LECTURE 0 FUNCTIONS

A function consists of THREE important parts: domain, expression and range.

Example 1. $f(x) = x + 1$ and $g(x) = \frac{x^2-1}{x-1}$.

Observe that
$$g(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1.$$
However, $f(x) \neq g(x)$ since $dom(f) = (-\infty, \infty) = \mathbb{R}$ while $dom(g) = (-\infty, 1) \cup (1, \infty) = \{x | x \neq 1\}$.

Fact. *1. For exponential, polynomials, sine and cosine functions, domains are* $(-\infty, \infty)$

2. For rational function $\frac{f(x)}{g(x)}$*, the domain is* $\{x | g(x) \neq 0\}$.

3. For logarithmic function $\log_a x$*, the domain is* $(0, \infty)$

4. For root function, $x^{\frac{1}{n}}$*, domain is* $(-\infty, \infty)$ *if* n *is odd and* $[0, \infty)$ *if* n *is even*

5. $dom(\sec(x)) = dom(\tan(x)) = \left\{x | x \neq k\pi + \frac{\pi}{2}\right\}$, $dom(\csc(x)) = dom(\cot(x)) = \{x | x \neq k\pi\}$

Example 2. $f(x) = \frac{\sin x}{\sqrt{e^x - 1}}$.

1. $e^x - 1 \geq 0 \Rightarrow x \geq 0$
2. $\sqrt{e^x - 1} \neq 0 \Rightarrow e^x - 1 \neq 0 \Rightarrow x \neq 0$

Therefore, $dom(f) = (0, \infty)$.

Steps for finding inverse functions:
(1) Write $y = f(x)$
(2) Solve for $x = f^{-1}(y)$
(3) Interchange x and y to obtain $y = f^{-1}(x)$

Example 3. $f(x) = e^{2x-1}$

1. Writhe $y = e^{2x-1}$
2. Solve: $\ln y = 2x - 1 \Rightarrow x = \frac{1}{2}(\ln y + 1)$
3. $f^{-1}(x) = \frac{1}{2}(\ln x + 1)$.

LECTURE 1 LIMIT AND DEFINITION OF DERIVATIVES

We shall compute the following different types of limits:

$$\begin{cases} \lim\limits_{x\to a} f(x) \\ \lim\limits_{x\to a^-} f(x) \\ \lim\limits_{x\to a^+} f(x) \end{cases} \qquad \begin{cases} \lim\limits_{x\to\infty} f(x) \\ \lim\limits_{x\to-\infty} f(x) \end{cases}$$

Remark. $\lim\limits_{x\to a} f(x)$ exists if and only if both $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist and $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x)$

Example 1. $f(x) = x+1$ and $g(x) = \frac{x^2-1}{x-1}$.

$$\begin{cases} \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} (x+1) = 2 \\ \lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1} \frac{(x-1)(x+1)}{x-1} = 2 \end{cases}$$

Remark. f may not have value at a, but $\lim\limits_{x\to a} f(x)$ could still exist.

Example 2.

$$f(x) = sgn(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$

Then,

$$\lim_{x\to 0^-} f(x) = -1,\ \lim_{x\to 0^+} f(x) = 1 \text{ so } \lim_{x\to 0} f(x) \text{ DNE (does not exist).}$$

Definition. A function $f(x)$ is continuous at $x=a$ if

$$f(a) = \lim_{x\to a} f(x).$$

In concrete, we need:
(1) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist
(2) $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) \Rightarrow \lim\limits_{x\to a} f(x)$ exists
(3) $f(a) = \lim\limits_{x\to a} f(x)$

Example 3. Find b such that $f(x)$ is continous, where

$$f(x) = \begin{cases} x+b & x\geq 1 \\ x^2-1 & x<1 \end{cases}$$

Solution. The only point we need to consider is $x=1$ since other points are OK

$$f(1) = b+1,\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(x^2-1\right) = 0,\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+b) = b+1.$$

In order to make $f(x)$ continous at $x=1$, we need

$$f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow b+1 = 0 = b+1 \Rightarrow b = -1.$$

Now, we consider the limit at infinity and asymptotes.
Limit at Infinity:

$$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$

Example 4. Here are some important examples:

$$\lim_{x\to-\infty} e^x = 0,\ \lim_{x\to\pm\infty} \frac{1}{x} = 0,\ \lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2},\ \lim_{x\to\infty} x^2 = \infty.$$

Example 5.

$$\begin{aligned} \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0. \end{aligned}$$

Definition. (1) If either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then $y=L$ is called a horizontal asymptote.
(2) If any of the four cases happens $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$, then $x=a$ is called a veritical asymptote.

Example 6. Find horizontal and vertical asymptotes of the graph of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$.

Solution. (1)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\sqrt{2x^2+1}\cdot\sqrt{\frac{1}{x^2}}}{3-\frac{5}{x}} = \lim_{x\to\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3}.$$

(2)

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\sqrt{2x^2+1}\cdot\left(-\sqrt{\frac{1}{x^2}}\right)}{3-\frac{5}{x}} = \lim_{x\to\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}.$$

For vertical asymptotes, the choices of a are only points that are not in the domain. Note that

$$dom(f) = \{x|3x-5\neq 0\} = \left(-\infty,\frac{5}{3}\right)\cup\left(\frac{5}{3},\infty\right).$$

The only possible choice is $a=\frac{5}{3}$. However, we still need to check the two limits:

(3)

$$\lim_{x\to\frac{5}{3}^+}\frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{\frac{59}{9}}}{0^+} = \infty \text{ and } \lim_{x\to\frac{5}{3}^-}\frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{\frac{59}{9}}}{0^-} = -\infty$$

Therefore, we have vertical asymptote $x=\frac{5}{3}$ and horizontal asymptotes $y=\pm\frac{\sqrt{2}}{3}$.

Derivatives:

Definition. For a function $f(x)$ the derivative at $x=a$ is defined by

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

It is also the slope of the tangent line at point $(a, f(a))$.

Example 7. (1) $(e^x)' = e^x$ (2) $(\sin x)' = \cos x$

Example 8. Find a and b such that $f(x)$ is continous and differentiable, where

$$f(x) = \begin{cases} ax+b & x\geq 1\\ x^2-1 & x<1\end{cases}$$

Solution. The only point we need to consider is $x=1$ since other points are OK

$$f(1) = a+b,\ \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(x^2-1\right) = 0,\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(ax+b) = b+a.$$

In order to make $f(x)$ continous at $x=1$, we need

$$f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \Rightarrow b+a=0.$$

Also, in order to make it differetiable we need

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

to exist, or namely both $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$ and $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$ exist and $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$. In fact noting that $f(1)=0$ and $b=-a$,

$$\begin{cases} \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{x^2-1}{x-1} = \lim_{x\to 1^-}\frac{(x+1)(x-1)}{x-1} = 2\\ \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{ax-a}{x-1} = \lim_{x\to 1^-}\frac{a(x-1)}{x-1}\end{cases} \Rightarrow a=2.$$

Thus, we have

$$\begin{cases} a+b=0\\ a=2\end{cases} \Rightarrow \begin{cases} a=2\\ b=-2\end{cases}.$$

Example 9. Find the tangent line to the curve $y=e^x+\sin x$ at point $(0,1)$.

Solution. Let $f(x) = e^x+\sin x$, then $f'(x) = e^x+\cos x$, so at $x=0$, $f'(0) = e^0+\cos(0) = 2$. Thus, the tangent line is

$$y-1 = 2(x-0) \Rightarrow y = 2x+1.$$

LECTURE 2 DERIVATIVES CALCULATION

Table of Derivatives: $f'(x) = \frac{df}{dx} = \frac{d}{dx}(f(x))$: c is a constant

$$\begin{array}{cccc}
(c)' = 0, & (x^n)' = nx^{n-1}, & (e^x)' = e^x, & (\ln x)' = \frac{1}{x}, \\
 & & (a^x)' = (\ln a)\, a^x, & (\log_a x)' = \frac{1}{(\ln a) x}, \\
(\sin x)' = \cos x, & (\tan x)' = \sec^2 x, & (\sec x)' = \sec x \tan x, & \\
(\cos x)' = -\sin x, & (\cot x)' = -\csc x, & (\csc x)' = -\csc x \cot x, & \\
\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}}, & \left(\tan^{-1} x\right)' = \frac{1}{1+x^2}, & \left(\sec^{-1} x\right)' = \frac{1}{x\sqrt{x^2-1}}, & \\
\left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1-x^2}}, & \left(\cot^{-1} x\right)' = -\frac{1}{1+x^2}, & \left(\csc^{-1} x\right)' = -\frac{1}{x\sqrt{x^2-1}}, &
\end{array}$$

$$\begin{array}{ccccc}
(1) & (cf)' = cf', & (f \pm g)' = f' \pm g' & (fg)' = f'g + fg' & \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \\
(2) & [f(g(x))]' = f'(g(x)) \cdot g'(x) & & &
\end{array}$$

Example 1. $y = \tan^{-1}\left(\frac{t^2}{1+\ln t}\right)$

$$\begin{aligned}
y' &= \left[\tan^{-1}\left(\underbrace{\square}_{\square = \frac{t^2}{1+\ln t}}\right)\right]' = \frac{1}{1+\square^2} \cdot \square' = \frac{1}{1+\left(\frac{t^2}{1+\ln t}\right)^2} \cdot \left(\frac{t^2}{1+\ln t}\right)' \\
&= \frac{(1+\ln t)^2}{t^4 + (1+\ln t)^2} \cdot \frac{\left(t^2\right)' \cdot (1+\ln t) - \left(t^2\right) \cdot (1+\ln t)'}{(1+\ln t)^2} = \frac{(1+\ln t)^2}{t^4 + (1+\ln t)^2} \cdot \frac{2t \cdot (1+\ln t) - \left(t^2\right) \cdot \frac{1}{t}}{(1+\ln t)^2} \\
&= \frac{t + 2t \ln t}{t^4 + (1+\ln t)^2}.
\end{aligned}$$

Implicit Differentiation. Sometime, we only have an equation to describe the relation between two variables:

Example 2. Find y' if $y \sec\left(x^2\right) = x \tan y$

Solution. The parts with underline are where Chain Rule is applied:

$$\begin{aligned}
y \sec x = x \tan y \quad &\Rightarrow \quad \left(y \sec\left(x^2\right)\right)' = (x \tan y)' \\
&\Rightarrow \quad (y)' \sec\left(x^2\right) + y\left(\sec\left(x^2\right)\right)' = (x)' \tan y + x (\tan y)' \\
&\Rightarrow \quad y' \sec\left(x^2\right) + y \sec\left(x^2\right) \tan\left(x^2\right) \cdot \underline{2x} = \tan y + x \sec^2 y \cdot \underline{y'} \\
&\Rightarrow \quad y' = \frac{\tan y - 2xy \sec\left(x^2\right) \tan\left(x^2\right)}{\sec\left(x^2\right) - x \sec^2 y}.
\end{aligned}$$

Logarithmic Technique.

Example 3. $y = x^{\sin x}$

$$\ln y = \ln\left(x^{\sin x}\right) = \sin x \cdot \ln x \Rightarrow (\ln y)' = (\sin x \cdot \ln x)' = \cos x \ln x + \frac{\sin x}{x} \Rightarrow \frac{y'}{y} = \cos x \ln x + \frac{\sin x}{x}$$

Therefore

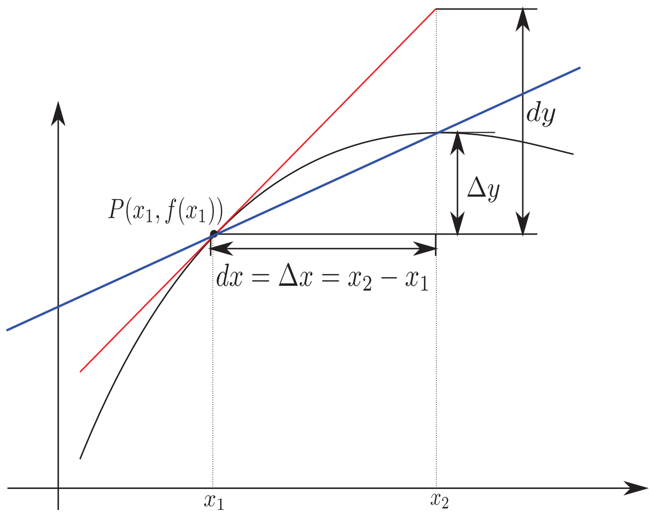
$$y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

Linear Approximation and Differentials.

Definition. (1)The linear approximation of f at a is

$$L(x) = f(a) + f'(a)(x-a).$$

(2) dx, dy are called differentials. If $y = f(x)$, then $dy = f'(x)\, dx$.



Example 4. Let $f(x) = \sqrt{x+3}$, then use linear approximation to find $\sqrt{3.98}$

Solution. Observe that $f(1) = \sqrt{4} = 2$ is close to $f(0.98) = \sqrt{3.98}$, so we pick $a = 1$, then

$$f'(x) = \left(\sqrt{x+3}\right)' = \left(\underbrace{\square^{\frac{1}{2}}}_{\square = x+3}\right)' = \frac{1}{2} \cdot (x+3)^{-\frac{1}{2}} \cdot (x+3)' = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Now,

$$L(x) = f(1) + f'(1)(x-1) = 2 + \frac{x-1}{4} \Rightarrow \sqrt{3.98} = f(0.98) \approx L(0.98) = 2 + \frac{0.98-1}{4} = 1.995.$$

LECTURE 3 APPLICATIONS OF DERIVATIVES

Derivatives in Physics (3.7).

$s = f(t)$ Position function	$v = f'(t)$ velocity function		t stands for time
	$a = f''(t)$ acceleration function		

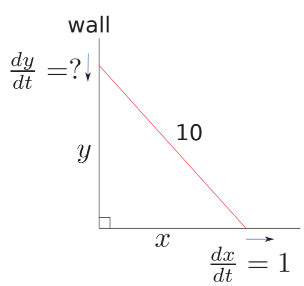
Example 1. The position of a particle is given by $s = f(t) = t^3 - 6t^2 + 9t$, where t is measured in seconds and s in meters.
(a) Find the velocity at time t. $v = f'(t) = 3t^2 - 12t + 9$
(b) What is the velocity after 2 s? $f'(2) = 3 * 2^2 - 12 * 2 + 9 = -3$, where the minus sign means the direction is negative.
(c) When is the particle at rest? $0 = v = f'(t) = 3\left(t^2 - 4t + 3\right) = 3(t-1)(t-3) \Rightarrow t = 1$ and $t = 3$.
(d) When is the particle moving forward? $v > 0 \Rightarrow 3(t-1)(t-3) > 0 \Rightarrow 0 \le t < 1$ and $t > 3$.
(e) Find the total distance traveled during the first five seconds.

$$\begin{cases} |f(1) - f(0)| = |4 - 0| = 4 \\ |f(3) - f(1)| = |0 - 4| = 4 \\ |f(5) - f(3)| = |20 - 0| = 20 \end{cases} \Rightarrow 4 + 4 + 20 = 28\,(\mathrm{m}).$$

(f) Find the acceleration at time t. $a = f''(t) = 6t - 12$.

Related Rates: Two things are related by simple geometric/physical formula. Then use chain rule (implicit differentiation) to find the "related rates".
KEY WORDS: speed, fast, rate, etc.
The problem will give you one (or more than one) rate or speed, and the question is asing you other rate.
Steps:
1. Read the problem and find key words to figure it out that this is about related rates.
2. Draw a diagram
3. Introduce variables and DO NOT PLUG IN ANY DATA.
4. Find the relation between (among) variables and use implicit differentiation to find the related rates' relation
5. Now, plug in the data to find the final answer.

Example 2. A 10-ft long ladder rests against a vertical wall. Its bottom slides away from the wall at rate 1ft/s, how fast is the top sliding down when the bottom is 6 ft from the wall?



Solution. As we known

$$x^2 + y^2 = 10^2 = 100.$$

Differentiate both sides WITH RESPECT TO time t to get

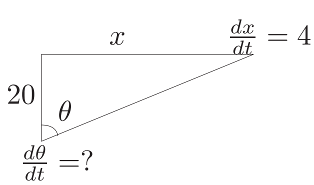
$$\frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$, $y = 8$, then

$$\frac{dy}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4}.$$

The negative sign means y is decreasing.

