# Research Statement

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My current research interests are Experiment Mathematics and Symbolic Computation, involving Special Functions, Combinatorics, Number Theory, Probability Theory, and Information Geometry. The following sections describe topics in details, where terms with ★ are obtained results while that with • refer to future work. (Some items contain both.)

## 1 The Bernoulli symbol

### 1.1 Introduction

The Bernoulli symbol $\mathcal{B}$ is a random variable $\mathcal{B} = \imath L_B - \frac{1}{2}$, where $\imath^2 = -1$ and $L_B$ has probability density function

$$p_{L_B}(t) = \frac{\pi}{2}\operatorname{sech}^2(\pi t) \text{ on } \mathbb{R}.$$

The Bernoulli numbers $(B_n)_{n=0}^{\infty}$ are defined and also can be computed in terms of moments [11, Thm. 2.3, pp. 384] as follows

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t-1} \text{ and } B_n = \mathbb{E}[\mathcal{B}^n] = \frac{\pi}{2}\int_{\mathbb{R}} \left(\imath t - \frac{1}{2}\right)^2 \operatorname{sech}^2(\pi t)\,dt, \tag{1.1}$$

which interprets the evaluation (eval) of umbral symbol [32] as the expectation: $eval(\mathcal{B}^n) = B_n = \mathbb{E}[\mathcal{B}^n]$. Therefore,

1. properties of umbral calculus are preserved, e.g., Bernoulli polynomials $B_n(x)$ are defined and symbolically expressed as
$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1} \Rightarrow B_n(x) = \mathbb{E}[(\mathcal{B}+x)^n] = \frac{\pi}{2}\int_{\mathbb{R}}\left(x + \imath t - \frac{1}{2}\right)^2 \operatorname{sech}^2(\pi t)\,dt;$$
2. formulae can be simplified into more compact form:
$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}B_l n^{m+1-l} = \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}\mathcal{B}^l n^{m+1-l} = \int_0^n (\mathcal{B}+x)^m\,dx;$$
3. formulae can also be visualized more directly, e.g., a direct calculus fact reveals the derivative of $B_n(x)$ as
$$B_n'(x) = B_{n-1}(x) \Leftrightarrow [(\mathcal{B}+x)^n]' = n(\mathcal{B}+x)^{n-1}.$$

More than traditional umbral calculus, Bernoulli symbol admits probabilistic technique to further explore as follows.

1. Conjugate variables, e.g., $X$ and $Y$, are defined as
$$\mathbb{E}[(X+Y)^n] = \delta_{0,n}, \text{ or equivalently, } \mathbb{E}\left(e^{tX}\right)\cdot\mathbb{E}\left(e^{tY}\right) = 1.$$
Easily, the uniform random variable (or uniform symbol) $\mathcal{U} \sim U[0,1]$ and $\mathcal{B}$ are conjugate, since
$$\mathbb{E}\left(e^{t\mathcal{U}}\right) = \int_0^1 e^{tu}\cdot 1du = \frac{e^t-1}{t} \text{ and } \mathbb{E}\left(e^{t\mathcal{B}}\right) = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = \frac{t}{e^t-1}.$$
Therefore, consider the case that $f(x) = x^n$. Since $\mathcal{B}$ and $\mathcal{U}$ cancel each other in powers,
$$f(x) = f(x+\mathcal{B}+\mathcal{U}) = \int_0^1 f(x+\mathcal{B}+u)\,du \Rightarrow B_n(x+1) - B_n(x) = nx^{n-1}.$$
2. Moment generating functions of independent random variables are the product of individual ones, i.e.,
$$\mathbb{E}\left[e^{t(X_1+\cdots+X_n)}\right] = \mathbb{E}\left[e^{tX_1}\right]\cdots\mathbb{E}\left[e^{tX_n}\right].$$
Thus, Nörlund polynomials $B_n^{(p)}(x)$ are defined and symbolically expressed as
$$\sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{t^n}{n!} = e^{tx}\left(\frac{t}{e^t-1}\right)^p \Rightarrow B_n^{(p)}(x) = (\mathcal{B}_1 + \cdots + \mathcal{B}_p + x)^n,$$
where $\{\mathcal{B}_k\}_{k=1}^{p}$ is a sequence of independent and identical distribution (i. i. d. s. ), with $\mathcal{B}_k \sim \mathcal{B}$ (throughout this statement, where $p$ may vary).

