MATRIX REPRESENTATION FOR MULTIPLICATIVE NESTED SUMS

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Abstract

We provide a computation for multiplicative nested sums, which are generalization of harmonic sums, through multiplication of matrices. Special cases of these matrices are stochastic transition matrices for random walks on finite number of sites. Diagonalization of such matrices leads to combinatorial identities and their generalizations.

Keywords and phrases: multiplicative nested sum, harmonic sum, matrix multiplication, random walk, combinatorial identity.

1. Introduction

We consider the following *multiplicative nested sums* (MNS): for $m \geq 1$,

$$\mathcal{S}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right), \tag{1}$$

and

$$\mathcal{A}\left(f_1,\ldots,f_k;N,m\right) := \sum_{N>n_1>\cdots>n_k\geq m} f_1\left(n_1\right)\cdots f_k\left(n_k\right). \tag{2}$$

i.e., the usual summand $f\left(n_1,\ldots,n_k\right)=f_1\left(n_1\right)\cdots f_k\left(n_k\right)$ is multiplicative, and the sum indices are nested. Here, f_l, for all $l=1,\ldots,k$, can be any functions defined on $\mathbb{N}$, unless $N=\infty$ when convergence has to be taken into consideration.

Typical examples, when taking $f_l(x) := \frac{\operatorname{sign}(a_l)^x}{x^{a_l}}$ and $m=1$, are the harmonic sums [3, eq. 4, pp. 1]

$$S_{a_1,\ldots,a_k}\left(N\right) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}\left(a_1\right)^{n_1}}{n_1^{|a_1|}}\times\cdots\times\frac{\operatorname{sign}\left(a_k\right)^{n_k}}{n_k^{|a_k|}}, \tag{3}$$

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which are connected to Mellin transforms and integrals [11], and [6, pp. 3]

$$H_{a_1,\ldots,a_k}(N) = \sum_{N>n_1>\cdots>n_k\geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}}. \tag{4}$$

Special case when $N \to \infty$, $k = 1$ and $a_1 > 1$ in (3) or (4) gives the Riemann-zeta value $\zeta(a_1)$. Ablinger [1, Chpt. 6] implemented the `Mathematica` package `HarmonicSums.m`,* based on the recurrence [2, eq. 2.1, pp. 21] inherited from the quasi-shuffle relation [9, eq. 1, pp. 51], to compute harmonic sums. In Section 2, We present an alternative computation for MNS, which also works for harmonic sums, by associating to each function f_l a matrix. Properties such as eigenvalues, eigenvectors, and diagonalization of the matrix follow naturally.

In fact, the matrix associated to sums was inspired by constructing random walks for special harmonic sums. Different types of random walks, from that on the plane with fixed length [4] to that on Riemannian matrix manifolds [7], appear in and connect various fields. For example, the coefficients connecting Euler polynomials and generalized Euler polynomials [10, eq. 3.8, pp. 781] appear in a random walk over a finite number of sites [10, Note 4.8, pp. 787]. In Section 3, special sum when $f_1 = \cdots = f_k = x^{-a}$ for $a \geq 1$ is interpreted as probability of certain event in a random walk, while the associated matrix for computation is the corresponding stochastic transition matrix.

Besides probability, special MNS, namely harmonic sums, also connect to combinatorics. For instance, Dilcher [5, Cor. 3, pp. 93] established

$$S_{\underbrace{1,\ldots,1}_{k}}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{n_1\cdots n_k} = \sum_{l=1}^{N} (-1)^{l-1}\binom{N}{l}\frac{1}{l^k}. \tag{5}$$

In the Section 4, through diagonalization of matrix, we provide examples of special sums that are connected to combinatorial identities including a generalization of (5) and also a general result on q-series. These examples can be viewed as alternative proofs of these identities.

