Introduction to Complex Variables
Course Notes for MATH 3080

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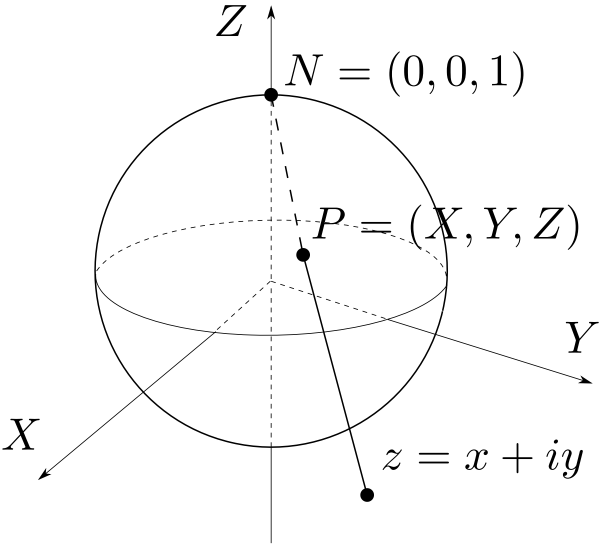
Z
$N = (0, 0, 1)$
$P = (X, Y, Z)$
Y
X
$z = x + iy$

Preface (by the Author)

Complex Variables has been one of the most important topics in mathematics since the mid- to late 1800's. A course in this topic, usually taught at the 3rd year university level, is a crucial part of an undergraduate mathematics education. At Dalhousie, MATH 3080 has long been such a course, taught annually.

These are the notes of this course, prepared more or less at the same time as I first taught the course, with a number of misprints corrected later on. These notes represent an almost word-by-word transcript of what I cover in class. They are therefore quite "skeletal", and are <u>not intended to replace a</u>

<u>good textbook. They are also not meant to replace class attendance</u>.

Instead, these notes are intended to serve as a record of the material covered and expected of the students to be mastered. It may also alleviate the chore of note-taking, and free up some energy for following the development of the subject matter in class. I would like to point out again, as I usually do in class, that studying these notes alone is not sufficient to master the material. Doing the weekly assignments, as independently as possible, is *essential*.

The contents of these notes are based on various sources, but are in some sections quite close to the book [6] (please see the list of references at the back of these notes). However, as secondary reading (on reserve) I have chosen the book by Fisher [2], for its relatively low price if you wish to purchase a copy, and for the large number of examples and problems with solutions.

The course material is such that almost every section, and most definitions and results, are related to further interesting results and concepts. I consciously resisted the temptation to make too many remarks to point these out to the reader. However the reader should be aware of the richness of the theory, and is encouraged to study it further, for instance by taking the follow-up course MATH 4010/5010, "Analytic Function Theory".

Finally, I thank the students who attended this class previously, for their enthusiasm and positive feedback throughout the terms. They found typo-

graphical errors and small mistakes which were corrected in preparing subsequent editions of these notes, including the latest (2018) edition.

Halifax — Karl Dilcher
January, 2018 — dilcher@mathstat.dal.ca

Picture credits

Some figures in these notes were copied from different textbooks. In particular

From [1]: Figures in Sections 2.3, 4.1 and in Example 4.26.

From [3]: Figures in Sections 2.4, 2.5, 3.5, Subsection 3.4.2 (2nd figure) and in Definition 4.27.

From [6]: Figures in Subsection 3.4.2 (1st figure), Sections 6.1 and 7.2

Cover picture:
The Riemann sphere (see Section 2.5)
(From [3, p. 11])

Instructor's Comments

I am deeply grateful to the author, as well as my current supervisor, for sharing the notes. This dramatically saves my time for preparation.

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January, 2019

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Contents

Chapter 1

Introduction

This preliminary chapter gives a very brief sketch of the history of complex variables (also known as complex analysis), and mentions some of the most important milestones and the mathematicians responsible for them. This is a version of the introductory power point presentation, without the accompanying pictures. For a more complete historical sketch, see, for instance, the final chapter in the book [1].

