

Recursion rules for the hypergeometric zeta function

Alyssa Byrnes*, Lin Jiu[†] and Victor H. Moll[‡]

*Department of Mathematics
 Tulane University
 New Orleans, LA 70118, USA
 *abyrnes1@tulane.edu
[†]ljui@tulane.edu
[‡]vhm@math.tulane.edu*

Christophe Vignat

*Department of Mathematics, Tulane University
 New Orleans, LA 70118, USA*

and

*Department of Mathematics, L.S.S. Supelec, Université Orsay
 Paris Sud, France
 cvignat@tulane.edu*

Received 3 December 2013

Accepted 20 February 2014

Published 3 April 2014

The hypergeometric zeta function is defined in terms of the zeros of the Kummer function $M(a, a+b; z)$. It is established that this function is an entire function of order 1. The classical factorization theorem of Hadamard gives an expression as an infinite product. This provides linear and quadratic recurrences for the hypergeometric zeta function. A family of associated polynomials is characterized as Appell polynomials and the underlying distribution is given explicitly in terms of the zeros of the associated hypergeometric function. These properties are also given a probabilistic interpretation in the framework of beta distributions.

Keywords: Zeta functions; hypergeometric functions; beta random variables; conjugate random variables.

Mathematics Subject Classification 2010: 11B83, 11B68, 60C05

1. Introduction

The zeta function attached to a collection of non-zero complex numbers $\mathbb{A} = \{a_n \neq 0 : n \in \mathbb{N}\}$ is defined by

$$\zeta_{\mathbb{A}}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s} \quad \text{for } \operatorname{Re} s > c. \quad (1.1)$$

The most common choice of sequences \mathbb{A} includes those coming from the zeros of a given function f :

$$\mathbb{A}(f) = \{z \in \mathbb{C} : f(z) = 0\} = \{z_n \in \mathbb{C} : f(z_n) = 0, n \in \mathbb{N}\}, \quad (1.2)$$

to produce the associated zeta function

$$\zeta_f(s) = \sum_{n=1}^{\infty} \frac{1}{z_n^s}. \quad (1.3)$$

The prototypical example is the classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.4)$$

coming from (half of) the zeros $\mathbb{A} = \{z_n = n > 0\}$ of the function $f(z) = \frac{\sin \pi z}{\pi z}$.

The literature contains a variety of zeta functions $\zeta_{\mathbb{A}}$ and their study is concentrated in reproducing the basic properties of (1.4). For example $\zeta(s)$, originally defined in the half-plane $\operatorname{Re} s > 1$, admits a meromorphic extension to the complex plane, with a single pole at $s = 1$. Moreover, the function $\zeta(s)$ admits special values

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad (1.5)$$

where the Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \quad (1.6)$$

Carlitz introduced in [5] coefficients β_n by

$$\frac{x^2}{e^x - 1 - x} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!} \quad (1.7)$$

and stated that *nothing is known about them*. Howard [15] used the notation $A_s = \frac{1}{2}\beta_s$, and in [16] he introduced the generalization $A_{k,r}$ by

$$\frac{x^k}{k!} \left(e^x - \sum_{s=0}^{k-1} \frac{x^s}{s!} \right)^{-1} = \sum_{r=0}^{\infty} A_{k,r} \frac{x^r}{r!}. \quad (1.8)$$

These numbers satisfy the recurrence

$$\sum_{r=0}^n \binom{n+k}{r} A_{k,r} = 0, \quad \text{for } n > 0 \quad (1.9)$$

with $A_{k,0} = 1$. It follows that $A_{k,n}$ is a rational number and some of their arithmetical properties are reviewed in Sec. 5.

The work presented here considers a zeta function constructed in terms of the *Kummer function*

$$M(a, b; z) = {}_1F_1 \left(\begin{matrix} a \\ b \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \quad (1.10)$$

The next definition introduces the main function considered here. The notation

$$\Phi_{a,b}(z) = {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z\right) = M(a, a+b; z), \quad \text{for } a, b \in \mathbb{R} \quad (1.11)$$

is employed throughout.

Definition 1.1. Let a, b be positive real numbers. The *hypergeometric zeta function* is defined by

$$\zeta_{a,b}^H(s) = \sum_{k=1}^{\infty} \frac{1}{z_{k;a,b}^s} \quad \text{for } \operatorname{Re} s > 1, \quad (1.12)$$

where $z_{k;a,b}$ is the sequence of complex zeros of the function $\Phi_{a,b}(z)$.

The special case $\Phi_{1,1}(z) = (e^z - 1)/z$ is the reciprocal of the generating function for the Bernoulli numbers (1.6). The coefficients $B_n^{(b)}$ are defined by

$$\frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \quad (1.13)$$

In the case $b \in \mathbb{N}$, these numbers are the coefficients $A_{k,r}$ defined by Howard in (1.8) (with $k = b$ and $r = n$). These numbers are discussed in Sec. 4. The function $\Phi_{1,2}(z)$ recently appeared in [7] is the asymptotic expansion of $n!$. Indeed, it can be shown that the coefficients a_k in the expansion

$$n! \sim \frac{n^n \sqrt{2\pi n}}{e^n} \sum_{k=0}^{\infty} \frac{a_k}{n^k} \quad \text{as } n \rightarrow \infty \quad (1.14)$$

are given by

$$a_k = \frac{1}{2^k k!} \left(\frac{d}{dz}\right)^{2k} \Phi_{1,2}^{-(k+1/2)} \Big|_{z=0}. \quad (1.15)$$

Dilcher [9, 8] considered the zeta function $\zeta_{a,b}^H$. In particular, he established an expression for $\zeta_{a,b}^H(m)$, for $a, b, m \in \mathbb{N}$, in terms of the hypergeometric Bernoulli numbers $B_{a,b}^n$ introduced in Sec. 7.

