BASIC PRE-CALCULUS FACTS

Part 1. Algebraic Operations

1. Basic Formulae

$$
\begin{cases}
ab+ac=a(b+c) & \text{Distribution Law} \\
\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd} & \\
\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} & \text{Treat it as } (a+b)\,c^{-1} \\
\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\left(\frac{c}{d}\right)^{-1}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc} & \\
a\left(\frac{b}{c}\right)=\frac{ab}{c} & \\
\frac{\frac{a}{b}}{c}=\frac{a}{bc},\ \frac{a}{\frac{b}{c}}=\frac{ac}{b} & \\
\frac{ab}{cb}=\frac{a}{c} \text{ if } b\neq 0 & \\
\frac{a-b}{c-d}=\frac{(-1)(b-a)}{(-1)(d-c)}=\frac{b-a}{c-d} & \\
a^2-b^2=(a+b)(a-b) & \\
\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)=x-y & \text{Treat it as } a=\sqrt{x}, b=\sqrt{y} \text{ in the last formula}
\end{cases}
$$

2. Erros to Avoid

$$
\begin{cases}
\frac{a}{b+c}\neq\frac{a}{b}+\frac{a}{c} & \\
\sqrt{x+y}\neq\sqrt{x}+\sqrt{y} & \\
(x+y)^2=x^2+y^2+2xy\neq x^2+y^2 & \text{In general, } (x+y)^n\neq x^n+y^n \\
a-b(x-1)=a-bx+b\neq a-bx-b & \\
\frac{\frac{x}{a}}{b}=\frac{x}{ab}\neq\frac{bx}{a} & \\
\sqrt{-x^2+a^2}\neq-\sqrt{x^2-a^2} & \\
\frac{a+bx}{a}=1+\frac{bx}{a}\neq 1+bx & \\
\sqrt{x^2}\neq x & \sqrt{x^2}=|x|=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}
\end{cases}
$$

Part 2. Numbers, Inequalities and Absolute Values

3. Numbers

$$
\begin{cases}
\text{Integers} & \cdots,-3,-2,-1,0,1,2,\cdots \\
\text{Rational Numbers } r=\frac{m}{n} & \frac{1}{2},-\frac{3}{7},0.17=\frac{17}{100},3=\frac{3}{1} \\
\text{Irrational Numbers} & \sqrt{3},\pi
\end{cases}
$$

4. Intevals

$$\begin{cases} (a,b) = \{x|a < x < b\} \\ [a,b] = \{x|a \le x \le b\} \\ [a,b) = \{x|a \le x < b\} \\ (a,b] = \{x|a < x \le b\} \\ (a,\infty) = \{x|x > a\} \\ [a,\infty) = \{x|x \ge a\} \\ (-\infty,b) = \{x|x < b\} \\ (-\infty,b] = \{x|x \le b\} \\ (-\infty,\infty) = \mathbb{R} & \text{SET OF ALL REAL NUMBERS} \end{cases}$$

5. Inequalities

<u>Rules:</u>

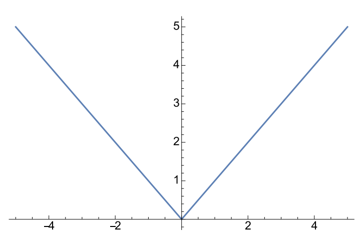
$$\begin{cases} a < b \Rightarrow a + c < b + c & (1) \\ a < b, c < d \Rightarrow a + c < b + d & (2) \\ a < b, c > 0 \Rightarrow ac < bd & (3) \\ a < b, c < 0 \Rightarrow ac > bd & (4) \\ 0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b} & (5) \end{cases}$$

Example. Solve $1 + x < 7x + 5$

$$\begin{aligned} & & 1 + x &< 7x + 5 \\ (1)\, c = -1 & \Rightarrow & x &< 7x + 4 \\ (1)\, c = -7x & \Rightarrow & -6x &< 4 \\ (4)\, c = -6 & \Rightarrow & x &> -\frac{2}{3}. \end{aligned}$$

6. Absolute Value

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$



Fact.

$$\begin{cases} |x| = a & \textit{if and only if } x = \pm a \\ |x| < a & \textit{if and only if } -a < x < a \\ |x| > a & \textit{if and only if } x > a \textit{ or } x < -a \end{cases}$$

7. Quadratic Forms

$$ax^2+bx+c=a\left(x-x_1\right)\left(x-x_2\right), \text{ WHERE } \begin{cases} x_1= & \frac{-b+\sqrt{b^2-4ac}}{2a} \\ x_2= & \frac{-b-\sqrt{b^2-4ac}}{2a} \end{cases}.$$

Example. Solve $x^3+3x^2-4x>0$

$$x\left(x-1\right)\left(x+4\right)>0$$

Interval	x	$x-1$	$x+4$	$x\left(x-1\right)\left(x+4\right)$
$x<-4$	-	-	-	-
$-4<x<0$	-	-	$+$	$+$
$0<x<1$	$+$	-	$+$	-
$x>1$	$+$	$+$	$+$	$+$

$$\Rightarrow \{x|-4<x<0 \text{ or } x>1\}=(-4,0)\cup(1,\infty)$$

To determine the table above, an easy way is substitution. For example, consider the second row, i.e., the interval $x<-4$. Then, we could substitute any number that is smaller than -4, say $x=-5$. Then,

$$x=-5<0,\ x-1=-6<0,\ x+4=-1<0.$$

Remark. $x\left(x-1\right)\left(x+4\right)=0 \Rightarrow x=-4,0,1.$

8. Coordinate Geometry

Distance Formula. For $P_1\left(x_1,y_1\right)$ and $P_2\left(x_2,y_2\right)$, the distance between is

$$\left|P_1P_2\right|=\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}.$$

Slope. For $P_1\left(x_1,y_1\right)$ and $P_2\left(x_2,y_2\right)$, the slope of nonvertical line passing through P_1 and P_2 is

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}.$$

Line Equation. $\begin{cases} \text{Point-Slope Form} & y-y_1=m\left(x-x_1\right) \begin{cases} \text{slope} & m \\ \text{point} & \left(x_1,y_1\right) \end{cases} \\ \text{Slope-Intercept Form} & y=mx+b \begin{cases} \text{slope} & m \\ y-\text{intercept} & b \end{cases} \end{cases}.$

Parallel and Perpendicular. For two lines with slopes m_1 and m_2,

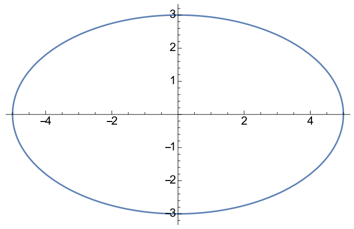
$$\begin{cases} \text{Parallel} & \text{if an only if } m_1=m_2 \\ \text{Perpendicular} & \text{if an only if } m_1m_2=-1 \end{cases}.$$

Circle. Center (h,k) radius r:

$$\left(x-h\right)^2+\left(y-k\right)^2=r^2.$$

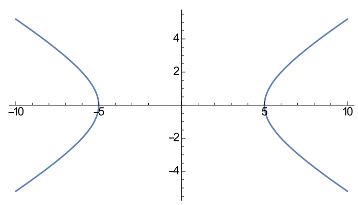
Ellipse in Standard Position.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Hyperbola in Standard Position.

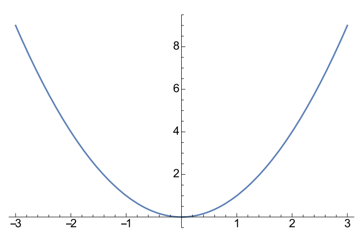
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



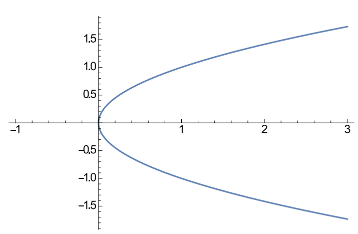
Parabolas.

$$y = ax^2 + bx + c \text{ or } x = ay^2 + by + c$$

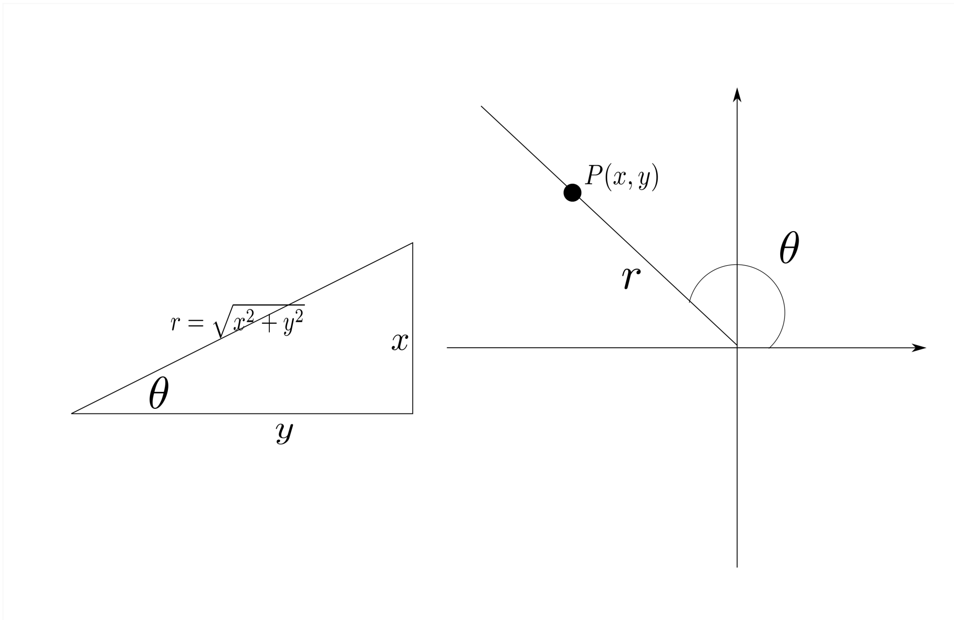
Example. $y = x^2$



Example. $x = y^2$



9. Trigonometry



$$\begin{cases}\sin\theta=\frac{y}{r},\cos\theta=\frac{x}{r},\tan\theta=\frac{y}{x},\csc\theta=\frac{r}{y},\sec\theta=\frac{r}{x},\cot\theta=\frac{x}{y}\\ \csc\theta=\frac{1}{\sin\theta},\sec\theta=\frac{1}{\cos\theta},\cot\theta=\frac{1}{\tan\theta},\tan\theta=\frac{\sin\theta}{\cos\theta},\cot\theta=\frac{\cos\theta}{\sin\theta}\\ \sin^2\theta+\cos^2\theta=1,\tan^2\theta+1=\sec^2\theta\\ \sin(-\theta)=-\sin\theta,\cos(-\theta)=\cos\theta\\ |\sin\theta|\le 1,|\cos\theta|\le 1\\ \sin(\theta+2\pi)=\sin\theta,\cos(\theta+2\pi)=\cos\theta,\tan(\theta+\pi)=\tan\theta,\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta\end{cases}$$

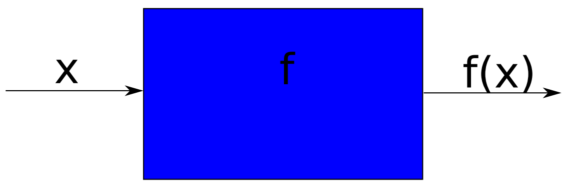
<u>Special Values.</u>

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

1.1 FOUR WAYS TO REPRESENT A FUNCTION

Definition. A function f is a rule that assigns to each element x in a set D EXACTLY one element, called $f(x)$, in a set E.

From the defintion, we could treat a function as a **machine** as the following picture.



Four ways to represent a function are:

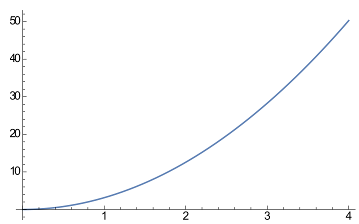
(1) Verbally: Description in Words. *The area of a circle is the square of the radius multiplied by the constant* π*.*

(2) Numerically: Table of Values.

Radius/m	1	2	3	4
Area/m^2	3.14	12.56	28.26	50.24

(3)Visually. Graph, the Set of ordered pair

$$\{(x, f(x)) \,|x \in D\}.$$



(4)**Algebraically**. Explicit Formula. $A(r) = \pi r^2$.

Definition. [Teminology] The set D is called the domain, and the set of all possible values of $f(x)$ is called the range.

Example. (1)For linear function $f(x) = x - 1$, the domain is $\mathbb{R}$.

(2)The denominator of a function cannot be zero. For $g(x) = \frac{x^2 - x}{x}$, the domain is $\{x|x \neq 0\} = (-\infty, 0) \cup (0, \infty)$. Although,

$$g(x) = \frac{x(x-1)}{x} = x - 1,$$

it does not mean that $f(x)$ and $g(x)$ are the same. In fact, $f(x) \neq g(x)$ because they have different domains.

(3)The expression under the square root (or even root) must be nonnegative. For $h(x) = \sqrt{x+2}$, the domain is $[-2, \infty)$.

Definition. (1)Functions defined by different formulae in different parts of their domains are called piecewise (defined) functions.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

$$f(x) = \begin{cases} 1-x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}.$$

(2)A function f is called an even function if $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the $y-$axis.

$$|x|, \cos x, x^2, x^4, \ldots$$

A function f is called an odd function if $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to about the origin.

$$x, -x, \sin x, x^3, \ldots$$

A function f is called increasing on an interval I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

A function f is called decreasing on an interval I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

Example. $f(x) = \sin x$. It is increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

1.2 MATHEMATICAL MODELS: A CATALOG OF ESSENTIAL FUNCTIONS

Linear Function: (graph is a straight line)

$$y = f(x) = mx + b$$

Polynomials:

$$y = P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer and $a_n \neq 0$. n is called the degree and $a_0, a_1, \ldots, a_n$ are called coefficients.

•$n = 0$: $P(x) = a_0$, constant function.
•$n = 1$: $P(x) = a_1 x + a_0$, linear function.
•$n = 2$: $P(x) = a_2 x^2 + a_1 x + a_0$ is called quadratic function.
•$n = 3$: $P(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is called cubic function.

Example. Find an expression for a cubic function f if $f(1) = 6$ and $f(-1) = f(0) = f(2) = 0$.

Solution. By definition of a cubic function, we could assume

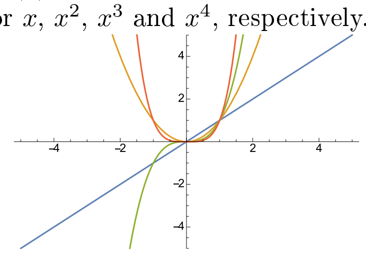
$$f(x) = ax^3 + bx^2 + cx + d.$$

Then,

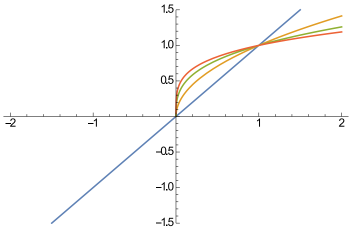
$$\begin{cases} f(1) = a + b + c + d = 6 \\ f(-1) = -a + b - c + d = 0 \\ f(0) = d = 0 \\ f(2) = 8a + 4b + 2c + d = 0 \end{cases} \Rightarrow \begin{cases} a = -3 \\ b = 3 \\ c = 6 \\ d = 0 \end{cases}$$

Power Functions. $f(x) = x^a$ for a constant a.

(1)If $a = n$, a positive integer, $f(x)$ is a polynomial. The following graphs are for x, x^2, x^3 and x^4, respectively.

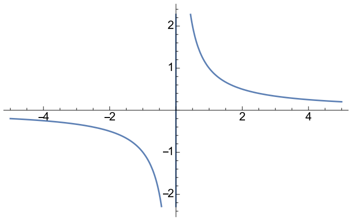


(2)If $a = 1/n$ for a positive integer n, then f is called a root function.



$$dom\,(f) = \begin{cases} \mathbb{R} & \text{if } n \text{ is odd} \\ [0,\infty) & \text{if } n \text{ is even} \end{cases}$$

(3)If $a = -1$, then f is called the reciprocal function. $dom\,(f) = \{x|x \neq 0\}$.



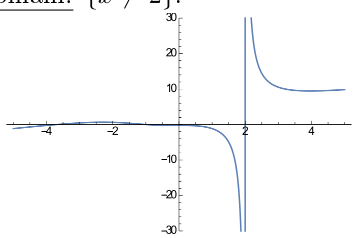
Rational Functions. Ratio of two polynomials

$$f\,(x) = \frac{P\,(x)}{Q\,(x)} \text{ and } dom\,(f) = \{x|Q\,(x) \neq 0\}\,.$$

Example.

$$f\,(x) = \frac{x^4 + 4x^3 + x^2 + 2}{x^3 - 8}.$$

Domain: $\{x \neq 2\}$.



Algebraic Functions. Any basic operations $(+, -, \times, \div, \bullet^k)$ of polynomials. For example

$$\sqrt[3]{x^2+2},\ x^2(\sqrt{x}-1) - \frac{\sqrt[3]{x}-\sqrt{x}}{x^2+1}.$$

[Questions]What are the domains?

1.3 NEW FUNCTIONS FROM OLD FUNCTIONS

1. Transformation

TABLE1. Vertical and Horizontal Shifts. Suppose $c > 0$,

New Function	Manipulation from $y = f(x)$
$y = f(x) + c$	Shift the graph of $y = f(x)$ a distance c units *upward.*
$y = f(x) - c$	Shift the graph of $y = f(x)$ a distance c units *downward.*
$y = f(x + c)$	Shift the graph of $y = f(x)$ a distance c units *to the left.*
$y = f(x - c)$	Shift the graph of $y = f(x)$ a distance c units *to the* right.

Example. The following picture shows the graph of x^2(blue), x^2+2 (yellow), x^2-2 (green), $(x+1)^2$ (red) and $(x-1)^2$ (purple)

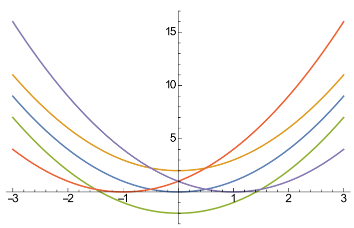
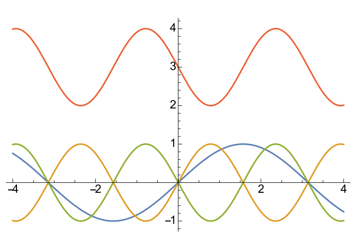


TABLE2. Vertical and Horizontal Stretching and Reflecting.
Suppose $c > 1$,

New Function	Manipulation from $y = f(x)$
$y = cf(x)$	*Stretch* the graph of $y = f(x)$ *vertically* by a factor of c.
$y = (1/c) f(x)$	*Shrink* the graph of $y = f(x)$ *vertically* by a factor of c.
$y = f(cx)$	*Shrink* the graph of $y = f(x)$ *horizontally* by a factor of c.
$y = f(x/c)$	*Stretch* the graph of $y = f(x)$ *horizontally* by a factor of c.
$y = -f(x)$	*Reflect* the graph of $y = f(x)$ about the $x-$*axis.*
$y = f(-x)$	*Reflect* the graph of $y = f(x)$ about the $y-$*axis.*

Example. $y = 3 - \sin(2x)$
The right order would be:

$$\sin x \mapsto \sin(2x) \mapsto -\sin(2x) \mapsto -\sin(2x) + 3$$



2. Combinatoins

Basic Operations.

$$f+g, f-g, fg, \frac{f}{g}.$$

Naturally, the domain will be the intersection of $dom\,(f)$ and $dom\,(g)$. In particular, for the quotient $\frac{f}{g}$, besides the intersection, we also need to exclude zeros of g.

Example. Consider $f\,(x) = \sqrt{x+1}$ and $g\,(x) = x^2 - 1$. Obviously, $dom\,(f) = [-1, \infty)$ and $dom\,(g) = \mathbb{R}$. Thus, their intersection would be $[-1, \infty)$. Therefore

$$dom\,(f+g) = dom\,(f-g) = dom\,(fg) = [-1, \infty).$$

And

$$dom\left(\frac{f}{g}\right) = (-1, 1) \cup (1, \infty)\,,$$

since the zeros of g are ± 1, which should be excluded from the domain.

Composition.

$$f \circ g\,(x) = f\,(g\,(x))\,.$$

Example. Let $f\,(x) = \sqrt{x}$ and $g\,(x) = \sqrt{2-x}$, then $dom\,(f) = [0, \infty)$ and $dom\,(g) = (-\infty, 2]$.

(1)$f \circ g\,(x) = f\,(g\,(x)) = \sqrt{g\,(x)} = \sqrt{\sqrt{2-x}} = (2-x)^{\frac{1}{4}}$, and $dom\,(f \circ g) = (-\infty, 2]$.