Example 3. A man walks along a straight path at speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Solution. By geometry

$$\tan\theta = \frac{x}{20} \Rightarrow x = 20\tan\theta.$$

Differentiate WITH RESPECT TO time t to get

$$\frac{dx}{dt} = 20\sec^2\theta \cdot \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{20\sec^2\theta} = \frac{dx}{dt} \cdot \frac{1}{20}\cos^2\theta.$$

Now, $dx/dt = 4$ and when $x = 15$, $\cos\theta = \frac{4}{5}$. Then

$$\frac{d\theta}{dt} = 4 \cdot \frac{1}{20} \cdot \left(\frac{4}{5}\right)^2 = \frac{16}{125}.$$

Exponential Growth & Decay. KEY: "Rate of change is proportional to the size"

$$\frac{dy}{dx} = ky, \text{ where } \begin{cases} k > 0 & \text{law of natural growth} \\ k < 0 & \text{law of natural decay} \end{cases}$$

Theorem. *The solution to* $\frac{dy}{dx} = ky$ *is*

$$y(x) = y(0) e^{kt}.$$

In concrete

$$\begin{cases} \textit{Population Growth} & P(t) = P(0) e^{kt}, k > 0 \\ \textit{Radioactive Decay} & m(t) = m_0 e^{kt}, k < 0 \\ & \textit{Half life:} t_{HL} = \frac{\ln\left(\frac{1}{2}\right)}{k} \\ \textit{Newton's Law of Cooling} & T(t) = T_s + (T(0) - T_s) e^{kt},\ k < 0 \\ & T_s : \textit{Surrounding Temperature} \end{cases}$$

Example 4. [Population Growth] Consider that

Year	World Population
1950	2560 Million
1960	3040 M

[Question]: What is the population in the year 2020?
Solution. Define the function of population by $P(t)$,

$$P(t) = P(0) e^{kt}.$$

Intepretate the table to get that

$$t = 0 \leftrightarrow \text{Year } 1950 \leftrightarrow P(0) = 2560 \Rightarrow P(t) = 2560 e^{kt}.$$

and

$$t = 10 \leftrightarrow \text{Year } 1960 \leftrightarrow P(10) = 3040.$$

So,

$$3040 = P(10) = 2560 e^{k10} \Rightarrow 10k = \ln\frac{3040}{2560} \Rightarrow k = \frac{1}{10}\ln\frac{3040}{2560} \approx 0.017.$$

Now,

$$P(70) = 2560 e^{70k} \approx 8524 \text{ Million}.$$

Example 5. [Radioactive decay]
$m(t)$= mass remaining at time t; $m_0 = m(0)$;
Half-life. Time required for any quantity decay to 1/2, denoted by t_{HL}

$$m(t) = m_0 e^{kt},\ t < 0.$$

Then

$$m(t_{HL}) = \frac{1}{2} m_0 = m_0 e^{kt_{HL}} \Leftrightarrow \frac{1}{2} e^{kt_{HL}} \Leftrightarrow t_{HL} = \frac{1}{k}\ln\frac{1}{2} = \frac{1}{k}\ln\left(2^{-1}\right) = -\frac{\ln 2}{k}.$$

Radium-226 has $t_{HL} = 1590$ years and we have a sample of 100 mg.
[Q]What is the mass after 1000 years? How long does it take to decay to 30 mg?
Solution. As we computed

$$1590 = t_{HL} = -\frac{\ln 2}{k} \Rightarrow k = -\frac{\ln 2}{1590}.$$

Then

$$m(t) = m_0 e^{kt} = 100 e^{-\frac{\ln 2}{1590} t}.$$

(1)$m(1000) = 100 e^{-\frac{1590}{\ln 2}\cdot 1000} \approx 65$ mg
(2)We need to solve

$$30 = m(t) = 100 e^{-\frac{\ln 2}{1590} t} \Leftrightarrow \frac{3}{10} = e^{-\frac{\ln 2}{1590} t} \Leftrightarrow -\frac{\ln 2}{1590} t = \ln\frac{3}{10} \Rightarrow t \approx 2762.$$

Example 6. [Newton's Law of Cooling]
$T(t)$= temperature at time t; T_s=surrounding temperature. Then

$$\frac{dT(t)}{dt} = k(T(t) - T_s) \Rightarrow T(t) = T_s + (T(0) - T_s) e^{kt},\ k < 0.$$

Suppose we have a bottle of soda at 72° F and placed in a refrigerator of temperatue 44°F. Then, we know that

$$T(0) = 72 \text{ and } T_s = 44.$$

Suppose we also know that after 30 minutes, the bottle becomes 61°F. Thus

$$61 = T(30) = 44 + (72 - 44) e^{k30} \Rightarrow k = \frac{1}{30}\ln\left(\frac{17}{28}\right) \approx -0.01663.$$

[Question]What will happen after an hour?
Solution.

$$T(60) = 4 + (72 - 44) e^{k60} \approx 56.3.$$

[Question]How long does it take to drop the tempurature to 50° F?
Solution.

$$50 = T(t) = T(60) = 4 + (72 - 44) e^{kt} \Rightarrow t \approx 92.6$$

LECTURE 4 MAX & MIN

	1st Derivative f'	2nd Derivative f''
Sign	Increasing/Decreasing Test: $\begin{cases} f' > 0 & \nearrow \\ f' < 0 & \searrow \end{cases}$	Concavity Test: $\begin{cases} f'' > 0 & \text{Concave Upwards } (y = x^2) \\ f'' < 0 & \text{Concave Downwards } (y = \ln x) \end{cases}$
Zeros	Critical Points: Point x such that $f'(x) = 0$ or $f'(x)$ DNE	Inflection Points: Point x such that $f''(x) = 0$ and f'' changes sign
at x	1st Derivative Test: $\begin{cases} f' : + \mapsto - : & \text{local max} \\ f' : - \mapsto + : & \text{local min} \end{cases}$	2nd Derivative Test: $\begin{cases} f' = 0, f'' < 0 : & \text{local max} \\ f' = 0, f'' > 0 & \text{local min} \end{cases}$

Example 1. $f(x) = x^4 - 4x^3$, where $-1 \le x \le 5$

(1)Find all local/global max/min: $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$, then $f'(x) = 0 \Rightarrow x = 0$ and $x = 3$. Thus, possible local max or min are points

$$x = -1, 0, 3, 5$$

1st D Test:

	$x=-1$	$(-1,0)$	$x=0$	$(0,3)$	$x=3$	$(3,5)$	$x=5$
f'		$-$	0	$-$	0	$+$	
f	5	$\searrow$	0, Nothing	$\searrow$	-27, Min	$\searrow$	125

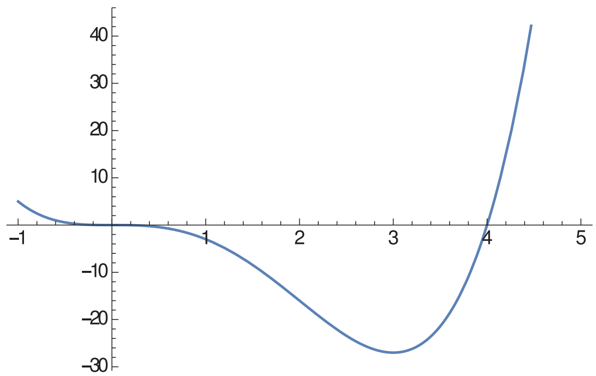
2nd D Test: $f''(x) = 12x^2 - 24x$

	$x=0$	$x=3$
f''	0	$66 > 0$
f	Nothing	Min

(2)Find all inflection points and intervals of concavity.

$f''(x) = 12x^2 - 24x = 12x(x-2) \Rightarrow f''(x) = 0$ has two solutions $x = 0$ and $x = 2$. Now

	$[-1,0)$	$x=0$	$(0,2)$	$x=2$	$(2,5]$
f''	$+$	0	$-$	0	$+$
f	CU	0 IP	CD	-16 IP	CU



Curve Sketching. **Guidelines**: Given a function $y = f(x)$

A. Domain

B. Intercepts:

$$\begin{cases} \bullet x\text{-intecept } (x, 0) & f(x) = 0 \\ \bullet y - \text{intecept } (0, y) & f(0) = y \end{cases}$$

C. Symmetry:

$$\begin{cases} \bullet \text{Even} : f(-x) = f(x) & \text{Example: } y = x^2 \\ \bullet \text{Odd} : f(-x) = -f(x) & \text{Example: } y = 3x^3 - x \\ \bullet \text{Periodic} : f(x+p) = f(x) & \text{Example: } y = \sin x, p = 2\pi \end{cases}$$

D. Asymptotes:

$$\begin{cases} \bullet \text{Vertical: } \lim_{x \to a^\pm} f(x) = \pm\infty & \text{Example: } y = \frac{1}{x} \Rightarrow \begin{cases} \lim_{x \to 0^+} \frac{1}{x} = \infty \\ \lim_{x \to 0^-} \frac{1}{x} = -\infty \end{cases} \\ \bullet \text{Horizontal: } \lim_{x \to \pm\infty} f(x) = L & \text{Example: } y = \frac{1}{x} \Rightarrow \lim_{x \to \infty} \frac{1}{x} = 0 = \lim_{x \to -\infty} \frac{1}{x} \\ \bullet \text{Slant} : \lim_{x \to \pm\infty} [f(x) - (mx + b)] = 0 & \text{KEY: Only happens when } \frac{p(x)}{q(x)} \ deg(p) - deg(q) = 1. \end{cases}$$

The slant asymptote is $y = mx + b$.

E. Interval of Increase or Decrease: [Increase/Decrease Test: Increase $f' > 0$; Decrease $f' < 0$]

F. Local Maximum and Minimum Value[First or Second Derivative Test]

G. Concavity and Points of Inflection: [Concave Upward: $f'' > 0$; Concave Downward $f'' < 0$]

H. DRAW!!!

Example. $f(x) = \frac{x^2}{\sqrt{x+1}}$

A. Domain: $dom(f) = \{x | x+1 > 0\} = (-1, \infty)$.

B. Intercepts:

$$\begin{cases} f(0) = 0 \Rightarrow y\text{-intercept is } 0 \text{ or } (0,0) \\ f(x) = 0 \Rightarrow x = 0 \Rightarrow x\text{-intercept is } 0 \text{ or } (0,0) \end{cases}$$

C. Symmetry:

$$\begin{cases} f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq f(x) & \text{NOT Even} \\ f(-x) = \frac{(-x)^2}{\sqrt{-x+1}} = \frac{x^2}{\sqrt{1-x}} \neq -f(x) & \text{NOT Odd} \\ \text{No Trig Part} & \text{NOT Periodic} \end{cases}$$

D. Asymptotes:

$$\begin{cases} \lim_{x \to -1^+} f(x) = \frac{1}{0^+} = \infty \\ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 \cdot \frac{1}{\sqrt{x}}}{\sqrt{x+1} \cdot \frac{1}{\sqrt{x}}} = \lim_{x \to \infty} \frac{x^{3/2}}{\sqrt{1 + \frac{1}{\sqrt{x}}}} = \infty \\ deg(numerator) = 2, deg(denominator) = 1/2 \quad \text{No Slant Asymptote} \end{cases}$$

E. Intervals of Increase or Decrease.

$$f'(x) = \frac{\left(x^2\right)' \sqrt{x+1} - x^2 \left((x+1)^{\frac{1}{2}}\right)'}{x+1} = \frac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}.$$

So,

$$f'(x) = 0 \Rightarrow x = 0, -\frac{4}{3} < -1, (\text{Not in the domain})$$

Intervals	$(-1,0)$	$x=0$	$(0,\infty)$
f'	$-$	0	$+$
f	$\searrow$	Min	$\nearrow$

F. Local Max and Min: $(0,0)$ is the only local min and there is no local max.

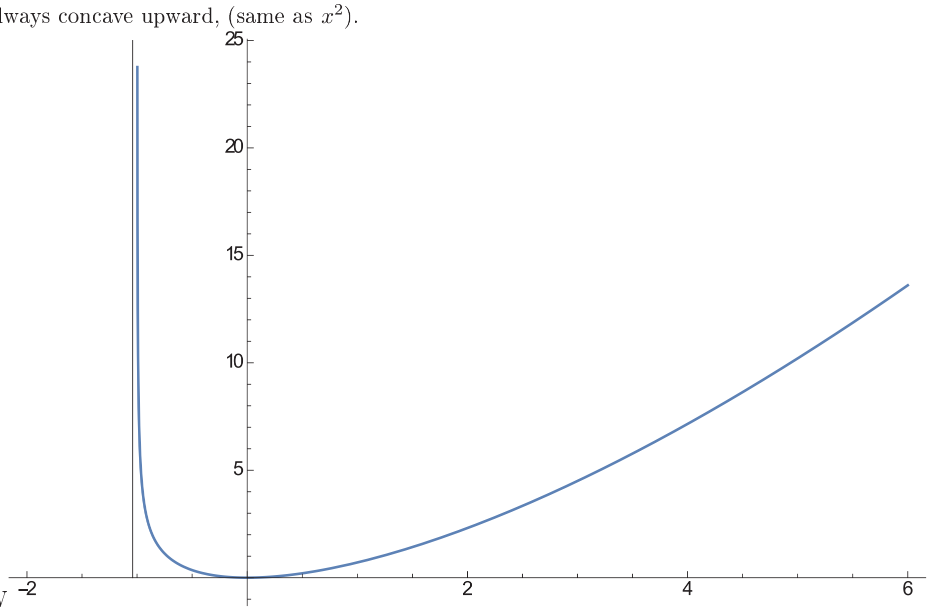
G. Concavity and Inflection Points

$$f''(x) = \frac{3x^2 + 8x + 8}{4(x+1)^{\frac{5}{2}}}.$$

The denominator is positive because of the square root while the numerator is always positive as well, since

$$3x^2 + 8x + 8 = 3x^2 + 8x + \frac{16}{3} + \frac{8}{3} = 3\left(x + \frac{4}{3}\right)^2 + \frac{8}{3}.$$

Thus, f is always concave upward, (same as x^2).



H. DRAW

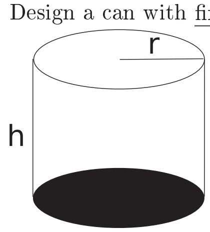
LECTURE 5 OPTIMIZATION & NEWTON'S METHOD

Optimization. STEPS:

1. Understand the Problem
2. Draw a Diagram (If Necessary)
3. Introduce Notations.
4. Establish the Function
5. Eliminate Extra Variables (If Needed)
6. Solve for Absolute/Global Max or Min

Example 1. A cylinder can is to made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

1. Design a can with fixed volume and minimal area of surface



2.
3. V: Volume $\left(cm^3\right)$; A: Area $\left(cm^2\right)$; r: Radius (cm); h: Height (cm)
4.

$$\begin{cases} V = \pi r^2 h & = 1000 \\ A = 2\pi r h + \pi r^2 \cdot 2 & [\text{Top, Bottom, Side}] \end{cases}$$

5. Since we have

$$h = \frac{1000}{\pi r^2},$$

$$A\left(r\right) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2\pi r^2 + \frac{2000}{r}.$$

6.

$$A'\left(r\right) = 4\pi r - \frac{2000}{r^2} \Rightarrow A'\left(r\right) = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}.$$

Note that

$$A''\left(r\right) = 4\pi + \frac{4000}{r^3} \Rightarrow A''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0.$$

Second derivative test shows that it is the local min. Also

$$h = \frac{1000}{\pi\left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

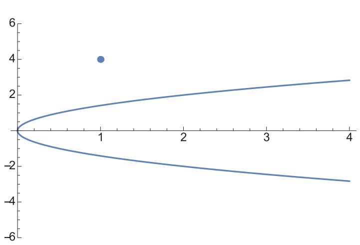
Thus the dimension is that

$$radius = \sqrt[3]{\frac{500}{\pi}}\ cm \text{ and } height = 2\sqrt[3]{\frac{500}{\pi}}\ cm$$

Example 2. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

1. Find the closed distance and recall that for $P\left(x, y\right)$ and $Q\left(s, t\right)$

$$|PQ| = \sqrt{\left(x-s\right)^2 + \left(y-t\right)^2}.$$



2.
3. Point on the parabola is (x, y) and the distance is d and $D = d^2$
4. We know that

$$D = d^2 = \left(x-1\right)^2 + \left(y-4\right)^2$$

5. Since $y^2 = 2x \Leftrightarrow x = \frac{y^2}{2}$, we have

$$D(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2 = \frac{y^4}{4} - 8y + 17.$$

6.

$$D'(y) = y^3 - 8 \Rightarrow D'(y) = 0 \Rightarrow y = 2.$$

Also,

$$D''(y) = 3y^2 \Rightarrow D''(2) = 12 > 0.$$

Thus, $y = 2$ and $x = \frac{y^2}{2} = 2$ is the minimum. Thus the point closest to $(1,4)$ is $(2,2)$.

Newton's Method. Question: How to find a solution of $f(x) = 0$ for complicated f. At least, an approximate root is acceptable.

Steps:

(1)Choose x_1

(2)Follow the recurrence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

and so on

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(3)The limit

$$r = \lim_{n\to\infty} x_n$$

is the root, i.e.

$$f(r) = 0.$$

Example 3. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation

$$x^3 - 2x - 5 = 0.$$

Solution. $f'(x) = 3x^2 - 2$, then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

(1)$x_1 = 2$

(2)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = \frac{21}{10}$$

and

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.0946.$$

LECTURE 6 INDEFINITE INTEGRALS AND ANTIDERIVATIVES

Table of Indefinite Integrals
$\int cf(x)\,dx = c\int f(x)\,dx$
$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$
$\int \tan x dx = \ln\lvert\sec x\rvert + C$
$\int \sec^2 x dx = \tan x + C$
$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$
$\int \csc x \cot x dx = -\csc x + C$
$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

Definition. A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x)$$

Definition. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C,$$

where C is an arbitary constant.

Definition. [Indefinite Integral] The operation for finding the most general antiderivative:

$$\int f(x)\,dx = F(x) + C \text{ or equivalently } F(x) = \int f(x)\,dx + C,$$

where F is any particular antiderivative of f.

Example 1. Find the indefinite integral

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 10\int x^4 dx - 2\int \sec^2 x dx.$$

By the table,

$$\begin{cases} \int x^4 dx = \frac{x^{4+1}}{4+1} + C_1 = \frac{x^5}{5} + C_1 & , \\ \int \sec^2 x dx = \tan x + C_2 & . \end{cases}$$

Thus,

$$\int \left(10x^4 - 2\sec^2 x\right) dx = 2x^5 - 2\tan x + \underline{C}.$$

Example 2. Find f if $f'(x) = e^x + 20\left(1+x^2\right)^{-1}$ and $f(0) = -2$.