### 1.2 Main results

★ The Bernoulli-Barnes polynomials $B_n(\mathbf{a};x)$, where $\mathbf{a}=(a_1,\ldots,a_k)$ satisfies $|\mathbf{a}|=\prod_{l=1}^{k} a_l \neq 0$, is defined and symbolically expressed [25, eq. 1.17, pp. 651] as, by letting $\boldsymbol{\mathcal{B}}=(\mathcal{B}_1,\ldots,\mathcal{B}_k)$:

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t}-1}=\sum_{n=0}^{\infty}B_n(\mathbf{a};x)\frac{t^n}{n!}\Rightarrow B_n(\mathbf{a};x)=\frac{1}{|\mathbf{a}|}\left(x+\mathbf{a}\cdot\boldsymbol{\mathcal{B}}\right)^n,\ \text{where } \mathbf{a}\cdot\boldsymbol{\mathcal{B}}=\sum_{l=1}^{k}a_l\mathcal{B}_l.$$

Based on this symbolic expression, we have,

1. for polynomial $P$, [25, Thm. 2.2, pp. 652]

$$P(x-\mathbf{a}\cdot\boldsymbol{\mathcal{B}})=\sum_{j=0}^{n}\sum_{|J|=j}|a|_{J^*}P^{(n-j)}\left(x+(\mathbf{a}\cdot\boldsymbol{\mathcal{B}})_J\right),\ \text{where } J\subset[n]:=\{1,\ldots,n\}\ \text{and } J^*=[n]\backslash J,$$

which, by taking $P(x)=\frac{x^m}{|\mathbf{a}|m!}$, gives the difference formula [6, Thm. 5.1]:

$$(-1)^m B_m(-x;\mathbf{a})-B_m(x;\mathbf{a})=m!\sum_{j=0}^{n-1}\sum_{|J|=j}\frac{B_{m-n+j}(x;\mathbf{a}_J)}{(m-n+j)!};$$

2. for $A:=a_1+\cdots+a_n\neq 0$, direct proofs of self-duality [25, Thm. 5.1, pp. 656] of sequence $\left(\left(-\frac{1}{A}\right)^n B_n(\mathbf{a})\right)_{n=0}^{\infty}$ and the symmetry formula [25, Thm. 5.2, pp. 656]

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}A^{m-k}B_{l+k}(x;\mathbf{a})=(-1)^l\sum_{k=0}^{l}\binom{l}{k}A^{l-k}B_{m+k}(-x;\mathbf{a}),$$

which Bayad and Beck [6] are looking for.

★ The multiple zeta values (MZV),

$$\zeta_r(n_1,\ldots,n_r)=\sum_{k_1,\ldots,k_r>0}\frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}\cdots(k_1+\cdots+k_r)^{n_r}}$$

have more than one analytic continuations at non-positive integers. For instance, Sadaoui [33, Thm. 1] used the Raabe's identity while Akiyama and Yanigawa [2, pp. 350] considered the Euler-Maclaurin summation. Since both results involve Bernoulli number, applying Bernoulli symbol reveals, to our surprise, that both analytic continuations coincide, symbolically expressed as [24, Thm. 2.1], for $n_1,\ldots,n_k\in\mathbb{Z}_-\cup\{0\}$,

$$\zeta_r(n_1,\ldots,n_r)=\prod_{k=1}^{r}(-1)^{n_k}\mathcal{C}_{1,\ldots,k}^{n_k+1},\ \text{where recursively } \mathcal{C}_1^n=\frac{(\mathcal{B}_1)^n}{n},\ \mathcal{C}_{1,\ldots,k+1}=\frac{(\mathcal{C}_{1,\ldots,k}+\mathcal{B}_{k+1})^n}{n},$$

Recurrence [24, Thm. 3.1]

$$\zeta_r(-n_1,\ldots,-n_r)=\frac{(-1)^{n_r}}{n_r+1}\sum_{k=0}^{n_r+1}\binom{n_r+1}{k}(-1)^k\zeta_{r-1}(-n_1,\ldots,-n_{r-2},-n_{r-1}-k)B_{n_r+1-k},$$

contiguity identities [24, Thm. 4.1] and generating functions [24, Thm. 5.1] follow naturally.

