## "Random Walks" for Harmonic Sums

Lin Jiu

**RISC** 

SFB Status-Seminar

November 29th 2016

# Acknowledgment



Dr. J. Ablinger



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Prof. C. Schneider



Prof. C. Vignat

## **Outlines**

■ "Random": Integral Representation of Special Harmonic Sums

2 Random: Random Walk for Harmonic Sums

3 !Random: Diagonalization & Combinatorics

## Beginning-Partition

Schneider Research in Number Theory (2016) 2:9

Research in Number Theory

Partition zeta functions

Open Access

Abstract

integer partitions, and continued fractions, to find partition-theoretic formulas to Remann zeta function, multiple zeta values, and other number-theoretic objects.

#### Keywords: Partitions, o-series, Zeta functions 1 Introduction, notations and central theorem

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function  $\mathcal{E}(x)$  emerged nearly fully-formed from Euler's pioneering work [3, 8]. Euler discovered the magical-seeming generating function for the partition function p(n)

$$\frac{1}{(q;q)_{\infty}} = \sum_{n=0}^{\infty} p(n)q^n,$$

in which the a-Pochhammer symbol is defined as  $(z; a)_0 := 1, (z; a)_n := \prod_{i=1}^{n-1} (1 - za^i)$ for  $n \ge 1$ , and  $(z;a)_\infty = \lim_{z\to\infty} (z;a)_n$  if the product converges, where we take  $z \in \mathbb{C}$  and  $q := e^{i2\pi \tau}$  with  $\tau \in \mathbb{H}$  (the upper half-plane). He also discovered the beautiful product formula relating the zeta function  $\mathcal{E}(s)$  to the set  $\mathbb{F}$  of primes

$$\frac{1}{\prod_{j \in \Gamma} \left(1 - \frac{1}{p^j}\right)} = \sum_{n=1}^\infty \frac{1}{n^j} := \zeta(\delta, \, \text{Re}(j) > 1. \endalign{ \begin{tabular}{l} \line (0,0) \l$$

 $x \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right) = \sin x$ to solve the so-called Basel problem, finding the exact value of c(2); he went on to find an exact formula for  $\xi(2k)$  for every  $k \in \mathbb{Z}^+$  [8]. Euler's approach to these problems,

interweaving infinite products, infinite sums and special functions, permeates number Very much in the spirit of Euler, here we consider certain series of the form  $\sum_{i,j,m} \phi(\lambda_i)$ , where the sum is taken over the set P of integer partitions  $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r), \lambda_1 \ge \lambda_2 \ge$ 



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 R. Schneider. Partition zeta functions. Research in Number Theory 2016, 2:8,

■ Let 
$$\varphi_{\infty}(f;q) = \prod_{n=1}^{\infty} (1 - f(n)q^n)$$
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- R. Schneider. Partition zeta functions. Research in Number Theory 2016, 2:8,
- Let  $\varphi_{\infty}(f;q) = \prod (1-f(n)q^n)$ :

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## Beginning-Partition

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RESEARCH Open Access
Partition zeta functions

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#### Abstract

We exploit transformations relating generalized q-series, infinite products, sums over integer partitions, and confirmed fractions, to find partition-eheoretic formulas to compute the values of constants such as x, and to connect usues over partitions to the Remain acts function, multiple zeto values, and other number-theoretic objects.

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One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function \( \xi(s) \) emerged nearly fully-formed from Euler's pioneering work (3, 8). Euler discovered the magical-seeming generating function for the partition function \( x(s) \).

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R. Schneider, Partition zeta functions. Research in Number Theory 2016, 2:8.

