Matrix Representation of Harmonic Sums*

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Abstract

We provide an alternative computation for harmonic sums through multiplication of matrices, special cases of which are interpreted as stochastic matrices associated to random walks, so that harmonic sums are recognized as probabilities of certain paths. Diagonalization of these matrices allows to recover and generalize a combinatorial identity. General matrix representation of multiplicative nested sums leads to more combinatorial identities.

Keywords: harmonic sum, index matrix, random walk, combinatorial identity, multiplicative nested sum

1 Introduction

The harmonic sums [4, eq. 4, pp. 1], defined by

$$S_{a_1,\ldots,a_k}(N)=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}}\times\cdots\times\frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}},\ N\in\mathbb{N}$$

are connected to Mellin transforms and integrals [11]. Here, $a_1,\ldots,a_k\in\mathbb{R}\setminus\{0\}$ are called indices. For simplicity, when all indices equal, we call these sums *equally indexed harmonic sums*, denoted by

$$S_{\boldsymbol{a}_k}(N)=S_{\underbrace{a,\ldots,a}_{k}}(N).$$

As a simple example, taking $N\to\infty$, $k=1$ and $a_1>0$ gives the Riemann-zeta function:

$$S_{a_1}(\infty)=\sum_{n_1\geq 1}\frac{1}{n_1^{a_1}}=\zeta(a_1).$$

Besides studies of properties and applications, Ablinger [1, Chpt. 6] additionally implemented the `Mathematica` package `HarmonicSums.m`[1], which has become a main tool for symbolic computation. This package is based on the recurrence [3, eq. 2.1, pp. 21]

$$\begin{aligned}S_{a_1,\ldots,a_n}(N)\cdot S_{b_1,\ldots,b_m}(N) &= \sum_{l_1=1}^{N}\frac{\operatorname{sign}(a_1)^{l_1}}{l_1^{|a_1|}}S_{a_2,\ldots,a_n}(l_1)\,S_{b_1,\ldots,b_m}(l_1)\\ &+\sum_{l_2=1}^{N}\frac{\operatorname{sign}(b_1)^{l_2}}{l_2^{|b_1|}}S_{a_1,\ldots,a_n}(l_2)\,S_{b_2,\ldots,b_m}(l_2)\\ &-\sum_{l=1}^{N}\frac{(\operatorname{sign}(a_1)\operatorname{sign}(b_1))^{l}}{l^{|a_1|+|b_1|}}S_{a_2,\ldots,a_n}(l)\,S_{b_2,\ldots,b_m}(l),\end{aligned}$$

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[1] http://www.risc.jku.at/research/combinat/software/HarmonicSums/index.php

inherited from the quasi-shuffle relation [9, eq. 1, pp. 51]. Therefore, as the number of indices increase, it takes more time to compute the sums. In Section 2, we provide an alternative computation by associating to each index a_l, $l = 1, \ldots, k$, an *index matrix*, the multiplication of which provides the values of harmonic sums. In addition, if the indices satisfy certain patterns, this method also provides symbolic results on the length of the indices.

Random walks, on the other hand, of different types, from that on the plane with fixed length [5] to that on Riemannian matrix manifolds [7], appear also in various fields. Some different topics, which seem to be completely unrelated, can often be connected to certain random walks. For example, the coefficients connecting Euler polynomials and generalized Euler polynomials [10, eq. 3.8, pp. 781] appear in a random walk over a finite number of sites [10, Note 4.8, pp. 787]. In Section 3, we set up random walk models with stochastic matrices coinciding with the index matrices for *positively equally indexed harmonic sums*, i.e., $S_{\boldsymbol{a}_k}(N)$ with $a > 0$, which in the models calculate the probabilities of certain paths.

Besides probability, harmonic sums also connect to combinatorics. For instance, Ablinger used [6, Cor. 3, pp. 93]:

$$\sum_{l=1}^{N} (-1)^{l-1} \binom{N}{l} \frac{1}{l^k} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k}, \tag{1.1}$$

to connect $S_{\boldsymbol{1}_k}(N)$ to the *complete Bell polynomials* Y_k [2, eq. 12.3.7, pp. 205] as follows [1, Prop. 2.7.2, pp. 35][2]:

$$S_{\boldsymbol{1}_k}(N) = \frac{1}{k!} Y_k\left(\ldots, (l-1)! S_l(N), \ldots\right), \tag{1.2}$$

which in fact extends similarly to $S_{\boldsymbol{a}_k}(N)$ [3, pp. 27–28] and $S_{\boldsymbol{a}_k}(\infty)$ [8, Thm. 2.1, pp. 277]. We first compute the diagonalization of the index matrix in Section 4. Then in Section 5, we apply the diagonalization to positively equally indexed harmonic sums, revealing an alternative proof of (1.1) and its generalization.

Finally in Section 6, we generalize this matrix representation to *multiplicative nested sums*, i.e., sums of the form

$$\mathcal{S}(f; k; N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

Namely, the factorization for the general summand $f(n_1, \ldots, n_k) = f_1(n_1) \cdots f_k(n_k)$ holds. Examples of obtained combinatorial identities involve binomial coefficients and q-series.