Note. The many examples of zeta functions discussed in the literature include the *Bessel zeta function*

$$\zeta_{Bes,a}(s) = \sum_{n=1}^{\infty} \frac{1}{j_{a,n}^s}, \quad (1.16)$$

where $\{j_{a,n}\}$ are the zeros of $J_a(z)/z^a$, with $J_a(z)$ the Bessel function of the first kind

$$J_a(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+a+1)} \left(\frac{z}{2}\right)^{2m+a}. \quad (1.17)$$

Papers considering $\zeta_{Bes,a}$ include [2, 11, 14, 21]. A second example is the *Airy-zeta function*, defined by

$$\zeta_{Ai}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}, \quad (1.18)$$

where $\{a_n\}$ are the zeros of the Airy function

$$Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right) dt. \quad (1.19)$$

This is considered by Crandall [6] in the so-called *quantum bouncer*. Special values include the remarkable

$$\zeta_{Ai}(2) = \frac{3^{5/3}}{4\pi^2} \Gamma^4\left(\frac{2}{3}\right). \quad (1.20)$$

A third example is the zeta function studied by Hassen and Nguyen [12, 13]. This is defined by the integral

$$\zeta_{HN,b}(s) = \frac{1}{\Gamma(s+b-1)} \int_0^{\infty} \frac{x^{s+b-2} dx}{e^x - 1 - z - z^2/2! - \dots - z^{b-1}/(b-1)!}. \quad (1.21)$$

The main results presented here include a relation among the Kummer function $\Phi_{a,b}(z)$ and the hypergeometric zeta function $\zeta_{a,b}^H$ defined in (1.12). This is

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k \quad (1.22)$$

and it appears in Proposition 3.1. It is shown that the function $\zeta_{a,b}^H$ satisfies a couple of linear recurrence relations:

$$\sum_{\ell=1}^p B(a+p-\ell, b) \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1) = -\frac{bp}{(a+b)(a+b+p)} B(a+p, b), \quad (1.23)$$

appearing in Theorem 3.2 and

$$(n-1)! \sum_{j=2}^n \frac{B_{n-j}^{(a,b)}}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B_{n-1}^{(a,b)} + B_n^{(a,b)}, \quad (1.24)$$

established in Theorem 7.11, where $B_n^{(a,b)}$ are the so-called *hypergeometric Bernoulli numbers* introduced in Sec. 7. The third recurrence is quadratic

$$\sum_{k=1}^p \zeta_{a,b}^H(k+1) \zeta_{a,b}^H(p-k+1) = (a+b+p+1) \zeta_{a,b}^H(p+2) + \left(\frac{a-b}{a+b}\right) \zeta_{a,b}^H(p+1). \quad (1.25)$$

This is given in Theorem 3.3.

Theorem 4.1 expresses the rational numbers $B_n^{(b)} := B_n^{(1,b)}$ in terms of the values $\zeta_{1,b}^H(n)$. This extends the classical result for the Bernoulli numbers $B_n = B_n^{(1,1)}$. Section 5 states some conjectures on arithmetical properties of the denominators of

$B_n^{(b)}$ extending the von Staudt–Clausen theorem for Bernoulli numbers. Section 6 introduces a probabilistic technique to approach these questions and Sec. 7 discusses a family of polynomials introduced by Dilcher and proposes a natural generalization.

This work provides linear identities linking three types of functions: the classical beta function, the hypergeometric Bernoulli polynomials and the hypergeometric zeta function. Explicitly, Theorem 3.2 gives a linear recurrence involving the beta function and the hypergeometric zeta function, Theorem 7.7 gives a linear recurrence involving the beta function (written as binomial coefficients) and the hypergeometric Bernoulli polynomials and, finally, Theorem 7.11 gives a relation between the hypergeometric Bernoulli numbers and the hypergeometric zeta function.

Notation. It is an unfortunate fact that many of the terms used in the present work are denoted by the letter B . The list below shows the symbols employed here.

B_n	Bernoulli number	(1.6)
$M(a, b; z)$	Kummer function	(1.10)
$\Phi_{a,b}(z)$	Kummer function	(1.11)
$\zeta_{a,b}^H(s)$	hypergeometric zeta function	(1.12)
$B_n^{(b)}$	hypergeometric Bernoulli number	(1.13)
$B(a, b)$	the beta function	(3.6)
$\mathfrak{B}_{a,b}$	a beta distributed random variable	(6.2)
$\mathfrak{Z}_{a,b}$	a complex random variable	(6.8)
$B_n^{(a,b)}(x)$	hypergeometric Bernoulli polynomial	(7.8)
$B_n^{(a,b)}$	hypergeometric Bernoulli number	(7.9)

2. Properties of the Kummer Function $\Phi_{a,b}(z)$

The function

$$\Phi_{a,b}(z) = {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z\right) = M(a, a+b; z), \quad \text{for } a, b \in \mathbb{R}, \quad (2.1)$$

defined in terms of the Kummer function $M(a, b; z)$ is the main object considered in the present work. The function $M(a, b; z)$ satisfies the differential equation

$$z \frac{d^2 M}{dz^2} + (b - z) \frac{dM}{dz} - aM = 0, \quad (2.2)$$

obtained from the standard hypergeometric equation

$$z \frac{dw^2}{dz^2} + [c - (a + b + 1)z] \frac{dw}{dz} - abw = 0 \quad (2.3)$$

by scaling $z \mapsto z/b$, letting $b \rightarrow \infty$ and replacing the parameter c by b .

The first result shows that the special case $a = 1$ gives the function considered by Howard [16].

Theorem 2.1. For $b \in \mathbb{N}$, the function $\Phi_{1,b}(z)$ is given by

$$\Phi_{1,b}(z) = \frac{b!}{z^b} \left(e^z - \sum_{k=0}^{b-1} \frac{z^k}{k!} \right). \quad (2.4)$$

Proof. This follows directly from the expansion

$${}_1F_1 \left(\begin{matrix} 1 \\ 1+b \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(1)_k z^k}{(1+b)_k k!} = \sum_{k=0}^{\infty} \frac{b!}{(b+k)!} z^k = \frac{b!}{z^b} \sum_{k=b}^{\infty} \frac{z^k}{k!}. \quad \square$$

Corollary 2.2. The zeta function $\zeta_{HA}(s)$ in (1.21) is given by

$$\zeta_{HN,b}(s) = \frac{b!}{\Gamma(s+b-1)} \int_0^{\infty} \frac{x^{s-2} dx}{\Phi_{1,b}(x)}. \quad (2.5)$$

The next property of $\Phi_{a,b}(z)$ is a representation as an infinite product. The result comes from the classical Hadamard factorization theorem for entire functions. A preliminary lemma is given first.

Lemma 2.3. The Kummer function $\Phi_{a,b}(z)$ satisfies

$$\frac{d}{dz} \Phi_{a,b}(z) = \frac{a}{a+b} \Phi_{a+1,b}(z). \quad (2.6)$$

Proof. This comes directly from formula

$$\frac{d}{dz} {}_1F_1 \left(\begin{matrix} a \\ b \end{matrix} \middle| z \right) = \frac{a}{b} {}_1F_1 \left(\begin{matrix} a+1 \\ b+1 \end{matrix} \middle| z \right), \quad (2.7)$$

which is entry 13.3.15 in [19, p. 325]. \square

The next step is to analyze the factorization of the function $\Phi_{a,b}(z)$. Recall that the *order* of an entire function $h(z)$ is defined as the infimum of $\alpha \in \mathbb{R}$ for which there exists a radius $r_0 > 0$ such that

$$|h(z)| < e^{|z|^\alpha} \quad \text{for } |z| > r_0. \quad (2.8)$$

The order of $h(z)$ is denoted by $\rho(h)$. See [4] for information on the order of entire functions. The main result used here is Hadamard's theorem stated below.

