APPLICATION OF ENTROPY IN RIEMANN MANIFOLDS

YONGGANG LI[1], BINGZHAO LI[1,2], HUAFEI SUN[1, 2,*], LIN JIU[3]

[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
[2]Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, China
[3] Department of Mathematics, Tulane University, New Orleans, LA 70118, U.S.A.

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Entropy refers to the degree of chaos in one system, which is widely used in thermodynamics and information theory. In this paper, we give the definition and some properties of entropy along the curve in Riemannian manifolds. Then we give the definition of entropy for function on Riemannian manifolds and obtain some properties. Last we define the von Neuman entropy and the Riemann-von Neuman entropy between two points for density matrix and obtain some properties.

Key words: entropy, Riemann manifold, von Neuman entropy.

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1. INTRODUCTION

Entropy refers to the degree of chaos in one system, which is proposed by R. Clausius and applied in thermodynamics [1]. Later on, entropy is introduced to information theory by C. E. Shannon [2] for the first time. In 1988, C. Tsallis proposed the Tsallis entropy [3] which is an extension of the Shannon entropy with one parameter. The Tsallis entropy is a new method of measuring information in statistical physics, and it is often used in the theory of the non-extensive thermodynamics and the image segmentation [4]. Hence, the concept of entropy is very important in signal processing.

In recent years, Riemannian geometry [5], an important branch of the foundations of mathematics, which is the theoretical basis of the general relativity, has been paid more and more attention and becomes an useful tool in signal processing such as radar signal processing [6] and face recognition [7]. With the rapid development of computer technology, Riemannian geometry has been also applied to the engineering calculations [8].

However, in engineering calculation, there is scarcely a good way to compare the Riemannian geometric method and the traditional method. It is known that entropy refers to the degree of randomness or disorder in one system, hence inspired by

*Email: huafeisun@bit.edu.cn. (Corresponding author)

this point, we can obtain a state function to measure the degree of change from one point to another.

In this paper, we first give the definition of entropy along the curve in Riemannian manifolds, and give some properties of this entropy in Riemannian manifolds and some examples. Next, we define entropy for function on Riemannian manifolds and obtain some results. Finally we define the von Neuman entropy and the Riemann-von Neuman entropy between two points for density matrices and obtain some results. The manuscript is organized as follows. Section 2 introduces some general definitions and important formulas. In section 3, we give some definitions of entropy in manifolds and obtain some results.

2. PRELIMINARIES

In this section, we first introduce some common knowledge. Even though this paper is based on Riemannian manifold, we do not list knowledge about Riemannian manifold, and ones can refer to [5].

2.1. RIESZ-THORIN THEOREM

The Riesz-Thorin theorem [9] is a interpolation theorem about linear operators and is given as following.

Let $(\mathbf{X},\mu)$ and $(\mathbf{Y},\nu)$ be two σ-finite measure spaces. Let $\mathcal{L}$ be a linear operator defined on the set of all the simple functions on $\mathbf{X}$ and take values in the set of measurable functions on $\mathbf{Y}$. Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and suppose that

$$\| \mathcal{L}(f) \|_{L^{q_0}} \leq M_0 \| f \|_{L^{p_0}}, \| \mathcal{L}(f) \|_{L^{q_1}} \leq M_1 \| f \|_{L^{p_1}}$$

hold for all simple functions on $\mathbf{X}$. Then we have

$$\| \mathcal{L}(f) \|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \| f \|_{L^p},$$

where $0 < \theta < 1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $M_0^{2q(\theta-1)} M_1^{-2q\theta}$ only depends on q.

From Riesz-Thorin theorem, one can obtain other inequality, such as Hausdorff-Young inequality.

2.2. VON NEUMAN ENTROPY FOR DENSITY MATRICES

A density matrix ρ is a symmetric positive-definite matrix satisfying $\mathrm{Tr}(\rho) = 1$, then the von Neuman entropy of ρ is defined as follows [10]

$$S(\rho) = -\,\mathrm{Tr}(\rho \log \rho), \tag{1}$$

where $\log\rho = -\sum\limits_{k=-\infty}^{+\infty}\frac{(I-A)^k}{k}$.

By the spectral theory, we see the property: $S(\rho) = -\Sigma\lambda_i\log\lambda_i$, where $\lambda_i{}'s$ are the eigenvalues of ρ, which shows that the von Neuman entropy is unitarily invariant.

2.3. TSALLIS ENTROPY

The Tsallis entropy is defined as [3]

$$E_T(\rho) = \frac{\int_{\mathbb{R}}(\rho(x))^q\,\mathrm{d}x - 1}{1-q}, \tag{2}$$

where $\rho(x) \geq 0$, $\int_{\mathbb{R}}\rho(x)\,\mathrm{d}x = 1$ and $q \in (0,1)$.

