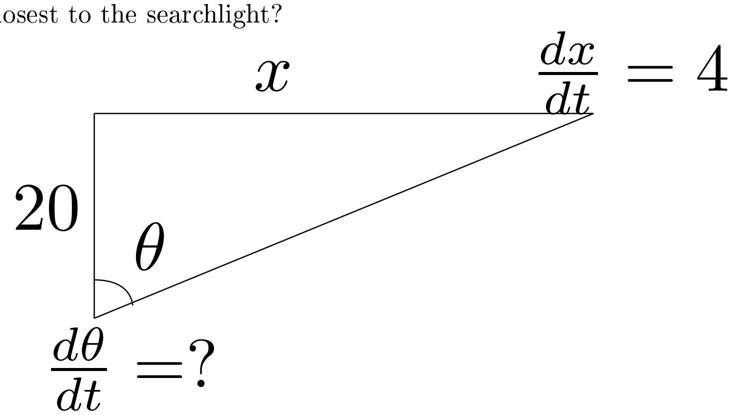
$$\frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$, $y = 8$, then

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4}.$$

The negative sign means y is decreasing.

Example. A man walks along a straight path at speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Solution. By geometry

$$\tan\theta = \frac{x}{20} \Rightarrow x = 20\tan\theta.$$

Differentiate WITH RESPECT TO time t to get

$$\frac{dx}{dt} = 20\sec^2\theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{20\sec^2\theta} = \frac{dx}{dt} \cdot \frac{1}{20}\cos^2\theta.$$

Now, $dx/dt = 4$ and when $x = 15$, $\cos\theta = \frac{4}{5}$. Then

$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{20} \cdot \left(\frac{4}{5}\right)^2 = \frac{16}{125}.$$

3.10 LINEAR APPROXIMATION AND DIFFERENTIALS & 3.11 HYPERBOLIC FUNCTIONS

Linear Approximation.

Definition. When x is close to a, the linear approximation of $f(x)$ at a is given by

$$L(x) = f(a) + f'(a)(x - a) \approx f(x).$$

.

Example. Compute the linear approximation of $f(x) = \sqrt{x}$ at $a = 4$.

Solution. $f'(x) = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2\sqrt{x}}$. Thus, $f(a) = f(4) = \sqrt{4} = 2$ and $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Therefore

$$L(x) = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1.$$

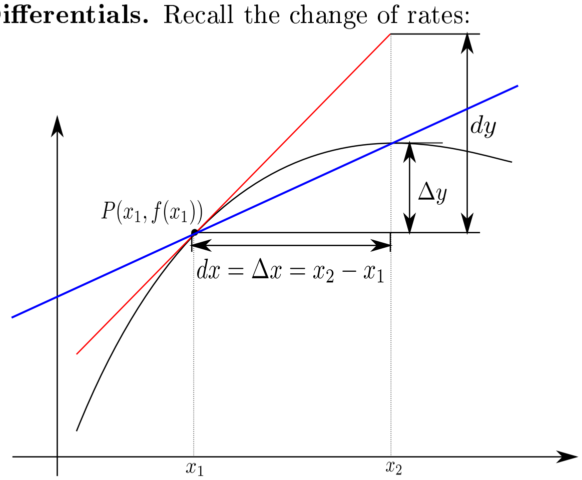
Example. Use linear approximation to compute $\sqrt{4.05}$.

Solution. $f(x) = \sqrt{x}$ and $a = 4$ since 4.05 is very close 4. Thus

$$\sqrt{4.05} = f(4.05) \approx L(4.05) = \frac{1}{4} \cdot 4.05 + 1 = 2.0125.$$

Remark. $\sqrt{4.05} = 2.01246117\cdots$.

Differentials. Recall the change of rates:



$$\begin{cases} \text{slope of secant line} = \frac{\Delta y}{\Delta x} \\ \text{slope of tangent line} = \frac{dy}{dx} = f'(x) \end{cases}$$

We call dx and dy differentials and they are connected by

$$dy = f'(x)\,dx.$$

Example. $f(x) = x^3 + x^2 - 2x + 1$ and $x_1 = 2$, $x_2 = 2.05$.
Then, $f'(x) = 3x^2 + 2x - 2$. Now, we have

$$dx = \Delta x = x_2 - x_1 = 0.05.$$

Also,

$$\begin{cases} \Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(2.05) - f(2) = 9.717625 - 9 = 0.717625 \\ dy = f'(x)\, dx = f'(2) \cdot 0.05 = 0.7 \end{cases}$$

Hyperbolic Functions.

Definition. We define

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

and then

$$\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{1}{\tanh x}, \mathrm{sech} x = \frac{1}{\cosh x}, \mathrm{csch} x = \frac{1}{\sinh x}.$$

<u>Identities</u> :

$$\begin{cases} \sinh(-x) = -\sinh x & , \sin(-x) = -\sin(x) \\ \cosh(-x) = \cosh x & , \cos(-x) = \cos(x) \\ \cosh^2 x - \sinh^2 x = 1 & , \cos^2 x + \sin^2 x = 1 \end{cases}$$

<u>Derivatives</u> :

$$\begin{cases} (\sinh x)' = \cosh x & , (\sin x)' = \cos x \\ (\cosh x)' = \sinh x & , (\cos x)' = -\sin x \end{cases}$$

<u>Inverse Functions</u>:

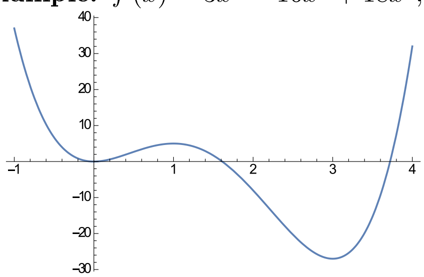
$$\begin{cases} \sinh^{-1} x & = \ln\left(x + \sqrt{x^2 + 1}\right) \\ \cosh^{-1} x & = \ln\left(x + \sqrt{x^2 - 1}\right) \\ \tanh^{-1} x & = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \end{cases}$$

4.1 MAXIMUM AND MINIMUM VALUES

Definition. Let c be a number in $dom\,(f)$. Then $f\,(c)$ is said to be
(1)ablsolute (global) maximum valule of f if for any x in $dom\,(f)$, $f\,(x) \leq f\,(c)$;
(2)ablsolute (global) minimum valule of f if for any x in $dom\,(f)$, $f\,(x) \geq f\,(c)$;
(3)local maximum valule of f if when x is near c, $f\,(x) \leq f\,(c)$;
(4)local minimum valule of f if when x is near c,, $f\,(x) \geq f\,(c)$.

Remark. When considering local max/min, we rule out endpoints

Example. $f\,(x) = 3x^4 - 16x^3 + 18x^2$, where $-1 \leq x \leq 4$



Local Max: $(1, 5)$
Global Max: $(-1, 37)$
Local Min:$(0, 0)$, $(3, -27)$
Global Min:$(3, -27)$

Theorem. *[Extreme Value Theorem] If f is continuous on a closed interval $[a, b]$, then f attains absolute max value $f\,(c)$ and absolute min value $f\,(d)$ for some c and d in $[a, b]$.*

Oberservation: $f'\,(x) = 12x^3 - 48x^2 + 36x = 12x\,(x-1)\,(x-3)$. And then

$$f'\,(0) = f'\,(1) = f'\,(3) = 0.$$

Theorem. *[Fermat's Theorem] If f has a local max or min at c, then $f'\,(c) = 0$.*

Definition. A *critical number* of a function f is the number c in $dom\,(f)$ such that either $f'\,(c) = 0$ or $f'\,(c)$ does not exist.

