Bessel Random Walks and Identities for Higher-order Bernoulli and Euler Polynomials

Lin Jiu

The 18th International Conference on Fibonacci Numbers and Their Applications
@ Dalhousie University

July 3rd, 2018

Joint Work with

loading...

Christophe Vignat

Euler polynomials

Generating function

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z + 1}e^{xz}$$

Euler polynomials

Generating function

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz}$$

generalization:

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz}$$

Euler polynomials

Generating function

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz}$$

generalization:

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz}$$

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x)E_{k_2}(0)\cdots E_{k_p}(0).$$

Euler polynomials

Generating function

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz}$$

generalization:

$$\sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^p e^{xz}$$

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x)E_{k_2}(0)\cdots E_{k_p}(0).$$

inversed formula

$$E_n(x) = P\left(E_{n_1}^{(p_1)}(x),\ldots,E_{n_k}^{(p_k)}(x)\right)?$$

Y-N-Y

Y-N-Y

Theorem(L. Jiu, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

Y-N-Y

Theorem(L. Jiu, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell,$$

Y-N-Y

Theorem(L. Jiu, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \quad T_N(\cos\theta) = \cos(N\theta) \quad \text{Chebyshev}$$

Y-N-Y

Theorem(L. Jiu, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \quad T_N(\cos\theta) = \cos(N\theta) \quad \text{Chebyshev}$$

$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1$

Y-N-Y

Theorem(L. Jiu, V. H. Moll and C. Vignat, 2014)

For any positive integer N,

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

where

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell, \quad T_N(\cos\theta) = \cos(N\theta) \quad \text{Chebyshev}$$

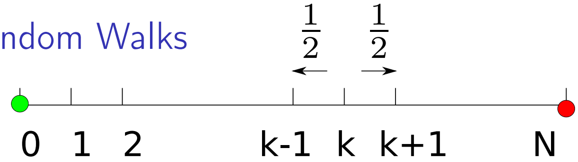
$\underline{N = 2:}$ $T_2(z) = 2z^2 - 1$

$$\frac{1}{T_2(1/z)} = \frac{z^2}{2 - z^2} \Rightarrow p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

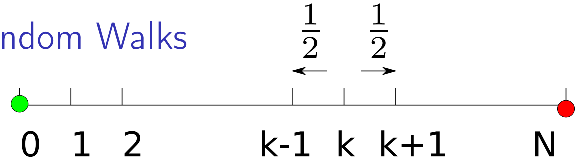
$$E_n(x) = \frac{1}{2^n} \sum_{\ell=2}^{\infty} p_\ell^{(2)} E_n^{(\ell)} \left(\frac{\ell}{2} - 1 + 2x \right).$$

Random Walks

Random Walks

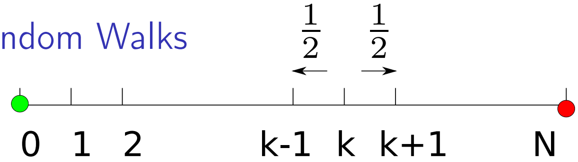


Random Walks



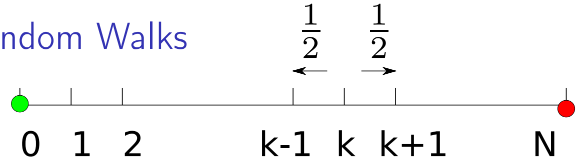
- 0 is the source and N is the sink;

Random Walks



- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk

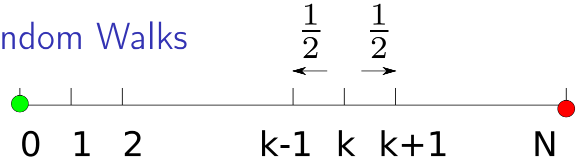
Random Walks



- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk

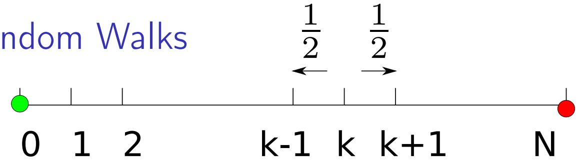
$$\mathbb{P}\left(k+1|k\right) = \mathbb{P}\left(k-1|k\right) = \frac{1}{2};$$

Random Walks



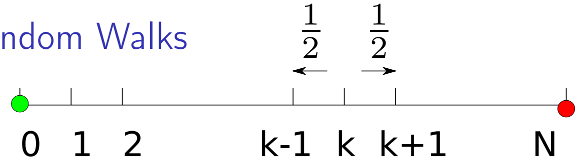
- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk
$$\mathbb{P}\,(k+1|k) = \mathbb{P}\,(k-1|k) = \frac{1}{2};$$
- let ν_N be the random number of steps to reach the sink starting from the source.