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Define the partition-theoretic generaliztion of Riemann-zeta function as

$$\zeta_{\mathcal{P}}\left(\left\{a\right\}^{k}\right) := \sum_{I(\lambda)=k} \frac{1}{n_{\lambda}^{a}}.$$



(1)

#### DEF: harmonic sum

$$S_{a_1,\ldots,a_k}(N) = \sum_{\substack{N \geq n_1 \geq \cdots \geq n_k \geq 1}} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \cdots \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}.$$

$$k = 1, \ a_1 > 0, \ N = \infty$$

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$$\zeta_{\mathcal{P}}\left(\left\{a\right\}^{k}\right) := \sum_{l(\lambda)=k} \frac{1}{n_{\lambda}^{a}} = \sum_{\lambda_{1} \geq \dots \geq \lambda_{k} \geq 1} \frac{1}{(\lambda_{1} \dots \lambda_{k})^{a}}$$

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#### Fact

$$\zeta_{\mathcal{P}}\left(\left\{a\right\}^{k}\right)=S_{a_{k}}\left(\infty\right)$$

### Prop.

$$\prod_{n=1}^{\infty} \frac{1}{1 - \frac{t^a}{n^a}} = \sum_{k=0}^{\infty} \sum_{l(\lambda) = k} \frac{t^{ak}}{n_{\lambda}^a} = \sum_{k=0}^{\infty} S_{a_k} \left(\infty\right) t^{ak}.$$

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$$B(N, 1+t) = \frac{1}{N} \sum_{k=0}^{\infty} (-t)^k S_{1_k}(N).$$

$$m=2$$

$$\begin{split} \sum_{k=0}^{\infty} S_{2_k} \left( \infty \right) t^{2k} &= \Gamma \left( 1 + t \right) \Gamma \left( 1 - t \right) = B \left( 1 + t, 1 - t \right) \\ &= \int_0^1 \lambda^{-t} \left( 1 - \lambda \right)^t d\lambda = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 \ln^k \left( \frac{1 - \lambda}{\lambda} \right) d\lambda . \\ S_{2_k} \left( \infty \right) &= \frac{1}{(2k)!} \int_0^1 \ln^{2k} \left( \frac{1 - \lambda}{\lambda} \right) d\lambda . \end{split}$$

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### Multiple Beta Function

$$B(\alpha_1,\ldots,\alpha_m):=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m)}=\int_{\Omega_m}\prod_{i=1}^m x_i^{\alpha_i-1}dx_i$$

where  $\Omega_m = \{(x_1, \dots, x_m) \in \mathbb{R}_+^m : x_1 + \dots + x_{m-1} < 1, \ x_1 + \dots + x_m = 1\}.$ 

## Prop

$$S_{m_k}(\infty) = \frac{(-1)^{mk} (m-1)!}{(mk)!} \int_{\Omega_m} \ln^{mk} \left( \prod_{j=0}^{m-1} x_{j+1}^{\xi_m^j} \right) dx, \ \xi_m = \exp\left(\frac{2\pi \iota}{m}\right)$$

$$\zeta \left( m \right) \quad = \quad \frac{(-1)^m}{m} \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \\ & \quad \ln^m \left( x_1^{\xi_m^0} \cdots x_{m-1}^{\xi_{m-2}^m} \left( 1-x_1-\cdots-x_{m-1} \right)^{\xi_m^{m-1}} \right) dx_{m-1} \cdots dx_1$$

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### Prop

$$S_{m_k}(\infty) = \frac{(-1)^{mk} (m-1)!}{(mk)!} \int_{\Omega_m} \ln^{mk} \left( \prod_{j=0}^{m-1} x_{j+1}^{\xi_m^j} \right) dx, \ \xi_m = \exp\left(\frac{2\pi \iota}{m}\right)$$

$$\zeta(m) = \frac{(-1)^m}{m} \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{m-2}} \\ \ln^m \left( x_1^{\xi_m^0} \cdots x_{m-1}^{\xi_{m-1}^{m-2}} \left( 1 - x_1 - \cdots - x_{m-1} \right) \xi_m^{m-1} \right) dx_{m-1} \cdots dx_1$$

### Multiple Beta Function

$$B(\alpha_1,\ldots,\alpha_m):=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\cdots+\alpha_m)}=\int_{\Omega_m}\prod_{i=1}^mx_i^{\alpha_i-1}dx,$$

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## **BREAK**

$$S_{\mathbf{1}_k}\left( \mathcal{N} 
ight) = \sum\limits_{\mathcal{N} \geq n_1 \geq \cdots \geq n_k \geq 1} rac{1}{n_1 \cdots n_k}$$