Theorem 2.4. For $p \in \mathbb{N}$, define the elementary factors

$$E_p(z) = \begin{cases} 1 - z & \text{if } p = 0, \\ (1 - z) \exp \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p} \right) & \text{otherwise.} \end{cases} \quad (2.9)$$

Assume h is an entire function of finite order $\rho = \rho(h)$. Let $\{a_n\}$ be the collection of zeros of h repeated according to multiplicity. Then h admits the factorization

$$h(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right), \quad (2.10)$$

where $g(z)$ is a polynomial of degree $q \leq \rho$, $p = \lfloor \rho \rfloor$ and $m \geq 0$ is the order of the zero of h at the origin.

The next result establishes the order of $\Phi_{a,b}(z)$.

Theorem 2.5. *Let $a, b > 0$. Then $\Phi_{a,b}$ is an entire function of order 1.*

Proof. The ratio test shows that the function $\Phi_{a,b}(z)$ is entire. Moreover

$$\frac{(a)_\ell}{(a+b)_\ell} = \prod_{k=0}^{\ell-1} \frac{a+k}{a+b+k} < 1, \quad (2.11)$$

therefore

$$|\Phi_{a,b}(z)| \leq \sum_{\ell=0}^{\infty} \frac{(a)_\ell}{(a+b)_\ell} \left| \frac{z^\ell}{\ell!} \right| \leq \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} = e^{|z|}. \quad (2.12)$$

This proves $\rho(h) \leq 1$.

To establish the opposite inequality, use the asymptotic behavior

$$M(a, b; z) \sim \frac{e^{-z} z^{a-b}}{\Gamma(a)}, \quad \text{as } z \rightarrow \infty \quad (2.13)$$

(see [19, p. 323]) to see that, for every $0 \leq \varepsilon < 1$ and $z \in \mathbb{R}$,

$$\lim_{z \rightarrow \infty} \frac{\Phi_{a,b}(z)}{\exp(z^\varepsilon)} = +\infty. \quad (2.14)$$

Hence, for any given $r_0 > 0$, there is $r > r_0$ such that

$$|\Phi_{a,b}(r)| = \Phi_{a,b}(r) > \exp(r^\varepsilon) = \exp(|r|^\varepsilon). \quad (2.15)$$

This proves $\rho(h) \geq 1$ and the proof is complete. \square

The factorization of $\Phi_{a,b}(z)$ in terms of its zeros $\{z_{a,b;k}\}$ is discussed next. Section 13.9 of [19] states that if a and $b \neq 0, -1, -2, \dots$, then $\Phi_{a,b}(z)$ has infinitely many complex zeros. Moreover, if $a, b \geq 0$, then there are no real zeros. The growth of the large zeros of $M(a, a+b; z)$ is given by

$$z_{a,b;n} = \pm(2n+a)\pi i + \ln\left(-\frac{\Gamma(a)}{\Gamma(b)}(\pm 2n\pi i)^{b-a}\right) + O(n^{-1} \ln n), \quad (2.16)$$

where n is a large positive integer, and the logarithm takes its principal value.

Theorem 2.6. *The function $\Phi_{a,b}(z)$ admits the factorization*

$$\Phi_{a,b}(z) = e^{az/(a+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}. \quad (2.17)$$

Proof. Hadamard's theorem shows the existence of two constants A, B such that

$$\Phi_{a,b}(z) = e^{Az+B} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}. \quad (2.18)$$

Evaluating at $z = 0$, using $\Phi_{a,b}(0) = 1$, gives $B = 0$. To obtain the value of the parameter A , take the logarithmic derivative of (2.18) and use Lemma 2.3 to produce

$$\frac{a}{a+b} \frac{{}_1F_1\left(\begin{matrix} a+1 \\ a+b+1 \end{matrix} \middle| z\right)}{{}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z\right)} = A + \sum_{k=0}^{\infty} \left[\frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right]. \quad (2.19)$$

Expanding both sides near $z = 0$ gives

$$\frac{a}{a+b} + \frac{ab}{(a+b)^2(1+a+b)}z + \mathcal{O}(z^2) = A - \sum_{k=1}^{\infty} \frac{z}{z_{a,b;k}^2} + \mathcal{O}(z^2). \quad (2.20)$$

This gives $A = a/(a+b)$, completing the proof. \square

Corollary 2.7. *The hypergeometric zeta function has the special value*

$$\zeta_{a,b}^H(2) = -\frac{ab}{(a+b)^2(1+a+b)}. \quad (2.21)$$

Proof. Compare the coefficients of z in (2.20). \square

The next statement presents additional properties of the Kummer function which will be useful in the next section. It appears as entries 13.4.12 and 13.4.13 in [1].

Lemma 2.8. *The Kummer function satisfies*

$$\frac{d}{dz} \Phi_{a,b}(z) = -\frac{b}{a+b} \Phi_{a,b+1}(z) + \Phi_{a,b}(z) \quad (2.22)$$

and

$$\frac{d}{dz} \Phi_{a,b+1}(z) = \frac{a+b}{z} (\Phi_{a,b}(z) - \Phi_{a,b+1}(z)). \quad (2.23)$$

3. Recurrence for the Hypergeometric Zeta Function

This section describes some recurrences for the values $\zeta_{a,b}^H(k)$. The proofs are based on a relation between the Kummer function $\Phi_{a,b}(z)$ and these values. It is the analog of standard result for the usual zeta function

$$\sum_{k=1}^{\infty} \zeta(k+1) z^k = -\gamma - \psi(1-z), \quad (3.1)$$

where $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ is the digamma function and γ is the Euler constant. This relation is obtained directly from the product representation of $\Gamma(z)$ (see [1, entry 6.3.14]).

