Circuits, Systems & Signal Processing

Matrix geometric means and uncertainty relation

--Manuscript Draft--

Manuscript Number:	
Full Title:	Matrix geometric means and uncertainty relation
Article Type:	Original Research
Keywords:	matrix geometric mean; uncertainty relation; entropy; trace class operator
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Funding Information:	the National Natural Science Foundation of China (61179031) — Dr. Huafei Sun
	the National Natural Science Foundation of China (61171195) — Bingzhao Li
Abstract:	The research of geometric means of matrices is attracting more and more attention from scholars since it has very important applications in mathematics and physics, especially in the von Neuman entropy involving the quantum physics and statistical thermal physical. There are many achievements on the entropy, among which the entropy uncertainty relation, the generalization of the Heisenberg uncertainty relation, has been applied in mathematics, physics and signal processing community. In this paper, we consider the entropy uncertainty relation in the geometric means type. By using the definition of the geometric means for the trace class operator, and associating with the R\'{e}nyi entropy uncertainty relation and the Shannon entropy uncertainty relation, we obtain the R\'{e}nyi entropy uncertainty relation for the density and the entropy uncertainty relation for the trace class operator.

Matrix geometric means and uncertainty relation *

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Abstract: The research of geometric means of matrices is attracting more and more attention from scholars since it has very important applications in mathematics and physics, especially in the von Neuman entropy involving the quantum physics and statistical thermal physical. There are many achievements on the entropy, among which the entropy uncertainty relation, the generalization of the Heisenberg uncertainty relation, has been applied in mathematics, physics and signal processing community. In this paper, we consider the entropy uncertainty relation in the geometric means type. By using the definition of the geometric means for the trace class operator, and associating with the Rényi entropy uncertainty relation and the Shannon entropy uncertainty relation, we obtain the Rényi entropy uncertainty relation for the density and the entropy uncertainty relation for the trace class operator.

1 Introduction

The mean, which can be commonly used to analyze the measurement data and to represent the typical characters (central tendency) and average of things, has been a widely used to statistical index in the process of statistical work. Thus an appropriate method should be selected to calculate the mean according to the application fields and the application conditions [1]. As a extension of the means of real number, the means of symmetric positive-definite matrices (SPD) have been paid increasing attention to the study with the development of physics in recently years, and one of the hotest researches is the geometric mean for SPD [2].

The geometric mean of SPD has arisen naturally in several fields and has possessed very important application in the quantum entropy [3]. It is well known that if the ranks of A and B are both 1, then the geometric mean of SPD is just the geometric mean of two real numbers. Suppose that A is the temporal resolution and B is the frequency resolution of a signal, then we could obtain a lower bound for the geometric mean of two real numbers A and B by using the Heisenberg uncertainty relation.

The Heisenberg uncertainty relation, proposed by the German physicist Heisenberg in 1927, is a basic principle of quantum mechanics, and it plays an essential role in modern applied mathematics. It states that the position and the momentum of a particle can not be determined simultaneously in a quantum

*This work is supported by the National Natural Science Foundation of China, No. 61179031 and No. 61171195

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mechanical systems. On the mathematical side, we can describe the Heisenberg uncertainty relation as that the product of the variances of f and $\mathcal{F}(f)$ (the Fourier transform (FT) of f) can not be infinitely small. We know that the variances of f and $\mathcal{F}(f)$ represent the temporal resolution and the frequency resolution of a signal respectively, and we can therefore see that they can not be infinitely improved simultaneously in signal processing community.

On the other hand, there are other uncertainty relations such as the logarithmic uncertainty relation [4] and the entropy uncertainty relation (Hirschman inequality) [5]-[11], which are the finer characterization of the Heisenberg uncertainty relation. In [8] and [9], the entropy uncertainty relation is an extension to the density of the trace class operator, where the density has important application in quantum mechanics. The uncertainty relation of the trace class operator is the essence of how to deal with the relationship between the matrices and the symmetric positive-definite Hilbert-Schmidt operators.

In this paper, jointing the definition of the geometric mean of the trace class operator, and using the Rényi entropy uncertainty relation and the Hirschman inequality, we obtain some uncertainty relation for the trace class operator in the type of the geometric mean. These results are the extension of the Rényi entropy uncertainty relation and the Hirschman inequality. The paper is organized as follows. Section 2 introduces some general definitions and gives some important conclusions. In section 3, we obtain one Rényi entropy uncertainty relation for the density of the trace class operator for unitary operator. Finally, we get one Shannon entropy uncertainty relation for some linear operator which satisfies the Riesz-Thorin conditions.

2 Preliminaries

First in this paper, some important definitions and results are introduced as follows.

2.1 Geometric means of matrices

For any two positive numbers a and b, the means of a and b should have the following properties [2].

1. $M(a,b) > 0$,
2. If $a \leq b$, then $a \leq M(a,b) \leq b$,
3. $M(a,b) = M(b,a)$,
4. $M(a,b)$ is monotone increasing in a and b,
5. For any $\alpha \in \mathbb{R}$, $M(\alpha a, \alpha b) = \alpha M(a,b)$ for all positive numbers $a,\ b$,
6. $M(a,b)$ is continuous in $a,\ b$.