1.1 What Does “Complex Variables" Mean?

This subject and the corresponding area of mathematics are also known as *Complex Analysis* or *The Theory of Functions of a Complex Variable*.

Sometimes it is useful to see what encyclopedias or reference books have to say:

“. . . the branch of mathematical analysis that investigates functions of complex numbers.” (*Wikipedia*)

“. . . one of the most beautiful as well as useful branches of Mathematics.” (Murray R. Spiegel, *Schaum’s Outline of Complex Variables*).

Complex analysis is useful in many branches of mathematics, including
– algebraic geometry
– number theory

– applied mathematics, etc.

It is also important in many branches of physics and engineering, such as

– nuclear,
– aeronautical,
– mechanical, and perhaps most importantly
– electrical engineering.

1.2 A Very Brief History of Complex Numbers

• Ancient Greek and Indian mathematicians noted the impossibility of taking the square root of a negative number.

• For instance, Mahavira Acharya, wrote around 850 AD:

“As in the nature of things, a negative (quantity) is not a square (quantity), it has therefore no square root.”

• Apparently the first appearance of complex numbers in a book was in *Ars Magna* (1545) by Gerolamo Cardano (1501--1576.)

Cardano referred to complex numbers as “mental tortures".

- Complex numbers also appear in Cardano’s famous solution of a cubic equation, but he dismissed the relevant cases as invalid.

  Still, real (and correct) solutions of cubics were obtained by way of using square roots of negative numbers.

- The next milestone was the book *L’Algebra* (1572) by Rafael Bombelli (1526–1573).

  Bombelli made sense of complex numbers in the solutions of cubics, and reconciled the outcome with what was apparently “meaningless”.

  He also developed a calculus of operations with complex numbers. Although his book *L’Algebra* was widely read, complex numbers were still shrouded in mystery, little understood, and often entirely ignored. This is evident in the following quotes by some famous mathematians of the time:

- Simon Stevin (1548–1620):

  “*There is enough legitimate matter, even infinitely much, to exercise oneself without occupying oneself and wasting time on uncertainties.*” (1585).

- John Wallis (1616–1703):

  "These Imaginary Quantities (as they are commonly called) arising from the supposed root of a negative square (when they happen) are reputed to imply that the case proposed is impossible."

- Gottfried Wilhelm Leibniz (1646—1716):

  "The imaginary numbers are a fine and wonderful refuge of the Divine Spirit, almost an amphibian between being and non-being."(1702)

- Christiaan Huygens (1629–1695):

  "One would never have believed that

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$$

and there is something hidden in this which is incomprehensible to us."

Similar doubts concerning the meaning and legitimacy of complex numbers persisted for 2 1/2 centuries. Nevertheless, during the same period complex numbers were extensively used, and a considerable amount of theoretical work was done by such distinguished mathematicians as:

- René Descartes (1596–1650)

  ... who coined the term *imaginary number*. (Before him these numbers were called *sophisticated* or *subtle*).

- Leonhard Euler (1707–1783)

  ... who introduced the notation i for $\sqrt{-1}$. He also used complex numbers extensively and with ease.

- Abraham de Moivre (1667–1754)

  ... who in 1730 noted that complicated trigonometric identities could be obtained in a simple manner by using complex numbers.

- Johann Heinrich Lambert (1728–1777)

  ... who found applications of complex numbers in map projection.

- Jean le Rond d'Alembert (1717–83)

  ... who used complex numbers in hydrodynamics.

- Carl Friedrich Gauss (1777–1855)

… overcame all scruples concerning complex numbers. He was also the one who introduced the term *complex number*.

In 1831 Gauss published his results on the geometric representation of complex numbers as points in the plane. However, his mathematical diary showed that already by 1797 he was aware of this interpretation.

There were similar geometric representations by the following scholars, which were largely ignored:

– Caspar Wessel (1745–1818), a Norwegian surveyor, published in 1797;

– Jean-Robert Argand (1768–1822), a Swiss clerk, published in 1806.

