Matrix Representations for Multiplicative Nested Sums

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Abstract

We study the multiplicative nested sums, which are generalization of harmonic sums, and provide a computation through multiplication of index matrices. Special cases interpret the index matrices as stochastic transition matrices of random walks on finite number of sites. Relations among multiplicative nested sums, which are generalization of relations between harmonic series and multiple zeta functions, can be easily derived from identities of the index matrices. Combinatorial identities and their generalizations can also be derived from this computation.

Keywords: harmonic sum, multiple zeta function, random walk, combinatorial identity
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1 Introduction

We consider the following *multiplicative nested sums* (MNS): for $m, N \in \mathbb{N}$,

$$S\left(f_1,\ldots,f_k;N,m\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right), \tag{1}$$

and

$$A\left(f_1,\ldots,f_k;N,m\right) := \sum_{N>n_1>\cdots>n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right). \tag{2}$$

Namely, the usual summand $f\left(n_1,\ldots,n_k\right) = f_1\left(n_1\right)\cdots f_k\left(n_k\right)$ is multiplicative, and the sum indices are nested. Here, for all $l = 1,\ldots,k$, f_l can be any function defined on $\{m, m+1, \ldots, N\}$, unless $N = \infty$ when convergence needs to be taken into consideration.

Example 1.1. The first examples, by taking $f_l\left(n_l\right) = \operatorname{sign}\left(i_l\right)^{n_l}/n_l^{|i_l|}$ and $m = 1$, are the harmonic sums [3, eq. 4, pp. 1],

$$S_{i_1,\ldots,i_k}\left(N\right) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}\left(i_1\right)^{n_1}}{n_1^{|i_1|}} \times\cdots\times \frac{\operatorname{sign}\left(i_k\right)^{n_k}}{n_k^{|i_k|}}, \tag{3}$$

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and [7, pp. 168]

$$H_{i_1,\ldots,i_k}(N) = \sum_{N>n_1>\cdots>n_k\geq 1} \frac{\operatorname{sign}(i_1)^{n_1}}{n_1^{|i_1|}} \times \cdots \times \frac{\operatorname{sign}(i_k)^{n_k}}{n_k^{|i_k|}}. \tag{4}$$

Ablinger [1, Chpt. 6] implemented the `Mathematica` package `HarmonicSums.m`[1], based on recurrence [2, eq. 2.1, pp. 21] that is inherited from the quasi-shuffle relation [10, eq. 1, pp. 51], for computation.

In Section 2, we present the main theorem, namely an alternative computation, different from quasi-shuffle relation, for MNS, which naturally works for harmonic sums, by associating to each function f_l an *index matrix* and considering the multiplications. Properties such as inverse, identities, eigenvalues, eigenvectors, diagonalization of the index matrix follow.

Applications of this matrix expression vary in different fields, and appear in Section 3. Originally, this idea was inspired by constructing random walks for special harmonic sums. Different types of random walks from that on the plane with fixed length [4] to that on Riemannian matrix manifolds [8], appear in and connect to various fields. For example, the coefficients connecting Euler polynomials and generalized Euler polynomials [11, eq. 3.8, pp. 781] appear in a random walk over a finite number of sites [11, Note 4.8, pp. 787]. In Subsection 3.1, special sum when $f_1 = \cdots = f_k = x^{-a}$ for $a \geq 1$ is interpreted as probability of certain event of a random walk, while the index matrix is exactly the corresponding stochastic matrix.

Limit case of harmonic sums when $N \to \infty$ includes, by further assuming $i_1, \ldots, i_k > 0$, harmonic series [9, pp. 275]

$$\mathrm{S}(i_1, \ldots, i_k) := \sum_{n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}},$$

and multiple zeta values (by the notation of Hoffman [9])

$$\mathrm{A}(i_1, \ldots, i_k) := \sum_{n_1>\cdots>n_k\geq 1} \frac{1}{n_1^{i_1}\cdots n_k^{i_k}}.$$

For instance, the fact $\mathrm{S}(2,1) = 2\mathrm{A}(3)\,(= 2\zeta(3))$ has been well studied and rediscovered many times. Hoffman [9, Thm. 2.1, pp. 277, Thm. 2.2, pp. 278] further obtained the symmetric sums of $\mathrm{S}(i_1, \ldots, i_k)$ and $\mathrm{A}(i_1, \ldots, i_k)$ in terms of the Riemann-zeta function. In particular when $k = 2$ and $k = 3$, the relation between $\mathrm{S}(i_1, \ldots, i_k)$ and $\mathrm{A}(i_1, \ldots, i_k)$ [9, pp. 276] is also easy to see. In Subsection 3.2, we provide the truncated and generalized version of this relation, easily derived from identities of index matrices.