Solution. Since

$$f'(x) = e^x + 20\frac{1}{1+x^2},$$

f has the form

$$f(x) = \int f'(x)\,dx = e^x + 20\tan^{-1} x + C$$

for some constant C. Also,

$$-2 = f(0) = e^0 + 20\tan^{-1} 0 + C = 1 + C \Rightarrow C = -3.$$

Therefore,

$$f(x) = e^x + 20\tan^{-1} x - 3.$$

Change of Variables:

Recall that $f(g(x))' = f'(g(x)) \cdot g'(x)$, therefore $\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$. [Counterpart of the Chain Rule]

STEPS:

1. Choose the right substitution $u = g(x)$
2. Calculate the differentials: $du = g'(x)\,dx$
3. Replace $g'(x)\,dx = du$ and rewrite the function(integrand) as a function of u
4. Get the result in terms of u.
5. Substitute u by $g(x)$

Example 3. Compute

$$\int 2x\sqrt{x^2+1}dx$$

1. $u = g(x) = x^2 + 1$
2. $du = g'(x)\,dx = 2xdx$
3. $$\int 2x\sqrt{x^2+1}dx = \int \left(x^2+1\right)^{\frac{1}{2}} \underbrace{2xdx} = \int u^{\frac{1}{2}} du$$
4. $$\int 2x\sqrt{x^2+1}dx = \int u^{\frac{1}{2}} du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}u^{\frac{3}{2}} + C.$$

5.
$$\int 2x\sqrt{x^2+1}dx = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C$$

Example 4. Compute
$$\int x^3 \cos\left(x^4+2\right)dx.$$

1. $u = g(x) = x^4 + 2$
2. $du = g'(x) = 4x^3 dx$.
3. [Important Trick]
$$\int x^3\cos\left(x^4+2\right)dx = \int \cos\left(x^4+2\right)x^3dx = \int \cos\left(x^4+2\right)\cdot\frac{4}{4}\cdot x^3dx = \frac{1}{4}\int \cos u \cdot du.$$
4.
$$\int x^3\cos\left(x^4+2\right)dx = \frac{1}{4}\int \cos u\cdot du = \frac{1}{4}\sin u + C$$
5.
$$x^3\cos\left(x^4+2\right)dx = \frac{1}{4}\sin u + C = \frac{1}{4}\sin\left(x^4+2\right) + C$$

Example 5. Compute
$$\int \sqrt{2x+1}dx$$

Let $u = 2x+1$, then $du = 2dx$. Now
$$\int \sqrt{2x+1}dx = \int \sqrt{2x+1}\cdot\frac{2}{2}dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\cdot\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

Example 6. Find
$$\int \frac{x}{\sqrt{1-4x^2}}dx$$

Let $u = 1-4x^2$, then $du = -8xdx$. Thus,
$$\int \frac{x}{\sqrt{1-4x^2}}dx = \int \frac{x}{\sqrt{1-4x^2}}\cdot\frac{-8}{-8}\cdot dx = -\frac{1}{8}\int u^{-\frac{1}{2}}du = -\frac{1}{8}\cdot 2u^{\frac{1}{2}} + C = -\frac{1}{4}\sqrt{1-4x^2} + C.$$

Example 7. Evaluate
$$\int e^{5x}dx$$

Let $u = 5x \Rightarrow du = 5dx$, then
$$\int e^{5x}dx = \int e^{5x}\cdot\frac{5}{5}\cdot dx = \frac{1}{5}\int e^u du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C.$$

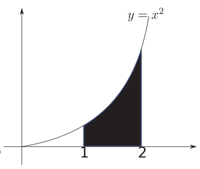
Example 8. Calculate
$$\int \tan x dx$$

Since $\tan x = \frac{\sin x}{\cos x}$ and recall that $(\cos x)' = \sin x$, we try that $u = \cos x$ and $du = -\sin x dx$. Then
$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{1}{\cos x}(-\sin x)dx = -\int u^{-1}du = -\ln|u| + C = -\ln|\cos x| + C.$$

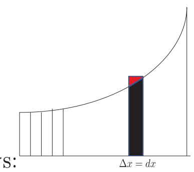
In addition,
$$-\ln|\cos x| = \ln|\cos|^{-1} = \ln\frac{1}{|\cos x|} = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|.$$

Therefore, we have a new formula that
$$\int \tan x dx = \ln|\sec x| + C.$$

LECTURE 7 DEFINITE INTEGRALS



DEF. & Properties. [Question] What is the area of the shadow region?



[Answer]:<u>This is a very important approach</u>. We try to cut the region vertically as follows:
Now we concentrate on each SLICE, which is a rectangle.

$$\begin{cases} \text{width} & = \Delta x = dx \\ \text{height} & = f(x), \text{ for some } x \text{ in this interval} \end{cases}.$$

Thus,

$$\text{Area} \approx \sum \text{rectangles} = \sum f(x)\, dx.$$

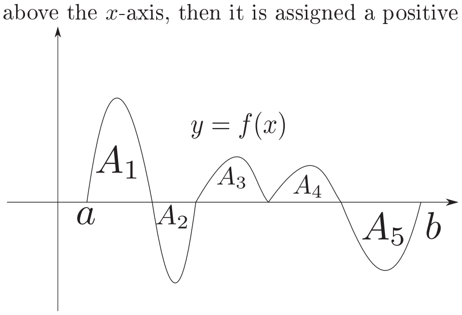
Now taking the limit that $\Delta x = dx \to 0$ then the error goes away and by the following notation

$$\text{Area} = \lim_{dx \to 0} \sum f(x)\, dx = \int_1^2 f(x)\, dx.$$

Definition. [Definite Integral] The definite integral

$$\int_a^b f(x)\, dx$$

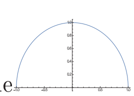
is the SIGNED area formed by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis. The signs are assigned in the way that if part of the region is above the x-axis, then it is assigned a positive sign; while if it is under the x-axis, it has negative sign. For example.



Then,

$$\int_a^b f(x)\, dx = A_1 - A_2 + A_3 + A_4 - A_5.$$

Example 1. $\int_{-1}^{1} \sqrt{1 - x^2} dx$.

Consider $f(x) = \sqrt{1 - x^2}$, the graph is a semicircle.

Thus,

$$\int_{-1}^{1} \sqrt{1 - x^2} dx = \text{Area} = \frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}.$$

Properties:

$$\begin{cases} 1. & \int_a^b c dx = c(b-a) \ \textbf{for any constant } c \\ 2. & \int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx \\ 3. & \int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx \ \textbf{for any constant } c \\ 4. & \int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx. (\star) \\ 5. & \textbf{If } f(x) \geq 0 \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\, dx \geq 0 \\ 6. & \textbf{If } f(x) \geq g(x) \ \textbf{ for } a \leq x \leq b\textbf{, then } \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx \\ 7. & \textbf{If } m \leq f(x) \leq M \textbf{ for } a \leq x \leq b\textbf{, then } m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a) \end{cases}$$

.

Fundamental Theorem of Calculus.

Theorem. *Suppose f is continuous on $[a,b]$.*

(1)The function defined by

$$g(x)=\int_a^x f(t)\,dt \quad a\le x\le b$$

is continuous on $[a,b]$ and differentiable on (a,b) and

$$g'(x)=f(x).$$

(2) Let F be any antiderivative of f, i.e. $F'=f$, then

$$\int_a^b f(x)\,dx=F(b)-F(a)=F(x)\Big|_{x=a}^{x=b}.$$

Example 2. $\int_{-1}^{2}\left(x^2+|x-1|\right)dx=?$

$$\begin{aligned}\int_{-1}^{2}\left(x^2+|x-1|\right)dx &= \int_{-1}^{1}\left(x^2+|x-1|\right)dx+\int_{1}^{2}\left(x^2+|x-1|\right)dx\\ &= \int_{-1}^{1}\left(x^2+1-x\right)dx+\int_{1}^{2}\left(x^2+x-1\right)dx\\ &= \left(\frac{x^3}{3}+x-\frac{x^2}{2}\right)\Big|_{x=-1}^{x=1}+\left(\frac{x^3}{3}+\frac{x^2}{2}-x\right)\Big|_{x=1}^{x=2}\\ &= \left[\left(\frac{5}{6}\right)-\left(-\frac{11}{6}\right)\right]+\left[\frac{8}{3}-\left(-\frac{1}{6}\right)\right]=\frac{11}{2}.\end{aligned}$$

Example 3. (1) $S(x)=\int_0^x \sin\left(\pi t^2/2\right)dt \Rightarrow S'(x)=\sin\left(\pi x^2/2\right)$

(2) $\frac{d}{dx}\int_1^{x^4}\sec t dt=\left(\underbrace{\int_1^{\square}\sec t dt}_{\square=x^4}\right)'=\sec(\square)\cdot\square'=\sec\left(x^4\right)\cdot\left(x^4\right)'=4x^3\sec\left(x^4\right).$

Remark. $g(x)=\int_a^{h(x)}f(t)\,dt\Rightarrow g'(x)=f(h(x))\cdot h'(x)$.

Change of Variables.

Example 4. Evaluate $\int_0^4\sqrt{2x+1}dx$

Let $u=g(x)=2x+1$, then

$$\begin{cases}du=g'(x)\,dx=2xdx & ,\\ g(0)=1 & ,\\ g(4)=9 & .\end{cases}$$

Thus,

$$\int_0^4\sqrt{2x+1}dx=\int_0^4\sqrt{2x+1}\cdot\frac{2}{2}\cdot dx=\frac{1}{2}\int_1^9 u^{\frac{1}{2}}du=\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=9}=\frac{1}{3}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)=\frac{26}{3}.$$

Last Theorem:

Theorem. *Let $A>0$.*

(1) If $f(x)$ is an odd function, then

$$\int_{-A}^{A}f(x)\,dx=0.$$

(2) If $f(x)$ is an even function, then

$$\int_{-A}^{A}f(x)\,dx=2\int_0^A f(x)\,dx.$$

Example 5.

$$\int_{-\pi}^{\pi}\frac{\sin t}{1+t^{2024}}dt=0.$$

6.2 VOLUMES

What is the volume obtained by rotating a region R about a straight line L?
Usually, L is parallel to either x-axis or y-axis, i.e., either vertical or horizontal.

Example 1. Find the volume of the solid obtained by rotation about the x-axis the region under $y = \sqrt{x}$ from 0 to 1.

1. Because L is the x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.
We draw the diagram
2. Now, each slice becomes a "pie" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the radius of the disc} = y & = f(x) = \sqrt{x} \end{cases}$$

3. We see the small pie has a volume

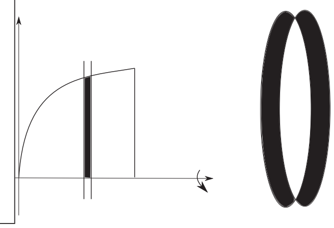
$$V_{pie} = \pi \cdot [f(x)]^2 \, dx = \pi x dx.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{pie}$$

5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{pie} = \int_0^1 \pi x dx = \pi \int_0^1 x dx = \pi \frac{x^2}{2}\bigg|_{x=0}^{x=1} = \frac{\pi}{2}.$$



Example 2. Find the volume of the solid obtained by roatating the region bouned by $y = x^3$, $y = 8$, $x = 0$ about the y-axis.

1. Because L is the y-axis, horizontal, we first cut the region R along the y-axis into vertical slices.
We draw the diagram
2. Now, each slice becomes a "pie" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dy \\ \text{the radius of the disc} = x & = y^{\frac{1}{3}} \end{cases}$$

3. We see the small pie has a volume

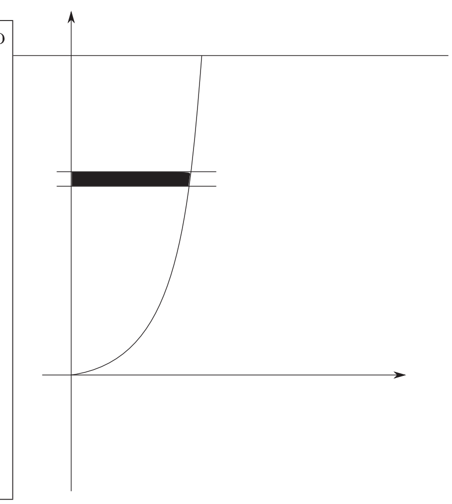
$$V_{pie} = \pi \cdot \left[y^{\frac{1}{3}}\right]^2 dy = \pi y^{\frac{2}{3}} dy.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{pie}$$

5. Take the limit such that the thickness of each slice is getting very small, then

$$V = \lim \sum V_{pie} = \int_0^8 \pi y^{\frac{2}{3}} dy = \pi \int_0^8 y^{\frac{2}{3}} dy = \pi \frac{y^{\frac{5}{3}}}{\frac{5}{3}}\bigg|_{y=0}^{y=8} = \frac{96\pi}{5}.$$



Example 3. Find the volume of the solid obtained by roatating the region bouned by $y = x$, $y = x^2$ about the x-axis.

1. Because L is the x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.
We draw the diagram
2. Now, each slice becomes a "donat" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the side is a ring rather than a disc with} & \begin{cases} \text{inner radius} & = x^2 \\ \text{outer radius} & = x \end{cases} \end{cases}$$

3. We see the small pie has a volume
$V_{donat} = (\text{Area of the ring}) \cdot \text{thickness} = \pi \left[(\text{outer})^2 - (\text{inner})^2 \right] dx = \pi \left(x^2 - x^4 \right) dx.$
4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{donat}$$

5. Take the limit such that the thickness of each slice is getting very small, then

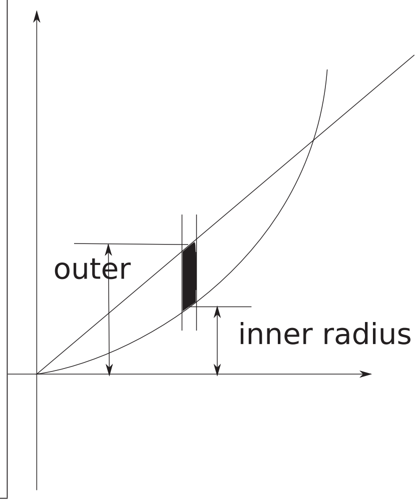
$$V = \lim \sum V_{donate} = \int_a^b \pi \left(x^2 - x^4 \right) dx.$$

6. a and b are determined by the intersections. Solve

$$\begin{cases} y = x^2 \\ y = x \end{cases} \Rightarrow x^2 = x \Rightarrow x = 0, 1 \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$$

7.

$$V = \int_0^1 \pi \left(x^2 - x^4 \right) dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Bigg|_{x=0}^{x=1} = \frac{2\pi}{15}.$$



Example 4. Find the volume of the solid obtained by roatating the region bouned by $y = x$, $y = x^2$ about the line $y = 2$.

1. Because L is parallel to x-axis, horizontal, we first cut the region R along the x-axis into vertical slices.
We draw the diagram
2. Now, each slice becomes a "donat" with:

$$\begin{cases} \text{thickness=thickness of slice} & = dx \\ \text{the side is a ring rather than a disc with} & \begin{cases} \text{inner radius} & = 2 - x \\ \text{outer radius} & = 2 - x^2 \end{cases} \end{cases}$$

3. We see the small pie has a volume

$$V_{donat} = \pi \left[(\text{outer})^2 - (\text{inner})^2 \right] \cdot \text{thickness} = \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx.$$

4. We shall sum the volumes of all small pies obtained from all the slices, so

$$V \approx \sum V_{donat}$$

5. Take the limit such that the thickness of each slice is getting very small, then

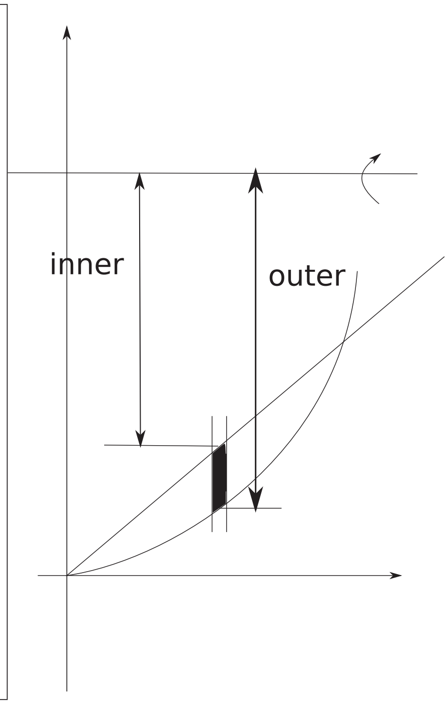
$$V = \lim \sum V_{donat} = \int_a^b \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx.$$

6. a and b are determined by the intersections. Solve

$$\begin{cases} y = x^2 \\ y = x \end{cases} \Rightarrow x^2 = x \Rightarrow x = 0, 1 \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases}$$

7.

$$V = \int_0^1 \pi \left[\left(2 - x^2\right)^2 - (2 - x)^2 \right] dx = \pi \int_0^1 \left(x^4 - 5x^2 + 4x \right) = \frac{8\pi}{15}.$$



6.4 WORK

Definition. Work=Force×Distance. Unit: Newton-Meter=Joule and ft-lb.

Example 1. How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high?

Solution. The force is to counteract the gravity. Thus,

$$F = mg = 1.2 \cdot 9.8 = 11.76\,(N)\,.$$

Therefore

$$W = F \times d = 11.76 \cdot 0.7 = 8.232\,(J)\,.$$

Theorem. *If $F = f(x)$, a function on the position x, then the work done in moving the object from a to b is*

$$W = \int_a^b f(x)\,dx.$$

Example 2. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

Solution.

$$W = \int_1^3 \left(x^2 + 2x\right) dx = \left(\frac{x^3}{3} + x^2\right)\Big|_{x=1}^{x=3} = \frac{50}{3}\,(\text{ft-lb})\,.$$

Example 3. A force of 40N is required to hold a spring that has been stretched from its natural length of 10cm(=0.1m) to a length of 15cm(=0.15m). How much work is done in stretching the spring from 15cm to 18cm(=0.18m)?