Following the means for positive numbers, means for operations are of interest in the context of matrices as well. One of the important means is the geometric mean of SPD, which is defined by [2]

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}},$$

where $A, B \in$ SPD.

If A, B are symmetric positive-definite Hilbert-Schmidt operators, we can also define the geometric mean of symmetric positive-definite Hilbert-Schmidt operators as above.

The important property of the geometric mean of SPD is the algebraic-geometric-harmonic inequality which also holds for the real numbers:

$$2(A^{-1}+B^{-1})^{-1} \leq A\sharp B \leq \frac{A+B}{2}.$$

2.2 The trace class operator and its density

The general trace class operator [8], [9] is defined in the sigma-finite measure spaces. Let (X, μ) and (Y, ν) be two sigma-finite measure spaces, and $L^2(X)$ and $L^2(Y)$ be the corresponding spaces of square-integrable functions, respectively. The trace class operator in (X, μ) satisfies

$$\gamma = \sum_j \lambda_j |f_j\rangle\langle f_j|,$$

where $\lambda_j \in [0,1]$, $\sum_j \lambda_j = 1$, f_j is the unit orthonormal functions in $L^2(X)$. Then the density ρ_γ of γ is defined as

$$\rho_\gamma(x) = \sum_j \lambda_j \left|f_j(x)\right|^2,$$

from which we see that

$$\operatorname{tr}\gamma = \int_{\mathbb{R}} \rho_\gamma(x)\,\mathrm{d}\,x = \sum_j \lambda_j = 1.$$

For a unitary operator $\mathcal{U} : L^2(X) \mapsto L^2(Y)$, we define an operator $\widehat{\gamma}$ as

$$\widehat{\gamma} = \mathcal{U}\gamma\mathcal{U}^*,$$

where $\parallel \mathcal{U} \parallel_{L_1 \to L_\infty} = sup\{t : (\mu\times\nu)(\{(x,y) : |\mathcal{U}(x,y)| > t\}) > 0\} < \infty$. And the density $\rho_{\widehat{\gamma}}$ of $\widehat{\gamma}$ is defined as

$$\rho_{\widehat{\gamma}}(u) = \sum_j \lambda_j \left|\mathcal{U}f_j(u)\right|^2.$$

From the quantum mechanical viewpoint, The trace class operator is positive linear combination of pure states, which are the orthogonal projections on functions.

2.3 The Shannon entropy

In information theory, the entropy is the average amount of information contained in each message received. Here, message stands for an event, sample or character drawn from a distribution or data stream. The entropy thus characterizes our uncertainty about our source of information. The definition of the Shannon entropy is defined as [10]

$$E(f) = -\int_{\mathbb{R}} f(x)\ln f(x)\,\mathrm{d}\,x,$$

where $f(x)$ is a probability density function such that $\int_{\mathbb{R}} f(x)\,\mathrm{d}\,x = 1$ and $f(x) \geq 0$.

The Shannon entropy is a basic concept both in the information theory and mathematics especially in the classical harmonic analysis. In the classical harmonic analysis, there is an important inequality called

Hausdorff-Young inequality [5], and from sharp Hausdorff-Young inequality [7], we can obtain Hirschman inequality or entropy uncertainty relation in Fourier domain

$$E(f) + E(\widehat{f}) \geq 1 - \ln 2.$$

The entropy uncertainty relation is the finer characterization of the Heisenberg uncertainty relation which implies the temporal resolution and the frequency resolution of a signal can not be too small at the same time. In the filtering theory, the classical tool is the Fourier transform. Using the concept of information entropy, for the original signal f, we know that the entropy of f is known. Then, by the filtering process, we get a new signal $\widehat{f}$. The entropy uncertainty relation shows that the filtering process has the loss of the information entropy [12]-[21].

2.4 The Rényi entropy and the Tsallis entropy

As the extension of the Shannon entropy, the Rényi entropy also has very important application [11]. The definition of the Rényi entropy is [11]

$$E_R(f) = \frac{\ln \int_{\mathbb{R}} f^p(x)\, \mathrm{d}\, x}{1-p},$$

where $p \in (0,1)$ and $f(x)$ is a probability density function such that $\int_{\mathbb{R}} f(x)\, \mathrm{d}\, x = 1$ and $f(x) \geq 0$.

In the case $p = 1$, using the limit of the Rényi entropy, one can obtain that

$$\begin{aligned} \lim_{p \to 1} E_R(f) &= \lim_{p \to 1} \frac{\ln \int_{\mathbb{R}} f^p(x)\, \mathrm{d}\, x}{1-p} \\ &= -\int_{\mathbb{R}} f(x) \ln f(x)\, \mathrm{d}\, x \\ &= E(f). \end{aligned}$$

As the Rényi entropy, in the field of the statistical physics, the definition of the Tsallis entropy, which can provide a satisfactory explanation for the non-additive system with the mixed or the irregular shapes of debris, is defined by the following equality [16]

$$E_T(\rho) = \frac{\int_{\mathbb{R}} (\rho(x))^p\, \mathrm{d}\, x - 1}{1-p}, \tag{2.1}$$

where $\rho(x) \geq 0$ and $\int_{\mathbb{R}} \rho(x)\, \mathrm{d}\, x = 1$, $p \in (0,1)$.