Finally, we focus on combinatorial identities, where harmonic sums also appear. For instance, Dilcher [6, Cor. 3, pp. 93] established, for special harmonic sum,

$$S_{\underbrace{1,\ldots,1}_{k}}(N) = \sum_{l=1}^{N} \binom{N}{l} \frac{(-1)^{l-1}}{l^k}, \tag{5}$$

from q-series of divisor functions. In the Subsection 3.3, we present examples that either by direct matrix expression, or through diagonalization of the index matrix, including a generalization of (5). These examples can be viewed as alternative proofs.

2 Matrix computation and properties

2.1 Index matrices and computations for MNS

Definition 2.1. Given a positive integer N and a function f on $\{1, \ldots, N\}$, we define the following $N \times N$ (lower triangular) *index matrices*:

$$\mathbf{S}_f := \begin{pmatrix} f(1) & 0 & 0 & \cdots & 0 \\ f(2) & f(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(N) & f(N) & f(N) & \cdots & f(N) \end{pmatrix}$$

[1] http://www.risc.jku.at/research/combinat/software/HarmonicSums/index.php

and

$$\mathbf{A}_f := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f(1) & 0 & 0 & \cdots & 0 & 0 \\ f(2) & f(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f(N-1) & f(N-1) & f(N-1) & \cdots & f(N-1) & 0 \end{pmatrix}.$$

Remark 2.1.

1. Shifting $\mathbf{S}_f$ downward by one row gives $\mathbf{A}_f$, i.e.,

$$\mathbf{A}_f = (\delta_{i-1,j})_{N\times N}\,\mathbf{S}_f, \text{ where } \delta_{a,b} = \begin{cases} 1, & \text{if } a = b; \\ 0, & \text{otherwiese.} \end{cases} \tag{6}$$

   For simplicity, we further denote $\mathbf{\Delta} := (\delta_{i-1,j})_{N\times N}$ so that $\mathbf{A}_f = \mathbf{\Delta}\mathbf{S}_f$.

2. When the dimension of index matrices needs to be clarified, we add N to the sub-indices as $\mathbf{S}_{N|f}$ and $\mathbf{A}_{N|f}$.

Theorem 2.1. *Let*

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{N\times N}.$$

Then, it holds that

$$S(f_1, \ldots, f_k; N, m) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right)_{N,m}, \tag{7}$$

and

$$A(f_1, \ldots, f_k; N, m) = \left(\mathbf{P}. \prod_{l=1}^{k} \mathbf{A}_{f_l}\right)_{N,m}, \tag{8}$$

where $\mathbf{M}_{i,j}$ denotes the entry located at the i^{th} row and j^{th} column of a matrix $\mathbf{M}$.

Proof. Since the proof for A is similar, we shall only prove a stronger result for S that

$$S(f_1, \ldots, f_k; i, j) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right)_{i,j}.$$

1. When $k = 1$, it is trivial to see that $(\mathbf{P} \cdot \mathbf{S}_{f_1})_{i,j} = \sum_{l=j}^{i} f_1(l) = S(f_1; i, j)$.

2. Suppose $S(f_1, \ldots, f_k; i, j) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right)_{i,j}$. Then,

$$\begin{aligned} S(f_1, \ldots, f_k; i, j) = \left(\mathbf{P} \cdot \prod_{l=1}^{k+1} \mathbf{S}_{f_l}\right)_{i,j} &= \left(\left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right) \cdot \mathbf{S}_{f_{k+1}}\right)_{i,j} \\ &= \sum_{l=j}^{i} S(f_1, \ldots, f_k; i, l)\, f_{k+1}(l) \\ &= \sum_{l=j}^{i} f_{k+1}(l) \sum_{i\geq n_1 \geq \cdots \geq n_k \geq l} f_1(n_1)\cdots f_k(n_k) \\ &= \sum_{i\geq n_1 \geq \cdots \geq n_k \geq l \geq j} f_1(n_1)\cdots f_k(n_k)\, f_{k+1}(l) \\ &= S(f_1, \ldots, f_{k+1}; i, j). \end{aligned}$$

□

2.2 Properties of the index matrix S

To simplify expressions in this section, we denote

$$\mathbf{S}_a := \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_N & a_N & a_N & \cdots & a_N \end{pmatrix}$$

and $\mathbf{A}_a := \mathbf{\Delta}\mathbf{S}_a$, i.e., rewriting evaluation $a(i)$ as sub-indices a_i. Next, we give some properties of $\mathbf{S}_a$.