Example. (1)$f\,(x) = |x|$, $x = 0$ is critical number since $f'\,(0)$ DNE.
(2) $f\,(x) = x^2 \Rightarrow f'\,(x) = 2x \Rightarrow x = 0$ is a critical number. Moreover, $(0, 0)$ is local & global min.
(3) $f\,(x) = x^3 \Rightarrow f'\,(x) = 2x^2 \Rightarrow x = 0$ is a critical number, but $(0, 0)$ is neither max nor min.

$$\text{Min/Max} \Rightarrow \text{Critical, but Critical} \not\Rightarrow \text{Max/Min}$$

[Question] How to find global max and min of a continous function on $[a, b]$?
The Closed Interval Method:
1. Find all the values of critical numbers of f in (a, b), i.e. solve $f'\,(x) = 0$ and also number c such that $f'\,(c)$ DNE, then evaluate the value.

2. Find $f(a)$ and $f(b)$.

3. Compare all the values. The smallest is the global min and the largest is the global max.

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.

1. $f'(x) = 12x^3 - 48x^2 + 36x = 12x\left(x^2 - 4x + 3\right) = 12x(x-1)(x-3)$. Thus,

$$f'(x) = 0 \Rightarrow x = 0, 1, 3.$$

Then,

$$f(0) = 0,\ f(1) = 5,\ f(3) = -27.$$

2. $f(-1) = 37$ and $f(4) = 32$

3. We get that the global max is $f(-1) = 37$ and global min is $f(3) = -27$.

Example. $f(x) = \ln\left(x^2 + x + 1\right)$ on $[-1, 1]$.

1. $f'(x) = -\frac{2x+1}{x^2+x+1}$, then $f'(x) = 0 \Rightarrow -\frac{2x+1}{x^2+x+1} = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$. Then, $f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right) < 0$.

2. $f(1) = \ln 3 > 0$. $f(-1) = \ln 1 = 0$.

3. So, the global max is $f(1) = \ln 3$ and global min is $f\left(-\frac{1}{2}\right) = \ln\frac{3}{4}$.

4.3 DERIVATIVES AFFECTS THE SHAPE OF THE GRAPH

f' Affects f

Example. $f(x) = x^2 \Rightarrow f'(x) = 2x$, thus,

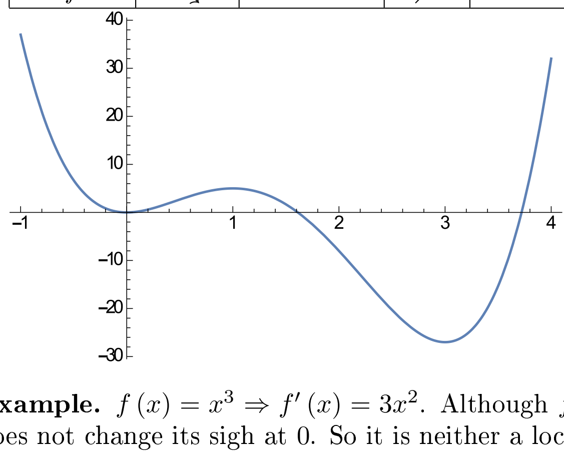
$$f'(x)\begin{cases} >0 & x>0 \\ =0 & x=0 \\ <0 & x<0 \end{cases}$$

Theorem. *[Increasing/Decreasing Test]*
(1) If $f'(x) > 0$ on an interval, then f is increasing on it.
(2) If $f'(x) < 0$ on an interval, then f is decreasing on it.

Theorem. *[The First Derivative Test] Suppose c is a critical number of a continuous function $f(x)$.*
(1)If f' changes from positive (+) to negative (-) at c, then f has a local maximum at c.
(2)If f' changes from negative (-) to positive (+) at c, then f has a local minimum at c.
(3)If f' does not change its sigh at c, then NOTHING happens.

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.
$f'(x) = 12x^3 - 48x^2 + 36x = 12x\left(x^2 - 4x + 3\right) = 12x(x-1)(x-3)$. So $f'(x) = 0 \Rightarrow x = 0, 1, 3$. Now

Intervals	$[-1, 0)$	$x = 0$	$(0, 1)$	$x = 1$	$(1, 3)$	$x = 3$	$(3, 4]$
f'	-	0	+	0	-	0	+
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$	Local Min	$\nearrow$



Example. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$. Although $f'(0) = 0$, but $f'(x) \geq 0$. So, f' does not change its sigh at 0. So it is neither a local max nor min.

f'' Affects f

Definition. (1) If the graph of f lies above all its tangents on an interval I, then it is called concave upward on I.

(2) If the graph of f lies below all its tangents on an interval I, then it is called concave downward on I.

Example. $f(x) = x^2$ is concave upward.

Example. $g(x) = \sqrt{x}$ and $h(x) = \ln x$ are concave downward.

Remark. $f''(x) = 2 > 0$. $h''(x) = -\frac{1}{x^2} < 0$

Theorem. *[Concavity Test]*
(1) If $f''(x) > 0$ for all x in I, then the graph of f on I is concave upward on I.
(2) If $f''(x) < 0$ for all x in I, then the graph of f on I is concave downward on I.

Recall: If f' changes its sign at c, then c is a local max or min. If f'' changes its sign at point c, then the concavity changes.

Definition. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous at the point and the graph changes its concavity at P.

Example. $f(x) = x^3 \Rightarrow f''(x) = 6x \begin{cases} > 0 & x > 0 \\ = 0 & x = 0 \\ < 0 & x < 0 \end{cases}$. Then, by the concavity test, point $(0,0)$ is an inflection point.

Example. $f(x) = x^4 \Rightarrow f''(x) = 12x^2 \geq 0$. Although $f''(0) = 0$, but the concavity at $(0,0)$ does not change, so it is NOT an inflection point.

Example. $f(x) = x^4 - 4x^3$. Then $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ and $f''(x) = 12x^2 - 24x = 12x(x-2)$.

Intervals	$(-\infty, 0)$	$x = 0$	$(0,2)$	$x = 2$	$(2, \infty)$
f''	$+$	0	-	0	$+$
f	Concave Upward	Inflection Point	Concave Downward	Inflection Point	Concave Upward

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.
$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-1)(x-3)$. So $f'(x) = 0 \Rightarrow x = 0, 1, 3$. $f''(x) = 36x^2 - 96x + 36$.

Intervals	$[-1, 0)$	$x = 0$	$(0,1)$	$x = 1$	$(1,3)$	$x = 3$	$(3,4]$
f'	-	0	$+$	0	-	0	$+$
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$	Local Min	$\nearrow$
f''		36>0		-24<0		264>0	

Theorem. *[The Second Derivative Test] Suppose f'' is continuous near c.*
(1)If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
(2)If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

4.4 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Recall:

$$\lim_{x\to 0}\frac{x^2-x}{x}=\frac{0}{0}\text{[Factorization]}$$

and

$$\lim_{x\to\infty}\frac{x^2-3x}{2x^2+2}=\frac{\infty}{\infty}\text{[Divistion]}$$

[Question]

$$\lim_{x\to\infty}\frac{\ln x}{x-1}=?$$

Theorem. *If*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\ \textit{or}\ \frac{\infty}{\infty},$$

i.e., an interminate form, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that the derivatives and limit on the right exist.