Solution. Hooke's Law: $F = kx$ where k is the spring constant and x is the distance away from its NATURE length.
Thus, we have

$$40 = k0.05 \Rightarrow k = 800.$$

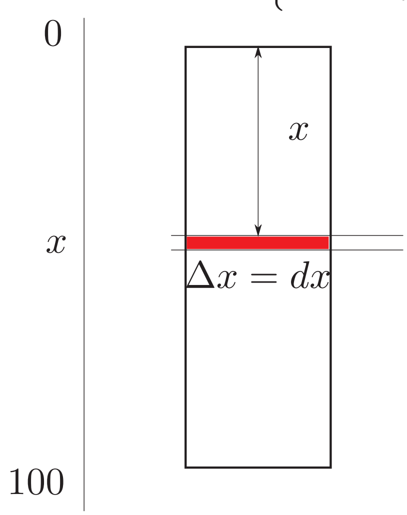
Thus,

$$W = \int_{\underline{0.05}}^{\underline{0.08}} 800x dx = 1.56\,(J)\,.$$

Example 4. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Solution. Density: $\rho = m/L = 200/100 = 2\text{lb/ft}$. We need to "cut" the cable vertically to see that for each small slice of thickness $\Delta x = dx$:

$$\begin{cases} \text{Force=Gravity} = \rho \cdot dx = 2dx \\ \text{Distance} = x \end{cases} \Rightarrow W = \sum 2dx \cdot x = \int_0^{100} 2xdx = 10,000(\text{ft-lb})$$



7.1 INTEGRATION BY PARTS

Recall: The Product Rule

$$(fg)' = f'g + fg' \Rightarrow (fg)' - f'g = fg'$$

Thus,

$$\int fg'dx = \int (fg)'\,dx - \int f'gdx = fg - \int f'gdx.$$

Also, if letting $u = f(x) \Rightarrow du = f'dx$ and $v = g(x) \Rightarrow dv = g'dx$, then

$$\int udv = uv - \int vdu.$$

Example 1. Find $\int x \sin x dx$

Solution. (I)Let $f(x) = x$ and $g(x) = -\cos x$, then $f'(x) = 1$ and $g'(x) = \sin x$.

$$\int x\sin xdx = \int fg'dx = fg - \int f'gdx = x(-\cos x) - \int 1(-\cos x)\,dx = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C.$$

(II) Establish the following table

$$\begin{array}{ll} u = x & dv = \sin xdx \\ du = dx & v = -\cos x \end{array}$$

to get that

$$\int x\sin xdx = x(-\cos x) - \int (-\cos x)\,dx = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C.$$

Remark. If let $u = \sin x \Rightarrow du = \cos xdx$ and $dv = xdx \Rightarrow v = \frac{x^2}{2}$, then

$$\int x\sin xdx = \frac{x^2\cos x}{2} - \int \frac{x^2}{2}\cos xdx,$$

which makes the integral more complicated.

Example 2. Evaluate $\int \ln xdx$

Solution.

$$\begin{array}{ll} u = \ln x & dv = dx \\ du = \frac{dx}{x} & v = x \end{array} \Rightarrow \int \ln xdx = x\ln x - \int x\cdot\frac{dx}{x} = x\ln x - x + C.$$

Example 3. Calculate $\int t^2e^tdt$.

Solution.

$$\begin{array}{ll} u = t^2 & dv = e^tdt \\ du = 2tdt & v = e^t \end{array} \Rightarrow \int t^2e^tdt = t^2e^t - \int e^t2tdt = t^2e^t - 2\int te^tdt.$$

Similar method shows that

$$\begin{array}{ll} u = t & dv = e^tdt \\ du = dt & v = e^t \end{array} \Rightarrow \int te^tdt = te^t - \int e^tdt = te^t - e^t + C.$$

Therefore

$$\int t^2e^tdt = t^2e^t - 2te^t - 2e^t + C.$$

For Definite Integrals:

$$\int fg'dx = fg - \int f'gdx \Rightarrow \int_a^b fg'dx = (fg)\Big|_{x=a}^{x=b} - \int_a^b f'gdx$$

Example 4. $\int_0^1 \tan^{-1} xdx$.

Solution.

$$\begin{array}{ll} u = \tan^{-1} x & dv = dx \\ du = \frac{dx}{1+x^2} & v = x \end{array} \Rightarrow \int_0^1 \tan^{-1} xdx = x\tan^{-1} x\Big|_{x=0}^{x=1} - \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx.$$

To compute $\int_0^1 \frac{x}{1+x^2}dx$, consider the substitution that $w = h(x) = x^2 + 1 \Rightarrow \begin{cases} dw = h'(x)\,dx = 2xdx \\ h(0) = 1 \\ h(1) = 2 \end{cases}$ to get

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^1 \left(1+x^2\right)^{-1} \underline{2xdx} = \frac{1}{2}\int_1^2 w^{-1}dw = \frac{1}{2}\ln|w|\Big|_{w=1}^{w=2} = \frac{\ln 2}{2}.$$

Therefore

$$\int_0^1 \tan^{-1} xdx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

Example 5. [An Interesting Example]$\int e^x \sin x dx$.

Solution. Note that

$$\begin{cases} \begin{matrix} u = e^x & dv = \sin x dx \\ du = e^x du & v = -\cos x \end{matrix} & \Rightarrow \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx, \\ \begin{matrix} u = e^x & dv = \cos x dx \\ du = e^x du & v = \sin x \end{matrix} & \Rightarrow \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx. \end{cases}$$

Thus,

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx == -e^x \cos x + e^x \sin x - \int e^x \sin x dx \Rightarrow \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

7.2 TRIGONOMETRIC INTEGRALS

Type I $\int \sin^m x \cos^n x dx$**.**

$$\begin{cases} \int \sin^m x \cos^{2k+1} x dx & = \int \sin^m x \left(\cos^2 x\right)^k \cos x dx \overset{u=\sin x}{=\!=\!=\!=} \int u^m \left(1-u^2\right)^k du \\ \int \sin^{2k+1} x \cos^n x dx & = \int \left(\sin^2 x\right)^k \cos^n x \sin x dx \overset{u=\cos x}{=\!=\!=\!=} -\int \left(1-u^2\right)^k u^n du \\ \int \sin^{2k} x \cos^{2l} x dx & \text{Use } \begin{cases} \sin^2 x = \frac{1}{2}\left(1-\cos(2x)\right) \\ \cos^2 x = \frac{1}{2}\left(1+\cos(2x)\right) \\ \sin x \cos x = \frac{1}{2}\sin(2x) \end{cases} \end{cases}$$

Example 1. Evaluate $\int \sin^5 x \cos^2 x dx$

Solution. Let $u = \cos x \Rightarrow du = -\sin x dx$

$$\begin{aligned} \int \sin^5 x \cos^2 x dx &= -\int \left(\sin^2 x\right)^2 \cos^2 x \underline{(-\sin x dx)} \\ &= -\int \left(1-u^2\right)^2 u^2 du = \int \left(-u^6 + 2u^4 - u^2\right) du \\ &= -\frac{u^7}{7} + \frac{2}{5}u^5 - \frac{u^3}{3} + C \\ &= \frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C. \end{aligned}$$

Example 2. Find $\int_0^\pi \sin^4 x dx$

Solution.

$$\begin{aligned} \int_0^\pi \sin^4 x dx &= \int_0^\pi \left(\sin^2 x\right)^2 dx = \int_0^\pi \left(\frac{1-\cos(2x)}{2}\right) dx \\ &= \frac{1}{4}\int_0^\pi \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx \\ &= \frac{1}{4}\int_0^\pi \left[1 - 2\cos(2x) + \frac{1}{2}\left(1+\cos(4x)\right)\right] dx \\ &= \frac{1}{4}\left(x - \sin(2x) + \frac{x}{2} + \frac{1}{8}\sin(4x)\right)\Bigg|_{x=0}^{x=\pi} \\ &= \frac{3}{8}\pi \end{aligned}$$

Type II $\int \tan^m x \sec^n x dx$

$$\begin{cases} \int \tan^m x \sec^{2k} x dx = \int \tan^m \left(\sec^2 x\right)^{k-1} \sec^2 x dx \overset{u=\tan x}{=\!=\!=\!=} \int u^m \left(1+u^2\right)^{k-1} du. \\ \int \tan^{2k+1} x \sec^n x dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x \cdot \tan x \sec x dx \overset{u=\sec x}{=\!=\!=\!=} \int \left(u^2-1\right)^k u^{n-1} du \end{cases}$$

.

Example 3. Compute $\int \tan^5 \theta \sec^7 \theta d\theta$

Solution. Let $u = \sec\theta \Rightarrow du = \sec\theta \tan\theta d\theta$

$$\begin{aligned} \int \tan^5 \theta \sec^7 \theta d\theta &= \int \left(\tan^2 \theta\right)^2 \sec^6 \theta \underline{\tan\theta \sec\theta d\theta} = \int \left(u^2-1\right)^2 u^6 du \\ &= \int \left(u^{10} - 2u^8 + u^6\right) du = \frac{u^{11}}{11} - \frac{2}{9}u^9 + \frac{u^7}{8} + C \\ &= \frac{1}{11}\sec^{11}\theta - \frac{2}{9}\sec^9\theta + \frac{1}{8}\sec^7\theta + C \end{aligned}$$

Example 4. Recall that $\int \tan x = \ln|\sec x| + C$ through $u = \cos x$. Now consider $\int \sec x dx$

$$\int \sec x dx = \int \sec \frac{\sec x + \tan x}{\sec x + \tan x} dx \overset{u=\sec x + \tan x}{=\!=\!=\!=\!=\!=\!=} \int \frac{du}{u} = \ln|\sec x + \tan x| + C.$$

Example 5. Calculate $\int \tan^3 x dx$

Solution.

$$\int \tan^3 x dx = \int \tan x \tan^2 x dx = \int \tan x \left(\sec^2 x - 1\right) dx = \int \tan x \sec^2 x dx - \int \tan x dx.$$

Since

$$\begin{cases} \int \tan x \sec^2 x dx \overset{u=\tan x}{=\!=\!=\!=\!=} \int u du = \frac{1}{2}\tan^2 x + C & , \\ \int \tan x dx = \ln|\sec x| + C & , \end{cases}$$

we have

$$\int \tan^3 x dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

Example 6. Find $\int \sec^3 x dx$

Solution.

$$\begin{array}{cc} u = \sec x & dv = \sec^2 x dx \\ du = \sec x \tan x dx & v = \tan x \end{array} \Rightarrow \int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

Thus,

$$\int \sec^3 x = \frac{1}{2}\left(\sec x \tan x + \int \sec x dx\right) = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$

Useful Formulas.

$$\begin{cases} \sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \\ \sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\ \cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \end{cases}$$

Example 7. $\int \sin 4x \cos 5x dx$

Solution.

$$\int \sin 4x \cos 5x dx = \frac{1}{2}\int \left[\sin(-x) + \sin 9x\right] dx = \frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + C.$$

7.3 TRIGONOMETRIC SUBSTITUTION

Recall that

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \overset{u=\cos x}{=\!=\!=\!=} -\int \frac{du}{u} = -\ln|u| + C = \ln|\sec x| + C.$$

Important Table

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta \Rightarrow \sqrt{a^2-x^2} = \lvert a\rvert\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta \Rightarrow \sqrt{a^2+x^2} = \lvert a\rvert\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta \Rightarrow \sqrt{a^2-x^2} = \lvert a\rvert\tan\theta$

Example 1. $\int \frac{\sqrt{9-x^2}}{x^2}dx$

Solution. If simply apply $u = 9 - x^2 \Rightarrow du = -2xdx$, then

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{1}{2}\int \sqrt{u}\underline{(x^{-3})}du$$

.

Thus, we try $x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta$, then,

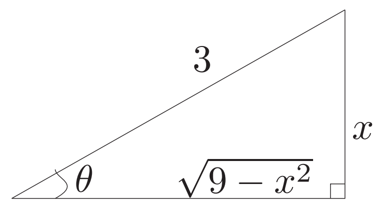
$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta d\theta = \int \cot^2\theta d\theta = \int \left(\csc^2\theta - 1\right) d\theta = -\cot\theta - \theta + C.$$

Now, $x = 3\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ implies $\theta = \sin^{-1}\left(\frac{x}{3}\right)$. How about $\cot\theta$? From the diagram on the right

$$\cot\theta = \frac{\sqrt{9^2-x^2}}{x}.$$

Therefore

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9^2-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C.$$



$$\boxed{\sin\theta = \frac{x}{3}}$$

Example 2. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Solve for y to get

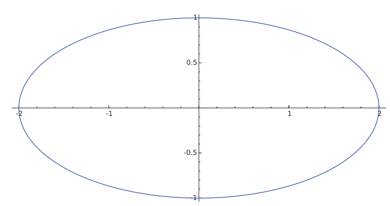
$$y = \pm\frac{b}{a}\sqrt{a^2-x^2} \Rightarrow A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}dx.$$

Now, let $x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$, then

$$\begin{cases} x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0 & , \\ x = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2} & . \end{cases}$$

Thus,

$$\begin{aligned} A &= 4\int_0^{\frac{\pi}{2}} \frac{b}{a}\cdot a\cos\theta\cdot a\cos\theta d\theta = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 2ab\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta \\ &= 2ab\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{2}} = \pi ab. \end{aligned}$$



Example 3. Find $\int \frac{x}{\sqrt{x^2+4}}dx$.

Solution. (I) Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta d\theta$, then

$$\int \frac{x}{\sqrt{x^2+4}}dx = \int \frac{2\tan\theta}{2\sec\theta}\cdot 2\sec^2\theta d\theta = 2\int \tan\theta\sec\theta d\theta = 2\sec\theta + C = 2\cdot\frac{\sqrt{x^2+4}}{2} + C = \sqrt{x^2+4} + C.$$

(II) Let $u = x^2 + 4 \Rightarrow du = 2xdx$, then

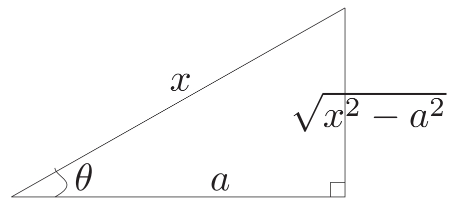
$$\int \frac{x}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{2xdx}{\sqrt{x^2+4}} = \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

Example 4. Formula

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left|x + \sqrt{x^2-a^2}\right| \underbrace{-\ln|a| + C}_{+C} = \ln\left|x + \sqrt{x^2-a^2}\right| + C.$$

Proof. Let $x = a\sec^2\theta \Rightarrow dx = a\sec\theta\tan\theta d\theta$, then

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C.$$



$$\boxed{\sec\theta = \frac{x}{a}}$$

From the diagram on the right, we see

$$\begin{cases} \sec\theta = \frac{x}{a} & , \\ \tan\theta = \frac{\sqrt{x^2-a^2}}{a} & , \end{cases}$$

which implies

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left|x + \sqrt{x^2-a^2}\right| + C.$$

Example 5. $\int \frac{x}{\sqrt{3-2x-x^2}}dx$.

Solution. First notice that

$$3 - 2x - x^2 = 4 - \left(1 + 2x + x^2\right) = 4 - (x+1)^2.$$

So use $u = x + 1 \Rightarrow du = dx$ to get that

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx = \int \frac{u-1}{\sqrt{4-u^2}}du.$$

Now, let $u = 2\sin\theta \Rightarrow du = 2\cos\theta d\theta$, then

$$\begin{aligned} \int \frac{x}{\sqrt{3-2x-x^2}}dx &= \int \frac{u-1}{\sqrt{4-u^2}}du = \int \frac{2\sin\theta - 1}{2\cos\theta} \cdot 2\cos\theta d\theta = -2\cos\theta - \theta + C \\ &= -2 \cdot \frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C. \end{aligned}$$

7.4 INTEGRATION OF RATINONAL FUNCTIONS BY PARTIAL FRACTIONS

$$\int \frac{P(x)}{Q(x)} dx = ?$$

STEPS:
1. Division if $deg(P) \geq deg(Q)$
2. Factorization of the denominator Q
3. Partial Fractional Decomposition of the correct form
4. Coefficients
5. Integrations

Form: If $Q(x) = (ax+b)(cx+d)(ex+f)^3 \left(gx^2+hx+i\right)\left(jx^2+kx+l\right)^2$, then we set up

$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f} + \frac{D}{(ex+f)^2} + \frac{E}{(ex+f)^3} + \frac{Gx+H}{gx^2+hx+i} + \frac{Ix+J}{jx^2+kx+l} + \frac{Kx+L}{(jx^2+kx+l)^2}.$$

Example 1. Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$

1.
$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

2. $x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x-1)\left(x^2-1\right) = (x-1)^2(x+1)$.
3. Set up
$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

4. To compute A, B, and C, we first get
$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2.$$
$$\begin{cases} x=1 \Rightarrow 4=2B & \Rightarrow B=2 \\ x=-1 \Rightarrow -4=4C & \Rightarrow C=-1 \\ x=0 \Rightarrow 0=-A+B+C & \Rightarrow A=1 \end{cases}$$

5.
$$\begin{aligned} \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx &= \int \left(x+1+\frac{4x}{x^3-x^2-x+1}\right) dx \\ &= \int \left(x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right) dx \\ &= \frac{x^2}{2}+x+\ln|x-1|-\frac{2}{x-1}-\ln|x+1|+C. \end{aligned}$$

Example 2. Find $\int \frac{2x^2-x+4}{x^3+4x} dx$

1. No need
2. $x^3+4x = x\left(x^2+4\right)$
3.
$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

4. $2x^2-x+4 = A\left(x^2+4\right) + x(Bx+C) = (A+B)x^2 + Cx + 4A$.
$$\begin{cases} A+B=2 \\ C=-1 \\ 4A=4 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=-1 \end{cases}$$

5.
$$\int \frac{2x^2-x+4}{x^3+4x} dx = \int \left(\frac{1}{x}+\frac{x-1}{x^2+4}\right) dx = \int \left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right) dx$$

$$\begin{cases} \int \frac{1}{x} dx = \ln|x| + C \\ \int \frac{x}{x^2+4} dx \overset{u=x^2+4}{=\!=\!=\!=\!=} \frac{1}{2}\int \frac{2x\,dx}{x^2+4} = \frac{1}{2}\int u^{-1} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4\right| + C \\ \int \frac{1}{x^2+4} dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \end{cases}$$

Consider $\underline{u = \frac{x}{a} \Rightarrow du = \frac{1}{a} dx}$

$$\int \frac{1}{x^2+a^2} dx = \int \frac{a}{x^2+a^2} \cdot \frac{dx}{a} = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2} \frac{dx}{a} = \frac{1}{a}\int \frac{1}{1+u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Thus,

$$\int \frac{2x^2-x+4}{x^3+4x}dx = \ln|x| + \frac{1}{2}\ln\left(x^2+4\right) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Example 3. Evaluate $\int \frac{4x^2-3x+2}{4x^2-4x+3}dx$

1.

$$\frac{4x^2-3x+2}{4x^2-4x+3} = \frac{4x^2-4x+3+x-1}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}$$

2. $4x^2-4x+3 = 4x^2-4x+1+2 = (2x-1)^2+2$. Now, let $u = 2x-1 \Rightarrow du = 2dx$

$$\int \frac{4x^2-3x+2}{4x^2-4x+3}dx = \int \left(1 + \frac{x-1}{4x^2-4x+3}\right)dx = x + \frac{1}{2}\int \frac{x-1}{4x^2-4x+3}2dx = x + \frac{1}{2}\int \frac{\frac{u+1}{2}-1}{u^2+2}du = x + \frac{1}{4}\int \frac{u-1}{u^2+2}du$$

3.4. No need

5.

$$\begin{aligned}\int \frac{4x^2-3x+2}{4x^2-4x+3}dx &= x + \frac{1}{4}\int \frac{u}{u^2+2}du - \frac{1}{4}\int \frac{1}{u^2+2}du = x + \frac{1}{8}\ln\left(u^2+2\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C \\ &= x + \frac{1}{8}\ln\left(4x^2-4x+3\right) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C\end{aligned}$$

Example 4. Calculate $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$

1.