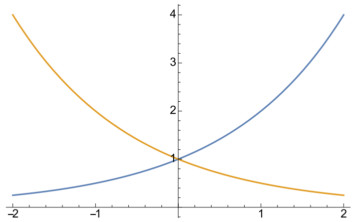
(2)$g \circ f\,(x) = g\,(f\,(x)) = \sqrt{2-\sqrt{x}}$ and $dom\,(g \circ f) = [0, 4]$.

(3)$f \circ f \circ f\,(x) = x^{\frac{1}{8}}$, $dom\,(f \circ f \circ f) = [0, \infty)$.

1.5 EXPONENTIAL FUNCTIONS

$$y = f(x) = a^x, \ (a > 0, a \neq 1).$$

Example. (1) $y = 2^x$(2) $y = \left(\frac{1}{2}\right)^x$: The blue curve is the graph of $y = 2^x$ and the yellow one is $y = \left(\frac{1}{2}\right)^x$



In general, for $y = a^x$, where $a > 0$ and $a \neq 1$.
(1)Domain: $(-\infty, \infty)$
(2)Range: $(0, \infty)$
(3)If $a > 1$, then $y = a^x$ is increasing while if $a < 1$, it is decreasing.
(4)The graphs of $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$ are symmetric with respect to the $y-$axis.
Reason:

$$a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x.$$

Laws.

$$\begin{cases} a^x a^y = a^{x+y} \\ a^{x-y} = \frac{a^x}{a^y} \\ (a^x)^y = a^{xy} & \left(2^2\right)^2 = 4^2 = 16 = 2^4 \\ (ab)^x = a^x b^x \end{cases}$$

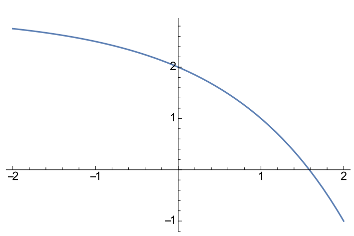
Example.

$$\sqrt[3]{a^9} = a^3, \text{ since } \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

<u>Important Facts</u>.

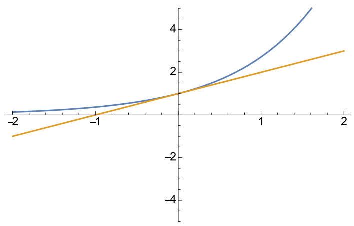
$$a^{-x} = \frac{1}{a^x} \text{ and } a^{\frac{1}{x}} = \sqrt[x]{a}.$$

Example. $y = 3 - 2^x$.



Natural Exponential& Number e. For the following graph the slope of the tangent line(yellow) is 1. Such choice of a is denoted by e. And $y = e^x$ is called natural exponential.

$$e = 2.172818\cdots$$



1.6 INVERSE FUNCTIONS AND LOGARITHMS

Definition. A function is said to be one-to-one if it never takes on the same value more than once; that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

Or if $f(x_1) = f(x_2)$, then necessarily, $x_1 = x_2$.

Horizontal Line Test. A function is one-to-one if and only if no horizontal line intersects its graph more that once.

Example. (1)$f(x) = x$, $g(x) = x^3$ are one-to-one
(2)$h(x) = \sin x$, $l(x) = x^2$ are not one-to-one

Definition. Let f be a one-to-one function with domain A and range B. Then its inverse function, denoted by f^{-1} , has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y.$$

Example. $f(x) = x^3$, $dom(f) = \mathbb{R}$ and $range(f) = \mathbb{R}$. $f^{-1}(x) = x^{\frac{1}{3}}$. If $y = x^3$, then $f^{-1}(y) = f^{-1}(x^3) = x$.

Cancellation Equations:

$$\begin{cases} f^{-1}(f(x)) = x & \text{for every } x \text{ in } dom(f) \\ f(f^{-1}(x)) = x & \text{for every } x \text{ in } range(f) = dom(f^{-1}). \end{cases}$$

Steps for finding inverse function.

STEP0: Determine that $f(x)$ is one-to-one
STEP1: Write $y = f(x)$
STEP2: Solve the equation for x in terms of y
STEP3: Interchange x and y to get $y = f^{-1}(x)$

Example. $f(x) = x^3$
(1)$f(x)$ is one-to-one by horizontal line test.
(2)Write $y = f(x) = x^3$.
(3)$x = y^{\frac{1}{3}}$.
(4)Interchange to get $y = f^{-1}(x) = x^{\frac{1}{3}}$.

Logarithmic Functions

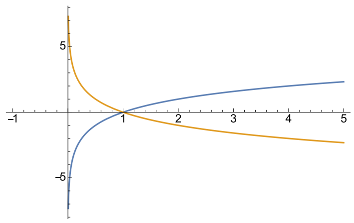
Definition. The inverse function of exponential function is called logarithmic function. In concrete, if

$$f(x) = a^x \Rightarrow f^{-1}(x) = \log_a x.$$

This is called the logarithmic function with base a. Namely,

$$y = a^x \Leftrightarrow x = \log_a y.$$

Example. $y = \log_2 x$ (blue) and $y = \log_{\frac{1}{2}} x$ (yellow).



Example.

$$\log_{10} 0.0001 = -4$$

Properties&Laws.
(1)Domain: $(0, \infty)$; Range $(-\infty, \infty)$.
(2)If $a > 1$, $\log_a x$ is increasing and if $0 < a < 1$, $\log_a x$ is decreasing.
(3)By the properties of inverse functions,

$$\begin{cases} \log_a (a^x) = x & \text{for any } x \in \mathbb{R} \\ a^{\log_a x} = x & \text{for any } x > 0 \end{cases}$$

(4)Laws:

$$\begin{cases} \log_a (xy) = \log_a x + \log_a y \\ \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \\ \log_a x^r = r \log_a x \end{cases}$$

Remark.

$$\log_a (x + y) \neq \log_a x + \log_a y.$$

Example.

$$\log_2 80 - \log_2 5 = \log_2 \left(\frac{80}{5}\right) = \log_2 16 = \log_2 2^4 = 4.$$

Natural Log: $a = e$.

$$\log_e x = \ln x.$$

Example. Solve for x if $\ln x + \ln (x - 1) = 1$.
Solution. By laws of logarithmics,

$$\begin{aligned} \ln x + \ln (x-1) = 1 \quad &\Leftrightarrow \quad \ln [x(x-1)] = 1 \\ &\Leftrightarrow \quad x(x-1) = e \\ &\Leftrightarrow \quad x^2 - x - e = 0 \end{aligned}$$

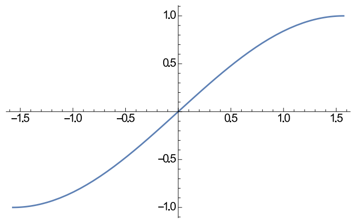
Now by the formulae of quadratic form,

$$x = \frac{1 \pm \sqrt{1 + 4e}}{2}.$$

Change of Base Formula:

$$\log_a x = \frac{\log_b x}{\log_b a}, \text{ in particular, } \log_a x = \frac{\ln x}{\ln a}.$$

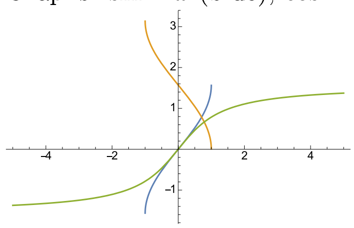
Inverse Trogonometric Functions



The inverse function of $\sin x$ is denoted by $\sin^{-1} x = \arcsin x$.

Functions	Definition	Domain	Range
$y = \sin^{-1} x$	$\sin x = y$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$\cos y = x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	$\tan y = x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Graphs: $\sin^{-1} x$ (blue), $\cos^{-1} x$ (yellow), $\tan^{-1} x$ (green).



Example. (1) $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

(2)$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$.

(3)$\cos^{-1}(-1) = \pi$

2.2 THE LIMIT OF A FUNCTION

Example. $f(x) = x - 1$ near $x = 2$

x	1.9	1.99	1.999	2.1	2.01	2.001
$f(x)$	0.9	0.99	0.999	1.1	1.01	1.001

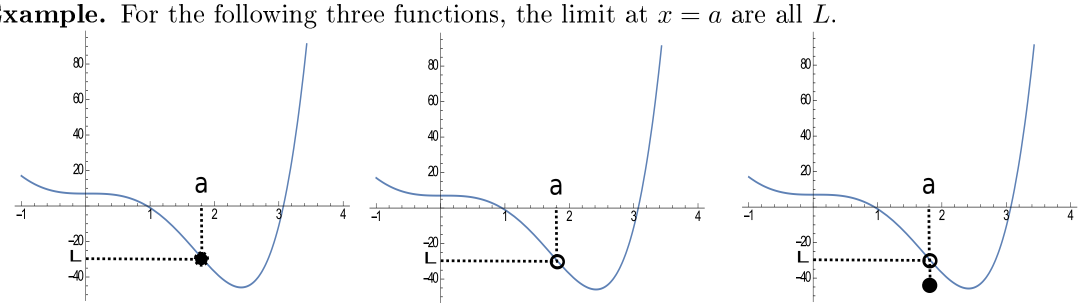
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} (x-1) = 1.$$

Definition. We denote that

$$\lim_{x\to a} f(x) = L,$$

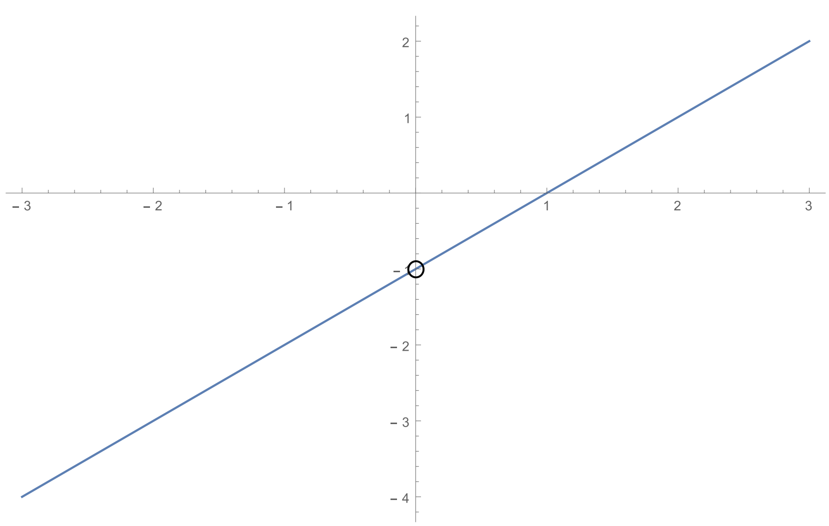
if the value $f(x)$ can be made arbitarily close to L when letting x sufficiently close to a but not equal to a.

Example. For the following three functions, the limit at $x = a$ are all L.



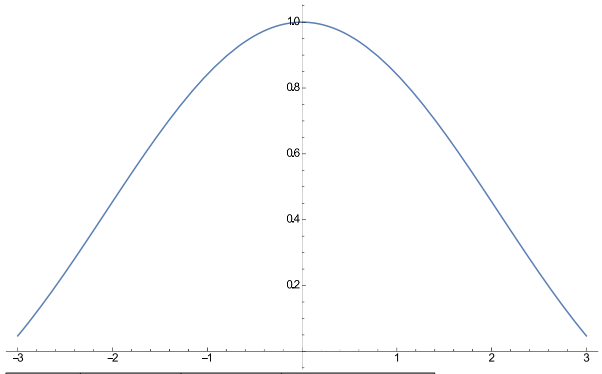
Example. $f(x) = \frac{x^2 - x}{x} = x - 1$ where $x \neq 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2 - x}{x} = -1.$$



Example.

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

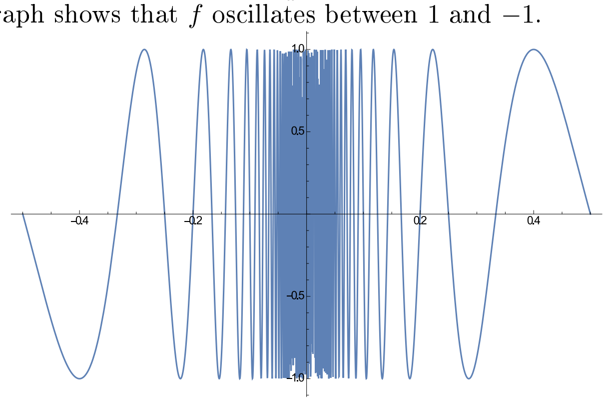


x	1	0.1	0.001
$\frac{\sin x}{x}$	0.8414	0.9983	0.99999983

Example.

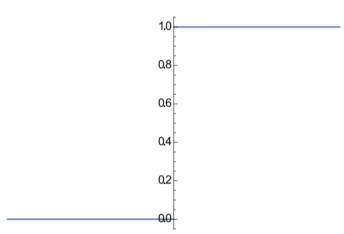
$$\lim_{x\to 0}\sin\frac{\pi}{x}\text{ does not exist (DNE).}$$

Although, for $f(x)=\sin\frac{\pi}{x}$, $f(1/n)=\sin(n\pi)=0$ if n is an integer, the actual graph shows that f oscillates between 1 and -1.



Example. The Heaviside function

$$H(t)=\begin{cases}0 & t<0\\ 1 & t\ge 0\end{cases}$$



Obviously, as $t\to 0$ for $t<0$, $H(t)\to 0$ while as $t\to 0$ for $t>0$, $H(t)\to 1$. Thus, $\lim_{t\to 0}H(t)$ DNE.

Definition. $\lim_{x\to a^-} f(x) (\lim_{x\to a^+} f(x))$ stands for the limit of $f(x)$ when x approaches to a from the left (right).

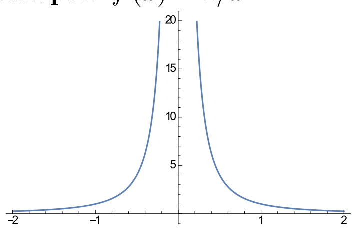
Example.

$$\lim_{t\to 0^-} H(t) = 0 \text{ and } \lim_{t\to 0^+} H(t) = 1.$$

Remark.

$$\lim_{x\to a} f(x) = L \Leftrightarrow \begin{cases} \lim_{x\to a^-} f(x) = L \\ \lim_{x\to a^+} f(x) = L \end{cases}$$

Example. $f(x) = 1/x^2$



$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Definition. [Infinite Limits/Vertical Asympototes]

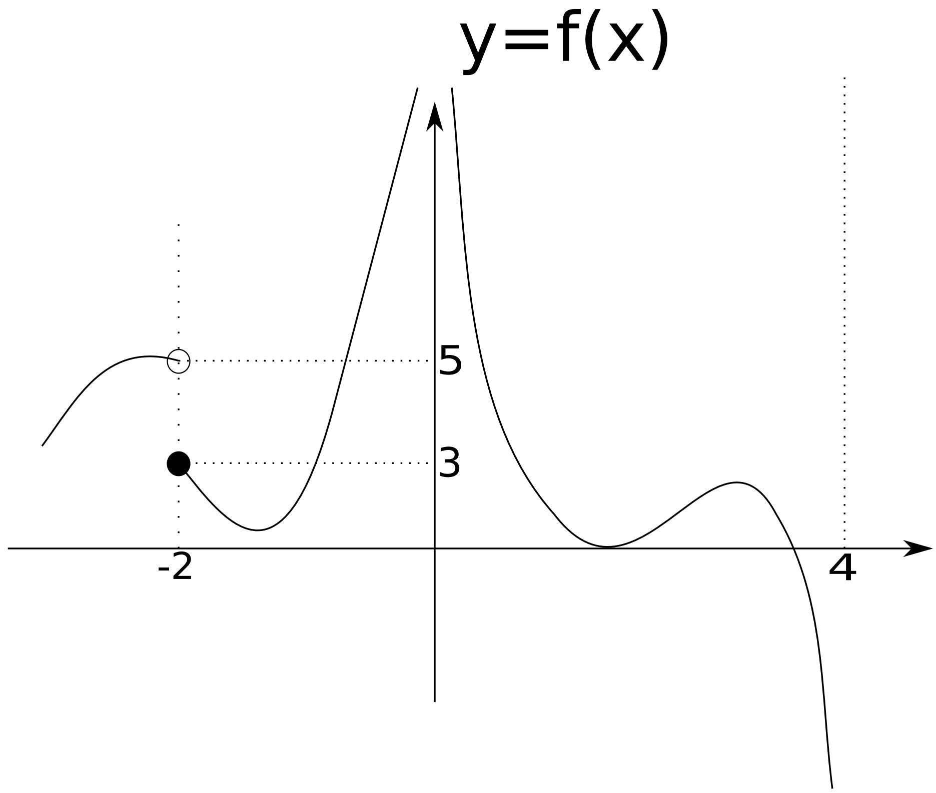
$$\lim_{x\to a} f(x) = \infty, \lim_{x\to a^-} f(x) = \infty, \lim_{x\to a^+} f(x) = \infty, \lim_{x\to a} f(x) = -\infty, \lim_{x\to a^-} f(x) = -\infty, \lim_{x\to a^+} f(x) = -\infty.$$

Example. $f(x) = \frac{1}{x}$.

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

Example. $f(x) = \ln x$.

$$\lim_{x\to 0^+} \ln x = -\infty.$$



Example.

$$\lim_{x\to-2^-} f(x) = 5, \lim_{x\to-2^+} f(x) = 3, \lim_{x\to 0} f(x) = \infty, \lim_{x\to 4^-} f(x) = -\infty$$

2.3 CALCULATING LIMITS USING THE LIMIT LAWS

Theorem. *[Laws] Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then*

$$\begin{cases} \lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\ \lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) \\ \lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) \\ \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} & \text{if } \lim_{x\to a} g(x)\neq 0 \end{cases}$$

Corollary. *If n is a positive integer and $\lim_{x\to a} f(x)$ exists,*

$$\begin{cases} \lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n & , \\ \lim_{x\to a}[f(x)]^{\frac{1}{n}} = \left[\lim_{x\to a} f(x)\right]^{\frac{1}{n}} & , \text{ where assume } \lim_{x\to a} f(x) > 0 \text{for } n \text{even.} \end{cases}$$

Theorem. *If f is a polynomial or rational function with a in the domain of f, then $f(x)$ satisfies the Direct Substitution Property, i.e.,*

$$\lim_{x\to a} f(x) = f(a).$$

Example.

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt{x+4}}{3x^2+5x+4} &= \frac{\lim_{x\to 0}\sqrt{x+4}}{\lim_{x\to 0}(3x^2+5x+4)} \\ &= \frac{\sqrt{\lim_{x\to 0}(x+4)}}{3\lim_{x\to 0}x^2+5\lim_{x\to 0}x+\lim_{x\to 0}4} \\ &= \frac{\sqrt{4}}{4} \\ &= \frac{1}{2}. \end{aligned}$$

Method1. [Factorization]

$$\lim_{x\to 1}\frac{x-1}{x^2-1} = \lim_{x\to 1}\frac{x-1}{(x+1)(x-1)} = \lim_{x\to 1}\frac{1}{x+1} = \frac{1}{2}.$$

Remark. Recall that $\lim_{x\to 0}\frac{x^2-x}{x} = -1$.