We start a random walk at site "N", with the rules: (as a pawn)

$$\mathbb{P}\left(i \to j\right) = \text{the probability from site "}i\text{" to site "}j\text{"} = \begin{cases} 1/i, & \text{if } j \leq i; \\ 0, & \text{if } j > i. \end{cases}$$

namely

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

For example, suppose we are at site "6":

$$\mathbb{P}\left(6 \to 6\right) = \mathbb{P}\left(6 \to 5\right) = \mathbb{P}\left(6 \to 4\right) = \mathbb{P}\left(6 \to 3\right) = \mathbb{P}\left(6 \to 2\right) = \mathbb{P}\left(6 \to 1\right) = \frac{1}{6}$$

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- <u>STEP k+1</u>: walk from site " $n_k$ " to site " $n_{k+1}$  ( $\leq n_k$ )", with  $\mathbb{P}(n_k \to n_{k+1}) = \frac{1}{n_k}$ .

$$\mathbb{P}\left(n_{k+1}=1\right) = \sum_{\text{All possible paths}} \frac{1}{N} \frac{1}{n_1} \cdots \frac{1}{n_k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{N n_1 \cdots n_k} = \frac{S_{1_k}\left(N\right)}{N}.$$

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Therefore, a typical walk is as follows:

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## $S_{\mathbf{1}_{k}}(N)$

On the other hand, the transition matrix of sites  $\{1,\ldots,N\}$  is

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}$$

i.e.

$$P_{N|1} = \left(p_{i,j}^{(1)}\right) \text{ with } p_{ij}^{(1)} = \mathbb{P}\left(i \to j\right) = \frac{1}{i}.$$

Therefore, after k+1 steps, entries of  $P_{N|1}^{k+1}$  give the transition probabilities among sites. In particular,

$$\left(P_{N|1}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N}S_{1_{k}}(N)$$

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ightarrow S_{m_{k}}\left(N\right) 
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ightarrow S_{a_{1},...,a_{k}}\left(N\right)$$

Recal

$$S_{a_1,...,a_k}(N) = \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{\text{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\text{sign}(a_k)^{n_k}}{n_k^{|a_k|}}$$

For l = 1, ..., k

$$P_{N|a_{I}} = \begin{pmatrix} sign(a_{I}) & 0 & \cdots & 0 \\ \frac{1}{2|a_{I}|} & \frac{1}{2|a_{I}|} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{sign(a_{I})^{N}}{N^{|a_{I}|}} & \frac{sign(a_{I})^{N}}{N^{|a_{I}|}} & \cdots & \frac{sign(a_{I})^{N}}{N^{|a_{I}|}} \end{pmatrix}$$

#### THM

$$S_{a_1,...,a_k}(N) = N \cdot \left( P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k} \right)_{N,1} = N \cdot \left( \prod_{l=0}^k P_{N|a_l} \right)_{N,1}$$

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Recall

$$S_{a_1,\ldots,a_k}\left(N\right) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{\operatorname{sign}\left(a_1\right)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}\left(a_k\right)^{n_k}}{n_k^{|a_k|}}$$

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$$P_{N|a_{l}} = \begin{pmatrix} sign(a_{l}) & 0 & \cdots & 0\\ \frac{1}{2^{|a_{l}|}} & \frac{1}{2^{|a_{l}|}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \cdots & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} \end{pmatrix}$$

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ightarrow S_{m{m}_{k}}\left(N
ight)
ightarrow S_{m{a}_{k}}\left(N
ight)
ightarrow S_{m{a}_{1},...,m{a}_{k}}\left(N
ight)$$

Recall

$$S_{a_1,\ldots,a_k}(N) = \sum_{N>n_1>\cdots>n_k>1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}$$

For  $l = 1, \ldots, k$ 

$$P_{N|a_{l}} = \begin{pmatrix} sign(a_{l}) & 0 & \cdots & 0 \\ \frac{1}{2^{|a_{l}|}} & \frac{1}{2^{|a_{l}|}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \cdots & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} \end{pmatrix}.$$