Proposition 2.1. *The following statements are true.*

1. *The inverse of* $\mathbf{S}_a$ *is given by*

$$\mathbf{S}_a^{-1} = \begin{pmatrix} 1/a_1 & 0 & 0 & \cdots & 0 & 0 \\ -1/a_1 & 1/a_2 & 0 & \cdots & 0 & 0 \\ 0 & -1/a_2 & 1/a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1/a_{N-1} & 1/a_N \end{pmatrix}.$$

2. *It holds that*

$$\mathbf{S}_a^{-1}\mathbf{S}_{ab}\mathbf{S}_b^{-1} = \mathbf{I} - \mathbf{\Delta}, \tag{9}$$

and

$$\mathbf{S}_a\mathbf{\Delta}\mathbf{S}_b\mathbf{\Delta}\mathbf{S}_c + \mathbf{S}_{ab}\mathbf{\Delta}\mathbf{S}_c + \mathbf{S}_a\mathbf{\Delta}\mathbf{S}_{bc} + \mathbf{S}_{abc} = \mathbf{S}_a\mathbf{S}_b\mathbf{S}_c. \tag{10}$$

3. $\mathbf{S}_a$ *has eigenvalues* $\{a_1, \ldots, a_N\}$. *Suppose all the* a_j'*s are distinct, then define* $\mathbf{D}_a = (d_{i,j})_{N\times N}$ *and* $\mathbf{E}_a := (e_{i,j})_{N\times N}$ *by*

$$d_{i,j} := \frac{a_i}{a_N}\prod_{k=i+1}^{N}\left(1 - \frac{a_k}{a_j}\right) \text{ and } e_{i,j} := \begin{cases} 0, & \text{if } i < j; \\ \frac{a_N}{a_i}\prod\limits_{\substack{k=j \\ k\neq i}}^{N}\frac{1}{1-\frac{a_k}{a_i}}, & \text{if } i \geq j. \end{cases}$$

It follows that $(d_{1,j}, \ldots, d_{N,j})^T$ *is an eigenvector with respect to* a_j, *and* $\mathbf{D}_a^{-1} = \mathbf{E}_a$, *implying*

$$\mathbf{S}_a = \mathbf{D}_a \operatorname{diag}(a_1, \ldots, a_N)\,\mathbf{E}_a, \tag{11}$$

where $\operatorname{diag}(a_1, \ldots, a_N)$ *means the diagonal matrix with entries* $\{a_1, \ldots, a_n\}$ *on the diagonal.*

Proof. We omit the straightforward computation and only sketch the idea here.

1. The inverse can be easily computed.

2. Denote $\mathbf{I}_a := \operatorname{diag}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_N}\right)$ and $\mathbf{\Delta}_a = \mathbf{\Delta}\mathbf{I}_a$ so that $\mathbf{S}_a^{-1} = \mathbf{I}_a - \mathbf{\Delta}_a$. Since $\mathbf{I}_a\mathbf{S}_{ab} = \mathbf{S}_b$ (but $\mathbf{S}_{ab}\mathbf{I}_b \neq \mathbf{S}_a$) and $\mathbf{\Delta}_a\mathbf{S}_{ab} = \mathbf{A}_b = \mathbf{\Delta}\mathbf{S}_b$, easily one obtains

$$\mathbf{S}_a^{-1}\mathbf{S}_{ab}\mathbf{S}_b^{-1} = (\mathbf{I}_a - \mathbf{\Delta}_a)\,\mathbf{S}_{ab}\mathbf{S}_b^{-1} = \mathbf{S}_b\mathbf{S}_b^{-1} - \mathbf{\Delta}\mathbf{S}_b\mathbf{S}_b^{-1} = \mathbf{I} - \mathbf{\Delta}.$$