Method2.[Conjugate]

$$\begin{aligned}
\lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4} &= \lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}\cdot\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}\\
&= \lim_{x\to-4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}\\
&= \lim_{x\to-4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}\\
&= \lim_{x\to-4}\frac{x-4}{\sqrt{x^2+9}+5}\\
&= -\frac{4}{5}
\end{aligned}$$

Method3.[Left & Right Limits]

Recall the following theorem that

$$\lim_{x\to a} f(x) = L \Leftrightarrow \begin{cases}\lim\limits_{x\to a} f(x) = L\\ \lim\limits_{x\to a} f(x) = L\end{cases}$$

Example.

$$\left.\begin{array}{r}\lim\limits_{x\to0^-}|x| = 0\\ \lim\limits_{x\to0^+}|x| = 0\end{array}\right\} \Rightarrow \lim_{x\to0}|x| = 0.$$

Example.

$$\left.\begin{array}{r}\lim\limits_{x\to0^-}\dfrac{|x|}{x} = \lim\limits_{x\to0^-}\dfrac{-x}{x} = -1\\ \lim\limits_{x\to0^+}\dfrac{|x|}{x} = \lim\limits_{x\to0^-}\dfrac{x}{x} = 1\end{array}\right\} \Rightarrow \lim_{x\to0}\frac{|x|}{x}\ \text{DNE}.$$

Theorem. *If when x is near a we have $f(x) \le g(x)$ and both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Theorem. *[Squeeze Theorem] If $f(x) \le g(x) \le h(x)$ when x is near a and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$.*

Method4.[Squeeze Theorem] Consider

$$\lim_{x\to0} x^2\sin\frac{1}{x}.$$

Note that

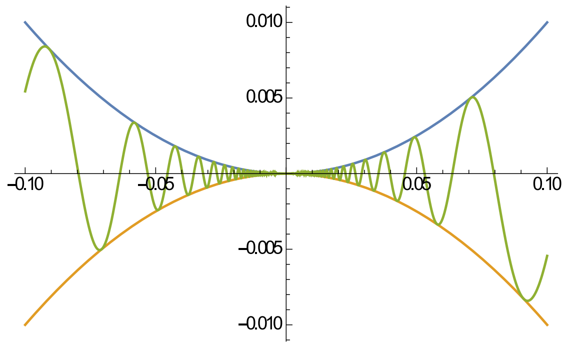
$$-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -x^2 \le x^2\sin\frac{1}{x} \le x^2,$$

and

$$\lim_{x\to0}\left(-x^2\right) = 0 = \lim_{x\to0} x^2.$$

By Squeeze Theorem,

$$\lim_{x\to0} x^2\sin\frac{1}{x} = 0.$$



0.010
0.005
−0.10
−0.05
0.05
0.10
−0.005
−0.010

2.5 CONTINUITY

Definition. A function f is continuous at a number a if

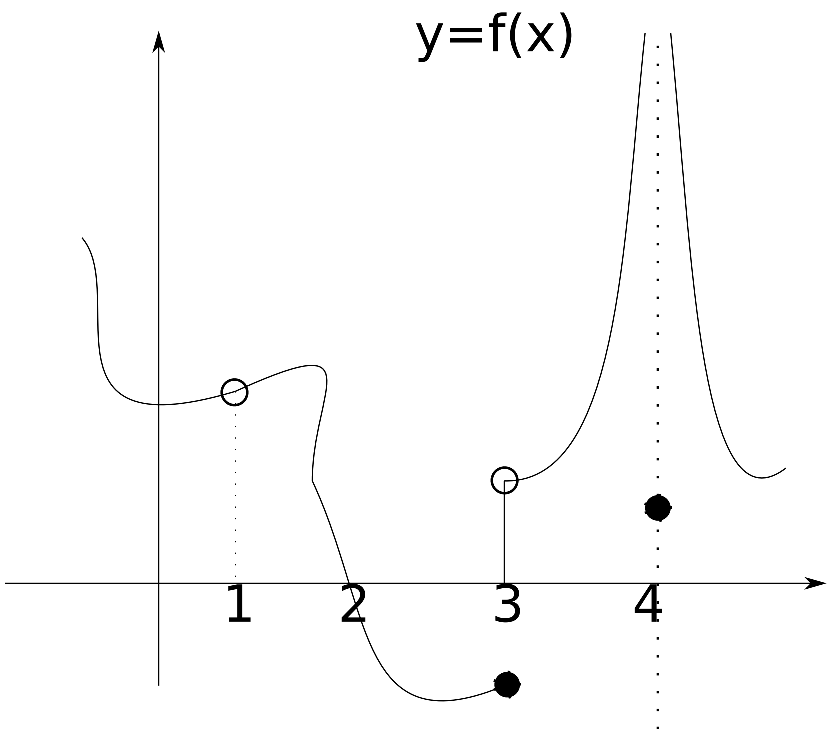
$$\lim_{x \to a} f(x) = f(a).$$

In concrete, it requires that

(1) $f(a)$ is defined, i.e. , $a \in dom(f)$

(2) $\lim_{x \to a} f(x)$ exists,

(3) $\lim_{x \to a} f(x) = f(a)$



Example.

$\underline{x = 1}$: $\lim_{x \to 1} f(x)$ exists but $f(1)$ is not defined.

$\underline{x = 2}$: $\lim_{x \to 2} f(x) = f(2) = 0$, so f is continuous at 1.

$\underline{x = 3}$: $\lim_{x \to 3} f(x)$ does not exists since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\underline{x = 4}$: $f(4)$ is defined but $\lim_{x \to 4} f(x) = \infty$.

Example. $f(x) = x - 1$, then , $\lim_{x \to 0} f(x) = -1 = f(0)$, so f is continoust at 0. On the other hand, $g(x) = \frac{x^2 - x}{x}$. $\lim_{x \to 0} g(x) = -1$, but $g(0)$ is not defined, so it is not continous at 0.

Definition. (1)f is continuous FROM THE RIGHT at a number a if

$$\lim_{x\to a^+} f(x) = f(a).$$

(2)f is continuous FROM THE LEFT at a number a if

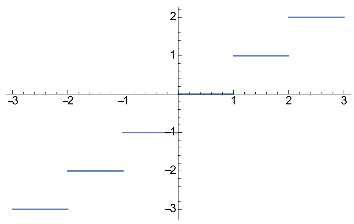
$$\lim_{x\to a^-} f(x) = f(a).$$

Example. The greatest integer function

$$[\![x]\!] := \text{largest integer } \le x.$$

For instance,

$$[\![\frac{1}{2}]\!] = 0, [\![1]\!] = 1, [\![-\frac{1}{2}]\!] = -1.$$



It is continous from the right, but not continous from the left at integers. For example,

$$\lim_{x\to 0^-} [\![x]\!] = -1 \neq 0 = [\![0]\!] = \lim_{x\to 0^+} [\![x]\!].$$

Definition. f is continuous on an interval $[a,b]$ if f is continous on any point in (a,b) and continuous from the right at b and continuous from the left at a.

Theorem. *[Rules of Continuous Functions]*

(1)If f and g are continuous at $x=a$, and c is a constant, then

$$f\pm g, f\cdot g, cf, \frac{f}{g}(\textit{provided } g(a)\neq 0)$$

are all continous at $x=a$.

(2)If f is continous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(b) = f\left(\lim_{x\to a} g(a)\right).$$

(3)If g is continous at a and f is continuous at $g(a)$, then, $f\circ g$ is continuous at a.

(4)Polynomials, Rational Functions, Root Functions, Trigonometric and Inverse Trigonometric Functions, Exponential and Logarithmic Functions are continous on their domains.

Example. Evaluate

$$\lim_{x\to\pi} \frac{\sin\left(\frac{x}{2}\right)}{2+\frac{x+1}{x}}.$$

Define $f(x) = \frac{\sin\left(\frac{x}{2}\right)}{2+\frac{x+1}{x}}$, which is continous at $x=\pi$ by the previous theorem. Thus,

$$\lim_{x\to\pi} f(x) = f(\pi) = \frac{\sin\left(\frac{\pi}{2}\right)}{2+\frac{\pi+1}{\pi}} = \frac{1}{\frac{2\pi+\pi+1}{\pi}} = \frac{\pi}{3\pi+1}.$$

Example. Find the value of a such that $f(x)$ is continous everywhere, where

$$f(x) = \begin{cases} ax^2 - 4 & \text{if } x \leq 0 \\ 3x + 2a & \text{if } x > 0 \end{cases}.$$

From the theorem, we know that $f(x)$ is continous on $(-\infty, 0)$ and $(0, \infty)$. The only thing is to make it at $x = 0$. Obviously,

$$\begin{cases} \lim_{x \to 0^-} f(x) = 2a \\ \lim_{x \to 0^+} f(x) = -4 \\ f(0) = -4 \end{cases}$$

Thus, we only need

$$\lim_{x \to 0^-} f(x) = 2a = -4 \Rightarrow a = -2.$$

Therefore,

$$f(x) = \begin{cases} -2x^2 - 4 & \text{if } x \leq 0 \\ 3x - 4 & \text{if } x > 0 \end{cases}.$$

Theorem. *[The Intermediate Value Theorem] Suppose that f is continuous on the interval $[a, b]$ where $f(a) \neq f(b)$. Let N be any number between $f(a)$ and $f(b)$, then there always exists a number c in (a, b), such that*

$$f(c) = N.$$

Example. Show that there is a root of $f(x) = 4x^3 - 6x^2 + 3x - 2$ between 1 and 2, i.e., there exist an x in $(1, 2)$ such that $f(x) = 0$.

Proof. (1)$f(x)$ is a polynomial therefore continous on $[1, 2]$.

(2) $f(1) = -1 < 0 < 12 = f(2)$.

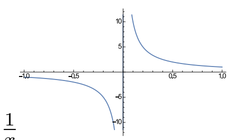
(3) By the Intermediate Value Theorem, there exists an x in $(1, 2)$, such that $f(x) = 0$. □

2.6 LIMITS AT INFINITY, HORIZONTAL ASYMPTOTES

Recall: we have learnt that

$$\lim_{\substack{x\to a \\ (x\to a^{\pm})}} f(x) = L \text{ and } \lim_{\substack{x\to a \\ (x\to a^{\pm})}} f(x) = \pm\infty.$$

[Question] How about $\lim_{x\to\pm\infty} f(x) = ?$



Example. $f(x) = \frac{1}{x}$

Observation:

$$\lim_{x\to\infty} f(x) = 0 \text{ and } \lim_{x\to-\infty} f(x) = 0.$$

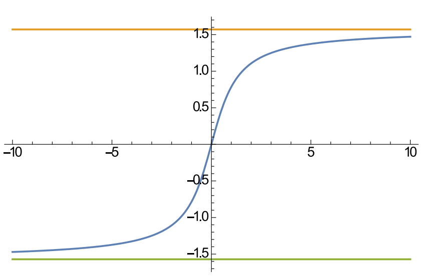
Definition. (1)Let f be a function defined on some interval (a,∞), then $\lim_{x\to\infty} f(x) = L$ can be made arbitarily close to L by taking x sufficiently large.

(2)Let f be a function defined on some interval $(-\infty, a)$, then $\lim_{x\to-\infty} f(x) = L$ can be made arbitarily close to L by taking x sufficiently large NAGATIVE.

(3)The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either (1) or (2) happens.

Example.

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \text{ and } \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}.$$



Theorem. *If $r > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$; if $r > 0$ is a rational number that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$.*

Method 5. [Division]

Example.

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1} &= \lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \text{ (degree of the denominator)}\\
&= \lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+4\frac{1}{x}+\frac{1}{x^2}}\\
&= \frac{\lim\limits_{x\to\infty}3-\lim\limits_{x\to\infty}\frac{1}{x}-2\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}5+4\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}\\
&= \frac{3}{5}.
\end{aligned}$$

Example.

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2-1}-x\right) &= \lim_{x\to\infty}\left(\sqrt{x^2-1}-x\right)\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}+x}\\
&= \lim_{x\to\infty}\frac{-1}{\sqrt{x^2-1}+x}\\
&= 0.
\end{aligned}$$

Example. $\lim\limits_{x\to-\infty}e^x=0$

Method 6. [Substitution]

$$\lim_{x\to0^-}e^{\frac{1}{x}}\overset{t=\frac{1}{x}}{=\!=}\lim_{t\to-\infty}e^t=0.$$

Example. Find the horizontal and vertical asymptotes of the graph of the function

$$f(x)=\frac{\sqrt{2x^2+1}}{3x-5}.$$

Solution. The domain is $\{x|3x-5\neq0\}=\left(-\infty,\frac{5}{3}\right)\cup\left(\frac{5}{3},\infty\right)$. We need to compute

$$\lim_{x\to\pm\infty}f(x)\text{ and }\lim_{x\to\frac{5}{3}^{\pm}}f(x).$$

(i)When $x\to\frac{5}{3}^+$, the numerator, $\sqrt{2x^2+1}>0$ and $3x-5\to0^+$, i.e. $3x-5>0$. Thus,

$$\lim_{x\to\frac{5}{3}^+}f(x)=\infty.$$

Similiarly, when $x\to\frac{5}{3}^-$, the numerator is still positive but $3x-5<0$. Thus

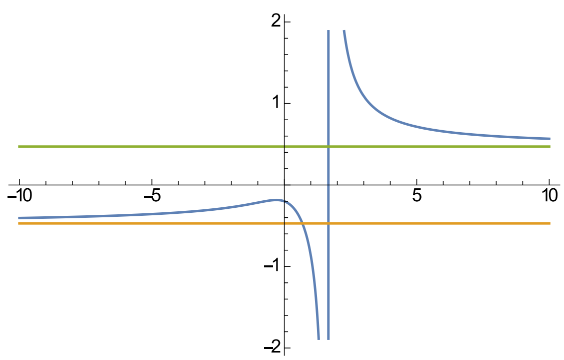
$$\lim_{x\to\frac{5}{3}^-}f(x)=-\infty.$$

(ii)When $x \to \infty$, $x > 0$ and $\sqrt{x^2} = |x| = x$, then

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \\
&= \frac{\sqrt{2}}{3}.
\end{aligned}$$

Now, when $x \to -\infty$, $x < 0$, then $\sqrt{x^2} = |x| = -x$. So

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{-\frac{1}{x}}{-\frac{1}{x}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\sqrt{\frac{1}{x^2}}}{-\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{-3+\frac{5}{x}} \\
&= -\frac{\sqrt{2}}{3}.
\end{aligned}$$



Infinite Limits at Infinity :

$$\lim_{x\to\pm\infty} f(x) = \pm\infty.$$

Example. $f(x) = x^3$

$$\lim_{x\to\infty} x^3 = \infty \text{ and } \lim_{x\to-\infty} x^3 = -\infty$$

Example.

$$\lim_{x\to\infty} \left(x^2 - x\right) = \lim_{x\to\infty} x(x-1) = \infty$$

Example.

$$\lim_{x\to-\infty}\frac{x^2+x}{3-x}=\lim_{x\to-\infty}\frac{x+1}{\frac{3}{x}-1}=\frac{-\infty}{-1}=\infty$$

2.7 DERIVATIVES AND RATES OF CHANGE

Definition. The derivative of a function f at a number a, denoted by $f'(a)$ is given by the following limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if it exists. Or alternatively, if write $x = a + h$ then $x - a = h$ and

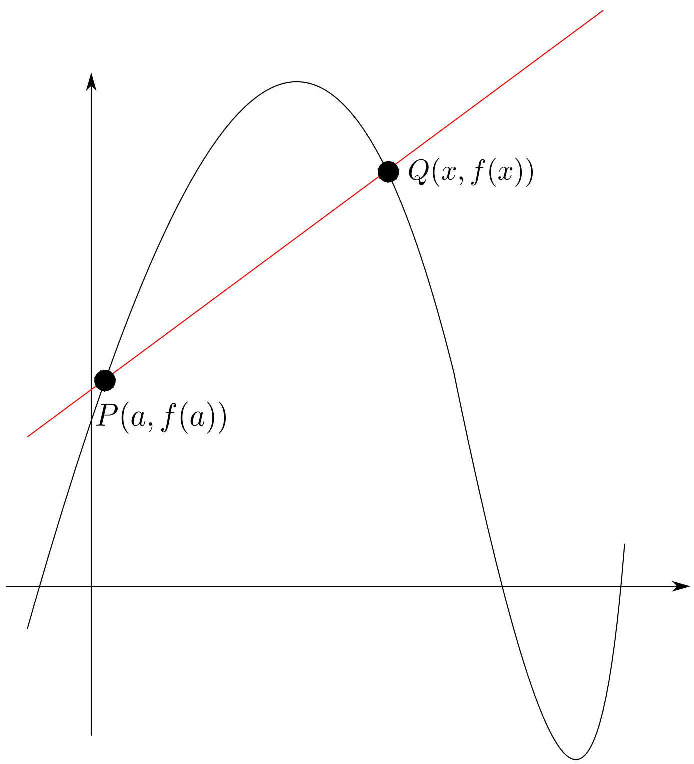
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example. $f(x) = x^2$ $a = 1$:

$$f'(1) = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} (x+1) = 2.$$

Geometric Intepretation:

Fact. *$f'(a)$ is the slope of tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.*



Example. Find the equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

Solution. The slope of the tangen line is

$$m = \lim_{x \to 3} \frac{\frac{3}{x} - 1}{x - 3} = \lim_{x \to 3} \frac{\frac{3-x}{x}}{x - 3} = \lim_{x \to 3} \frac{-1}{x} = -\frac{1}{3}.$$

Thus the tangent line is given by

$$y - 1 = -\frac{1}{3}(x - 1) \text{ or equivalently } x + 3y - 6 = 0.$$

In general, the equation of the tangent line to the curve $y = f(x)$ is given by

$$y - f(a) = f'(a)(x - a).$$

Physical Intepretation. When $s = f(t)$ is the position function with variable t standing for time. Then, the instantaneous velocity at the time $t = a$ is given by

$$v(a) = f'(a).$$

Example. A ball is dropped from a tower, 450 m above the groud. The position equation is

$$s = f(t) = \frac{1}{2}gt^2 = 4.9t^2.$$

(1)The velocity at $t = 5$ s is

$$v(5) = f'(5) = \lim_{t \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{t \to 5} \frac{4.9\left(x^2 - 5^2\right)}{x - 5} = 9.8\ (m/s)$$

(2)In general, when $t = a$

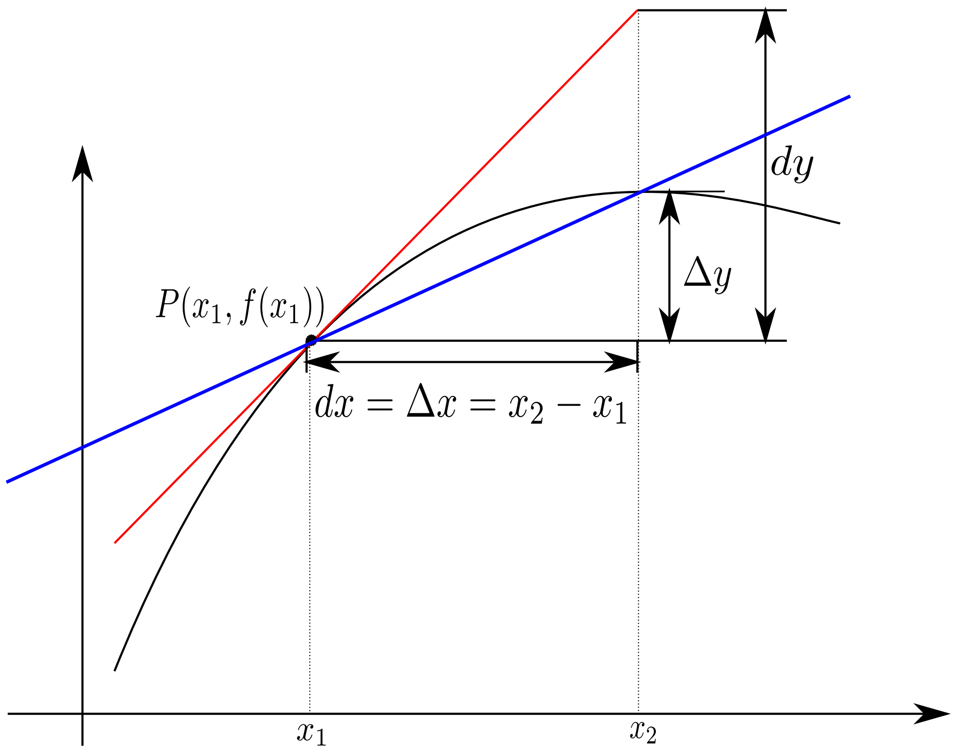
$$v(a) = f'(a) = \lim_{t \to a} \frac{4.9\left(x^2 - a^2\right)}{x - a} = 9.8a$$

(3)What is the velocity when it hits the ground?

Solve the equation that $450 = 4.9t^2$ to get that $t = \sqrt{\frac{450}{4.9}} \approx 9.6$ s. Thus,

$$v\left(\sqrt{\frac{450}{4.9}}\right) = 9.8\sqrt{\frac{450}{4.9}} \approx 94.$$

Rates of Change (Only Notations). From the following graph



$$\begin{cases} \text{slope of secent line} = \frac{\Delta y}{\Delta x} \\ \text{slope of tangent line} = \frac{dy}{dx} \end{cases}$$

2.8 THE DERIVATIVES AS A FUNCTION

Recall:

$$s = f(t) = 4.9t^2 \Rightarrow v(a) = f'(a) = 9.8a.$$

Definition. (1)

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(2) f is said to be differentiable at a is $f'(a)$ exists. f is called differentiable on (a, b) if it is differentiable at any point in (a, b).

(3) *Notations*: If $y = f(x)$

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(y) = \frac{df}{dx} = \frac{d}{dx}(f(x)).$$

Recall that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

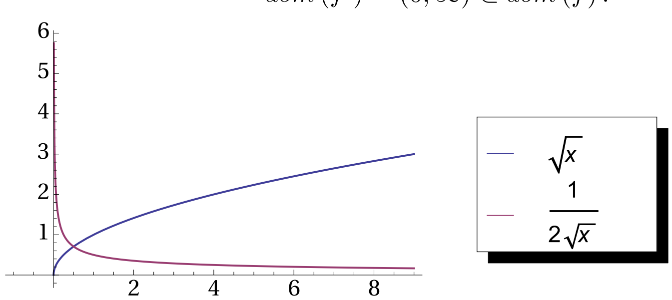
$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a}.$$

Example. $f(x) = \sqrt{x}$. $dom(f) = [0, \infty)$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}.$$

Also

$$dom(f') = (0, \infty) \subset dom(f).$$



Theorem. *If f is differentiable at a, then f is continuous at a.*

Remark. The converse is not true.