Example. (1)Consider

$$\lim_{x\to\infty}\frac{\ln x}{x-1}.$$

Since $\ln\infty=\infty$ and $\infty-1=\infty$, we have an indeterminate form, thus

$$\lim_{x\to\infty}\frac{\ln x}{x-1}\underset{\frac{\infty}{\infty}}{\overset{L'}{=}}\lim_{x\to\infty}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

(2)

$$\lim_{x\to 1}\frac{\ln x}{x-1}\underset{\frac{0}{0}}{\overset{L'}{=}}\lim_{x\to 1}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{1}=1.$$

(3)

$$\lim_{x\to 0^+}\frac{\ln x}{x-1}=\frac{\ln 0}{0-1}=\frac{-\infty}{-1}=\infty.$$

Indeterminate Product: If we have $0\cdot\infty$ form, it is also an indeterminate form and L'Hospital's Rule applies.

Example.

$$\lim_{x\to 0^+}x\ln x\underset{(-\infty)\cdot 0}{=}=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\underset{\frac{\infty}{\infty}}{\overset{L'}{=}}\lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'}=\lim_{x\to 0^+}\frac{1/x}{-x^{-2}}=\lim_{x\to 0^+}(-x)=0.$$

Remark.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{x}{(\ln x)^{-1}}=\underset{\frac{0}{0}}{\overset{L'}{=}}\lim_{x\to 0^+}\frac{1}{-1(\ln x)^{-2}\cdot\frac{1}{x}}=\lim_{x\to 0^+}-x(\ln x)^2=\cdots$$

Indeterminate Differences: $\infty-\infty$

Example.

$$\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)\underset{\infty-\infty}{=}=\lim_{x\to\frac{\pi}{2}^-}\left(\frac{1-\sin x}{\cos x}\right)\overset{L'}{\underset{\frac{0}{0}}{=}}\lim_{x\to\frac{\pi}{2}^-}\frac{(1-\sin x)'}{(\cos x)'}=\lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x}=0.$$

<u>Indeterminate Power</u>: 0^0, ∞^0, 1^∞.
Method: Logarithmic

Example.

$$\lim_{x\to 0^+}(1+\sin(4x))^{\cot x}.$$

Since $\lim_{x\to 0^+}(1+\sin(4x))=1$ and $\lim_{x\to 0^+}\cot x=\infty$. Let $y=(1+\sin(4x))^{\cot x}$, then

$$\ln y=\cot x\cdot\ln(1+\sin(4x))=\frac{\ln(1+\sin(4x))}{\tan x}.$$

Thus,

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(1+\sin(4x))}{\tan x}\overset{L'}{\underset{\frac{0}{0}}{=}}\frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2 x}=4.$$

Therefore

$$\lim_{x\to 0^+}y=e^{\lim\limits_{x\to 0^+}\ln y}=e^4.$$

Example. $\lim\limits_{x\to 0^+}x^x$. Let $y=x^x\Rightarrow\ln y=x\ln x$. Thus,

$$\lim_{x\to 0}\ln y=\lim_{x\to 0^+}x\ln x=0\Rightarrow\lim_{x\to 0^+}y=e^0=1.$$

4.5 SUMMARY OF CURVE SKETCHING

<u>**Guidelines**</u>: Given a function $y = f(x)$

A. Domain

B. Intercepts:

$$\begin{cases} \bullet x\text{-intecept}\,(x,0) & f(x) = 0 \\ \bullet y-\text{intecept}\,(0,y) & f(0) = y \end{cases}$$

C. Symmetry:

$$\begin{cases} \bullet\text{Even} : f(-x) = f(x) & \text{Example: } y = x^2 \\ \bullet\text{Odd} : f(-x) = -f(x) & \text{Example: } y = 3x^3 - x \\ \bullet\text{Periodic} : f(x+p) = f(x) & \text{Example: } y = \sin x, p = 2\pi \end{cases}$$

D. Asymptotes:

$$\begin{cases} \bullet\text{Vertical: } \lim\limits_{x\to a^{\pm}} f(x) = \pm\infty & \text{Example: } y = \frac{1}{x} \Rightarrow \begin{cases} \lim\limits_{x\to 0^+} \frac{1}{x} = \infty \\ \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty \end{cases} \\ \bullet\text{Horizontal: } \lim\limits_{x\to\pm\infty} f(x) = L & \text{Example: } y = \frac{1}{x} \Rightarrow \lim\limits_{x\to\infty} \frac{1}{x} = 0 = \lim\limits_{x\to-\infty} \frac{1}{x} \\ \bullet\text{Slant} : \lim\limits_{x\to\pm\infty} [f(x) - (mx+b)] = 0 & \text{KEY: Only happens when } \frac{p(x)}{q(x)}\ deg(p) - deg(q) = 1. \end{cases}$$

The slant asymptote is $y = mx + b$.

E. Interval of Increase or Decrease: [Increase/Decrease Test: Increase $f' > 0$; Decrease $f' < 0$]

F. Local Maximum and Minimum Value[First or Second Derivative Test]

G. Concavity and Points of Inflection: [Concave Upward: $f'' > 0$; Concave Downward $f'' < 0$]

H. DRAW!!!

Example. $f(x) = \frac{x^2}{\sqrt{x+1}}$

A. Domain: $dom(f) = \{x | x+1 > 0\} = (-1, \infty)$.

B. Intercepts:

$$\begin{cases} f(0) = 0 \Rightarrow y\text{-intercept is } 0 \text{ or } (0,0) \\ f(x) = 0 \Rightarrow x = 0 \Rightarrow x\text{-intercept is } 0 \text{ or } (0,0) \end{cases}$$

C. Symmetry:

$$\begin{cases} f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq f(x) & \text{NOT Even} \\ f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq -f(x) & \text{NOT Odd} \\ \text{No Trig Part} & \text{NOT Periodic} \end{cases}$$

D. Asymptotes:

$$\begin{cases} \lim\limits_{x\to -1^+} f(x) = \frac{1}{0^+} = \infty \\ \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{x^2\cdot\frac{1}{\sqrt{x}}}{\sqrt{x+1}\cdot\frac{1}{\sqrt{x}}} = \lim\limits_{x\to\infty} \frac{x^{3/2}}{\sqrt{1+\frac{1}{\sqrt{x}}}} = \infty \\ deg\,(numerator) = 2, deg(denominator) = 1/2 \quad \text{No Slant Asymptote} \end{cases}$$

E. Intervals of Increase or Decrease.

$$f'(x) = \frac{\left(x^2\right)'\sqrt{x+1} - x^2\left(\left(x+1\right)^{\frac{1}{2}}\right)'}{x+1} = \frac{x\left(3x+4\right)}{2\left(x+1\right)^{\frac{3}{2}}}.$$