Random Walks



- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk
$$\mathbb{P}\,(k+1|k) = \mathbb{P}\,(k-1|k) = \frac{1}{2};$$
- let ν_N be the random number of steps to reach the sink starting from the source.
$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

Random Walks



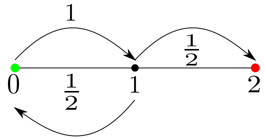
- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk

$$\mathbb{P}\left(k+1|k\right) = \mathbb{P}\left(k-1|k\right) = \frac{1}{2};$$

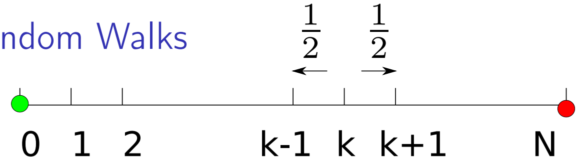
- let ν_N be the random number of steps to reach the sink starting from the source.

$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

$\underline{N = 2:}$



Random Walks



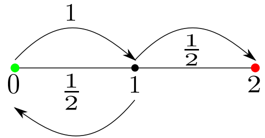
- 0 is the source and N is the sink;
- at each $k = 1, \ldots, N-1$, it is a "fair coin" walk

$$\mathbb{P}\left(k+1|k\right) = \mathbb{P}\left(k-1|k\right) = \frac{1}{2};$$

- let ν_N be the random number of steps to reach the sink starting from the source.

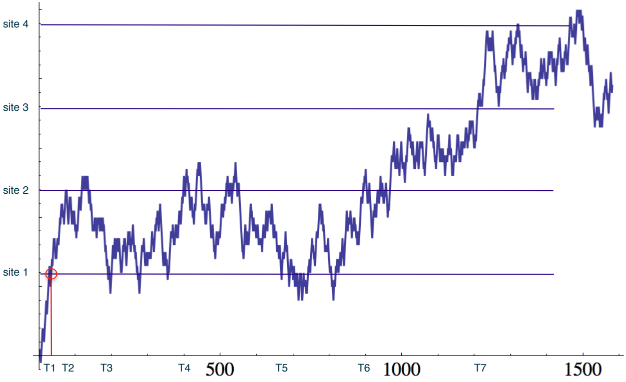
$$p_\ell^{(N)} = \mathbb{P}(\nu_N = \ell)$$

$\underline{N = 2:}$

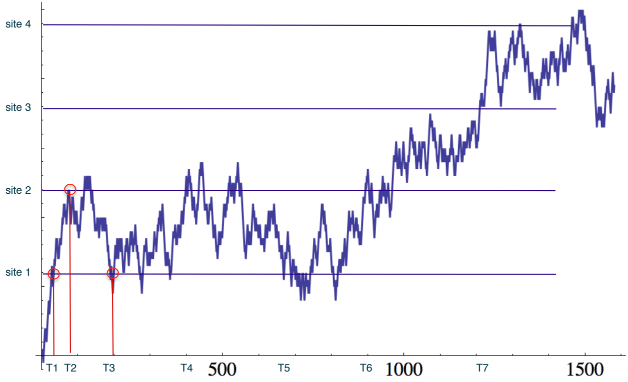


$$p_\ell^{(2)} = \begin{cases} \frac{1}{2^k}, & \text{if } \ell = 2k; \\ 0, & \text{otherwise} \end{cases}$$