2. Matrix computation and properties

THEOREM 2.1. *Define, for $l = 1, \ldots, k$, three $N \times N$ matrices as follows.*

$$\mathbf{P}_N := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

*http://www.risc.jku.at/research/combinat/software/HarmonicSums/index.php

$$\mathbf{S}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix},$$

and

$$\mathbf{A}_{N|f_l} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f_l(1) & 0 & 0 & \cdots & 0 & 0 \\ f_l(2) & f_l(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_l(N-1) & f_l(N-1) & f_l(N-1) & \cdots & f_l(N-1) & 0 \end{pmatrix}.$$

Then, it holds that

$$\mathcal{S}(f_1, \ldots, f_k; N, m) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{N,m}, \tag{6}$$

and

$$\mathcal{A}(f_1, \ldots, f_k; N, m) = \left(\mathbf{P}_N . \prod_{l=1}^{k} \mathbf{A}_{N|f_l}\right)_{N,m}, \tag{7}$$

where $\mathbf{M}_{i,j}$ denotes the entry at i^{th} row and j^{th} column of a matrix $\mathbf{M}$.

Proof. Since the proof for $\mathcal{A}$ is similar, we shall only prove a stronger result for $\mathcal{S}$ that, for $i, j = 1, \ldots, N$, (noting when $j > i$ the sum is 0,)

$$\mathcal{S}(f_1, \ldots, f_k; i, j) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{i,j}.$$

1. When $k = 1$, it is trivial to see

$$\left(\mathbf{P}_N \cdot \mathbf{S}_{N|f_1}\right)_{i,j} = \sum_{l=j}^{i} f_1(l) = \mathcal{S}(f_1; i, j).$$

2. Suppose $\mathcal{S}(f_1, \ldots, f_k; i, j) = \left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right)_{i,j}$. Then, for $k+1$, from

$$\left(\mathbf{P}_N \cdot \prod_{l=1}^{k+1} \mathbf{S}_{N|f_l}\right)_{i,j} = \left(\left(\mathbf{P}_N \cdot \prod_{l=1}^{k} \mathbf{S}_{N|f_l}\right) \cdot \mathbf{S}_{N|f_{k+1}}\right)_{i,j},$$

we have

$$\begin{aligned}
\left(\mathbf{P}_N\cdot\prod_{l=1}^{k+1}\mathbf{S}_{N|f_l}\right)_{i,j} &= \sum_{l=j}^{i}\mathcal{S}\left(f_1\cdots f_k;i,l\right)f_{k+1}\left(l\right)\\
&= \sum_{l=j}^{i}f_{k+1}\left(l\right)\sum_{i\geq n_1\geq\cdots\geq n_k\geq l}f_1\left(n_1\right)\cdots f_k\left(n_k\right)\\
&= \sum_{i\geq n_1\geq\cdots\geq n_k\geq l\geq j}f_1\left(n_1\right)\cdots f_k\left(n_k\right)f_{k+1}\left(l\right)\\
&= \mathcal{S}\left(f_1\cdots f_{k+1};i,j\right).
\end{aligned}$$

□

REMARK. Shifting $\mathbf{S}_{N|f}$ downward by one row gives $\mathbf{A}_{N|f}$, i.e.,

$$\mathbf{A}_{N|f}=\left(\delta_{i-1,j}\right)_{N\times N}\mathbf{S}_{N|f},\text{ where }\delta_{a,b}=\begin{cases}1, & \text{if } a=b;\\ 0, & \text{otherwise.}\end{cases}$$

Apparently, $\mathbf{A}_{N|f}$ always degenerates, while $\mathbf{S}_{N|f}$ is invertible provided that $f\left(j\right)\neq 0$, $j=1,\ldots,N$. Moreover, if $\left(f\left(j\right)\right)_{j=1}^{N}$, are mutually distinct, $\mathbf{S}_{N|f}$ is diagonalizable with eigenvalues $f(j)$. Next, we compute the eigenvectors.