Example. $f(x) = |x|$. It is continous everywhere, even $x = 0$,

$$\begin{cases} \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \\ \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0 \\ |0| = 0. \end{cases}$$

However, we compute the following limit

$$\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x},$$

which does not exist since

$$\lim_{x\to 0^+}\frac{|x|}{x}=\lim_{x\to 0^+}\frac{x}{x}=1 \text{ while } \lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=-1.$$

Thus, $f(x)$ is continous at $x=0$ but not differentiable at $x=0$.

How Can a Function Fail to Be Differentiable?
(1)CaseI: Discontinuity.
(2)CaseII: Limits on the left and right do exist but do not match. (Ex. $|x|$)
(3)CaseIII: The limit is infinity (i.e. A Vertical Tangent). For example $f(x)=\sqrt{x}$,

$$\lim_{x\to 0}\frac{\sqrt{x}-\sqrt{0}}{x-0}=\lim_{x\to 0}\frac{1}{x^{\frac{1}{2}}}=\infty.$$

Higher Derivatives.

Example. $f(x)=\sqrt{x}$ and $f'(x)=\frac{1}{2\sqrt{x}}$.

$$\begin{aligned}
f''(x) &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{2\sqrt{x+h}}-\frac{1}{2\sqrt{x}}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{2h\sqrt{(x+h)x}}\\
&= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{2h\sqrt{(x+h)x}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\\
&= \lim_{h\to 0}-\frac{1}{2}\cdot\frac{1}{\sqrt{(x+h)x}\left(\sqrt{x}+\sqrt{x+h}\right)}\\
&= -\frac{1}{4\sqrt{x^3}}
\end{aligned}$$

Notations:

$$y''=f''(x)=\frac{d^2y}{dx^2}=\frac{d^2y}{dxdx}=f^{(2)}(x),\ f^{(3)}(x),\ f^{(10)}(x).$$

Physical Intepretation.

$$\begin{cases} s=f(x) & \text{Position}\\ v=f'(x) & \text{Velocity}\\ a=f''(x) & \text{Acceleration.}\end{cases}$$

3.1 DERIVATIVES OF POLYNOMIALS AND EXPONENTIAL FUNCTIONS

Power Functions.

Theorem. *In general*

$$(x^n)' = nx^{n-1} = \frac{d}{dx}(x^n).$$

Proof.

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{(x-a)\left(x^{n-1}+x^{n-2}a+\cdots+a^{n-1}\right)}{x-a} = na^{n-1}.$$

$\square$

Example. (1)$n=0$: $x^0=1$ is a constant, so

$$(1)' = \frac{d}{dx}\left(x^0\right) = 0\cdot x^{-1} = 0.$$

(2)$n=1$:

$$(x)' = 1\cdot x^0 = 1.$$

(3)$n=1/2$:

$$\left(\sqrt{x}\right)' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}.$$

(4)$n=-1$:

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = (-1)\,x^{-2} = -\frac{1}{x^2}.$$

Example. Find the equations of the tangent line and normal line to the curve $y=x\sqrt{x}$ at the point $(1,1)$.

DEF. The normal line to a curve C at a point P is the line that passes through P and is perpendicular to the tangent line at P.

$$y=f(x)=x\sqrt{x}=x^{\frac{3}{2}} \Rightarrow y'=f'(x)=\left(x^{\frac{3}{2}}\right)'=\frac{3}{2}\sqrt{x}.$$

So the slope of tangent line at $(1,1)$ is

$$m=f'(1)=\frac{3}{2}$$

and the equation of tangent line is

$$y-1=\frac{3}{2}(x-1).$$

If the slope of normal line is m', then

$$m'\cdot m=-1 \Rightarrow m'=-\frac{1}{m}=-\frac{2}{3}.$$

Therefore the equation of the normal line is

$$y-1=-\frac{2}{3}(x-1).$$

Other Rules.

Theorem. *Let c be a constant and f, g are differentiable, then*

$$\begin{cases} \frac{d}{dx}(cf(x)) = (cf(x))' = cf'(x) = c\frac{d}{dx}(f(x)) \\ \frac{d}{dx}(f(x) \pm g(x)) = (f(x) \pm g(x))' = f'(x) \pm g'(x) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x)) \end{cases}$$

Remark.

$$(c)' = c(1)' = c \cdot 0 = 0.$$

Example. (1)For a polynomial $f(x) = x^4 - 6x^2 + 4$,

$$f'(x) = \left(x^4 - 6x^2 + 4\right)' = \left(x^4\right)' - \left(6x^2\right)' + (4)' = \left(x^4\right)' - 6\left(x^2\right)' + (4)' = 4x^3 - 12x.$$

(2)Find points on the curve $y = f(x)$ where tangents are horizontal.

Solution. It requires to solve the equation

$$f'(x) = 0 = 4x^3 - 12x = 4x\left(x^2 - 3\right) = 4x\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) \Leftrightarrow x = 0, \pm\sqrt{3}.$$

So points with horizontal tangents are

$$(0,4), \left(\sqrt{3}, -5\right), \left(-\sqrt{3}, -5\right).$$

Exponential Functions. $f(x) = a^x$, then

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)\, a^x.$$

Recall that when $a = e$, $f'(0) = 1$. Thus,

(1)

$$1 = \lim_{h \to 0} \frac{e^h - 1}{h}$$

(2)

$$\frac{d}{dx}\left(e^x\right)' = e^x.$$

Example. $f(x) = e^x - x$, then

$$f'(x) = (e^x - x)' = (e^x)' - (x)' = e^x - 1$$

and

$$f''(x) = (f'(x))' = (e^x - 1)' = (e^x)' - (1)' = e^x.$$

3.2 THE PRODUCT AND QUOTIENT RULES

Theorem.

$$\begin{cases} (fg)' = f'g + fg', & \frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)] \\ \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, & \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]g(x) - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \end{cases}$$

Example. $f(x) = xe^x$.

(1)

$$f'(x) = (xe^x)' = (x)'e^x + x(e^x)' = e^x + xe^x = (x+1)e^x.$$

(2)

$$\begin{aligned} f''(x) &= [(x+1)e^x]' = (x+1)'e^x + (x+1)(e^x)' = (x+2)e^x \\ f'''(x) &= [(x+2)e^x]' = (x+2)'e^x + (x+2)(e^x)' = (x+3)e^x \\ &\cdots \\ f^{(n)}(x) &= (x+n)e^x \end{aligned}$$

Example. $f(x) = \frac{\sqrt{x}}{g(x)}$, where $g(4) = 2$ and $g'(4) = 3$. Find $f'(4)$.

Solution. By the quotient rule

$$f'(x) = \frac{(\sqrt{x})'g(x) - \sqrt{x}[g(x)]'}{[g(x)]^2} = \frac{\frac{1}{2}x^{-\frac{1}{2}}g(x) - \sqrt{x}g'(x)}{[g(x)]^2}.$$

So,

$$f'(4) = \frac{\frac{1}{2}4^{-\frac{1}{2}}g(4) - \sqrt{4}g'(4)}{[g(4)]^2} = \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{4}}\cdot 2 - 2\cdot 3}{2^2} = -\frac{11}{8}.$$

Remark. Sometimes, simplification helps to avoid product & quotient rules.

Example. $f(t) = \sqrt{t}(at+b)$.

$$f'(t) = \left(\sqrt{t}\right)'(at+b) + \sqrt{t}(at+b)' = \frac{1}{2\sqrt{t}}(at+b) + \sqrt{t}a = \frac{3at+b}{2\sqrt{t}}.$$

Also,

$$f'(t) = \left(at^{\frac{3}{2}} + bt^{\frac{1}{2}}\right)' = a\left(t^{\frac{3}{2}}\right)' + b\left(t^{\frac{1}{2}}\right)' = \frac{3a\sqrt{t}}{2} + \frac{b}{2\sqrt{t}}.$$

3.3 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Example. $f(x) = \sin x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\frac{\sin x (\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right] \\ &= \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

Now here are two import limits:

$$\begin{cases} \lim_{h\to 0} \frac{\sin h}{h} = 1 \\ \lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \end{cases}$$

Then,

$$f'(x) = (\sin x)' = \cos x$$

Theorem.

$$\begin{cases} (\sin x)' = \frac{d}{dx}(\sin x) = \cos x & \\ (\cos x)' = \frac{d}{dx}(\cos x) = -\sin x & \\ (\tan x)' = \frac{d}{dx}(\tan x) = \sec^2 x & , \sec x = \frac{1}{\cos x} \\ (\csc x)' = \frac{d}{dx}(\csc x) = -\csc x \cot x & , \csc x = \frac{1}{\sin x} \\ (\sec x)' = \frac{d}{dx}(\sec x) = \sec x \tan x & \\ (\cot x)' = \frac{d}{dx}(\cot x) = -\csc^2 x & \end{cases}$$

Example.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x,$$

since

$$\cos^2 \theta + \sin^2 \theta = 1.$$

3.4 THE CHAIN RULE

Question. How to compute

$$\frac{d}{dx}\left[f \circ g\left(x\right)\right] = \left(f \circ g\right)' = ?$$

Theorem. *If g is differetiable at x and f is differentiable at $g(x)$, then $F(x) = f \circ g(x)$ is differentiable at x and*

$$F'(x) = f' \circ g(x) \cdot g'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notations. If $u = g(x)$ and $y = f(u) = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example. $y = \sqrt{x^2+1}$. $f(u) = \sqrt{u}$ and $g(x) = x^2+1$ then $y = f(g(x)) = \sqrt{x^2+1}$. Thus,

$$y' = f'(g(x))\, g'(x) = \frac{1}{2\sqrt{g(x)}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Example. $y = \sin\left(x^2\right)$. $f(u) = \sin u$, $g(x) = x^2$. Then,

$$y' = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot 2x = 2x\cos\left(x^2\right).$$

Theorem. *If $u = g(x)$ is differentiable at x, then*

$$\frac{d}{dx}\left[\left(g(x)\right)^n\right] = n\left(g(x)\right)^{n-1} \cdot g'(x).$$

Example. $f(t) = \left(\frac{t-2}{2t+1}\right)^9$. $u = g(x) = \frac{t-2}{2t+1}$, then

$$f'(t) = 9\left(g(x)\right)^8 \cdot g'(x) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{(t-2)'(2t+1) - (t-2)(2t+1)'}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

Theorem. *If $a > 0$, then*

$$\frac{d}{dx}\left(a^x\right) = \left(a^x\right)' = a^x \cdot \ln a.$$

Proof. $a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)\cdot x}$, so $g(x) = (\ln a) \cdot x$ By the Chain Rule

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{(\ln a)\cdot x}\right) = e^{(\ln a)\cdot x} \cdot g'(x) = a^x \ln a.$$

□

Question.

$$\frac{d}{dx}\left[f \circ g \circ h(x)\right] = ?$$

Example. $f(x) = \sin(\cos(\tan(x)))$.

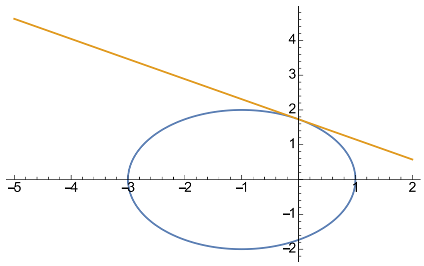
$$\begin{aligned}
f'(x) &= \left[\sin\left(\underbrace{\square}_{\cos(\tan(x))}\right)\right]' \\
&= \cos(\square)\cdot\square' \\
&= \cos(\cos(\tan(x)))\cdot[\cos(\tan(x))]' \\
&= \cos(\cos(\tan(x)))\cdot\left[\cos\left(\underbrace{\square}_{\tan(x)}\right)\right]' \\
&= \cos(\cos(\tan(x)))\cdot[-\sin(\square)]\cdot\square' \\
&= \cos(\cos(\tan(x)))\cdot[-\sin(\tan x)]\cdot(\tan x)' \\
&= -\cos(\cos(\tan(x)))\cdot\sin(\tan x)\cdot\sec^2 x.
\end{aligned}$$

Exercise.

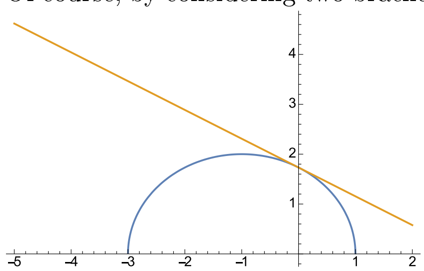
$$(\sin(\sin(\sin(\sin(x)))))' = ?$$

3.5 IMPLICIT DIFFERENTIATION

Example. Find the tangent line to the circle $x^2 + 2x + y^2 = 3$ at point $P\left(0, \sqrt{3}\right)$.



Of course, by considering two braches, we could only take the upper branch that



$$y = \sqrt{3 - x^2 - 2x}$$

and by Chain Rule to see that

$$y' = \frac{1}{2}\left(3 - x^2 - 2x\right)^{-\frac{1}{2}} \cdot \left(3 - x^2 - 2x\right)' = \frac{-2x - 2}{2\sqrt{3 - x^2 - 2x}}.$$

Then

$$y'\Big|_{x=0} = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

and therefore the tangent line is given by

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}x.$$

However, we could first assume $y = f(x)$ for some x and then by considering the equation

$$x^2 + 2x + y^2 = 3$$

and find derivatives of both sides, i.e. in general if two functions

$$g(x) = h(x),$$

then

$$g'(x) = h'(x).$$

Now the Right Hand Side (RHS)$= 3$, so

$$(\text{RHS})' = \frac{d}{dx}(\text{RHS}) = \frac{d}{dx}(3) = 0.$$

While

$$\frac{d}{dx}(\text{LHS}) = \frac{d}{dx}\left(x^2+2x+y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(2x) + \frac{d}{dx}\left(y^2\right).$$

It is easy to get

$$\frac{d}{dx}\left(x^2\right) = 2x \text{ and } \frac{d}{dx}(2x) = 2.$$

However, keep in mind that $y = f(x)$ is a function of x and then

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(\underbrace{\square^2}_{\square=y}\right) = 2\square \cdot \frac{d}{dx}(y) = 2y\frac{dy}{dx} = 2y \cdot y'.$$

Now we get

$$2x+2+2y\cdot y' = 0 \Rightarrow y' = -\frac{x+1}{y}.$$

Then

$$y'\Big|_{P=\left(0,\sqrt{3}\right)} = -\frac{0+1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

Example. Find y' if $\sin(x+y) = y^2\cos x$.

There is no hope to solve $y = f(x)$ as the previous problem. Thus we have to use implicit differentiation that

$$\frac{d}{dx}(\sin(x+y)) = \frac{d}{dx}\left(y^2\cos x\right).$$

(1)

$$\begin{aligned}
\frac{d}{dx}(\sin(x+y)) &= \frac{d}{dx}\left(\sin\left(\underbrace{\square}_{\square=x+y}\right)\right) \\
&= \cos(\square)\cdot\frac{d}{dx}(x+y) \\
&= \cos(x+y)\cdot\left[\frac{d}{dx}(x)+\frac{d}{dx}(y)\right] \\
&= \cos(x+y)\cdot(1+y').
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{d}{dx}\left(y^2\cos x\right) &= \frac{d}{dx}\left(y^2\right)\cos x + y^2\frac{d}{dx}(\cos x) \\
&= 2y\cdot y'\cos x + y^2(-\sin x).
\end{aligned}$$

Thus, we have

$$(1+y')\cos(x+y) = (2y\cdot\cos x)\,y' - y^2\sin x,$$

which implies

$$y' = \frac{\cos(x+y)+y^2\sin x}{2y\cos x - \cos(x+y)}.$$

Example. Find y'' if $x^4+y^4=16$.

(1) First of all

$$\frac{d}{dx}\left(x^4+y^4\right)=\frac{d}{dx}(16)\Rightarrow 4x^3+4y^3\cdot y'=0\Rightarrow y'=-\frac{x^3}{y^3}$$

(2)Also, we have

$$\frac{d^2}{dx^2}\left(x^4+y^4\right)=\frac{d^2}{dx^2}(16),$$

i.e.,

$$\frac{d}{dx}\left(4x^3+4y^3\cdot y'\right)=\frac{d}{dx}(0)=0.$$

Now

$$\begin{aligned}\frac{d}{dx}\left(4x^3+4y^3\cdot y'\right) &= \frac{d}{dx}\left(4x^3+4y^3\cdot y'\right)\\ &= 4\frac{d}{dx}\left(x^3\right)+4\frac{d}{dx}\left(y^3\cdot y'\right)\\ &= 4\cdot 3x^2+4\left[\frac{d}{dx}\left(y^3\right)y'+y^3\frac{d}{dx}\left(y'\right)\right]\\ &= 12x^2+4\left[3y^2\cdot\left(y'\right)^2+y^3\cdot y''\right]\\ &= 12x^2+12y^2\left(y'\right)^2+4y^3y''.\end{aligned}$$

Thus, we have

$$0=12x^2+12y^2\left(y'\right)^2+4y^3y''$$

and then

$$y''=-\frac{3x^2+12y^2y'}{y^3}=-\frac{3x^2+3y^2\left(-\frac{x^3}{y^3}\right)^2}{y^3}=-\frac{3x^2\left(y^4+x^4\right)}{y^7}=-\frac{48x^2}{y^7}.$$

Derivatives of Inverse Trigonometric Functions.

Example. $y=\sin^{-1}x$, find y'.

$$\begin{aligned}y=\sin^{-1}x &\Leftrightarrow x=\sin y\\ &\Leftrightarrow \frac{d}{dx}(x)=\frac{d}{dx}(\sin y)\\ &\Leftrightarrow 1=\cos y\cdot y'\\ &\Leftrightarrow y'=\frac{1}{\cos y}.\end{aligned}$$

Recall that $range\left(\sin^{-1}x\right)=range\,(y)=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\cos y\geq 0$. From

$$\sin^2 y+\cos^2 y=1,$$

we see

$$\cos y=\sqrt{1-\sin^2 y}\Rightarrow y'=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}.$$

Theorem.

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}},\ \frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}.$$

If $y = \tan^{-1} x$, then $x = \tan y$, then

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y) = \sec^2 y \cdot y' \Rightarrow y' = \frac{1}{\sec^2 y}.$$

Note that

$$\tan^2 y + 1 = \sec^2 y.$$

Thus

$$y' = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

3.6 DERIVATIVES OF LOGARITHMIC FUNCTIONS

Theorem.

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \textit{ and in particular, } \frac{d}{dx}(\ln x) = \frac{1}{x}, \textit{ since } \ln e = 1.$$

Proof. $y = \log_a x$ then $x = a^y$. Now

$$\frac{d}{dx}(x) = \frac{d}{dx}(a^y) = (\ln a)\, a^y \cdot y' \Rightarrow y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}.$$

$\square$

Example. $y = \ln(\sin x)$

$$y' = \frac{d}{dx}\left(\ln \underbrace{\square}_{\square = \sin x}\right) = \frac{1}{\square} \cdot \frac{d}{dx}(\square) = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{\cos x}{\sin x} = \cot x.$$

Theorem. *By Chain Rule, we have*

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx} \textit{ or } \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}.$$

Example. $f(x) = \ln|x|$ and Find f'.

Since

$$f(x) = \ln|x| = \begin{cases} \ln x & x \geq 0 \\ \ln(-x) & x < 0 \end{cases},$$

then

$$f'(x) = \begin{cases} \frac{1}{x} & x \geq 0 \\ \frac{-1}{-x} = \frac{1}{x} & x < 0 \end{cases}.$$

Thus

$$f'(x) = \frac{d}{dx}[\ln|x|] = \frac{1}{x}.$$

Application I: Simplify Computations

Example. Consider

$$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}.$$

By Quotient Rule,

$$y' = \frac{\left(x^{\frac{3}{4}}\sqrt{x^2+1}\right)'(3x+2)^5 - x^{\frac{3}{4}}\sqrt{x^2+1}\left[(3x+2)^5\right]'}{(3x+2)^{10}},$$

which is not easy to compute. On the other hand,

$$y = x^{\frac{3}{4}}\left(x^2+1\right)^{\frac{1}{2}}(3x+2)^{-5} \Rightarrow \ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 5\ln(3x+2).$$

By implicit differentiation,

$$\frac{y'}{y}=\frac{d}{dx}(\ln y)=\frac{3}{4}\frac{d}{dx}(\ln x)+\frac{1}{2}\frac{d}{dx}\left(\ln\left(x^2+1\right)\right)-5\frac{d}{dx}(\ln(3x+2))=\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}.$$

Now,

$$y'=y\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right)=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right).$$

Application II: Compute Derivatives for New Type of Functions

Example. $y=x^{\sqrt{x}}$.