### 1.3 Future work

- Explanation for the coincidence of two analytic continuations mentioned above;
- Application to other types of zeta functions, e.g., Witten-zeta function, where Bernoulli number are also involved;
- Further probabilistic study on Bernoulli symbol in the probabilistic approach;
- ★ Symbolic expressions and results on hypergeometric Bernoulli numbers [9, eq. 1.13, pp. 1763], polynomials [9, eq. 7.8, pp. 1778] and hypergeometric zeta functions [9, eq. 1.12, pp. 1763];
- ★ Analogue on Euler numbers, denoted by $E_n$, which similarly satisfies [26, Prop. 2.1, pp. 778]:

$$E_n=\int_{\mathbb{R}}\left(\iota t-\frac{1}{2}\right)^2\operatorname{sech}(\pi t)\,dt. \tag{1.2}$$

## 2 Matrix presentation of harmonic sums and multiplicative nested sums

### 2.1 Introduction

Nested sums are given by $\mathcal{S}(f;k;N) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} f(n_1,\ldots,n_k)$, and we focus on the special case that $f(n_1,\ldots,n_k) = f_1(n_1)\cdots f_k(n_k)$, called *multiplicative nested sums*. A large class of multiple nested sums is the harmonic sums [8, eq. 4, pp. 1]: for indices $a_1,\ldots,a_k \in \mathbb{R}\setminus\{0\}$,

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times\cdots\times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}},\ N\in\mathbb{N}, \tag{2.1}$$

which connect to a variety of topics, including combinatorics, quantum field theory, and naturally zeta functions, e.g., taking $N\to\infty$, $k=1$ and $a_1>0$: $S_{a_1}(\infty) = \zeta(a_1)$. Instead of the recurrence [7, eq. 2.1, pp. 21] that inherited from the quasi-shuffle relation [21, eq. 1, pp. 51], which the `Mathematica` package `HarmonicSums.m`[1] [1, Chpt. 6] uses, we associate each factor $f_l$, $l=1,\ldots,k$, an *index matrix*:

$$\mathcal{P}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix}$$

to provide alternative computations and interpretations.

### 2.2 Main results

★ If additionally define $f_0(x) = \frac{1}{x}$, then [23, Thm. 15]:

$$\mathcal{S}(f;k;N) = N\cdot\left(\prod_{l=0}^{k}\mathcal{P}_{N|f_l}\right)_{N,1}, \tag{2.2}$$

the $N^{\text{th}}$ row and $1^{\text{st}}$ column entry of the product matrix. Taking $f_l(n_l) = \operatorname{sign}(a_l)^{n_l}/n_l^{|a_l|}$, the harmonic sums admit the same result [23, Thm. 1], which is more direct than a recurrence.

★ For the special case of harmonic sums when $a_1=\cdots=a_k=a\geq 1$ and $N<\infty$, a random walk over finite number of sites [23, Sec. 3] interprets the index matrix as the stochastic matrix and the harmonic sums as probabilities of certain paths.