2 Matrix representation

To each index a_l of a harmonic sum $S_{a_1,\ldots,a_k}(N)$, $l = 1, \ldots, k$, we associate the $N \times N$, lower triangular index matrix $P_{N|a_l} := \left(p_{i,j}^{(a_l)}\right)$, defined as

$$p_{i,j}^{(a_l)} = \begin{cases} \frac{\operatorname{sign}(a_l)^i}{i^{|a_l|}}, & \text{if } i \geq j; \\ 0, & \text{otherwise.} \end{cases}$$

Namely,

$$P_{N|a_l} = \begin{pmatrix} \operatorname{sign}(a_l) & 0 & \cdots & 0 \\ \frac{1}{2^{|a_l|}} & \frac{1}{2^{|a_l|}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} & \cdots & \frac{\operatorname{sign}(a_l)^N}{N^{|a_l|}} \end{pmatrix}. \tag{2.1}$$

Now, we express harmonic sums as entries of the multiplication of these index matrices.

Theorem 1. *Denote $a_0 = 1$, then*

$$S_{a_1,\ldots,a_k}(N) = N \cdot \left(P_{N|a_0} P_{N|a_1} \cdots P_{N|a_k}\right)_{N,1} = N \cdot \left(\prod_{l=0}^{k} P_{N|a_l}\right)_{N,1}, \tag{2.2}$$

where for a matrix A, $(A)_{i,j}$ stands for the entry at the i^{th} row and j^{th} column.

[2] We correct a typo here.

Proof. We shall show a more general result for the last row of $\prod_{l=0}^{k} P_{N|a_l}$. Define

$$S_{a_1,\ldots,a_k}(N; j) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq j} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}},$$

then $S_{a_1,\ldots,a_k}(N) = S_{a_1,\ldots,a_k}(N;1)$. We claim that the last row of $\prod_{l=0}^{k} P_{N|a_l}$ is given by

$$\left(\frac{1}{N} S_{a_1,\ldots,a_k}(N;1), \frac{1}{N} S_{a_1,\ldots,a_k}(N;2), \ldots, \frac{1}{N} S_{a_1,\ldots,a_k}(N;N)\right),$$

and shall prove it by induction on k.

1) When $k=1$, let the vector $\left(u_1^{(1)}, \ldots, u_N^{(1)}\right)$ denote the last row of $P_{N|a_0} \cdot P_{N|a_1}$. From the matrix multiplication of $P_{N|a_0} \cdot P_{N|a_1}$, we see, for $j = 1, \ldots, N$,

$$u_j^{(1)} = \frac{1}{N}\left(\frac{\operatorname{sign}(a_1)^j}{j^{|a_1|}} + \cdots + \frac{\operatorname{sign}(a_1)^N}{N^{|a_1|}}\right) = \frac{1}{N} S_{a_1}(N;j).$$

2) Assume the claim holds for $k-1$, i.e., the last row of $\prod_{l=0}^{k-1} P_{N|a_l}$ is

$$\left(\frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;1), \frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;2), \ldots, \frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;N)\right).$$

Then, for the next index a_k, we denote the last row of $\prod_{l=0}^{k} P_{N|a_l}$ by $\left(u_1^{(k)}, \ldots, u_N^{(k)}\right)$. It follows,

$$u_j^{(k)} = \sum_{m=j}^{N} \frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;m) \frac{\operatorname{sign}(a_k)^m}{m^{|a_k|}}, \quad j = 1, \ldots, N.$$

To show $u_j^{(k)} = \frac{1}{N} S_{a_1,\ldots,a_k}(N;j)$, we first observe that both share the same type of summands, i.e.,

$$\frac{1}{N} \cdot \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}} \text{ with } N \geq n_1 \geq \cdots \geq n_k \geq j.$$

Then, since all summands in $\frac{1}{N} S_{a_1,\ldots,a_k}(N;j)$ have no multiplicity, it remains to show that $u_j^{(k)}$ has the same property. Suppose $u_j^{(k)}$ adds one term twice, denoted by

$$s_1 = s_2 = \frac{\operatorname{sign}(a_1)^{t_1} \cdots \operatorname{sign}(a_k)^{t_k}}{N t_1^{|a_1|} \cdots t_k^{|a_k|}}.$$

Further denote $t_k = L \in \{j, \ldots, N\}$, then both s_1 and s_2 belong to the part $\frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;L) \frac{\operatorname{sign}(a_k)^L}{L^{|a_k|}}$, implying the following term

$$\frac{\operatorname{sign}(a_1)^{t_1} \cdots \operatorname{sign}(a_{k-1})^{t_{k-1}}}{N t_1^{|a_1|} \cdots t_{k-1}^{|a_{k-1}|}}$$

appears in $\frac{1}{N} S_{a_1,\ldots,a_{k-1}}(N;L)$ twice, which is a contradiction. □

Remark 2. Applying a similar argument to the upper-left square submatrices of $\prod_{l=0}^{k} P_{N|a_l}$, we see that

$$\prod_{l=0}^{k} P_{N|a_l} = \left(p_{i,j}^{(a_0,\ldots,a_k)}\right) \text{ with } p_{i,j}^{(a_0,\ldots,a_k)} = \frac{1}{N} S_{a_1,\ldots,a_k}(i;j). \tag{2.3}$$

Note that when $j > i$, the empty sum gives 0, maintaining the lower triangular form.