The Tsallis entropy is a new method of measuring information‘non-extensive thermodynamics and is frequently used in the image segmentation [17].

When $p \to 1$, we obtain that

$$\begin{aligned} \lim_{p \to 1} E_T(\rho) &= \lim_{p \to 1} \frac{\int_{\mathbb{R}} (\rho(x))^p\, \mathrm{d}\, x - 1}{1-p} \\ &= -\int_{\mathbb{R}} f(x) \ln f(x)\, \mathrm{d}\, x \\ &= E(f). \end{aligned}$$

This implies that the Tsallis entropy can be viewed as the extension of the Shannon entropy.

The Rényi entropy and the Tsallis entropy are two important generalizations extension and the adjustable parameter p provdes flexibility comparing to the Shannon entropy.

2.5 Entropy uncertainty relation of unitary operator

In this subsection, we review one important theorem for unitary operator [9].

Lemma 1. Under the above assumptions, let $\gamma \geq 0$ be an operator in $\mathbf{L}^2(X)$ with $\operatorname{Tr}\gamma = 1$ such that

$$\int\limits_X \rho_\gamma(x) \ln_+ \rho_\gamma(x) \, \mathrm{d}\,\mu(x) < \infty \quad \text{and} \quad \int\limits_Y \rho_{\widehat{\gamma}}(u) \ln_+ \rho_{\widehat{\gamma}}(u) \, \mathrm{d}\,\nu(u) < \infty,$$

where $\ln_+ \rho = max\{\ln \rho, 0\}$. Then

$$- \int\limits_X \rho_\gamma(x) \ln_+ \rho_\gamma(x) \, \mathrm{d}\,\mu(x) - \int\limits_Y \rho_{\widehat{\gamma}}(u) \ln_+ \rho_{\widehat{\gamma}}(u) \, \mathrm{d}\,\nu(u)$$

$$\geq - \operatorname{tr}\gamma \ln \gamma - 2 \ln \| \, \mathcal{U} \, \|_{L_1 \to L_\infty} .$$

3 Geometric means and the Rényi entropy uncertainty relation

The entropy is a measurement of the lost of information for a system. For a system, the higher for degree, the less for entropy, and also the greater for the amount of information. In the information theory, the basic concept is the Shannon entropy. The Rényi entropy, a generalization of the Shannon entropy, has very important application in information theory especially in image threshold segmentation [15]. We know that for $f(x)$ and its Fourier transform $\widehat{f}(u)$, using the sharp Hausdorff-Young inequality, the Rényi entropy uncertainty relation holds [11]

$$E_R(|f|^2) + E_R(|\widehat{f}|^2) \geq \frac{p}{1-p} \ln\Big(\frac{q^{\frac{1}{2q}}}{p^{\frac{1}{2p}}}\Big),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{2} \leq p \leq 1$.

From the definition of the Rényi entropy, we have

$$E_R(|f|^2) + E_R(|\widehat{f}|^2) = \frac{\ln \int_{\mathbb{R}} f^p(x) \, \mathrm{d}\,x \cdot \int_{\mathbb{R}} \widehat{f}^p(u) \, \mathrm{d}\,u}{1-p}.$$

If $A = \int_{\mathbb{R}} f^p(x) \, \mathrm{d}\,x, \; B = \int_{\mathbb{R}} \widehat{f}^p(u) \, \mathrm{d}\,u$ with the pure states for the trace class operator such as the case $j = 1$, then $\frac{\ln(A\sharp B)}{2(1-p)} = E_R(|f|^2) + E_R(|\widehat{f}|^2)$.

For the mixture states, using the Jensen inequality and the Rényi entropy uncertainty relation, one can obtain a lower bound $\frac{p}{1-p} \ln(\frac{q^{\frac{1}{2q}}}{p^{\frac{1}{2p}}})$ for the density of the trace class operator $\rho_\gamma(x)$ and $\rho_{\widehat{\gamma}}(u)$. In the Fourier domain, this result can be improved [9]. In this section, we consider the unitary transform with $\| \, \mathcal{U} \, \|_{L_1 \to L_\infty} < \infty$, and we obtain the following result.

Theorem 1. For the trace class operators γ and $\widehat{\gamma}$, the geometric mean of A and $\widehat{A}$ satisfies the following inequality

$$\frac{\ln(A\sharp B)}{1-p} \geq \frac{-\operatorname{tr}\gamma \ln \gamma - 2 \ln \| \, \mathcal{U} \, \|_{L_1 \to L_\infty}}{2},$$

where $A = \int_{\mathbb{R}} (\rho_\gamma(x))^p \, \mathrm{d}\,x$, $B = \int_{\mathbb{R}} (\rho_{\widehat{\gamma}}(u))^p \, \mathrm{d}\,u$ and $p \in (0, 1)$.