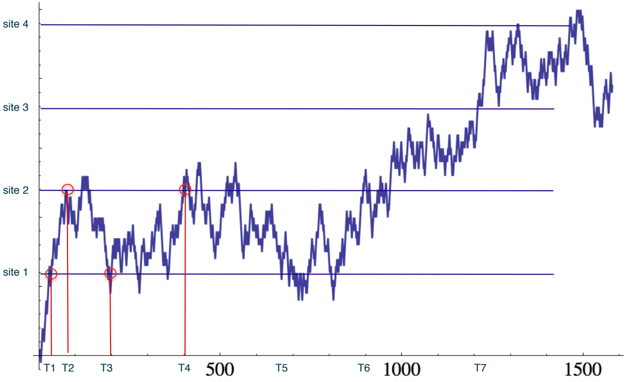
Reflected Brownian Motion



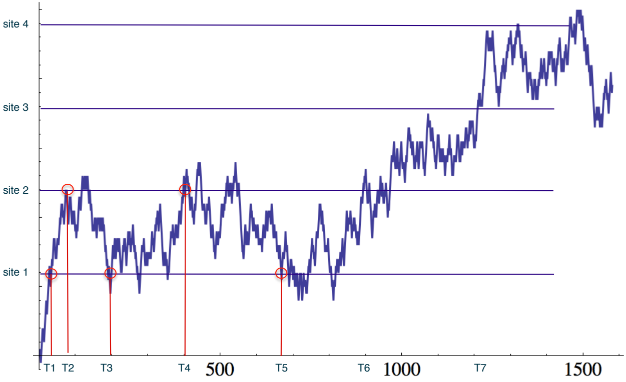
Reflected Brownian Motion



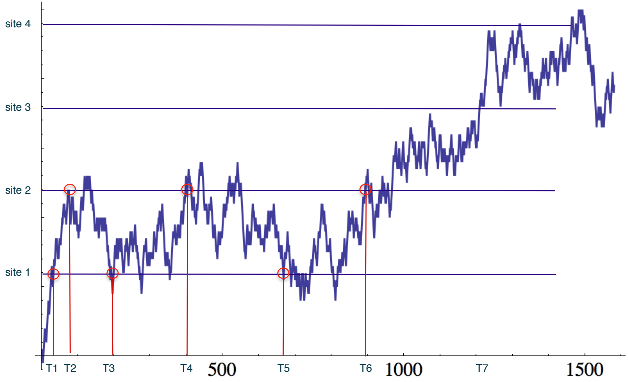
Reflected Brownian Motion



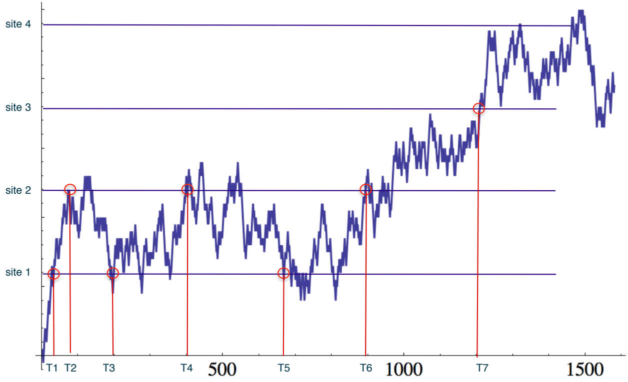
Reflected Brownian Motion



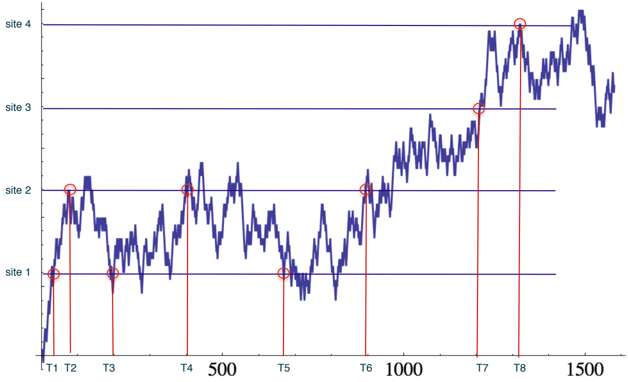
Reflected Brownian Motion



Reflected Brownian Motion



Reflected Brownian Motion



Random Sum

- L_j are independent and identically distributed with hyperbolic secant density

Random Sum

- L_j are independent and identically distributed with hyperbolic secant density

$$\mathbb{E}\left[L_j^n\right] = \int_{\mathbb{R}} t^n \operatorname{sech}(\pi t)\mathrm{d}t;$$

Random Sum

- L_j are independent and identically distributed with hyperbolic secant density

$$\mathbb{E}\left[L_j^n\right] = \int_{\mathbb{R}} t^n \operatorname{sech}(\pi t)\mathrm{d}t;$$

- ν_N is an integer valued random variable independent of the L_j 's:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)},$$

Random Sum

- L_j are independent and identically distributed with hyperbolic secant density

$$\mathbb{E}\left[L_j^n\right] = \int_{\mathbb{R}} t^n \operatorname{sech}(\pi t)\mathrm{d}t;$$

- ν_N is an integer valued random variable independent of the L_j 's:

$$\mathbb{E}\left[z^{\nu_N}\right] = \frac{1}{T_N\left(\frac{1}{z}\right)},$$

Theorem. (Klebanov et al. 1996). The random variable

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j$$

has the same hyperbolic secant distribution.