Proposition 3.1. *The Kummer function $\Phi_{a,b}(z)$ and the hypergeometric zeta function are related by*

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k. \quad (3.2)$$

Proof. The relation (2.22) gives

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left[\frac{\Phi'_{a,b}(z)}{\Phi_{a,b}(z)} - 1 \right]. \quad (3.3)$$

The fraction on the right-hand side is the logarithmic derivative of the product in Theorem 2.6. This yields

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left(\frac{a}{a+b} + \sum_{k=0}^{\infty} \left[\frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right] - 1 \right). \quad (3.4)$$

To establish the result, use the expansion

$$\frac{1}{z_{a,b;k} - z} = \frac{1}{z_{a,b;k}} \frac{1}{1 - z/z_{a,b;k}} \quad (3.5)$$

and expand the last term as the sum of a geometric series. Since $\min_k \{|z_{a,b;k}|\} \neq 0$, this series has a positive radius of convergence. \square

The next result gives a linear recurrence for the hypergeometric zeta function. This involves the beta function

$$B(u, v) = \int_0^1 x^{u-1} (1-x)^{v-1} dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (3.6)$$

with values

$$B(u, v) = \frac{u+v}{uv} \binom{u+v}{u}^{-1} \quad \text{for } u, v \in \mathbb{N}. \quad (3.7)$$

Theorem 3.2. *The hypergeometric zeta function satisfies the linear recurrence*

$$\sum_{\ell=1}^p B(a+p-\ell, b) \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1) = -\frac{bp}{(a+b)(a+b+p)} B(a+p, b). \quad (3.8)$$

Proof. Proposition 3.1 gives

$$\Phi_{a,b+1}(z) = \Phi_{a,b}(z) + \frac{a+b}{b} \Phi_{a,b}(z) \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k. \quad (3.9)$$

Matching the coefficient of z^p gives the identity

$$\frac{(a)_p}{(a+b+1)_p} = \frac{(a)_p}{(a+b)_p} + \frac{a+b}{b} \sum_{\ell=1}^p \frac{(a)_{p-\ell}}{(a+b)_{p-\ell}} \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1). \quad (3.10)$$

This simplifies to produce the result. \square

Note. The recurrence (3.8) is written as

$$\zeta_{a,b}^H(p) = -\frac{bB(a+p-1, b)}{(a+b)(a+b+p-1)(p-2)!B(a, b)} - \sum_{r=1}^{p-2} \frac{B(a+r, b)}{B(a, b)r!} \zeta_{a,b}^H(p-r).$$

The initial condition given in (2.21) shows that, for $p \in \mathbb{N}$, the value $\zeta_{a,b}^H(p)$ is a rational function of a, b . The first few values are

$$\begin{aligned} \zeta_{a,b}^H(2) &= -\frac{ab}{(a+b)^2(1+a+b)}, \\ \zeta_{a,b}^H(3) &= \frac{ab(a-b)}{(a+b)^3(a+b+1)(a+b+2)}, \\ \zeta_{a,b}^H(4) &= -\frac{abP_4(a, b)}{(a+b)^4(a+b+1)^2(a+b+2)(a+b+3)}, \end{aligned}$$

where

$$P_4(a, b) = a^2 + a^3 - 4ab - 2a^2b + b^2 - 2ab^2 + b^3.$$

The authors were unable to discern the patterns in $\zeta_{a,b}^H(p)$ for p positive integer.

The next result presents a different type of recurrence for $\zeta_{a,b}^H(s)$. It is the analogue of the classical identity

$$\left(n + \frac{1}{2}\right) \zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k) \zeta(2n-2k), \quad \text{for } n \geq 2 \quad (3.11)$$

(see [19, 25.6.16]).

Theorem 3.3. *The hypergeometric zeta function satisfies the quadratic recurrence*

$$\begin{aligned} &\sum_{k=1}^p \zeta_{a,b}^H(k+1) \zeta_{a,b}^H(p-k+1) \\ &= (a+b+p+1) \zeta_{a,b}^H(p+2) + \left(\frac{a-b}{a+b}\right) \zeta_{a,b}^H(p+1). \end{aligned} \quad (3.12)$$

Proof. The function $f(z) = \Phi_{a,b+1}(z)/\Phi_{a,b}(z)$ satisfies the differential equation

$$f'(z) = \frac{a+b}{z}(1-f(z)) - f(z) + \frac{b}{a+b}f^2(z). \quad (3.13)$$

This can be verified directly using the results of Lemma 2.8. Now use Theorem 3.1 to match the coefficients of z^p and produce

$$\begin{aligned} \frac{a+b}{b}(p+1)\zeta_{a,b}^H(p+2) &= -\frac{(a+b)^2}{b}\zeta_{a,b}^H(p+2) - \frac{a+b}{b}\zeta_{a,b}^H(p+1) \\ &\quad + \frac{a+b}{b} \sum_{k=1}^p \zeta_{a,b}^H(k+1)\zeta_{a,b}^H(p-k+1) + 2\zeta_{a,b}^H(p+1), \end{aligned}$$

which, after simplification, yields the result. \square

Note. Matching the constant terms recovers the value of $\zeta_{a,b}^H(2)$ in (2.21).

4. The Hypergeometric Bernoulli Numbers

This section considers properties of the *hypergeometric Bernoulli numbers* $B_n^{(b)}$, defined by the relation

$$\frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \quad (4.1)$$

These are precisely the numbers $A_{b,n}$ studied by Howard [16]. This follows from Theorem 2.1 and (1.8). The special case $b = 1$ corresponds to the Bernoulli numbers.

The next result appears in [16] in the case $b = 2$.

Theorem 4.1. *Let $b \in \mathbb{N}$. The hypergeometric Bernoulli numbers $B_n^{(b)}$ are expressed in terms of the hypergeometric zeta function as*

$$B_n^{(b)} = \begin{cases} 1 & \text{for } n = 0, \\ -1/(1+b) & \text{for } n = 1, \\ -n!\zeta_{1,b}^H(n)/b & \text{for } n \geq 2. \end{cases} \quad (4.2)$$

Proof. The product representation of $\Phi_{1,b}(z)$ given in Theorem 2.6 is

$$\Phi_{1,b}(z) = e^{z/(1+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{1,b;k}}\right) e^{z/z_{1,b;k}}. \quad (4.3)$$

Logarithmic differentiation yields

$$\begin{aligned} \frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \left[\frac{1}{z_{1,b;k}} - \frac{1}{z_{1,b;k} - z} \right] \\ &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \frac{1}{z_{1,b;k}} \left(1 - \sum_{\ell=0}^{\infty} \left(\frac{z}{z_{1,b;k}} \right)^{\ell} \right) \\ &= \frac{1}{1+b} - \sum_{\ell=1}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell+1). \end{aligned}$$

On the other hand, Theorem 2.1 gives

$$\Phi_{1,b}(z) = \frac{b!}{z^b} \left(e^z - \sum_{j=0}^{b-1} \frac{z^j}{j!} \right) \quad (4.4)$$

and logarithmic differentiation produces

$$\frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} = 1 + \frac{b}{z} \cdot \frac{1}{\Phi_{1,b}(z)} - \frac{b}{z}. \quad (4.5)$$

Therefore,

$$\frac{1}{\Phi_{1,b}(z)} = 1 - \frac{z}{1+b} - \frac{1}{b} \sum_{\ell=2}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell) = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}. \quad (4.6)$$