★ When all $f_l$'s are identical, the product (2.2) becomes powers. In particular, if $\{f_l(1),\ldots,f_l(N)\}$ are all distinct, then $\mathcal{P}_{N|f_l}$ is diagonalizable, so that

$$\mathcal{P}_{N|f_l} = \mathcal{Q}_{N|f_l}\operatorname{diag}\{f_l(1),\ldots,f_l(N)\}\mathcal{Q}_{N|f_l}^{-1} \Rightarrow \mathcal{P}_{N|f_l}^m = \mathcal{Q}_{N|f_l}\operatorname{diag}\{f_l(1)^m,\ldots,f_l(N)^m\}\mathcal{Q}_{N|f_l}^{-1},$$

the alternative computation for its powers lead to combinatorial identities:

1. $f_1(x) = x$ and $k=1$ gives binomial-type identity:

$$\sum_{l=1}^{N}(-1)^{N-l}\,l^{N+1}\binom{N}{l} = \frac{N(N+1)!}{2};$$

2. $f_1\equiv\cdots\equiv f_k = f$ with $f(m) = a_m$ leads to [35, eq. 2, pp. 313]:

$$\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} a_{n_1}\cdots a_{n_k} = \sum_{j=1}^{N}\left(\prod_{\substack{m=1\\ m\neq j}}^{N}\frac{1}{1-\frac{a_m}{a_j}}\right)a_j^k,$$

which, when $a_j = \frac{a-bq^{j+i-1}}{c-zq^{j+i-1}}$ and $N = n-i+1$, "turns out to be a common source of several $q$-identities" [35, pp. 314];

3. special harmonic sum (2.1) when $a_1=\cdots=a_k=1$ recovers [10, Cor. 3, pp. 93]:

$$\sum_{l=1}^{N}(-1)^{l-1}\binom{N}{l}\frac{1}{l^k} = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{1}{n_1\cdots n_k}.$$

### 2.3 Future work

- Matrix interpretation of the quasi-shuffle relation, i.e., the recurrence of harmonic sums;
- Modification of the index matrices to visualize the limit $N\to\infty$;
- Systematic algorithms for finding and proving combinatorial identities through index matrices.

[1] http://www.risc.jku.at/research/combinat/software/HarmonicSums/index.php

## 3 The Method of Brackets (MoB)

### 3.1 Introduction

*The method of brackets* (MoB), developed by I. Gonzalez [16, 17], has its origin on the evaluation of definite integrals arising from the Schwinger parametrization of Feynman diagrams. Besides examples of its application appearing in [5, 12, 13, 14, 15], software implementation has been produced by K. Kohl in [27] using Sage with internal use of Mathematica. In concrete, MoB evaluates the integral $\int_0^\infty f(x)\,dx$, by the following rules:

Rule $P_1$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)dx = \sum_n a_n \langle \alpha n + \beta \rangle.$$

Rule $P_2$: For $\alpha \notin \mathbb{N} \cup \{0\}$,

$$(a_1 + \cdots + a_r)^\alpha = \sum_{m_1,\ldots,m_r} \phi_{m_1,\ldots,m_r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle -\alpha + a_1 + \cdots + a_r \rangle}{\Gamma(-\alpha)},$$

where $\phi_{m_1,\ldots,m_r} = \prod_{i=1}^{r} \phi_{m_i}$ is the product of *indicators* defined by $\phi_n := (-1)^n / \Gamma(n+1)$.

Rule $E_1$:

$$\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*), \text{ where } n^* \text{ solves } \alpha n + \beta = 0$$

Rule $E_2$:

$$\sum_{n_1,\ldots n_r} \phi_{n_1,\ldots,n_r} \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = f(n_1^*, \ldots, n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*).$$

where $(n_1^*, \ldots, n_r^*)$ solves $A(n_1, \ldots n_r)^T + (c_1, \ldots, c_r)^T = \vec{0}^T$ for non-singular matrix $A = (a_{ij})_{r\times r}$.

Rule $E_3$: Each representation of an integral by a bracket series has associated an index as

$$\text{index} = \text{number of sums} - \text{number of brackets}.$$

The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule $E_2$. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is discarded too.

Rule $E_4$: Fix $k \in \mathbb{N}$, and for $m \in \mathbb{N}$, Pochhammer symbols with negative index and negative integer base are evaluated as

$$(-km)_{-m} = \frac{k}{k+1} \cdot \frac{(-1)^m (km)!}{((k+1)m)!} \tag{3.1}$$

Basic results and examples can be found in [12, 13, 14, 15].

