Bernoulli and Euler Symbols: Umbral Calculus, Random Variables, and Multiple Zeta Values

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DKU-SJTU Workshop

Jan. 5th, 2022

Bernoulli polynomials & Bernoulli numbers

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$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad \text{and} \quad \frac{t\,e^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

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- Euler-Maclaurin Summation Formula

$$\sum_{j=a}^{n} f(j) = \int_a^n f(x)dx + \frac{f(a)+f(n)}{2} + \sum_{s=1}^{m} \frac{B_{2s}}{(2s)!} \left(f^{(2s-1)}(n) - f^{(2s-1)}(a) \right)$$
$$+ \int_a^n \frac{B_{2m} - B_{2m}(\{x\})}{(2m)!} f^{(2m)}(x)dx.$$

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- Euler-Maclaurin Summation Formula
$$\sum_{j=a}^{n}f(j)=\int_a^n f(x)dx+\frac{f(a)+f(n)}{2}+\sum_{s=1}^{m}\frac{B_{2s}}{(2s)!}\left(f^{(2s-1)}(n)-f^{(2s-1)}(a)\right)$$
$$+\int_a^n\frac{B_{2m}-B_{2m}(\{x\})}{(2m)!}f^{(2m)}(x)dx.$$
- Modular forms/Eisenstein series:
$$G_{2k}(\tau)=2\zeta(2k)\left(1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sum_{d|n}d^{2k-1}e^{2\pi in\tau}\right).$$

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Treat $t = \partial_x$, and

$$\frac{t}{e^t - 1} \bullet x^n = B_n(x)$$

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- Bernoulli-Barnes Polynomials: let $\mathbf{a} = (a_1, \ldots, a_k)$

  $\vec{\mathcal{B}} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ and $|\mathbf{a}| = \prod_{i=1}^{k} a_i$

  $$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_i t} - 1} = \sum_{n=0}^{\infty} B_n(\mathbf{a}; x)\frac{t^n}{n!} \Leftrightarrow B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|}\left(x + \mathbf{a}\cdot\vec{\mathcal{B}}\right)^n;$$

Theorem (A. Bayad and M. Beck, 2014)

Difference Formula: Suppose $A = \sum_{k=1}^{n} a_k \neq 0$*, then*

$$(-1)^m B_m(\mathbf{a}; -x) - B_m(\mathbf{a}; x) = m! \sum_{l=1}^{n-1} \sum_{|L|=l} \frac{B_{m-n+l}(\mathbf{a}_L; x)}{(m-n+l)!},$$

where $L \subset \{1, \ldots, n\}$ *and* $B_m(\mathbf{a}_L; x) = x^m$ *if* $L = \emptyset$*.*

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Theorem (**L. Jiu**, V. Moll and C. Vignat, 2016)

$$f\left(x - \mathbf{a} \cdot \vec{\mathcal{B}}\right) = \sum_{\ell=0}^{n} \sum_{|L|=\ell} |\mathbf{a}|_{L^*} f^{(n-\ell)}\left(x + \left(\mathbf{a} \cdot \vec{\mathcal{B}}\right)_L\right).$$

$L^* = \{1, \ldots, n\} \setminus L$.

Corollary

Pick $f(x) = x^m / m!$.

Analytic Continuation: for $n_1, \ldots, n_r$ positive integers

Theorem (Sadaoui, 2014)

$$\zeta_r(-n_1, \ldots, -n_r) = (-1)^r \sum_{k_2, \ldots, k_r} \frac{1}{\bar{n} + r - \bar{k}} \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1}$$

$$\times \sum_{l_1, \ldots, l_r} \binom{\bar{n} + r - \bar{k}}{l_1} \binom{k_2}{l_2} \cdots \binom{k_r}{l_r} B_{l_1} \cdots B_{l_r},$$

$\bar{n} = \sum_{j=1}^{n} n_j$, $\bar{k} = \sum_{j=2}^{r} k_j$, $k_2, \ldots k_r \geq 0$, $l_j \leq k_j$ *for* $2 \leq j \leq r$ *and* $l_1 \leq \bar{n} + r + \bar{k}$.