The conclusion follows by comparing coefficients of powers of z . \square

Note. The previous theorem suggests the definition

$$\zeta_{1,b}^H(1) = \frac{b}{1+b}. \quad (4.7)$$

5. Some Arithmetical Conjectures

Theorem 3.2 and the relation (4.2) show that the coefficients $B_n^{(b)}$ are rational numbers. For example:

$$\begin{aligned} B_n^{(1)} &= \left\{ 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots \right\}, \\ B_n^{(2)} &= \left\{ 1, -\frac{1}{3}, \frac{1}{18}, \frac{1}{90}, -\frac{1}{270}, -\frac{5}{1134}, -\frac{1}{5670}, \frac{7}{2430}, \dots \right\}, \\ B_n^{(3)} &= \left\{ 1, -\frac{1}{4}, \frac{1}{40}, \frac{1}{160}, \frac{1}{5600}, -\frac{1}{896}, -\frac{13}{19200}, \frac{7}{76800}, \dots \right\}. \end{aligned}$$

The arithmetic properties of the Bernoulli numbers $B_n = B_n^{(1)}$ are very intriguing. It is well known that the Bernoulli numbers of odd index vanish (except $B_1 = -1/2$), leaving B_{2n} for consideration. Write $B_{2n} = N_{2n}/D_{2n}$ in reduced form. The arithmetic properties of D_{2n} include the von Staudt–Clausen theorem stated below. The numerators N_{2n} are much more difficult to analyze. Chapter 15 of [17] contains information about their relation to Wiles theorem (previously known as Fermat’s last theorem).

Theorem 5.1. *The denominator of the Bernoulli number B_{2n} is given by*

$$D_{2n} = \prod_{(p-1)|2n} p. \quad (5.1)$$

In particular, the denominator of B_{2n} is even (actually always divisible by 6) and it is square-free. In the case of the hypergeometric Bernoulli numbers, computer

experiments suggest an extension of these properties. Let

$$\mathfrak{D}(b) = \{\text{denominator}(B_n^{(b)}) : n \geq 0\}. \quad (5.2)$$

The examples

$$\begin{aligned} \mathfrak{D}(2) &= \{1, 3, 18, 90, 270, 1134, 5670, 2430, \dots\}, \\ \mathfrak{D}(3) &= \{1, 4, 40, 160, 5600, 896, 19200, 76800, \dots\}, \\ \mathfrak{D}(4) &= \{1, 5, 75, 875, 26250, 78750, 918750, 3093750, \dots\} \end{aligned} \quad (5.3)$$

show that $\mathfrak{D}(b)$ contains an initial segment of *odd* numbers.

Conjecture 5.2. *Let $\alpha(b)$ be the number of odd terms at the beginning of $\mathfrak{D}(b)$. Then*

$$\alpha(b) = \begin{cases} \nu_2(b) + 1 & \text{if } b \not\equiv 0 \pmod{4}, \\ 2^{\nu_2(b)} & \text{if } b \equiv 0 \pmod{4}. \end{cases} \quad (5.4)$$

The prime factorization of $D_{2n} = \text{den}(B_{2n})$ shows that if p is a prime dividing D_{2n} , then $p \leq 2n + 1$. Numerical evidence of the corresponding statement for $B_n^{(b)}$ leads to the next conjecture.

Conjecture 5.3. *Every prime p dividing the denominator of $B_n^{(b)}$ satisfies $p \leq n + b$.*

These two conjectures have been verified up to $b = 1000$.

Note. It is no longer true that the denominators are square-free. For example $B_7^{(3)} = 7/76800$ and $76800 = 2^{10} \cdot 3 \cdot 5^2$.

6. A Probabilistic Approach

This section presents an interpretation of the Kummer function $\Phi_{a,b}(z)$ as the expectation of a complex random variable. For a random variable \mathfrak{R} with a continuous distribution function $r(x)$, the expectation operator is defined by

$$\mathbb{E}u(\mathfrak{R}) = \int_{\mathbb{R}} u(x)r(x)dx, \quad (6.1)$$

for the class of functions u for which the integral is finite.

The techniques employed here were recently used in [3] to solve a problem proposed by Zeilberger related to the Narayana polynomials discussed in [18].

Definition 6.1. Let $a, b > 0$. The random variable $\mathfrak{B}_{a,b}$ is called *beta distributed*, or simply *a beta random variable*, if its distribution function is given by

$$f_{\mathfrak{B}_{a,b}}(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (6.2)$$

Here $B(a, b)$ is the beta function (3.6).

The integral representation [1, 13.2.1]

$$\Phi_{a,b}(z) = {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z\right) = \frac{1}{B(a,b)} \int_0^1 e^{tz} t^{a-1} (1-t)^{b-1} dt \quad (6.3)$$

shows that $\Phi_{a,b}(z)$ is the moment generating function of a beta random variable $\mathfrak{B}_{a,b}$:

$$\mathbb{E}(e^{z\mathfrak{B}_{a,b}}) = \Phi_{a,b}(z), \quad (6.4)$$

where \mathbb{E} is the expectation operator.

This representation is used to construct a new random variable. Some preliminary discussion is given first.

A random variable Γ is said to be *exponentially distributed* if its distribution function is given by

$$f_{\Gamma}(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{elsewhere.} \end{cases} \quad (6.5)$$

The moment generating function of an exponentially distributed random variable Γ is

$$\mathbb{E}[e^{z\Gamma}] = \frac{1}{1-z}, \quad \text{for } |z| < 1. \quad (6.6)$$

Note. The authors hope that not too much confusion will be created by using the symbol Γ for this kind of random variables. The choice of name is clear: if Γ is exponentially distributed, then

$$\mathbb{E}[\Gamma^{\alpha-1}] = \Gamma(\alpha). \quad (6.7)$$

Consider a sequence $\{\Gamma_k\}_{k \geq 1}$ of independent identically distributed random variables, each with the same exponential distribution (6.5).

Definition 6.2. Let $\{z_{a,b;k} : k \in \mathbb{N}\}$ be the collection of zeros of the Kummer function $\Phi_{a,b}(z)$. The complex-valued random variable $\mathfrak{Z}_{a,b}$ is defined by

$$\mathfrak{Z}_{a,b} = -\frac{a}{a+b} + \sum_{k=1}^{\infty} \frac{\Gamma_k - 1}{z_{a,b;k}}. \quad (6.8)$$

Some properties of $\mathfrak{Z}_{a,b}$ are given below. The main relation between $\mathfrak{Z}_{a,b}$ and the Kummer function $\Phi_{a,b}(z)$ is stated first.