Example 3. Consider $S_{2,\frac{1}{2},-\sqrt{2}}(3)$, namely $a_1 = 2$, $a_2 = \frac{1}{2}$, $a_3 = -\sqrt{2}$ and $N = 3$. We use the index matrices

$$P_{3|1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, P_{3|2} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}, P_{3|\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}, P_{3|-\sqrt{2}} = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{pmatrix}$$

to obtain

$$P_{3|1}P_{3|2}P_{3|\frac{1}{2}}P_{3|-\sqrt{2}} = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{5}{8} - \frac{1}{8\sqrt{2}} + 2^{-\frac{7}{2}-\sqrt{2}} & * & 0 \\ -\frac{49}{108} - \frac{13}{108\sqrt{2}} - \frac{1}{27\sqrt{3}} - 3^{-\frac{7}{2}-\sqrt{2}} + 2^{-\sqrt{2}}\left(\frac{13}{108\sqrt{2}} + \frac{1}{27\sqrt{3}}\right) & * & * \end{pmatrix}.$$

On the other hand, the package `HarmonicSums.m` provides

```
HarmonicSums by Jakob Ablinger — © RISC — Version 1.0 (27 / 01 / 15)

In[2]=S[2,1/2,-Sqrt[2],3]
```

Out[2]= $-\frac{49}{36} - \frac{13}{36\sqrt{2}} + \frac{13}{9}2^{-\frac{5}{2}-\sqrt{2}} - \frac{1}{9\sqrt{3}} + \frac{2^{-\sqrt{2}}}{9\sqrt{3}} - 3^{-\frac{5}{2}-\sqrt{2}}$

Therefore, we confirm (2.2) for this specific example:

$$\begin{aligned} & -\frac{49}{36} - \frac{13}{36\sqrt{2}} + \frac{13}{9}2^{-\frac{5}{2}-\sqrt{2}} - \frac{1}{9\sqrt{3}} + \frac{2^{-\sqrt{2}}}{9\sqrt{3}} - 3^{-\frac{5}{2}-\sqrt{2}} \\ = \; & 3 \cdot \left(-\frac{49}{108} - \frac{13}{108\sqrt{2}} - \frac{1}{27\sqrt{3}} - 3^{-\frac{7}{2}-\sqrt{2}} + 2^{-\sqrt{2}}\left(\frac{13}{108\sqrt{2}} + \frac{1}{27\sqrt{3}}\right)\right). \end{aligned}$$

3 Random walk interpretation

In this section, we interpret special index matrices as the stochastic matrices of certain random walks, with the corresponding harmonic sums calculating probabilities of certain paths for this walk.

3.1 The random walk model for $S_{\mathbf{1}_k}(N)$

Beginning with a simple case, we first recall the associated index matrix with respect to $a = 1$:

$$P_{N|1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix}. \tag{3.1}$$

Label N sites as follows:

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet \\ 1 & 2 & 3 & & N-1 & N \end{array}$$

We start a random walk at site "N", with the rules:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

In general, denote

$$\mathbb{P}(i \to j) = \text{the probability from site "}i\text{" to site "}j\text{".}$$

For example, suppose we are at site "6":

$$\begin{array}{cccccccccc} & & & & & \text{here} & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & N \end{array}$$

Then, the next step only allows to walk to sites $\{1,2,3,4,5,6\}$, with probabilities:

$$\mathbb{P}(6 \to 6) = \mathbb{P}(6 \to 5) = \mathbb{P}(6 \to 4) = \mathbb{P}(6 \to 3) = \mathbb{P}(6 \to 2) = \mathbb{P}(6 \to 1) = \frac{1}{6}.$$

Therefore, a typical walk is as follows:

STEP 1: walk from site "N" to some site "n_1 $(\le N)$", with $\mathbb{P}(N \to n_1) = \frac{1}{N}$;
STEP 2: walk from site "n_1" to some site "n_2 $(\le n_1)$", with $\mathbb{P}(n_1 \to n_2) = \frac{1}{n_1}$;

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

STEP $k+1$: walk from site "n_k" to some site "n_{k+1} $(\le n_k)$", with $\mathbb{P}(n_k \to n_{k+1}) = \frac{1}{n_k}$.