Proof. By the definition of the Rényi entropy, we see that

$$E_R(\rho_\gamma) = \frac{\ln \int_{\mathbb{R}} (\rho_\gamma(x))^p \, \mathrm{d}\,x}{1-p}, \quad E_R(\rho_{\widehat{\gamma}}) = \frac{\ln \int_{\mathbb{R}} (\rho_{\widehat{\gamma}}(u))^p \, \mathrm{d}\,u}{1-p},$$

are respectively the Rényi entropies of the densities for the trace class operators $\rho_\gamma(x)$ and $\rho_{\widehat{\gamma}}(u)$.

Hence

$$\frac{\ln(A\sharp B)}{1-p}=\frac{\ln\int_{\mathbb{R}}(\rho_\gamma(x))^p\,\mathrm{d}\,x}{1-p}+\frac{\ln\int_{\mathbb{R}}(\rho_{\widehat{\gamma}}(u))^p\,\mathrm{d}\,u}{1-p}$$

is the the Rényi entropy uncertainty relation for $\rho_\gamma(x)$ and $\rho_{\widehat{\gamma}}(u)$.

Using the trace of the trace class operator, we have

$$A=\int_{\mathbb{R}}(\rho_\gamma(x))^p\,\mathrm{d}\,x=\mathrm{tr}(\gamma\rho_\gamma^{p-1}).$$

Thus we obtain

$$\frac{\ln(A\sharp B)}{1-p}=\frac{\ln(\mathrm{tr}(\gamma\rho_\gamma^{p-1})\sharp\,\mathrm{tr}(\widehat{\gamma}\rho_{\widehat{\gamma}}^{p-1}))}{1-p}.$$

Since $e^{\ln A}=A$ for $A\in SPD$, we get

$$\mathrm{tr}(\gamma\rho_\gamma^{p-1})=\mathrm{tr}\,e^{\ln(\gamma\rho_\gamma^{p-1})}.$$

Notice that $\gamma\rho_\gamma^{p-1}=\gamma\cdot I\rho_\gamma^{p-1}$, where I is the identity element, we have

$$\mathrm{tr}(\gamma\rho_\gamma^{p-1})=\mathrm{tr}\,e^{\ln(\gamma\rho_\gamma^{p-1})}=\mathrm{tr}\,e^{\ln(\gamma)+(p-1)I\ln\rho_\gamma}.$$

Using Peierls-Bogoliubov Inequality [3], we have

$$\ln(\frac{\mathrm{tr}\,e^{C+D}}{\mathrm{tr}\,e^{C}})\geq\frac{\mathrm{tr}\,De^{C}}{\mathrm{tr}\,e^{C}}.$$

Setting that $C=\ln(\gamma)$ and $D=(p-1)I\ln\rho_\gamma$, we have

$$\begin{aligned}\ln\mathrm{tr}(\gamma\rho_\gamma^{p-1})&=\ln(\mathrm{tr}\,e^{\ln(\gamma)+(p-1)I\ln\rho_\gamma})\\&=\ln(\frac{\mathrm{tr}\,e^{\ln(\gamma)+(p-1)I\ln\rho_\gamma}}{\mathrm{tr}\,e^{\ln(\gamma)}})\\&\geq\frac{\mathrm{tr}((p-1)I\ln\rho_\gamma e^{\ln(\gamma)})}{\mathrm{tr}\,e^{\ln(\gamma)}}\\&=(p-1)\,\mathrm{tr}(\gamma\ln\rho_\gamma)\\&=(1-p)E(\rho_\gamma),\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\ln(A\sharp B)}{1-p}&=\frac{\ln(\mathrm{tr}(\gamma\rho_\gamma^{p-1})\sharp\,\mathrm{tr}(\widehat{\gamma}\rho_{\widehat{\gamma}}^{p-1}))}{1-p}\\&=\frac{\ln(\mathrm{tr}(\gamma\rho_\gamma^{p-1}))}{2(1-p)}+\frac{\ln(\mathrm{tr}(\widehat{\gamma}\rho_{\widehat{\gamma}}^{p-1}))}{2(1-p)}\\&\geq\frac{(p-1)\,\mathrm{tr}(\gamma\ln\rho_\gamma)}{2(1-p)}+\frac{(p-1)\,\mathrm{tr}(\widehat{\gamma}\ln\rho_{\widehat{\gamma}})}{2(1-p)}\\&=\frac{E(\rho_\gamma)+E(\rho_{\widehat{\gamma}})}{2}.\end{aligned}$$

From Lemma 1,

$$\frac{\ln(A\sharp B)}{1-p}\geq\frac{-\,\mathrm{tr}\,\gamma\ln\gamma-2\ln\parallel\mathcal{U}\parallel_{L_1\to L_\infty}}{2}.$$

This completes the proof of the theorem.

