The Bernoulli Symbol $\mathfrak{B}$ and Its Applications

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Lin Jiu

Tulane University @ Nankai University

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Prof. Christophe Vignat

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Outlines

1. Introduction
   - Bernoulli Symbol $\mathfrak{B}$
   - Probabilistic Interpretation
   - Together with $\mathfrak{U}$, the uniform symbol
2. Applications, Extensions and Results
   - Generalized Bernoulli Numbers/Polynomials
   - Bernoulli-Barnes Polynomial
   - Multi-Zeta Values
3. Future Work

Easy Beginning

The simple evaluation rule is

$$eval(\mathfrak{B}^n) = B_n, \text{ the } n^{\text{th}} \text{ Bernoulli number}$$

or more simply

$$\mathfrak{B}^n = B_n,$$

and together with

$$e^{\mathfrak{B}t} = \frac{t}{e^t - 1}.$$

Proof.

$$e^{\mathfrak{B}t} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}^n t^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.$$

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Example1

$$S_m(n) := \sum_{k=1}^{n} k^m = 1^m + \cdots + n^m$$

$$= \frac{1}{m+1} \sum_{l=0}^{m} \binom{m+1}{l} B_l n^{m+1-l}$$

$$= \frac{1}{m+1} \sum_{l=0}^{m} \binom{m+1}{l} \mathfrak{B}^l n^{m+1-l}$$

$$= \frac{1}{m+1} \left[(\mathfrak{B}+n)^{m+1} - \mathfrak{B}^{m+1} \right]$$

Operator

- $(\Delta_n \circ \int) \circ (\mathfrak{B}^m)$

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Definition

Bernoulli polynomials are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

which is equivalent to

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

or with the symbol $\mathfrak{B}$,

$$B_n(x) = (\mathfrak{B} + x)^n.$$

Example

$$B_n'(x) = nB_{n-1}(x) \Leftrightarrow [(\mathfrak{B} + x)^n]' = n(\mathfrak{B} + x)^{n-1}$$

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Probabilistic Interpretation (Not Umbral Calculus)

Recall: Random Variable $X \sim p(x)$

$$P[X < x] = \int_{-\infty}^{x} p(t)\,dt.$$

$\mathfrak{B} \sim B(x)$ such that

$$\begin{cases} \mathbb{E}[\mathfrak{B}^n] = \int_{\mathbb{R}} x^n B(x)\,dx = B_n & , \\ \mathbb{E}\left[e^{\mathfrak{B}t}\right] = \int_{\mathbb{R}} e^{xt} B(x)\,dx = \frac{t}{e^t - 1} & . \end{cases}$$

Theorem[Density of $\mathfrak{B}$](A .Dixit, V. H. Moll, and C. Vignat)

$\mathfrak{B} \sim \iota L_B - \frac{1}{2}$, where

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Probabilistic Interpretation (Continued)

Any $f \in L^1(\mathbb{R}), f(\mathfrak{B})$.

$$eval[f(\mathfrak{B})] = \mathbb{E}[f(\mathfrak{B})] = \int_{\mathbb{R}} f(t) B(t) dt.$$

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Uniform Symbol $\mathfrak{U}$

$\mathfrak{U} \sim U[0,1]$.

$$\mathbb{E}[f(\mathfrak{U})] = \int_0^1 f(x)\,dx$$

The following fact is easy but important

$$\mathbb{E}\left[e^{t\mathfrak{U}}\right] = \int_0^1 e^{tx}\,dx = \frac{e^t - 1}{t},$$

the reciprocal of $\mathbb{E}\left[e^{t\mathfrak{B}}\right]$!

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Important dual $(\mathfrak{B}, \mathfrak{U})$

Recall: For independent random variables X and Y, if

$$\begin{cases} \mathbb{E}\left[e^{tX}\right] = F(x) & , \\ \mathbb{E}\left[e^{tY}\right] = G(x) & , \end{cases}$$

then

$$\mathbb{E}\left[e^{t(X+Y)}\right] = F(x)\,G(x).$$

Fact

$$\mathbb{E}\left[e^{t(\mathfrak{B}+\mathfrak{U})}\right] = 1$$

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$(\mathfrak{B}, \mathfrak{U})$

$$f(x+\mathfrak{B}+\mathfrak{U}) = \sum_{n\geq 0} a_n (x+\mathfrak{B}+\mathfrak{U})^n$$

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$(\mathfrak{B},\mathfrak{U})$–Example

