The Method of Brackets (MoB) and Integrating by Differentiating (IbD) Method

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Outlines

1 The method of brackets (MoB)
- Rules
- Ramanujan's Master Theorem (RMT)
- Examples
- Recent result

2 Integration by Differentiating
- Formulas
- Recent proofs
- Connection

Rules

Idea

MoB evaluates the definite integral

$$\int_0^\infty f(x)\,dx$$

(most of the time) in terms of SERIES, with *ONLY SIX* rules:

Defintion [Indicator]

$$\phi_n := \frac{(-1)^n}{n!} = \frac{(-1)^n}{\Gamma(n+1)}$$

and

$$\phi_{1,\ldots,r} := \phi_{n_1,\ldots,n_r} = \phi_{n_1}\phi_{n_2}\cdots\phi_{n_r} = \prod_{i=1}^{r}\phi_{n_i}.$$

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Rules (P-Production; E-Evaluation) $I = \int_0^\infty f(x)\,dx$

P_1: $f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n+\beta-1} \Rightarrow \int_0^\infty f(x)\,dx \mapsto \sum_n a_n \langle \alpha n + \beta \rangle$—Bracket Series;

P_2: For $\alpha < 0$, $(a_1 + \cdots + a_r)^\alpha \mapsto \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}$;

P_3: For each bracket series, we assign *index=# of sums− # of brackets*;

E_1: $\sum_n \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*)$, where n^* solves $\alpha n + \beta = 0$;

E_2: $$\sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} f(n_1,\ldots,n_r) \prod_{i=1}^{r} \langle a_{i1} n_1 + \cdots + a_{ir} n_r + c_i \rangle = \frac{f(n_1^*,\ldots,n_r^*) \prod_{i=1}^{r} \Gamma(-n_i^*)}{|\det A|},$$

$$(n_1^*,\ldots,n_r^*) \text{ solves } \begin{cases} a_{11} n_1 + \cdots + a_{1r} n_r + c_1 & = 0 \\ \ldots & \cdots \\ a_{r1} n_1 + \cdots + a_{rr} n_r + c_r & = 0 \end{cases};$$

E_3: The value of a multi-dimensional bracket series of ***POSITIVE*** index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded. Any series producing a non-real contribution is also discarded.

Ramanujan's Master Theorem[RMT]

Theorem[RMT]

$$\int_0^\infty x^{s-1}\left\{a(0)-\frac{a(1)}{1!}x+\frac{a(2)}{2!}x^2-\cdots\right\}dx=a(-s)\Gamma(s)$$

(1)

$$\int_0^\infty x^{s-1}\left(\sum_{n=0}^\infty \phi_n a(n)x^n\right)dx=a(-s)\Gamma(s)$$

(2) [Hardy]

•$H(\delta):=\{s=\sigma+\iota t:\sigma\geq-\delta, 0<\delta<1\}$;

•$\psi(x)\in C^\infty(H(\delta))$; $\exists C,P,A,\ A<\pi$ such that $|\psi(s)|\leq Ce^{P\delta+A|t|}$, $\forall s\in H(\delta)$;

•$0<c<\delta,\ \Psi(x):=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}\frac{\pi}{\sin(\pi s)}\psi(-s)x^{-s}dx \overset{0<x<e^{-P}}{=\!=\!=\!=} \sum_{k=0}^\infty\psi(k)(-x)^k$;

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Theorem[RMT]

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(2) Keep Track of s;
(3) Apply the Formula;
(4) Multiple Integrals;

$$\int_0^\infty\int_0^\infty \sum_{n,m} a(m,n)\,x^m y^n\,dxdy = ?$$

(5) More Sums than Integrals (brackets);

$$\int_0^\infty f_1(x)\,f_2(x)\,dx = \int_0^\infty \sum_{m,n} a(m,n)\,x^{m+n}dx = \sum_{m,n} a(m,n)\,\langle m+n+1\rangle = ?$$