Notice that if $p\to1$, then the Rényi entropy is just the Shannon entropy, hence Theorem 1 can be viewed as the generalization of the Shannon entropy uncertainty relation for the density of the trace class operator. In fact, by using the fact that if $a_n \geq b_n$, then $a \geq b$ ($a_n \to a,\ b_n \to b$), and we have

$$
\begin{aligned}
\lim_{q\to 1}(E_R(\rho_\gamma)+E_R(\rho_{\widehat{\gamma}})) &= \lim_{q\to 1}\frac{\ln(\operatorname{tr}(\gamma\rho_\gamma^{p-1})\sharp\operatorname{tr}(\widehat{\gamma}\rho_{\widehat{\gamma}}^{p-1}))^2}{1-p} \\
&= E(\rho_\gamma)+E(\rho_{\widehat{\gamma}}) \\
&\geq \lim_{q\to 1}(-\operatorname{tr}\gamma\ln\gamma - 2\ln\parallel\mathcal{U}\parallel_{L_1\to L_\infty}) \\
&= -\operatorname{tr}\gamma\ln\gamma - 2\ln\parallel\mathcal{U}\parallel_{L_1\to L_\infty},
\end{aligned}
$$

which is just the Lemma 1. If we use the Fourier transform, then the theorem can be improved to [8]

$$
-\operatorname{tr}\rho(\ln\operatorname{tr}\rho+\ln\|\rho\|_\infty).
$$

For more information about Rényi entropy, one can refer to [18], [21-27]. As discussion the above for the Rényi entropy, we can obtain the entropy uncertainty relation of the Tsallis entropy for the density of the class trace operator.

Corollary 1. For the trace class operators γ and $\widehat{\gamma}$, we have

$$
\begin{aligned}
E_T(\rho_\gamma)+E_T(\rho_{\widehat{\gamma}}) &= \frac{\int_{\mathbb{R}}\rho_\gamma^q(x)\,\mathrm{d}\,x-1+\int_{\mathbb{R}}\rho_{\widehat{\gamma}}^q(u)\,\mathrm{d}\,u-1}{1-q} \\
&\geq \frac{2\exp(\frac{(1-q)\cdot(-\operatorname{tr}\gamma\ln\gamma-2\ln\|\mathcal{U}\|_{L_1\to L_\infty})}{2})-2}{1-q},
\end{aligned}
$$

where $q \in (0,1)$.

Proof. From the definition of the Tsallis entropy, we can get

$$
E_T(\rho_\gamma)=\frac{\int_{\mathbb{R}}\rho_\gamma^q(x)\,\mathrm{d}\,x-1}{1-q}=\frac{\operatorname{tr}\gamma\rho_\gamma^{q-1}-1}{1-q}.
$$

By the relation of the exponential function and the logarithmic function for the operator, we have that

$$
\gamma\rho_\gamma^{q-1}=\exp(\ln\gamma\rho_\gamma^{q-1})=\exp(\ln\gamma\cdot\rho_\gamma^{q-1}I),
$$

where I is the identity.

Since γ commutates with $\rho_\gamma^{q-1}I$, we obtain that

$$
\gamma\rho_\gamma^{q-1}=\exp(\ln\gamma+\ln\rho_\gamma^{q-1}I).
$$

From the following inequality

$$
\ln(\frac{\operatorname{tr}e^{C+D}}{\operatorname{tr}e^C})\geq\frac{\operatorname{tr}De^C}{\operatorname{tr}e^C},
$$

we get

$$
\frac{\operatorname{tr}e^{C+D}}{\operatorname{tr}e^C}\geq\exp(\frac{\operatorname{tr}De^C}{\operatorname{tr}e^C}).
$$

Hence, for the Tsallis entropy, we obtain

$$
\begin{aligned}
E_T(\rho_\gamma) &= \frac{\int_{\mathbb{R}} \rho_\gamma^q(x)\,\mathrm{d}\,x - 1}{1-q} \\
&= \frac{\operatorname{tr}\gamma\rho_\gamma^{q-1} - 1}{1-q} \\
&= \frac{\operatorname{tr}\exp(\ln\gamma + \ln\rho_\gamma^{q-1} I) - 1}{1-q} \\
&= \frac{\frac{\operatorname{tr}\exp(\ln\gamma + \ln\rho_\gamma^{q-1} I)}{\operatorname{tr}\exp\ln\gamma} - 1}{1-q} \\
&\geq \frac{\exp(\frac{\operatorname{tr}\ln\rho_\gamma^{q-1} I e^{\ln\gamma}}{\operatorname{tr} e^{\ln\gamma}}) - 1}{1-q} \\
&= \frac{\exp((q-1)\cdot\operatorname{tr}\gamma\ln\rho_\gamma)) - 1}{1-q} \\
&= \frac{\exp((1-q)\cdot E(\rho_\gamma)) - 1}{1-q}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
E_T(\rho_\gamma) + E_T(\rho_{\widehat{\gamma}}) &\geq \frac{\exp((1-q)\cdot E(\rho_\gamma)) - 1}{1-q} + \frac{\exp((1-q)\cdot E(\rho_\gamma)) - 1}{1-q} \\
&= \frac{\exp((1-q)\cdot E(\rho_\gamma)) + \exp((1-q)\cdot E(\rho_\gamma)) - 2}{1-q}.
\end{aligned}
$$