Similarly, multiplying by $\mathbf{S}_a^{-1}$ from the left and $\mathbf{S}_c^{-1}$ from the right on (10), we obtain

$$\mathbf{\Delta}\mathbf{S}_b\mathbf{\Delta} + \mathbf{S}_a^{-1}\mathbf{S}_{ab}\mathbf{\Delta} + \mathbf{\Delta}\mathbf{S}_{bc}\mathbf{S}_c^{-1} + \mathbf{S}_a^{-1}\mathbf{S}_{abc}\mathbf{S}_c^{-1} = \mathbf{S}_b,$$

which reduces to $\mathbf{I} - \mathbf{\Delta} = \mathbf{S}_b^{-1}\mathbf{S}_{bc}\mathbf{S}_c^{-1}$, i.e., (9).

3. Eigenvalues are trivial to see. To verify eigenvectors, noting that $d_{i,j} = 0$ if $j > i$, due to the zero term $k = j - 1$ in the product, it is equivalent to prove $\forall i = j, \ldots, N$,

$$\sum_{l=j}^{i} a_l \frac{a_i}{a_N}\prod_{k=l+1}^{N}\left(1 - \frac{a_k}{a_j}\right) = a_j\frac{a_i}{a_N}\prod_{k=i+1}^{N}\left(1 - \frac{a_k}{a_j}\right), \tag{12}$$

which can be directly computed by induction on i. The inverse $\mathbf{D}_a^{-1} = \mathbf{E}_a$ is equivalent to

$$\delta_{ij} = \sum_{t=j}^{i} \frac{a_i}{a_t} \left(\prod_{k=i+1}^{N} \left(1 - \frac{a_k}{a_t}\right)\right) \cdot \left(\prod_{\substack{k=j \\ k\neq t}}^{N} \frac{1}{1 - \frac{a_k}{a_t}} \right), \tag{13}$$

which reduces to

$$\sum_{t=j}^{i} \left(\prod_{\substack{k=j \\ k\neq t}}^{i} \frac{1}{a_t - a_k} \right) = \delta_{i,j},$$

following from [12, eq. 1, pp. 313], by multiplying both sides by z and then letting $z \to \infty$.

□

Remark 2.2. The diagonalization leads to easy computation of powers that

$$(\mathbf{S}_a)^k = \mathbf{D}_a \operatorname{diag}\left(a_1^k, \dots, a_N^k\right) \mathbf{E}_a. \tag{14}$$

3 Applications

3.1 Random walks

In this subsection, we let, for $l = 1, \dots, k$, $f_l(x) \equiv H_a(x) := 1/x^a$, where $a \geq 1$. When $a = 1$,

$$\mathbf{S}_{H_1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \tag{15}$$

Now, label N sites as follows:

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \cdots\ \underset{N-1}{\bullet}\ \underset{N}{\bullet}$$

and consider a random walk starting from site "N", with the rules:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

Let $\mathbb{P}(i \to j)$ denote the probability from site "i" to site "j". For example, suppose we are at site "6":

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \underset{4}{\bullet}\ \underset{5}{\bullet}\ \overset{\text{here}}{\underset{6}{\bullet}}\ \underset{7}{\bullet}\ \underset{8}{\bullet}\ \cdots\ \underset{N}{\bullet}$$

then, the next step only allows to walk to sites $\{1, 2, 3, 4, 5, 6\}$, with probabilities:

$$\mathbb{P}(6 \to 6) = \cdots = \mathbb{P}(6 \to 1) = \frac{1}{6}.$$

Therefore, a typical walk is as follows:

STEP 1: walk from "N" to some site "$n_1 (\leq N)$", with $\mathbb{P}(N \to n_1) = \frac{1}{N}$;
STEP 2: walk from "n_1" to "$n_2 (\leq n_1)$", with $\mathbb{P}(n_1 \to n_2) = \frac{1}{n_1}$;

... ...

STEP $k+1$: walk $n_k \mapsto n_{k+1} (\leq n_k)$, with $\mathbb{P}(n_k \to n_{k+1}) = \frac{1}{n_k}$.