So,

$$f'(x) = 0 \Rightarrow x = 0, -\frac{4}{3} < -1, \text{(Not in the domain)}$$

Intervals	$(-1,0)$	$x=0$	$(0,\infty)$
f'	$-$	0	$+$
f	$\searrow$	Min	$\nearrow$

F. Local Max and Min: $(0,0)$ is the only local min and there is no local max.

G. Concavity and Inflection Points

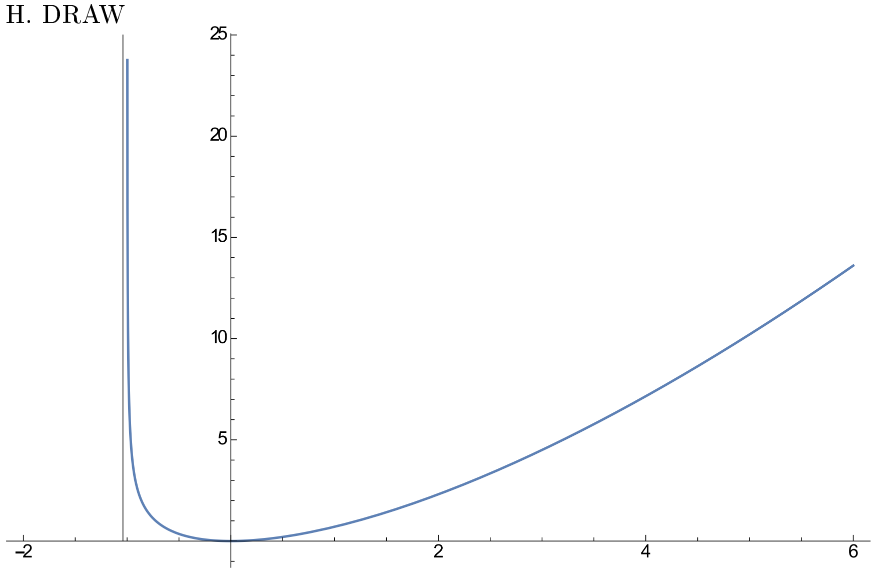
$$f''(x) = \frac{3x^2+8x+8}{4\left(x+1\right)^{\frac{5}{2}}}.$$

The denominator is positive because of the square root while the numerator is always positive as well, since

$$3x^2+8x+8 = 3x^2+8x+\frac{16}{3}+\frac{8}{3} = 3\left(x+\frac{4}{3}\right)^2+\frac{8}{3}.$$

Thus, f is always concave upward, (same as x^2).

H. DRAW



Example. $f(x) = \frac{x^3}{x^2+1}$

A. $dom\,(f) = (-\infty,\infty) = \mathbb{R}$

B. $f(0)=0$ and $f(x)=0\Rightarrow x=0$. Thus the both intercepts are 0.
C. $f(-x)=\frac{(-x)^3}{(-x)^2+1}=-\frac{x^3}{x^2+1}=-f(x)$, so it is odd.
D. Since domain is $\mathbb{R}$, so there is no vertical asymptotes. Also,

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x^3\cdot\frac{1}{x^2}}{(x^2+1)\cdot\frac{1}{x^2}}=\infty \text{ and similarly, } \lim_{x\to-\infty} f(x)=-\infty.$$

However, since the degree of numerator is 1 higher than the degree of denominator, it must have slant asymptote. Note that

$$f(x)=\frac{x^3}{x^2+1}=\frac{x^3+x-x}{x^2+1}=x-\frac{x}{x^2+1}.$$

Thus,

$$\lim_{x\to\pm\infty}[f(x)-x]=\lim_{x\to\pm\infty}\left[-\frac{x}{x^2+1}\right]=\lim_{x\to\pm\infty}\left[-\frac{\frac{1}{x}}{1+\frac{1}{x^2}}\right]=0.$$

So the slant asymptote is $y=x$.
E.

$$f'(x)=\frac{x^2\left(x^2+3\right)}{\left(x^2+1\right)^2}\geq 0.$$

Thus, f is always increasing
F. Since f is always increasing, so there is no local min or max.
G.

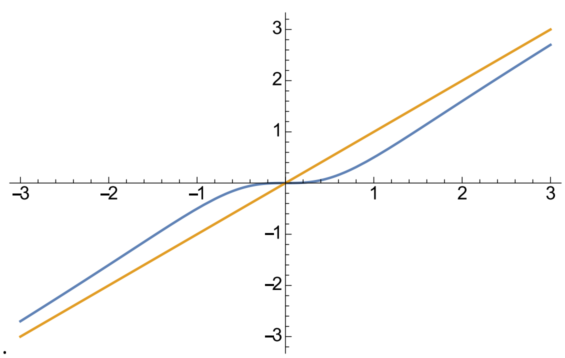
$$f''(x)=\frac{2x\left(3-x^2\right)}{\left(x^2+1\right)^3}\Rightarrow f''(x)=0 \text{ when } x=0,\pm\sqrt{3}$$

So

Intervals	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},\infty)$
f''	$+$	$-$	$+$	$-$
f	Concave Upward	Concave Downward	Concave Upward	Concave Downward

, and inflection points are

$$\left(-\sqrt{3},-\frac{3}{4}\sqrt{3}\right),(0,0),\left(\sqrt{3},\frac{3}{4}\sqrt{3}\right).$$

H.


Example. $f(x)=\frac{\cos x}{2+\sin x}$.
A. $dom(f)=\{x|2+\sin x\neq 0\}=(-\infty,\infty)$.
B. $f(0)=\frac{\cos 0}{2+\sin 0}=\frac{1}{2}$, which is the y-intercept and solve $f(x)=0\Leftrightarrow\cos x=0$ to see that $x=\frac{2n+1}{2}\pi$ for integer n.

C. $f(-x) = \frac{\cos x}{2-\sin x}$, which is neither $f(x)$ nor $-f(x)$. However,

$$f(x+2\pi) = f(x),$$

so we only need to consider the interval $[0, 2\pi]$, in which the x-intercepts are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

D. The domain shows no possible vertical asymptotes and $\lim_{x\to\pm\infty} f(x)$ DNE. So, there is no horizontal asymptote either.

E.

$$f'(x) = -\frac{2\sin x + 1}{(2+\sin x)^2},$$

so $f'(x) = 0$ implies $2\sin x + 1 = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$. (Note that we only consider $[0, 2\pi]$). Thus

Intervals	$\left(0, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$-$	0	$+$	0	$-$
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$

F. As shown above, local min is $\left(\frac{7\pi}{6}, -\frac{1}{\sqrt{3}}\right)$ and Local Max is $\left(\frac{11\pi}{6}, \frac{1}{\sqrt{3}}\right)$.