By the mean value inequality, we have

$$
\begin{aligned}
E_T(\rho_\gamma) + E_T(\rho_{\widehat{\gamma}}) &\geq \frac{2\exp(\frac{(1-q)\cdot E(\rho_\gamma) + (1-q)\cdot E(\rho_\gamma)}{2}) - 2}{1-q} \\
&= \frac{2\exp(\frac{(1-q)\cdot(E(\rho_\gamma) + E(\rho_\gamma))}{2}) - 2}{1-q}.
\end{aligned}
$$

Hence, from Lemma 1, we have the following inequality

$$
E_T(\rho_\gamma) + E_T(\rho_{\widehat{\gamma}}) \geq \frac{2\exp(\frac{(1-q)\cdot(-\operatorname{tr}\gamma\ln\gamma - 2\ln\|\mathcal{U}\|_{L_1\to L_\infty})}{2}) - 2}{1-q},
$$

where $q \in (0,1)$.

This completes the proof of the corollary.

Notice that the Tsallis entropy is just the Shannon entropy when $q \to 1$, hence Corollary 1 should be the generalization of the Shannon entropy uncertainty relation for the density of the trace class operator. When

$$
\begin{aligned}
E(\rho_\gamma) + E(\rho_{\widehat{\gamma}}) &= \lim_{q\to 1} E_T(\rho_\gamma) + E_T(\rho_{\widehat{\gamma}}) \\
&\geq \lim_{q\to 1} \frac{2\exp(\frac{(1-q)\cdot(-\operatorname{tr}\gamma\ln\gamma - 2\ln\|\mathcal{U}\|_{L_1\to L_\infty})}{2}) - 2}{1-q},
\end{aligned}
$$

from the fact that

$$
\lim_{x\to 1} \frac{e^{ax} - 1}{x} = a,
$$

we can obtain

$$
\begin{aligned}
&E(\rho_\gamma) + E(\rho_{\widehat{\gamma}}) \\
&\geq \lim_{q\to 1} \frac{2\exp(\frac{(1-q)\cdot(-\operatorname{tr}\gamma\ln\gamma - 2\ln\|\mathcal{U}\|_{L_1\to L_\infty})}{2}) - 2}{1-q} \\
&= 2\frac{-\operatorname{tr}\gamma\ln\gamma - 2\ln\parallel\mathcal{U}\parallel_{L_1\to L_\infty}}{2} \\
&= -\operatorname{tr}\gamma\ln\gamma - 2\ln\parallel\mathcal{U}\parallel_{L_1\to L_\infty}.
\end{aligned}
$$

Thus Corollary 1 implies the Shannon entropy uncertainty relation for the density of the trace class operator. For the FT, the improved results is

$$E_T(\rho_\gamma) + E_T(\rho_{\widehat{\gamma}}) \geq \frac{2\exp(\frac{(1-q)\cdot(-\operatorname{tr}\rho(\ln\operatorname{tr}\rho+\ln\|\rho\|_\infty))}{2}) - 2}{1-q}.$$

4 Geometric means and the Shannon entropy uncertainty relation

The entropy plays a very important role in physics and information theory, and it describes the degree of a system of chaos. As a basic concept, the Shannon entropy has wide applications in signal processing and harmonic analysis. As the generalization of the Heisenberg uncertainty relation, which is a basic principle in quantum mechanical systems, the entropy uncertainty relation is significant in Fourier analysis and also satisfies the following.

For any $f \in \mathbb{S}$ ($\mathbb{S}$ is the Schwartz space) and $\widehat{f}(u) = \int_{-\infty}^{\infty} f(t)e^{-2\pi tu}\,\mathrm{d}\,t$, we have

$$E(f) + E(\widehat{f}) \geq 1 - \ln 2,$$

and when f is the Gauss function, the equality holds. This inequality implies that for the Fourier optical system, the loss of information can not be too small at the same time. Also existence of loss of information for the Fourier optical system can be seen easily.

For any $a,\ b \in \mathbb{R}$, we have that

$$\ln(e^{\frac{a}{2}}(e^{-\frac{a}{2}}e^{b}e^{-\frac{a}{2}})^{\frac{1}{2}}e^{\frac{a}{2}}) = \ln e^{\frac{a+b}{2}} = \frac{a+b}{2},$$

and if $a = \gamma\ln\rho_\gamma,\ b = \widehat{\gamma}\ln\rho_{\widehat{\gamma}}$ with $j = 1$, then we have

$$\operatorname{tr}(\ln(e^{\frac{a}{2}}(e^{-\frac{a}{2}}e^{b}e^{-\frac{a}{2}})^{\frac{1}{2}}e^{\frac{a}{2}})) = -\frac{E(f)+E(\widehat{f})}{2} \leq \frac{\ln 2 - 1}{2}.$$

This is the Hirschman inequality [12] in Fourier domain.