However, the Cartesian coordinate system called the *complex plane* is now also called *Argand diagram*.

General acceptance of the theory is to a large Part due to

– Augustin Louis Cauchy (1789--1857) and

– Niels Henrik Abel (1802--1829).

By the latter part of the 19th century, all vestiges of mystery and distrust of complex numbers had disappeared. A crowning achievement of the theory in the 19th century was the proof of the prime number theorem, a spectacular breakthrough by

- Jacques Hadamard (1865–1963) and
- Charles de la Vallée Poussin (1866–1962).

In addition to Cauchy and Hadamard,

- Karl Weierstrass (1815–1897),
- Bernhard Riemann (1826–1866)

played important roles in the development of complex analysis to where it stands now.

I conclude this brief introduction with a famous quotation by Jacques Hadamard (1865–1963): *"The shortest path between two truths in the real domain passes through the complex domain."*

1.3 Some Number Sets

We recall some important number sets.

$$
\begin{array}{rcl}
\mathbb{N} = \{1, 2, \ldots\} : & & 1 + 2 = 3 \in \mathbb{N} \quad \text{but} \quad 2 - 1 = -1 \notin \mathbb{N} \\
\Downarrow & & \\
\mathbb{Z} = \{0, \pm 1, \ldots\} : & & \forall a,\ b \in \mathbb{Z},\ a \pm b,\ a \cdot b \in \mathbb{Z}, \quad \text{but} \quad \dfrac{a}{b} \notin \mathbb{Z} \\
\Downarrow & & \\
\mathbb{Q} = \{\text{rational number}\} : & & \text{Pythagorean triples } (1, 1, \sqrt{2}) \text{ with } \sqrt{2} \notin \mathbb{Q} \\
 & \Leftrightarrow & x^2 - 2 = 0 \text{ has no solution in } \mathbb{Q} \\
\Downarrow & & \\
\mathbb{R} = \{\text{real number}\} : & & x^2 + 1 = 0 \text{ has no solution} \Leftrightarrow \sqrt{-1} \notin \mathbb{R} \\
\Downarrow & & \\
\mathbb{C} = \{\text{complex number}\} & &
\end{array}
$$

Chapter 2

Complex Numbers

In this chapter, we will construct complex numbers, introduce some notation and concepts used throughout this course, and see how complex numbers can be represented geometrically.

2.1 Construction of the Complex Numbers

Definition 2.1. A *complex number* is defined to be an ordered pair (x, y) of real numbers. Addition and multiplication of complex numbers are defined by

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2),\\ (x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).\end{aligned}$$

The main properties of complex numbers, as defined above, are summarized in the following theorem.

Theorem 2.2. *The set of complex numbers, with "+" and "·" defined above, is a field. That is, the following properties hold: Let* $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ *and* $z_3 = (x_3, y_3)$ *be complex numbers. Then,*

(1) *The commutative law holds*

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1.$$

(2) *The associative law holds*

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3).$$

(3) *The distributive law holds*

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

(4) *There is an additive identity* $(0,0)$ *such that*

$$z_1 + (0,0) = z_1.$$

(5) *There is a multiplicative identity* $(1,0)$ *such that*

$$z_1(1,0) = z_1.$$

(6) *Each element has an additive inverse*

$$(x,y) + (-x,-y) = (0,0).$$

(7) *Each element that is not* $(0,0)$ *has a multiplicative inverse*

$$(x,y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = (1,0).$$

Proof. Exercise. □

Remark. The set (and field) of complex numbers is denoted by the symbol $\mathbb{C}$.

2.2 Further Properties

2.2.1 Notation and Terminology

We begin this section with a number of important remarks.

(1) We write (x, y) as $x + iy$.

(2) We identify the complex number $(x, 0)$ with the real number x. This makes sense and is consistent since by Definition 2.1, we have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

(3) Note that $i = 0 + i \cdot 1 = (0, 1)$. Thus,

$$i^2 = (0,1)(0,1) = (0^2 - 1^2, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$$

In short, this is crucial that

$$i^2 = -1, \tag{2.2.1}$$

which also allows us to write $i = \sqrt{-1}$.