We consider the event that after $k+1$ steps, we arrive the final destination site "1", i.e., $\mathbb{P}(n_{k+1} = 1)$, due to the fact that site "1" is a *sink* that once walks to, one never gets out. Since the steps are independent,

$$\mathbb{P}(n_{k+1} = 1) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k} = \frac{S(H_1, \dots, H_1; N, 1)}{N}. \tag{16}$$

Meanwhile, the stochastic transition matrix is exactly given by $\mathbf{S}_{H_1}$, i.e., $\mathbf{S}_{H_1} = (\mathbb{P}(i \to j))_{N\times N}$. Thus,

$$\left((\mathbf{S}_{H_1})^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1}=1) = \frac{1}{N} S(H_1,\ldots,H_1;N,1), \tag{17}$$

which is a probabilistic interpretation of (7) with slight difference that $\mathbf{P}$ in (7) replaced by $\mathbf{S}_{H_1}$.

Remark 3.1. When $a > 1$, we could also form a similar random walk by

- adding another sink "$\mathfrak{N}$", to the right of "N", with $\mathbb{P}(\mathfrak{R} \to n) = \delta_{\mathfrak{R},n}$ for $n \in \{1,\ldots,N,\mathfrak{R}\}$;
- defining for $l = 1,2,\ldots,N$,

$$\mathbb{P}(l \to j) = \begin{cases} 0, & \text{if } l < j \le N; \\ \frac{1}{l^a}, & \text{if } 1 \le j \le l; \\ 1 - \frac{1}{l^{a-1}}, & \text{if } j = \mathfrak{N}. \end{cases}$$

Similar computation for $\mathbb{P}(n_{k+1}=1)$ reveals that

$$S\left(\underbrace{H_a,\ldots,H_a}_{k};N,1\right) = N^a\left((\mathbf{S}_{H_a})^{k+1}\right)_{N,1}. \tag{18}$$

3.2 Relations between S and A

Recall the harmonic series [9, pp. 275]

$$\mathrm{S}(i_1,\ldots,i_k) := S\left(\frac{1}{x^{i_1}},\ldots,\frac{1}{x^{i_k}};\infty,1\right)$$

and the multiple zeta values [9, pp. 276]

$$\mathrm{A}(i_1,\ldots,i_k) := A\left(\frac{1}{x^{i_1}},\ldots,\frac{1}{x^{i_k}};\infty,1\right).$$

Examples of relation between them are [9, pp. 276]

$$\mathrm{S}(i_1,i_2) = \mathrm{A}(i_1,i_2) + \mathrm{A}(i_1+i_2), \tag{19}$$

and

$$\mathrm{S}(i_1,i_2,i_3) = \mathrm{A}(i_1,i_2,i_3) + \mathrm{A}(i_1+i_2,i_3) + \mathrm{A}(i_1,i_2+i_3) + \mathrm{A}(i_1+i_2+i_3). \tag{20}$$

Next, we will establish the truncated and generalized version of (19) and (20), in the sense that we truncate the series (from both above and below) into sums, which at the same time allows flexibility for general summand, not being restricted to be negative powers.

Theorem 3.1. *It holds that*

$$S(f,g;N-1,m) = A(f,g;N,m) + A(fg;N,m) \tag{21}$$

and

$$S(f,g,h;N-1,m) = A(f,g,h;N,m) + A(fg,h;N,m) + A(f,gh;N,m) + A(fgh;N,m). \tag{22}$$

Proof. By Theorem 2.1, the right-hand side of (21) is given by

$$(\mathbf{P}\mathbf{A}_f\mathbf{A}_g)_{N,m} + (\mathbf{P}\mathbf{A}_{fg})_{N,m} = (\mathbf{P}\boldsymbol{\Delta}(\mathbf{S}_f\boldsymbol{\Delta}\mathbf{S}_g + \mathbf{S}_{fg}))_{N,m}.$$

From (9), we see

$$\mathbf{I} - \boldsymbol{\Delta} = (\mathbf{S}_f)^{-1}\mathbf{S}_{fg}(\mathbf{S}_g)^{-1} \Leftrightarrow \mathbf{S}_f\boldsymbol{\Delta}\mathbf{S}_g + \mathbf{S}_{fg} = \mathbf{S}_f\mathbf{S}_g.$$

An easy observation shows that (noticing the different dimensions of matrices)

$$\left((\mathbf{P}_N\boldsymbol{\Delta}_N)\left(\mathbf{S}_{N|f}\mathbf{S}_{N|g}\right)\right)_{N,m} = \left(\mathbf{P}_{N-1}\mathbf{S}_{N-1|f}\mathbf{S}_{N-1|g}\right)_{N-1,m} = S(f,g;N-1,m).$$

Similarly, (22) is equivalent to (10), by replacing $(a,b,c) \mapsto (f,g,h)$. □

Remark 3.2. We are still seeking for proofs of the analogue of general results [9, Thm. 2.1, pp. 277, Thm. 2.2, pp. 278] for symmetric sums.

