Recursion Rules for The Hypergeometric Zeta Function

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Joint Work with Alyssa Byrnes, Victor H. Moll and Christophe Vignat

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Outlines

Question

Definition

$$\Phi_{a,b}(z) := M(a, a+b, z) = {}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n} \cdot \frac{z^n}{n!},$$

where

$$(a)_n := a(a+1)\cdots(a+n-1)$$

is the Pochhammer symbol.

Objects

The sequence $\{A_{a,b}\}$ defined by the generating function

$$\frac{1}{\Phi_{a,b}(z)} = \sum_{n=0}^{\infty} A_{a,b} \frac{z^n}{n!}.$$

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Question (Continued)

Objects

The roots of $\Phi_{a,b}(z)$, denoted by

$$\left\{z_{a,b;n}^{s} : n \in \mathbb{N}\right\}$$

and the corresponding hypergeometric-zeta function

$$\zeta_{a,b}^{H}(s) = \sum_{n=0}^{\infty} \frac{1}{z_{a,b;n}^{s}}, .$$

Zeta Functions

Example

Riemann-Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \; Re(s) > 1.$$

$$\mathbb{A} := \{1, 2, \ldots\} = \{a_n\}_{n \in \mathbb{N}}, \text{ where } a_n = n$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}.$$

$$f(z) := \frac{\sin(\pi z)}{\pi z} \Longrightarrow \mathbb{A} = \{z \in \mathbb{C} : f(z) = 0, z > 0\}.$$

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Zeta Functions (Continued)

Example

Bessel-Zeta Function

$$\zeta_{Bes,a}(s) := \sum_{n=1}^{\infty} \frac{1}{j_{a,n}^{s}},$$

where $\mathbb{A} := \{j_{a,n}\}$ are the zeros of $J_a(z)/z^a$ for the Bessel function of the first kind

$$J_a(z) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+a+1)} \left(\frac{z}{2}\right)^{2m+a}.$$

Zeta Functions (Continued)

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Zeta Functions (Continued)

In general, given (/selected) function f, we could define

$$\{a_n\} := \mathbb{A}(f) := \{z \in \mathbb{C} : z \neq 0,\ f(z) = 0\}$$

and

$$\zeta_f(s) = \zeta_{\mathbb{A}}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}.$$

Bernoulli Numbers

Definition

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \text{ for } |x| < 2\pi.$$

Theorem

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}.$$

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F. T. Howard 1967

$$\frac{\frac{x^k}{k!}}{e^x - \sum_{s=0}^{k-1} \frac{x^s}{s!}} = \sum_{r=0}^{\infty} A_{k,r} \frac{x^r}{r!} \Longrightarrow A_{1,r} = \frac{1}{2}\beta_r$$

$$\sum_{r=0}^{n} \binom{n+k}{r} A_{k,r} = 0.$$

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$$\zeta_{a,b}^{H}(s)=\sum_{n=0}^{\infty}\frac{1}{z_{a,b;n}^{s}},$$

where $\mathbb{A}:=\{z_{a,b;n}:n\in\mathbb{N}\}$ is the set of zeros of $\Phi_{a,b}(z)$.

Connection

Fact

{A. Byrnes, -, V. Moll, C. Vignat} *By applying Hadamard Factorization Theorem,*

$$\Phi_{a,b}(z) = e^{\frac{a}{a+b}z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}.$$

Fact

{A. Byrnes, -, V. Moll, C. Vignat} *By Contiguous Relation*

$$\frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^{H}(k+1) z^k.$$

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Recurrences

Theorem

{*A. Byrnes, -, V. Moll, C. Vignat*} *[Linear Recurrence]*

$$\sum_{k=1}^{p} B(a+p+k,b)\frac{p!}{(p-k)!}\zeta_{a,b}^{H}(k+1) = -\frac{bp}{(a+b)(a+b+p)}B(a+p,b),$$

where

$$B(u,v) := \int_0^1 x^{u-1}(1-x)^{v-1}\,dx$$

is the Beta function.

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Recurrences (Continued)

Theorem

{A. Byrnes, -, V. Moll, C. Vignat} [Quadratic Recurrence]

$$\sum_{k=1}^{p} \zeta_{a,b}^{H}(k+1)\,\zeta_{a,b}^{H}(p-k+1) = (a+b+p+1)\,\zeta_{a,b}^{H}(p+2) + \left(\frac{a-b}{a+b}\right)\zeta_{a,b}^{H}(p+1),$$

which is the analogue of

$$(n+1)\,\zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k)\,\zeta(2n-2k)\,,\ \text{for } n \geq 2.$$

Hypergeometric Bernoulli Numbers

Definitions

$$\frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}.$$

Theorem

{A. Byrnes, -, V. Moll, C. Vignat}

$$B_n^{(b)} = \begin{cases} 1 & n = 0 \\ -\frac{1}{b+1} & n = 1 \\ -\frac{n!}{b}\zeta_{1,b}^{H}(n) & n \geq 2 \end{cases}.$$

Remark

It suggests

$$\zeta_{1,b}^{H}(1) = \frac{b}{1+b}.$$

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A Probabilistic Approach

Moments & Cumulants

For a given random variable X with continuous density function $r(x)$,
(I) the expectation operator is defined by

$$\mathbb{E}u(X) := \int_{\mathbb{R}} u(x)\, r(x)\, dx;$$

(II) the moments are $\mathbb{E}[X^n]$ and the moment generating function is given by

$$\varphi_X(t) := \mathbb{E}\left[e^{tX}\right] = \int_{\mathbb{R}} e^{tx} r(x)\, dx = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!};$$