When $q \to 1$, we find that $E_T(\rho) \to E(\rho)$, where $E(\rho)$ is the Shannon entropy defined as

$$E(\rho) = -\int_{\mathbb{R}}\rho(x)\ln\rho(x)\,\mathrm{d}x. \tag{3}$$

Hence the Tsallis entropy can be regarded as the extension of the Shannon entropy.

3. THE MAIN RESULTS

3.1. ENTROPY ALONG CURVE ON RIEMANNIAN MANIFOLD

Suppose that (M,g) is a Riemannian manifold with the Riemannian metric g. Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma(0) = a$ and $\gamma(1) = b$. Then we give the following definition.

Definition 3.1 *Entropy along γ is defined as*

$$E_\gamma = -\int_0^1 \frac{l(t)}{l(1)}\ln\frac{l(t)}{l(1)}\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t, \tag{4}$$

where $l(t) = \int_0^t\sqrt{g(\gamma'(x),\gamma'(x))}\,\mathrm{d}x$ denotes the arc length of the curve γ.

By the above definition, we have

$$\begin{aligned} E_\gamma &= -\int_0^1 \frac{l(t)}{l(1)}\ln\frac{l(t)}{l(1)}\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t \\ &= -\int_0^1 \frac{l(t)}{l(1)}\ln\frac{l(t)}{l(1)}\,\mathrm{d}\,l(t) \\ &= -l(1)\int_0^1 \frac{l(t)}{l(1)}\ln\frac{l(t)}{l(1)}\,\mathrm{d}\,\frac{l(t)}{l(1)} \\ &= -l(1)\int_0^1 x\ln x\,\mathrm{d}x \\ &= \frac{1}{4}l(1). \end{aligned}$$

It is worth noting that the definition of entropy along γ is a quarter of the arc length of γ.

If $\widetilde{\gamma}(t)$ is the segment [5] connecting a and b, then we see that

$$\widetilde{l}(1) = \int_0^1 \sqrt{g(\widetilde{\gamma}'(x), \widetilde{\gamma}'(x))}\,\mathrm{d}\,x \le l(1),$$

hence we obtain

$$E_\gamma = \frac{1}{4}l(1) \ge \frac{1}{4}\widetilde{l}(1) = E_{\widetilde{\gamma}} \triangleq E_a^b. \tag{5}$$

Here we call E_a^b the manifold entropy from a to b. Based on this definition, the following properties can be obtained.

Proposition 3.1 *(1) E_a^b only depends on a and b, does not depend on the choice of the curves connecting a and b. (2) E_a^b is non-negative, and $E_a^b = 0$ if and only if $a = b$.*

Associated with the properties of entropy, the above two properties imply that E_a^b can be called as entropy. Having the above definition, we give some examples.

Example 1 If $M = \mathbb{R}^n$ with canonical metrics, we see that the segment connecting a and b is $\gamma(t) = (1-t)a + tb$, where $t \in [0,1]$, hence $E_a^b = \frac{1}{4}\sqrt{\sum\limits_i^n (a_i - b_i)^2}$.

Example 2 If M is a unit sphere $S^n(1)$ with the canonical metrics on $S^n(1)$, then $E_a^b = \frac{1}{2}\arcsin\left(\frac{1}{2}\sqrt{\sum\limits_i^n (a_i - b_i)^2}\right)$.

Example 3 If M is a set $PD(n)$ consists of all symmetric positive-definite matrices with the Riemannian metric g [11] defined by $g_P(X,Y) = \mathrm{Tr}(P^{-1}XP^{-1}Y)$ at $P \in M$ where $X, Y \in T_P PD(n)$, and Tr denotes the trace of a matrix, then

$$E_A^B = \frac{1}{4}\mathrm{Tr}((\log(A^{-1}B))^T \log(A^{-1}B)) = \frac{1}{4}(\sum_i^n \log^2 \lambda_i)^2,$$

where $\lambda_i{}'s$ are the eigenvalues of $A^{-1}B$.

In this subsection, we give the definition of entropy along the curve in Riemannian manifolds, and give some properties.

3.2. ENTROPY OF FUNCTION ON RIEMANNIAN MANIFOLD

We have analyzed entropy along a curve which is a special case of function on Riemannian manifold, thus it is worthy considering entropy of general functions on Riemannian manifold. For a given function f on Riemannian manifold (M, g), we can define entropy of f along the curve γ with $\gamma(0) = a$ and $\gamma(1) = b$.