2.

3.

$$\frac{1-x+2x^2-x^3}{x\left(x^2+1\right)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{\left(x^2+1\right)^2}$$

4. $1-x+2x^2-x^3 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + (Dx+E)x = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

$$A=1, B=-1, C=-1, D=1, E=0$$

5.

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x\left(x^2+1\right)^2}dx &= \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{\left(x^2+1\right)^2}\right)dx \\ &= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1}(x) - \frac{1}{2\left(x^2+1\right)} + C.\end{aligned}$$

Example 5. $\int \frac{\sqrt{x+4}}{x}dx$

Let $u = \sqrt{x+4} \Rightarrow x = u^2-4 \Rightarrow dx = 2udu$.

$$\begin{aligned}\int \frac{\sqrt{x+4}}{x}dx &= \int \frac{u}{u^2-4}2udu = \int \frac{2u^2}{u^2-4}du = 2\int du + 8\int \frac{du}{u^2-4} = 2u + 4\ln\left|\frac{u-2}{u+2}\right| + C \\ &= 2\sqrt{x+4} + 4\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C\end{aligned}$$

7.7 APPROXIMATION INTEGRATION

Reason:
Some functions do not have elementary antiderivatives:

$$\int e^{x^2}dx,\ \int \sin\left(x^2\right)dx,\ \int \frac{\sin x}{x}dx$$

Rules	Formulas	Error
Midpoint	$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x_1}) + \cdots + f(\bar{x_n})\right]$ $\begin{cases}\Delta x = \frac{b-a}{n}\\ \bar{x_i} = \frac{1}{2}(x_{i-1}+x_i)\end{cases}$	If $\lvert f''(x)\rvert \le K$ for $a \le x \le b$, then $\lvert E_M\rvert \le \frac{K(b-a)^3}{24n^2}$
Trapezoidal	$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$ $\begin{cases}\Delta x = \frac{b-a}{n}\\ x_i = a + i\Delta x\end{cases}$	If $\lvert f''(x)\rvert \le K$ for $a \le x \le b$, then $\lvert E_T\rvert \le \frac{K(b-a)^3}{12n^2}$
Simpson's	$\int_a^b f(x)\,dx \approx S_n$ $= \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots$ $+2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$ $\begin{cases}\Delta x = \frac{b-a}{n} & ,n \text{ must be even}\\ x_i = a + i\Delta x\end{cases}$	If $\left\lvert f^{(4)}(x)\right\rvert \le K$ for $a \le x \le b$, then $\lvert E_S\rvert \le \frac{K(b-a)^5}{180n^2}$

Example 1. $\int_1^5 \frac{1}{x}dx$, i.e, $f(x) = \frac{1}{x}$, $n = 4$

Thus, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $x_4 = 5$. $\Delta x = \frac{5-1}{4} = 1$. Also, $\bar{x_1} = \frac{3}{2}$, $\bar{x_2} = \frac{5}{2}$, $\bar{x_3} = \frac{7}{2}$ and $\bar{x_4} = \frac{9}{2}$

$$\begin{cases} M_n = 1\cdot\left(\frac{1}{\frac{3}{2}} + \frac{1}{\frac{5}{2}} + \frac{1}{\frac{7}{1}} + \frac{1}{\frac{9}{2}}\right) = \frac{496}{315} \approx 1.574603\\ T_n = \frac{1}{2}\cdot\left[\frac{1}{1} + 2\cdot\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \frac{1}{5}\right] = \frac{101}{60} \approx 1.683333\\ S_n = \frac{1}{3}\cdot\left(\frac{1}{1} + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}\right) = \frac{73}{45} \approx 1.622222\end{cases}$$

Note that

$$\int_1^5 \frac{dx}{x} = \ln|x|\Big|_{x=1}^{x=5} = \ln 5 \approx 1.6094379$$

4.4 L'Hospital's Rule:

Theorem.

$$\lim_{x\to a^{\pm}(or\ \pm\infty)} \frac{f(x)}{g(x)} \overset{L'}{\underset{\frac{0}{0}\ or\ \frac{\pm\infty}{\pm\infty}}{=\!=\!=\!=\!=}} \lim_{x\to a^{\pm}(or\ \pm\infty)} \frac{f'(x)}{g'(x)}.$$

Example 2. (1)

$$\lim_{x\to\infty}\frac{e^x}{x} \overset{L'}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

(2)

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty}\frac{x}{e^{-x}} \overset{L'}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty}\frac{1}{-e^{-x}} = \lim_{x\to-\infty}\left(-e^x\right) = 0.$$

(3)

$$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} \overset{L'}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0^+}(-x) = 0.$$

7.8 IMPROPER INTEGRALS

Type I: Infinite Interval.

$$\begin{cases} \int_a^\infty f(x)\,dx = \lim\limits_{t\to\infty} \int_a^t f(x)\,dx \\ \int_{-\infty}^b f(x)\,dx = \lim\limits_{t\to-\infty} \int_t^b f(x)\,dx \end{cases}$$

If the limit exists, then the imporper integral is called *convergent*, otherwise it is called *divergent*.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

if the both imporper integrals of the RHS are convergent.

Example 1. $\int_{-\infty}^0 xe^x dx$

Solution.

$$\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}\int_t^0 xe^x dx \overset{\substack{u=x \quad dv=e^x dx \\ du=dx \quad v=e^x}}{=\!=\!=\!=\!=\!=} \lim_{t\to-\infty}\left(xe^x\Big|_{x=t}^{x=0} - \int_t^0 e^x dx\right) = \lim_{t\to-\infty}\left(te^t - 1 + e^t\right).$$

L'Hospital's Rule:

$$\lim_{x\to a^{\pm}(or\ \pm\infty)} \frac{f(x)}{g(x)} \underset{\frac{0}{0}\ or\ \frac{\pm\infty}{\pm\infty}}{\overset{L'}{=\!=\!=}} \lim_{x\to a^{\pm}(or\ \pm\infty)} \frac{f'(x)}{g'(x)}.$$

Now

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} \underset{\frac{\infty}{\infty}}{\overset{L'}{=\!=}} \lim_{t\to-\infty}\frac{1}{-e^{-t}} = \lim_{t\to-\infty}\left(-e^t\right) = 0.$$

Thus,

$$\int_{-\infty}^0 xe^x dx = \lim_{t\to-\infty}\left(te^t - 1 + e^t\right) = -1.$$

Example 2. $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$

Solution. First of all $\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx$, then

$$\begin{cases} \int_{-\infty}^0 \frac{1}{1+x^2}dx = \lim\limits_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}dx = \lim\limits_{t\to-\infty}\left(\tan^{-1}x\Big|_{x=t}^{x=0}\right) = \lim\limits_{t\to-\infty}\left(0-\tan^{-1}t\right) = \frac{\pi}{2} \\ \int_0^\infty \frac{1}{1+x^2}dx = \lim\limits_{t\to\infty}\int_0^t \frac{1}{1+x^2}dx = \lim\limits_{t\to\infty}\left(\tan^{-1}x\Big|_{x=0}^{x=t}\right) = \lim\limits_{t\to\infty}\left(\tan^{-1}t - 0\right) = \frac{\pi}{2} \end{cases}$$

Thus,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \int_{-\infty}^0 \frac{1}{1+x^2}dx + \int_0^\infty \frac{1}{1+x^2}dx = \pi.$$

Example 3. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}dx$ convergent?

Solution.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\int_1^t x^{-p}dx = \begin{cases} \lim\limits_{t\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_{x=1}^{x=t} = \lim\limits_{t\to\infty}\frac{1}{1-p}\left(\frac{1}{t^{p-1}}-1\right) = \begin{cases}\frac{1}{p-1} & if\ p>1 \\ \infty & if\ p<1\end{cases} & when\ p\neq 1 \\ \lim\limits_{t\to\infty}\ln|x|\Big|_{x=1}^{x=t} = \lim\limits_{t\to\infty}\ln t = \infty & when\ p=1 \end{cases}$$

Theorem. *$\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p>1$ and divergent if $p\le 1$.*

Type 2: Discontiuous Integrands.

$$\int_a^b f(x)\,dx = \begin{cases} \lim\limits_{t\to b^-}\int_a^t f(x)\,dx & \text{if } f \text{ is only discontinous at } b \\ \lim\limits_{t\to a^+}\int_t^b f(x)\,dx & \text{if } f \text{ is only discontinous at } a \\ \int_a^c f(x)\,dx + \int_c^b f(x)\,dx & \text{if } f \text{ is only discontinous at } c, \text{ where } a<c<b \text{ and both integrals are convergent.} \end{cases}$$

Example 4. $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

Solution.

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to2^+}\int_t^5 \frac{1}{\sqrt{x-2}}dx \overset{u=x-2}{=\!=\!=\!=} \lim_{t\to2^+}\int_{t-2}^3 u^{-\frac{1}{2}}du = \lim_{t\to2^+} 2\sqrt{u}\Big|_{u=t-2}^{u=3} = \lim_{t\to2^+} 2\left(\sqrt{3}-\sqrt{t-2}\right) = 2\sqrt{3}.$$

Example 5. $\int_0^3 \frac{dx}{x-1}$

Solution. WARNING

$$\int_0^3 \frac{dx}{x-1} \overset{u=x-1}{=\!=\!=\!=\!=} \int_{-1}^2 \frac{du}{u} = \ln|u| \Big|_{u=-1}^{u=2} = \ln 2 \textbf{WORNG ANWSER}$$

First of all $\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$, then

$$\int_0^1 \frac{dx}{x-1} = \lim_{t\to 1^-} \int_0^t \frac{dx}{x-1} \overset{u=x-1}{=\!=\!=\!=} \lim_{t\to 1^-} \int_{-1}^{t-1} \frac{du}{u} = \lim_{t\to 1^-} \ln|u| \Big|_{u=-1}^{u=t-1} = \lim_{t\to 1^-} \ln(1-t) = -\infty$$

So the integral is divergent.

Theorem. *[Comparison Theorem] Suppose both f and g are continous for $x \geq a$ with $f(x) \geq g(x) \geq 0$.*
(a) If $\int_a^\infty f(x)\,dx$ is convergent, so is $\int_a^\infty g(x)\,dx$.
(b) If $\int_a^\infty g(x)\,dx$ is divergent, so is $\int_a^\infty f(x)\,dx$.

Example 6. Show that $\int_0^\infty e^{-x^2} dx$ is convergent $\left(\frac{1}{2\sqrt{\pi}}\right)$

Proof. First of all, we write

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx,$$

where the first integral is ordinary. Now, when $x \geq 1$, we have

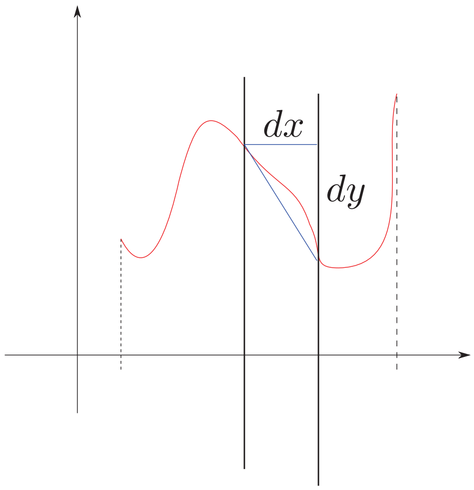
$$x \geq 1 \Rightarrow x^2 \geq 1 \Rightarrow -x^2 \leq -x \Rightarrow 0 \leq e^{-x^2} \leq e^{-x}.$$

Also note that

$$\int_1^\infty e^{-x} dx = \lim_{t\to\infty} \int_1^t e^{-x} dx = \lim_{t\to\infty} \left[\left(-e^{-t}\right) - \left(-e^{-1}\right)\right] = \frac{1}{e},$$

which is convergent. So by Comparision Theorem, the second integral is convergent, impliying $\int_0^\infty e^{-x^2} dx$ is convergent. $\square$

8.1 Arc Length



$$L = \lim \sum |\text{Line Segment}| = \int \sqrt{(dx)^2 + (dy)^2}$$

(1)If the curve is $y = f(x)$, $a \le x \le b$, then, $\frac{dy}{dx} = f'(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

(2)If the curve is $x = g(y)$, $c \le y \le d$, then $g'(y) = \frac{dx}{dx}$

$$L = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 1. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the point $(1,1)$ and $(4,8)$

Solution. (1) $y = f(x) = x^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = f'(x) = \frac{3}{2}\sqrt{x}$. Then,

$$L = \int_1^4 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2} dx = \int_1^4 \sqrt{1 + \frac{9}{4}x} dx$$

Let $u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx$. Thus,

$$L = \frac{4}{9}\int_{\frac{13}{4}}^{10} \sqrt{u} du = \frac{80\sqrt{10} - 13\sqrt{13}}{27}.$$

(2) $x = g(y) = y^{\frac{2}{3}} \Rightarrow \frac{dx}{dy} = g'(y) = \frac{2}{3}y^{-\frac{1}{3}}$.

$$L = \int_1^8 \sqrt{1 + \frac{4}{9}y^{-\frac{2}{3}}} dy = \frac{1}{27}\sqrt{9 + \frac{4}{y^{\frac{2}{3}}}} \left(4y^{\frac{1}{3}} + 9y\right)\Bigg|_1^8 = \frac{80\sqrt{10} - 13\sqrt{13}}{27}.$$

Arc Length Function.

Definition.

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt.$$

Remark. (1) $s(b)$ is the length from a to b

(2) $\frac{ds}{dx} = s'(x) = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow (ds)^2 = (dx)^2 + (dy)^2$.

(3) "Parametrization" (Coming Soon)

Example 2. $y = x^2 - \frac{1}{8}\ln x$ and $P_0(1,1)$.

Let $f(x) = x^2 - \frac{1}{8}\ln x \Rightarrow f'(x) = 2x - \frac{1}{8x}$. Then,

$$s(x) = \int_1^x \sqrt{1 + [f'(t)]^2} dt = \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2} dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt = x^2 + \frac{1}{8}\ln x - 1.$$

For example, the length from $P_0(1,1)$ to $P(3, f(3))$ is

$$L = s(3) = \int_1^3 \sqrt{1 + [f'(t)]^2} dt = 8 + \frac{1}{8}\ln 3.$$

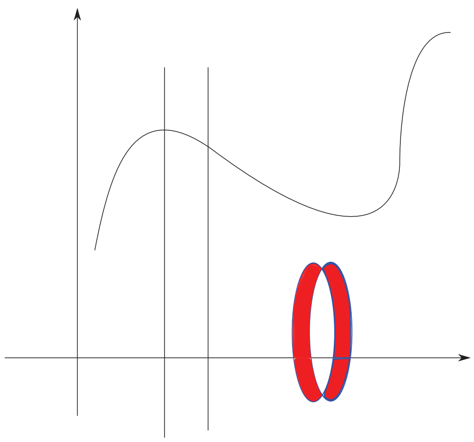
8.2 Area of a Surface of Revolution.