(III) the cumulants $\kappa_X(n)$ are given by

$$\log \varphi_X(t) = \sum_{m=1}^{\infty} \kappa_X(m) \frac{z^m}{m!}.$$

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Moments & Cumulants (Continued)

Example

[Exponential Distribution] Random variable Γ has density function

$$f_{\Gamma}(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \Longrightarrow \mathbb{E}\left[\Gamma^{a}\right] = \Gamma(a+1),$$

then

$$\mathbb{E}\left[e^{t\Gamma}\right] = \frac{1}{1-t}, \text{ for } |t| < 1.$$

Moments & Cumulants (Continued)

Example

[Beta Distribution] Random variable $\mathcal{B}_{a,b}$ has density function

$$f_{\mathcal{B}_{a,b}}(x) = \begin{cases} \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\mathbb{E}\left[e^{t\mathcal{B}_{a,b}}\right] = \frac{1}{B(a,b)} \int_0^1 e^{tx} x^{a-1} (1-x)^{b-1} \, dx = \Phi_{a,b}(t).$$

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Theorem

For a general random variable X,

$$\kappa_X(n) = \mathbb{E}[X^n] - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_X(j)\, \mathbb{E}\left[X^{j-1}\right].$$

Theorem

{*A. Byrnes, -, V. Moll, C. Vignat*}

$$(n-1)! \sum_{k=2}^{n} \frac{B(a+n-k,b)}{(n-k)!} \zeta_{a,b}^{H}(k) = \frac{a}{a+b} B(a+n-k,b) - B(a+n,b),$$

which is just the linear recurrence mentioned before.

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Recall

{A. Byrnes, -, V. Moll, C. Vignat}[Linear Recurrence]

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Conjugate Random Variables

Definition

Two random variables X and Y are called conjugate random variables if

$$\mathbb{E}\left[(X+Y)^n\right] = \delta_n = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}.$$

Equivalently,

$$1 = \mathbb{E}\left[e^{t(X+Y)}\right] = \mathbb{E}\left[e^{tX}\right]\mathbb{E}\left[e^{tY}\right].$$

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$$\mathcal{Z}_{a,b} := -\frac{a}{a+b} + \sum_{k=1}^{\infty} \frac{\Gamma_k - 1}{z_{a,b;k}},$$

where $\{\Gamma_k\}_{k=0}^{\infty}$ is a sequence of i.i.d of exponential distributions, is the conjugate of $\mathcal{B}_{a,b}$, namely

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Conjugate Random Variables (Continued)

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{A. Byrnes, -, V. Moll, C. Vignat} *Let X and Y be conjugate random variables and define polynomials*

$$P_n(z) = \mathbb{E}\left[(z+X)^n\right] \text{ and } Q_n(z) = \mathbb{E}\left[(z+Y)^n\right],$$

then

$$\begin{cases} P_{n+1}(z) - zP_n(z) = \sum_{j=0}^{n} \binom{n}{j} \kappa_X(j+1) P_{n-j}(z) \\ Q_{n+1}(z) - zQ_n(z) = -\sum_{j=0}^{n} \binom{n}{j} \kappa_X(j+1) Q_{n-j}(z) \end{cases}$$

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Theorem

{A. Byrnes, -, V. Moll, C. Vignat} $[X = \mathcal{B}_{a,b}\ Y = \mathcal{Z}_{a,b}]$

$$(n-1)!\sum_{j=2}^{n}\frac{B_{n-j}^{(a,b)}}{(n-j)}\zeta_{a,b}^{H}(j) = \frac{a}{a+b}B_{n-1}^{(a,b)} + B_{n}^{(a,b)}, \ (**)$$

where $B_n^{(a,b)}$ *are the generalized Bernoulli numbers defined by*

$$\sum_{n=0}^{\infty} B_n^{(a,b)}\frac{z^n}{n!} = \frac{1}{\Phi_{a,b}(z)}.$$

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Conjugate Random Variables (Continued)

Remark

Identity (**) is different from the linear recurrence obtained through moments and cumulants (*).

$$\begin{cases} (n-1)!\sum_{k=2}^{n}\frac{B(a+n-k,b)}{(n-k)!}\zeta_{a,b}^{H}(k)=\frac{a}{a+b}B(a+n-k,b)-B(a+n,b). & (*) \\ (n-1)!\sum_{j=2}^{n}\frac{B_{n-j}^{(a,b)}}{(n-j)}\zeta_{a,b}^{H}(j)=\frac{a}{a+b}B_{n-1}^{(a,b)}+B_{n}^{(a,b)} & (**) \end{cases}$$

Example

Let $a=5$, $b=3$ and $n=3$,

$$\begin{cases} 2\zeta_{5,3}^{H}(3)+\frac{5}{4}\zeta_{5,3}^{H}(2)+\frac{1}{32}=0 \\ 2\zeta_{5,3}^{H}(3)-\frac{5}{4}\zeta_{5,3}^{H}(2)-\frac{13}{384}=0 \end{cases}$$

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We also have some results on generalized Bernoulli polynomials defined by

$$\sum_{n=0}^{\infty} B_n^{(a,b)}(x)\frac{z^n}{n!} = \frac{e^{xz}}{\Phi_{a,b}(z)}.$$

These polynomials satisfy

$$\sum_{k=0}^{n} \binom{a+b+n-1}{k}\binom{a+n-k-1}{a-1} B_k^{(a,b)}(x) = (a+b)_n \frac{x^n}{n!}.$$

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Future Work

1. Explanation for two different linear recurrences.
2. From ${}_1F_1$ to ${}_2F_1$ or even ${}_pF_q$.
3. Analogue for "Euler Case"

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The End

Thank you for your patience!