Suppose $f(x)=x^n$
On the other hand,

$$f(x)=f(x+\mathfrak{B}+\mathfrak{U})=\int_0^1 f(x+\mathfrak{B}+u)\,du.$$

If letting $F(x)=\frac{x^{n+1}}{n+1}\Rightarrow F'=f$, then

$$x^n=F(x+1+\mathfrak{B})-F(x+\mathfrak{B})=\frac{1}{n+1}\left[B_{n+1}(x+1)-B_{n+1}(x)\right].$$

Namely,

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If letting $F(x) = \frac{x^{n+1}}{n+1} \Rightarrow F' = f$, then

$$x^n = F(x + 1 + \mathfrak{B}) - F(x + \mathfrak{B}) = \frac{1}{n+1}\left[B_{n+1}(x+1) - B_{n+1}(x)\right].$$

Namely,

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$(\mathfrak{B},\mathfrak{U})$–Example

Suppose $f(x)=x^n$
On the other hand,

$$f(x)=f(x+\mathfrak{B}+\mathfrak{U})=\int_0^1 f(x+\mathfrak{B}+u)\,du.$$

If letting $F(x)=\frac{x^{n+1}}{n+1}\Rightarrow F'=f$, then

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Generalized Bernoulli Numbers

Definition

$$\left(\frac{t}{e^t-1}\right)^p=\sum_{n=0}^{\infty}B_n^{(p)}\frac{t^n}{n!}.$$

Theorem[Recurrence & Lucas Formula(1878)]

$$B_n^{(p+1)}=\left(1-\frac{n}{p}\right)B_n^{(p)}-nB_{n-1}^{(p)}$$

and

$$B_n^{(p+1)}=(-1)^p\,p\binom{n}{p}\beta^{n-p}\left(\beta\right)_p,$$

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Generalized Bernoulli Numbers(Continued)

Proof of Recurrence

We shall prove a polynomial version

$$B_n^{(p+1)}(x) = \left(1 - \frac{n}{p}\right) B_n^{(p)}(x) - n\left(1 - \frac{x}{p}\right) B_{n-1}^{(p)}(x),$$

where

$$\left(\frac{t}{e^t - 1}\right)^p e^{tx} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!}.$$

Symbolically,

$$B_n^{(p)}(x) = \left(\mathfrak{B}_1 + \cdots + \mathfrak{B}_p + x\right)^n$$

for an i.i.d. sequence $\{\mathfrak{B}_i\}$. First consider $p = 1$, i.e.,

$$B_n^{(2)}(x) = (1 - n) B_n(x) - n(1 - x) B_{n-1}(x)$$

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Generalized Bernoulli Numbers(Continued)

Proof of Recurrence(Continued)

Note that

$$f(x+1)-f(x)=\int_0^1 f'(x+u)\,du=f'(x+\mathfrak{U}).$$

KEY: $f(x):=xB_n(x)\Rightarrow f'(x)=B_n(x)+nxB_{n-1}(x)$, then

$$LHS=(x+1)B_n(x+1)-xB_n(x)=nx^n+nx^{n-1}+B_n(x),$$

and

$$RHS=B_n(x+\mathfrak{U})+n(x+\mathfrak{U})B_{n-1}(x+\mathfrak{U}).$$

Now, substitution $x\mapsto x+\mathfrak{B}'$ yields

Generalized Bernoulli Numbers(Continued)

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Now, substitution $x\mapsto x+\mathfrak{B}'$ yields

Generalized Bernoulli Numbers(Continued)

Proof of Recurrence(Continued)

$$LHS = n\left(x+\mathfrak{B}'\right)^{n} + n\left(x+\mathfrak{B}'\right)^{n-1} + B_n\left(x+\mathfrak{B}'\right) = nB_n(x) + nB_{n-1}(x) + B_n^{(2)}(x)$$

and

$$RHS = B_n(x+\mathfrak{B}+\mathfrak{U}) + n(x+\mathfrak{B}+\mathfrak{U})B_{n-1}(x+\mathfrak{B}+\mathfrak{U}) = B_n(x) + nxB_{n-1}(x).$$

Matching both sides gives the result. Now, for inductive step, in

$$B_n^{(p+1)}(x) = \left(1-\frac{n}{p}\right)B_n^{(p)}(x) - n\left(1-\frac{x}{p}\right)B_{n-1}^{(p)}(x)$$

replace x by $x+\mathfrak{B}$, then

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Generalized Bernoulli Numbers(Continued, Tricky Part)