Theorem 6.3. *The complex-valued random variable $\mathfrak{Z}_{a,b}$ satisfies*

$$\mathbb{E}e^{z\mathfrak{Z}_{a,b}} = \frac{1}{\Phi_{a,b}(z)}. \quad (6.9)$$

Proof. The independence of the family $\{\Gamma_k\}$ and Theorem 2.6 give

$$\mathbb{E}e^{z\mathfrak{Z}_{a,b}} = e^{-az/(a+b)} \prod_{k=1}^{\infty} \frac{1}{1 - z/z_{a,b;k}} e^{-z/z_{a,b;k}}. \quad (6.10)$$

This is the stated result. \square

Lemma 6.4. *The symmetry property*

$$\mathfrak{Z}_{a,b} = -1 - \mathfrak{Z}_{b,a}, \quad (6.11)$$

holds in the sense of distribution.

Proof. The symmetry property follows from Kummer transformation

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z\right) = e^z {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix} \middle| -z\right). \quad (6.12)$$

□

Now consider $\mathfrak{B}_{a,b}$ to be a beta-distributed random variable independent of $\mathfrak{Z}_{a,b}$. Their moment generating functions satisfy

$$1 = \mathbb{E}[e^{z\mathfrak{B}_{a,b}}] \times \mathbb{E}[e^{z\mathfrak{Z}_{a,b}}] = \mathbb{E}[e^{z(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})}]. \quad (6.13)$$

Expanding in a power series yields the identity

$$\mathbb{E}[(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^n] = \delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \quad (6.14)$$

This is now used to provide a probabilistic representation of an analytic function.

Theorem 6.5. *Let $\mathfrak{B}_{a,b}$ and $\mathfrak{Z}_{a,b}$ as before. Then, if f is an analytic function,*

$$\mathbb{E}f(z + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b}) = f(z), \quad (6.15)$$

provided the expectation is finite.

Proof. It suffices to establish the result for $f(z) = z^p$, with $p \in \mathbb{N}$. This follows directly from (6.14):

$$\begin{aligned} \mathbb{E}[(z + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^p] &= \sum_{k=0}^p \binom{p}{k} z^{p-k} \mathbb{E}[(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^k] \\ &= z^p. \end{aligned}$$

□

Note. Hassen and Nguyen [12] show that

$$\int_0^1 x^{a-1} (1-x)^{b-1} B_n^{(a,b)}(x) dx = \begin{cases} B(a,b) & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \quad (6.16)$$

This follows directly from Theorem 6.5 by taking $f(z) = z^n$ and then $z = 0$.

The next item in this section gives a probabilistic point of view of the linear recurrence in Theorem 3.2. Some preliminary background is discussed first.

Definition 6.6. The independent random variables X and Y are called *conjugate* if

$$\mathbb{E}[(X + Y)^n] = \delta_n. \quad (6.17)$$

For example, (6.14) shows that $\mathfrak{B}_{a,b}$ and $\mathfrak{Z}_{a,b}$ are conjugate.

Definition 6.7. Let X be a random variable. The *moment generating function* of X is

$$\varphi_X(t) = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!}. \quad (6.18)$$

The sequence of *cumulants* $\kappa_X(n)$ is defined by

$$\log \varphi_X(z) = \sum_{m=1}^{\infty} \kappa_X(m) \frac{z^m}{m!}. \quad (6.19)$$

Example 6.8. The cumulant generating function for the $\mathfrak{B}_{a,b}$ distribution is

$$\log \Phi_{a,b}(z) = \frac{a}{a+b} z - \sum_{p=2}^{\infty} \frac{z^p}{p} \zeta_{a,b}^H(p), \quad (6.20)$$

so the cumulants are

$$\kappa_{\mathfrak{B}_{a,b}}(1) = \frac{a}{a+b} \quad \text{and} \quad \kappa_{\mathfrak{B}_{a,b}}(p) = -(p-1)! \zeta_{a,b}^H(p) \quad \text{for } p \geq 2. \quad (6.21)$$

For a general random variable X , the moments $\mathbb{E}X^n$ and its cumulants $\kappa_X(n)$ are related by

$$\kappa_X(n) = \mathbb{E}X^n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_X(j) \mathbb{E}X^{n-j}. \quad (6.22)$$

See [20] for details. In the special case of a random variable with a beta distribution, this gives

$$(n-1)! \sum_{j=2}^n \frac{B(a+n-j, b)}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B(a+n-1, b) - B(a+n, b) \quad (6.23)$$

that is equivalent to the linear recurrence identity in Theorem 3.2.

7. The Generalized Bernoulli Polynomials

Dilcher introduced in [9] the generalized Bernoulli polynomials by

$$\frac{e^{xz}}{\Phi_{1,b}(z)} = \sum_{k=0}^{\infty} B_k^{(b)}(x) \frac{z^k}{k!}. \quad (7.1)$$

These polynomials are now interpreted as moments.

Theorem 7.1. *The generalized Bernoulli polynomials are given by*

$$B_k^{(b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{1,b})^k. \quad (7.2)$$

Proof. This follows directly from Theorem 6.3. □

Dilcher [9] used generating functions to provide the following recursion for these polynomials.

Theorem 7.2 (Dilcher). *The generalized Bernoulli polynomials $B_k^{(b)}(x)$ satisfy*

$$B_k^{(b)}(x+1) = \sum_{p=0}^{b-1} \binom{k}{p} B_{k-p}^{(b)}(x) + \binom{k}{b} x^{k-b}. \quad (7.3)$$

A probabilistic proof is presented next. A preliminary result is stated first.

Lemma 7.3. *Let $\mathfrak{B}_{a,b}$ be a random variable with a beta distribution and $g \in C^1[0, 1]$. Then, for $a, b > 1$,*

$$\mathbb{E}g'(x + \mathfrak{B}_{a,b}) = (a + b - 1)[\mathbb{E}g(x + \mathfrak{B}_{a,b-1}) - \mathbb{E}g(x + \mathfrak{B}_{a-1,b})]. \quad (7.4)$$

For $a = 1$ and $b > 1$

$$\mathbb{E}g'(x + \mathfrak{B}_{1,b}) = -bg(x) + b\mathbb{E}g(x + \mathfrak{B}_{1,b-1}). \quad (7.5)$$

Proof. A direct calculation shows

$$\begin{aligned} B(a, b)\mathbb{E}g'(x + \mathfrak{B}_{a,b}) &= \int_0^1 g'(x+t)t^{a-1}(1-t)^{b-1}dt \\ &= g(x+t)t^{a-1}(1-t)^{b-1}|_0^1 \\ &\quad - (a-1) \int_0^1 g(x+t)t^{a-2}(1-t)^{b-1}dt \\ &\quad + (b-1) \int_0^1 g(x+t)t^{a-1}(1-t)^{b-2}dt. \end{aligned}$$

The result follows by simplification. The case $a = 1$ is straightforward. \square

The formula (7.5) can be extended directly to higher-order derivatives.