PROPOSITION 2.2. *If $\left(f\left(n\right)\right)_{n=1}^{N}$ are mutually distinct, we form two matrices $\mathbf{D}_{N|f}=\left(a_{i,j}\right)_{N\times N}$ and $\mathbf{E}_{N|f}=\left(b_{i,j}\right)_{N\times N}$ where*

$$a_{i,j}=\frac{f\left(i\right)}{f\left(N\right)}\prod_{k=i+1}^{N}\left(1-\frac{f\left(k\right)}{f\left(j\right)}\right)\ \text{ and }b_{i,j}=\begin{cases}0, & \text{if } i<j;\\ \frac{f(N)}{f(i)}\prod\limits_{\substack{k=j\\ k\neq i}}^{N}\frac{1}{1-\frac{f(k)}{f(i)}}, & \text{if } i\geq j.\end{cases}$$

Then, $\left(a_{1,j},\ldots,a_{N,j}\right)^T$ is an eigenvector of $f(j)$, i.e., columns of $\mathbf{D}_{N|f}$ are eigenvectors of $\mathcal{S}_{N|f}$, and $\left(\mathbf{D}_{N|f}\right)^{-1}=\mathbf{E}_{N|f}$.

PROOF. Note that $a_{i,j}=0$ if $j>i$, due to the zero term $k=j-1$ in the product. For eigenvector of $f\left(j\right)$, it suffices to prove $\forall i=1,\ldots,N$,

$$\sum_{l=j}^{i}f\left(l\right)\frac{f\left(i\right)}{f\left(N\right)}\prod_{k=l+1}^{N}\left(1-\frac{f\left(k\right)}{f\left(j\right)}\right)=f\left(j\right)\frac{f\left(i\right)}{f\left(N\right)}\prod_{k=i+1}^{N}\left(1-\frac{f\left(k\right)}{f\left(j\right)}\right),$$

which can be directly computed by induction on i. To prove the inverse matrix $\left(\mathbf{D}_{N|f}\right)^{-1}=\mathbf{E}_{N|f}$, it is equivalent to show for $i,j=1,\ldots,N$,

$$\sum_{s=j}^{i}\frac{f\left(i\right)}{f\left(s\right)}\prod_{k=s+1}^{N}\left(1-\frac{f\left(k\right)}{f\left(s\right)}\right)\prod_{\substack{l=j\\ l\neq s}}^{N}\frac{1}{1-\frac{f(l)}{f(s)}}=\delta_{i,j}. \tag{8}$$

This reduces to a general result that given a sequence of distinct real numbers $(a_t)_{t=m}^n$, it holds that

$$\sum_{t=m}^{n}\prod_{\substack{l=m\\ l\neq t}}^{n}\frac{1}{(a_l-a_t)}=\delta_{m,n},$$

which can be obtained easily from [12, eq. 1, pp. 313] by taking $z\to\infty$. Additionally note that the case $m=n$ of the identity above holds due to empty product on the left hand side. □

Remark. Now, it follows that $\forall k\in\mathbb{N}$,

$$\left(\mathbf{S}_{N|f}\right)^k=\mathbf{D}_{N|f}\mathrm{diag}\left(f(1)^k,\ldots,f(N)^k\right)\left(\mathbf{D}_{N|f}\right)^{-1}. \tag{9}$$

3. Random walks

Let $f_l(x)\equiv H_a(x):=1/x^a$ where $a\geq 1$, $l=1,\ldots,k$. When $a=1$,

$$\mathbf{S}_{N|H_1}=\begin{pmatrix}1&0&\cdots&0\\ \frac{1}{2}&\frac{1}{2}&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ \frac{1}{N}&\frac{1}{N}&\cdots&\frac{1}{N}\end{pmatrix}. \tag{10}$$

Now, label N sites as

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \cdots\ \underset{N-1}{\bullet}\ \underset{N}{\bullet}$$

We start a random walk at site "N", with the rules:

1. one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
2. steps are independent.