(4) With the notation $x + iy$, we can make computations, using the normal algebraic rules, and in addition the identity (2.2.1).

Example 2.3. Multiplication:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2 \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2),\end{aligned}$$

which is consistent with Definition 2.1.

By Theorem 2.2 (7), a quotient of two complex numbers, provided that the denominator is nonzero, can always be written in the form $x + iy$. To do so, we need to multiple the numerator and denominator by the conjugate of the denominator.

Example 2.4.

$$\frac{3+2i}{2-i} = \frac{(3+2i)(2+i)}{(2-i)(2+i)} = \frac{3\cdot 2 - 2\cdot 1 + i(3\cdot 1 + 2\cdot 2)}{2^2 - i^2} = \frac{4+7i}{5} = \frac{4}{5} + i\frac{7}{5}.$$

Definition 2.5. (*a*) The *(complex) conjugate* of $z = x + iy$ is $\bar{z} = x - iy$.

(*b*) If $z = x + iy$, the *real part* of z is $Re(z) = x$, and the *imaginary part* of $Im(z) = y$ (<u>Important: it is not iy!</u>).

(*c*) The *absolute value (*or *modulus)* of a complex number $z = x + iy$ is defined by $|z| = |x + iy| = \sqrt{x^2 + y^2}$.

(*d*) Two complex numbers $z = x + iy$ and $w = u + iv$ are *equal* iff $x = u$ and $y = v$.

The concepts just defined have the following properties:

Theorem 2.6. (1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \bar{\bar{z}} = z.$

(2) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

(3) $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$.

(4) $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

(5) $|z|^2 = z\bar{z}$.

(6) *z is real iff $\bar{z} = z$.*

(7) $-|z| \le \mathrm{Re}(z) \le |z|, \qquad -|z| \le \mathrm{Im}(z) \le |z|.$

(8) $|z_1 z_2| = |z_1|\,|z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ *(if $z_2 \neq 0$).*

(9) $|z| = 0$ *iff* $z = 0$

(10) $|z| = |-z| = |\bar{z}|.$

(11) $|z_1 + z_2| \le |z_1| + |z_2|.$

Proof. (1)–(10): Exercise. Proof of (11): We first square of left-hand side and expand, using the various properties from above:

$$\begin{aligned} |z_1 + z_2|^2 &\overset{(5)}{=} (z_1 + z_2)(\overline{z_1 + z_2}) \\ &\overset{(1)}{=} (z_1 + z_2)\overline{(z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\ &\overset{(5)}{=} |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1}z_2) \\ &\overset{(1),(2)}{=} |z_1|^2 + |z_2|^2 + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) \\ &\overset{(3)}{=} |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\overline{z_2}). \end{aligned}$$

Now, we have

$$\operatorname{Re}(z_1\overline{z_2}) \overset{(7)}{\leq} |z_1\overline{z_2}| \overset{(8),(10)}{=} |z_1|\,|z_2|\,.$$

Therefore,

$$|z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| = (|z_1|+|z_2|)^2\,.$$

The desired inequality is now obtained by taking the square roots of both sides, noting that, as absolute values, everything is nonnegative. □

Remark. For a geometric representation of the triangle inequality, see the end of Subsection 2.3.1.

2.2.2 Applications to Polynomials

In this brief interlude we obtain a number of important consequences for the zeros (or roots) of polynomials.