Recall:

region $\to$ solid

Now

curve $\to$ SURFACE



$$A = \lim \sum \text{perimeter} \times \text{thickness} = \int 2\pi \text{radius} ds$$

(1) If $y = f(x)$ for $a \le x \le b$, rotating about x-axis, then

$$A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx = \int 2\pi y ds$$

(2) If $x = g(y)$ for $c \le y \le d$, rotating about y-axis, then

$$A = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy = \int 2\pi x ds$$

Example 3. $y = \sqrt{4 - x^2}$ $-1 \le x \le 1$ (an arc of $x^2 + y^2 = 4$) about x-axis

Solution.

$$f(x) = \sqrt{4 - x^2} \Rightarrow f'(x) = \frac{1}{2\sqrt{4 - x^2}} (-2x) = -\frac{x}{\sqrt{4 - x^2}}.$$

Thus,

$$A = \int 2\pi y ds = \int_{-1}^{1} 2\pi \sqrt{4 - x^2} \cdot \sqrt{1 + \frac{x^2}{4 - x^2}} dx = 2\pi \int_{-1}^{1} 2 dx = 8\pi$$

Example. $y = x^2$ about y-axis from $(1, 1)$ to $(2, 4)$

Solution. (I) $x = g(y) = y^{\frac{1}{2}} \Rightarrow \frac{dx}{dy} = g'(y) = \frac{1}{2\sqrt{y}}$.

$$A = \int_1^4 2\pi y^{\frac{1}{2}} \sqrt{1 + \frac{1}{4y}} dy = \pi \int_1^4 \sqrt{4y + 1} dy \overset{u=4y+1}{=\!=\!=\!=\!=} \frac{\pi}{4} \int_5^{17} \sqrt{u} du = \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5}\right).$$

(II) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$$A = \int 2\pi x ds = \int_1^2 2\pi x \sqrt{1 + 4x^2} dx \overset{u=1+4x^2}{=\!=\!=\!=\!=\!=} 2\pi \int_5^{17} \frac{1}{8} \sqrt{u} du = \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5}\right).$$

9.1 MODELING WITH DIFFERNTIAL EQUATIONS

Example 1. Exponential Growth/Decay. "The rate of change is proportional to the size".

$$\frac{dy}{dt} = y'(t) = ky.$$

In Chapter 3, we claimed that

$$y = y(0)e^{kt}.$$

It is a solution since it satisfies the equation.

$$y' = y(0)e^{kt} \cdot k = ky.$$

Example 2. Motion of a Spring.

Hooke's Law:

$$F(x) = -kx,$$

where the negative sign implies that the directions of the force and change of position are opposite. By Newton's Law:

$$F = ma = m\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Example 3. For population growth in Example 1:

$$P(t) = P(0)e^{kt},\, k > 0 \Rightarrow \lim_{t\to\infty} P(t) = \infty.$$

This is the ideal situation since usually there is a carrying capacity M. If $P \to M$, the population levels off; if $P > M$, it decreases.

Logistic Model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

Equilibrium Solutions: Constant solutions that makes the derivative 0. Here are two $P = 0$ and $P = M$.

Definition. Some necessary definitions.

(1) Order: the order of the highest derivative in the equation
(2) Solution:
(3) Inintial Condition: When learning indefinite integrals, we saw such example:

Example 4. Find f if $f'(x) = e^x + 20\left(1 + x^2\right)^{-1}$ and $f(0) = -2$.

$$\begin{cases} f(x) = \int f'(x)\,dx = e^x + 20\tan^{-1} x + C \\ -2 = f(0) = e^0 + 20\tan^{-1} 0 + C = 1 + C \Rightarrow C = -3. \end{cases} \Rightarrow f(x) = e^x + 20\tan^{-1} x - 3$$

- The condition that $f(0) = -2$ is the initial condition.
- Differential equation + initial condition = initial problem.

Example 5. Consider the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$

(1) Show that every member in the family $y = (1 + ce^t)/(1 - ce^t)$ is a solution
(2) Find the solution with initial condition $y(0) = 2$

Solution.(1)

$$\begin{cases} y' = \frac{ce^t\left(1-ce^t\right)-\left(1+ce^t\right)\left(-ce^t\right)}{\left(1-ce^t\right)^2} = \frac{2ce^t}{\left(1-ce^t\right)^2} \\ \frac{1}{2}\left(y^2-1\right) = \frac{1}{2}\cdot\frac{\left(1+ce^t\right)^2-\left(1-ce^t\right)^2}{\left(1-ce^t\right)^2} == \frac{2ce^t}{\left(1-ce^t\right)^2} \end{cases} \Rightarrow y' = \frac{y^2-1}{2}.$$

(2)

$$2 = y(0) = \frac{1+c}{1-c} \Rightarrow c = \frac{1}{3} \Rightarrow y(t) = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}.$$

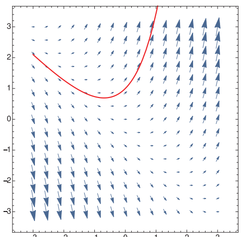
9.2 DIRECTION FIELD AND EULER'S METHOD

Consider the general first order differential equation

$$y' = F(x, y).$$

Sometimes it is not easy to solve explicitly. So we at each point of (x, y), we compute $y' = F(x, y)$, which is the slope of the tangent line to the solution curve.

We draw <u>short line segments with slope $F(x, y)$ at points (x, y)</u>, then the result is called direction field.



Example 1. $y' = x + y$, $y(0) = 1$.
At each point, we could draw the direction vector:
$(0, 1) : y' = 0 + 1 = 1 \Rightarrow \tan\left(\theta|_{(0,1)}\right) = 1$
$(1, 2) : y' = 1 + 2 = 3$

Euler's Method: Numerical Apporximation.

Theorem. *If $y' = F(x, y)$ and $y(x_0) = y_0$, then the approximate values for the solution at $x_n = x_{n-1} + h$ with step h are*

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}).$$

To understand this formula, we do the following

$$y_n - y_{n-1} + hF(x_{n-1}, y_{n-1}) \Rightarrow \Delta y \approx (\Delta x)\, y' = y'dx = dy.$$

Example 2. $y' = x + y$, $y(0) = 1$.

n	1	2	3	4	5	6	7	8	9	10
x_n	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_n	1.1	1.22	1.362	1.5282	1.72102	1.94312	2.19743	2.48718	2.81595	3.18748

(1) $y_1 = y_0 + h(x_0 + y_0) = 1 + 0.1 \times (0 + 1) = 1.1$
(2) $y_2 = y_1 + h(x_1 + y_1) = 1.1 + 0.1 \times (0.1 + 1.1) = 1.1 + 0.12 = 1.22.$

Remark. The solution we will learn to solve by the end of this chapter is

$$y = 2e^x - x - 1 \Rightarrow y' = 2e^x - 1 = x + y$$

and

$$y(1) = 2e^1 - 1 - 1 \approx 3.43656.$$

9.3 SEPARABLE EQUATIONS

KEY:

$$y' = \frac{dy}{dx} = F(x,y) = f(x)\,g(y) \Rightarrow \frac{dy}{g(y)} = f(x)\,dx \Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Example 1. Solve the initial problem

$$\begin{cases} y' = \frac{x^2}{y^2} & , \\ y(0) = 2 & . \end{cases}$$

Solution. Since $y' = \frac{dy}{dx} = \frac{x^2}{y^2}$, we have

$$y^2dy = x^2dx \Rightarrow \int y^2dy = \int x^2dx \Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + C \Rightarrow y = \sqrt[3]{x^3+K}, \text{ where } K = 3C.$$

Now,

$$2 = y(0) = \sqrt[3]{K} \Rightarrow K = 8 \Rightarrow y = \sqrt[3]{x^3+8}.$$

Example 2. $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$

Solution. Separating variables to get

$$(2y+\cos y)\,dy = 6x^2dx \Rightarrow \int (2y+\cos y)\,dy = \int 6x^2dx \Rightarrow y^2 + \sin y = 2x^3 + C.$$

Definition. An *orthogonal trajectory* of a family of curves is a curve that intersects each of the member orthogonally.

Example 3. Find the orthogonal trajectories of the family of curves $x = ky^2$, where k is an arbitary constant.

Solution. Remark. Because of the constant k, we have a family of curves rather than only one curve.

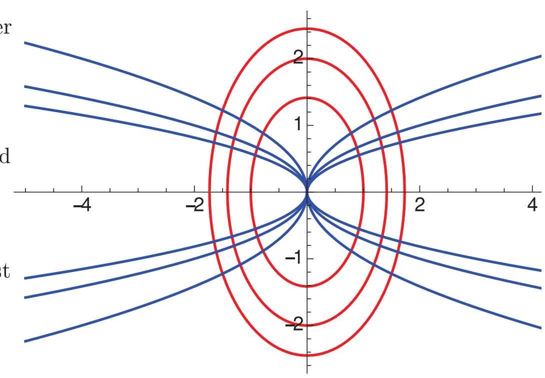
$$x = ky^2 \Rightarrow 1 = 2ky\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2ky}.$$

This is the tangents of the curves. But, we need an expression that does not depand on k, therefore

$$k = \frac{x}{y^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y\frac{x}{y^2}} = \frac{y}{2x}.$$

Since the orthogonal trajectories are perpendicular to the given curves, they must satisfy

$$\frac{dy}{dx} = -\frac{2x}{y} \Rightarrow ydy = -2xdx \Rightarrow \int ydy = -\int 2xdx \Rightarrow \frac{y^2}{2} = -x^2 + C.$$



9.4 POPULATION GROWTH

Natural Growth

Recall that

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = kdt \Rightarrow \int \frac{dP}{P} = \int kdt \Rightarrow \ln|P| = kt + C \Rightarrow P = Ae^{kt}, \text{ where } A = e^{C}.$$

Also, note that

$$P_0 = P(0) = A \Rightarrow P(t) = P_0 e^{kt}.$$

(1) $\lim_{t\to\infty} P(t) = \infty$

Logistic Model

Recall: We introduce *carrying capacity* M

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

Now, we solve it:

$$\begin{aligned}
\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \quad &\Rightarrow \quad \frac{dP}{P\left(1 - \frac{P}{M}\right)} = kdt \\
&\Rightarrow \quad \int \frac{MdP}{P(M-P)} = \int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kdt \\
&\Rightarrow \quad \ln|P| - \ln|M-P| = kt + C \\
&\Rightarrow \quad \ln\left|\frac{P}{M-P}\right| = kt + C \\
&\Rightarrow \quad \frac{P}{M-P} = Ae^{kt} \\
&\Rightarrow \quad P = \frac{M}{1 + Ae^{-kt}}.
\end{aligned}$$

- $\lim_{t\to\infty} P(t) = M.$

9.5 LINEAR EQUATIONS

Recall that

$$\frac{dy}{dx} = x + y$$

First order linear differential equations.

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Define

$$I(x) = e^{\int P(x)dx} \Rightarrow I(x)\frac{dy}{dx} + P(x)I(x)y = I(x)Q(x).$$

Observe the LHS. Note that

$$\frac{d}{dx}(yI(x)) = \frac{dy}{dx}I(x) + y\frac{d}{dx}\left(e^{\int P(x)dx}\right) = I(x)\frac{dy}{dx} + y\frac{d}{dx}\left(\underset{\square=\int P(x)dx}{e^{\square}}\right) = I(x)\frac{dy}{dx} + ye^{\square}\cdot\frac{d}{dx}(\square) = I(x)\frac{dy}{dx} + P(x)I(x)y.$$

Thus,

$$LHS = \frac{d}{dx}(I(x)y) = RHS \Rightarrow I(x)y = \int\left(I(x)\frac{dy}{dx} + P(x)I(x)y\right)dx = \int I(x)Q(x)dx.$$

This $I(x)$ is alled the <u>integrating factor</u>.

Example 1. $y' = x + y$.

Solution.

$$\begin{aligned} y' - y = x \quad &\Rightarrow \quad \begin{cases} P(x) = -1 \\ Q(x) = x \end{cases} \\ &\Rightarrow \quad I(x) = e^{\int -1dx} = e^{-x} \\ &\Rightarrow \quad \frac{d}{dx}(e^{-x}y) = e^{-x}y' - e^{-x}y = e^{-x}x \\ &\Rightarrow \quad e^{-x}y = \int e^{-x}xdx. \end{aligned}$$

Integration by parts yeilds

$$\begin{matrix} u = x & v = e^{-x}dx \\ du = dx & -e^{-x} \end{matrix} \quad \Rightarrow \int e^{-x}xdx = -xe^{-x} - \int\left(-e^{-x}\right)dx = -xe^{-x} - e^{-x} + C.$$

Thus

$$e^{-x}y = -xe^{-x} - e^{-x} + C \Rightarrow y = Ce^{x} - x - 1.$$

Example 2. $y' + 3x^2y = 6x^2$

Solution.$P(x) = 3x^2 \Rightarrow I(x) = e^{\int P(x)dx} = e^{\int 3x^2dx} = e^{x^3}$. Thus,

$$\frac{d}{dx}\left(e^{x^3}y\right) = e^{x^3}y' + ye^{x^3}3x^2 = e^{x^3}6x^2 \Rightarrow e^{x^3}y = \int e^{x^3}6x^2dx \overset{u=x^3}{=\!=\!=\!=} 2\int e^{u}du = 2e^{x^3} + C$$

Therefore,

$$y = Ce^{-x^3} + 2.$$

Example 3. $x^2y' + xy = 1$, $x > 0$, $y(1) = 2$.

Solution. Since $x > 0$

$$\begin{aligned} x^2y' + xy = 1 \quad &\Rightarrow \quad y' + \frac{1}{x}y = \frac{1}{x^2} \\ &\Rightarrow \quad P(x) = \frac{1}{x} \Rightarrow I(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x > 0 \\ &\Rightarrow \quad \frac{d}{dx}(xy) = xy' + y = \frac{1}{x} \Rightarrow xy = \int \frac{1}{x}dx = \ln x + C \\ &\Rightarrow \quad y = \frac{\ln x + C}{x}. \end{aligned}$$

Now,

$$2 = y(1) = \frac{\ln 1 + C}{1} \Rightarrow C = 2 \Rightarrow y = \frac{\ln x + 2}{x}.$$

10.1&10.2

Part 1. 10.1 Curves Defined by Parametric Equations

Recall: Curves we have learnt before are mostly either $y = f(x)$ or $x = g(y)$.

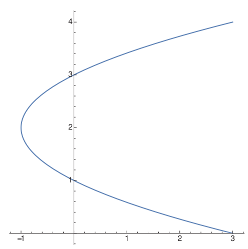
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \text{, where } t \in [a, b] \text{ is called the parameter.}$$

Aims

$$\begin{cases} \text{(1) Recognize curves given by parametric equations} \\ \text{(2) Write curves in terms of parametric equations} \end{cases}$$

Example 1. $x = t^2 - 2t$ and $y = t + 1$ for $t \in (-\infty, \infty)$

t	x	y
-1	3	0
0	0	1
1	-1	2
2	0	3



$$t = y - 1 \Rightarrow x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$$

Example 2. $x = \cos t$ and $y = \sin t$

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

The peroid is 2π. That is when t travels from 0 to 2π, we have the whole circle.

If $x = \cos(2t)$ and $y = \sin(2t)$, then it is still the unit circle but with period π

[Bonus Question] Figure it out

$$\begin{cases} x = \frac{1-t^2}{1+t^2} \\ y = \frac{2t}{1+t^2} \end{cases} \quad t \in (-\infty, \infty)$$

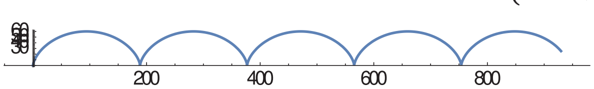
Example 3. Find parametric equations for the circle with center (h, k) and radius r

Solution.

$$(x-h)^2 + (y-k)^2 = r^2 \Rightarrow \begin{cases} x = h + r\cos\theta \\ y = k + r\sin\theta \end{cases}$$

Example 4. Cycloid

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$



Part 2. 10.2 Calculus with Parametric Curves

Tangents

KEY WORD: Chain Rule

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{g''f' - g'f''}{f'^2}}{f'} = \frac{g''f' - g'f''}{f'^3}.$$

Example 5. A curve C is defined by the parametric equations $x = t^2$ $y = t^3 - 3t$

(1) Show that C has two tangents at the point $(3, 0)$ and find their equations.

(2) Find the points on C where the tangent is horizontal or vertical

(3) Determin where the curve is concave upward or downward

Solution. (1) $\begin{cases} x = t^2 = 3 \\ y = t^3 - 3t = 0 \end{cases} \Rightarrow t = \pm\sqrt{3}$. Since

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right),$$

we have

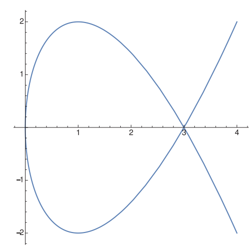
$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \sqrt{3} \text{ and } \left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = -\sqrt{3}.$$

Thus the two equations are

$$y = \pm\sqrt{3}\,(x - 3)\,.$$

(2) i>Horizontal$\Leftrightarrow \frac{dy}{dx} = 0$.

$$\frac{3}{2}\left(t - \frac{1}{t}\right) = 0 \Rightarrow t^2 = 1 \Rightarrow t \pm 1 \text{ where } \left.\frac{dx}{dt}\right|_{t=\pm 1} \neq 0. \Rightarrow C\,(1, \pm 2)\,.$$



ii>Vertical$\Leftrightarrow \frac{dy}{dx} = \infty \Leftrightarrow \frac{dx}{dt} = 0$

$$2t = 0 \Rightarrow t = 0 \Rightarrow (0, 0)$$

(3)

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left[\frac{3}{2}\left(t - \frac{1}{t}\right)\right]'}{2t} = \frac{3\left(1 + \frac{1}{t^2}\right)}{4t} = \frac{3\left(t^2 + 1\right)}{4t^3} \Rightarrow \begin{cases} \frac{d^2y}{dx^2} > 0 \Leftrightarrow t > 0 & CU \\ \frac{d^2y}{dx^2} < 0 \Leftrightarrow t < 0 & CD \end{cases}$$

Areas

Recall that

$$A = \int_a^b f(x)\,dx$$

Thus,

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)\,\underline{f'(t)\,dt}$$

Remark. This is the area of the region under the curve.