Focus on $\mathbb{P}(n_{k+1} = 1)$, which also includes the case that one hits site "1" before step $k+1$. Since the steps are independent,

$$\mathbb{P}(n_{k+1} = 1) = \sum_{\text{All possible paths}} \frac{1}{N}\frac{1}{n_1}\cdots\frac{1}{n_k} = \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{N n_1 \cdots n_k} = \frac{S_{\mathbf{1}_k}(N)}{N}. \tag{3.2}$$

On the other hand, the transition matrix of sites $\{1,\ldots,N\}$ is exactly given by $P_{N|1}$, i.e,

$$P_{N|1} = \left(p_{i,j}^{(1)}\right) \text{ with } p_{ij}^{(1)} = \mathbb{P}(i \to j) = \frac{1}{i}.$$

Therefore, after $k+1$ steps, entries of $P_{N|1}^{k+1}$ give the transition probabilities among sites. In particular,

$$\left(P_{N|1}^{k+1}\right)_{N,1} = \mathbb{P}(n_{k+1} = 1) = \frac{1}{N} S_{\mathbf{1}_k}(N), \tag{3.3}$$

which recovers (2.2) for the special case $a_1 = \cdots = a_k = 1$.

3.2 Model for $S_{\boldsymbol{a}_k}(N)$ with $a > 1$

Assume $a > 1$ so that for all $l = 1,2,\ldots,N$, $l \cdot \frac{1}{l^a} = \frac{1}{l^{a-1}} \le 1$. Now, modify the walk in Subsection 3.1, as follows:

- artificially add an extra site, denoted by "$\mathfrak{N}$", to the right of the site "N";
- make site "$\mathfrak{N}$" a *sink* , the same as site "1", i.e., from site "$\mathfrak{N}$", one can only walk to site "$\mathfrak{N}$" again, with probability 1;
- for other sites, $l = 1,2,\ldots,N$,

$$\mathbb{P}(l \to j) = \begin{cases} 0, & \text{if } l < j \le N; \\ \frac{1}{l^a}, & \text{if } 1 \le j \le l; \\ 1 - \frac{1}{l^{a-1}}, & \text{if } j = \mathfrak{N}. \end{cases}$$

Therefore, the probability of a typical $(k+1)$-step walk, starting from site "N" without falling into sink "$\mathfrak{N}$", is given by $\frac{1}{N^a} \cdot \frac{1}{n_1^a} \cdots \frac{1}{n_k^a}$. Hence,

$$\mathbb{P}(n_{k+1} = 1) = \sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{N^a} \cdot \frac{1}{n_1^a} \cdots \frac{1}{n_k^a} = \frac{1}{N^a} S_{\boldsymbol{a}_k}(N).$$

Meanwhile, the $(N+1) \times (N+1)$ stochastic matrix is given by

$$M_{(N+1)|a} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{2^a} & \frac{1}{2^a} & \cdots & 0 & 1 - \frac{1}{2^{a-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N^a} & \frac{1}{N^a} & \cdots & \frac{1}{N^a} & 1 - \frac{1}{N^{a-1}} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} P_{N|a} & \overrightarrow{\left(1 - \frac{1}{l^{a-1}}\right)} \\ \hline \underbrace{(0,\ldots,0)}_{N} & 1 \end{array}\right),$$

where

$$\overrightarrow{\left(1-\frac{1}{l^{a-1}}\right)} = \left(0, 1-\frac{1}{2^{a-1}}, \ldots, 1-\frac{1}{N^{a-1}}\right)^T.$$

Since powers of the upper-left $N \times N$ submatrix are independent of the other blocks, we finally obtain

$$\left(P_{N|a}^{k+1}\right)_{N,1} = \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \mathbb{P}\left(n_{k+1}=1\right) = \frac{1}{N^a} S_{\boldsymbol{a}_k}(N). \tag{3.4}$$

Remark 4. Formula (3.4) is slightly different from (2.2). This is due to the fact that the initial index matrix in Section 2 is $P_{N|1}$, while here we uniformly choose $P_{N|a}$.

Remark 5. The role of the additional site "$\mathfrak{N}$" is to "collect" extra probabilities of walks from each site; and the requirement $a > 1$ is to guarantee that all probabilities are between 0 and 1. For any $a \neq 0$, forming matrix in the same way leads to similar result:

$$M_{(N+1)|a} = \left(\begin{array}{c|c} P_{N|a} & \overrightarrow{\left(1-\frac{1}{l^{a-1}}\right)} \\ \hline \underbrace{(0,\ldots,0)}_{N} & 1 \end{array}\right) \Rightarrow \left(M_{(N+1)|a}^{k+1}\right)_{N,1} = \left(P_{N|a}^{k+1}\right)_{N,1} = \frac{\operatorname{sign}(a)^N}{N^{|a|}} S_{\boldsymbol{a}_k}(N).$$

4 Diagonalization of the index matrices

Since any index $a \neq 0$, the index matrix $P_{N|a}$ obviously has N distinct eigenvalues $\frac{\operatorname{sign}(a)}{j^{|a|}}$, $j = 1, \ldots, N$, the index matrices are diagonalizable. Explicit computation of the powers of $P_{N|a}$ can be easily obtained from its diagonalization. The following proposition gives the values of the eigenvectors.