G.

$$f''(x) = -\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}.$$

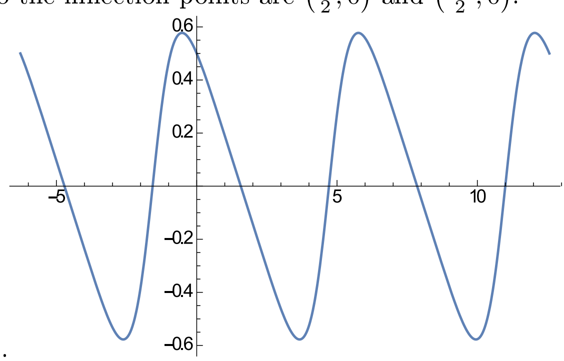
Now, $f''(x) = 0 \Leftrightarrow 2\cos x(1-\sin x) = 0$.

$$\begin{cases} \cos x = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \\ 1 - \sin x = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \end{cases} \Rightarrow f''(x) = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}.$$

Intervals	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, 2\pi\right)$
f''	$-$	0	$+$	0	$-$
f	Concave Downward	Inflection	Concave Downward	Inflection	Concave Downward

So the inflection points are $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$.

H.



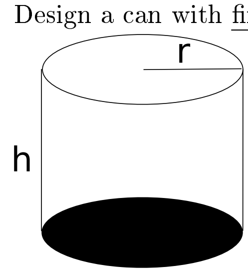
4.7 OPTIMIZATION PROBLEMS

STEPS:.

1. Understand the Problem
2. Draw a Diagram (If Necessary)
3. Introduce Notation.
4. Establish the Function
5. Eliminate Extra Variables (If Needed)
6. Solve for Absolute/Global Max or Min

Example. A cylinder can is to made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

1. Design a can with <u>fixed volume</u> and <u>minimal area of surface</u>



2.
3. V: Volume (cm^3); A: Area (cm^2); r: Radius (cm); h: Height (cm)
4.

$$\begin{cases} V = \pi r^2 h & = 1000 \\ A = 2\pi r h + \pi r^2 \cdot 2 & \text{[Top, Bottom, Side]} \end{cases}$$

5. Since we have

$$h = \frac{1000}{\pi r^2},$$

$$A(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r}.$$

6.

$$A'(r) = 4\pi r - \frac{2000}{r^2} \Rightarrow A'(r) = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}.$$

Note that

$$A''(r) = 4\pi + \frac{4000}{r^3} \Rightarrow A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0.$$

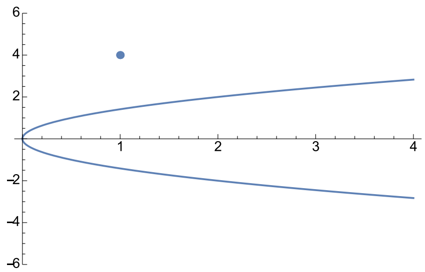
Second derivative test shows that it is the local min. Also

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Thus the dimension is that

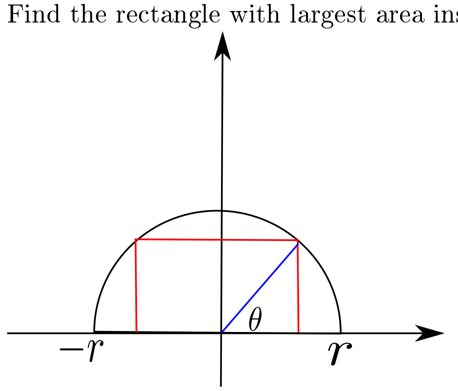
$$radius = \sqrt[3]{\frac{500}{\pi}}\ cm \text{ and } height = 2\sqrt[3]{\frac{500}{\pi}}\ cm$$

Example. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

1. Find the closed distance and recall that for $P(x, y)$ and $Q(s, t)$
$$|PQ| = \sqrt{(x-s)^2 + (y-t)^2}.$$
2. 
3. Point on the parabola is (x, y) and the distance is d and $D = d^2$
4. We know that
$$D = d^2 = (x-1)^2 + (y-4)^2$$
5. Since $y^2 = 2x \Leftrightarrow x = \frac{y^2}{2}$, we have
$$D(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{y^4}{4} - 8y + 17.$$
6.
$$D'(y) = y^3 - 8 \Rightarrow D'(y) = 0 \Rightarrow y = 2.$$

Also,
$$D''(y) = 3y^2 \Rightarrow D''(2) = 12 > 0.$$
Thus, $y = 2$ and $x = \frac{y^2}{2} = 2$ is the minimum. Thus the point closest to $(1, 4)$ is $(2, 2)$.

Example. Find the ara of the largest rectangle that can be inscribed in a semicircle of radius r.

1. Find the rectangle with largest area inscribed in a semicircle.
2. 

3. The angle is θ and the area is A:
4. Note that $0 \le \theta \le \pi/2$
$$A = (r \cdot \sin\theta) \cdot (2r\cos\theta) = r^2 (2\sin\theta\cos\theta) = r^2 \sin 2\theta.$$
5. No need.

6. There is no need for differentiation since $0 \leq 2\theta \leq \pi$ and $\sin x$ has a max when $x = \frac{\pi}{2}$, i.e.,

$$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}.$$

So the dimension is that

$$\begin{cases} \text{one side} = r \sin \frac{\pi}{4} = \frac{r}{\sqrt{2}} \\ \text{the other side} = 2r \cos \frac{\pi}{4} = \sqrt{2} r \end{cases}$$

4.8 NEWTON'S METHOD

Question: How to find a solution of $f(x) = 0$ for complicated f. At least, an approximate root is acceptable.

Steps:

(1)Choose x_1

(2)Follow the recurrence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and so on

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(3)The limit

$$r = \lim_{n\to\infty} x_n$$

is the root, i.e.

$$f(r) = 0.$$

Example. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation

$$x^3 - 2x - 5 = 0.$$

Solution. $f'(x) = 3x^2 - 2$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

(1)$x_1 = 2$

(2)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = \frac{21}{10}$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.0946.$$

Example. Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places

Solution. We first need to intepretate this question into FINDING A ROOT OF A FUNCTION. Consider

$$f(x) = x^6 - 2,$$

then

$$r = \sqrt[6]{2} \Rightarrow f(r) = 0.$$

Now choose $x_1 = 1$ and follow

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_{n^5}} = \frac{5}{6}x_n + \frac{1}{3x_n^5}$$

to get

$$\begin{cases} x_2 \approx & 1.6666667 \\ x_3 \approx & 1.12644368 \\ x_4 \approx & 1.12249707 \\ x_5 \approx & 1.12246205 \\ x_6 \approx & 1.12246205 \end{cases}$$

4.9 ANTIDERIVATIVES & (PART OF) 5.3 INDIFINITE INTEGRALS

Definition. A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$

for all x in I.

Example. $f(x) = \cos x$ then

$$\begin{cases} F(x) = \sin x, & (\sin x)' = \cos x \\ F_1(x) = \sin x + 1, & F_1'(x) = f(x) \\ F_2(x) = \sin x + 2, & F_2'(x) = f(x) \\ F_c(x) = \sin x + C & \text{for any constant } C. \end{cases}$$

Theorem. *[DEF.] If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C,$$

where C is an arbitary constant.

Definition. [Indefinite Integral] The operation for finding the most general antiderivative:

$$\int f(x)\,dx = F(x) + C \text{ or equivalently } F(x) = \int f(x)\,dx + C,$$

where F is any particular antiderivative of f.

Remark. The "dx" part in the notation is very important.