Definition 3.2 *Entropy of f along γ is defined as*

$$E_\gamma(f) = -\int_0^1 \frac{|(f\circ\gamma)(t)|}{l(f\circ\gamma)} \ln \frac{|(f\circ\gamma)(t)|}{l(f\circ\gamma)} \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t, \tag{6}$$

where $l(f\circ\gamma) = \int_0^1 |(f\circ\gamma)(t)|\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t$.

When $M = \mathbb{R}^n$ with the canonical metrics and $\gamma(t) = (t,0,0,\cdots,0)$, we have

$$E_\gamma(f) = -\int_0^1 \frac{|f(t)|}{\|f\|_1} \ln \frac{|f(t)|}{\|f\|_1}\,\mathrm{d}t.$$

This is the classical Shannon entropy.

When $M = S^1(1)$ with the canonical metrics and $\gamma(t)$ is geodesic, we see that $(f\circ\gamma)(t)$ is a periodic function, and $\sqrt{g(\gamma'(t),\gamma'(t))}$ is a constant. Hence we give the definition of entropy for the periodic function and have the following theorem.

Theorem 3.1 *Suppose that f is a continuous periodic function, such that $f(x) = f(x+1)$ for $x\in\mathbb{R}$, $f(x) = \sum\limits_{k=-\infty}^{+\infty} F_k e^{-ikx}$, and $\|f\|_{L^2[0,1]} = (\int_0^1 |f(x)|^2\,\mathrm{d}x)^{\frac{1}{2}} = 1$ with $(\sum\limits_{k=-\infty}^{+\infty} |F_k|^p)^{\frac{1}{p}} < +\infty$ ($p\geq 2$). Then we have the following inequality*

$$-\int_0^1 |f(t)|^2 \ln |f(t)|^2\,\mathrm{d}t - \sum_{k=-\infty}^{+\infty} F_k^2 \ln F_k^2 \geq 0, \tag{7}$$

where $F_k = \int_0^1 f(x)e^{ikx}\,\mathrm{d}x$.

Proof : Let $L(f)(x) = \sum\limits_{k=-\infty}^{+\infty} F_k e^{-ikx}$. Then, it is easy to prove that $L(f)$ is a linear operator. And the L^p norm of $L(f)$ satisfies

$$\|L(f)\|_{L^p} = (\sum_{k=-\infty}^{+\infty} |F_k|^p)^{\frac{1}{p}}.$$

It is easy to verify that $\|L(f)\|_{L^p}$ is a norm. By $F_k = \int_0^1 f(x)e^{ikx}\,\mathrm{d}x$, we have

$$|F_k| = |\int_0^1 f(x)e^{ikx}\,\mathrm{d}x| \leq \int_0^1 |f(x)|\,\mathrm{d}x.$$

Hence $\|L(f)\|_{L^\infty} = \max\limits_{k\in\mathbb{Z}} |F_k| \leq \int_0^1 |f(x)|\,\mathrm{d}x = |f|_{L^1}[0,1]$.

When $p=2$, we have the Parseval identity

$$\|f\|_{L^2[0,1]} = \sum_{k=-\infty}^{+\infty} |F_k|^2.$$

By the Riesz-Thorin theorem, we have

$$\|L(f)\|_{L^q} \leq \|f\|_{L^p[0,1]}, \quad 2 \leq p, \quad \frac{1}{q} + \frac{1}{p} = 1$$

and thus

$$-\int_0^1 |f(t)|^2 \ln |f(t)|^2 \,\mathrm{d}t - \sum_{k=-\infty}^{+\infty} F_k^2 \geq 0.$$

When $f(x) = e^{ikx},\ k \in \mathbb{Z}$, the inequality becomes equality.

This inequality can be considered as the Hirschman inequality [12] for periodic function, and this is different with the Hirschman inequality for continuous function.

Suppose that (M,g) is geodesically complete [5] and

$$l(|f|^2) = (\int_{-\infty}^{+\infty} |(f\circ\gamma)(t)|^2 \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)^{\frac{1}{2}} < \infty,$$

where $\gamma(t)$ is a geodesic. Setting

$$E_{\gamma(t)}(|f|^2) = -\int_{-\infty}^{+\infty} |\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2 \ln |\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2 \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t,$$

we obtain the following theorem.