Theorem (Akiyama and Tanigawa, 2001)

$$\begin{aligned} \zeta_r(-n_1, \ldots, -n_r) = & -\frac{\zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r - 1)}{1 + n_r} \\ & -\frac{\zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r)}{2} \\ & + \sum_{q=1}^{n_r} (-n_r)_q \frac{B_{q+1}}{(q+1)!} \zeta_{r-1}(-n_1, \ldots, -n_{r-2}, -n_{r-1} - n_r + q) \end{aligned}$$

Theorem (L.Jiu, V.H.Moll and C.Vignat, 2018)

$$\zeta_r\left(-n_1,\ldots,-n_r\right)=\prod_{k=1}^{r}\left(-1\right)^{n_k}\mathcal{C}_{1,\ldots,k}^{n_k+1},$$

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$$\mathcal{C}_1^n=\frac{\mathcal{B}_1^n}{n},\ \mathcal{C}_{1,2}^n=\frac{\left(\mathcal{C}_1+\mathcal{B}_2\right)^n}{n},\ldots,\mathcal{C}_{1,\ldots,k+1}^n=\frac{\left(\mathcal{C}_{1,\ldots,k}+\mathcal{B}_{k+1}\right)^n}{n}.$$

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-
$$\begin{aligned}
\zeta_2(-n,0) &= (-1)^n \mathcal{C}_1^{n+1} \cdot (-1)^0 \mathcal{C}_{1,2}^{0+1} \\
&= (-1)^n \frac{\mathcal{C}_1+\mathcal{B}_2}{1} \cdot \mathcal{C}_1^{n+1} \\
&= (-1)^n \left(\mathcal{C}_1^{n+2} + \mathcal{B}_2 \mathcal{C}_1^{n+1}\right) \\
&= (-1)^n \left(\frac{\mathcal{B}_1^{n+2}}{n+2} + \mathcal{B}_2 \frac{\mathcal{B}_1^{n+1}}{n+1}\right) \\
&= (-1)^n \left[\frac{B_{n+2}}{n+2} - \frac{1}{2}\frac{B_{n+1}}{n+1}\right]
\end{aligned}$$

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- $$\begin{aligned}\zeta_2\left(-n,0\right) &= \left(-1\right)^n\mathcal{C}_1^{n+1}\cdot\left(-1\right)^0\mathcal{C}_{1,2}^{0+1}\\ &= \left(-1\right)^n\frac{\mathcal{C}_1+\mathcal{B}_2}{1}\cdot\mathcal{C}_1^{n+1}\\ &= \left(-1\right)^n\left(\mathcal{C}_1^{n+2}+\mathcal{B}_2\mathcal{C}_1^{n+1}\right)\\ &= \left(-1\right)^n\left(\frac{\mathcal{B}_1^{n+2}}{n+2}+\mathcal{B}_2\frac{\mathcal{B}_1^{n+1}}{n+1}\right)\\ &= \left(-1\right)^n\left[\frac{B_{n+2}}{n+2}-\frac{1}{2}\frac{B_{n+1}}{n+1}\right].\end{aligned}$$
- $$\zeta\left(-n\right)=\left(-1\right)^n\mathcal{C}^{n+1}=\left(-1\right)^n\frac{B_{n+1}}{n+1}.$$

Lattice/Motzkin Path

Theorem (L. Jiu and D. Y. H. Shi, 2019)

Define $(M_{n,k}(x))_{n,k=0}^{\infty}$, *by* $M_{0,0} = 1$, $M_{n,k} = 0$ *if* $k > n$, *or one of* k *and* n *is negative, and the recurrence*

$$M_{n+1,k}(x) = M_{n,k-1}(x) - \left(-x + \frac{1}{2}\right) M_{n,k}(x) - \frac{(k+1)^4}{4\,(2k+1)\,(2k+3)} M_{n,k+1}(x),$$

