1 Overview

My current research interests involve number theory, combinatorics, probability theory and special function. Main tools used here are experimental mathematics and symbolic computation. More precisely, large part of my work was achieved with the help of mathematical software, such as SageMath, Mathematica and Maple.

2 The Bernoulli and Euler symbols

2.1 Introduction

The *Bernoulli and Euler symbols*, denoted by $\mathcal{B}$ and $\mathcal{E}$, respectively, are tools to study Bernoulli and Euler polynomials and their generalizations, whose definitions and basic properties can be found in, e.g., [18, Chpt. 24]. Each of the two symbols satisfies a simple evaluation rule:

$$eval\left(\mathcal{B}^n\right)=\mathcal{B}^n=B_n \quad \text{and} \quad eval\left(\mathcal{E}^n\right)=\mathcal{E}^n=E_n/2^n, \tag{2.1}$$

where B_n and E_n are *Bernoulli and Euler numbers*, respectively. For simplicity, the "*eval*" operator is usually omitted during evaluation.

Originally, both $\mathcal{B}$ and $\mathcal{E}$ arise from the traditional umbral calculus (see, e.g., [20]). Meanwhile, recent results interpret them as certain random variables. More specifically, by letting, for $t\in\mathbb{R}$,

$$p_B(t):=\pi\operatorname{sech}^2(\pi t)/2 \quad \text{and} \quad p_E(t):=\operatorname{sech}(\pi t),$$

we define two random variables L_B and L_E with density functions p_B and p_E, respectively. Then, with $i^2=-1$, we have

$$\mathcal{B}=iL_B-1/2 \quad \text{and} \quad \mathcal{E}=iL_E-1/2. \tag{2.2}$$

Note that, (2.2) implies the evaluation rule in (2.1) is exactly taking the expectation, i.e., $eval=\mathbb{E}$. In addition, we shall use the symbols $\mathcal{B}^{(p)}$ and $\mathcal{E}^{(p)}$, defined next. Let $(\mathcal{B}_i)_{i=1}^p$ be a sequence of independent and identically distributed (i. i. d.) random variables with each $\mathcal{B}_i$ having the same distribution as $\mathcal{B}$, denoted as $\mathcal{B}_i\sim\mathcal{B}$. Similarly define a sequence of i. i. d. random variables $(\mathcal{E}_i)_{i=1}^p$ with $\mathcal{E}_i\sim\mathcal{E}$. Then,

$$\mathcal{B}^{(p)}:=\mathcal{B}_1+\cdots+\mathcal{B}_p \quad \text{and} \quad \mathcal{E}^{(p)}:=\mathcal{E}_1+\cdots+\mathcal{E}_p.$$

Based on umbral symbol and random variable interpretations, we have the symbolic expressions for the following objects.

- Bernoulli and Euler numbers, B_n and E_n:
$$B_n=\mathbb{E}[\mathcal{B}^n] \quad \text{and} \quad E_n=\mathbb{E}[(2\mathcal{E})^n];$$
- Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$:
$$B_n(x)=\mathbb{E}[(\mathcal{B}+x)^n] \quad \text{and} \quad E_n=\mathbb{E}[(\mathcal{E}+x)^n];$$
- Bernoulli and Euler polynomials of order p, $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$:
$$B_n^{(p)}(x)=\mathbb{E}\left[\left(x+\mathcal{B}^{(p)}\right)^n\right] \quad \text{and} \quad E_n^{(p)}(x)=\mathbb{E}\left[\left(x+\mathcal{E}^{(p)}\right)^n\right];$$
- Bernoulli-Barnes polynomials $B_n(\mathbf{a};x)$, with $\mathbf{a}=(a_1,\ldots,a_k)$ and $a_l\neq0$:
$$B_n(\mathbf{a};x)=\mathbb{E}\left[\left(x+a_1\mathcal{B}_1+\cdots+a_k\mathcal{B}_k\right)^n/\left(a_1a_2\cdots a_k\right)\right],$$
where $\boldsymbol{\mathcal{B}}=(\mathcal{B}_1,\ldots,\mathcal{B}_k)$ and $\mathbf{a}\cdot\boldsymbol{\mathcal{B}}=\sum_{l=1}^k a_l\mathcal{B}_l$.