Lemma 7.4. *Let $g \in C^k[0, 1]$ and $\mathfrak{B}_{1,b}$ be as before. Then, provided $b \geq k$,*

$$\mathbb{E}g^{(k)}(x + \mathfrak{B}_{1,b}) = - \sum_{\ell=1}^k \frac{b!}{(b-\ell)!} g^{(k-\ell)}(x) + \frac{b!}{(b-k)!} \mathbb{E}g(x + \mathfrak{B}_{1,b-k}). \quad (7.6)$$

To prove Dilcher's theorem, replace x by $x + \mathfrak{B}_{1,b}$ and take the expectation with respect to $\mathfrak{B}_{1,b}$ to see that Theorem 7.2 is equivalent to the identity

$$(x+1)^k = \sum_{p=0}^{b-1} \binom{k}{p} x^{k-p} + \mathbb{E}(x + \mathfrak{B}_{1,b-k})^k. \quad (7.7)$$

This is precisely the statement of Lemma 7.4 for $g(x) = x^k$.

The expression for the polynomials $B_k^{(b)}(x)$ given in Theorem 7.1 provides a natural way to extend them to a two-parameter family.

Definition 7.5. The *hypergeometric Bernoulli polynomials* are defined by

$$B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{a,b})^n \quad (7.8)$$

and the *hypergeometric Bernoulli numbers* by

$$B_n^{(a,b)} = B_n^{(a,b)}(0) = \mathbb{E}(\mathfrak{Z}_{a,b})^n. \quad (7.9)$$

Note. The case considered by Howard is $B_n^{(b)} = B_n^{(1,b)}$.

Proposition 7.6. The exponential generating function for the polynomials $B_n^{(a,b)}(x)$ is given by

$$\sum_{n=0}^{\infty} B_n^{(a,b)}(x) \frac{z^n}{n!} = \frac{e^{xz}}{\Phi_{a,b}(z)}. \quad (7.10)$$

The next result appears, in the special case $x = 0$, as Proposition 2.1 in [9]. The result gives a change of basis formula from $\{B_n^{(a,b)}(x) : n = 0, 1, 2, \dots\}$ to $\{x^n : n = 0, 1, 2, \dots\}$.

Theorem 7.7. The polynomials $B_n^{(a,b)}(x)$ satisfy $B_0^{(a,b)}(x) = 1$ and, for $n \geq 1$,

$$\sum_{k=0}^n \binom{a+b+n-1}{k} \binom{a-1+n-k}{a-1} B_k^{(a,b)}(x) = (a+b)_n \frac{x^n}{n!}. \quad (7.11)$$

Proof. Theorem 6.5, with $f(x) = x^n$ gives

$$\mathbb{E}(x + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^n = x^n. \quad (7.12)$$

The binomial theorem now gives

$$\sum_{k=0}^n \binom{n}{k} \mathbb{E}(x + \mathfrak{Z}_{a,b})^k \mathbb{E}\mathfrak{B}_{a,b}^{n-k} = x^n. \quad (7.13)$$

The moments of the beta random variable $\mathfrak{B}_{a,b}$ are

$$\begin{aligned} \mathbb{E}\mathfrak{B}_{a,b}^p &= \frac{1}{B(a,b)} \int_0^1 x^p \cdot x^{a-1} (1-x)^{b-1} dx \\ &= \frac{B(a+p,b)}{B(a,b)} = \frac{\Gamma(a+p)}{\Gamma(a+b+p)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)}. \end{aligned}$$

The expression (7.13) is now expressed as

$$\frac{\Gamma(a+b)}{\Gamma(a)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(a+n-k)}{\Gamma(a+b+n-k)} B_k^{(a,b)}(x) = x^n, \quad (7.14)$$

and this is equivalent to the stated result. \square

The probabilistic approach presented here provides a direct proof of a symmetry property established in [9]. It extends the classical relation $B_n(1-x) = (-1)^n B_n(x)$ of the Bernoulli polynomials.

Theorem 7.8. *The polynomials $B_n^{(a,b)}(x)$ satisfy the symmetry*

$$B_n^{(a,b)}(1-x) = (-1)^n B_n^{(b,a)}(x). \quad (7.15)$$

Proof. The moment representation

$$B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{a,b})^n, \quad (7.16)$$

and using (6.11), yields

$$\begin{aligned} B_n^{(a,b)}(1-x) &= \mathbb{E}(1-x + \mathfrak{Z}_{a,b})^n \\ &= \mathbb{E}(-x - \mathfrak{Z}_{b,a})^n \\ &= (-1)^n \mathbb{E}(x + \mathfrak{Z}_{b,a})^n. \end{aligned} \quad \square$$

The next result presents a linear recurrence for the polynomials $B_n^{(a,b)}(x)$.

Theorem 7.9. *Let X and Y be conjugate random variables. Define the polynomials*

$$P_n(z) = \mathbb{E}(z + X)^n \quad \text{and} \quad Q_n(z) = \mathbb{E}(z + Y)^n. \quad (7.17)$$

Then P_n and Q_n satisfy the recurrences

$$P_{n+1}(z) - zP_n(z) = \sum_{j=0}^n \binom{n}{j} \kappa_X(j+1) P_{n-j}(z) \quad (7.18)$$

and

$$Q_{n+1}(z) - zQ_n(z) = - \sum_{j=0}^n \binom{n}{j} \kappa_X(j+1) Q_{n-j}(z). \quad (7.19)$$

Proof. Let X_1 and X_2 be two independent random variables distributed as X and let

$$\begin{aligned} f(z) &= \mathbb{E}[X_1(X_1 + Y + z + X_2)^n - X_1(z + X_2)^n] \\ &= \sum_{j=0}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + Y)^j (z + X_2)^{n-j}] - \mathbb{E}X_1(z + X_2)^n \\ &= \sum_{j=1}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + Y)^j] \mathbb{E}(z + X_2)^{n-j}. \end{aligned}$$

Theorem 3.3 in [10] shows that the cumulants satisfy

$$\kappa_X(p) = \mathbb{E}[X(X + Y)^{p-1}] \quad \text{for } p \geq 1. \quad (7.20)$$

Therefore

$$f(z) = \sum_{j=1}^n \binom{n}{j} \kappa_X(j+1) P_{n-j}(z). \quad (7.21)$$

The function $f(z)$ may also be expressed as

$$f(z) = \sum_{j=0}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + z)^{n-j}(Y + X_2)^j] - \mathbb{E}X_1 \mathbb{E}(z + X_2)^n. \quad (7.22)$$

The relation $\mathbb{E}(Y + X_2)^j = \delta_j$ holds since X_2 and Y are conjugate random variables. This reduces the previous expression for f to

$$f(z) = \mathbb{E}X_1(X_1 + z)^n - \mathbb{E}X_1 P_n(z). \quad (7.23)$$

This can be simplified using

$$\begin{aligned} \mathbb{E}X_1(X_1 + z)^n &= \mathbb{E}(X_1 + z)^{n+1} - z\mathbb{E}(X_1 + z)^n \\ &= P_{n+1}(z) - zP_n(z). \end{aligned}$$

The function f has been expressed as

$$f(z) = P_{n+1}(z) - (z + \kappa_X(1))P_n(z) \quad (7.24)$$

using $\mathbb{E}(X) = \kappa_X(1)$. The recurrence for P_n comes by comparing (7.21) and (7.24).