#### THM

$$S_{a_1,...,a_k}(N) = N \cdot \left(P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k}\right)_{N,1} = N \cdot \left(\prod_{l=0}^k P_{N|a_l}\right)_{N,1}$$



$$S_{\mathbf{1}_{k}}\left(N
ight)
ightarrow S_{m{m}_{k}}\left(N
ight)
ightarrow S_{m{a}_{k}}\left(N
ight)
ightarrow S_{m{a}_{1},...,m{a}_{k}}\left(N
ight)$$

Recall

$$S_{a_1,\ldots,a_k}(N) = \sum_{N>n_1>\cdots>n_k>1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}$$

For  $l = 1, \ldots, k$ 

$$P_{N|a_{l}} = \begin{pmatrix} sign(a_{l}) & 0 & \cdots & 0 \\ \frac{1}{2^{|a_{l}|}} & \frac{1}{2^{|a_{l}|}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} & \cdots & \frac{sign(a_{l})^{N}}{N^{|a_{l}|}} \end{pmatrix}$$

#### THM.

$$S_{a_1,...,a_k}(N) = N \cdot \left(P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k}\right)_{N,1} = N \cdot \left(\prod_{l=0}^k P_{N|a_l}\right)_{N,1}.$$

$$M_{(N+1)|_{\partial}} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^{g}} & \frac{1}{2^{g}} & \cdots & 0 & 1 - \frac{1}{2^{g-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^{g}} & \frac{1}{N^{g}} & \cdots & \frac{1}{N^{g}} & 1 - \frac{1}{N^{g-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{N|a} & 1 & \vdots \\ \hline (0, \dots, 0) & 1 & \vdots \\ \hline (0, \dots, 0) & 1 & \vdots \end{pmatrix}$$

$$\mathbb{P}\left(\mathfrak{N} 
ightarrow \mathfrak{N}
ight) = 1 ext{ and } \mathbb{P}\left(i 
ightarrow \mathfrak{N}
ight) = 1 - rac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N^a}S_{a_k}(N)$$

$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^{a}} & \frac{1}{2^{a}} & \cdots & 0 & 1 - \frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^{a}} & \frac{1}{N^{a}} & \cdots & \frac{1}{N^{a}} & 1 - \frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{N|a} & \boxed{1 - \frac{1}{I^{a-1}}} \\ \boxed{0, \dots, 0} & 1 \\ \boxed{0, \dots, 0} & 1 \end{pmatrix}$$

$$\mathbb{P}\left(\mathfrak{N} 
ightarrow \mathfrak{N}
ight) = 1 ext{ and } \mathbb{P}\left(i 
ightarrow \mathfrak{N}
ight) = 1 - rac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N^a}S_{a_k}\left(N\right)$$



$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^{a}} & \frac{1}{2^{a}} & \cdots & 0 & 1 - \frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^{a}} & \frac{1}{N^{a}} & \cdots & \frac{1}{N^{a}} & 1 - \frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{N|a} & \frac{1}{(1 - \frac{1}{J^{a-1}})} \\ \frac{(0, \dots, 0)}{N} & 1 \end{pmatrix}$$

$$\mathbb{P}\left(\mathfrak{N} 
ightarrow \mathfrak{N}
ight) = 1 ext{ and } \mathbb{P}\left(i 
ightarrow \mathfrak{N}
ight) = 1 - rac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N^a}S_{a_k}(N)$$



$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^{a}} & \frac{1}{2^{a}} & \cdots & 0 & 1 - \frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^{a}} & \frac{1}{N^{a}} & \cdots & \frac{1}{N^{a}} & 1 - \frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{N|a} & 1 & \frac{1}{N^{a-1}} \\ \hline (0, \dots, 0) & 1 \\ \hline \end{pmatrix}$$

$$\mathbb{P}(\mathfrak{N} \to \mathfrak{N}) = 1 \text{ and } \mathbb{P}(i \to \mathfrak{N}) = 1 - \frac{1}{i^{a-1}}.$$