2.2 Main results

2.2.1 Orthogonal polynomials

Since $E_n^{(p)}(x)$ can be viewed as the nth moment of certain random variable, it is natural to consider other probabilistic objects, such as the monic orthogonal polynomials, denoted by $\Omega_n^{(p)}(y)$. Here, orthogonality means, for any integers r and n with $0\le r<n$,

$$y^r\Omega_n^{(p)}(y)\Big|_{y^k=E_k^{(p)}(x)}=0,$$

where the left-hand sides means expanding the polynomial and evaluating at $y^k=E_k^{(p)}(x)$ for each power of y. In [13], we not only give the recurrence of $\Omega_n^{(p)}(y)$ as

$$\Omega_{n+1}^{(p)}(y)=\left(y-x+\frac{p}{2}\right)\Omega_n^{(p)}(y)+\frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y), \tag{2.3}$$

but also recognize that $\Omega_n^{(p)}(y)=i^nn!P_n^{\left(\frac{p}{2}\right)}\left(-i\left(y-x+\frac{p}{2}\right);\frac{\pi}{2}\right)/2^n$ for Meixner-Pollaczek polynomial $P_n^{(\lambda)}(y;\phi)$ [16, eq. 9.7.1]. In fact, (2.3) connects $E_n^{(p)}(x)$ to generalized Motzkin numbers through a continued fraction expression (see [13, Thm. 2]). Namely, define a sequence $\mathfrak{E}_{n,k}^{(p)}$ by $\mathfrak{E}_{0,0}^{(p)}=1$ and for $n>0$, by the recurrence

$$\mathfrak{E}_{n+1,k}^{(p)}=\mathfrak{E}_{n,k-1}^{(p)}+\left(x-\frac{p}{2}\right)\mathfrak{E}_{n,k}^{(p)}-\frac{(k+1)\,(k+p)}{4}\mathfrak{E}_{n,k+1}^{(p)}.$$

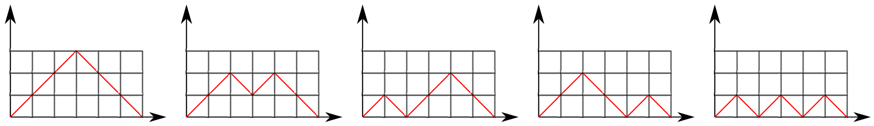
Then, when $k=0$, we have $\mathfrak{E}_{n,0}^{(p)}=E_n^{(p)}(x)$. The weighted lattice paths interpretation [6, p. 319] of generalized Motzkin numbers now can be endowed on $E_n^{(p)}(x)$, giving it a matrix representation [13, Thm. 14]. See also an example [13, Ex. 16], which shows connections between E_n and Catalan number C_n, through weighted lattice paths. Analogue for $B_n(x)$ are also obtained [13, Thm. 17]: let $\varrho_n(y)$ be the monic orthogonal polynomials with respect to $B_n(x)$, i.e., for integers r and n, with $0\le r<n$, $y^r\varrho_n(y)|_{y^k=B_k(x)}=0$. Then,

$$\varrho_{n+1}(y)=\left(y-x+\frac{1}{2}\right)\varrho_n(y)+\frac{n^4}{4(2n+1)(2n-1)}\varrho_{n-1}(y).$$

In particular, $\varrho_n(y)=n!p_n\left(y;\frac12,\frac12,\frac12,\frac12\right)/(n+1)_n$, where $p_n(y;a,b,c,d)$ is the continuous Hahn polynomial [16, pp. 200–202].