Example. $f(x) = 1/x$. $dom(f) = (-\infty, 0) \cup (0, \infty)$

$$F(x) = \int f(x)\,dx = \ln|x| + C.$$

Example. $f(x) = x^n$ for $n \neq -1$.

$$F(x) = \int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Suppose $F'(x) = f(x)$ and $G'(x) = g(x)$

Function	Particular Antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$1/x$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/\left(1+x^2\right)$	$\tan^{-1} x$
$\sinh x$	$\cosh x$

Thus, if we rewrite the table, we will have

Table of Indefinite Integrals
$\int cf(x)\,dx = c\int f(x)\,dx$
$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$
$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$
$\int \csc x \cot x dx = -\csc x + C$
$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\int \sinh x dx = \cosh x + C$

Example. Find the indefinite integral

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 10\int x^4 dx - 2\int \sec^2 x dx.$$

By the table,

$$\begin{cases} \int x^4 dx = \frac{x^{4+1}}{4+1} + C_1 = \frac{x^5}{5} + C_1 & , \\ \int \sec^2 x dx = \tan x + C_2 & . \end{cases}$$

Thus,

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 2x^5 - 2\tan x + (10C_1 - 2C_2) = 2x^5 - 2\tan x + C,$$

where

$$C = 10C_1 - 2C_2$$

also stands for arbitary constant term.

Remark. From now on, we could first write down the particular antiderivative for each term first and finally add the constant C in the end when computing indefinite integrals. Namely,

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 10 \int x^4 dx - 2 \int \sec^2 x dx = 10 \cdot \frac{x^5}{5} - 2\tan x + C.$$

Example. It also does not matter if we change the variable from x to some others.

$$\int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} d\theta = \int \csc\theta \cdot \cot\theta d\theta = -\csc\theta + C.$$

Example. Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Solution. Symplification shows

$$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}.$$

Thus,

$$g(x) = \int g(x) dx = 4 \int \sin x dx + 2 \int x^4 dx - \int x^{-\frac{1}{2}} dx = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C.$$

Example. Find f if $f'(x) = e^x + 20\left(1 + x^2\right)^{-1}$.

Solution. Since

$$f'(x) = e^x + 20\frac{1}{1 + x^2},$$

f has the form

$$f(x) = \int f'(x) dx = e^x + 20\tan^{-1} x + C$$

for some constant C. Also,

$$-2 = f(0) = e^0 + 20\tan^{-1} 0 + C = 1 + C \Rightarrow C = -3.$$

Therefore,

$$f(x) = e^x + 20\tan^{-1} x - 3.$$

Example. A partical moves in a straight line has acceleration given by

$$a(t) = 6t + 4.$$

Its initial velocity is $v(0) = -6$ cm/s and initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution. Recall the important physicial fact that

$$s'(t) = v(t) \text{ and } v'(t) = a(t).$$

Thus,

$$v(t) = \int a(t)dt = 3t^2 + 4t + C_1$$

for some constant C_1. Note that

$$-6 = v(0) = C_1.$$

Therefore,

$$v(t) = 3t^2 + 4t - 6$$

and then

$$s(t) = \int v(t)dt = t^3 + 2t^2 - 6t + C_2,$$

where the constant C_2 is given by

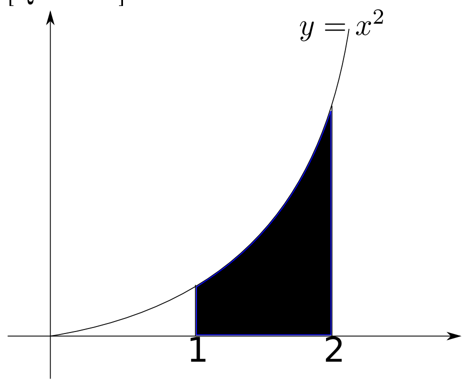
$$9 = s(0) = C_2.$$

Finally,

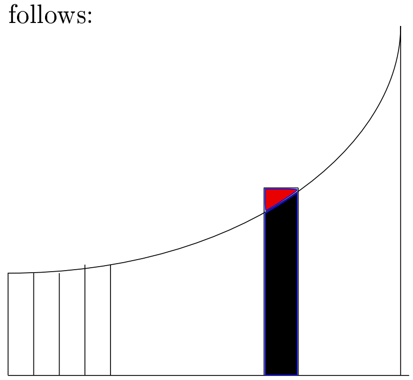
$$s(t) = t^3 + 2t^2 - 6t + 9.$$

5.1 & 5.2 DEFINITE INTEGRALS AND AREA

[Question] What is the area of the shadow region?



[Answer]:This is a very important approach. We try to cut the region vertically as follows:



Now we concentrate on each SLIGCE, which is a rectangle.

$$\begin{cases} \text{width} & = \Delta x = dx \\ \text{height} & = f(x), \text{ for some } x \text{ in this interval} \end{cases}.$$

Thus,

$$\text{Area} \approx \sum \text{rectangles} = \sum f(x)\,dx.$$

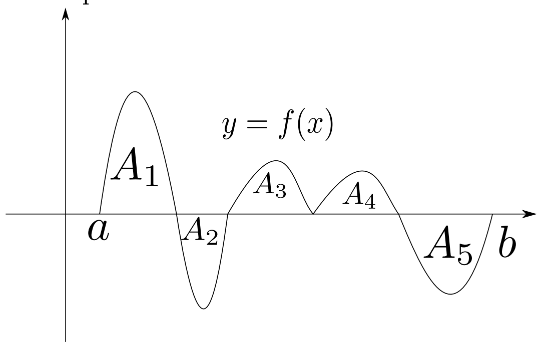
Now taking the limit that $\Delta x = dx \to 0$ then the error goes away and by the following notation

$$\text{Area} = \lim_{dx \to 0} \sum f(x)\,dx = \int_1^2 f(x)\,dx.$$

Definition. [Definite Integral] The definite integral

$$\int_a^b f(x)\,dx$$

is the SIGNED area formed by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis. The signs are assigned in the way that if part of the region is above the x-axis, then it is assigned a positive sign; while if it is under the x-axis, it has negative sign. For example.

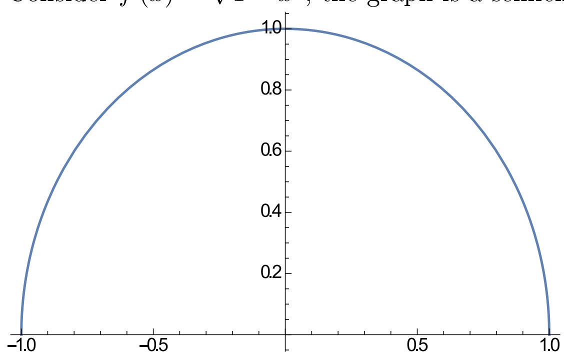


Then,

$$\int_a^b f(x)\, dx = A_1 - A_2 + A_3 + A_4 - A_5.$$

Example. $\int_{-1}^{1} \sqrt{1-x^2}dx$.