### 3.2 Main results

- ★ The rule $E_4$ was recently introduced [18, eq. 27, pp. 685] from the analytic continuation of Pochhammer symbol with negative integer index and negative integer base, while calculating entry **6.671.7** in [19].
- ★ We verify that this method does not depend on factorization of the integrand [22, Thm. 4.2.2, pp. 24], i.e., if $f(x) = \prod_{i=1}^{r} g_i(x)$, applying MoB to compute $\int_0^\infty f(x)\,dx$ and $\int_0^\infty \prod_{i=1}^{r} g_i(x)\,dx$ leads to the same results.
- ★ We compared MoB with the negative dimension integration method [22, Sec. 8.1], showing two methods are completely different; and also with the integrating by differentiating method [22, Sec. 8.2], revealing their formal connections.
- ★ A pure Mathematica package [22, Chpt. 9] is developing and current version recovers examples in [14, 15].

### 3.3 Future work

- Further study on Ramanujan's master theorem [20, pp. 186, eq. B], which is the funding theorem of MoB;
- Application of the `Sigma.m`[2] package developed by Schneider [34] to the obtained series from MoB;
- Development of a purely Sage package.

[2] http://www.risc.jku.at/research/combinat/software/Sigma/index.php

## 4 Differential Geometry and Information Geometry

### 4.1 Introduction

In order to apply differential geometry on probability and statistics, Amari [3, 4] initiated the theory of information geometry working over statistical manifold

$$S := \{p(x;\theta)\,|\theta \in \Theta \subseteq \mathbb{R}^n \text{ and } \ p \text{ is a probability density function.}\}$$

A series of concepts such as $\alpha$-connections, dual connections and Fisher metrics are consequently introduced and studied. In particular, with the help of the Fisher information matrix, which is defined as

$$g_{ij}(\theta) := \mathbb{E}[(\partial_i l)(\partial_j l)] = \int (\partial_i l)(\partial_j l) p(x;\theta) dx, \quad i,j = 1,2,\ldots,n, \text{ where } \partial_i := \frac{\partial}{\partial \theta_i} \text{ and } l := \log p(x;\theta),$$

the geometric structure is constructed and also highly connected to information theory. A remarkable result which motivated Amari is that the sectional curvature of the manifold consisting of normal distributions is $-\frac{1}{2}$, the hyperbolic geometry.

Currently, information geometry can be well applied to exponential family (including normal distributions), whose the probability density function $p(x)$ can be expressed in terms of functions $\{C, F_1, \ldots, F_n\}$ and a convex function $\phi$ on $\Theta$ as:

$$p(x;\theta) = \exp\left\{C(x) + \sum_i \theta_i F_i(x) - \phi(\theta)\right\}.$$

### 4.2 Main results

★ Both 1st [36, Thm. 1.1, pp. 1102] and 2nd [36, Thm. 1.2, 1.3, pp. 1102–1103] arc length variational formulae are given.
★ The $\alpha$-structure of frame bundles [30, Thm. 5.8] is obtained.
★ A list of possible holonomy groups [31, Thm. 5.3] of exponential family is calculated.
★ A natural gradient algorithm [37, Thm. 1, pp. 4342] is introduced to design the controller of an open-loop stochastic distribution control system (SDCS) of multi-input and single output.
★ An algorithm [29, Algorithm 1, pp. 6] for optimal control on $SE(n)$ is studied and simulated.
★ While considering the geometric means of matrices for the trace class operator, the Rényi entropy [28, Thm. 1], Tsallis entropy [28, Cor. 1], and Shannon entropy [28, Thm. 2] uncertainty relations are studied.

### 4.3 Future work

Currently, I am mainly interested in

- exploring the application of information geometry technique to hyperbolic secant family, namely the probability density function of the form $C(s)\operatorname{sech}^t(x)$,

since both Bernoulli symbol (1.1) and Euler symbol (1.2) belong to this family.

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