2.2.2 Combinatorial interpretation as weighted lattice paths

In general, moments of a random variable can be connected to the generalized Motzkin numbers, whose combinatorial interpretation is the weighted lattice paths (see, e.g., [6, p. 319]). An interesting example [13, Ex. 3] shows the connection between the Euler number E_n and the Catalan number C_n. There are $C_3 := \frac{1}{4}\binom{6}{3} = 5$ Dyck paths, from $(0,0)$ to $(6,0)$ listed as follows:



Meanwhile, E_6 counts the weighted Dyck paths, by associating each path from (j,k) to $(j+1,k-1)$ to weight $-k^2$. Therefore, we have

$$-61 = E_6 = (-1)^3\left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right). \tag{2.4}$$

2.2.3 Random walk, Brownian motion, and Bessel process

As a convolution, it is not hard to see that

$$E_n^{(p)}(x) = \sum_{k_1+\cdots+k_p=n} \binom{n}{k_1,\ldots,k_p} E_{k_1}(x)E_{k_2}(0)\cdots E_{k_p}(0).$$

In [12], based on the probabilistic interpretation, we obtained, for any positive integer N,

$$E_n(x) = \frac{1}{N^n}\sum_{l=N}^{\infty} p_l^{(N)} E_n^{(l)}\left(\frac{l-N}{2}+Nx\right),$$

where the coefficients $p_l^{(N)}$ appear in the series expansion of the reciprocal of the Nth Chebychev polynomial of the first kind T_N

$$\frac{1}{T_N(1/z)} = \sum_{\ell=0}^{\infty} p_\ell^{(N)} z^\ell.$$

Moreover, $p_\ell^{(N)}$ can also be viewed as transition probabilities in the context of a random walk over a finite number of sites [12, Note 4.8]. Inspired by this idea, we further explored another probabilistic method in [15]. By considering both 1-dim reflected Brownian motion and 3-dim Bessel process and especially studying the hitting times for consecutive level sites, we obtained several identities involving $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$. For example, [15, Thm. 7], for any positive integer n,

$$3^n B_n\left(\frac{x+4}{6}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k}E_n^{(2k+2)}\left(\frac{x+2k+3}{2}\right).$$

2.2.4 Bernoulli-Barnes polynomial

For the Bernoulli-Barnes polynomials $B_n(\mathbf{a};x)$, Bayad and Beck [2] obtained several properties such as the difference formula

$$(-1)^m B_m(\mathbf{a};-x) - B_m(\mathbf{a};x) = m!\sum_{j=0}^{n-1}\sum_{|J|=j}\frac{B_{m-n+j}(\mathbf{a}_J;x)}{(m-n+j)!}; \tag{2.5}$$

and the self-duality of sequence $\left((-1/A)^n B_n(\mathbf{a};0)\right)_{n=0}^{\infty}$, where $A := a_1+\cdots+a_n \neq 0$, and the symmetry formula. In [11], based on the symbolic expression, we show that, for polynomial P,

$$P(x-\mathbf{a}\cdot\boldsymbol{\mathcal{B}}) = \sum_{j=0}^{n}\sum_{|J|=j}|a|_{J^*}\,P^{(n-j)}(x+(\mathbf{a}\cdot\boldsymbol{\mathcal{B}})_J),$$

where $J\subset[n] := \{1,\ldots,n\}$ and $J^* = [n]\backslash J$, which, by taking $P(x) = x^m/(|\mathbf{a}|\,m!)$ gives (2.5). Namely, the symbolic expression of $B_n(\mathbf{a};x)$ gives a more general result. Also, for the self-duality of $\left((-1/A)^n B_n(\mathbf{a};0)\right)_{n=0}^{\infty}$, we gave a direct proof that Bayad and Beck asked for.