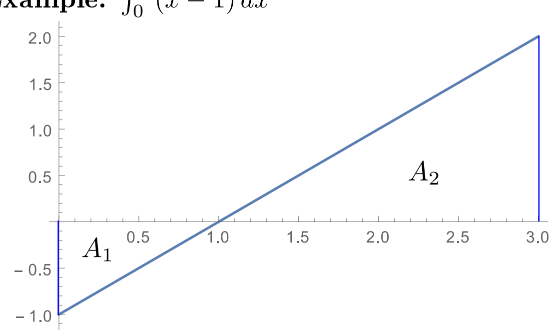
Consider $f(x) = \sqrt{1-x^2}$, the graph is a semicircle



Thus,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \text{Area} = \frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}.$$

Example. $\int_0^3 (x-1)\, dx$



$$\int_0^3 (x-1)\,dx = -A_1 + A_2 = -\left(\frac{1}{2}\cdot 1\cdot 1\right) + \left(\frac{1}{2}\cdot 2\cdot 2\right) = 1.5.$$

Properties:

$$\begin{cases}
1. & \int_a^b c dx = c\,(b-a) \ \ \textbf{for any constant } c \\
2. & \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \\
3. & \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \ \textbf{for any constant } c \\
4. & \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \\
5. & \textbf{If } f(x) \geq 0 \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\,dx \geq 0 \\
6. & \textbf{If } f(x) \geq g(x) \ \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \\
7. & \textbf{If } m \leq f(x) \leq M \ \textbf{ for } a \leq x \leq b\textbf{, then } m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a)
\end{cases}$$

.

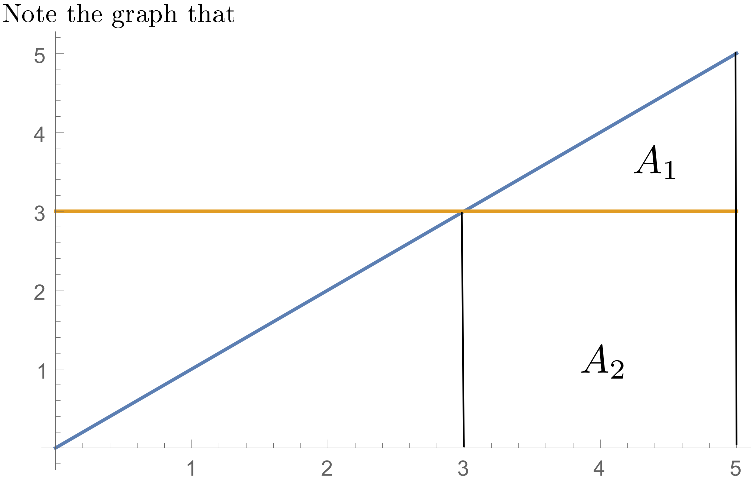
Example. Consider

$$f(x) = \begin{cases} 3 & x < 3 \\ x & x \geq 3 \end{cases},$$

then

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_0^3 3dx + \int_3^5 xdx = 9 + \int_3^5 xdx.$$

Note the graph that



We see

$$\int_3^5 xdx = A_1 + A_2 = \frac{1}{2}\cdot 2\cdot 2 + \cdot 2\cdot 3 = 8.$$

Therefore

$$\int_0^5 f(x)\,dx = 9 + 8 = 17.$$

5.3 FUNDAMENTAL THEOREM OF CALCULUS

Theorem. *Suppose f is continuous on $[a,b]$.*

(1) The function defined by

$$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b) and

$$g'(x) = f(x).$$

(2) Let F be any antiderivative of f, i.e. $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}.$$

Example. Define $g(x) = \int_0^x \sqrt{1+t^2}dt$, then find $g'(x)$.

Solution. Let $f(t) = \sqrt{1+t^2}$, then $g(x) = \int_0^x f(t)\,dt$. By the fundamental theorem of calculus, part 1,

$$g'(x) = f(x) = \sqrt{1+x^2}.$$

Example. $S(x) = \int_0^x \sin\left(\pi t^2/2\right)dt$, then if letting $f(t) = \sin\left(\pi t^2/2\right)$

$$S(x) = \int_0^x f(t)\,dt \Rightarrow S'(x) = f(x) = \sin\left(\frac{\pi x^2}{2}\right).$$

Example. Find

$$\frac{d}{dx}\int_1^{x^4} \sec t dt.$$

Solution. Let

$$g(x) = \int_1^{x^4} \sec t dt \text{ and } f(t) = \sec t \Rightarrow g(x) = \int_1^{x^4} f(t)\,dt.$$

Now since the upper bound of this integral is x^4 instead of x, we cannot apply the Fundamental Theorem of Calculus directly. Let

$$u = h(x) = x^4 \Rightarrow g(u) = \int_1^u f(t)\,dt.$$

By Fundamental Theorem of Calculus,

$$\frac{dg}{du} = f(u).$$

By Chain Rule,

$$\frac{dg}{dx} = \frac{dg}{du}\cdot\frac{du}{dx} = f(u)\cdot h'(x) = \sec u \cdot \left(x^4\right)' = 4x^3 \sec\left(x^4\right).$$

By the "Box" notation, we could write it as

$$\frac{d}{dx}\int_1^{x^4}\sec t dt = \Bigg(\underbrace{\int_1^{\square}\sec t dt}_{\square=x^4}\Bigg)' = \sec(\square)\cdot\square' = \sec\left(x^4\right)\cdot\left(x^4\right)' = 4x^3\sec\left(x^4\right).$$

Example.

$$\int_1^3 e^x dx = ?$$

Solution. By the formula

$$\int e^x dx = e^x + C,$$

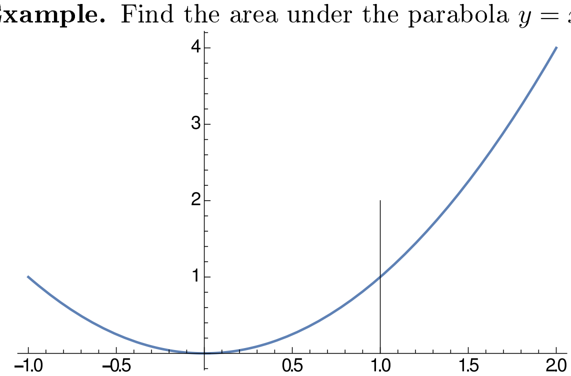
we have

$$\int_1^3 e^x dx = e^x\Big|_{x=1}^{x=3} = e^3 - e^1$$

Example.

$$\int_3^6 \frac{dx}{x} = \ln|x|\Big|_{x=3}^{x=6} = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2.$$

Example. Find the area under the parabola $y = x^2$ from 0 to 1.



$$A = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Example. Find the area under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.

$$A = \int_0^b \cos x dx = \sin x\Big|_{x=0}^{x=b} = \sin b.$$

Example.

$$\int_{-1}^3 |x|\, dx = \int_{-1}^0 (-x)\, dx + \int_0^3 x dx = -\frac{x^2}{2}\Big|_{x=-1}^{x=0} + \frac{x^2}{2}\Big|_{x=0}^{x=3} = \frac{1}{2} + \frac{9}{2} = 5.$$

Example.

$$\begin{aligned}\int_0^3 \left(x^3 - 6x\right) dx &= \int_0^3 x^3 dx - 6\int_0^3 x dx \\ &= \left.\frac{x^4}{4}\right|_{x=0}^{x=3} - 6 \cdot \left.\frac{x^2}{2}\right|_{x=0}^{x=3} \\ &= \frac{3^4}{4} - 3 \cdot 9 \\ &= -\frac{27}{4}.\end{aligned}$$

Example.