Recall that we have $\frac{d}{dx}(x^n)=nx^{n-1}$ and $\frac{d}{dx}(a^x)=a^x\ln a$. But right now, x appears on both base and power, so neither formula applies.

$$y=x^{\sqrt{x}}\Rightarrow \ln y=\ln x^{\sqrt{x}}=\sqrt{x}\ln x---\text{Product Form!}$$

By implicit differentiation,

$$\frac{y'}{y}=\frac{d}{dx}(\ln y)=\frac{d}{dx}\left(\sqrt{x}\ln x\right)=\frac{d}{dx}\left(\sqrt{x}\right)\ln x+\sqrt{x}\frac{d}{dx}(\ln x)=\frac{1}{2\sqrt{x}}\ln x+\frac{\sqrt{x}}{x}=\frac{\ln x+2}{2\sqrt{x}}.$$

Then,

$$y'=y\cdot\frac{\ln x+2}{2\sqrt{x}}=x^{\sqrt{x}}\cdot\frac{\ln x+2}{2\sqrt{x}}.$$

Exercise. Consider $y=\ln(5x)$, *then*

$$y'=\frac{d}{dx}(\ln 5x)=\frac{1}{5x}(5x)'=\frac{5}{5x}=\frac{1}{x}=\frac{d}{dx}(\ln x).$$

Why?

The Number e as a Limit. Recall that we have two ways to define e

(1)The special number a such that the slope of the tangent line of $y=a^x$ at $(0,1)$ is 1.

(2)The special number a such that the limit holds

$$\lim_{h\to0}\frac{a^h-1}{h}=1.$$

(3)

$$e=\lim_{x\to0}(1+x)^{\frac{1}{x}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

3.8 EXPONENTIAL GROWTH AND DECAY

KEY: "Rate of change is proportional to the size"

$$\frac{dy}{dx} = ky, \text{ where } \begin{cases} k > 0 & \text{law of natural growth} \\ k < 0 & \text{law of natural decay} \end{cases}$$

Theorem. *The solution to* $\frac{dy}{dx} = ky$ *is*

$$y(x) = y(0)\, e^{kt}.$$

Example. [Population Growth] Consider that

Year	World Population
1950	2560 Million
1960	3040 M

[Question]: What is the population in the year 2020?

Solution. Define the function of population by $P(t)$, then

$$\frac{dP}{dt} = kt,$$

for some $k > 0$. Thus

$$P(t) = P(0)\, e^{kt}.$$

Intepretate the table to get that

$$t = 0 \leftrightarrow \text{Year } 1950 \leftrightarrow P(0) = 2560 \Rightarrow P(t) = 2560 e^{kt}.$$

and

$$t = 10 \leftrightarrow \text{Year } 1960 \leftrightarrow P(10) = 3040.$$

So,

$$3040 = P(10) = 2560 e^{k10} \Rightarrow 10k = \ln \frac{3040}{2560} \Rightarrow k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017.$$

Now,

$$P(70) = 2560 e^{70k} \approx 8524 \text{ Million}.$$

Example. [Radioactive decay]

$m(t)$= mass remaining at time t; $m_0 = m(0)$;

Half-life. Time required for any quantity decay to 1/2, denoted by t_{HL}

$$m(t) = m_0 e^{kt},\ t < 0.$$

Then

$$m(t_{HL}) = \frac{1}{2} m_0 = m_0 e^{kt_{HL}} \Leftrightarrow \frac{1}{2} e^{kt_{HL}} \Leftrightarrow t_{HL} = \frac{1}{k} \ln \frac{1}{2} = \frac{1}{k} \ln \left(2^{-1}\right) = -\frac{\ln 2}{k}.$$

Radium-226 has $t_{HL} = 1590$ years and we have a sample of 100 mg.

[Q]What is the mass after 1000 years? How long does it take to decay to 30 mg?

Solution. As we computed

$$1590 = t_{HL} = -\frac{\ln 2}{k} \Rightarrow k = -\frac{\ln 2}{1590}.$$

Then

$$m(t) = m_0 e^{kt} = 100 e^{-\frac{\ln 2}{1590} t}.$$

(1)$m(1000) = 100 e^{-\frac{1590}{\ln 2} \cdot 1000} \approx 65$ mg

(2)We need to solve

$$30 = m(t) = 100 e^{-\frac{\ln 2}{1590} t} \Leftrightarrow \frac{3}{10} = e^{-\frac{\ln 2}{1590} t} \Leftrightarrow -\frac{\ln 2}{1590} t = \ln \frac{3}{10} \Rightarrow t \approx 2762.$$

Example. [Newton's Law of Cooling]

$T(t)$= temperature at time t; T_s=surrounding temperature. Then

$$\frac{dT(t)}{dt} = k(T(t) - T_s) \Rightarrow T(t) = T_s + (T(0) - T_s) e^{kt}, \ k < 0.$$

Suppose we have a bottle of soda at 72° F and placed in a refrigerator of temperatue 44°F. Then, we know that

$$T(0) = 72 \text{ and } T_s = 44.$$

Suppose we also know that after 30 minutes, the bottle becomes 61°F. Thus

$$61 = T(30) = 44 + (72 - 44) e^{k30} \Rightarrow k = \frac{1}{30} \ln\left(\frac{17}{28}\right) \approx -0.01663.$$

[Question]What will happen after an hour?

Solution.

$$T(60) = 4 + (72 - 44) e^{k60} \approx 56.3.$$

[Question]How long does it take to drop the tempurature to 50° F?

Solution.

$$50 = T(t) = T(60) = 4 + (72 - 44) e^{kt} \Rightarrow t \approx 92.6$$

Example. [Contiuously Compounded Interest]

$1000 is invested at 6%, compounded annually, then

$$\begin{cases} f(1) = 1000 \times 1.06 = 1060 & \text{after 1 year} \\ f(2) = 1000 \times 1.06^2 = 1123.60 & \text{after 2 years} \\ f(t) = 1000 \times (1.06)^t & \text{in general} \end{cases}$$

Thus, we have the formula

$$A_0 (1 + r)^t.$$

However, if the interest is compounded n times a year, it becomes

$$A_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

For instance if it is daily compounded, then

$$f(t) = 1000 \times \left(1 + \frac{0.06}{365}\right)^{365t}.$$

Now, if let $n \to \infty$, then

$$A(t) = \lim_{n \to \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}.$$

3.9 RELATED RATES

KEY: Two things are related by simple geometric/physical formula. Then use chain rule (implicit differentiation) to find the "related rates".

Example. Air is being pumped into a spherical balloon so that its volume increases at a rate $100\mathrm{cm}^3/\mathrm{s}$. How fast is the radius of balloon increasing when the diameter is 50 cm?

Solution. Two things: volume: V and radius r (diameter) with

$$V = \frac{4}{3}\pi r^3.$$

It is given that

$$\frac{dV}{dt} = 100.$$

We want to compute

$$\frac{dr}{dt} =? \text{ when } r = 25.$$

Now from

$$V = \frac{4}{3}\pi r^3.$$

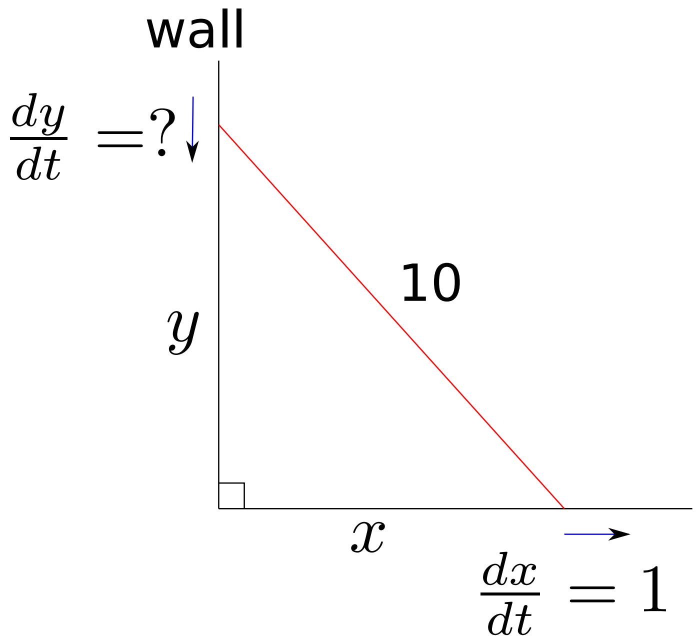
Taking derivatives WITH RESPECT TO time t gives

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}.$$

Thus, when $r = 25$

$$\frac{dr}{dt} = \frac{100}{4\pi (25)^2} = \frac{1}{25\pi}\,(\mathrm{cm/s}).$$

Example. A 10-ft long ladder rests against a vertical wall. It the bottom slides away from the way at rate 1ft/s, how fast is the top sliding down when the bottom is 6 ft from the wall?



Solution. As we known

$$x^2 + y^2 = 10^2 = 100.$$

Differentiate both sides WITH RESPECT TO time t to get

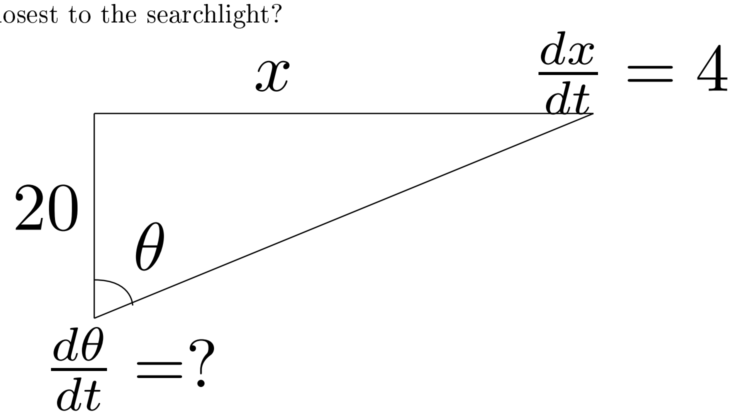
$$\frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$, $y = 8$, then

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4}.$$

The negative sign means y is decreasing.

Example. A man walks along a straight path at speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Solution. By geometry

$$\tan\theta = \frac{x}{20} \Rightarrow x = 20\tan\theta.$$

Differentiate WITH RESPECT TO time t to get

$$\frac{dx}{dt} = 20\sec^2\theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{20\sec^2\theta} = \frac{dx}{dt} \cdot \frac{1}{20}\cos^2\theta.$$

Now, $dx/dt = 4$ and when $x = 15$, $\cos\theta = \frac{4}{5}$. Then

$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{20} \cdot \left(\frac{4}{5}\right)^2 = \frac{16}{125}.$$

3.10 LINEAR APPROXIMATION AND DIFFERENTIALS & 3.11 HYPERBOLIC FUNCTIONS

Linear Approximation.

Definition. When x is close to a, the linear approximation of $f(x)$ at a is given by

$$L(x) = f(a) + f'(a)(x-a) \approx f(x).$$

.

Example. Compute the linear approximation of $f(x) = \sqrt{x}$ at $a = 4$.

Solution. $f'(x) = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2\sqrt{x}}$. Thus, $f(a) = f(4) = \sqrt{4} = 2$ and $f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. Therefore

$$L(x) = 2 + \frac{1}{4}(x-4) = \frac{1}{4}x + 1.$$

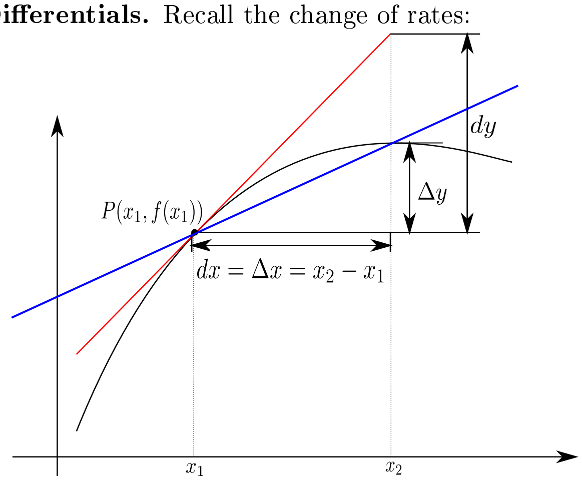
Example. Use linear approximation to compute $\sqrt{4.05}$.

Solution. $f(x) = \sqrt{x}$ and $a = 4$ since 4.05 is very close 4. Thus

$$\sqrt{4.05} = f(4.05) \approx L(4.05) = \frac{1}{4} \cdot 4.05 + 1 = 2.0125.$$

Remark. $\sqrt{4.05} = 2.01246117\cdots$.

Differentials. Recall the change of rates:



$$\begin{cases} \text{slope of secant line} = \frac{\Delta y}{\Delta x} \\ \text{slope of tangent line} = \frac{dy}{dx} = f'(x) \end{cases}$$

We call dx and dy differentials and they are connected by

$$dy = f'(x)\,dx.$$

Example. $f(x) = x^3 + x^2 - 2x + 1$ and $x_1 = 2$, $x_2 = 2.05$.
Then, $f'(x) = 3x^2 + 2x - 2$. Now, we have

$$dx = \Delta x = x_2 - x_1 = 0.05.$$

Also,

$$\begin{cases} \Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(2.05) - f(2) = 9.717625 - 9 = 0.717625 \\ dy = f'(x)\, dx = f'(2) \cdot 0.05 = 0.7 \end{cases}$$

Hyperbolic Functions.

Definition. We define

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

and then

$$\tanh x = \frac{\sinh x}{\cosh x}, \coth x = \frac{1}{\tanh x}, \mathrm{sech} x = \frac{1}{\cosh x}, \mathrm{csch} x = \frac{1}{\sinh x}.$$

Identities :

$$\begin{cases} \sinh(-x) = -\sinh x & , \sin(-x) = -\sin(x) \\ \cosh(-x) = \cosh x & , \cos(-x) = \cos(x) \\ \cosh^2 x - \sinh^2 x = 1 & , \cos^2 x + \sin^2 x = 1 \end{cases}$$

Derivatives :

$$\begin{cases} (\sinh x)' = \cosh x & , (\sin x)' = \cos x \\ (\cosh x)' = \sinh x & , (\cos x)' = -\sin x \end{cases}$$

Inverse Functions:

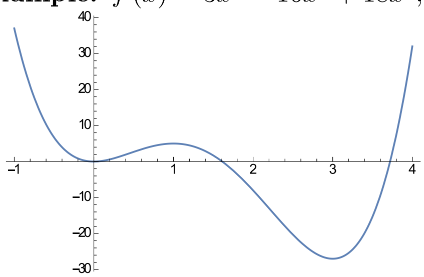
$$\begin{cases} \sinh^{-1} x & = \ln\left(x + \sqrt{x^2+1}\right) \\ \cosh^{-1} x & = \ln\left(x + \sqrt{x^2-1}\right) \\ \tanh^{-1} x & = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \end{cases}$$

4.1 MAXIMUM AND MINIMUM VALUES

Definition. Let c be a number in $dom(f)$. Then $f(c)$ is said to be
(1)ablsolute (global) maximum value of f if for any x in $dom(f)$, $f(x) \leq f(c)$;
(2)ablsolute (global) minimum value of f if for any x in $dom(f)$, $f(x) \geq f(c)$;
(3)local maximum value of f if when x is near c, $f(x) \leq f(c)$;
(4)local minimum value of f if when x is near c,, $f(x) \geq f(c)$.

Remark. When considering local max/min, we rule out endpoints

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$, where $-1 \leq x \leq 4$



Local Max: $(1, 5)$
Global Max: $(-1, 37)$
Local Min:$(0, 0)$, $(3, -27)$
Global Min:$(3, -27)$

Theorem. *[Extreme Value Theorem] If f is continuous on a closed interval $[a, b]$, then f attains absolute max value $f(c)$ and absolute min value $f(d)$ for some c and d in $[a, b]$.*

Oberservation: $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$. And then

$$f'(0) = f'(1) = f'(3) = 0.$$

Theorem. *[Fermat's Theorem] If f has a local max or min at c, then $f'(c) = 0$.*

Definition. A *critical number* of a function f is the number c in $dom(f)$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example. (1)$f(x) = |x|$, $x = 0$ is critical number since $f'(0)$ DNE.
(2) $f(x) = x^2 \Rightarrow f'(x) = 2x \Rightarrow x = 0$ is a critical number. Moreover, $(0, 0)$ is local & global min.
(3) $f(x) = x^3 \Rightarrow f'(x) = 2x^2 \Rightarrow x = 0$ is a critical number, but $(0, 0)$ is neither max nor min.

$$\text{Min/Max} \Rightarrow \text{Critical, but Critical} \not\Rightarrow \text{Max/Min}$$

[Question] How to find global max and min of a continous function on $[a, b]$?
The Closed Interval Method:
1. Find all the values of critical numbers of f in (a, b), i.e. solve $f'(x) = 0$ and also number c such that $f'(c)$ DNE, then evaluate the value.

2. Find $f(a)$ and $f(b)$.

3. Compare all the values. The smallest is the global min and the largest is the global max.

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.

1. $f'(x) = 12x^3 - 48x^2 + 36x = 12x\left(x^2 - 4x + 3\right) = 12x(x-1)(x-3)$. Thus,

$$f'(x) = 0 \Rightarrow x = 0, 1, 3.$$

Then,

$$f(0) = 0,\ f(1) = 5,\ f(3) = -27.$$

2. $f(-1) = 37$ and $f(4) = 32$

3. We get that the global max is $f(-1) = 37$ and global min is $f(3) = -27$.

Example. $f(x) = \ln\left(x^2 + x + 1\right)$ on $[-1, 1]$.

1. $f'(x) = -\frac{2x+1}{x^2+x+1}$, then $f'(x) = 0 \Rightarrow -\frac{2x+1}{x^2+x+1} = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$. Then, $f\left(-\frac{1}{2}\right) = \ln\left(\frac{3}{4}\right) < 0$.

2. $f(1) = \ln 3 > 0$. $f(-1) = \ln 1 = 0$.

3. So, the global max is $f(1) = \ln 3$ and global min is $f\left(-\frac{1}{2}\right) = \ln\frac{3}{4}$.

4.2 THE MEAN VALUE THEOREM

Theorem. *[Mean Value]Let f be a function that satisfies the following hypotheses:*
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

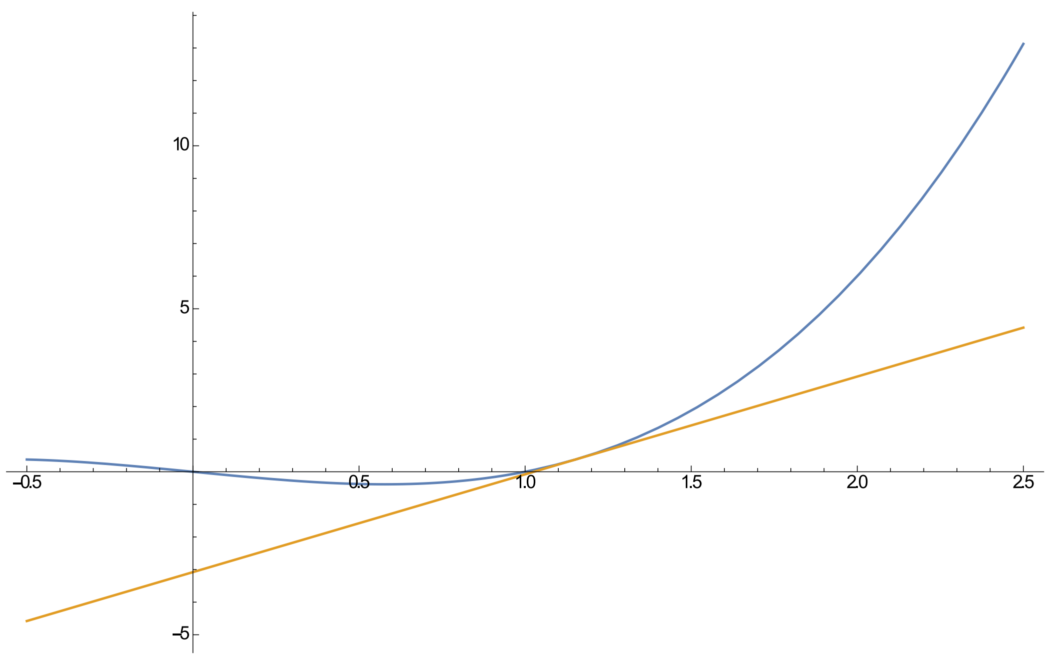
Theorem. *[Rolle's]Let f be a function that satisfies the following hypotheses:*
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.
Then there is a number c in (a, b) such that $f'(c) = 0$.