Let $\mathbb{P}(i\to j)$ = the probability from site "i" to site "j". For example, suppose we are at site "6":

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \underset{4}{\bullet}\ \underset{5}{\bullet}\ \overset{\text{here}}{\underset{6}{\bullet}}\ \ \underset{7}{\bullet}\ \underset{8}{\bullet}\ \cdots\ \underset{N}{\bullet}$$

Then, the next step only allows to walk to sites $\{1,2,3,4,5,6\}$, with

$$\mathbb{P}(6\to 6)=\cdots=\mathbb{P}(6\to 1)=\frac{1}{6}.$$

Therefore, a typical walk is as follows:

<u>STEP 1</u>: walk from "N" to some site "$n_1\,(\leq N)$", with $\mathbb{P}(N\to n_1)=\frac{1}{N}$;

STEP 2: walk from "n_1" to "$n_2\,(\le n_1)$", with $\mathbb{P}\,(n_1 \to n_2) = \frac{1}{n_1}$;

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

STEP $k+1$: walk "$n_k \mapsto n_{k+1}\,(\le n_k)$" with $\mathbb{P}\,(n_k \to n_{k+1}) = \frac{1}{n_k}$.

Focus on $\mathbb{P}\,(n_{k+1} = 1)$. Since the steps are independent,

$$\mathbb{P}\,(n_{k+1} = 1) = \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{N n_1 \cdots n_k} = \frac{\mathcal{S}\,(H_1, \ldots, H_1; N, 1)}{N}. \tag{11}$$

On the other hand, the transition matrix is exactly $\mathbf{S}_{N|H_1}$, i.e,

$$\mathbf{S}_{N|H_1} = (\alpha_{i,j}) \text{ with } \alpha_{i,j} = \mathbb{P}\,(i \to j) = \begin{cases} 1/i, & \text{if } j \le i; \\ 0, & \text{if } j > i. \end{cases}$$

Therefore,

$$\left(\left(\mathbf{S}_{N|H_1} \right)^{k+1} \right)_{N,1} = \mathbb{P}\,(n_{k+1} = 1) = \frac{1}{N} \mathcal{S}\,(H_1, \ldots, H_1; N, 1)\,, \tag{12}$$

which is a probabilistic interpretation of (6), with slight difference that $\mathbf{P}_N$ in (6) replaced by $\mathbf{S}_{N|H_1}$.

Remark. When $a > 1$, we could also form a similar random walk by

- artificially adding an extra site, denoted by "$\mathfrak{N}$", to the right of "N";
- making site "$\mathfrak{N}$" a *sink* , i.e., $\mathbb{P}\,(\mathfrak{N} \to n) = \delta_{\mathfrak{N},n}$, $n \in \{1, \ldots, N, \mathfrak{N}\}$;
- defining for $l = 1, 2, \ldots, N$,

$$\mathbb{P}\,(l \to j) = \begin{cases} 0, & \text{if } l < j \le N; \\ \frac{1}{l^a}, & \text{if } 1 \le j \le l; \\ 1 - \frac{1}{l^{a-1}}, & \text{if } j = \mathfrak{N}. \end{cases}$$

Similar computation for $\mathbb{P}\,(n_{k+1} = 1)$ reveals that

$$\mathcal{S}\,(H_a, \ldots, H_a; N, 1) = N^a \left(\left(\mathbf{S}_{N|H_a} \right)^{k+1} \right)_{N,1}. \tag{13}$$

4. Combinatorial identities

The matrix computation in section 2, especially the diagonalization for computing matrix power, leads to alternative proofs for some combinatorial identities and their generalizations.