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N^a}S_{a_k}\left(N\right)$$



$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^{a}} & \frac{1}{2^{a}} & \cdots & 0 & 1 - \frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^{a}} & \frac{1}{N^{a}} & \cdots & \frac{1}{N^{a}} & 1 - \frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} P_{N|a} & \frac{1}{(1 - \frac{1}{J^{a-1}})} \\ \frac{(0, \dots, 0)}{N} & 1 \end{pmatrix}$$

$$\mathbb{P}\left(\mathfrak{N} o\mathfrak{N}
ight)=1$$
 and  $\mathbb{P}\left(i o\mathfrak{N}
ight)=1-rac{1}{i^{s-1}}.$ 

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1} = 1\right) = \frac{1}{N^a}S_{a_k}\left(N\right).$$



#### **BREAK**

a = 1

$$\begin{split} P_{N|1} &= \left( \begin{array}{ccc} \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{array} \right) \text{ and } \left( P_{N|1}^{k+1} \right)_{N,1} = \frac{1}{N} S_{1_k} \left( N \right) \\ P_{N|1} &= Q_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{N} \right\} Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{N^{k+1}} \right\} Q_{N|1}^{-1} \\ Q_{N|1} &= \left( \frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}} \right) \text{ and } Q_{N|1}^{-1} = \left( (-1)^{i+j} \binom{N-1}{i-1} \binom{i-1}{j-1} \right) \\ \frac{1}{N} S_{1_k} \left( N \right) = \left( P_{N|1}^{k+1} \right)_{N,1} = \sum_{l=1}^{N} \frac{1}{l^{k+1}} \left( -1 \right)^{l-1} \binom{N-1}{l-1}, \end{split}$$

$$\sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^{k}} = \sum_{N \ge n_{1} \ge \dots \ge n_{k} \ge 1} \frac{1}{n_{1} \cdots n_{k}} = S_{1_{k}} (N)$$

a = 1

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \text{ and } \begin{pmatrix} P_{N|1}^{k+1} \end{pmatrix}_{N,1} = \frac{1}{N} S_{1_k} (N)$$

$$P_{N|1} = Q_{N|1} \operatorname{diag} \left\{ 1, \dots, \frac{1}{N} \right\} Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag} \left\{ 1, \dots, \frac{1}{N^{k+1}} \right\} Q_{N|1}^{-1}$$

$$Q_{N|1} = \left( \frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}} \right) \text{ and } Q_{N|1}^{-1} = \left( (-1)^{i+j} \binom{N-1}{i-1} \binom{i-1}{j-1} \right)$$

$$\frac{1}{N}S_{1_{k}}\left(N\right)=\left(P_{N\mid1}^{k+1}\right)_{N,1}=\sum_{l=1}^{N}\frac{1}{l^{k+1}}\left(-1\right)^{l-1}\binom{N-1}{l-1},$$

$$\sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^{k}} = \sum_{N \ge n_{1} \ge \dots \ge n_{k} \ge 1} \frac{1}{n_{1} \cdots n_{k}} = S_{1_{k}} (N)$$

a = 1

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \text{ and } \begin{pmatrix} P_{N|1}^{k+1} \end{pmatrix}_{N,1} = \frac{1}{N} S_{1_k} (N)$$

$$P_{N|1} = \mathit{Q}_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{\mathit{N}} \right\} \mathit{Q}_{N|1}^{-1} \Rightarrow P_{N|1}^{\mathit{k}+1} = \mathit{Q}_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{\mathit{N}^{\mathit{k}+1}} \right\} \mathit{Q}_{N|1}^{-1}$$

$$Q_{N|1} = \left( \frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}} \right) \text{ and } Q_{N|1}^{-1} = \left( (-1)^{i+j} \binom{N-1}{i-1} \binom{i-1}{j-1} \right)$$

$$\frac{1}{N}S_{1_{k}}\left(N\right) = \left(P_{N|1}^{k+1}\right)_{N,1} = \sum_{l=1}^{N} \frac{1}{l^{k+1}} \left(-1\right)^{l-1} \binom{N-1}{l-1},$$