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Rule P_2

$$
\begin{aligned}
& \frac{\Gamma(-\alpha)}{(a_1+\cdots+a_r)^{-\alpha}}. \\
= & \int_0^\infty x^{-\alpha-1} e^{-(a_1+\cdots+a_r)x} dx \\
= & \int_0^\infty x^{-\alpha-1} e^{-a_1 x} e^{-a_2 x} \cdots e^{-a_r x} dx \\
= & \int_0^\infty x^{-\alpha-1} \prod_{i=1}^{r} \left(\sum_{n_i=0}^{\infty} \phi_{n_i} (ax)^{n_i} \right) dx \\
= & \int_0^\infty \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} x^{n_1+\cdots+n_r-\alpha-1} dx \\
= & \sum_{n_1,\ldots,n_r} \phi_{1,\ldots,r} a_1^{n_1} \cdots a_r^{n_r} \langle -\alpha + n_1 + \cdots + n_r \rangle
\end{aligned}
$$

Examples

$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay} \quad [y>0\ Re(a)>0]$$

Rule P_2:

$$\frac{1}{\sqrt{a^2+x^2}} = \left(a^2+x^2\right)^{-\frac{1}{2}} = \sum_{n_1,n_2}\phi_{1,2}a^{2n_1}x^{2n_2}\frac{\left\langle \frac{1}{2}+n_1+n_2\right\rangle}{\Gamma\left(\frac{1}{2}\right)}$$

$J_0\left(xy\right)$

$$J_0\left(xy\right) = \sum_{n_3}\phi_{n_3}\frac{y^{2n_3}}{\Gamma\left(n_3+1\right)2^{2n_3}}x^{2n_3}$$

Rule P_1

$$\begin{aligned} I &= \int_0^\infty \sum_{n_1,n_2,n_3}\phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{\Gamma\left(n_3+1\right)\Gamma\left(\frac{1}{2}\right)2^{2n_3}}\left\langle n_1+n_2+\frac{1}{2}\right\rangle x^{2n_2+2n_3+1}dx \\ &= \sum \phi_{1,2,3}\frac{y^{2n_3}a^{2n_1}}{\Gamma\left(n_3+1\right)\Gamma\left(\frac{1}{2}\right)2^{2n_3}}\left\langle n_1+n_2+\frac{1}{2}\right\rangle \left\langle 2n_2+2n_3+2\right\rangle \end{aligned}$$

Examples

$$I := \int_0^\infty xJ_0(xy)\frac{dx}{\sqrt{a^2+x^2}} = y^{-1}e^{-ay} \quad [y>0\ Re(a)>0]$$

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Multi-dim

$$I=\int_{\mathbb{R}^m} f\left(x_1^2+\cdots+x_m^2\right)dx_1\cdots dx_m$$

(I) Usual method: By the n-dim spherical coordinate that $r=x_1^2+\cdots+x_m^2$ and

$$\begin{cases} x_1=r\cos\left(\phi_1\right), & 0\leq\phi_1\leq\pi,\\ x_2=r\sin\left(\phi_2\right)\cos\left(\phi_2\right), & 0\leq\phi_2\leq\pi,\\ \cdots & \cdots\\ x_{n-2}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-3}\right)\cos\left(\phi_{m-2}\right), & 0\leq\phi_{m-2}\leq\pi,\\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\cos\left(\phi_{m-1}\right), & 0\leq\phi_{m-1}\leq 2\pi,\\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\sin\left(\phi_{m-1}\right), & 0\leq r<\infty, \end{cases}$$

we have

$$dx_1\cdots dx_m=r^{m-1}\sin^{m-2}\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)drd\phi_1\cdots d\phi_{m-1}.$$

Thus,

$$I\quad=\quad 2\pi^{\frac{m}{2}}\left[\int_0^\infty r^{m-1}f\left(r^2\right)dr\right]\frac{1}{\Gamma\left(\frac{m}{2}\right)}.$$