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\int_0^2 x^3 dx - 6\int_0^2 x dx + 3\int_0^2 \frac{1}{1+x^2} dx \\ &= 2 \cdot \left.\frac{x^4}{4}\right|_{x=0}^{x=2} - 6 \cdot \left.\frac{x^2}{2}\right|_{x=0}^{x=2} + 3\tan^{-1} x\Big|_{x=0}^{x=2} \\ &= \frac{1}{2} \cdot 2^4 - 3 \cdot 2^2 + 3\tan^{-1} 2 \\ &= -4 + 3\tan^{-1} 2.\end{aligned}$$

5.5 THE SUBSTITUTION RULE

Example. Calculate

$$\int 2x\sqrt{x^2+1}dx$$

This integral cannot be found in the table. Now consider another problem: if

$$F(x) = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}},$$

Then

$$F'(x) = \frac{2}{3}\Bigg[\underbrace{\Box^{\frac{3}{2}}}_{\Box = x^2+1}\Bigg]' = \frac{2}{3}\cdot\frac{3}{2}\Box^{\frac{1}{2}}\cdot\Box' = \sqrt{x^2+1}\cdot 2x.$$

It suggests that

$$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C.$$

In theory, integral is the inverse operation of derivative and substitution rule is the counterpart of chain rule.

Recall: Differential. If $y = f(x)$, then the differentials, dy and dx satisfy that

$$dy = f'(x)\,dx.$$

STEPS:

1. Choose the right substitution $u = g(x)$
2. Calculate the differentials: $du = g'(x)\,dx$
3. Replace $g'(x)\,dx = du$ and rewrite the function(integrand) as a function of u
4. Get the result in terms of u.
5. Substitute u by $g(x)$

Example. Compute

$$\int 2x\sqrt{x^2+1}dx$$

1. $u = g(x) = x^2+1$
2. $du = g'(x)\,dx = 2xdx$
3. $$\int 2x\sqrt{x^2+1}dx = \int \left(x^2+1\right)^{\frac{1}{2}}\underbrace{2xdx} = \int u^{\frac{1}{2}}du$$
4. $$\int 2x\sqrt{x^2+1}dx = \int u^{\frac{1}{2}}du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{\frac{3}{2}} + C.$$
5. $$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C$$

Example. Compute
$$\int x^3 \cos\left(x^4+2\right) dx.$$

1. $u = g(x) = x^4 + 2$
2. $du = g'(x) = 4x^3 dx.$
3. [Important Trick]

$$\int x^3 \cos\left(x^4+2\right) dx = \int \cos\left(x^4+2\right) x^3 dx = \int \cos\left(x^4+2\right) \cdot \frac{4}{4} \cdot x^3 dx = \frac{1}{4} \int \cos u \cdot du.$$

4.
$$\int x^3 \cos\left(x^4+2\right) dx = \frac{1}{4} \int \cos u \cdot du = \frac{1}{4} \sin u + C$$

5.
$$x^3 \cos\left(x^4+2\right) dx = \frac{1}{4} \sin u + C = \frac{1}{4} \sin\left(x^4+2\right) + C$$

Example. Compute
$$\int \sqrt{2x+1} dx$$

Let $u = 2x+1$, then $du = 2dx$. Now
$$\int \sqrt{2x+1} dx = \int \sqrt{2x+1} \cdot \frac{2}{2} dx = \frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{2} \cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C.$$

Example. Find
$$\int \frac{x}{\sqrt{1-4x^2}} dx$$

Let $u = 1-4x^2$, then $du = -8xdx$. Thus,
$$\int \frac{x}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}} \cdot \frac{-8}{-8} \cdot dx = -\frac{1}{8} \int u^{-\frac{1}{2}} du = -\frac{1}{8} \cdot 2u^{\frac{1}{2}} + C = -\frac{1}{4} \sqrt{1-4x^2} + C.$$

Example. Evaluate
$$\int e^{5x} dx$$

Let $u = 5x \Rightarrow du = 5dx$, then
$$\int e^{5x} dx = \int e^{5x} \cdot \frac{5}{5} \cdot dx = \frac{1}{5} \int e^u du = \frac{1}{5} e^u + C = \frac{1}{5} e^{5x} + C.$$

Example. Calculate
$$\int \tan x dx$$

Since $\tan x = \frac{\sin x}{\cos x}$ and recall that $(\cos x)' = \sin x$, we try that $u = \cos x$ and $du = -\sin x dx$. Then
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} (-\sin x) dx = -\int u^{-1} du = -\ln|u| + C = -\ln|\cos x| + C.$$

In addition,
$$-\ln|\cos x| = \ln|\cos|^{-1} = \ln \frac{1}{|\cos x|} = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|.$$

Therefore, we have a new formula that
$$\int \tan x dx = \ln|\sec x| + C.$$

Definite Integrals' Substitution Rule

STEPS:
1. Choose the right substitution $u = g(x)$
2. Calculate the differentials: $du = g'(x)\,dx$; and the end points $g(a)$ and $g(b)$
3. Replace $g'(x)\,dx = du$ and rewrite the function(integrand) as a function of u and also the endpoints, i.e.,

$$\int_a^b [\text{function of } x]\,dx = \int_{g(a)}^{g(b)} [\text{function of } u]\,du$$

4. Evalute the definite integral.

Example. Evaluate $\int_0^4 \sqrt{2x+1}dx$

Let $u = g(x) = 2x+1$, then

$$\begin{cases} du = g'(x)\,dx = 2xdx & , \\ g(0) = 1 & , \\ g(4) = 9 & . \end{cases}$$

Thus,

$$\int_0^4 \sqrt{2x+1}dx = \int_0^4 \sqrt{2x+1}\cdot\frac{2}{2}\cdot dx = \frac{1}{2}\int_1^9 u^{\frac{1}{2}}du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=9} = \frac{1}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{26}{3}.$$

Example. Find$\int_1^2 \frac{dx}{(3-5x)^2}$

Let $u = g(x) = 3-5x$, then $du = g'(x)\,dx = -5dx$ and $g(1) = -2$, $g(2) = -7$. Thus,

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(3-5x)^2}\cdot\frac{-5}{-5}\cdot dx = -\frac{1}{5}\int_{-2}^{-7} u^{-2}du = -\frac{1}{5}\cdot\frac{u^{-1}}{-1}\Big|_{-2}^{-7} = \frac{1}{5}\left(\frac{1}{-7} - \frac{1}{-2}\right) = \frac{1}{14}.$$

Example. Calculate $\int_1^e \frac{\ln x}{x}dx$

Let $u = g(x) = \ln x \Rightarrow du = \frac{1}{x}dx$ and $g(1) = 0$, $g(e) = 1$, then

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 udu = \frac{u^2}{2}\Big|_{u=0}^{u=1} = \frac{1}{2}.$$

Last Theorem:

Theorem. *Let $A > 0$.*

(1) If $f(x)$ is an odd function, then

$$\int_{-A}^{A} f(x)\,dx = 0.$$

(2) If $f(x)$ is an even function, then

$$\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx.$$

Example.

$$\int_{-\pi}^{\pi} \frac{\sin t}{1+t^{2024}}dt = 0.$$