Then, $M_{n,0}(x) = B_n\,(x)$. *In addition, the lattice path interpretation allows us to define the infinite-dimensional matrix*

$$R := \begin{pmatrix} x-\frac{1}{2} & -\omega_1 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{1}{2} & -\omega_2 & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\omega_n & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{1}{2} & -\omega_{n+1} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix}$$

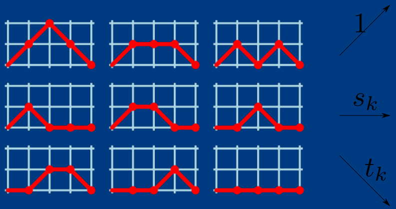
$$\boxed{\begin{matrix} \omega_n \\ || \\ \frac{n^4}{4(2n+1)(2n-1)} \end{matrix}}$$

$$(m_n)_{n=0}^{\infty} \sim m_n = \int_{\mathbb{R}} x^n \,\mathrm{d}\mu(x) \quad \overset{?}{\Rightarrow} \quad (P_n(x))_{n=1}^{\infty} \sim \int_{\mathbb{R}} P_n(x)\, P_m(x)\, \mathrm{d}\mu(x) = C_n \delta_{m,n}$$

$$\Rightarrow \quad P_{n+1}(x) = (x + s_n)\, P_n(x) + t_n P_{n-1}(x)$$

$$\Rightarrow \quad \boxed{\sum_{n=0}^{\infty} m_n x^n = \cfrac{m_0}{1 + s_0 x + \cfrac{t_1 x^2}{1 + s_1 x + \cfrac{t_2 x^2}{1 + \cdots}}}}$$

$$M_{n+1,k} = M_{n,k-1} + s_k M_{n,k} + t_{k+1} M_{n,k+1}$$



$$\boxed{\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}}$$

Main Results: $H_n(a_k)$ for the following sequences

$$B_{2k+1}\left(\frac{x+1}{2}\right), E_{2k}\left(\frac{x+1}{2}\right), E_{2k+1}\left(\frac{x+1}{2}\right), E_{2k+2}\left(\frac{x+1}{2}\right),$$

$$B_k\left(\frac{x+r}{q}\right) - B_k\left(\frac{x+s}{q}\right), E_k\left(\frac{x+r}{q}\right) \pm E_k\left(\frac{x+s}{q}\right),$$

$$kE_{k-1}(x), B_{k+1,x_{8,1}}(x), B_{k+1,x_{8,2}}(x), B_{k+1,x_{12,1}}(x), B_{k+1,x_{12,2}}(x),$$

$$(2k+1)E_{2k}, (2^{2k+2}-1)B_{2k+2}, (2k+1)B_{2k}\left(\frac{1}{2}\right), (2k+3)B_{2k+2},$$

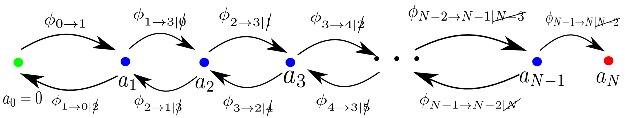
$a_k,\ k \geq 1$	B_{k-1}	B_{2k}	$(2k+1)B_{2k}$	$(2^{2k}-1)B_{2k}$
a_0	0	1	1	0

$a_k,\ k \geq 1$	E_{2k-2}	$E_{k-1}(1)$	$E_{k+3}(1)$	$E_{2k-1}(1)$	$E_{2k+5}(1)$	$\frac{E_k(1)}{k!}$
a_0	0	0	$-\frac{1}{4}$	0	$\frac{1}{2}$	1

$a_k,\ k \geq 1$	$\frac{E_{2k-1}(1)}{(2k-1)!}$	$E_{2k-2}\left(\frac{x+1}{2}\right)$	$(2k+1)E_{2k}$		
a_0	0	0	0		

$$\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}, \frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}, \frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}$$