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(I) Usual method: By the n-dim spherical coordinate that $r=x_1^2+\cdots+x_m^2$ and

$$\begin{cases} x_1=r\cos\left(\phi_1\right), & 0\leq\phi_1\leq\pi, \\ x_2=r\sin\left(\phi_2\right)\cos\left(\phi_2\right), & 0\leq\phi_2\leq\pi, \\ \cdots & \cdots \\ x_{n-2}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-3}\right)\cos\left(\phi_{m-2}\right), & 0\leq\phi_{m-2}\leq\pi, \\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\cos\left(\phi_{m-1}\right), & 0\leq\phi_{m-1}\leq 2\pi, \\ x_{n-1}=r\sin\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)\sin\left(\phi_{m-1}\right), & 0\leq r<\infty, \end{cases}$$

we have

$$dx_1\cdots dx_m=r^{m-1}\sin^{m-2}\left(\phi_1\right)\cdots\sin\left(\phi_{m-2}\right)drd\phi_1\cdots d\phi_{m-1}.$$

Thus,

$$I = 2\pi^{\frac{m}{2}}\left[\int_0^\infty r^{m-1}f\left(r^2\right)dr\right]\frac{1}{\Gamma\left(\frac{m}{2}\right)}.$$

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Direct computation shows:

$$\begin{aligned}
I &= 2^m \int_{\mathbb{R}_+^m} \left[\sum_{l=0}^{\infty} \phi_l a(l) \left(x_1^2 + \cdots + x_m^2 \right)^l \right] dx_1 \cdots dx_m \\
&= 2^m \int_{\mathbb{R}_+^m} \sum_{l=0}^{\infty} \phi_l a(l) \sum_{\substack{n_1,\ldots,n_m \\ n_1+\cdots+n_m=l}} \binom{l}{n_1,\cdots,n_m} x_1^{2n_1} \cdots x_m^{2n_m} dx_1 \cdots dx_m \\
&= 2^m \sum_{l=n_1+\cdots+n_m} \phi_l a(l) \sum_{n_1,\ldots,n_m} \phi_{1,\ldots,m} \binom{l}{n_1,\cdots,n_m} \frac{1}{\phi_{1,\ldots,m}} \prod_{j=1}^{m} \langle 2n_j + 1 \rangle \\
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as desired.

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Multi-dim

$$I=\int_{\mathbb{R}_+^m}\frac{x_1^{p_1-1}\cdots x_m^{p_m-1}dx_1\cdots dx_m}{(r_0+r_1x_1+\cdots+r_mx_m)^s}=\frac{\Gamma(p_1)\cdots\Gamma(p_m)\Gamma(s-p_1-p_2-\cdots-p_n)}{r_1^{p_1}\cdots r_m^{p_m}r_0^{s-p_1-\cdots-p_m}\Gamma(s)}.$$

$$(r_0+r_1x_1+\cdots+r_mx_m)^{-s}=\sum_{n_0,n_1,\ldots,n_m}\phi_{0,1,\ldots,m}r_0^{n_0}r_1^{n_1}x_1^{n_1}\cdots r_m^{n_m}x_m^{n_m}\frac{\langle s+n_0+\cdots+n_m\rangle}{\Gamma(s)}$$

$$I=\frac{1}{\Gamma(s)}\sum_{n_0,n_1,\ldots,n_m}\phi_{0,1,\ldots,m}r_0^{n_0}\cdots r_m^{n_m}\langle s+n_0+\cdots+n_m\rangle\prod_{j=1}^{m}\langle n_m+p_m\rangle.$$

$$\begin{bmatrix}1&1&1&\cdots&1\\0&1&0&\cdots&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}n_0\\n_1\\\cdots\\n_m\end{bmatrix}+\begin{bmatrix}s\\p_1\\\cdots\\p_m\end{bmatrix}=\begin{bmatrix}0\\0\\0\\\cdots\\0\end{bmatrix},$$

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Null/Divergent Series

$K_0(x) = \int_0^\infty \frac{\cos(tx)dt}{\sqrt{1+t^2}}$.

$$K_0(x) = \frac{1}{2}\sum_n \phi_n \Gamma(-n)\frac{x^{2n}}{4^n} \text{ and } K_0(x) = \sum_n \phi_n \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma(-n)} \cdot \frac{4^n}{x^{2n+1}}$$

$$\mathcal{M}(K_0)(s) = \int_0^\infty x^{s-1} K_0(x)\, dx = 2^{s-2}\Gamma\left(\frac{s}{2}\right)^2$$