Theorem 2.7. (*a*) *A polynomial of degree n with complex coefficients has at most n zeros in $\mathbb{C}$.*

(b) *If f is a polynomial with real coefficients and $f(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then also $f(\overline{z_0}) = 0$.*

Proof. (a) We prove this by induction. The base case $n = 1$ is true since a linear polynomial always has exactly one zero. (We could also use $n = 0$ as base case). For the induction hypothesis we assume that a polynomial of degree $n-1$ has at most $n-1$ zeros. Now let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 2$, and let z_0 be a zero of f. Then, since $f(z_0) = 0$, we have

$$f(z) = f(z) - f(z_0) = a_n(z^n - z_0^n) + a_{n-1}(z^{n-1} - z_0^{n-1}) + \cdots + a_1(z - z_0).$$

Now, since for each $j = 1, \ldots, n$ the term $z - z_0$ divides $z^j - z_0^j$ (note that $z = z_0$ is a zero of this last expression), we can write

$$f(z) = (z - z_0)g(z)$$

for some polynomial $g(z)$ of degree at most $n-1$. Since by induction hypothesis $g(z)$ has at most $n-1$ zeros, $f(z)$ has at most n zeros. This completes the proof by induction.

(b) We write out the equation $f(z_0) = 0$,

$$0 = a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0,$$

and take the complex conjugate of both sides:

$$\begin{aligned} 0 &= \overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0} \\ &= \overline{a_n}\,\overline{z_0}^n + \overline{a_{n-1}}\,\overline{z_0}^{n-1} + \cdots + \overline{a_1}\,\overline{z_0} + \overline{a_0}. \end{aligned}$$

But since the coefficients of f are real, we have $\overline{a_n} = a_n$, $\overline{a_{n-1}} = a_{n-1}, \ldots$, $\overline{a_0} = a_0$. Therefore,

$$0 = a_n \overline{z_0}^n + a_{n-1} \overline{z_0}^{n-1} + \cdots + a_1 \overline{z_0} + a_0 = f(\overline{z_0}),$$

which finishes the proof. □

Remarks. (*a*) We will see later that a polynomial of degree n has in fact *exactly* n complex roots (counting multiplicities). This is the important *Fundamental Theorem of Algebra.*

(*b*) This part of the theorem implies that the non-real zeros of a polynomial with real coefficients occur in conjugate pairs.

Terminology: (a) A number iy with $y \in \mathbb{R}$, $y \neq 0$, is called *pure imaginary* (or *purely imaginary*).

(b) To refer to a number $x + iy$ with $y \neq 0$, use the terms "not real" or "nonreal" (rather than "complex" which, by definition, also contains the real numbers).

2.2.3 Can $\mathbb{C}$ be Ordered?

We saw at the beginning of this section that the complex numbers have the same basic algebraic properties as the reals and the rationals, i.e., all three are *fields*. However, one very important difference between $\mathbb{C}$ and on the one hand and $\mathbb{Q}$ and $\mathbb{R}$ on the other hand is the fact that **the complex numbers cannot be ordered**.

Without going into details of the theory of ordered fields, we note that an ordering "$>$" of the real numbers has (among others) the following basic properties:

(1) If $x \neq 0$ then either $x > 0$ or $-x > 0$, but not both.

(2) If $x > 0$ and $y > 0$ then $xy > 0$ and $x + y > 0$.

Note that this relation "$>$" should be seen as an arbitrary relation satisfying the two conditions (1) and (2), and not necessarily the "natural ordering" we are used to.

Now, any ordering "$>$" on $\mathbb{C}$ should preserve the corresponding ordering on $\mathbb{R}$ since $\mathbb{R}$ is a subset of $\mathbb{C}$. To obtain a contradiction, suppose that we can order the complex numbers.

Since $i \neq 0$, then either $i > 0$ or $-i > 0$. Condition (2) then gives either

$$-1 = i \cdot i > 0 \quad \text{or} \quad -1 = (-i)(-i) > 0.$$

But then, again, by (2), we have

$$1 = (-1)(-1) > 0.$$

So altogether we have both $1 > 0$ and $-1 > 0$, which is a contradiction to condition (1). This shows that the complex numbers cannot be ordered.

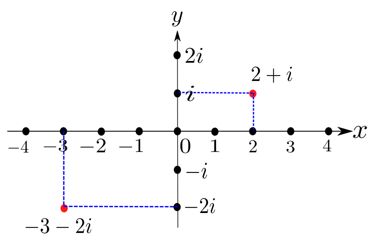
Remark. If $z \in \mathbb{C} \setminus \mathbb{R}$ then writing, e.g., "$z > 0$" makes no sense. However, we may have order relations such as $\mathrm{Re} z > 0$, $\mathrm{Im} z > 0$, or $|z| > 0$. If we write, for instance, $\varepsilon > 0$, then it is automatically assumed that $\varepsilon \in \mathbb{R}$.