Hyperbolic Secant

Let $L \sim \text{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) \quad = \quad \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \text{sech}\,(\pi t)\, \mathrm{d}t.$$

Hyperbolic Secant

Let $L \sim \text{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \text{sech}\left(\pi t\right) \mathrm{d}t.$$

The proof uses the integral:

$$\int_{\mathbb{R}} t^k \text{sech}\left(\pi t\right) dt = \frac{|E_k|}{2^k}.$$

Hyperbolic Secant

Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) \;=\; \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\,\mathrm{d}t.$$

The proof uses the integral:

$$\int_{\mathbb{R}} t^k \operatorname{sech}(\pi t)\, dt = \frac{|E_k|}{2^k}. \quad \left(\mathbb{E}\left[e^{z\left(x+iL-\frac{1}{2}\right)}\right] = \frac{2}{e^z+1} e^{xz}\right)$$

Hyperbolic Secant

Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\,\mathrm{d}t.$$

The proof uses the integral:

$$\int_{\mathbb{R}} t^k \operatorname{sech}(\pi t)\, dt = \frac{|E_k|}{2^k}. \quad \left(\mathbb{E}\left[e^{z\left(x+iL-\frac{1}{2}\right)}\right] = \frac{2}{e^z+1} e^{xz}\right)$$

Let $\{L_j\}_{1 \leq j \leq p}$ be p independent random variables $L_j \sim \operatorname{sech}(\pi t)$. The generalized Euler polynomial is given by

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \left(iL_1 - \frac{1}{2}\right) + \cdots + \left(iL_p - \frac{1}{2}\right)\right)^n\right].$$

Hyperbolic Secant

Let $L \sim \operatorname{sech}(\pi t)$, then the Euler polynomial is given by

$$E_n(x) = \mathbb{E}\left[\left(x + iL - \frac{1}{2}\right)^n\right] = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\,\mathrm{d}t.$$

The proof uses the integral:

$$\int_{\mathbb{R}} t^k \operatorname{sech}(\pi t)\, dt = \frac{|E_k|}{2^k}. \quad \left(\mathbb{E}\left[e^{z\left(x+iL-\frac{1}{2}\right)}\right] = \frac{2}{e^z+1}e^{xz}\right)$$

Let $\{L_j\}_{1\le j\le p}$ be p independent random variables $L_j \sim \operatorname{sech}(\pi t)$. The generalized Euler polynomial is given by

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \left(iL_1 - \frac{1}{2}\right) + \cdots + \left(iL_p - \frac{1}{2}\right)\right)^n\right].$$

L. B. Klebanov, A. V. Kakosyan, S. T. Rachev, and G. Temnov. On a class of distributions stable under random summation. J. Appl. Prob., 49:303–318, 2012.

Probabilistic Interpretation

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

Probabilistic Interpretation

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

$$L \sim \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \Rightarrow x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N} \sum_{j=1}^{\nu_N} iL_j \right) - \frac{1}{2}$$

$$\Rightarrow x + iL - \frac{1}{2} \sim \frac{1}{N} \sum_{j=1}^{\nu_N} \left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2} \right)$$

Probabilistic Interpretation

$$E_n(x) = \frac{1}{N^n} \sum_{\ell=N}^{\infty} p_\ell^{(N)} E_n^{(\ell)} \left(\frac{\ell - N}{2} + Nx \right),$$

$$L \sim \frac{1}{N} \sum_{j=1}^{\nu_N} L_j \Rightarrow x + iL - \frac{1}{2} \sim x + \left(\frac{1}{N} \sum_{j=1}^{\nu_N} iL_j \right) - \frac{1}{2}$$

$$\Rightarrow x + iL - \frac{1}{2} \sim \frac{1}{N} \sum_{j=1}^{\nu_N} \left(iL_j - \frac{\nu_N}{2} + Nx - \frac{N}{2} + \frac{\nu_N}{2} \right)$$

Take moments.