Proof of Recurrence(Continued)

$$\begin{aligned} n\left(1-\frac{x+\mathfrak{B}}{p}\right)B_{n-1}^{(p)}(x+\mathfrak{B}) &= n\left(1-\frac{x}{p}\right)B_{n-1}^{(p)}(x+\mathfrak{B})-\frac{n}{p}\mathfrak{B}B_{n-1}^{(p)}(x+\mathfrak{B}) \\ &= n\left(1-\frac{x}{p}\right)B_{n-1}^{(p+1)}(x)-\frac{n}{p}\underline{\mathfrak{B}B_{n-1}^{(p)}(x+\mathfrak{B})} \end{aligned}$$

By symmetry,

$$\begin{aligned} n\mathfrak{B}B_{n-1}^{(p)}(x+\mathfrak{B}) &= n\mathfrak{B}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+x+\mathfrak{B}\right)^{n-1} \\ &= \frac{n}{p+1}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+x+\mathfrak{B}\right)^{n-1} \\ &= \frac{n}{p+1}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{k+1}x^{n-1-k} \\ [l=k+1] \quad &= \frac{n}{p+1}\sum_{l=1}^{n}\binom{n-1}{l-1}\left(\mathfrak{B}_1+\cdots+\mathfrak{B}_p+\mathfrak{B}\right)^{l}x^{n-l} \end{aligned}$$

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The Lucas' Formula follows by induction

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B_n^{(p+1)} = (-1)^p\, p\binom{n}{p}\beta^{n-p}(\beta)_p
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Bernoulli-Barnes Polynomial

- Bernoulli:

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}$$

- Generalized (Norlünd):

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- Bernoulli-Barnes

$$e^{tx}\prod_{i=1}^{k}\frac{t}{e^{a_it}-1}=\sum_{n=0}^{\infty}B_n(\mathbf{a};x)\frac{t^n}{n!},$$

for $\mathbf{a}=(a_1,\ldots,a_k)$.

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Symbolic Expressions

- Bernoulli:

$$B_m(x) = (x + \mathfrak{B})^m$$

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- Bernoulli-Barnes ($\forall l = 1, \ldots, n,\ a_l \neq 0$)

$$B_n(\mathbf{a}; x) = \frac{1}{|\mathbf{a}|}\left(x + \mathbf{a} \cdot \vec{\mathfrak{B}}\right)^n, \text{ where } \begin{cases} \mathbf{a} = (a_1, \ldots, a_k) \\ \vec{\mathfrak{B}} = (\mathfrak{B}_1, \ldots, \mathfrak{B}_k) \\ \mathbf{a} \cdot \vec{\mathfrak{B}} = \sum_{l=1}^{k} a_l \mathfrak{B}_l \\ |\mathbf{a}| = \prod_{l=1}^{k} a_l \end{cases}$$

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Main Results

Theorem(A. Bayad and M. Beck)

(1)Difference Formula: Suppose $A=\sum_{k=1}^{n}a_k\neq 0$, then

$$(-1)^m B_m(\mathbf{a};-x)-B_m(\mathbf{a};x)=m!\sum_{l=1}^{n-1}\sum_{|L|=l}\frac{B_{m-n+l}(\mathbf{a}_L;x)}{(m-n+l)!},$$

where $L\subset\{1,\ldots,n\}$ and $B_m(\mathbf{a}_L;x)=x^m$ if $L=\emptyset$.

(2) Sequence $\{(-1)^n A^{-n}B_n(\mathbf{a})\}$ is self-dual. A self-dual sequence $\{s_n\}$ satisfies

$$s_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k s_k.$$

Remark. Authors ask for direct proof of (2).

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(2) Sequence $\{(-1)^n A^{-n}B_n(\mathbf{a})\}$ is self-dual. A self-dual sequence $\{s_n\}$ satisfies

$$s_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k s_k.$$

Remark. Authors ask for direct proof of (2).

Main Results

Theorem(A. Bayad and M. Beck)

(1)Difference Formula: Suppose $A=\sum_{k=1}^{n} a_k \neq 0$, then

$$(-1)^m B_m(\mathbf{a};-x) - B_m(\mathbf{a};x) = m! \sum_{l=1}^{n-1} \sum_{|L|=l} \frac{B_{m-n+l}(\mathbf{a}_L;x)}{(m-n+l)!},$$

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Main Results(More General/Direct Cases)

Theorem(L. Jiu, V. H. Moll and C. Vignat)