2.2.5 Multiple zeta value at non-positive integers

The multiple zeta function

$$\zeta_r(n_1,\ldots,n_r) = \sum_{k_1,\ldots,k_r>0}\frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}\cdots(k_1+\cdots+k_r)^{n_r}}$$

has more than one analytic continuations at non-positive integers. For instance, Sadaoui [21, Thm. 1] used the Raabe's identity while Akiyama and Yanigawa [1, p. 350] considered the Euler-Maclaurin summation formula. Since both results involve Bernoulli number, applying Bernoulli symbol reveals, to our surprise, that both analytic continuations coincide. More precisely, for non-negative integers $n_1,\ldots,n_k$, we have, symbolically,

$$\zeta_r(-n_1,\ldots,-n_r) = \prod_{k=1}^{r}(-1)^{n_k}\mathcal{C}_{1,\ldots,k}^{n_k+1} \tag{2.6}$$

where $\mathcal{C}_1^n = \mathcal{B}_1^n/n$ and recursively, $\mathcal{C}_{1,\ldots,k+1} = (\mathcal{C}_{1,\ldots,k}+\mathcal{B}_{k+1})^n/n$. This shows the two approaches, by Raabe's identity and Euler-Maclaurin summation formula that lead to analytic continuations of MZVs, coincide on non-positive integer values.

Other results such as recurrence [10, Thm. 3.1], contiguity identities [10, Thm. 4.1], and generating functions [10, Thm. 5.1] follow naturally from (2.6).

2.3 Future work

- For the monic orthogonal polynomials for $B_n^{(p)}(x)$, we are still working on the corresponding orthogonal polynomials for general p. In the last section of [13], we present some calculation and a conjecture.
- For multiple zeta values, besides studying the explanation for the coincidence of two analytic continuations, I am also interested in applying Bernoulli symbols to other types of zeta functions, such as Tornheim zeta-function

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r n^s (m+n)^t}.$$

- In early work [4], we studied the hypergeometric Bernoulli polynomials $B_n^{(a,b)}(x)$, defined by

$$\frac{e^{xt}}{{}_1F_1\left(\begin{smallmatrix} a \\ a+b \end{smallmatrix} \middle| t\right)} = \sum_{n=0}^{\infty} B_n^{(a,b)}(x)\frac{t^n}{n!}. \tag{2.7}$$

  and hypergeometric zeta functions. In [4], it is obtained that $B_n^{(a,b)}(x) = \mathbb{E}\left[\left(x+\mathfrak{Z}_{a,b}\right)^n\right]$, where $\mathfrak{Z}_{a,b}$ is the conjugate of the Beta distribution. but not through the Bernoulli symbol $\mathcal{B}$. To apply $\mathcal{B}$, it might need some modification, such as the $\mathcal{C}$ symbol in (2.6).
- Besides the combinatorial interpretations of $B_n^{(p)}(x)$ and $E_n^{(p)}(x)$, to find some new properties, or to interpret combinatorial identities, a more recent project a general formula connecting E_{2n} and C_n, inspired by (2.4).
- The area of computer proofs for combinatorial identities has developed dramatically. See, e.g., [19]. The properties of orthogonal polynomials yield a different criterion for identities. For example, consider $B_n(x+1) - B_n(x) = nx^{n-1}$. By the Bernoulli symbol (2.1), we can consider the polynomials $P(n;y) = (y+x+1)^n - (y+x)^n - nx^{n-1}$. The orthogonality implies that

$$P(n;y) = \sum_{k=1}^{n-1} \alpha_{n,k}\varrho_k(y) \tag{2.8}$$

  for some constant $\alpha_{n,k}$ that are independent on y. Computation shows

$$P(2;y) = 2\varrho_1(y), \quad P(3;y) = 3\varrho_2(y) + 3x\varrho_1(y), \quad \text{and} \quad P(4;y) = 4\varrho_3(y) + 12x\varrho_2(y) + \left(12x^2 - \frac{2}{5}\right)\varrho_1(y).$$

  1. It can be shown that (2.8) holds in general for identities on Bernoulli polynomials. Therefore, to find an executable criterion to determine whether an identity holds or not gives the computer proof.
  2. It is obvious that the identity mention in the goal above is of polynomial type. Once this family is proven, we shall try to extend it to analytic function identities.