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DEF

A. Kempf et al. study Dirac-delta function to obtain the following formulas:

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\int_a^b f(x)\,dx &= \lim_{\varepsilon\to 0} f(\partial_\varepsilon)\frac{e^{\varepsilon b}-e^{\varepsilon a}}{\varepsilon},\\
\int_{-\infty}^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} 2\pi f(-\iota\partial_\varepsilon)\,\delta(\varepsilon) = 2\pi\delta(\iota\partial_\varepsilon)\,f(\varepsilon),\\
\int_0^{\infty} f(x)\,dx &= \lim_{\varepsilon\to 0} f(-\partial_\varepsilon)\frac{1}{\varepsilon},\\
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Example

$$I=\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

$$f(x)=\frac{1}{x}\cdot\frac{1}{2\iota}\left(e^{\iota x}-e^{-\iota x}\right)$$

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Note that $1/\partial_\varepsilon$ is the inverse operation of derivative, i.e., integration.

$$I=\frac{1}{2\iota}\lim_{\varepsilon\to0}\left(e^{-\iota\partial_\varepsilon}-e^{\iota\partial_\varepsilon}\right)\circ(\ln\varepsilon+c)$$

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$$e^{a\partial_{\varepsilon}}\circ f\left(\varepsilon\right)=f\left(\varepsilon+a\right).$$

$$\begin{aligned}I &= \frac{1}{2\iota}\lim_{\varepsilon\to0}\left[\left(\ln\left(\varepsilon-\iota\right)+c\right)-\left(\ln\left(\varepsilon+\iota\right)+c\right)\right]\\ &= \frac{1}{2\iota}\lim_{\varepsilon\to0}\left[\ln\left(\varepsilon-\iota\right)-\ln\left(\varepsilon+\iota\right)\right]=\frac{1}{2\iota}\left(\frac{-\iota\pi}{2}-\frac{\iota\pi}{2}\right)=\frac{\pi}{2}.\end{aligned}$$

Remark

$$
\begin{aligned}
I &= \int_0^\infty \frac{1}{x} \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} x^{2n+1} dx \\
&= \sum_n \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} \langle 2n+1 \rangle \\
&= \frac{1}{2} \cdot \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(2\left(-\frac{1}{2}\right)+2\right)} \Gamma\left(\frac{1}{2}\right) \\
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Proofs

D. Jia , E. Tang, A. Kempf, "present a list of propositions that put the above integration by differentiation methods on a rigorous footing."

$$\int_0^\infty f(x)\, e^{-xy}\, dx = \lim_{a\to\infty} f(-\partial_y)\, \frac{1-e^{-ay}}{y},$$

provided that $f:\mathbb{R}\to\mathbb{R}$ is entire and Laplace transformable on $\mathbb{R}_+$. Formal/Key idea:

$$\begin{aligned}\int_0^\infty f(x)\, e^{-xy}\, dx &= \int_0^\infty \sum_{n=0}^\infty c_n x^n e^{-xy}\, dx \\ &= \sum_{n=0}^\infty c_n \lim_{a\to\infty} \int_0^a x^n e^{-xy}\, dx \\ &= \sum_{n=0}^\infty c_n \lim_{a\to\infty} \int_0^a (-\partial_y)^n\, e^{-xy}\, dx.\end{aligned}$$

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$$f(x)=\sum_{n=0}^{\infty}a_nx^{\alpha n+\beta-1}\Rightarrow\int_0^{\infty}f(x)\,dx=\lim_{\varepsilon\to0}\int_0^{\infty}e^{-\varepsilon x}f(x)\,dx.$$

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Recall P_1:

$$f(x)=\sum_{n=0}^{\infty}a_nx^{\alpha n+\beta-1}\Rightarrow\int_0^{\infty}f(x)\,dx=\sum_n a_n\langle\alpha n+\beta\rangle.$$

Formally,

$$\langle a\rangle:=\int_0^{\infty}x^{a-1}dx.$$

and

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Therefore

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 & =\lim_{\varepsilon\to0}\sum_{n}a_{n}\int_{0}^{\infty}x^{\alpha n+\beta-1}e^{-\varepsilon x}dx\\
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Possible Future Work

- Connect MoB to IbD, and provide rigorous proofs of the former.
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End

Thank you!