Hitting Time

Consider

- a linear Brownian motion W_t starting from 0
- the hitting time T of level $z = 1$, denoted by W_t
- another independent Brownian motion ω_t.

Then

$$\omega_T \sim \operatorname{sech}(x).$$

Denote

$$T_1 < T_2 < \cdots < T_I = T$$

the successive epochs at which W_t visits the sites $z_i = \frac{i}{N}$, $0 \leq i \leq N$.

Hitting Time

This defines a random walk with

$$p_\ell^{(N)} = \mathbb{P}\{W_t \text{ reaches the sink in } \ell \text{ steps}\}.$$

Now write

$$T = (T - T_{\ell-1}) + (T_{\ell-1} - T_{\ell-2}) + \cdots + (T_1 - 0)$$

and

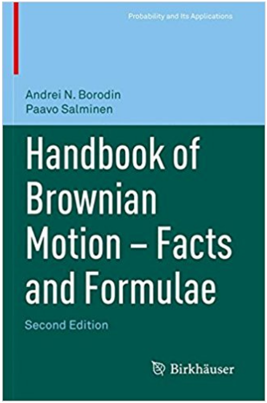
$$\omega_T \sim \omega_{T-T_{\ell-1}} + \omega_{T_{\ell-1}-T_{\ell-2}} + \cdots + \omega_{T_1-0},$$

each term $\sim \operatorname{sech}(x)$.
This corresponds Klebanov's random sum decomposition

$$Z_N = \frac{1}{N}\sum_{j=1}^{\nu_N} L_j.$$

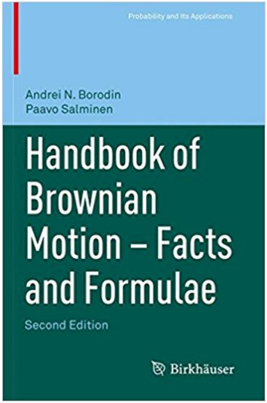
Generalization

Generalization

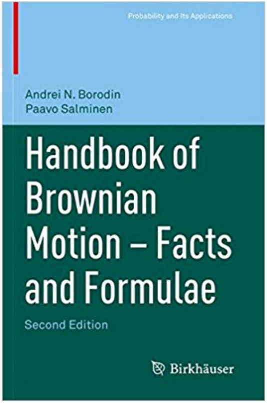


Generalization

- Bessel process in $\mathbb{R}^n$:
$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$



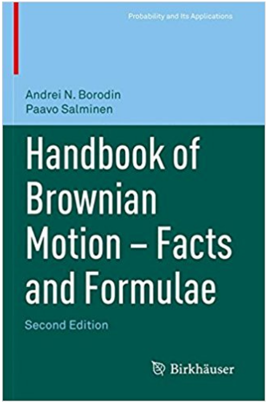
Generalization



- Bessel process in $\mathbb{R}^n$:
$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$
- Moment generating functions for hitting times: $H_z := \min_s \left\{R_s^{(n)} = z\right\}$.

$$\mathbb{E}_x\left(e^{-\alpha H_z};\ \sup_{0\le s\le H_z} R_s^{(n)} < y\right) = \begin{cases} \frac{x^{-\nu} I_\nu(x\sqrt{2\alpha})}{z^{-\nu} I_\nu(z\sqrt{2\alpha})}, & 0 \le x \le z \le y; \\ \frac{S_\nu\left(y\sqrt{2\alpha}, x\sqrt{2\alpha}\right)}{S_\nu\left(y\sqrt{2\alpha}, z\sqrt{2\alpha}\right)}, & z \le x \le y, \end{cases} \quad (2.1.4)$$

Generalization

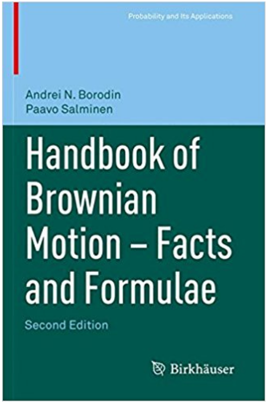


- Bessel process in $\mathbb{R}^n$:
$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$
- Moment generating functions for hitting times: $H_z := \min_s \left\{ R_s^{(n)} = z \right\}$.