In this section, associated with the Shannon entropy, when $\mathcal{L}$ is a linear operator satisfying (here we call this properties Riesz-Thorin conditions)

$$\| \mathcal{L}(f) \|_{L^2} = \| f \|_{L^2}, \| \mathcal{L}(f) \|_{L^\infty} \leq M \| f \|_{L^1}.$$

By the Riesz-Thorin theorem [13], we have

$$\| \mathcal{L}(f) \|_{L^q} \leq M^\theta \| f \|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q \geq 2$, $\theta \in (0,1)$ and M^θ only depends on q.

Using the definition of 2.2 and the geometric means, we obtain the following inequality for Shannon entropy.

Theorem 2. Suppose that $\mathcal{L}$ is a linear operator satisfying the Riesz-Thorin conditions with $\lim\limits_{q\to 2} M^\theta = 1$. Then the following inequality holds

$$\operatorname{tr}(\ln(e^{\frac{\gamma_1}{2}}(e^{-\frac{\gamma_1}{2}}e^{\widehat{\gamma_1}}e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}}e^{\frac{\gamma_1}{2}})) \leq \frac{1}{2}\cdot\frac{\partial M^\theta}{\partial q}\,|_{q=2},$$

where $\gamma_1 = \gamma \ln \rho_\gamma$ and $\widehat{\gamma_1} = \widehat{\gamma} \ln \rho_{\widehat{\gamma}}$.

Proof. From the Baker-Campbell-Hausdorff formula [14]

$$\begin{aligned}\ln(e^A e^B) &= A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) + \cdots \\ &= A + B + f(A, B),\end{aligned}$$

where $\operatorname{tr} f(A, B) = 0$. By setting

$$Z = A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) + \cdots,$$

we obtain

$$\begin{aligned}\ln(e^A e^B e^C) &= Z + C + \frac{1}{2}[Z, C] + \frac{1}{12}([Z, [Z, C]] + [C, [C, Z]]) + \cdots \\ &= A + B + C + h(A, B, C),\end{aligned}$$

where $\operatorname{tr} h(A, B, C) = 0$.

Notice that

$$\ln(e^{\frac{\gamma_1}{2}} (e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}}) = \ln(e^{\frac{\gamma_1}{2}} e^{\frac{\ln(e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})}{2}} e^{\frac{\gamma_1}{2}}),$$

then, letting $A = \frac{\gamma_1}{2}$, $B = \frac{\ln(e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})}{2}$, and $C = \frac{\gamma_1}{2}$, by the Baker-Campbell-Hausdorff formula for the case of three operators obtained above, we have

$$B = \frac{\ln(e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})}{2} = \frac{-\gamma_1 + \widehat{\gamma_1}}{2} + \bar{f}(\gamma_1, \widehat{\gamma_1}),$$

where $\operatorname{tr} \bar{f}(\gamma, \widehat{\gamma}) = 0$.

Thus, by the Baker-Campbell-Hausdorff formula again,

$$\ln(e^{\frac{\gamma_1}{2}} (e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}}) = \frac{\gamma_1 + \widehat{\gamma_1}}{2} + g(\gamma_1, \widehat{\gamma_1}),$$

where $\operatorname{tr} g(\gamma_1, \widehat{\gamma_1}) = 0$.

Let $\gamma_1 = \gamma \ln \rho_\gamma$ and $\widehat{\gamma_1} = \widehat{\gamma} \ln \rho_{\widehat{\gamma}}$, we have

$$\begin{aligned}\operatorname{tr}(\ln(e^{\frac{\gamma_1}{2}} (e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}})) &= \operatorname{tr}(\frac{\gamma \ln \rho_\gamma + \widehat{\gamma} \ln \rho_{\widehat{\gamma}}}{2}) \\ &= \frac{1}{2}(\int_{\mathbb{R}} \rho_\gamma(x) \ln \rho_\gamma(x) \, \mathrm{d}\, x + \int_{\mathbb{R}} \rho_{\widehat{\gamma}}(u) \ln \rho_{\widehat{\gamma}}(u) \, \mathrm{d}\, u).\end{aligned}$$

By the definitions of $\rho_\gamma(x)$ and $\rho_{\widehat{\gamma}}(u)$ and the Jensen inequality, we obtain

$$\begin{aligned}\operatorname{tr}(\ln(e^{\frac{\gamma_1}{2}} (e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}})) &= \frac{1}{2}(\int_{\mathbb{R}} \rho_\gamma(x) \ln \rho_\gamma(x) \, \mathrm{d}\, x + \int_{\mathbb{R}} \rho_{\widehat{\gamma}}(u) \ln \rho_{\widehat{\gamma}}(u) \, \mathrm{d}\, u) \\ &\leq \frac{1}{2}(\sum_j \lambda_j \int_{\mathbb{R}} \mid f_j(x) \mid^2 \ln \mid f_j(x) \mid^2 \, \mathrm{d}\, x \\ &\quad + \sum_j \lambda_j \int_{\mathbb{R}} \mid \mathcal{L}(f_j)(u) \mid^2 \ln \mid \mathcal{L}(f_j)(u) \mid^2 \, \mathrm{d}\, u).\end{aligned}$$