Example. $f(x) = x^3 - x$ on $[0, 2]$. Of course it is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Note that $f(0) = 0$ and $f(2) = 4$. Thus, there exists a number c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{6 - 0}{2 - 0} = 3.$$

In fact, consider the point $c = 2/\sqrt{3} \approx 1.1547$, then,

$$f'(x) = 3x^2 - 1 \Rightarrow f'\left(\frac{2}{\sqrt{3}}\right) = 3 \cdot \frac{4}{3} - 1 = 3.$$



Theorem. *If $f'(x) = 0$ for all x in (a,b), then f is a constant on (a,b). In particular, if $f'(x) = g'(x)$ for all x in (a,b), then $f-g$ is a constant on (a,b), i.e., $f(x) = g(x) + c$ for some constant c.*

Example. Show that

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

Solution. (1) Define the right function(put the constant on the other side): Define

$$f(x) = \tan^{-1} x + \cot^{-1} x.$$

(2) Verify that $f'(x) = 0$ to get that $f = c$:

$$f'(x) = \frac{1}{1+x^2} + \left(-\frac{1}{1+x^2}\right) = 0.$$

Thus,

$$f(x) = c, \text{where } c \text{ is a constant.}$$

(3)Compute the constant c:

$$c = f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

4.3 DERIVATIVES AFFECTS THE SHAPE OF THE GRAPH

f' Affects f

Example. $f(x) = x^2 \Rightarrow f'(x) = 2x$, thus,

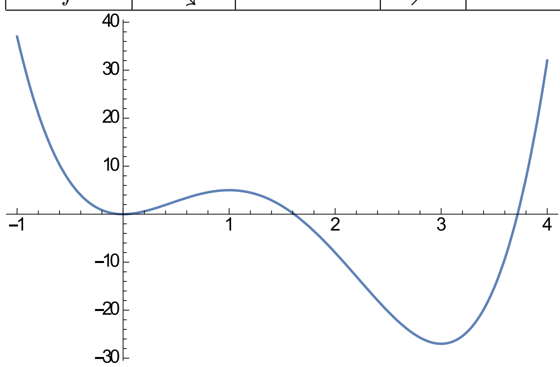
$$f'(x) \begin{cases} > 0 & x > 0 \\ = 0 & x = 0 \\ < 0 & x < 0 \end{cases}$$

Theorem. *[Increasing/Decreasing Test]*
(1) If $f'(x) > 0$ on an interval, then f is increasing on it.
(2) If $f'(x) < 0$ on an interval, then f is decreasing on it.

Theorem. *[The First Derivative Test] Suppose c is a critical number of a continuous function $f(x)$.*
(1)If f' changes from positive (+) to negative (-) at c, then f has a local maximum at c.
(2)If f' changes from negative (-) to positive (+) at c, then f has a local minimum at c.
(3)If f' does not change its sigh at c, then NOTHING happens.

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.
$f'(x) = 12x^3 - 48x^2 + 36x = 12x\left(x^2 - 4x + 3\right) = 12x(x-1)(x-3)$. So $f'(x) = 0 \Rightarrow x = 0, 1, 3$. Now

Intervals	$[-1,0)$	$x=0$	$(0,1)$	$x=1$	$(1,3)$	$x=3$	$(3,4]$
f'	-	0	+	0	-	0	+
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$	Local Min	$\nearrow$



Example. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$. Although $f'(0) = 0$, but $f'(x) \geq 0$. So, f' does not change its sigh at 0. So it is neither a local max nor min.

f'' Affects f

Definition. (1) If the graph of f lies above all its tangents on an interval I, then it is called concave upward on I.

(2) If the graph of f lies below all its tangents on an interval I, then it is called concave downward on I.

Example. $f(x) = x^2$ is concave upward.

Example. $g(x) = \sqrt{x}$ and $h(x) = \ln x$ are concave downward.

Remark. $f''(x) = 2 > 0$. $h''(x) = -\frac{1}{x^2} < 0$

Theorem. *[Concativity Test]*
(1) If $f''(x) > 0$ for all x in I, then the graph of f on I is concave upward on I.
(2) If $f''(x) < 0$ for all x in I, then the graph of f on I is concave downward on I.

Recall: If f' changes its sign at c, then c is a local max or min. If f'' changes its sign at point c, then the concavity changes.

Definition. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous at the point and the graph changes its concavity at P.

Example. $f(x) = x^3 \Rightarrow f''(x) = 6x \begin{cases} > 0 & x > 0 \\ = 0 & x = 0 \\ < 0 & x < 0 \end{cases}$. Then, by the concavity test, point $(0,0)$ is an inflection point.

Example. $f(x) = x^4 \Rightarrow f''(x) = 12x^2 \geq 0$. Although $f''(0) = 0$, but the concavity at $(0,0)$ does not change, so it is NOT an inflection point.

Example. $f(x) = x^4 - 4x^3$. Then $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ and $f''(x) = 12x^2 - 24x = 12x(x-2)$.

Intervals	$(-\infty, 0)$	$x = 0$	$(0, 2)$	$x = 2$	$(2, \infty)$
f''	$+$	0	-	0	$+$
f	Concave Upward	Inflection Point	Concave Downward	Inflection Point	Concave Upward

Example. $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 4]$.
$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x-1)(x-3)$. So $f'(x) = 0 \Rightarrow x = 0, 1, 3$. $f''(x) = 36x^2 - 96x + 36$.

Intervals	$[-1, 0)$	$x = 0$	$(0, 1)$	$x = 1$	$(1, 3)$	$x = 3$	$(3, 4]$
f'	-	0	$+$	0	-	0	$+$
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$	Local Min	$\nearrow$
f''		36>0		-24<0		264>0	

Theorem. *[The Second Derivative Test] Suppose f'' is continuous near c.*
(1)If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.
(2)If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

4.4 INDETERMINATE FORMS AND L'HOSPITAL'S RULE

Recall:

$$\lim_{x\to 0}\frac{x^2-x}{x}=\frac{0}{0}[\text{Factorization}]$$

and

$$\lim_{x\to\infty}\frac{x^2-3x}{2x^2+2}=\frac{\infty}{\infty}[\text{Divistion}]$$

[Question]

$$\lim_{x\to\infty}\frac{\ln x}{x-1}=?$$

Theorem. *If*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}\ \textit{or}\ \frac{\infty}{\infty},$$

i.e., an interminate form, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that the derivatives and limit on the right exist.

Example. (1)Consider

$$\lim_{x\to\infty}\frac{\ln x}{x-1}.$$

Since $\ln\infty=\infty$ and $\infty-1=\infty$, we have an indeterminate form, thus

$$\lim_{x\to\infty}\frac{\ln x}{x-1}\overset{L'}{\underset{\frac{\infty}{\infty}}{=}}\lim_{x\to\infty}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

(2)

$$\lim_{x\to 1}\frac{\ln x}{x-1}\overset{L'}{\underset{\frac{0}{0}}{=}}\lim_{x\to 1}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{1}=1.$$

(3)

$$\lim_{x\to 0^+}\frac{\ln x}{x-1}=\frac{\ln 0}{0-1}=\frac{-\infty}{-1}=\infty.$$

Indeterminate Product: If we have $0\cdot\infty$ form, it is also an indeterminate form and L'Hospital's Rule applies.

Example.

$$\lim_{x\to 0^+}x\ln x\underset{(-\infty)\cdot 0}{=}=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}\overset{L'}{\underset{\frac{\infty}{\infty}}{=}}\lim_{x\to 0^+}\frac{(\ln x)'}{(x^{-1})'}=\lim_{x\to 0^+}\frac{1/x}{-x^{-2}}=\lim_{x\to 0^+}(-x)=0.$$

Remark.

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{x}{(\ln x)^{-1}}\overset{L'}{\underset{\frac{0}{0}}{=}}\lim_{x\to 0^+}\frac{1}{-1(\ln x)^{-2}\cdot\frac{1}{x}}=\lim_{x\to 0^+}-x(\ln x)^2=\cdots$$

Indeterminate Differences: $\infty-\infty$

Example.

$$\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)\underset{\infty-\infty}{=}=\lim_{x\to\frac{\pi}{2}^-}\left(\frac{1-\sin x}{\cos x}\right)\overset{L'}{\underset{\frac{0}{0}}{=}}\lim_{x\to\frac{\pi}{2}^-}\frac{(1-\sin x)'}{(\cos x)'}=\lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x}=0.$$

Indeterminate Power: 0^0, ∞^0, 1^∞.
Method: Logarithmic

Example.

$$\lim_{x\to0^+}(1+\sin(4x))^{\cot x}.$$

Since $\lim_{x\to0^+}(1+\sin(4x))=1$ and $\lim_{x\to0^+}\cot x=\infty$. Let $y=(1+\sin(4x))^{\cot x}$, then

$$\ln y=\cot x\cdot\ln(1+\sin(4x))=\frac{\ln(1+\sin(4x))}{\tan x}.$$

Thus,

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(1+\sin(4x))}{\tan x}\overset{L'}{\underset{\frac{0}{0}}{=}}\frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2 x}=4.$$

Therefore

$$\lim_{x\to0^+}y=e^{\lim_{x\to0^+}\ln y}=e^4.$$

Example. $\lim_{x\to0^+}x^x$. Let $y=x^x\Rightarrow\ln y=x\ln x$. Thus,

$$\lim_{x\to0}\ln y=\lim_{x\to0^+}x\ln x=0\Rightarrow\lim_{x\to0^+}y=e^0=1.$$

4.5 SUMMARY OF CURVE SKETCHING

<u>**Guidelines**</u>: Given a function $y = f(x)$
A. Domain
B. Intercepts:

$$\begin{cases} \bullet x\text{-intecept}\,(x,0) & f(x) = 0 \\ \bullet y - \text{intecept}\,(0,y) & f(0) = y \end{cases}$$

C. Symmetry:

$$\begin{cases} \bullet\text{Even} : f(-x) = f(x) & \text{Example: } y = x^2 \\ \bullet\text{Odd} : f(-x) = -f(x) & \text{Example: } y = 3x^3 - x \\ \bullet\text{Periodic} : f(x+p) = f(x) & \text{Example: } y = \sin x, p = 2\pi \end{cases}$$

D. Asymptotes:

$$\begin{cases} \bullet\text{Vertical: } \lim\limits_{x\to a^{\pm}} f(x) = \pm\infty & \text{Example: } y = \frac{1}{x} \Rightarrow \begin{cases} \lim\limits_{x\to 0^+} \frac{1}{x} = \infty \\ \lim\limits_{x\to 0^-} \frac{1}{x} = -\infty \end{cases} \\ \bullet\text{Horizontal: } \lim\limits_{x\to\pm\infty} f(x) = L & \text{Example: } y = \frac{1}{x} \Rightarrow \lim\limits_{x\to\infty} \frac{1}{x} = 0 = \lim\limits_{x\to-\infty} \frac{1}{x} \\ \bullet\text{Slant} : \lim\limits_{x\to\pm\infty} [f(x) - (mx+b)] = 0 & \text{KEY: Only happens when } \frac{p(x)}{q(x)}\ deg(p) - deg(q) = 1. \end{cases}$$

The slant asymptote is $y = mx + b$.
E. Interval of Increase or Decrease: [Increase/Decrease Test: Increase $f' > 0$; Decrease $f' < 0$]
F. Local Maximum and Minimum Value[First or Second Derivative Test]
G. Concavity and Points of Inflection: [Concave Upward: $f'' > 0$; Concave Downward $f'' < 0$]
H. DRAW!!!

Example. $f(x) = \frac{x^2}{\sqrt{x+1}}$
A. Domain: $dom(f) = \{x | x + 1 > 0\} = (-1, \infty)$.
B. Intercepts:

$$\begin{cases} f(0) = 0 \Rightarrow y\text{-intercept is 0 or } (0,0) \\ f(x) = 0 \Rightarrow x = 0 \Rightarrow x\text{-intercept is 0 or } (0,0) \end{cases}$$

C. Symmetry:

$$\begin{cases} f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq f(x) & \text{NOT Even} \\ f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq -f(x) & \text{NOT Odd} \\ \text{No Trig Part} & \text{NOT Periodic} \end{cases}$$

D. Asymptotes:

$$\begin{cases} \lim_{x\to -1^+} f(x) = \frac{1}{0^+} = \infty \\ \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2\cdot\frac{1}{\sqrt{x}}}{\sqrt{x+1}\cdot\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{x^{3/2}}{\sqrt{1+\frac{1}{\sqrt{x}}}} = \infty \\ deg\,(numerator) = 2, deg(denominator) = 1/2 \quad \text{No Slant Asymptote} \end{cases}$$

E. Intervals of Increase or Decrease.

$$f'(x) = \frac{\left(x^2\right)'\sqrt{x+1} - x^2\left((x+1)^{\frac{1}{2}}\right)'}{x+1} = \frac{x\,(3x+4)}{2\,(x+1)^{\frac{3}{2}}}.$$

So,

$$f'(x) = 0 \Rightarrow x = 0, -\frac{4}{3} < -1, (\text{Not in the domain})$$

Intervals	$(-1,0)$	$x=0$	$(0,\infty)$
f'	$-$	0	$+$
f	$\searrow$	Min	$\nearrow$

F. Local Max and Min: $(0,0)$ is the only local min and there is no local max.

G. Concavity and Inflection Points

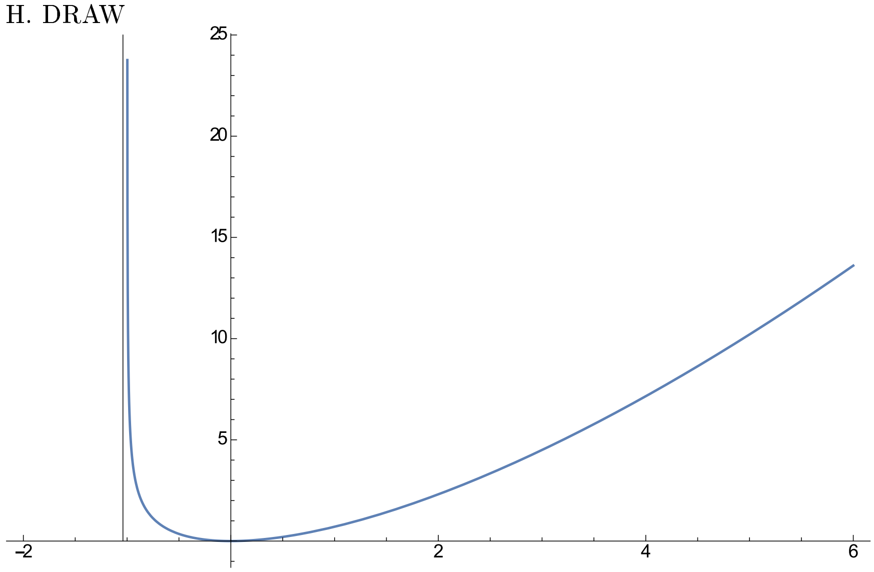
$$f''(x) = \frac{3x^2+8x+8}{4\,(x+1)^{\frac{5}{2}}}.$$

The denominator is positive because of the square root while the numerator is always positive as well, since

$$3x^2+8x+8 = 3x^2+8x+\frac{16}{3}+\frac{8}{3} = 3\left(x+\frac{4}{3}\right)^2+\frac{8}{3}.$$

Thus, f is always concave upward, (same as x^2).

H. DRAW



Example. $f(x) = \frac{x^3}{x^2+1}$

A. $dom\,(f) = (-\infty,\infty) = \mathbb{R}$

B. $f(0) = 0$ and $f(x) = 0 \Rightarrow x = 0$. Thus the both intercepts are 0.
C. $f(-x) = \frac{(-x)^3}{(-x)^2+1} = -\frac{x^3}{x^2+1} = -f(x)$, so it is odd.
D. Since domain is $\mathbb{R}$, so there is no vertical asymptotes. Also,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3 \cdot \frac{1}{x^2}}{(x^2+1)\cdot \frac{1}{x^2}} = \infty \text{ and similarly, } \lim_{x\to-\infty} f(x) = -\infty.$$

However, since the degree of numerator is 1 higher than the degree of denominator, it must have slant asymptote. Note that

$$f(x) = \frac{x^3}{x^2+1} = \frac{x^3+x-x}{x^2+1} = x - \frac{x}{x^2+1}.$$

Thus,

$$\lim_{x\to\pm\infty} [f(x) - x] = \lim_{x\to\pm\infty} \left[-\frac{x}{x^2+1}\right] = \lim_{x\to\pm\infty} \left[-\frac{\frac{1}{x}}{1+\frac{1}{x^2}}\right] = 0.$$

So the slant asymptote is $y = x$.
E.

$$f'(x) = \frac{x^2\left(x^2+3\right)}{\left(x^2+1\right)^2} \geq 0.$$

Thus, f is always increasing
F. Since f is always increasing, so there is no local min or max.
G.

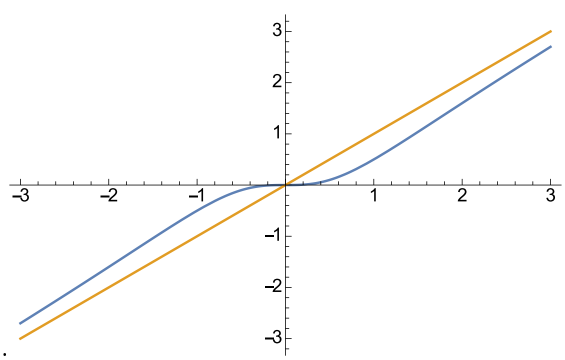
$$f''(x) = \frac{2x\left(3-x^2\right)}{\left(x^2+1\right)^3} \Rightarrow f''(x) = 0 \text{ when } x = 0, \pm\sqrt{3}$$

So

Intervals	$\left(-\infty, -\sqrt{3}\right)$	$\left(-\sqrt{3}, 0\right)$	$\left(0, \sqrt{3}\right)$	$\left(\sqrt{3}, \infty\right)$
f''	$+$	$-$	$+$	$-$
f	Concave Upward	Concave Downward	Concave Upward	Concave Downward

, and inflection points are

$$\left(-\sqrt{3}, -\frac{3}{4}\sqrt{3}\right), (0,0), \left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right).$$

H.


Example. $f(x) = \frac{\cos x}{2+\sin x}$.
A. $dom(f) = \{x | 2 + \sin x \neq 0\} = (-\infty, \infty)$.
B. $f(0) = \frac{\cos 0}{2+\sin 0} = \frac{1}{2}$, which is the y-intercept and solve $f(x) = 0 \Leftrightarrow \cos x = 0$ to see that $x = \frac{2n+1}{2}\pi$ for integer n.

C. $f(-x) = \frac{\cos x}{2-\sin x}$, which is neither $f(x)$ nor $-f(x)$. However,

$$f(x+2\pi) = f(x),$$

so we only need to consider the interval $[0, 2\pi]$, in which the x-intercepts are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

D. The domain shows no possible vertical asymptotes and $\lim_{x\to\pm\infty} f(x)$ DNE. So, there is no horizontal asymptote either.

E.

$$f'(x) = -\frac{2\sin x + 1}{(2+\sin x)^2},$$

so $f'(x) = 0$ implies $2\sin x + 1 = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$. (Note that we only consider $[0, 2\pi]$). Thus

Intervals	$\left(0, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right)$
f'	$-$	0	$+$	0	$-$
f	$\searrow$	Local Min	$\nearrow$	Local Max	$\searrow$

F. As shown above, local min is $\left(\frac{7\pi}{6}, -\frac{1}{\sqrt{3}}\right)$ and Local Max is $\left(\frac{11\pi}{6}, \frac{1}{\sqrt{3}}\right)$.

G.

$$f''(x) = -\frac{2\cos x\,(1-\sin x)}{(2+\sin x)^3}.$$

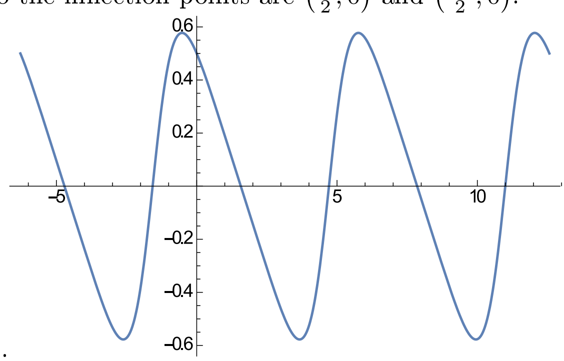
Now, $f''(x) = 0 \Leftrightarrow 2\cos x\,(1-\sin x) = 0$.

$$\begin{cases} \cos x = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \\ 1-\sin x = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \end{cases} \Rightarrow f''(x) = 0 \Rightarrow x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}.$$

Intervals	$\left(0, \frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	$\frac{3\pi}{2}$	$\left(\frac{3\pi}{2}, 2\pi\right)$
f''	$-$	0	$+$	0	$-$
f	Concave Downward	Inflection	Concave Downward	Inflection	Concave Downward

So the inflection points are $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$.

H.



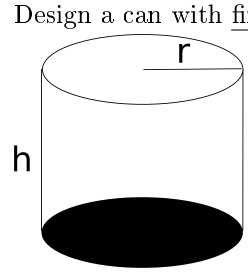
4.7 OPTIMIZATION PROBLEMS

STEPS:.