2.3 Geometric Representation

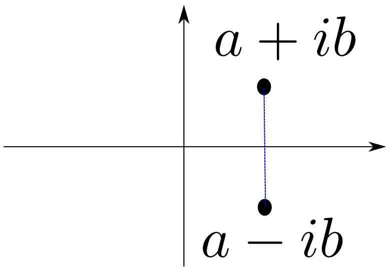
As already indicated earlier, the complex numbers have two important geometric representations that are consistent with corresponding representations of points in $\mathbb{R}^2$. A third representation will be briefly mentioned in Section 2.5 below.

2.3.1 Rectangular (or Cartesian) Coordinates

Consider the complex number $x + iy$ as a point in the plane $\mathbb{R}^2$. The x-axis is called the *real axis*, and the y-axis is called the *imaginary axis*. This representation is called the *Argand diagram* or the *Gauss plane* or the *complex plane*.



The modulus $|x+iy| = \sqrt{x^2+y^2}$ is the distance from the origin to the point (x,y). Also, $-z$ is the reflection of z about the origin, and the complex conjugate $\overline{z}$ is the reflection of z about the real axis, as shown below.



Example 2.8.

(1) $|z|=1$ describes a circle in the complex plane, the *unit circle*.

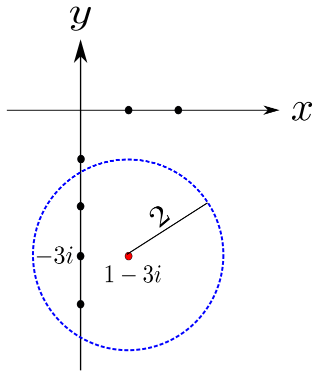
(2) $|z|<1$ is the *open unit disk*, $|z|\le 1$ the *closed unit disk*.

(3) Im $z>0$ is the *upper half-plane*.

(4) Re $z>0$ is the *right half-plane*

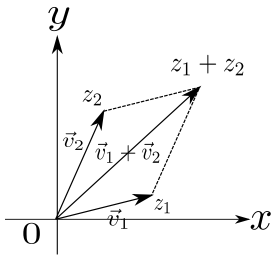
In general, a circle is given by $|z-a|=r$ and an open disk by $|z-a|<r$, where $a \in \mathbb{C}$ and $r>0$.

Example 2.9. The equation $|z-1+3i|=2$ represents the circle whose center is $a=1-3i$ and radius is $r=2$.



Example 2.10. A concept to which we will return later is that of a *neighbourhood* of a point $z_0 \in \mathbb{C}$; it is defined to be a disk of some positive radius, centered at z_0.

Addition of complex numbers: Identify a complex number z with the vector from 0 to z. Then, addition of complex numbers corresponds to addition of vectors, as shown here:

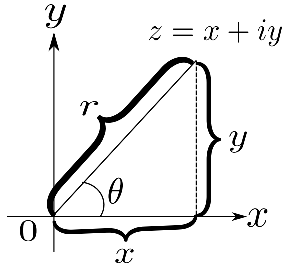


This also explains the name "triangle inequality" for $|z_1 + z_2| \le |z_1| + |z_2|$ (see Theorem 2.6 (11)).

2.3.2 Polar Coordinate

Given the complex number $z = x + iy$, we represent (x, y) in polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = |z| = \sqrt{x^2 + y^2}.$$



The angle θ (measured in radians) is called the *argument* of z, denoted by $\arg z$; it is unique up to a multiple of 2π. The unique value of $\arg z$ that lies in the interval $(-\pi, \pi]$ is referred to as the *principal argument* of z, denoted by $\operatorname{Arg} z$. The representation

$$z = r(\cos\theta + i\sin\theta)$$

is referred to as the *polar representation* of z. The main property of the argument is given by the following theorem.