$$\mathbb{E}_x \left(e^{-\alpha H_z};\ \sup_{0 \le s \le H_z} R_s^{(n)} < y \right)$$
$$= \begin{cases} \dfrac{x^{-\nu} I_\nu(x\sqrt{2\alpha})}{z^{-\nu} I_\nu(z\sqrt{2\alpha})}, & 0 \le x \le z \le y; \\ \dfrac{S_\nu\left(y\sqrt{2\alpha}, x\sqrt{2\alpha}\right)}{S_\nu\left(y\sqrt{2\alpha}, z\sqrt{2\alpha}\right)}, & z \le x \le y, \end{cases} \quad (2.1.4)$$

- dimension $n = 2 + 2\nu$ for $\nu \ge 0$

Generalization



- Bessel process in $\mathbb{R}^n$:
$$R_t^{(n)} := \sqrt{\left(W_t^{(1)}\right)^2 + \cdots + \left(W_t^{(n)}\right)^2}$$
- Moment generating functions for hitting times: $H_z := \min_s \left\{R_s^{(n)} = z\right\}$.

$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\le s\le H_z} R_s^{(n)} < y\right) = \begin{cases} \dfrac{x^{-\nu} I_\nu(x\sqrt{2\alpha})}{z^{-\nu} I_\nu(z\sqrt{2\alpha})}, & 0 \le x \le z \le y; \\ \dfrac{S_\nu\left(y\sqrt{2\alpha}, x\sqrt{2\alpha}\right)}{S_\nu\left(y\sqrt{2\alpha}, z\sqrt{2\alpha}\right)}, & z \le x \le y, \end{cases} \quad (2.1.4)$$

- dimension $n = 2 + 2\nu$ for $\nu \ge 0$

$$S_\nu(x, y) := (xy)^{-\nu}\left[I_\nu(x)K_\nu(y) - K_\nu(x)I_\nu(y)\right].$$

$n = 3 \Leftrightarrow \nu = 1/2$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)}$$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh(x)$$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh(x)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{t}{e^t - 1} e^{tx}$$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh\left(x\right)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}} \sinh\left(x\right)$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2} \sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

$n = 3 \Leftrightarrow \nu = 1/2$

- $$I_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2m+\frac{1}{2}}}{m!\Gamma\left(m+\frac{3}{2}\right)} = \sqrt{\frac{2}{x\pi}}\sinh(x)$$

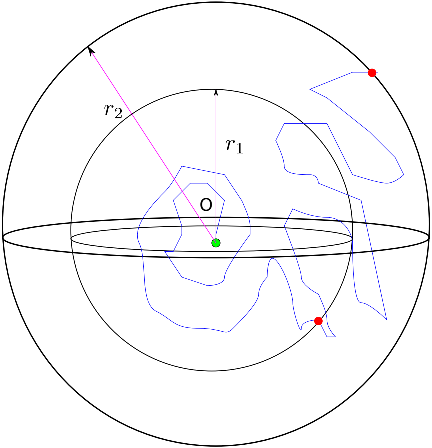
$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{t}{e^t - 1}e^{tx} = \frac{te^{tx}e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} = \frac{te^{t\left(x-\frac{1}{2}\right)}}{2}\sinh\left(\frac{t}{2}\right)$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$$

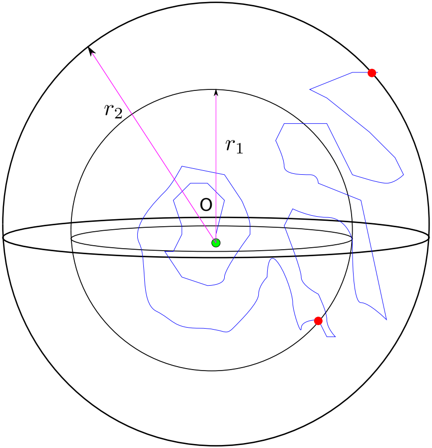
$$\mathbb{E}_x\left(e^{-\alpha H_z}; \sup_{0\leq s\leq H_z} R_s^{(3)} < y\right) = \begin{cases} \dfrac{z\sinh\left(x\sqrt{2\alpha}\right)}{x\sinh\left(z\sqrt{2\alpha}\right)}, & 0 \leq x \leq z \leq y \\ \dfrac{z\sinh\left((y-x)\sqrt{2\alpha}\right)}{x\sinh\left((y-z)\sqrt{2\alpha}\right)}, & z \leq x \leq y \end{cases}$$