Proposition 6. *A column eigenvector corresponding to the eigenvalue* $\frac{\operatorname{sign}(a)^j}{j^{|a|}}$, $j = 1, \ldots, N$, *is*

$$v_j^{(a)} = \left(\underbrace{0,\ldots,0}_{j-1}, \prod_{l=j}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}}, \ldots, \frac{\operatorname{sign}(a)^N N^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^{N-1}(N-1)^{|a|}}, 1\right)^T. \tag{4.1}$$

Proof. It suffices to show that for $i = j, \ldots, N$,

$$\sum_{m=j}^{i}\left(\prod_{l=m}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}}\right) = \frac{\frac{\operatorname{sign}(a)^j}{j^{|a|}}}{\frac{\operatorname{sign}(a)^i}{i^{|a|}}} \prod_{l=i}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \tag{4.2}$$

which can be proven by induction on i.

1) When $i = j$, both sides of (4.2) are the same:

$$\prod_{l=j}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}}.$$

2) Now suppose (4.2) holds for i. Then, for $i+1$, we have

$$\begin{aligned}
&\sum_{m=j}^{i+1}\left(\prod_{l=m}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}}\right) \\
= &\frac{\frac{\operatorname{sign}(a)^j}{j^{|a|}}}{\frac{\operatorname{sign}(a)^i}{i^{|a|}}} \prod_{l=i}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} + \prod_{l=i+1}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \\
= &\prod_{l=i+1}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \left(\frac{\frac{\operatorname{sign}(a)^j}{j^{|a|}}}{\frac{\operatorname{sign}(a)^i}{i^{|a|}}} \cdot \frac{\operatorname{sign}(a)^{i+1}(i+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^i i^{|a|}} + 1\right) \\
= &\frac{\frac{\operatorname{sign}(a)^j}{j^{|a|}}}{\frac{\operatorname{sign}(a)^{i+1}}{(i+1)^{|a|}}} \prod_{l=i+1}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}},
\end{aligned}$$

which completes the proof. □

Let $Q_{N|a} = \left(v_1^{(a)}, \ldots, v_N^{(a)}\right)$, the matrix consisting of all eigenvectors. From (4.1) we have, for $Q_{N|a} = \left(q_{i,j}^{(a)}\right)$,

$$q_{i,j}^{(a)} = \prod_{l=i}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^j j^{|a|}}{\operatorname{sign}(a)^l l^{|a|}},$$

which automatically gives 0 if $i < j$. Since

$$P_{N|a} = Q_{N|a} \operatorname{diag}\left\{\operatorname{sign}(a), \ldots, \frac{\operatorname{sign}(a)^N}{N^{|a|}}\right\} Q_{N|a}^{-1},$$

we then need to compute $Q_{N|a}^{-1}$, the inverse matrix of $Q_{N|a}$.

Proposition 7. *Let $Q_{N|a}^{-1} = \left(\bar{q}_{i,j}^{(a)}\right)$, then*

$$\bar{q}_{i,j}^{(a)} = \begin{cases} 0, & \text{if } i < j; \\ \dfrac{\prod_{l=j}^{N-1} \operatorname{sign}(a)^l l^{|a|}}{\prod_{\substack{l=j \\ l \neq i}}^{N} \left(\operatorname{sign}(a)^l l^{|a|} - \operatorname{sign}(a)^i i^{|a|}\right)}, & \text{if } i \geq j. \end{cases} \tag{4.3}$$

Proof. It is equivalent to show that, for $n \geq m$

$$\delta_{mn} = \sum_{t=m}^{n} \left(\left(\prod_{l=n}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^t t^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \right) \cdot \left(\frac{\prod_{l=m}^{N-1} \operatorname{sign}(a)^l l^{|a|}}{\prod_{\substack{l=m \\ l \neq t}}^{N} \left(\operatorname{sign}(a)^l l^{|a|} - \operatorname{sign}(a)^t t^{|a|}\right)} \right) \right).$$

1) If $n = m$, the sum reduces to the trivial identity:

$$\left(\prod_{l=n}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^n n^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \right) \cdot \left(\frac{\prod_{l=n}^{N-1} \operatorname{sign}(a)^l l^{|a|}}{\prod_{l=n+1}^{N} \left(\operatorname{sign}(a)^l l^{|a|} - \operatorname{sign}(a)^n n^{|a|}\right)} \right) = 1.$$