3.3 Combinatorial identities

The matrix computation in Section 2, especially the diagonalization for computing matrix power, leads to alternative proofs for some combinatorial identities and their generalizations.

Example 3.1. Butler and Karasik [5, Thm. 4, pp. 7] obtained if $G(n,k)$ satisfies $G(n,n) = 1$, $G(n,-k) = 0$ and for $k \geq 1$

$$G(n,k) = G(n-1,k-1) + a_k G(n-1,k),$$

then

$$S\left(\underbrace{a,\ldots,a}_{k}, N, 1\right) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} a_{n_1} \cdots a_{n_k} = G(N+k,N),$$

based on a proof related to Stirling numbers of the second kind. In fact through index matrices, we could provide a direct proof without using Stirling numbers.

- When $k = 1$, an induction on N shows directly that

$$\sum_{N \geq n_1 \geq 1} a_{n_1} = a_N + G(N,N-1) = a_N G(N,N) + G(N,N-1) = G(N+1,N).$$

- For inductive step in k, similarly from (18), we see

$$\begin{aligned} S\left(\underbrace{a,\ldots,a}_{k}, N, 1\right) &= a_N \left(\prod_{l=1}^{k} \mathbf{S}_a\right)_{N,1} = a_N \left(\mathbf{S}_a \left(\prod_{l=1}^{k-1} \mathbf{S}_a\right)\right)_{N,1} \\ &= \frac{1}{a_N} \sum_{m=1}^{N} a_N \cdot a_m G(m+k-1,m) \\ &= G(N+k,N), \end{aligned}$$

  by recurrence.

Example 3.2. Suppose $(a_m)_{m=1}^N$ are all distinct. An alternative expression of the previous example can be obtained by the diagonalization.

$$\begin{aligned} S\left(\underbrace{a,\ldots,a}_{k}, N, 1\right) &= \frac{1}{a_N} \left(\mathbf{D}_{H_a} \operatorname{diag}\left\{a_1^{k+1}, \ldots, a_N^{k+1}\right\} \mathbf{E}_{H_a}\right)_{N,1} \\ &= \frac{1}{a_N} \sum_{j=1}^{N} a_j^{k+1} \left(\frac{a_N}{a_j} \prod_{\substack{m=1 \\ m \neq j}}^{N} \frac{1}{1 - \frac{a_m}{a_j}}\right) \\ &= \sum_{j=1}^{N} \left(\prod_{\substack{m=1 \\ m \neq j}}^{N} \frac{1}{1 - \frac{a_m}{a_j}}\right) a_j^k. \end{aligned}$$

This recovers a general result [12, eq. 2, pp. 313], which, when taking $a_j = \frac{a - bq^{j+i-1}}{c - zq^{j+i-1}}$ and $N = n-i+1$, "turns out to be a common source of several q-identities" [12, pp. 314]. Special case when $a_m = m^a$, yields

$$S_{\underbrace{a,\ldots,a}_{k}}(N) = \sum_{l=1}^{N} \left(\prod_{\substack{n=1 \\ n \neq l}}^{N} \frac{n^a}{n^a - l^a}\right) \frac{1}{l^{ak}}, \tag{23}$$

which gives (5) when $a = 1$.

Remark 3.3. When $a = m \in \mathbb{Z}_+$, consider $\xi_m := \exp\left\{\frac{2\pi\imath}{m}\right\}$, where $\imath^2 = -1$, and the factorization

$$n^m - l^m = (n-l)(n-\xi_m l)\cdots\left(n-\xi_m^{m-1}l\right).$$

We could obtain the following binomial-type expression (like (5))

$$S_{\underbrace{a,\ldots,a}_{k}}(N) = \sum_{l=1}^{N}\left(\prod_{t=0}^{m-1}\binom{N}{\xi_m^t l}\frac{\pi\left(1-\xi_m^t\right)l}{\sin\left(\pi\xi_m^t l\right)}\right)\frac{1}{l^{mk}},$$

where the usual binomial coefficient is generalized as $\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(x+1)\Gamma(x-y+1)}$.

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