Example 6. Find the area under one arch of the cycloid.

Solution.

$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases} \quad 0 \leq \theta \leq 2\pi.$$

$$\begin{aligned} A &= \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta = r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta \\ &= r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \cos^2\theta\right)d\theta = r^2\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1 + \cos(2\theta)}{2}\right)d\theta \\ &= r^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\right)d\theta = r^2\left.\left(\frac{3}{2}\theta - 2\sin\theta - \frac{1}{4}\sin(4\theta)\right)\right|_{\theta=0}^{\theta=2\pi} \\ &= 3\pi r^2. \end{aligned}$$

Arc Length

Recall

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int \sqrt{(dx)^2 + (dy)^2} = \int ds.$$

Formula:

$$L = \int \sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example 7. Find the length of one arch of the cycloid.

Solution.
$$\begin{cases} x = r(\theta - \sin\theta) & \Rightarrow \frac{dx}{d\theta} = r(1-\cos\theta) \\ y = r(1-\cos\theta) & \Rightarrow \frac{dy}{d\theta} = r\sin\theta \end{cases} 0 \le \theta \le 2\pi.$$

Thus,

$$L = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta = r\int_0^{2\pi}\sqrt{2(1-\cos\theta)}\, d\theta$$

TRICK: Recall that

$$\sin x = \frac{1-\cos(2x)}{2} \overset{x=\frac{\theta}{2}}{\Longrightarrow} \sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2 x} = 2\sin\frac{\theta}{2},$$

where $0 \le \theta \le 2\pi \Rightarrow 0 \le \frac{\theta}{2} \le \pi \Rightarrow \sin\frac{\theta}{2} \ge 0$. Therefore

$$L = 2r\int_0^{2\pi}\sin\frac{\theta}{2}\, d\theta = 4r\int_0^{\pi}\sin x\, dx = -4r\cos x\Big|_{x=0}^{x=-\pi} = 8r.$$

Surface Area

Recall

$$S = \begin{cases} \int 2\pi y\, ds & \text{, if rotates the curve about } x-\text{axis} \\ \int 2\pi x\, ds & \text{, if rotates the curve about } y-\text{axis} \end{cases}$$

Then, use

$$y = g(t),\ x = f(t),\ ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example 8. Show that the surface area of a sphere of radius r is $4\pi r^2$.

Solution. $x = r\cos t$ and $y = r\sin t$ for $0 \le t \le \pi$. Then, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = r\, dt$

$$S = \int_0^{\pi} 2\pi y\, ds = 2\pi r\int_0^{\pi}(\sin t)\, r\, dt = 2\pi r^2\int_0^{\pi}\sin t\, dt = 4\pi r^2.$$

Or we could consider the interval $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and rotate the right half circle about y-axis. Then, mn

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi x\, ds = 2\pi r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\, dt = 4\pi r^2.$$

10.3&10.4 POLAR COORDINATES

Instead of having x-coordinate and y-coordinate, we focus on another way to describe the location of a point by

$$\begin{cases} r & = \text{distance from the point to the origin} \\ \theta & = \text{the angle between the direction of the point and the positive } x\text{-axis} \end{cases}$$

Transform Points

Theorem. *For (x,y) in Cartesina coordiate and (r,θ) in polar coordiate of the SAME point P, we have:*

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \textit{and} \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

Example 1. (1) $(2,\pi/3)$ (2) $(1,-1)$

Solution. (1) $x = 2\cos(\pi/3) = 1$ and $y = 2\sin(\pi/3) = \sqrt{3}$. So the point is $\left(1,\sqrt{3}\right)$ in Cartesian coordinates
(2) $r = \sqrt{1^2+(-1)^2} = \sqrt{2}$ and $\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = -\frac{\pi}{4}$ (or $\frac{7\pi}{4}$). So $\left(\sqrt{2},-\frac{\pi}{4}\right)$.

Curves

Recall that usually, we have curves $y = f(x)$ and sometimes $x = g(y)$ and sometimes $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$. Now USUALLY, curves in polar coordiates are given by

$$r = f(\theta).$$

Example 2. (1) $r = 2$ (2) $\theta = \frac{\pi}{4}$ (3) $r = 2\cos\theta$

Solution. (1) Since all the points on the curve have the fixed distance 2, so it is the circle centered at the origin with radius 2. Or,

$$4 = r^2 = x^2+y^2.$$

(2) Straight line. It can be seen also by

$$\frac{y}{x} = \tan\theta = 1 \Rightarrow y = x.$$

(3)TRICK: Multiplying both sides by r to get

$$r^2 = 2r\cos\theta \Rightarrow x^2+y^2 = 2x \Rightarrow (x-1)^2+y^2 = 1.$$

Tangents

KEY:

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 3. The cardioid has the equation $1+\sin\theta$.
(1) Find the slope of tangent line at $\theta = \frac{\pi}{3}$. (2) Find points where the tangent line is horizontal or vertical.

Solution. $f(\theta) = 1+\sin\theta \Rightarrow f'(\theta) = \cos\theta$. Thus,

$$\frac{dy}{dx} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.$$

(1) At $\theta = \frac{\pi}{3}$,

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\frac{1}{2}\left(1+2\cdot\frac{\sqrt{3}}{2}\right)}{\left(1+\frac{\sqrt{3}}{2}\right)\left(1-2\cdot\frac{\sqrt{3}}{2}\right)} = \frac{\sqrt{3}+1}{\left(\sqrt{3}+2\right)\left(1-\sqrt{3}\right)} = -1.$$

(2)

$$\begin{cases} \text{Horizontal} \Leftrightarrow \frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0 & \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \\ \text{Vertical} \Leftrightarrow \frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 & \Rightarrow \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \end{cases}.$$

The case $\theta = \frac{3\pi}{2}$ is special since both the denominator and numerator become 0.

$$\begin{aligned}\lim_{\theta \to \frac{3\pi}{2}^-} \frac{dy}{dx} &= \lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)\,(1-2\sin\theta)} = \lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta} \cdot \lim_{\theta \to \frac{3\pi}{2}^-} \frac{1+2\sin\theta}{1-2\sin\theta} = -\frac{1}{3} \lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta} \\ &\overset{L'}{\underset{\frac{0}{0}}{=}} -\frac{1}{3} \lim_{\theta \to \frac{3\pi}{2}^-} \frac{-\sin\theta}{\cos\theta} = \infty.\end{aligned}$$

Also, by symmetry, $\lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy}{dx} = -\infty$.

Area

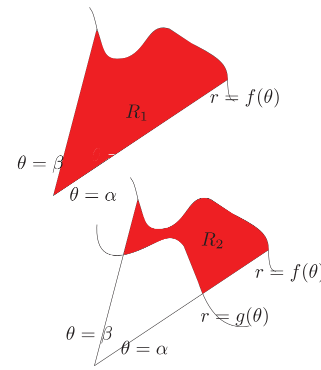
Remark. This is the area enclosed purely by the curve, namely without the x-axis.

Theorem. *For the diagrams on the right,*
(1)

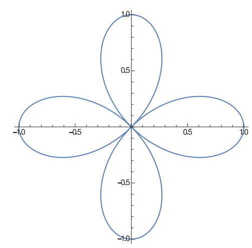
$$Area\,(R_1) = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta$$

(2)

$$\begin{aligned} Area\,(R_2) &= \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta - \frac{1}{2}\int_\alpha^\beta [g(\theta)]^2\, d\theta \\ &= \frac{1}{2}\int_\alpha^\beta \left\{[f(\theta)]^2 - [g(\theta)]^2\right\} d\theta \end{aligned}$$



Example 4. Find the are enclosed by one loop of the four leaved rose $r = \cos(2\theta)$



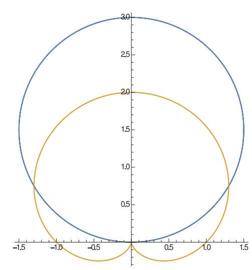
Solution. When $r = \cos(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ since $0 \le \theta \le 2\pi \Rightarrow 0 \le 2\theta \le 4\pi$. Thus, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Therefore,

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta = \int_0^{\pi/4} \frac{1+\cos(4\theta)}{2} d\theta = \frac{\pi}{8}.$$

Example 5. Find the area on the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$.

Solution. The key part is to find intersections.

$$\begin{cases} r = 3\sin\theta \\ r = 1+\sin\theta \end{cases} \Rightarrow 2\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$



Therefore,

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[8\sin^2\theta - 1 - 2\sin\theta\right] d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[4(1-\cos(2\theta)) - 1 - 2\sin\theta\right] d\theta = \frac{1}{2}\left[3\theta - 2\sin(2\theta) + 2\cos\theta\right]\Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{5\pi}{6}} \\ &= \pi. \end{aligned}$$

Arc Length

Recall that

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \Rightarrow L = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Now,

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases} \Rightarrow L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)} d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

Example 6. Find the length $r = 1+\sin\theta$

Solution.

$$A = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\,d\theta = \int_0^{2\pi} \sqrt{2+2\sin\theta}\,d\theta.$$

<u>TRICK</u>:

$$\begin{array}{rcl}
A &=& \displaystyle\int_0^{2\pi} \sqrt{2+2\sin\theta}\cdot\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}d\theta \\
&=& \displaystyle\int_0^{2\pi} \cdot\frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}d\theta = \int_0^{2\pi} \frac{2\left|\cos\theta\right|}{\sqrt{2-2\sin\theta}}d\theta \\
[u = 2-2\sin\theta] &=& \displaystyle\int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}d\theta - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}d\theta \\
&=& \displaystyle\int_2^0 -u^{-\frac{1}{2}}\,du - \int_0^4 -u^{\frac{1}{-2}}\,du + \int_4^2 -u^{-\frac{1}{2}}\,du \\
&=& 8.
\end{array}$$

11.1 SEQUENCES

Definition. $\{a_1, a_2, \dots, a_n, \dots\}$ or $\{a_n\}_{n=1}^{\infty}$

Example 1. (1) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

(2) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty} = \left\{0, 1, \sqrt{2}, \sqrt{3}, \dots\right\}$

(3) Fibonacci $\{f_n\}_{n=1}^{\infty}$

$$f_1 = f_2 = 1, f_{n+2} = f_{n+1} + f_n \Rightarrow \{1, 1, 2, 3, 5, \dots\} \Rightarrow f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(4) $\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}\right\} = \left\{(-1)^{n+1}\,\frac{n+2}{5^n}\right\}_{n=1}^{\infty}$

Definition. A convergent sequence is a sequence $\{a_n\}_{n=1}^{\infty}$ has the limit

$$\lim_{n\to\infty} a_n = L$$

Otherwise, it is called divergent. The way to find the limit usually follows the way of functions, i.e., if $f(x)$ satisfies $f(n) = a_n$ and

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L.$$

Theorem. *(1) If $r > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^r} = 0.$$

(2)

$$\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0.$$

This is the directly corollary of the sequence theorem. If $a_n \le b_n \le c_n$ for all $n \ge n_0$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then, $\lim_{n\to\infty} b_n = L$.

(3) Properties:

$$\begin{cases} \lim_{n\to\infty} (a_n \pm b_b) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \\ \lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n, \quad \lim_{n\to\infty} c = c \\ \lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) \\ \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} & , if \ \lim_{n\to\infty} b_n \ne 0 \\ \lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p & , if \ p > 0 \ and \ a_n > 0. \end{cases}$$

Example 2. (1)

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n \cdot \frac{1}{n}}{(n+1)\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

(2)

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0 \text{ since } \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

(3) $\{(-1)^n\}_{n=1}^{\infty}$ is divergent since the limit DNE

(4)

$$\lim_{n\to\infty} \frac{n}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n} + \frac{2}{n^2}}} = \infty$$

Definition 3.

Increasing: $a_n \le a_{n+1}$	Monotone: increasing OR decreasing
Decreasing: $a_n \ge a_{n+1}$	
Bounded above: $a_n \le M$	Bounded: bounded above AND bounded below: $\vert a_n\vert \le M$
Bounded below: $a_n \ge m$	

Theorem. *Every bounded monotone sequence is convergent.*

11.2 SERIES

$$\sum_{n=n_0}^{\infty} a_n$$

If define the *partial sum*

$$s_n := \sum_{k=1}^{n} a_k \Rightarrow \begin{cases} a_n = s_n - s_{n-1} \\ \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n. \end{cases}$$

Definition. A series is calle *convergent* if

$$\sum_{n=1}^{\infty} a_n = s \text{ exits.}$$

Otherwise, it is called *divergent*.

Theorem. *If $\sum a_n$ and $\sum b_n$ are both convergent, then so are $\sum (a_n \pm b_n)$ and $\sum ca_n$ for any constant c. And*

$$\begin{cases} \sum ca_n = c\sum a_n \\ \sum (a_n \pm b_n) = \sum a_n \pm \sum b_n \end{cases}$$

Example 1. Geometric Series.

$$a + ar + ar^2 + \cdots$$

So, $a_n = ar^{n-1}$ and $a_{n+1}/a_n = r$. Here r is called the common ratio. Thus,

$$s_n = \frac{a(1-r^n)}{1-r}.$$

The geoemtric sereis

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a}{1-r}(1-r^n)$$

is

$$\begin{cases} \text{convergent and } = \frac{a}{1-r} & \text{, if } |r| < 1; \\ \text{divergent} & \text{, if } |r| \geq 1. \end{cases}$$

Example 2. (1) Find $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

Consider $a_n = (-1)^{n-1}\frac{2^n}{3^n} \cdot 5$, which can be seen by computing the common ratio that

$$\frac{-\frac{10}{3}}{5} = \frac{\frac{20}{9}}{-\frac{10}{3}} = \frac{-\frac{40}{27}}{\frac{20}{9}} = -\frac{2}{3}.$$

Therefore

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \sum_{n=1}^{\infty} a_n = \frac{5}{1-(-\frac{2}{3})} = 3.$$

(2) Determine whether the series $\sum\limits_{n=1}^{\infty} 2^{2n}3^{1-n}$ converges or diverges?

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 3 \cdot \frac{4^n}{3^n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^n \text{—diverges since } \frac{4}{3} > 1.$$

(3)

$$2.3\bar{17} = 2.3171717171\cdots = 2.3 + \frac{17}{1000} + \frac{17}{10^5} + \cdots = 2.3 + \frac{\frac{17}{1000}}{1-\frac{1}{100}} = \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}.$$

Remark.

$$0.\bar{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1.$$

Theorem 3. $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution.

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Thus,

$$\lim_{n\to\infty} s_n = 1.$$

Example 4. Harmonic Sum $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges.

Proof. Since

$$\begin{cases} s_2 = 1 + \frac{1}{2} & , \\ s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} & , \\ s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) > 1 + \frac{3}{2} & , \\ \cdots & , \end{cases}$$

we have

$$s_{2^n} > 1 + \frac{n}{2} \Rightarrow \lim_{n\to\infty} s_n \text{ has no limit.}$$

□

Theorem. *If ther series* $\sum a_n$ *is convergent, then* $\lim a_n = 0$. *(Not the opposite direction)*

Corollary. *If* $\lim a_n$ *does not exist or* $\neq 0$, *then* $\sum a_n$ *is divergent.*

Example 5. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$. $\lim \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$.

Example 6. Find the sum $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$.

Solution. Since,

$$\begin{cases} \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 & , \\ \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 & , \end{cases} \Rightarrow \sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 4.$$

11.3 INTEGRAL TEST

Theorem. *Suppose f is continous, positive, decreasing function on $[1,\infty)$ and $a_n = f(n)$. The series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent.*

(1) If $\int_1^\infty f(x)\,dx$ is convergent, $\sum\limits_{n=1}^{\infty} a_n$ is convergent;

(2) and if $\int_1^\infty f(x)\,dx$ is divergent, $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Example 1. Test the series $\sum\limits_{n=1}^{\infty} \frac{1}{1+n^2}$.

Solution. Let $f(x) = 1/\left(1+x^2\right)$, then it is continous, positive and decrasing on $[1,\infty)$. Also $a_n = \frac{1}{1+n^2} = f(n)$. Now, since

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t f(x)\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{1+x^2}dx = \lim_{t\to\infty}\left(\tan^{-1}x\right)\Big|_{x=1}^{x=t} = \lim_{t\to\infty}\left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{4}$$

shows that $\int_1^\infty f(x)\,dx$ is convergent, so is $\sum\limits_{n=1}^{\infty} \frac{1}{1+n^2}$.

Example 2. Determine the <u>p-series</u> $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$.

Solution. (1) If $p<0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$. So it is divergent.
(2) If $p=0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \lim\limits_{n\to\infty} 1 = 1$. So it is divergent.
(3) If $p>0$, then consider $f(x) = 1/x^p$, which is continuous, positive and decreasing on $[1,\infty)$. In Chapter7, we know that

$$\int_1^\infty \frac{1}{x^p}dx \text{ is convergent if and only iff } p>1.$$

Therefore, we have

$$\text{the } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if and only if } p>1.$$

Remark. (1) It is not necessary that both series and integral start from 1:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \Leftrightarrow \int_4^\infty \frac{1}{(x-3)^2}dx.$$

(2) f is not necessary to be always decreasing. As long as it is ultimately decreasing, the test works.
(3) See the following truth.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6} \neq 1 = \int_1^\infty \frac{1}{x^2}dx.$$

Example 3. Test $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. Let $f(x) = \ln x/x$. It is obviously continous and positive when $x>1$. Now since

$$f'(x) = \frac{1-\ln x}{x^2} \Rightarrow f'(x) < 0 \text{ when } x>e.$$

So it is ultimately decreasing. Note that

$$\int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx \overset{u=\ln x}{=\!=\!=} \lim_{t\to\infty}\int_0^{\ln t} u\,du = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty,$$

so the series $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

11.4 COMPARISION TEST

Recall that

Theorem. *[Comparison Theorem] Suppose both f and g are continous for $x \geq a$ with $f(x) \geq g(x) \geq 0$.*
(a) If $\int_a^\infty f(x)\,dx$ is convergent, so is $\int_a^\infty g(x)\,dx$.
(b) If $\int_a^\infty g(x)\,dx$ is divergent, so is $\int_a^\infty f(x)\,dx$.