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$$f\left(x-\mathbf{a}\cdot\vec{\mathfrak{B}}\right)=\sum_{j=0}^{n}\sum_{|J|-j}|\mathbf{a}|_{J^*}f^{(n-j)}\left(x+\left(\mathbf{a}\cdot\vec{\mathfrak{B}}\right)_J\right),$$

where $J\subset\{1,\ldots,n\}$, $J^*=\{1,\ldots,n\}\setminus J$. In particular,

$$f\left(x+A+\mathbf{a}\cdot\vec{\mathfrak{B}}\right)=f\left(x-\mathbf{a}\cdot\vec{\mathfrak{B}}\right).$$

(1) is the special case for $f(x)=x^m/m!$.
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Important Fact

Fact

$$-\mathfrak{B} = \mathfrak{B} + 1.$$

Recall

$$\left(x + A + \mathbf{a} \cdot \vec{\mathfrak{B}}\right) = f\left(x - \mathbf{a} \cdot \vec{\mathfrak{B}}\right).$$

Proof

$$e^{-\mathfrak{B}t} = e^{\mathfrak{B}(-t)} = \frac{-t}{e^{-t} - 1} = e^{t}\frac{t}{e^{t} - 1} = e^{t}e^{\mathfrak{B}t} = e^{(\mathfrak{B}+1)t}.$$

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Multi-Zeta Functions

Definition

• $Re(n_r) \geq 1$ and $\sum_{j=1}^{k} Re(n_r + 1 - j) \geq k,\ 2 \leq k \leq r$

$$\zeta_r(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} = \sum_{k_1,\ldots,k_r} \frac{1}{k_1^{n_1}(k_1+k_2)^{n_2}\cdots(k_1+\cdots+k_r)^{n_r}}$$

• B. Sadaoui's analytic continuation is based on Raabe's identity by linking

$$Y_{\mathbf{a}}(\mathbf{n}) = \int_{[1,\infty)^r} \frac{d\mathbf{x}}{(x_1+a_1)\cdots(x_1+a_1+\cdots+x_r+a_r)^{n_r}}$$

to the multiple zeta function

$$Z(\mathbf{n}, \mathbf{z}) = \sum_{k_1,\ldots,k_r>0} \frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}$$

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•When $\mathbf{n} \mapsto -\mathbf{n}$, $Y_{\mathbf{a}}(-\mathbf{n})$ is a polynomial in $\mathbf{a}$.
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$$f(x+\mathcal{V}) = \int_1^\infty f(x+v)\, dv = \sum_{n=1}^\infty f(x+n+\mathfrak{U}).$$

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Main Results

Theorem(Sadaoui)

$$\begin{aligned}
\zeta_r\left(-n_1,\ldots,-n_r\right) &= (-1)^r \sum_{k_2,\ldots,k_r} \frac{1}{\bar{n}+r-\bar{k}} \\
&\times \prod_{j=2}^{r} \frac{\binom{\sum_{i=j}^{r} n_i - \sum_{i=j+1}^{n} k_i + r - j + 1}{k_j}}{\sum_{i=j}^{r} n_i - \sum_{i=j}^{n} k_i + r - j + 1} \\
&\times \sum_{l_1,\ldots,l_r} \binom{\bar{n}+r-\bar{k}}{l_1}\binom{k_2}{l_2}\cdots\binom{k_r}{l_r} B_{l_1}\cdots B_{l_r},
\end{aligned}$$

where $\bar{n}=\sum_{j=1}^{n} n_j$, $\bar{k}=\sum_{j=2}^{r} k_j$, $k_2,\ldots k_r \geq 0$, $l_j \leq k_j$ for $2 \leq j \leq r$ and $l_1 \leq \bar{n}+r+\bar{k}$.

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Main Results (Continued)

Theorem(L. Jiu, V. H. Moll and C. Christophe)

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Example

$$\begin{aligned}\zeta_2(-n,0) &= (-1)^n\,\mathfrak{C}_1^{n+1}\cdot(-1)^0\,\mathfrak{C}_{1,2}^{0+1}\\ &= (-1)^n\,\frac{\mathfrak{C}_1+\mathfrak{B}_2}{1}\cdot\mathfrak{C}_1^{n+1}\\ &= (-1)^n\left(\mathfrak{C}_1^{n+2}+\mathfrak{B}_2\mathfrak{C}_1^{n+1}\right)\\ &= (-1)^n\left[\frac{B_{n+2}}{n+2}-\frac{1}{2}\frac{B_{n+1}}{n+1}\right].\end{aligned}$$

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Theorem[Polynomial Case](L. Jiu, V. H. Moll and C. Christophe)