Theorem 3.2 *Suppose that f is a function on the geodesically complete Riemannian manifold (M,g), $\gamma(t)$ is a geodesic, and $l(|f|^2) < \infty$. Then the following inequality holds*

$$E_{\gamma(t)}(|f|^2) \leq \frac{\ln(2\pi)}{2} + \int_{-\infty}^{+\infty} \frac{l_\gamma^2(t)}{2} |\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2 \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t. \tag{8}$$

Proof : Since (M,g) is geodesically complete, letting

$$l_\gamma^2(x) = (\int_0^x \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)^2,$$

we have the following equalities

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\frac{(\int_0^x \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)^2}{2}} \sqrt{g(\gamma'(x),\gamma'(x))}\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} e^{-\frac{(\int_0^x \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)^2}{2}} \,\mathrm{d}(\int_0^x \sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t) \\
&= \sqrt{2\pi}.
\end{aligned}$$

Using the definition of $E_{\gamma(t)}(|f|^2)$, we have

$$\begin{aligned}
E_{\gamma(t)}(|f|^2) &= -\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\ln|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&= -\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\frac{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}\ln(|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\frac{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&= -\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\frac{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}\ln(|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&+\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\frac{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}\ln(\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t.
\end{aligned}$$

By the Jensen inequality, we get

$$\begin{aligned}
E_{\gamma(t)}(|f|^2) &\le -\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\cdot\ln(\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)\\
&+\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\frac{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}{\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}}}\ln(\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&= -\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\cdot\ln(\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t)\\
&+\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\ln(\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t.
\end{aligned}$$

By the definition of $l(|f|^2)$, we have

$$\begin{aligned}
E_{\gamma(t)}(|f|^2) &\le \int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\ln(\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&= \int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\ln(\sqrt{2\pi}e^{\frac{l_\gamma^2(t)}{2}})\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&= \frac{\ln(2\pi)}{2}\int_{-\infty}^{+\infty}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t\\
&+\int_{-\infty}^{+\infty}\frac{l_\gamma^2(t)}{2}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t.
\end{aligned}$$

Hence we obtain

$$E_{\gamma(t)}(|f|^2)\le\frac{\ln(2\pi)}{2}+\int_{-\infty}^{+\infty}\frac{l_\gamma^2(t)}{2}|\frac{(f\circ\gamma)(t)}{l(|f|^2)}|^2\sqrt{g(\gamma'(t),\gamma'(t))}\,\mathrm{d}t.$$

If (M,g) is $\mathbb{R}^n$ with canonical Euclidean metric, and $\gamma(t) = (t,0,0\cdots,0)$, we have

$$E_{\gamma(t)}(|f|^2) \leq \frac{\ln(2\pi)}{2} + \int_{-\infty}^{+\infty} \frac{t^2}{2} |\frac{f(t)}{\|f\|_2}|^2 \,\mathrm{d}t.$$

This is the classical Shannon entropy inequality [13].

3.3. VON NEUMAN ENTROPY FOR SYMMETRIC POSITIVE-DEFINITE MATRICES

Symmetric positive-definite matrices not only have important applications in mathematics but also have a very wide range of applications in physics and engineering. Using the von Neuman entropy of density matrix [10] which is very significant in quantum statistical mechanics, we define the von Neuman entropy between two symmetric positive-definite matrices in this section.

Definition 3.3 *If A and B are two density matrices, then $tB+(1-t)A$ for $t \in [0,1]$ is also density matrix. The geodesic between A and B is $\gamma(t) = tB+(1-t)A$, and the von Neuman entropy from A to B in the Frobenius inner product can be defined by*

$$S_A^B = -\int_0^1 \mathrm{Tr}[(tB+(1-t)A)\log(tB+(1-t)A)]\,\mathrm{d}t. \tag{9}$$

Then we have the following theorem.

Theorem 3.3 *If A and B are two density matrices, then we have the following inequalities*

$$\frac{1}{2}(S(B)+S(A)) \leq S_A^B \leq \log n. \tag{10}$$

Proof : *From the definition of S_A^B, we have*

$$\begin{aligned} S_A^B &= -\int_0^1 \mathrm{Tr}[(tB+(1-t)A)\log(tB+(1-t)A)]\,\mathrm{d}t \\ &= -\mathrm{Tr}\int_0^1 (tB+(1-t)A)\log(tB+(1-t)A)\,\mathrm{d}t. \end{aligned}$$

Since $X\log X$ (X is a symmetric positive-definite matrix) is a convex function, we have

$$\begin{aligned} S_A^B &\geq -\mathrm{Tr}\int_0^1 (tB\log B + (1-t)A\log A\,\mathrm{d}t \\ &= -\mathrm{Tr}(\frac{B\log B}{2} + \frac{A\log A}{2}) \\ &= \frac{1}{2}(S(B)+S(A)). \end{aligned}$$

In particular, when $A=B$, the equality holds.