3 Matrix presentations for special functions

Throughout this section, for given matrix M, $M_{i,j}$ denotes the entry of M at the ith row and jth column.

3.1 Multiplicative nested sums

Nested sums are given by

$$\mathcal{S}(f;k;N) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} f(n_1,\ldots,n_k),$$

and we focus on the special case that $f(n_1,\ldots,n_k) = f_1(n_1)\cdots f_k(n_k)$, called *multiplicative nested sums*. A large class of multiple nested sums is the *harmonic sums* [3, eq. 4]: for indices $a_1,\ldots,a_k \in \mathbb{R}\backslash\{0\}$,

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times\cdots\times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}},\ N\in\mathbb{N}, \tag{3.1}$$

which naturally connect to zeta functions. For instance, taking $N\to\infty$, $k=1$ and assuming $a_1>0$, we see $S_{a_1}(\infty) = \zeta(a_1)$. Now, associate each factor f_l an *index matrix*:

$$\mathcal{P}_{N|f_l} := \begin{pmatrix} f_l(1) & 0 & \cdots & 0 \\ f_l(2) & f_l(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_l(N) & f_l(N) & \cdots & f_l(N) \end{pmatrix}.$$

The main presentation is that by defining $f_0(x) = 1/x$, we have [14, Thm. 15]:

$$\mathcal{S}(f;k;N) = N\cdot\left(\prod_{l=0}^{k}\mathcal{P}_{N|f_l}\right)_{N,1}. \tag{3.2}$$

For the special case of harmonic sums when $a_1 = \cdots = a_k = a \geq 1$ and $N<\infty$, a random walk over finite number of sites [14, Sec. 3] interprets the index matrix as the stochastic matrix and the harmonic sums as probabilities of certain events. Moreover, this provides an easy proof of the limit [14, Prop. 3.1]

$$\lim_{k\to\infty}\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{1}{n_1 n_2\cdots n_k} = N. \tag{3.3}$$

When f_l's are identical and $\mathcal{P}_{N|f_l}$ is diagonalizable, (3.2) can also be calculated through the diagonalization. By this method, combinatorial identities can be derived or reproved, e.g., [5, Cor. 3]:

$$\sum_{l=1}^{N}(-1)^{l-1}\binom{N}{l}\frac{1}{l^k}=\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{1}{n_1\cdots n_k}.$$

Future work

- The limit result (3.3) indicates to explore these nested sums with probabilistic interpretations, e.g., interpreting more identities from the probabilistic point of view.
- The paper [14] only provides two examples of proving combinatorial identities through matrix expressions. Besides finding more examples, it is of great interest and importance whether it can be extended to a systematical method.

3.2 Bernoulli and higher-order Euler polynomials

As mentioned in Section 2, the monic orthogonal polynomials with respect to $E_n^{(p)}(x)$ and $B_n(x)$ lead to their connections to the generalized Motzkin numbers. Then, the weighted lattice path interpretation allows us to derive matrix representations. Define the infinite dimensional matrix

$$RE^{(p)}:=\begin{pmatrix} x-\frac{p}{2} & -\frac{p}{4} & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{p}{2} & -\frac{p+1}{2} & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{p}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\frac{n(n+p-1)}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{p}{2} & -\frac{(n+1)(n+p)}{4} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix},$$

and let $RE_m^{(p)}$ be the left upper $m\times m$ block of $RE^{(p)}$. Then for any nonnegative integer $n\leq m$, we have

$$\left[\left(RE_m^{(p)}\right)^n\right]_{1,1}=E_n^{(p)}(x).$$

Similarly result [14, Ex. 15] for Bernoulli polynomials is also obtained.

Future work

- Besides the orthogonal polynomial and Motzkin path interpretations, I would like to find other properties that related to the matrix $RE^{(p)}$. For instance, the first column of $\left(RE_m^{(p)}\right)^n$ gives the coefficients of expanding y^n in terms of $\Omega_{n+1}^{(p)}(y)$'s.