Now, we have

Theorem. *(I) [Comparision Test] Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, satisfying $0 < a_n \leq b_n$.*
(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent as well;
(ii) if $\sum a_n$ is divergent, then $\sum b_n$ is divergent as well.
(II) [The Limit Comparision Test] Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, satisfying

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0, \text{ a finite number i.e., not } \infty.$$

Then, either both $\sum a_n$ and $\sum b_n$ converge or both diverge.

Example 1. $\sum\limits_{n=1}^{\infty} \frac{1}{2^n+1}$.

Solution.

$$2^n + 1 > 2^n \Rightarrow \frac{1}{2^n+1} < \frac{1}{2^n} \text{ and note } \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1/2}{1-1/2} = 1.$$

Example 2. $\sum\limits_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.

Solution.

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2} \text{ and note } \sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} \left(= \frac{5\pi^2}{12}\right).$$

Example 3. $\sum\limits_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution. For $n > 3 > e \approx 2.718\cdots$, $\ln n > 1$. Then

$$\frac{\ln n}{n} > \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Example 4. $\sum\limits_{n=1}^{\infty} \frac{1}{2^n-1}$.

Solution. (1) Although, $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1$, we do not have $\frac{1}{2^n-1} < \frac{1}{2^n}$. Instead, $\frac{1}{2^n-1} > \frac{1}{2^n}$.

(2)

$$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n-1}} = \lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{1} = 1 \Rightarrow \text{Both converge.}$$

Example 5. $\sum\limits_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$.

Solution.

$$\lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2n^{\frac{5}{2}}+3n^{\frac{3}{2}}}{\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{1+\frac{5}{n^5}}} = 2 \Rightarrow \text{ Both diverge.}$$

11.5 ALTERNATING SERIES

Definition. $\sum (-1)^{n(\pm 1)} b_n$ for some $b_n \geq 0$.

Example 1. (1)

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n \pm 1} \frac{1}{n}.$$

(2)

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}.$$

Theorem. *For an alternating series $\sum (-1)^{n(-1)} b_n$, if (i) $b_{n+1} \leq b_n$ for all $n \geq \underline{n_0}$; and (ii) $\lim\limits_{n\to\infty} b_n = 0$, then the series converges.*

Example 2. (1) $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$, i.e., $b_n = \frac{1}{n}$.

(i) $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$; (ii) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. Thus, the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent.

(2) $\sum\limits_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$. Since $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$, the series diverges.

Example 3. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$

Solution. $b_n = \frac{n^2}{n^3+1}$. It is obvious that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0.$$

But to see whether it is decreasing is not easy. Consider $f(x) = \frac{x^2}{x^3+1}$.

$$f'(x) = \frac{2x\left(x^3+1\right) - x^2\left(3x^2\right)}{\left(x^3+1\right)^2} = \frac{x\left(2 - x^3\right)}{\left(x^3+1\right)^2} \Rightarrow f'(x) < 0 \text{ when } x > \sqrt[3]{2}.$$

Thus, choose $n_0 = 2$, for all $n \geq n_0 = 2$,

$$b_{n+1} = f(n+1) \leq f(n) = b_n.$$

Therefore, the series converges.

Example 4. $\boxed{\text{This example shows that condition (i) of the theorem that } b_n \text{ needs to be nonincreasing is important.}}$

Consider the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$, where $\{b_n\}_{n=1}^{\infty} = \left\{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \cdots\right\}$. It is true that $\lim\limits_{n\to\infty} b_n = 0$. However, it does not satisfy (i). Note that when n is even, $b_n = 0$. Therefore

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

11.6 ABSOLUTE CONVERGENCE

Definition. (1) Absolute Convergence (absolutely convergent): $\sum |a_n|$ converges.

(2) Conditional Convergence (conditionally convergent): converges, but does not absolutely converge, i.e. $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example 1. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Solution. $a_n = \frac{(-1)^n}{n^2} \Rightarrow |a_n| = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges. So $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Example 2. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ but } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by alternating series theorem.}$$

Theorem. *If a series is absolutely convergent, then it is convergent.*

Example 3. $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n^3}$

Solution. $a_n = \frac{\cos n}{n^3}$, then

$$|a_n| = \left| \frac{\cos n}{n^3} \right| \le \frac{1}{n^3}.$$

By Comparision Test and p-series theorem, $\sum |a_n|$ is convergent, so is $\sum a_n$.

Theorem. *[Ratio Test] If*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \begin{cases} < 1 & \Rightarrow \sum a_n \textit{ is absolutely convergent} \\ > 1 \textit{ or } = \infty & \Rightarrow \sum a_n \textit{ is divergent} \\ = 1 & \Rightarrow \textit{no conclusion.} \end{cases}$$

Example 4. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$.

Solution. $a_n = (-1)^n \frac{n^3}{3^n}$. So,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \frac{1}{3} \cdot \left(\frac{n+1}{n} \right)^3 = \frac{1}{3} < 1.$$

Theorem. *[Root Test] If*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \begin{cases} < 1 & \Rightarrow \sum a_n \textit{ is absolutely convergent} \\ > 1 \textit{ or } = \infty & \Rightarrow \sum a_n \textit{ is divergent} \\ = 1 & \Rightarrow \textit{no conclusion.} \end{cases}$$

Example 5. $\sum\limits_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$.

Solution. $a_n = \left(\frac{2n+3}{3n+2} \right)^n$. So,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left[\left| \left(\frac{2n+3}{3n+2} \right)^n \right| \right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1.$$

Rearrangement.

11.7 STRATEGY

(1) Remember that always first check whether

$$\lim_{n\to\infty} a_n = 0.$$

(2) Alternatinve series are relatively easy
(3) Factorials$\Rightarrow$Ratio Test; Pure n^{th} powers$\Rightarrow$ Root Test
(4) Basic facts on

Geometric Series, p-Series

helps in comparision. Rational or algebraic series$\Rightarrow$(Limit) Comparision Test with p-Series.
(5) Absolute Convergence$\Longrightarrow$Convergence and for positive series, they are the same.

Tests	Main Context	Hint
Integral Test	$a_n \geq 0,\ a_n = f(n),\ f(x)\ is \begin{cases} continuous \\ ultimately\ decreasing \\ nonnegative \end{cases}\quad then\ \sum_{n=1}^{\infty} a_n \Leftrightarrow \int_0^{\infty} f(x)dx$	functions like $\frac{e^{\frac{1}{n}}}{n^2}$
Comparision Test	$a_n \geq 0, b_n \geq 0 \begin{cases} \text{General:} \quad 0 \leq a_n \leq b_n \Rightarrow \begin{cases} \sum b_n C \Rightarrow \sum a_n C \\ \sum a_n D \Rightarrow \sum b_n D \end{cases} \\ \text{Limit} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = c > 0 \text{ then } \sum a_n \Leftrightarrow \sum b_n \end{cases}$	Rational, Algebraic
Alternating Test	$\sum (-1)^n b_n,\ b_n \geq 0 : If \begin{cases} \lim_{n\to\infty} b_n = 0 \\ b_{n+1} \leq b_n\ ultimately \end{cases}$	Alternating
Ratio and Root Test	$\text{Compute } L = \begin{cases} \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert & [Ratio] \\ \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} & [Root] \end{cases}, \text{ then } L \begin{cases} < 1 & \text{Absolutely Convergent} \\ > 1 \text{ or } = \infty & \text{Divergent} \\ = 1 & \text{No Conclusion} \end{cases}$	

11.8 POWER SREIES

Definition. $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$

(1) a is called the center; (2) $\{c_n\}_{n=0}^{\infty}$ is the sequence of coefficients.

Question: For what values of x is the series convergent?

The following method is a little different from the textbook.

For $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$, first compute

$$\rho := \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|.$$

Then we obtain the **radius of convergence** $R = 1/\rho$. Therefore, we conclude that

(1) for all $x \in (a-R, a+R) \Leftrightarrow |x-a| < R$, the series is absolutely convergent;
(2) for all $x \in (-\infty, a-R) \cup (a+R, \infty) \Leftrightarrow |x-a| > R$, the series is divergent;
(3) for the two boundary points $x = a-R$ and $x = a+R$, we need further discussion.
(4) If $\rho = \infty \Rightarrow R = 0$, then it only converges when $x = a$; if $\rho = 0 \Rightarrow R = \infty$, then it converges absolutely for all x.

Example 1. $\sum_{n=0}^{\infty} x^n$

Solution. $c_n = 1 \Rightarrow \rho = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = 1 \Rightarrow R = 1/\rho = 1$. Note that the center is $a = 0$.

(1) When $|x| < 1 \Leftrightarrow x \in (-1, 1)$, $\sum_{n=0}^{\infty} x^n$ converges absolutely (by root test).

(2) When $|x| > 1 \Leftrightarrow x \in (-\infty, -1) \cup (1, \infty)$, $\sum_{n=0}^{\infty} x^n$ diverges.

(3) When $|x| = 1$, both $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty} (-1)^n$ diverges.

Therefore the interval of convergence is $(-1, 1)$

Example 2. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$

Solution. $c_n = \frac{1}{n} \Rightarrow \rho = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1 \Rightarrow R = 1/\rho = 1$. Note that the center is $a = 2$.

(1) When $|x-2| < 1 \Leftrightarrow x \in (1, 3)$, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converges absolutely.

(2) When $|x-2| > 1 \Leftrightarrow x \in (-\infty, 1) \cup (3, \infty)$, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ diverges.

(3) When $|x-2| = 1$, $\sum_{n=1}^{\infty} \frac{(3-2)^n}{n}$ diverges but $\sum_{n=1}^{\infty} \frac{(1-2)^n}{n}$ converges.

Therefore the interval of convergence is $[1, 3)$.

Example 3. [Bessel Function of the First Kind]

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Solution. First write it as (letting $y = x^2$)

$$J_0(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{2^{2n} (n!)^2} \Rightarrow c_n = \frac{(-1)^n}{2^{2n} (n!)^2} \Rightarrow \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{2^{2(n+1)}((n+1)!)^2}}{\frac{(-1)^n}{2^{2n}(n!)^2}} \right| = \lim_{n\to\infty} \frac{1}{4(n+1)^2} = 0 \Rightarrow R = \infty.$$

Thus it converges for any y, therefore for any x. Therefore the interval of convergence is $(-\infty, \infty)$.

Example 4. $\sum_{n=0}^{\infty} \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$

Solution. $c_n = \frac{(-3)^n}{\sqrt{n+1}} \Rightarrow \rho = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-3)^{n+1}}{\sqrt{n+2}}}{\frac{(-3)^n}{\sqrt{n+1}}}\right| = \lim_{n\to\infty} 3\sqrt{\frac{n+1}{n+2}} = 3 \Rightarrow R = 1/\rho = \frac{1}{3}$. Note that the center is $a = 1$.

(1) When $|x-1| < \frac{1}{3} \Leftrightarrow x \in \left(\frac{2}{3}, \frac{4}{3}\right)$, $\sum_{n=0}^{\infty} \frac{(-3)^n(x-1)^n}{\sqrt{n+1}}$ converges absolutely.

(2) When $|x-1| > \frac{1}{3} \Leftrightarrow x \in \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$, $\sum_{n=0}^{\infty} \frac{(-3)^n(x-1)^n}{\sqrt{n+1}}$ diverges.

(3) When $|x-1| = \frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{2}{3}-1\right)^n}{\sqrt{n+1}}$ diverges but $\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{4}{3}-1\right)^n}{\sqrt{n+1}}$ converges.

Therefore, $(\frac{2}{3}, \frac{4}{3}]$.

11.9 REPRESENTATIONS OF FUNCTIONS AS POWER SERIES

Basic Formula

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^{n}=1+x+x^{2}+\cdots \text{ for } |x|<1.$$

Example 1.

$$\frac{1}{1+x^{2}}=\frac{1}{1-(-x^{2})}=\sum_{n=0}^{\infty}\left(-x^{2}\right)^{n}=\sum_{n=0}^{\infty}(-1)^{n}x^{2n} \text{ for } \left|-x^{2}\right|<1\Leftrightarrow|x|<1.$$

Example 2.

$$\frac{1}{2+x}=\frac{1}{2}\cdot\frac{1}{1-\left(-\frac{x}{2}\right)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^{n}=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n}}{2^{n+1}} \text{ for } \left|-\frac{x}{2}\right|<1\Leftrightarrow|x|<2.$$

Example 3.

$$\frac{x^{3}}{2+x}=x^{3}\cdot\frac{1}{2+x}=x^{3}\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n}}{2^{n+1}}=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n+3}}{2^{n+1}} \text{ for } \left|x^{2}\right|<1\Leftrightarrow|x|<1.$$

Theorem. *If $f(x)=\sum c_{n}(x-a)^{n}$ has radius R, then on the interval $(a-R,a+R)$*
(i) $f(x)$ is differentiable and

$$f'(x)=\sum_{n=0}^{\infty}c_{n}\frac{d}{dx}\left[(x-a)^{n}\right]=\sum_{n=1}c_{n}n(x-a)^{n-1};$$

(ii) $f(x)$ is continuous (since differentiable) and

$$\int f(x)\,dx=\sum_{n=0}^{\infty}c_{n}\int(x-a)^{n}\,dx=\sum_{n=0}^{\infty}c_{n}\frac{(x-a)^{n+1}}{n+1}+C.$$

The seires in both (i) and (ii) have the same radius of convergence, R.

Example 4.

$$\frac{1}{(1-x)^{2}}=\left(\frac{1}{1-x}\right)'=\left(\sum_{n=0}^{\infty}x^{n}\right)'=\sum_{n=0}^{\infty}(x^{n})'=\sum_{n=1}^{\infty}nx^{n-1}.$$

Example 5.

$$\ln(1+x)=\int\frac{dx}{1+x}=\int\sum_{n=0}^{\infty}(-1)^{n}x^{n}dx=\sum_{n=0}^{\infty}(-1)^{n}\int x^{n}dx=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n+1}}{n+1}+C.$$

Then, letting $x=0$ to get that

$$0=\ln(1+0)=LHS=RHS=C.$$

Therefore,

$$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n}\frac{x^{n+1}}{n+1}.$$

11.10 TAYLOR AND MACLAURIN SERIES

Theorem. *If f has a power series representation at a, i.e. for all x satisfying $|x-a|<R$,*

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n,$$

we have

$$c_n=\frac{f^{(n)}(a)}{n!}.$$

Definition. (1) Taylor series of the function f at a: $(0!=1)$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots$$

(2) Maclaurin Series: $a=0$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\cdots$$

Example 1. Find the Maclaurin series of the function $f(x)=e^x$ and its radius of convergence

Solution.

$$f(x)=e^x\Rightarrow f^{(n)}(x)=e^x\Rightarrow f^{(n)}(0)=1.$$

Therefore

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Since $c_n=\frac{1}{n!}$, then

$$\rho=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right|=\lim_{n\to\infty}\frac{1}{n+1}=0\Rightarrow R=\frac{1}{\rho}=\infty.$$

Example 2. Find the Taylor series of the function $f(x)=e^x$ at $a=2$.

Solution.

$$f^{(n)}(x)=e^x\Rightarrow f^{(n)}(2)=e^2\Rightarrow e^x=f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n=\sum_{n=0}^{\infty}\frac{e^2}{n!}(x-2)^n.$$

In addition,

$$\rho=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{e^2}{(n+1)!}}{\frac{e^2}{n!}}\right|=\lim_{n\to\infty}\frac{1}{n+1}=0\Rightarrow R=\frac{1}{\rho}=\infty.$$

Example 3. Find the Maclaurin series for $\sin x$.

Solution. Consider the following table

$$\begin{aligned}
f(x)=\sin x &\quad\Rightarrow\quad f(0)=0\\
f'(x)=\cos x &\quad\Rightarrow\quad f(0)=1\\
f''(x)=-\sin x &\quad\Rightarrow\quad f(0)=0\\
f'''(x)=-\cos x &\quad\Rightarrow\quad f(0)=-1\\
f^{(4)}(x)=\sin x &\quad\Rightarrow\quad f(0)=0
\end{aligned}$$

Now, we have

$$f(x)=\sin x=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},\ \Rightarrow R=\infty.$$

Theorem. *Important Maclaurin Series and Their Radii of Convergence*

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 \cdots, & R = 1 \\
\ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \tfrac{x^{n+1}}{n+1} = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots & R = 1 \\
e^x &= \sum_{n=0}^{\infty} \tfrac{x^n}{n!} = 1 + x + \tfrac{x^2}{2} + \cdots & R = \infty \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n+1}}{(2n+1)!} & R = \infty \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n}}{(2n)!} & R = \infty \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \tfrac{x^{2n+1}}{2n+1} & R = 1 \\
(1+x)^k &= \sum_{n=0}^{\infty} \tbinom{k}{n} x^n & R = 1
\end{aligned}$$

Remark. (1)

$$\tan^{-1} x = \int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

(2) For the last Binomial Series,

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-(n-1))}{n!}$$

Example 4. $f(x) = \frac{1}{\sqrt{4-x}}$

$$f(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^n}{4^n}$$

Example 5. Find the Maclaurin series for $f(x) = e^x \sin x$

Solution.

$$\begin{aligned}
f(x) &= e^x \sin x \\
&= \left(1 + x + \frac{x^2}{2!} + \cdots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= x + x^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right) x^3 + \cdots
\end{aligned}$$