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x).$$



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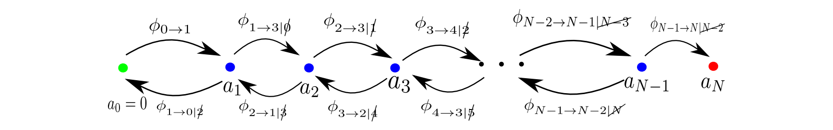
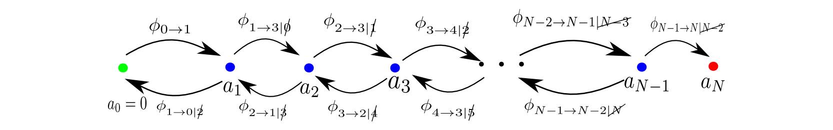


Theorem (L. Jiu, I. Simonelli, and H. Yue, 21'+)

$$\phi_{0\to a_n}=\phi_{0\to a_1}\phi_{a_1\to a_2|\cancel{0}}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

$$B_{n+1}\left(\frac{x+2}{5}\right) - B_{n+1}\left(\frac{x}{5}\right) = \frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x).$$



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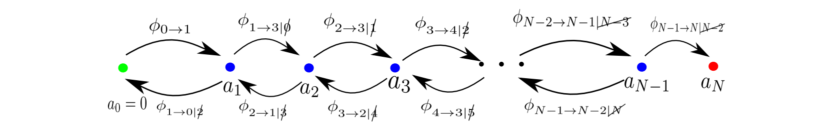
where

$$L_j = \phi_{a_{j-1}\to a_j|\cancel{a_{j-2}}}\phi_{a_j\to a_{j-1}|\cancel{a_{j+1}}}$$

$$P(L_1,\dots,L_n) = \sum_{*}(-1)^{\ell+1}L_{j_1}\cdots L_{j_\ell},$$

$$B_{n+1}\left(\frac{x+2}{5}\right)-B_{n+1}\left(\frac{x}{5}\right)=\frac{n+1}{5^n}\sum_{k=0}^{\infty}\frac{3^k}{2^{2k+3}}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\frac{1}{12^{\ell}}E_n^{(2k+2\ell+3)}(k+\ell+x).$$



Theorem (L. Jiu, I. Simonelli, and H. Yue, 21'+)

$$\phi_{0\to a_n}=\phi_{0\to a_1}\phi_{a_1\to a_2|\cancel{0}}\cdots\phi_{a_{n-1}\to a_n|\cancel{a_{n-2}}}\cdot\frac{1}{1-P(L_1,\ldots,L_n)},$$

where

$$L_j=\phi_{a_{j-1}\to a_j|\cancel{a_{j-2}}}\phi_{a_j\to a_{j-1}|\cancel{a_{j+1}}}$$

$$P\left(L_1,\ldots,L_n\right)=\sum_{*}(-1)^{\ell+1}L_{j_1}\cdots L_{j_\ell},$$

for the condition $*$ given by (1) $\ell=1,2,\ldots,n$; (2) and $j_1<j_2-1$, $j_2<j_3-1$,$\ldots$,$j_{\ell-1}<j_\ell-1$.

Future Work

- To apply this symbolic expression to the Tornheim zeta function, defined as

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r n^s (m+n)^t},$$

  and its multi-dimensional extension.

- Another type of zeta function to which we may apply the Bernoulli symbol is the hypergeometric zeta function $\zeta_{a,b}^H(s)$. Let $\left(z_{k;a,b}\right)_{k=1}^{\infty}$ be the sequence of complex zeros of ${}_1F_1\left(\begin{smallmatrix} a \\ a+b \end{smallmatrix}\middle| t\right)$, defined by

$$\zeta_{a,b}^H(s) := \sum_{k\geq 1} z_{k;a,b}^{-s} \quad \text{for} \quad \mathrm{Re}\, s > 1.$$

- Orthogonal polynomials w. r. t. $B_n^{(p)}(x)$.
- Computer proofs for identities involving Bernoulli and Euler polynomials
- Hyperbolic secant (square) distribution and information geometry
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Thank you!!!