1. Understand the Problem
2. Draw a Diagram (If Necessary)
3. Introduce Notation.
4. Establish the Function
5. Eliminate Extra Variables (If Needed)
6. Solve for Absolute/Global Max or Min

Example. A cylinder can is to made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

1. Design a can with <u>fixed volume</u> and <u>minimal area of surface</u>



2.
3. V: Volume (cm^3); A: Area (cm^2); r: Radius (cm); h: Height (cm)
4.

$$\begin{cases} V = \pi r^2 h & = 1000 \\ A = 2\pi r h + \pi r^2 \cdot 2 & \text{[Top, Bottom, Side]} \end{cases}$$

5. Since we have

$$h = \frac{1000}{\pi r^2},$$

$$A(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r}.$$

6.

$$A'(r) = 4\pi r - \frac{2000}{r^2} \Rightarrow A'(r) = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}.$$

Note that

$$A''(r) = 4\pi + \frac{4000}{r^3} \Rightarrow A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0.$$

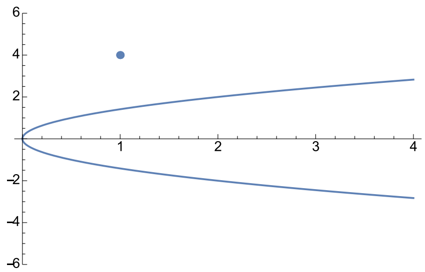
Second derivative test shows that it is the local min. Also

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

Thus the dimension is that

$$radius = \sqrt[3]{\frac{500}{\pi}}\ cm \text{ and } height = 2\sqrt[3]{\frac{500}{\pi}}\ cm$$

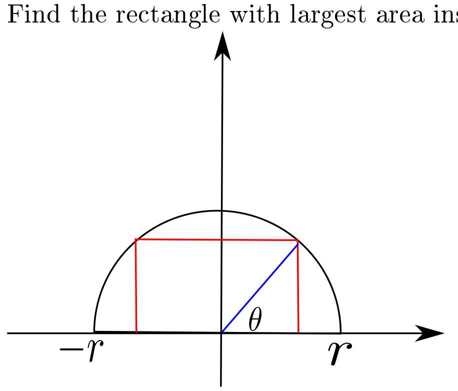
Example. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.

1. Find the closed distance and recall that for $P(x,y)$ and $Q(s,t)$
$$|PQ| = \sqrt{(x-s)^2 + (y-t)^2}.$$
2. 

3. Point on the parabola is (x,y) and the distance is d and $D = d^2$
4. We know that
$$D = d^2 = (x-1)^2 + (y-4)^2$$
5. Since $y^2 = 2x \Leftrightarrow x = \frac{y^2}{2}$, we have
$$D(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{y^4}{4} - 8y + 17.$$
6.
$$D'(y) = y^3 - 8 \Rightarrow D'(y) = 0 \Rightarrow y = 2.$$

Also,
$$D''(y) = 3y^2 \Rightarrow D''(2) = 12 > 0.$$
Thus, $y = 2$ and $x = \frac{y^2}{2} = 2$ is the minimum. Thus the point closest to $(1,4)$ is $(2,2)$.

Example. Find the ara of the largest rectangle that can be inscribed in a semicircle of radius r.

1. Find the rectangle with largest area inscribed in a semicircle.
2. 

3. The angle is θ and the area is A:
4. Note that $0 \le \theta \le \pi/2$
$$A = (r \cdot \sin\theta) \cdot (2r\cos\theta) = r^2 (2\sin\theta\cos\theta) = r^2 \sin 2\theta.$$
5. No need.

6. There is no need for differentiation since $0 \le 2\theta \le \pi$ and $\sin x$ has a max when $x = \frac{\pi}{2}$, i.e.,

$$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}.$$

So the dimension is that

$$\begin{cases} \text{one side} = r\sin\frac{\pi}{4} = \frac{r}{\sqrt{2}} \\ \text{the other side} = 2r\cos\frac{\pi}{4} = \sqrt{2}r \end{cases}$$

4.8 NEWTON'S METHOD

Question: How to find a solution of $f(x) = 0$ for complicated f. At least, an approximate root is acceptable.

<u>**Steps**</u>:

(1)Choose x_1

(2)Follow the recurrence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and so on

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(3)The limit

$$r = \lim_{n\to\infty} x_n$$

is the root, i.e.

$$f(r) = 0.$$

Example. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation

$$x^3 - 2x - 5 = 0.$$

Solution. $f'(x) = 3x^2 - 2$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

(1)$x_1 = 2$

(2)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = \frac{21}{10}$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.0946.$$

Example. Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places

Solution. We first need to intepretate this question into FINDING A ROOT OF A FUNCTION. Consider

$$f(x) = x^6 - 2,$$

then

$$r = \sqrt[6]{2} \Rightarrow f(r) = 0.$$

Now choose $x_1 = 1$ and follow

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_{n^5}} = \frac{5}{6}x_n + \frac{1}{3x_n^5}$$

to get

$$\begin{cases} x_2 \approx & 1.6666667 \\ x_3 \approx & 1.12644368 \\ x_4 \approx & 1.12249707 \\ x_5 \approx & 1.12246205 \\ x_6 \approx & 1.12246205 \end{cases}$$

4.9 ANTIDERIVATIVES & (PART OF) 5.3 INDIFINITE INTEGRALS

Definition. A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$

for all x in I.

Example. $f(x) = \cos x$ then

$$\begin{cases} F(x) = \sin x, & (\sin x)' = \cos x \\ F_1(x) = \sin x + 1, & F_1'(x) = f(x) \\ F_2(x) = \sin x + 2, & F_2'(x) = f(x) \\ F_c(x) = \sin x + C & \text{for any constant } C. \end{cases}$$

Theorem. *[DEF.] If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C,$$

where C is an arbitary constant.

Definition. [Indefinite Integral] The operation for finding the most general anti-derivative:

$$\int f(x)\,dx = F(x) + C \text{ or equivalently } F(x) = \int f(x)\,dx + C,$$

where F is any particular antiderivative of f.

Remark. The "dx" part in the notation is very important.

Example. $f(x) = 1/x$. $dom(f) = (-\infty, 0) \cup (0, \infty)$

$$F(x) = \int f(x)\,dx = \ln|x| + C.$$

Example. $f(x) = x^n$ for $n \neq -1$.

$$F(x) = \int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Suppose $F'(x) = f(x)$ and $G'(x) = g(x)$

Function	Particular Antiderivative
$cf(x)$	$cF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n\ (n\neq -1)$	$\frac{x^{n+1}}{n+1}$
$1/x$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/\left(1+x^2\right)$	$\tan^{-1} x$
$\sinh x$	$\cosh x$

Thus, if we rewrite the table, we will have

Table of Indefinite Integrals
$\int cf(x)\,dx = c\int f(x)\,dx$
$\int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n\neq -1$
$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$
$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$
$\int \csc x \cot x dx = -\csc x + C$
$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$
$\int \sinh x dx = \cosh x + C$

Example. Find the indefinite integral

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 10\int x^4 dx - 2\int \sec^2 x dx.$$

By the table,

$$\begin{cases} \int x^4 dx = \frac{x^{4+1}}{4+1} + C_1 = \frac{x^5}{5} + C_1 & , \\ \int \sec^2 x dx = \tan x + C_2 & . \end{cases}$$

Thus,

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 2x^5 - 2\tan x + (10C_1 - 2C_2) = 2x^5 - 2\tan x + C,$$

where

$$C = 10C_1 - 2C_2$$

also stands for arbitary constant term.

Remark. From now on, we could first write down the particular antiderivative for each term first and finally add the constant C in the end when computing indefinite integrals. Namely,

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 10\int x^4 dx - 2\int \sec^2 x dx = 10 \cdot \frac{x^5}{5} - 2\tan x + C.$$

Example. It also does not matter if we change the variable from x to some others.

$$\int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} d\theta = \int \csc\theta \cdot \cot\theta d\theta = -\csc\theta + C.$$

Example. Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Solution. Symplification shows

$$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}.$$

Thus,

$$g(x) = \int g(x)\, dx = 4\int \sin x dx + 2\int x^4 dx - \int x^{-\frac{1}{2}} dx = -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C.$$

Example. Find f if $f'(x) = e^x + 20\left(1 + x^2\right)^{-1}$.

Solution. Since

$$f'(x) = e^x + 20\frac{1}{1+x^2},$$

f has the form

$$f(x) = \int f'(x)\, dx = e^x + 20\tan^{-1} x + C$$

for some constant C. Also,

$$-2 = f(0) = e^0 + 20\tan^{-1} 0 + C = 1 + C \Rightarrow C = -3.$$

Therefore,

$$f(x) = e^x + 20\tan^{-1} x - 3.$$

Example. A partical moves in a straight line has acceleration given by

$$a(t) = 6t + 4.$$

Its initial velocity is $v(0) = -6$ cm/s and initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution. Recall the important physicial fact that

$$s'(t) = v(t) \text{ and } v'(t) = a(t).$$

Thus,

$$v(t) = \int a(t)dt = 3t^2 + 4t + C_1$$

for some constant C_1. Note that

$$-6 = v(0) = C_1.$$

Therefore,

$$v(t) = 3t^2 + 4t - 6$$

and then

$$s(t) = \int v(t)dt = t^3 + 2t^2 - 6t + C_2,$$

where the constant C_2 is given by

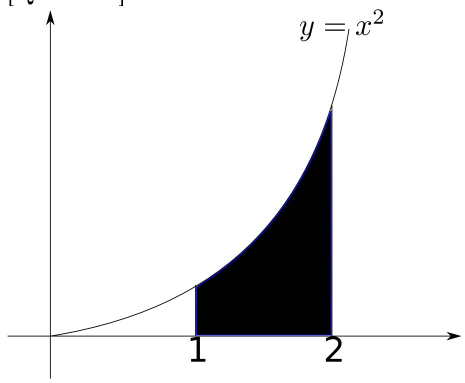
$$9 = s(0) = C_2.$$

Finally,

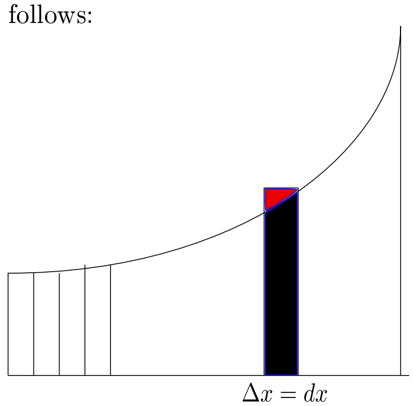
$$s(t) = t^3 + 2t^2 - 6t + 9.$$

5.1 & 5.2 DEFINITE INTEGRALS AND AREA

[Question] What is the area of the shadow region?



[Answer]:This is a very important approach. We try to cut the region vertically as follows:



Now we concentrate on each SLIGCE, which is a rectangle.

$$\begin{cases} \text{width} & = \Delta x = dx \\ \text{height} & = f(x), \text{ for some } x \text{ in this interval} \end{cases}.$$

Thus,

$$\text{Area} \approx \sum \text{rectangles} = \sum f(x)\, dx.$$

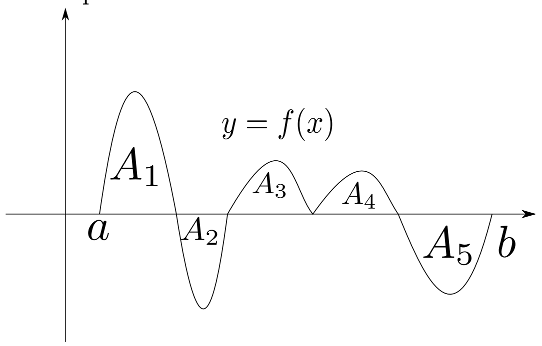
Now taking the limit that $\Delta x = dx \to 0$ then the error goes away and by the following notation

$$\text{Area} = \lim_{dx \to 0} \sum f(x)\, dx = \int_1^2 f(x)\, dx.$$

Definition. [Definite Integral] The definite integral

$$\int_a^b f(x)\, dx$$

is the SIGNED area formed by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis. The signs are assigned in the way that if part of the region is above the x-axis, then it is assigned a positive sign; while if it is under the x-axis, it has negative sign. For example.

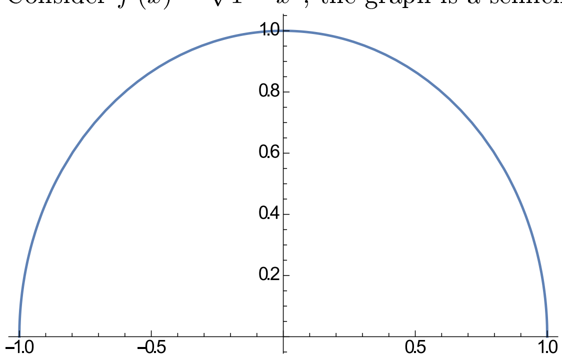


Then,

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 + A_4 - A_5.$$

Example. $\int_{-1}^{1} \sqrt{1-x^2}dx$.

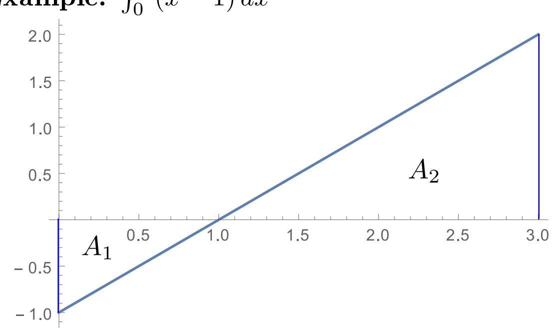
Consider $f(x) = \sqrt{1-x^2}$, the graph is a semicircle



Thus,

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \text{Area} = \frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}.$$

Example. $\int_0^3 (x-1)\,dx$



$$\int_0^3 (x-1)\,dx = -A_1 + A_2 = -\left(\frac{1}{2}\cdot 1\cdot 1\right) + \left(\frac{1}{2}\cdot 2\cdot 2\right) = 1.5.$$

Properties:

$$\begin{cases} 1. & \int_a^b c dx = c\,(b-a) \ \ \textbf{for any constant } c \\ 2. & \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \\ 3. & \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx \ \textbf{for any constant } c \\ 4. & \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \\ 5. & \textbf{If } f(x) \geq 0 \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\,dx \geq 0 \\ 6. & \textbf{If } f(x) \geq g(x) \ \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \\ 7. & \textbf{If } m \leq f(x) \leq M \ \textbf{ for } a \leq x \leq b\textbf{, then } m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a) \end{cases}$$

.

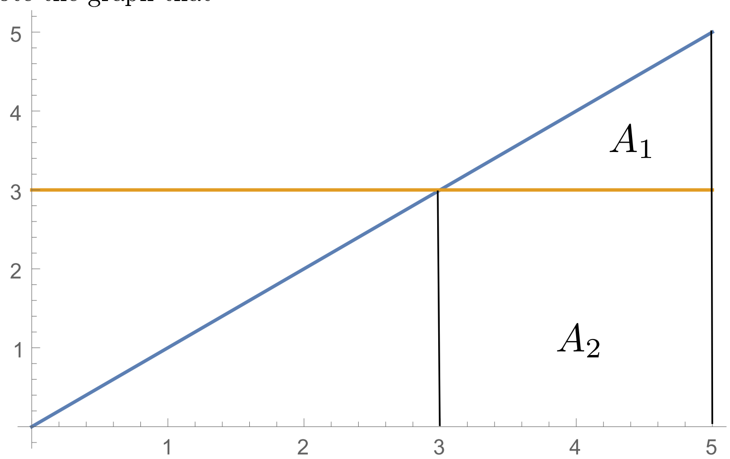
Example. Consider

$$f(x) = \begin{cases} 3 & x < 3 \\ x & x \geq 3 \end{cases},$$

then

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_0^3 3dx + \int_3^5 xdx = 9 + \int_3^5 xdx.$$

Note the graph that



We see

$$\int_3^5 xdx = A_1 + A_2 = \frac{1}{2}\cdot 2\cdot 2 + \cdot 2\cdot 3 = 8.$$

Therefore

$$\int_0^5 f(x)\,dx = 9 + 8 = 17.$$

5.3 FUNDAMENTAL THEOREM OF CALCULUS

Theorem. *Suppose f is continuous on $[a,b]$.*

(1)The function defined by

$$g(x)=\int_a^x f(t)\,dt \quad a\le x\le b$$

is continuous on $[a,b]$ and differentiable on (a,b) and

$$g'(x)=f(x).$$

(2) Let F be any antiderivative of f, i.e. $F'=f$, then

$$\int_a^b f(x)\,dx=F(b)-F(a)=F(x)\Big|_{x=a}^{x=b}.$$

Example. Define $g(x)=\int_0^x\sqrt{1+t^2}dt$, then find $g'(x)$.

Solution. Let $f(t)=\sqrt{1+t^2}$, then $g(x)=\int_0^x f(t)\,dt$. By the fundamental theorem of calculus, part 1,

$$g'(x)=f(x)=\sqrt{1+x^2}.$$

Example. $S(x)=\int_0^x\sin\left(\pi t^2/2\right)dt$, then if letting $f(t)=\sin\left(\pi t^2/2\right)$

$$S(x)=\int_0^x f(t)\,dt\Rightarrow S'(x)=f(x)=\sin\left(\frac{\pi x^2}{2}\right).$$

Example. Find

$$\frac{d}{dx}\int_1^{x^4}\sec t dt.$$

Solution. Let

$$g(x)=\int_1^{x^4}\sec t dt \text{ and } f(t)=\sec t\Rightarrow g(x)=\int_1^{x^4}f(t)\,dt.$$

Now since the upper bound of this integral is x^4 instead of x, we cannot apply the Fundamental Theorem of Calculus directly. Let

$$u=h(x)=x^4\Rightarrow g(u)=\int_1^u f(t)\,dt.$$

By Fundamental Theorem of Calculus,

$$\frac{dg}{du}=f(u).$$

By Chain Rule,

$$\frac{dg}{dx}=\frac{dg}{du}\cdot\frac{du}{dx}=f(u)\cdot h'(x)=\sec u\cdot\left(x^4\right)'=4x^3\sec\left(x^4\right).$$

By the "Box" notation, we could write it as

$$\frac{d}{dx}\int_1^{x^4} \sec t dt = \Big(\underbrace{\int_1^{\square} \sec t dt}_{\square = x^4}\Big)' = \sec(\square)\cdot\square' = \sec\left(x^4\right)\cdot\left(x^4\right)' = 4x^3\sec\left(x^4\right).$$

Example.

$$\int_1^3 e^x dx = ?$$

Solution. By the formula

$$\int e^x dx = e^x + C,$$

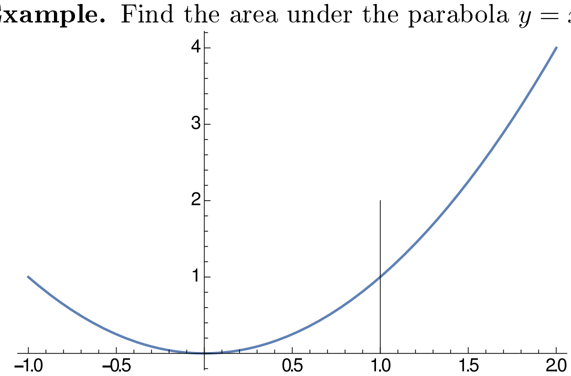
we have

$$\int_1^3 e^x dx = e^x\Big|_{x=1}^{x=3} = e^3 - e^1$$

Example.

$$\int_3^6 \frac{dx}{x} = \ln|x|\Big|_{x=3}^{x=6} = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2.$$

Example. Find the area under the parabola $y = x^2$ from 0 to 1.



$$A = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_{x=0}^{x=1} = \frac{1}{3}.$$

Example. Find the area under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.

$$A = \int_0^b \cos x dx = \sin x\Big|_{x=0}^{x=b} = \sin b.$$

Example.

$$\int_{-1}^3 |x|\, dx = \int_{-1}^0 (-x)\, dx + \int_0^3 x dx = -\frac{x^2}{2}\Big|_{x=-1}^{x=0} + \frac{x^2}{2}\Big|_{x=0}^{x=3} = \frac{1}{2} + \frac{9}{2} = 5.$$

Example.

$$\begin{aligned}
\int_0^3 \left(x^3 - 6x\right) dx &= \int_0^3 x^3 dx - 6\int_0^3 x dx \\
&= \left.\frac{x^4}{4}\right|_{x=0}^{x=3} - 6 \cdot \left.\frac{x^2}{2}\right|_{x=0}^{x=3} \\
&= \frac{3^4}{4} - 3 \cdot 9 \\
&= -\frac{27}{4}.
\end{aligned}$$

Example.

$$\begin{aligned}
\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= 2\int_0^2 x^3 dx - 6\int_0^2 x dx + 3\int_0^2 \frac{1}{1+x^2} dx \\
&= 2 \cdot \left.\frac{x^4}{4}\right|_{x=0}^{x=2} - 6 \cdot \left.\frac{x^2}{2}\right|_{x=0}^{x=2} + 3\tan^{-1} x\Big|_{x=0}^{x=2} \\
&= \frac{1}{2} \cdot 2^4 - 3 \cdot 2^2 + 3\tan^{-1} 2 \\
&= -4 + 3\tan^{-1} 2.
\end{aligned}$$

5.5 THE SUBSTITUTION RULE

Example. Calculate

$$\int 2x\sqrt{x^2+1}dx$$

This integral cannot be found in the table. Now consider another problem: if

$$F(x) = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}},$$

Then

$$F'(x) = \frac{2}{3}\left[\underbrace{\square^{\frac{3}{2}}}_{\square = x^2+1}\right]' = \frac{2}{3}\cdot\frac{3}{2}\square^{\frac{1}{2}}\cdot\square' = \sqrt{x^2+1}\cdot 2x.$$

It suggests that

$$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C.$$

In theory, integral is the inverse operation of derivative and substitution rule is the counterpart of chain rule.