Recall that

$$\zeta_r(n_1,\ldots,n_r,z_1,\ldots,z_r)=\sum_{k_1,\ldots,k_r>0}\frac{1}{(k_1+z_1)^{n_1}\cdots(k_1+z_1+\cdots+k_r+z_r)^{n_r}}.$$

Then,

$$\zeta_r(-n_1,\ldots,-n_r,z_1,\ldots,z_r)=\prod_{k=1}^{r}(-1)^{n_k}\mathfrak{C}_{1,\ldots,k}^{n_k+1}(z_1,\ldots,z_i),$$

where

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Main Results (Continued)

Theorem[Recurrence](L. Jiu, V. H. Moll and C. Christophe)

$$\begin{aligned}
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&\times\zeta_{r-1}\left(-n_1,\ldots,-n_{r-1}-k;z_1,\ldots,z_{r-1}\right)B_{n_r+1-k}\left(z_r\right).
\end{aligned}$$

Symbolically,

$$\zeta_r\left(-\mathbf{n};\mathbf{z}\right)=(-1)^{n_r}\frac{\left(\mathfrak{B}-\mathcal{Z}_{r-1}\right)^{n_r+1}}{n_r+1}=\zeta_1\left(-n_r;-\mathcal{Z}_{r-1}\right),$$

where

$$\mathcal{Z}_r^k=\zeta_r\left(-n_1,\ldots,-n_{r-1},-n_r-k;\mathbf{z}\right)$$

Main Results (Continued)

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$$\zeta_r\left(-\mathbf{n};\mathbf{z}\right)=(-1)^{n_r}\frac{\left(\mathfrak{B}-\mathcal{Z}_{r-1}\right)^{n_r+1}}{n_r+1}=\zeta_1\left(-n_r;-\mathcal{Z}_{r-1}\right),$$

where

$$\mathcal{Z}_r^k=\zeta_r\left(-n_1,\ldots,-n_{r-1},-n_r-k;\mathbf{z}\right)$$

Main Results (Continued)

Theorem[Contiguity identities](L. Jiu, V. H. Moll and C. Christophe)

$$\begin{aligned} & \zeta_r\left(-n_1,\ldots,n_r;z_1,\ldots,z_{r-1},z_r+1\right) \\ = \; & \zeta_r\left(-n_1,\ldots,n_r;z_1,\ldots,z_{r-1},z_r\right) \\ & +(-1)^{n_r}\left(z_r-\mathcal{Z}_{r-1}\right)^{n_r}. \end{aligned}$$

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Main Results (Continued)

Theorem[Generating Function](L. Jiu, V. H. Moll and C. Christophe)

Define that

$$F_r\left(w_1,\ldots,w_r\right):=\sum_{n_1,\ldots,n_r\geq 0}\frac{w_1^{n_1}\cdots w_r^{n_r}}{n_1!\cdots n_r!}\zeta_r\left(-n_1,\ldots,-n_r\right)$$

and also denote

$$F_{\mathfrak{B}}\left(w\right)=\sum_{n\geq 0}B_n\frac{w^n}{n!}=\frac{w}{e^w-1}.$$

Then, recursively,

$$\begin{aligned}F_r\left(w_1,\ldots,w_r\right) &= \frac{1}{w_r}\Big[F_{r-1}\left(w_1,\ldots,w_{r-1}\right)-F_{\mathfrak{B}}\left(-w_r\right)\\ &\quad \times F_{r-1}\left(w_1,\ldots,w_{r-2},w_{r-1}+w_r\right)\Big]\end{aligned}$$

Future Work

- Code: Mathematica Sage
  [Good News] Rules are direct.
  [Bad News] Choice of functions are tricky.
- Hypergeometric Bernoulli Numbers:

$$\frac{\frac{t^N}{N!}}{e^t-1-t-\cdots-\frac{t^{N-1}}{(N-1)!}}=\frac{1}{{}_1F_1\left(\left.\begin{matrix}1\\N+1\end{matrix}\right|t\right)}=\sum_{n=0}^{\infty}B_{N,n}\frac{t^n}{n!}$$

  [Good News] A. Byrnes, L. Jiu, V. H. Moll and C. Vignat, *Recursion Rules for the Hypergeometric Zeta Functions*, International Journal of Number Theory, vol. 10, No 7, 1761-1782, 2014
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Future Work

- Euler Version:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

[Good News]

$$\mathfrak{B} \sim \iota L_B - \frac{1}{2}, \text{ where } L_B \sim \frac{\pi}{2\cosh^2(\pi x)}$$

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End

Thank You!

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