$$\sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^{k}} = \sum_{N \ge n_{1} > \dots > n_{k} \ge 1} \frac{1}{n_{1} \cdots n_{k}} = S_{1_{k}} (N)$$

a = 1

$$P_{N|1} = \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{array} \right) \text{ and } \left( P_{N|1}^{k+1} \right)_{N,1} = \frac{1}{N} \mathbf{S}_{\mathbf{1}_k} \left( \mathbf{N} \right)$$

$$P_{N|1} = Q_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{N} \right\} \, Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag} \left\{ 1, \ldots, \frac{1}{N^{k+1}} \right\} \, Q_{N|1}^{-1}$$

$$Q_{N|1} = \left(\frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}}\right) \text{ and } Q_{N|1}^{-1} = \left((-1)^{i+j} \binom{N-1}{i-1} \binom{i-1}{j-1}\right)$$

$$\frac{1}{N}S_{1_{k}}(N) = \left(P_{N|1}^{k+1}\right)_{N,1} = \sum_{l=1}^{N} \frac{1}{l^{k+1}} \left(-1\right)^{l-1} \binom{N-1}{l-1},$$

$$\sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^k} = \sum_{\substack{N \ge n_1 \ge \dots \ge n_k \ge 1}} \frac{1}{n_1 \cdots n_k} = S_{1_k} (N)$$

a = 1

$$\begin{split} P_{N|1} &= \left( \begin{array}{ccc} \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{array} \right) \text{ and } \left( P_{N|1}^{k+1} \right)_{N,1} = \frac{1}{N} S_{1_k} \left( N \right) \\ \\ P_{N|1} &= Q_{N|1} \operatorname{diag} \left\{ 1, \dots, \frac{1}{N} \right\} Q_{N|1}^{-1} \Rightarrow P_{N|1}^{k+1} = Q_{N|1} \operatorname{diag} \left\{ 1, \dots, \frac{1}{N^{k+1}} \right\} Q_{N|1}^{-1} \\ \\ Q_{N|1} &= \left( \frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}} \right) \operatorname{and} Q_{N|1}^{-1} = \left( (-1)^{i+j} \binom{N-1}{i-1} \binom{i-1}{j-1} \right) \\ \\ \frac{1}{N} S_{1_k} \left( N \right) = \left( P_{N|1}^{k+1} \right)_{N,1} = \sum_{i=1}^{N} \frac{1}{j_{k+1}} \left( -1 \right)^{i-1} \binom{N-1}{l-1}, \end{split}$$

$$\sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^{k}} = \sum_{N > n_{1} > \dots > n_{k} > 1} \frac{1}{n_{1} \cdots n_{k}} = S_{1_{k}}(N).$$

a > 0

$$P_{N\mid a} = \left( \begin{array}{cccc} \frac{1}{2^{3}} & 0 & \cdots & 0 \\ \frac{1}{2^{3}} & \frac{1}{2^{3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N^{3}} & \frac{1}{N^{3}} & \cdots & \frac{1}{N^{3}} \end{array} \right) \text{ and } \left(P_{N\mid a}^{k+1}\right)_{N,a} = \frac{1}{N^{3}}S_{a_{k}}\left(N\right).$$

Diagonalization implies

$$S_{a_k}(N) = \sum_{l=1}^{N} \left( \prod_{\substack{n=1\\n\neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}.$$

Docal

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{j^k}.$$

When a - 1

$$\prod_{n=1\atop n=1}^{N}\frac{n}{n-1}=\left(-1\right)^{l-1}\binom{N}{l}$$

#### *a* > 0

$$P_{N|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2^{3}} & \frac{1}{2^{3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{M^{2}} & \frac{1}{M^{2}} & \cdots & 1 \\ \frac{1}{M^{2}} & \frac{1}{M^{2}} & \cdots & \frac{1}{M^{2}} \end{pmatrix} \text{ and } \begin{pmatrix} P_{N|a}^{k+1} \end{pmatrix}_{N,a} = \frac{1}{N^{a}} S_{a_{k}}(N).$$

