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Integral representations of equally positive integer-indexed harmonic sums at infinity

Lin Jiu

*Correspondence:
ljiu@risc.uni-linz.ac.at
Research Institute for Symbolic Computation, Johannes Kepler University, 4040 Linz, Austria

Abstract

We identify a partition-theoretic generalization of Riemann zeta function and the equally positive integer-indexed harmonic sums at infinity, to obtain the generating function and the integral representations of the latter. The special cases coincide with zeta values at positive integer arguments.

Keywords: Harmonic sum, Integral representation, Zeta value

1 Background

The harmonic sum of *indices* $a_1, \ldots, a_k \in \mathbb{R}\setminus\{0\}$ is defined as (see [1, eq. 4, pp. 1])

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}(a_1)^{n_1}}{n_1^{|a_1|}} \times \cdots \times \frac{\operatorname{sign}(a_k)^{n_k}}{n_k^{|a_k|}},$$

which is naturally connected to the Riemann zeta function, by noting that $N = \infty$, $k = 1$ and $a_1 > 0$ gives $S_{a_1}(\infty) = \zeta(a_1)$. A variety of the study can be found in the literature. For instance, Hoffman [4] established the connection between harmonic sums and multiple zeta values. We especially focus on the *equally positively indexed harmonic sums*, given by the case $a_1 = \cdots = a_k = a > 0$

$$S_{a_k}(N) := S_{\underbrace{a,\ldots,a}_{k}}(N) = \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{1}{(n_1\cdots n_k)^a}, \tag{1.1}$$

and also the *equally positive integer-indexed harmonic sums* (EPIIHS), namely $a = m \in \mathbb{Z}_{>0}$. If $N = \infty$, we additionally assume $m \in \mathbb{Z}_{>1}$ for convergence.

Recently, Schneider [7] studied the generalized q-Pochhammer symbol and obtained [7, pp. 3]

$$\prod_{n\in X} \frac{1}{1 - f(n)\,q^n} = \sum_{\lambda\in\mathcal{P}_X} q^{|\lambda|} \prod_{\lambda_i\in\lambda} f(\lambda_i), \tag{1.2}$$

where

- $X \subseteq \mathbb{Z}_{>0}$ and $f : \mathbb{Z}_{>0} \longrightarrow \mathbb{C}$ such that if $n \notin X$ then $f(n) = 0$;
- $\mathcal{P}_X$ is the set of partitions into elements of X;
- $\lambda \vdash n$ means λ is a partition of n, the size $|\lambda|$ is the sum of the parts of λ, i.e., the number n being partitioned, and $\lambda_i \in \lambda$ means $\lambda_i \in \mathbb{Z}_{>0}$ is a part of partition λ.

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Further define $l(\lambda) := k$, $n_\lambda := \lambda_1 \cdots \lambda_k$ and denote $\mathcal{P} := \mathcal{P}_{\mathbb{Z}_{>0}}$. Noting $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$, a *partition-theoretic generalization of Riemann zeta function* [7, eq. 11, pp. 4] is defined and identified as

$$\zeta_{\mathcal{P}}\left(\{a\}^k\right) := \sum_{l(\lambda)=k} \frac{1}{n_\lambda^a} = \sum_{\lambda_1 \geq \cdots \geq \lambda_k \geq 1} \frac{1}{\lambda_1^a \cdots \lambda_k^a} = S_{\boldsymbol{a}_k}(\infty), \tag{1.3}$$

which leads to the generating function and the integral representation of $S_{\boldsymbol{m}_k}(\infty)$, presented in the next section.

2 Main results

We first apply (1.2) to the case $X = \{1, 2, \ldots, N\}$ and $f(n) := \frac{t^a}{n^a}$, obtaining

$$\prod_{n=1}^{N} \frac{1}{1 - \frac{t^a}{n^a} q^n} = \sum_{\lambda \in \mathcal{P}_X} q^{|\lambda|} \prod_{\lambda_i \in \lambda} \frac{t^a}{\lambda_i^a} = \sum_{\lambda \in \mathcal{P}_X} q^{|\lambda|} \frac{t^{l(\lambda)a}}{n_\lambda^a},$$

which, by further letting $q \to 1$, yields the following generating function.

Theorem 1 *The generating function of $S_{\boldsymbol{a}_k}(N)$ is given by*

$$\sum_{k=0}^{\infty} S_{\boldsymbol{a}_k}(N)\, t^{ak} = \prod_{n=1}^{N} \frac{n^a}{n^a - t^a}. \tag{2.1}$$

Remark 2 The special case for $a = 1$ is [2, eq. 9, pp. 1272]

$$\sum_{k=0}^{\infty} t^k S_{\boldsymbol{1}_k}(N) = \frac{N!}{(1-t)\cdots(N-t)} = N \cdot B(N, 1-t), \tag{2.2}$$

involving the beta function B, defined by

$$B(x, y) := \int_0^1 z^{x-1} (1-z)^{y-1}\, dz = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{2.3}$$

where the integral representation holds for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