2) If $n > m$,

$$\sum_{t=m}^{n} \left(\left(\prod_{l=n}^{N-1} \frac{\operatorname{sign}(a)^{l+1}(l+1)^{|a|} - \operatorname{sign}(a)^t t^{|a|}}{\operatorname{sign}(a)^l l^{|a|}} \right) \cdot \left(\frac{\prod_{l=m}^{N-1} \operatorname{sign}(a)^l l^{|a|}}{\prod_{\substack{l=m \\ l \neq t}}^{N} \left(\operatorname{sign}(a)^l l^{|a|} - \operatorname{sign}(a)^t t^{|a|}\right)} \right) \right)$$

$$= \left(\prod_{l=m}^{n-1} \operatorname{sign}(a)^l l^{|a|} \right) \sum_{t=m}^{n} \frac{1}{\prod_{\substack{l=m \\ l \neq t}}^{n} \left(\operatorname{sign}(a)^l l^{|a|} - \operatorname{sign}(a)^t t^{|a|}\right)},$$

which gives 0 as a special case of $a_t = \operatorname{sign}(a)^t t^{|a|}$ in the next proposition. $\square$

Proposition 8. *For a sequence $(a_t)_{t=m}^{n}$ with $n > m$, of distinct real numbers, it holds*

$$\sum_{t=m}^{n} \frac{1}{\prod_{\substack{l=m \\ l \neq t}}^{n} (a_l - a_t)} = 0.$$

Proof. Given a sequence $(a_t)_{t=1}^N$ of distinct real numbers, the partial fraction decomposition [12, eq. 1, pp. 313]

$$\frac{1}{(1-a_1z)\cdots(1-a_Nz)} = \sum_{t=1}^{N} \frac{1}{1-a_tz}\left(\prod_{\substack{l=1\\l\neq t}}^{N} \frac{a_t}{a_t-a_l}\right), \tag{4.4}$$

can also be applied to its subsequence $(a_t)_{t=m}^n$ with both sides multiplied by z, i.e.,

$$\frac{z}{(1-a_mz)\cdots(1-a_nz)} = \sum_{t=m}^{n}\left(\prod_{\substack{l=m\\l\neq t}}^{n} \frac{1}{a_l-a_t}\right)\frac{-a_tz}{1-a_tz}.$$

Noting $n>m$, we let $z\to\infty$ to obtain the desired result. $\square$

Remark 9. We consider the diagonalization of a single index matrix, since it allows an easy computation for its powers, i.e.,

$$P_{N|a}^k = Q_{N|a}\operatorname{diag}\left\{\operatorname{sign}(a)^k,\ldots,\left(\frac{\operatorname{sign}(a)^N}{N^{|a|}}\right)^k\right\}Q_{N|a}^{-1}.$$

In fact, from (2.2), we only need the last row of $Q_{N|a}$ and first column of $Q_{N|a}^{-1}$ to compute $S_{\boldsymbol{a}_k}(N)$. Due to empty product, the last row of $Q_{N|a}$ is always the constant vector

$$\mathbf{1} = (\underbrace{1,\ldots,1}_{N}), \tag{4.5}$$

while the first column of $Q_{N|a}^{-1}$ has a more complicated form:

$$\left(\ldots,\frac{\prod_{l=1}^{N-1}\operatorname{sign}(a)^l\, l^{|a|}}{\prod_{\substack{l=1\\l\neq i}}^{N}\left(\operatorname{sign}(a)^l\, l^{|a|}-\operatorname{sign}(a)^i\, i^{|a|}\right)},\ldots\right)^T. \tag{4.6}$$

Remark 10. Similarly, harmonic sums with indices of certain pattern, e.g.

$$S_{\underbrace{a,b,c,a,b,c,\ldots,a,b,c}_{k \text{ copis of } a,b,c}}(N),$$

can also be computed through the diagonalization of the product $P_{N|a}P_{N|b}P_{N|c}$. However, from (2.3), we see that the product, though still of the lower triangular form, does not have the property that nonzero entries of each row equal. We pay more attention to the power of single index matrix rather than their product, not only because computations are easier, but also because the corresponding results, presented in the next section, are connected to combinatorial identities.

5 Connections to combinatorial identities

5.1 $S_{\mathbf{1}_k}(N)$ and $P_{N|1}$

Application of (4.1) and (4.3) yields the following result.

Corollary 11. *If we denote $Q_{N|1}=\left(q_{i,j}^{(1)}\right)$ and $Q_{N|1}^{-1}=\left(\bar{q}_{i,j}^{(1)}\right)$, then*

$$q_{i,j}^{(1)} = \frac{\binom{i-1}{j-1}}{\binom{N-1}{j-1}} \text{ and } \bar{q}_{i,j}^{(1)} = (-1)^{i+j}\binom{N-1}{i-1}\binom{i-1}{j-1}.$$

Remark 12. Computation of $Q_{N|1}$ is straightforward and that of $Q_{N|1}^{-1}$ needs to split the product:

$$\prod_{\substack{l=j\\l\neq i}}^{N}(l-i)=\left(\prod_{j\leq l<i}(l-i)\right)\left(\prod_{i<l\leq N}(l-i)\right)=(-1)^{i+j}(i-j)!\cdot(N-i)!. \tag{5.1}$$

Now, we see

$$\frac{1}{N}S_{\mathbf{1}_k}(N)=\left(P_{N|1}^{k+1}\right)_{N,1}=\left(Q_{N|1}\operatorname{diag}\left\{1,\ldots,\frac{1}{N^{k+1}}\right\}Q_{N|1}^{-1}\right)_{N,1}=\sum_{l=1}^{N}\frac{1}{l^{k+1}}(-1)^{l-1}\binom{N-1}{l-1},$$

which is exactly (1.1), using

$$\binom{N-1}{l-1}\cdot\frac{N}{l}=\binom{N}{l}.$$

5.2 $S_{\boldsymbol{a}_k}(N)$ and $P_{N|a}$ with $a>0$

Similar discussion on arbitrary positive a yields the following generalization of (1.1).