Diagonalization implies

$$S_{a_k}(N) = \sum_{l=1}^{N} \left( \prod_{\substack{n=1\\n\neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}$$

Danel

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{jk}$$

When a - 1

$$\prod_{n=1}^{N} \frac{n}{n-l} = (-1)^{l-1} \binom{N}{l}$$

#### a > 0

$$P_{N|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2^{3}} & \frac{1}{2^{3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{M^{2}} & \frac{1}{M^{2}} & \cdots & \frac{1}{M^{2}} \end{pmatrix} \text{ and } \begin{pmatrix} P_{N|a}^{k+1} \end{pmatrix}_{N,a} = \frac{1}{N^{3}} S_{a_{k}}(N).$$

Diagonalization implies

$$S_{a_k}(N) = \sum_{l=1}^{N} \left( \prod_{\substack{n=1\\n\neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}$$

Recal

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^k}$$

When a=1.

$$\prod_{n=1}^{N} \frac{n}{n-l} = (-1)^{l-1} \binom{N}{l}$$

#### a > 0

$$P_{N\mid a} = \left( \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \frac{1}{2^{2}} & \frac{1}{2^{3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N^{2}} & \frac{1}{N^{2}} & \cdots & \frac{1}{N^{2}} \end{array} \right) \text{ and } \left(P_{N\mid a}^{k+1}\right)_{N,a} = \frac{1}{N^{a}} S_{a_{k}}\left(N\right).$$

Diagonalization implies:

$$S_{a_k}(N) = \sum_{l=1}^{N} \left( \prod_{\substack{n=1\\n\neq l}}^{N} \frac{n^a}{n^a - l^a} \right) \frac{1}{l^{ak}}.$$

Danel

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k}$$

When a = 1

$$\prod_{n=1}^{N} \frac{n}{n-l} = (-1)^{l-1} \binom{N}{l}$$

#### a > 0

$$P_{N|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2^{a}} & \frac{1}{2^{a}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4^{1/3}} & \frac{1}{1^{1/3}} & \cdots & \frac{1}{4^{1/3}} \end{pmatrix} \text{ and } \begin{pmatrix} P_{N|a}^{k+1} \end{pmatrix}_{N,a} = \frac{1}{N^{a}} S_{a_{k}}(N).$$

Diagonalization implies:

$$S_{a_{k}}(N) = \sum_{l=1}^{N} \left( \prod_{\substack{n=1\\n\neq l}}^{N} \frac{n^{a}}{n^{a} - l^{a}} \right) \frac{1}{l^{a_{k}}}.$$

Recall

$$S_{1_k}(N) = \sum_{l=1}^{N} (-1)^{l-1} {N \choose l} \frac{1}{l^k}.$$

When a = 1.

$$\prod_{\substack{n=1\\n\neq l}}^{N} \frac{n}{n-l} = (-1)^{l-1} \binom{N}{l}$$

$$S(f;k;N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

Define  $f_0(x) = \frac{1}{x}$  and for l = 0, ..., k

$$\mathcal{P}_{N|f_{1}} := \begin{pmatrix} f_{1}(1) & 0 & \cdots & 0 \\ f_{1}(2) & f_{1}(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{1}(N) & f_{1}(N) & \cdots & f_{1}(N) \end{pmatrix}$$

#### THM

1 It holds that

$$S(f; k; N) = N \cdot \left(\prod_{l=0}^{k} \mathcal{P}_{N|f_l}\right)_{N,1}$$

2 If  $\{f_l(1), \ldots, f_l(N)\}$  are all distinct, then

$$\mathcal{P}_{N,f_{l}} = \mathcal{Q}_{N,f_{l}} \operatorname{diag} \left\{ f_{l} \left( 1 \right), \ldots, f_{l} \left( N \right) \right\} \mathcal{Q}_{N,f_{l}}^{-1}.$$