<u>Recall</u>: Differential. If $y = f(x)$, then the differentials, dy and dx satisfy that

$$dy = f'(x)\,dx.$$

STEPS:

1. Choose the right substitution $u = g(x)$
2. Calculate the differentials: $du = g'(x)\,dx$
3. Replace $g'(x)\,dx = du$ and rewrite the function(integrand) as a function of u
4. Get the result in terms of u.
5. Substitute u by $g(x)$

Example. Compute

$$\int 2x\sqrt{x^2+1}dx$$

1. $u = g(x) = x^2+1$
2. $du = g'(x)\,dx = 2xdx$
3. $$\int 2x\sqrt{x^2+1}dx = \int \left(x^2+1\right)^{\frac{1}{2}}\underbrace{2xdx} = \int u^{\frac{1}{2}}du$$
4. $$\int 2x\sqrt{x^2+1}dx = \int u^{\frac{1}{2}}du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{\frac{3}{2}} + C.$$
5. $$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C$$

Example. Compute

$$\int x^3 \cos\left(x^4+2\right) dx.$$

1. $u = g(x) = x^4 + 2$
2. $du = g'(x) = 4x^3 dx.$
3. [Important Trick]

$$\int x^3 \cos\left(x^4+2\right) dx = \int \cos\left(x^4+2\right) x^3 dx = \int \cos\left(x^4+2\right) \cdot \frac{4}{4} \cdot x^3 dx = \frac{1}{4} \int \cos u \cdot du.$$

4.

$$\int x^3 \cos\left(x^4+2\right) dx = \frac{1}{4} \int \cos u \cdot du = \frac{1}{4} \sin u + C$$

5.

$$x^3 \cos\left(x^4+2\right) dx = \frac{1}{4} \sin u + C = \frac{1}{4} \sin\left(x^4+2\right) + C$$

Example. Compute

$$\int \sqrt{2x+1} dx$$

Let $u = 2x+1$, then $du = 2dx$. Now

$$\int \sqrt{2x+1} dx = \int \sqrt{2x+1} \cdot \frac{2}{2} dx = \frac{1}{2} \int u^{\frac{1}{2}} du = \frac{1}{2} \cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (2x+1)^{\frac{3}{2}} + C.$$

Example. Find

$$\int \frac{x}{\sqrt{1-4x^2}} dx$$

Let $u = 1 - 4x^2$, then $du = -8x dx$. Thus,

$$\int \frac{x}{\sqrt{1-4x^2}} dx = \int \frac{x}{\sqrt{1-4x^2}} \cdot \frac{-8}{-8} \cdot dx = -\frac{1}{8} \int u^{-\frac{1}{2}} du = -\frac{1}{8} \cdot 2u^{\frac{1}{2}} + C = -\frac{1}{4} \sqrt{1-4x^2} + C.$$

Example. Evaluate

$$\int e^{5x} dx$$

Let $u = 5x \Rightarrow du = 5dx$, then

$$\int e^{5x} dx = \int e^{5x} \cdot \frac{5}{5} \cdot dx = \frac{1}{5} \int e^u du = \frac{1}{5} e^u + C = \frac{1}{5} e^{5x} + C.$$

Example. Calculate

$$\int \tan x dx$$

Since $\tan x = \frac{\sin x}{\cos x}$ and recall that $(\cos x)' = \sin x$, we try that $u = \cos x$ and $du = -\sin x dx$. Then

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} (-\sin x) dx = -\int u^{-1} du = -\ln |u| + C = -\ln |\cos x| + C.$$

In addition,

$$-\ln |\cos x| = \ln |\cos|^{-1} = \ln \frac{1}{|\cos x|} = \ln \left| \frac{1}{\cos x} \right| = \ln |\sec x|.$$

Therefore, we have a new formula that

$$\int \tan x dx = \ln |\sec x| + C.$$

Definite Integrals' Substitution Rule

STEPS:
1. Choose the right substitution $u = g(x)$
2. Calculate the differentials: $du = g'(x)\,dx$; and the end points $g(a)$ and $g(b)$
3. Replace $g'(x)\,dx = du$ and rewrite the function(integrand) as a function of u and also the endpoints, i.e.,

$$\int_a^b [\text{function of } x]\,dx = \int_{g(a)}^{g(b)} [\text{function of } u]\,du$$

4. Evalute the definite integral.

Example. Evaluate $\int_0^4 \sqrt{2x+1}dx$

Let $u = g(x) = 2x+1$, then

$$\begin{cases} du = g'(x)\,dx = 2xdx & , \\ g(0) = 1 & , \\ g(4) = 9 & . \end{cases}$$

Thus,

$$\int_0^4 \sqrt{2x+1}dx = \int_0^4 \sqrt{2x+1}\cdot\frac{2}{2}\cdot dx = \frac{1}{2}\int_1^9 u^{\frac{1}{2}}du = \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=9} = \frac{1}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{26}{3}.$$

Example. Find$\int_1^2 \frac{dx}{(3-5x)^2}$

Let $u = g(x) = 3-5x$, then $du = g'(x)\,dx = -5dx$ and $g(1) = -2$, $g(2) = -7$.
Thus,

$$\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(3-5x)^2}\cdot\frac{-5}{-5}\cdot dx = -\frac{1}{5}\int_{-2}^{-7} u^{-2}du = -\frac{1}{5}\cdot\frac{u^{-1}}{-1}\Big|_{-2}^{-7} = \frac{1}{5}\left(\frac{1}{-7} - \frac{1}{-2}\right) = \frac{1}{14}.$$

Example. Calculate $\int_1^e \frac{\ln x}{x}dx$

Let $u = g(x) = \ln x \Rightarrow du = \frac{1}{x}dx$ and $g(1) = 0$, $g(e) = 1$, then

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 udu = \frac{u^2}{2}\Big|_{u=0}^{u=1} = \frac{1}{2}.$$

Last Theorem:

Theorem. *Let $A > 0$.*

(1) If $f(x)$ is an odd function, then

$$\int_{-A}^{A} f(x)\,dx = 0.$$

(2) If $f(x)$ is an even function, then

$$\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx.$$

Example.

$$\int_{-\pi}^{\pi} \frac{\sin t}{1+t^{2024}}dt = 0.$$

6.1 AREAS BETWEEN CURVES

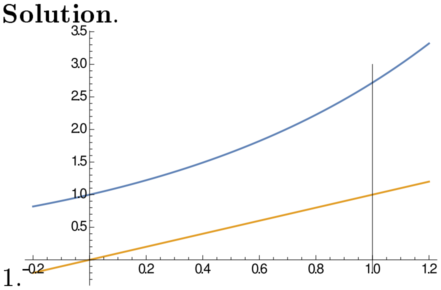
Theorem. *The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by*

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Steps:
1. Draw a diagram
2. If necessary, find the intersections i.e., intervals $[a, b]$ then determine $f(x)$ and $g(x)$
3. Apply the formula
4. Evaluate

Example. Find the area of region bounded by $y = e^x$, $y = x$, $x = 0$ and $x = 1$.

Solution.



1.

2. $a = 0$, $b = 1$, $f(x) = e^x$ and $g(x) = x$. Since $f(x) \geq g(x)$,

$$|f(x) - g(x)| = (e^x - x).$$

3.

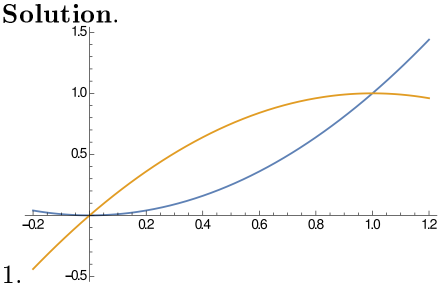
$$A = \int_0^1 |f(x) - g(x)|\, dx = \int_0^1 (e^x - x)\, dx.$$

4.

$$A = \int_0^1 (e^x - x)\, dx = \int_0^1 e^x dx. - \int_0^1 x dx = e^x \Big|_{x=0}^{x=1} - \frac{x^2}{2}\Big|_{x=0}^{x=1} = e - \frac{3}{2}.$$

Example. Find the area of region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution.



1.

2. We need to solve that

$$\begin{cases} y = x^2 \\ y = 2x - x^2 \end{cases} \Rightarrow x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0 \Rightarrow x\,(x-1) = 0 \Rightarrow x = 0, 1$$

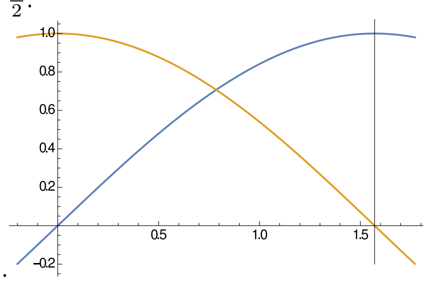
Thus,

$$a = 0, b = 1, f\,(x) = 2x - x^2, g\,(x) = x^2 \text{ and } f\,(x) \geq g\,(x) \text{ when } 0 \leq x \leq 1$$

3.
$$A = \int_0^1 |f\,(x) - g\,(x)|\, dx = \int_0^1 \left[2x - x^2 - x^2\right] dx = \int_0^1 \left(2x - 2x^2\right) dx.$$

4.
$$A = 2\int_0^1 x dx - 2\int_0^1 x^2 dx = 2\frac{x^2}{2}\Big|_{x=0}^{x=1} - 2\frac{x^3}{3}\Big|_{x=0}^{x=1} = 1 - \frac{2}{3} = \frac{1}{3}.$$

Example. Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$.

1. 

2. We need to compute the intersection:

$$\begin{cases} y = \sin x \\ y = \cos x \end{cases} \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}.$$

Then,

$$a = 0, b = \frac{\pi}{2}, f\,(x) = \sin x, g\,(x) = \cos x \text{ and } |f\,(x) - g\,(x)| = \begin{cases} \cos x - \sin x & \text{if } 0 \leq x \leq \frac{\pi}{4} \\ \sin x - \cos x & \text{if } \frac{\pi}{4} < x \leq \frac{\pi}{2} \end{cases}$$

3.
$$A = \int_0^{\frac{\pi}{2}} |f\,(x) - g\,(x)|\, dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx.$$

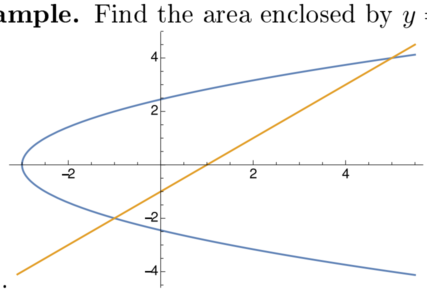
4.
$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx \\ &= \int_0^{\frac{\pi}{4}} \cos x dx - \int_0^{\frac{\pi}{4}} \sin x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x dx \\ &= \sin x\Big|_{x=0}^{x=\frac{\pi}{4}} - (-\cos x)\Big|_{x=0}^{x=\frac{\pi}{4}} - \cos x\Big|_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} - \sin x\Big|_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} \\ &= \left(\frac{\sqrt{2}}{2} - 0\right) + \left(\frac{\sqrt{2}}{2} - 1\right) - \left(0 - \frac{\sqrt{2}}{2}\right) - \left(1 - \frac{\sqrt{2}}{2}\right) \\ &= 2\sqrt{2} - 2. \end{aligned}$$

Special Case for Curves that $x = f(y)$ and $x = g(y)$.

Theorem. *The area A of the region bounded by the curves $x = f(y)$, $x = g(y)$ and the lines $y = c$, $y = d$ is given by*

$$A = \int_c^d |f(y) - g(y)|\, dy = \int_c^d (\textit{curve on the right-curve on the left})\, dy.$$

Example. Find the area enclosed by $y = x - 1$ and $y^2 = 2x + 6$.

1. 

2. Solve

$$\begin{cases} y = x - 1 & \Rightarrow x = y + 1 \\ y^2 = 2x + 6 & \Rightarrow x = \frac{1}{2}\left(y^2 - 6\right) \end{cases} \Rightarrow y+1 = \frac{1}{2}\left(y^2 - 6\right) \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow y = -2, 4$$

Then

$c = -2, d = 4, f(y) = \text{curve on the right} = y+1, g(y) = \text{curve on the left} = \dfrac{1}{2}\left(y^2 - 6\right)$

3.

$$A = \int_c^d (\text{curve on the right-curve on the left})\, dy = \int_{-2}^4 \left[(y+1) - \frac{1}{2}\left(y^2 - 6\right)\right] dy = \int_{-2}^4 \left(y - \frac{y^2}{2} + 4\right) dy$$

4.

$$A = \int_{-2}^4 \left(y - \frac{y^2}{2} + 4\right) dy = \left[\frac{y^2}{2} - \frac{1}{2} \cdot \frac{y^3}{3} + 4y\right] \Bigg|_{y=-2}^{y=4} = 18.$$

6.2 VOLUMES

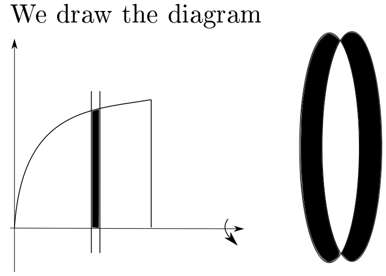
Problem. What is the volume obtained by rotating a region R about a straight line L?

Usually, L is parallel to either x-axis or y-axis, i.e., either vertical or horizontal.

Example. Find the volume of the solid obtained by rotation about the x-axis the region under $y = \sqrt{x}$ from 0 to 1.

1. Because L is the x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.

We draw the diagram



2. Now, each slice becomes a "pie" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the radius of the disc} = y & = f(x) = \sqrt{x} \end{cases}$$

3. We see the small pie has a volume

$$V_{pie} = \pi \cdot [f(x)]^2 \, dx = \pi x dx.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{pie}$$

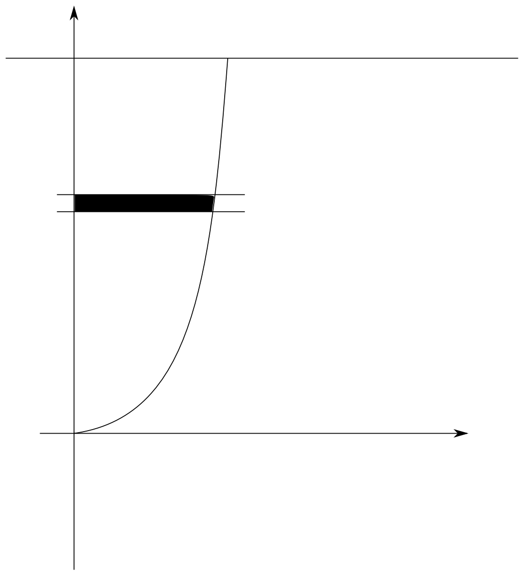
5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{pie} = \int_0^1 \pi x dx = \pi \int_0^1 x dx = \pi \frac{x^2}{2} \Big|_{x=0}^{x=1} = \frac{\pi}{2}.$$

Example. Find the volume of the solid obtained by roatating the region bouned by $y = x^3$, $y = 8$, $x = 0$ about the y-axis.

1. Because L is the y-axis, horizontal, we first cut the region R along the y-axis into vertical slices.

We draw the diagram



2. Now, each slice becomes a “pie” with:

$$\begin{cases} \text{thickness=thickness of slice} & = dy \\ \text{the radius of the disc} = x & = y^{\frac{1}{3}} \end{cases}$$

3. We see the small pie has a volume

$$V_{pie} = \pi \cdot \left[y^{\frac{1}{3}}\right]^2 dy = \pi y^{\frac{2}{3}} dy.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{pie}$$

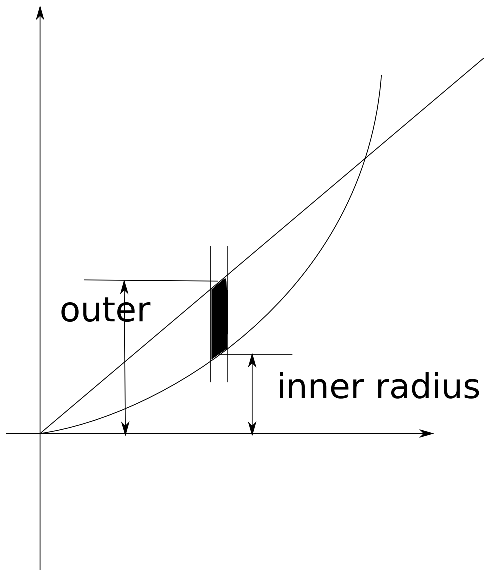
5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{pie} = \int_0^8 \pi y^{\frac{2}{3}} dy = \pi \int_0^8 y^{\frac{2}{3}} dy = \pi \left. \frac{y^{\frac{5}{3}}}{\frac{5}{3}} \right|_{y=0}^{y=8} = \frac{96\pi}{5}.$$

Example. Find the volume of the solid obtained by roatating the region bouned by $y = x$, $y = x^2$ about the x-axis.

1. Because L is the x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.

We draw the diagram



2. Now, each slice becomes a "donat" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the side is a ring rather than a disc with} & \begin{cases} \text{inner radius} & = x^2 \\ \text{outer radius} & = x \end{cases} \end{cases}$$

3. We see the small pie has a volume

$$V_{donat} = (\text{Area of the ring})\cdot\text{thickness} = \pi\left[\left(\text{outer}^2\right) - (\text{inner})^2\right] dx = \pi\left(x^2 - x^4\right) dx.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{donat}$$

5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{donate} = \int_a^b \pi\left(x^2 - x^4\right) dx.$$

6. a and b are determined by the intersections. Solve

$$\begin{cases} y = x^2 \\ y = x \end{cases} \Rightarrow x^2 = x \Rightarrow x = 0, 1 \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$$

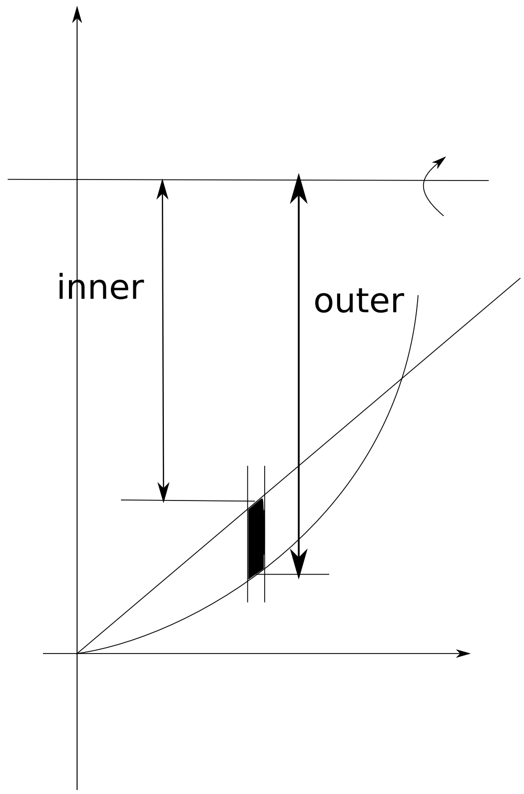
7.

$$V = \int_0^1 \pi\left(x^2 - x^4\right) dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{x=0}^{x=1} = \frac{2\pi}{15}.$$

Example. Find the volume of the solid obtained by roatating the region bouned by $y = x$, $y = x^2$ about the line $y = 2$.

1. Because L is parallel to x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.

We draw the diagram



2. Now, each slice becomes a "donat" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the side is a ring rather than a disc with} & \begin{cases} \text{inner radius} & = 2 - x \\ \text{outer radius} & = 2 - x^2 \end{cases} \end{cases}$$

3. We see the small pie has a volume

$$V_{donat} = \pi \left[\left(\text{outer}^2\right) - (\text{inner})^2 \right] \cdot \text{thickness} = \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{donat}$$

5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{donat} = \int_a^b \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx.$$

6. a and b are determined by the intersections. Solve

$$\begin{cases} y = x^2 \\ y = x \end{cases} \Rightarrow x^2 = x \Rightarrow x = 0, 1 \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$$

7.

$$V = \int_0^1 \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx = \pi \int_0^1 \left(x^4 - 5x^2 + 4x\right) = \pi \left(\frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + 4\frac{x^2}{2} \right) \Bigg|_{x=0}^{x=1} = \frac{8\pi}{15}.$$