Corollary 3 *For $m \in \mathbb{Z}_{>1}$, denote $\xi_m := \exp\left(\frac{2\pi i}{m}\right)$ with $i^2 = -1$. Then,*

$$\sum_{k=0}^{\infty} S_{\boldsymbol{m}_k}(\infty)\, t^{mk} = \prod_{j=0}^{m-1} \Gamma\left(1 - \xi_m^j t\right). \tag{2.4}$$

Proof From (2.1) and (2.2), we have

$$\sum_{k=0}^{\infty} S_{\boldsymbol{m}_k}(N)\, t^{mk} = \prod_{n=1}^{N} \frac{n^m}{\left(n - \xi_m^0 t\right) \cdots \left(n - \xi_m^{m-1} t\right)} = \prod_{j=0}^{m-1} N \cdot B\left(N, 1 - \xi_m^j t\right).$$

Then, apply the limit (see [6, pp. 254, ex. 5]) $\Gamma(z) = \lim_{n \to \infty} N^z B(N, z)$ to $z_j = 1 - \xi_m^j t$, $j = 0, \ldots, m-1$, by noting $\xi_m^0 + \cdots + \xi_m^{m-1} = 0$, to complete the proof. □

Remark 4 An alternative proof can be given by letting $N = \infty$ in (2.1) and applying [3, Thm. 1.1, pp. 547].

Remark 5 For general $a > 0$, we failed to obtain a closed form of $\prod_{n=1}^{\infty} \frac{n^a}{n^a - t^a}$.

Example 6 When $m = 2$, we apply (2.4) to get

$$B(1+t, 1-t) = \Gamma(1+t)\,\Gamma(1-t) = \sum_{k=0}^{\infty} S_{\mathbf{2}_k}(\infty)\, t^{2k}.$$

From the integral representation (2.3), we obtain (also see Remark 7)

$$B(1+t, 1-t) = \int_0^1 z^t (1-z)^{-t}\, dz = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 \log^k\left(\frac{z}{1-z}\right) dz. \tag{2.5}$$

Then it follows, by comparing coefficients of t,

$$S_{\mathbf{2}_k}(\infty) = \frac{1}{(2k)!} \int_0^1 \log^{2k}\left(\frac{z}{1-z}\right) dz.$$

In particular, $k = 1$ yields

$$\frac{\pi^2}{6} = \zeta(2) = S_2(\infty) = \frac{1}{2} \int_0^1 \log^2\left(\frac{z}{1-z}\right) dz.$$

Remark 7 We may interchange the integral and the sum of the series in (2.5), by restricting t to a closed compact set, e.g., $\left[-\frac{1}{2}, \frac{1}{2}\right]$, satisfying $\mathrm{Re}(1-t)$, $\mathrm{Re}(1+t) > 0$ as that in (2.3), in order to guarantee uniform convergence of the integral representation. (Similar discussion is omitted for the multiple beta function, defined next.)

Definition 8 The *multiple beta function* [5, Ch. 49] is defined as

$$B(\alpha_1, \ldots, \alpha_m) := \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_m)}{\Gamma(\alpha_1 + \cdots + \alpha_m)} = \int_{\Omega_m} \prod_{i=1}^{m} x_i^{\alpha_i - 1} d\mathbf{x}, \tag{2.6}$$

where $\Omega_m = \left\{(x_1, \ldots, x_m) \in \mathbb{R}^m_{>0} : x_1 + \cdots + x_{m-1} < 1,\ x_1 + \cdots + x_m = 1\right\}$ and the integral representation requires $\mathrm{Re}(\alpha_1), \ldots, \mathrm{Re}(\alpha_m) > 0$.

Following the same idea as that in Example 6, we first have, from (2.4),

$$B\left(1 - \xi_m^0 t, \ldots, 1 - \xi_m^{m-1} t\right) = \frac{1}{(m-1)!} \sum_{k=0}^{\infty} S_{\mathbf{m}_k}(\infty)\, t^{mk}.$$

Then, apply the integral representation (2.6), expand the integrand as a power series in t, and compare coefficients of t, to obtain the following integral representation.

Theorem 9 *For all $m, k \in \mathbb{Z}_{>0}$ with $m \geq 2$,*

$$S_{\mathbf{m}_k}(\infty) = \frac{(-1)^{mk}(m-1)!}{(mk)!} \int_{\Omega_m} \log^{mk}\left(\prod_{j=0}^{m-1} x_{j+1}^{\xi_m^j}\right) d\mathbf{x}.$$

Corollary 10 *In particular, the case $k = 1$ implies for integer $m \in \mathbb{Z}_{>1}$ that*

$$\zeta(m) = \frac{(-1)^m}{m} \int_{\Omega_m} \log^m\left(\prod_{j=0}^{m-1} x_{j+1}^{\xi_m^j}\right) d\mathbf{x},$$

or alternatively

$$\begin{aligned}\zeta(m) = \frac{(-1)^m}{m}\int_0^1\int_0^{1-x_1} & \\ \cdots\int_0^{1-x_1-\cdots-x_{m-2}} & \log^m\left(x_1^{\xi_m^0}\cdots x_{m-1}^{\xi_m^{m-2}}(1-x_1-\cdots-x_{m-1})^{\xi_m^{m-1}}\right) \\ \times dx_{m-1}\cdots dx_1. &\end{aligned}$$

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