EXAMPLE 4.1. *Recall when $f_l(x) \equiv H_a(x) := 1/x^a$. Using (9), we have*

$$\left(\mathbf{S}_{N|H_a}\right)^{k+1} = \mathbf{D}_{N|H_a} \mathrm{diag}\left(1, \ldots, \frac{1}{N^{ak}}\right) \left(\mathbf{D}_{N|H_a}\right)^{-1}.$$

Then, by (13),

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^a \cdots n_k^a} = \mathcal{S}\left(H_a, \ldots, H_a; N, 1\right) = \sum_{l=1}^{N} \left(\prod_{\substack{n=1 \\ n \neq l}}^{N} \frac{n^a}{n^a - l^a}\right) \frac{1}{l^{ak}},$$

which gives (5) *when $a = 1$.*

REMARK. When $a = m \in \mathbb{Z}_+$, considering the primitive root of unity $\xi_m := \exp\left\{\frac{2\pi \mathrm{i}}{m}\right\}$, where $\mathrm{i}^2 = -1$, and the factorization

$$n^m - l^m = (n - l)(n - \xi_m l) \cdots \left(n - \xi_m^{m-1} l\right),$$

we could obtain the following binomial-type expression (like (5))

$$\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^a \cdots n_k^a} = \sum_{l=1}^{N} \left(\prod_{t=0}^{m-1} \binom{N}{xi_m^t} \frac{\pi\left(1 - \xi_m^t\right) l}{\sin\left(\pi \xi_m^t l\right)}\right) \frac{1}{l^{mk}},$$

where the usual binomial coefficients are generalized as

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.$$

EXAMPLE 4.2. *Let $f_l(m) = f(m) = a_m$ with $(a_m)_{m=1}^N$ distinct. Then,*

$$\begin{aligned}
&\sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} a_{n_1} \cdots a_{n_k} \\
&= \mathcal{S}\left(f, \ldots, f; N, 1\right) \\
&= \frac{1}{f(N)} \left(\mathbf{D}_{N|H_a} \mathrm{diag}\left\{f(1)^{k+1}, \ldots, f(N)^{k+1}\right\} \left(\mathbf{D}_{N|H_a}\right)^{-1}\right)_{N,1} \\
&= \frac{1}{f(N)} \sum_{j=1}^{N} f(j)^{k+1} \left(\frac{f(N)}{f(j)} \prod_{\substack{m=1 \\ m \neq j}}^{N} \frac{1}{1 - \frac{f(m)}{f(j)}}\right) \\
&= \sum_{j=1}^{N} \left(\prod_{\substack{m=1 \\ m \neq j}}^{N} \frac{1}{1 - \frac{a_m}{a_j}}\right) a_j^k,
\end{aligned}$$

recovering a general result [12, eq. 2, pp. 313], which, when taking $a_j = \frac{a - bq^{j+i-1}}{c - zq^{j+i-1}}$ and $N = n - i + 1$, "turns out to be a common source of several q-identities" [12, pp. 314].

5. Future work

In the literature, a lot of results are obtained on the relations between the following two sums: [8, pp. 275]

$$S\left(i_1,\ldots,i_k\right) := \mathcal{S}\left(\frac{1}{x^{i_1}},\ldots,\frac{1}{x^{i_k}};\infty,1\right),$$

and [8, pp. 276]

$$A\left(i_1,\ldots,i_k\right) := \mathcal{A}\left(\frac{1}{x^{i_1}},\ldots,\frac{1}{x^{i_k}};\infty,1\right),$$

where the latter is multiple zeta function. Typical examples include [8, pp. 276]

$$S(1,2) = A(3) = \zeta(3) \text{ and } S(i_1,i_2) = A(i_1,i_2) + A(i_1+i_2).$$

We would like to the following identity, viewed as a finite version of the second one above, that

$$\mathcal{S}\left(f_1,f_2;N,m\right) = \mathcal{A}\left(f_1,f_2;N+1,m\right) + \mathcal{A}\left(f_1\cdot f_2;N+1,m\right),$$

through the matrix multiplication. We did not search in the literature for proofs, but it does hold in computations for some concrete N and m. Difficulty comes from the facts that

1. dimensions of matrices for $\mathcal{S}$ and $\mathcal{A}$ differ by 1;
2. transition matrix $\left(\delta_{i-1,j}\right)_{N\times N}$ in (2) does not commute with $\mathcal{S}$.

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