Corollary 13. *It holds, for $a>0$*

$$S_{\boldsymbol{a}_k}(N)=\sum_{l=1}^{N}\left(\prod_{\substack{n=1\\n\neq l}}^{N}\frac{n^a}{n^a-l^a}\right)\frac{1}{l^{ak}}. \tag{5.2}$$

Proof. Apply (4.5) and (4.6) to see that the last row of $Q_{N|a}$ is $\mathbf{1}=(1,\ldots,1)$ and the first column of $Q_{N,a}^{-1}$ is

$$\left(\frac{\prod\limits_{n=1}^{N-1}n^a}{\prod\limits_{\substack{n=1\\n\neq 1}}^{N}(n^a-1^a)},\ldots,\frac{\prod\limits_{n=1}^{N-1}n^a}{\prod\limits_{\substack{n=1\\n\neq i}}^{N}(n^a-1^a)},\ldots,\frac{\prod\limits_{n=1}^{N-1}n^a}{\prod\limits_{\substack{n=1\\n\neq N}}^{N}(n^a-N^a)}\right)^T.$$

Therefore,

$$\left(P_{N|a}^{k+1}\right)_{N,1}=\left(Q_{N|a}\operatorname{diag}\left\{1,\ldots\frac{1}{N^{a(k+1)}}\right\}Q_{N|a}^{-1}\right)_{N,1}=\sum_{l=1}^{N}\left(\frac{\prod\limits_{n=1}^{N-1}n^a}{\prod\limits_{\substack{n=1\\n\neq l}}^{N}(n^a-l^a)}\right)\frac{1}{l^{a(k+1)}},$$

which, together with (3.4), implies the desired result. □

Remark 14. Choosing $a=1$ in (5.2) gives (1.1), by using the same trick in (5.1), i.e., $\forall N\in\mathbb{Z}_+$ and $\forall l\in\{1,2,\ldots,N\}$,

$$\prod_{\substack{n=1\\n\neq l}}^{N}\frac{n}{n-l}=(-1)^{l-1}\binom{N}{l}.$$

In addition, when $a=m\in\mathbb{Z}_+$, considering the primitive root of unity $\xi_m:=\exp\left\{\frac{2\pi\iota}{m}\right\}$, where $\iota^2=-1$, and the factorization $n^m-l^m=(n-l)(n-\xi_m l)\cdots\left(n-\xi_m^{m-1}l\right)$, we deduce

$$S_{\boldsymbol{m}_k}(N)=\sum_{l=1}^{N}\left(\prod_{t=0}^{m-1}\frac{\Gamma(N+1)}{\Gamma(N-\xi_m^t l+1)\Gamma(1+\xi_m^t l)}\cdot\frac{\pi(1-\xi_m^t)l}{\sin(\pi\xi_m^t l)}\right)\frac{1}{l^{mk}},$$

where, if $\xi_m^t=1$, we take the limit $\lim\limits_{\xi_m^t\to 1}\frac{\pi\left(1-\xi_m^t\right)l}{\sin(\pi\xi_m^t l)}=(-1)^l$. Recognizing

$$\frac{\Gamma(N+1)}{\Gamma(N-\xi_m^t l+1)\Gamma(1+\xi_m^t l)}=\binom{N}{\xi_m^t l},$$

as a generalization of the usual binomial coefficients

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\,\Gamma(k+1)},$$

we finally obtain the following binomial-type expression

$$S_{\boldsymbol{m}_k}(N) = \sum_{l=1}^{N} \left(\prod_{t=0}^{m-1} \binom{N}{\xi_m^t l} \frac{\pi(1-\xi_m^t)\,l}{\sin(\pi \xi_m^t l)} \right) \frac{1}{l^{mk}}.$$

6 General matrix representation for multiplicative nested sums

In this section, we consider the general multiplicative nested sum

$$\mathcal{S}(f;k;N) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} f_1(n_1) \cdots f_k(n_k).$$

When $f_l(n_l) = \operatorname{sign}(a_l)^{n_l} / n_l^{|a_l|}$, we recover the harmonic sums. The proof of the following result is omitted due to similarity to the previous approaches, which only needs the substitution:

$$\operatorname{sign}(a_l)^{n_l} n_l^{|a_l|} \mapsto \frac{1}{f_l(n_l)}.$$

Theorem 15. *Define* $f_0(x) = \frac{1}{x}$ *and for* $l = 0, \ldots, k$

$$\mathcal{P}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix}.$$

1. *It holds that*

$$\mathcal{S}(f;k;N) = N \cdot \left(\prod_{l=0}^{k} \mathcal{P}_{N|f_l} \right)_{N,1}.$$