3.3 Zonal polynomials

Zonal polynomial $C_\lambda(y_1,\ldots,y_m)$ is a symmetric polynomial in variables $y_1,\ldots,y_m$. Here, λ is a partition of an integer n, namely $\lambda=(\lambda_1,\ldots\lambda_k)$, such that

$$\lambda_1\geq\cdots\geq\lambda_k\geq 1 \quad\text{and}\quad \lambda_1+\cdots+\lambda_k=n.$$

For any partitions λ and μ of n, we can consider the Lexicographical order, i.e,

$$\lambda>\mu\Leftrightarrow\lambda_i=\mu_i \text{ for } i=1,\ldots,t, \text{ and } \lambda_t>\mu_t.$$

Another form of λ is $\lambda=1^{s_1}2^{s_2}\cdots n^{s_n}$ such that

$$n=\lambda_1+\cdots+\lambda_k=\underbrace{n+\cdots+n}_{s_n}+\cdots+\underbrace{1+\cdots+1}_{s_1}.$$

One can define C_λ through different aspects, including

- transform involving Wishart distribution [22];
- eigenfunction of the Laplace-Beltrami operator on the space of $m\times m$ symmetric, positive-definite matrices, denoted by $SPD(m)$ [8];
- representation theory and invariant subspaces [**?** , p. 611];
- and as a special case of Jack polynomials, which is also a special case of Macdonald polynomial.

However, in practical computation, a transition matrix (see, e.g., [17]) appears to be the key. More precisely, define the *monomial symmetric function* as [22, eq. 6]:

$$M_\lambda(y_1\ldots,y_m)=M_{(1^{s_1}2^{s_2}\ldots)}(y_1\ldots,y_k)=\left(\prod_{i=1}^{n}\frac{1}{s_i!}\right)\sum_{i_1,\ldots i_l}y_{i_1}^{\lambda_1}\cdots y_{i_l}^{\lambda_l}. \tag{3.4}$$

A general result by James [8] shows that

$$C_\kappa(Y) = \sum_{\lambda\le\kappa} c_{\kappa,\lambda} M_\lambda(Y). \tag{3.5}$$

Coefficients are defined by

$$c_{\kappa,\lambda} = \sum_{\lambda<\mu\le\kappa} \frac{(\lambda_i+t)-(\lambda_j-t)}{\rho_\kappa-\rho_\lambda} c_{\kappa,\mu}, \tag{3.6}$$

where for $\lambda=(\lambda_1,\dots,\lambda_l)$, the sum is over all $\mu=(\lambda_1,\dots,\lambda_i+t,\dots,\lambda_j-t,\dots,\lambda_l)$ for $t=1,\dots,\lambda_j$ such that by rearranging tuple μ in a descending order, it lies as $\lambda<\mu\le\kappa$. In fact, this leads to the matrix transition

$$\begin{pmatrix} C_{(n)} \\ C_{(n,n-1)} \\ \vdots \\ C_{(1,\dots,1)} \end{pmatrix} = \left(c_{\kappa,\lambda}\right) \begin{pmatrix} M_{(n)} \\ M_{(n,n-1)} \\ \vdots \\ M_{(1,\dots,1)} \end{pmatrix}.$$

Based on this matrix transition, we now have both a `Mathematica` and `Sage` package for computation of C_λ.

Future work

- The continuation of implementation and modification of the codes is part of future work.
- C_λ is the key to define hypergeometric functions of matrix argument (see, e.g., [9]):

$${}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\middle| Y\right) := \sum_{n=0}^{\infty}\sum_{\lambda\in\mathcal{P}_n} \frac{(a_1)_\lambda\cdots(a_p)_\lambda}{(b_1)_\lambda\cdots(b_q)_\lambda}\cdot\frac{\mathcal{C}_\lambda(Y)}{n!}.$$

  When $p=q=1$, this generalization, by comparing with (2.7), may lead to Bernoulli polynomials of matrix argument. With all aspects for defining C_λ, this generalization may lead to fruitful results.

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