From the following inequality

$$\parallel \mathcal{L}(f) \parallel_{L^q} \leq M^\theta \parallel f \parallel_{L^p},$$

we see that if $\lim_{q\to 2} M^\theta = 1$ and $\| \mathcal{L}(f) \|_{L^2} = \| f \|_{L^2}$, and there for we have

$$E(f) + E(\mathcal{L}f) \geq -\frac{\partial M^\theta}{\partial q} \mid_{q=2} .$$

Hence, we get

$$\begin{aligned}
\operatorname{tr}(\ln(e^{\frac{\gamma_1}{2}}(e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}})) &\leq \frac{1}{2}(\sum_j \lambda_j \int_{\mathbb{R}} \mid f_j(x) \mid^2 \ln \mid f_j(x) \mid^2 \mathrm{d}\, x \\
&+ \sum_j \lambda_j \int_{\mathbb{R}} \mid \mathcal{L}(f_j)(u) \mid^2 \ln \mid \mathcal{L}(f_j)(u) \mid^2 \mathrm{d}\, u) \\
&= \frac{1}{2} \sum_j \lambda_j (\int_{\mathbb{R}} \mid f_j(x) \mid^2 \ln \mid f_j(x) \mid^2 \mathrm{d}\, x \\
&+ \int_{\mathbb{R}} \mid \mathcal{L}(f_j)(u) \mid^2 \ln \mid \mathcal{L}(f_j)(u) \mid^2 \mathrm{d}\, u) \\
&\leq \frac{1}{2} \sum_j \lambda_j (\frac{\partial M^\theta}{\partial q} \mid_{q=2}) \\
&= \frac{1}{2} \cdot \frac{\partial M^\theta}{\partial q} \mid_{q=2} .
\end{aligned}$$

This completes the proof of the theorem.

We see that the geometric means of the trace class operator has a lower bound. It means that if $\frac{1}{2} \cdot \frac{\partial M^\theta}{\partial q} \mid_{q=2} > 0$, then the trace of the geometric means of γ and $\widehat{\gamma}$ can not be too small at the same time. This theorem only provides a lower bound, and for $j > 1$ we see that the inequality is strictly greater than $\frac{1}{2} \cdot \frac{\partial M^\theta}{\partial q} \mid_{q=2}$. For $j = 1$, Theorem 2 implies that the entropy uncertainty relation holds for the linear operator which satisfies the Riesz-Thorin conditions, especially for the Fourier transform. And in the Fourier domain, this results can be improved to $1 - \ln 2$ which can be obtained by the sharp Hausdorff-Young inequality [7].

The condition that $\lim_{q\to 2} M^\theta = 1$ in Theorem 2 is necessary, because the entropy uncertainty principle is obtained by derivation

$$\| \mathcal{L}(f) \|_{L^q} \leq M^\theta \| f \|_{L^p} .$$

For the unitary transform with $\| \mathcal{U} \|_{L_1 \to L_\infty} < \infty$, we obtain the following corollary.

Corollary 2. Suppose that $\mathcal{U}$ is a unitary operator satisfying $\| \mathcal{U} \|_{L_1 \to L_\infty} < \infty$. Then the following inequality holds

$$\operatorname{tr}(\ln(e^{\frac{\gamma_1}{2}}(e^{-\frac{\gamma_1}{2}} e^{\widehat{\gamma_1}} e^{-\frac{\gamma_1}{2}})^{\frac{1}{2}} e^{\frac{\gamma_1}{2}})) \leq -\operatorname{tr} \gamma \ln \gamma - 2 \ln \| \mathcal{U} \|_{L_1 \to L_\infty},$$

where $\gamma_1 = \gamma \ln \rho_\gamma$ and $\widehat{\gamma_1} = \widehat{\gamma} \ln \rho_{\widehat{\gamma}}$. Especially for the Fourier transform the result can be improved to

$$-\operatorname{tr} \rho(\ln \operatorname{tr}(\rho) + \ln \|\rho\|_\infty).$$

5 Conclusion

In this paper, first, by using the definition of geometric mean of symmetric positive-definite matrices and the defined density of the trace class operator, we obtain one Rényi entropy uncertainty relation for the density. Then, using the geometric mean of symmetric positive-definite matrices and the linear operator which satisfies the Riesz-Thorin conditions, we obtain one Shannon entropy uncertainty relation by the help of the Baker-Campbell-Hausdorff formula for the trace class operator. Although these results rely on the geometric mean of SPD, one can find that the uncertainty relation does not change.

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