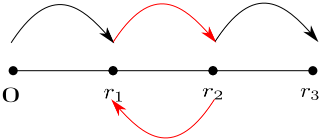
$n = 3 \Leftrightarrow \nu = 1/2$

$n = 3 \Leftrightarrow \nu = 1/2$

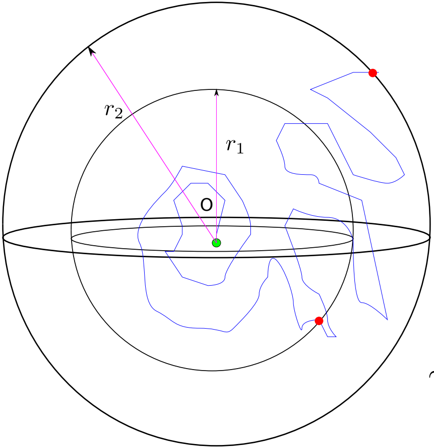


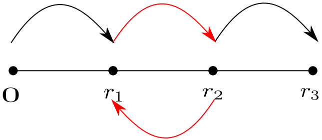
$n = 3 \Leftrightarrow \nu = 1/2$





$n = 3 \Leftrightarrow \nu = 1/2$

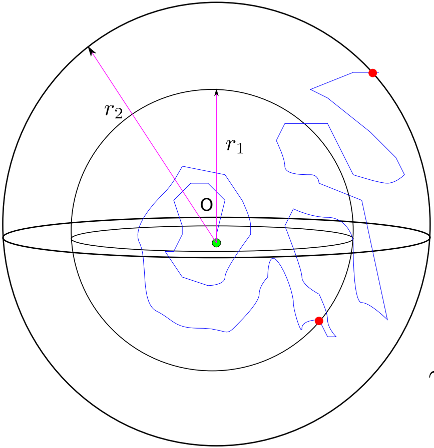


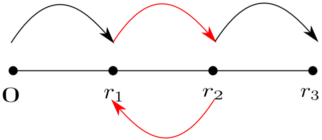


$$0 \to r_3$$

$$\sim 0 \to r_1 + \boxed{r_1 \leftrightarrow r_2} + r_2 \to r_3$$

$n = 3 \Leftrightarrow \nu = 1/2$



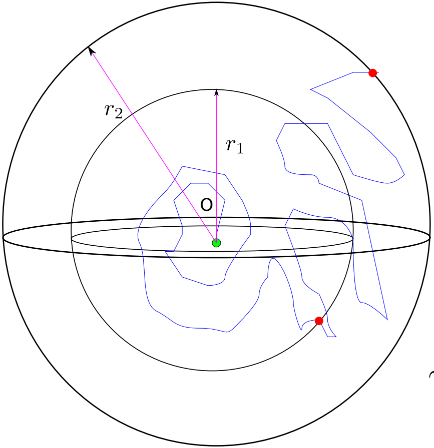


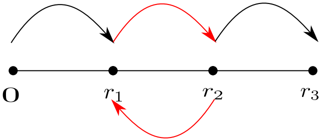
$$0 \to r_3$$

$$\sim 0 \to r_1 + \boxed{r_1 \leftrightarrow r_2} + r_2 \to r_3$$

$r_1 = 1,\ r_2 = 2$, and $r_3 = 3$

$n = 3 \Leftrightarrow \nu = 1/2$





$$0 \to r_3$$

$$\sim 0 \to r_1 + \boxed{r_1 \leftrightarrow r_2} + r_2 \to r_3$$

$r_1 = 1$, $r_2 = 2$, and $r_3 = 3$

$$\frac{3^n}{n+1}\left[B_{n+1}\left(\frac{x+5}{6}\right) - B_{n+1}\left(\frac{x+3}{6}\right)\right] = \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz} : \quad E_n(x) = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}(\pi t)\,\mathrm{d}t.$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2}{e^z+1}e^{xz} : \quad E_n\left(x\right) = \int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}\left(\pi t\right) \mathrm{d}t.$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{t}{e^t-1}e^{tx} : \quad B_n\left(x\right) = \frac{\pi}{2}\int_{\mathbb{R}} \left(x + it - \frac{1}{2}\right)^n \operatorname{sech}^2\left(\pi t\right) \mathrm{d}t.$$