Entries are calculated explicitly

$$S(f;k;N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

Define  $f_0(x) = \frac{1}{x}$  and for  $l = 0, \dots, k$ 

$$\mathcal{P}_{N|f_{i}} := \begin{pmatrix} f_{i}\left(1\right) & 0 & \cdots & 0 \\ f_{i}\left(2\right) & f_{i}\left(2\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{i}\left(N\right) & f_{i}\left(N\right) & \cdots & f_{i}\left(N\right) \end{pmatrix}$$

#### THM

1 It holds that

$$S(f; k; N) = N \cdot \left(\prod_{l=0}^{k} \mathcal{P}_{N|f_l}\right)_{N,1}$$

2 If  $\{f_l(1), \ldots, f_l(N)\}$  are all distinct, then

$$\mathcal{P}_{N,f_{l}} = \mathcal{Q}_{N,f_{l}} \operatorname{diag} \left\{ f_{l} \left( 1 \right), \ldots, f_{l} \left( N \right) \right\} \mathcal{Q}_{N,f_{l}}^{-1}.$$

Entries are calculated explicitly

$$S(f;k;N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

Define  $f_0(x) = \frac{1}{x}$  and for  $l = 0, \dots, k$ 

$$\mathcal{P}_{N|f_{j}} := \left( \begin{array}{cccc} f_{j}(1) & 0 & \cdots & 0 \\ f_{j}(2) & f_{j}(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{j}(N) & f_{j}(N) & \cdots & f_{j}(N) \end{array} \right).$$

#### THM

1 It holds that

$$S(f; k; N) = N \cdot \left(\prod_{l=0}^{k} \mathcal{P}_{N|f_l}\right)_{N,1}$$

2 If  $\{f_l(1), \ldots, f_l(N)\}$  are all distinct, then

$$\mathcal{P}_{N,f_l} = \mathcal{Q}_{N,f_l} \operatorname{diag} \left\{ f_l \left( 1 \right), \dots, f_l \left( N \right) \right\} \mathcal{Q}_{N,f_l}^{-1}.$$

Entries are calculated explicitly

$$S(f;k;N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

Define  $f_0(x) = \frac{1}{x}$  and for  $l = 0, \dots, k$ 

$$\mathcal{P}_{N|f_i} := \left( \begin{array}{cccc} f_i\left(1\right) & 0 & \cdots & 0 \\ f_i\left(2\right) & f_i\left(2\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_i\left(N\right) & f_i\left(N\right) & \cdots & f_i\left(N\right) \end{array} \right).$$

#### THM.

1 It holds that

$$S(f;k;N) = N \cdot \left(\prod_{i=0}^{k} \mathcal{P}_{N|f_i}\right)_{N,1}.$$

2 If  $\{f_l(1), \ldots, f_l(N)\}$  are all distinct, then

$$\mathcal{P}_{\textit{N},\textit{f}_{\textit{I}}} = \mathcal{Q}_{\textit{N},\textit{f}_{\textit{I}}} \, \mathsf{diag} \, \{\textit{f}_{\textit{I}} \, (1) \, , \ldots, \textit{f}_{\textit{I}} \, (\textit{N}) \} \, \mathcal{Q}_{\textit{N},\textit{f}_{\textit{I}}}^{-1}.$$

Entries are calculated explicitly.

$$\mathcal{S}\left(f;k;N\right):=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}f_1\left(n_1\right)\cdots f_k\left(n_k\right).$$

 $k = 1 \text{ and } f_1(x) = x, \text{ i.e.}$ 

$$\sum_{N \ge n_1 \ge 1} n_1 = \frac{N(N+1)}{2} \Rightarrow \sum_{l=1}^{N} (-1)^{N-l} l^{N+1} {N \choose l} = \frac{N(N+1)!}{2};$$

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a general result by Zeng: which, when taking  $a_j=\frac{a-bq^{j+i-1}}{c-zq^{j+i-1}}$  and N=n-i+1, "turns out to be a common source of several q-identities"

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- The integral representation leads to

$$B_{2k} = \frac{(-1)^{k+1}}{(1-2^{1-2k})(2\pi)^{2k}} \int_0^1 \ln^{2k} \left(\frac{\lambda}{1-\lambda}\right) d\lambda.$$

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#### End

# Thank You!