2. *If* $\{f_l(1), \ldots, f_l(N)\}$ *are all distinct, then* $\mathcal{P}_{N|f_l}$ *is diagonalizable as*

$$\mathcal{P}_{N,f_l} = \mathcal{Q}_{N,f_l} \operatorname{diag}\{f_l(1), \ldots, f_l(N)\}\, \mathcal{Q}_{N,f_l}^{-1},$$

where, denoting $\mathcal{Q}_{N|f_l} = \left(\vartheta_{i,j}^{(f_l)}\right)$ *and* $\mathcal{Q}_{N|f_l}^{-1} = \left(\bar{\vartheta}_{i,j}^{(f_l)}\right)$, *it holds that*

$$\vartheta_{i,j}^{(f_l)} = \prod_{m=i}^{N-1} \frac{\frac{1}{f_l(m+1)} - \frac{1}{f_l(j)}}{\frac{1}{f_l(m)}} = \prod_{k=i}^{N-1} \left(\frac{f_l(m)}{f_l(m+1)} - \frac{f_l(m)}{f_l(j)} \right),$$

and

$$\bar{\vartheta}_{i,j}^{(f_l)} = \begin{cases} 0, & \text{if } i < j; \\ \frac{f_l(N)}{f_l(i)} \prod\limits_{\substack{m=j \\ m \neq i}}^{N} \frac{1}{1 - \frac{f_l(m)}{f_l(i)}}, & \text{if } i \geq j. \end{cases}$$

In particular, the last row of $\mathcal{Q}_{N|f_l}$ *is the constant row vector* $\mathbf{1} = (\underbrace{1, \ldots, 1}_{N})$, *due to empty products.*

Example 16. Let $f_1(j) = j$, for $j = 1, \ldots, N$. Then

$$\mathcal{S}(f;1;N) = \sum_{N \geq n_1 \geq 1} n_1 = \frac{N(N+1)}{2}.$$

On the other hand,

$$\begin{aligned}
\left(\mathcal{P}_{N|f_1}^2\right)_{N,1} &= \left(\mathcal{Q}_{N,f_1}\,\mathrm{diag}\left\{1\ldots,N^2\right\}\mathcal{Q}_{N,f_1}^{-1}\right)_{N,1}\\
&= \sum_{l=1}^{N} l^2\left(\frac{N}{l}\prod_{\substack{k=1\\k\neq l}}^{N}\frac{1}{1-\frac{k}{l}}\right)\\
&= N\sum_{l=1}^{N}\frac{l^N}{\prod_{k<l}(l-k)\prod_{k>l}(l-k)}\\
&= N\sum_{l=1}^{N}(-1)^{N-l}\,\frac{l^{N+1}}{N!}\binom{N}{l}.
\end{aligned}$$

Therefore,

$$\sum_{l=1}^{N}(-1)^{N-l}\,l^{N+1}\binom{N}{l}=\frac{N\,(N+1)!}{2}.$$

Example 17. Take $f_1\equiv\cdots\equiv f_k=f$ such that $f(m)=a_m$, mapping positive integers to a sequence $(a_m)_{m=1}^{N}$ of distinct numbers, then

$$\begin{aligned}
\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} a_{n_1}\cdots a_{n_k} &= \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} f(n_1)\cdots f(n_k)\\
&= \mathcal{S}(f;k;N)\\
&= \frac{1}{f(N)}\left(\mathcal{P}_{N|f}^{k+1}\right)_{N,1}\\
&= \frac{1}{f(N)}\left(\mathcal{Q}_{N|f}\,\mathrm{diag}\left\{f(1)^{k+1},\ldots,f(N)^{k+1}\right\}\mathcal{Q}_{N|f}^{-1}\right)_{N,1}\\
&= \frac{1}{f(N)}\sum_{j=1}^{N} f(j)^{k+1}\left(\frac{f(N)}{f(j)}\prod_{\substack{m=1\\m\neq j}}^{N}\frac{1}{1-\frac{f(m)}{f(j)}}\right)\\
&= \sum_{j=1}^{N}\left(\prod_{\substack{m=1\\m\neq j}}^{N}\frac{1}{1-\frac{f(m)}{f(j)}}\right)f(j)^k\\
&= \sum_{j=1}^{N}\left(\prod_{\substack{m=1\\m\neq j}}^{N}\frac{1}{1-\frac{a_m}{a_j}}\right)a_j^k.
\end{aligned}$$

We recover a general result [12, eq. 2, pp. 313], which, when taking $a_j=\frac{a-bq^{j+i-1}}{c-zq^{j+i-1}}$ and $N=n-i+1$, "turns out to be a common source of several q-identities" [12, pp. 314].

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