The second identity is obtained by replacing X and Y and remarking that $\kappa_X(p) = -\kappa_Y(p)$ and $\mathbb{E}(X + Y) = 0$, since X and Y are conjugate random variables. \square

Theorem 7.10. *The hypergeometric Bernoulli polynomials $B_n^{(a,b)}(z)$ and the companion family $C_n^{(a,b)}(z)$ defined by*

$$B_n^{(a,b)}(z) = \mathbb{E}(z + \mathfrak{Z}_{a,b})^n \quad \text{and} \quad C_n^{(a,b)}(z) = \mathbb{E}(z + \mathfrak{B}_{a,b})^n \quad (7.25)$$

satisfy the recurrences

$$B_{n+1}^{(a,b)}(z) - zB_n^{(a,b)}(z) = \sum_{j=0}^n \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) B_{n-j}^{(a,b)}(z) \quad (7.26)$$

and

$$C_{n+1}^{(a,b)}(z) - zC_n^{(a,b)}(z) = - \sum_{j=0}^n \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) C_{n-j}^{(a,b)}(z). \quad (7.27)$$

Proof. The result now follows from Theorem 7.9 and the cumulants for the beta distribution given in Example 6.8. \square

Our last result provides a probabilistic approach to the linear recurrences for the hypergeometric zeta function.

For a random variable X , the moments $\mathbb{E}X^n$ and its cumulants $\kappa_X(n)$ satisfy the relation (6.22). This is now used to produce a linear recurrence for the hypergeometric zeta function.

Theorem 7.11. *The hypergeometric zeta function $\zeta_{a,b}^H$ satisfies*

$$(n-1)! \sum_{j=2}^n \frac{B_{n-j}^{(a,b)}}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B_{n-1}^{(a,b)} + B_n^{(a,b)}. \quad (7.28)$$

Proof. Use the identity (6.22) to the random variable $\mathfrak{Z}_{a,b}$. Its moments are the hypergeometric Bernoulli numbers

$$\mathbb{E}\mathfrak{Z}_{a,b}^p = B_p^{(a,b)} \quad (7.29)$$

and its cumulants are

$$\kappa_{\mathfrak{Z}_{a,b}}(n) = \begin{cases} (n-1)! \zeta_{a,b}^H(n) & \text{for } n \geq 2, \\ -\frac{a}{a+b} & \text{for } n = 1, \end{cases} \quad (7.30)$$

since $\mathfrak{Z}_{a,b}$ and $\mathfrak{B}_{a,b}$ are conjugate random variables. A second proof is obtained by letting $z = 0$ in (7.26). \square

Note. Surprisingly, the two linear recurrences for $\zeta_{a,b}^H$, given in Theorem 3.2 and in Theorem 7.11 are different. For example, choosing $a = 5, b = 3$ these produce for $n = 3$ the relations

$$2\zeta_{5,3}^H(3) + \frac{5}{4}\zeta_{5,3}^H(2) + \frac{1}{32} = 0,$$

$$2\zeta_{5,3}^H(3) - \frac{5}{4}\zeta_{5,3}^H(2) - \frac{13}{384} = 0.$$

Acknowledgments

The work of the third author was partially supported by NSF-DMS 0070567. The first author is an undergraduate student and the second one is a graduate student at Tulane University, both partially supported by the same grant.

References

- [1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Dover, New York, 1972).
- [2] A. Actor and I. Bender, The zeta function constructed from the zeros of the Bessel function, *J. Phys. A* **29** (1996) 6555–6580.
- [3] T. Amdeberhan, V. Moll and C. Vignat, A probabilistic interpretation of a sequence related to the Narayana polynomials, *Online J. Anal. Combin.* **8** (2013) 1–25.
- [4] R. P. Boas, *Entire Functions* (Academic Press, New York, 1954).
- [5] L. Carlitz, The Staudt–Clausen theorem, *Math. Mag.* **34** (1961) 131–146.
- [6] R. E. Crandall, On the quantum zeta function, *J. Phys. A* **29** (1996) 6795–6816.

- [7] V. De Angelis, Stirling's series revisited, *Amer. Math. Monthly* **116** (2009) 839–843.
- [8] K. Dilcher, Bernoulli numbers and confluent hypergeometric functions, in *Number Theory for the Millenium I*, eds. B. C. Berndt, N. Boston, H. G. Diamond, A. J. Hildebrand and W. Phillipp (AK Peters, Natick, MA, 2002), pp. 343–363.
- [9] K. Dilcher and L. Malloch, Arithmetic properties of Bernoulli–Padé numbers and polynomials, *J. Number Theory* **92** (2002) 330–347.
- [10] E. Di Nardo, P. Petrullo and D. Senato, Cumulants and convolutions via Abel polynomials, *European J. Combin.* **31** (2010) 1792–1804.
- [11] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Lecture Notes in Physics (Springer-Verlag, New York, 1995).
- [12] A. Hassen and H. Nguyen, Hypergeometric Bernoulli polynomials and Appell sequences, *Int. J. Number Theory* **4** (2008) 767–774.
- [13] ———, Hypergeometric zeta functions, *Int. J. Number Theory* **6** (2010) 99–126.
- [14] J. Hawkins, On a zeta function associated with Bessel's equation, Ph.D. thesis, University of Illinois (1983).
- [15] F. T. Howard, A sequence of numbers related to the exponential function, *Math. Comp.* **34** (1967) 599–615.
- [16] ———, Some sequences of numbers related to the exponential function, *Math. Comp.* **34** (1967) 701–716.
- [17] K. Ireland and M. Rosen, *A Classical Introduction to Number Theory*, 2nd edn. (Springer-Verlag, New York, 1990).
- [18] M. Lassalle, Two integer sequences related to Catalan numbers, *J. Combin. Theory Ser. A* **119** (2012) 923–935.
- [19] D. W. Lozier, The NIST digital library of mathematical functions project, *Ann. Math. Artif. Intell.* **38**(1–3) (2003) 105–119.
- [20] P. J. Smith, A recursive formulation of the old problem of obtaining moments from cumulants and viceversa, *Amer. Statist.* **49** (1995) 217–218.
- [21] K. Stolarski, Singularities of Bessel-zeta functions and Hawkins' polynomials, *Mathematika* **32** (1985) 96–103.