For density matrix ρ, we have the following inequality [10]

$$S(B) \leq \log n.$$

Hence we obtain

$$S_A^B \leq \log n.$$

When $A=B=\frac{1}{n}I_{n\times n}$, the equality holds. Thus we finish the proof of the theorem.

The Riemannian inner product defined by $g_P(X,Y)=\mathrm{Tr}(P^{-1}XP^{-1}Y)$ at $P\in M$ where $X,Y\in T_P PD(n)$ for symmetric positive-definite metrices [11] is frequently used for Riemannian mean of symmetric positive-definite matrices. The geodesic connecting A and B is given by

$$\gamma(t)=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \quad t\in[0,1].$$

Here, we give the Riemann-von Neuman entropy between two points.

Definition 3.4 *Let A and B be two density matrices, then the Riemann-von Neuman entropy from A to B is*

$$R_A^B=-\int_0^1 \mathrm{Tr}[(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})\log(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})]\,\mathrm{d}t. \quad (11)$$

Even though for any t, $\gamma(t)$ may not always the density matrices, we can obtain the following theorem. Before given the theorem, we introduce some basis concepts about the matrixes to easily prove that if $A\geq B$, then $x^TAx\geq x^TBx$ for all $x\in\mathbb{R}^n$.

Theorem 3.4 *Suppose that A and B are two density matrices, then the following inequality holds*

$$R_A^B \leq S_A^B \quad (12)$$

and $R_A^B=S_A^B$ if and only if $A=B$.

Proof : First we know that $f(t)=a^t$ is a convex function for any $a>0$, we have

$$a^t \leq at+1-t, \quad t\in[0,1].$$

Hence we have

$$\begin{pmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^t \end{pmatrix}_{n\times n} \leq \begin{pmatrix} \lambda_1 t+1-t & 0 & \cdots & 0 \\ 0 & \lambda_2 t+1-t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n t+1-t \end{pmatrix}_{n\times n},$$

where $t \in [0,1]$, and above inequality is equivalent to

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}_{n\times n}^{t} \leq \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}_{n\times n} t + (1-t)I,$$

where $t \in [0,1]$.

If $\lambda_i{}'s$ are the eigenvalues of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then we have

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t \leq tA^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-t)I,$$

where $t \in [0,1]$.

Then we obtain

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \leq tB + (1-t)A,$$

where $t \in [0,1]$.

For $q \in [0,1]$, and $f(A) = A^q$ is a monotone increasing function, we obtain that

$$(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})^q \leq (tB + (1-t)A)^q,$$

where $t \in [0,1]$.

Hence the following inequality holds

$$\int_0^1 (A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})^q \,\mathrm{d}t \leq \int_0^1 (tB + (1-t)A)^q \,\mathrm{d}t.$$

Therefore, we have

$$\frac{\int_0^1 (A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})^q \,\mathrm{d}t - I}{I - qI} \leq \frac{\int_0^1 (tB + (1-t)A)^q \,\mathrm{d}t - I}{I - qI}.$$

Similarly as the relations of the Shannon entropy and the Tsallis entropy, we see that if $a_n \geq b_n$, then $a \geq b$ $(a_n \to a,\ b_n \to b)$, hence we obtain

$$\begin{aligned} &-\int_0^1 (A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}) \log(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}) \,\mathrm{d}t \\ &\leq -\int_0^1 (tB + (1-t)A) \log(tB + (1-t)A) \,\mathrm{d}t. \end{aligned}$$

We see that if $f(t)$ is monotone increasing, then $\mathrm{Tr}\, f(A)$ is also monotone increasing [10]. Hence, we have

$$\begin{aligned} R_A^B &= -\int_0^1 \mathrm{Tr}(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})\log(A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}})\,\mathrm{d}t \\ &\leq -\int_0^1 \mathrm{Tr}(tB+(1-t)A)\log(tB+(1-t)A)\,\mathrm{d}t. \\ &= S_A^B \end{aligned}$$

This completes the proof of the theorem.

From the steps of the proof of Theorem 3.3, we find that if A and B are two symmetric positive-definite matrices we can also obtain the similar inequality as Theorem 3.3. From Theorem 3.3, the degree of chaos for the set of symmetric positive-definite matrices with Riemannian inner product is lower than the set of symmetric positive-definite matrices with Frobenius inner product.

4. CONCLUSION

In this paper, first we give the definition of entropy along the curve in Riemannian manifolds, and give some properties of entropy in manifolds. Then we give the definition of entropy for function on manifolds and obtain some properties. last we define the von Neuman entropy between two points for density matrices. All the results are based on two points in one Riemannian manifold, and the manifold entropy defined in this paper can be considered as a tool to measure capability of the numerical algorithms.

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