Theorem 2.11. *For all complex numbers z and w we have*

$$\arg(zw) = \arg z + \arg w,$$

and for $w \neq 0$,

$$\arg\left(\frac{z}{w}\right) = \arg z - \arg w,$$

where the arguments are interpreted as holding up to multiples of 2π.

Proof. We write $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$, so that $\arg z = \theta$ and $\arg w = \phi$. Then

$$\begin{aligned} zw &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)] \\ &= rs[\cos(\theta+\phi) + i\sin(\theta+\phi)], \end{aligned}$$

where we have used the addition formulas for sine and cosine. This identity implies, in particular, that $\arg(zw) = \theta + \phi = \arg z + \arg w$, as desired. □

For the second part, we find similarly that

$$\frac{z}{w} = \frac{r}{s}[\cos(\theta-\phi) + i\sin(\theta-\phi)].$$

If follows that $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$, as desired.

We have already seen how we can multiply complex numbers using their Cartesian representation. Theorem 2.11 shows us how to multiply complex numbers in polar representation:

Corollary 2.12. *Complex numbers can be multiplied by multiplying their moduli and adding their arguments.*

Similarly, the polar representation allows us to find the quotient of two complex numbers without having to multiply by the conjugate of the denominator. Indeed, we simply form the quotient of their moduli and the difference of their arguments.

In particular, multiplying (respectively dividing) by a complex number z with $|z| = 1$ means rotating by $\arg z$ in the positive (respectively negative) direction.

Example 2.13. Multiplication by i is equivalent to a counterclockwise rotation by $\pi/2$. Indeed, the net effect is to multiply the modulus by $1 = |i|$ and increase its argument by $\pi/2 = \arg i$.

2.4 Powers and Roots

In this section we consider two very important operations on complex numbers, namely raising it to an arbitrary integer power, and taking the nth root.

2.4.1 Powers

Given $z = r(\cos\theta + i\sin\theta)$, we can use Theorem 2.11 to multiply this number by itself, obtaining

$$z^2 = z \cdot z = r^2(\cos(2\theta) + i\sin(2\theta)).$$

This is true in general:

Theorem 2.14. *(De Morvre's fomula) Let $z = r(\cos\theta + i\sin\theta)$ be nonzero. Then for any integer n we have*

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)). \tag{2.4.1}$$

Proof. For positive $n \geq 1$ we prove this by induction. The induction beginning, $n = 1$, is simply the given number in its polar representation, $z = r(\cos\theta +$

$i\sin\theta)$. Now assume that the identity (2.4.1) holds for some $n \geq 1$. Given this, we have

$$\begin{aligned} z^{n+1} = z \cdot z^n &= [r(\cos\theta + i\sin\theta)] \cdot [r^n(\cos(n\theta) + i\sin(n\theta))] \\ &= r^{n+1}(\cos(\theta + n\theta) + i\sin(\theta + n\theta)) \\ &= r^{n+1}(\cos((n+1)\theta) + i\sin((n+1)\theta)), \end{aligned}$$

where we have used Theorem 2.11 and its proof. This proves (2.4.1) for $n \geq 1$. □

For $n = 0$ this is simply $z^0 = r^0(\cos 0 + i\sin 0)$ which is obviously true as both sides are 1 (note that $z \neq 0$). Finally, suppose that n is negative, say $n = -m, m \geq 1$. Then, using the second part of Theorem 2.11 and its proof, we have

$$\begin{aligned} z^n = \frac{1}{z^m} &= \frac{1}{r^m}(\cos(0 - m\theta) + i\sin(0 - m\theta)) \\ &= r^n(\cos(n\theta) + i\sin(n\theta)), \end{aligned}$$

as desired.

An important special case of Theorem 2.14 is given by numbers that lie on the unit circle. In this case we have $|z| = 1$, and so we obtain the following:

Corollary 2.15. *For any integer n we have*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{2.4.2}$$

This last identity, and more generally Theorem 2.14, is of great theoretical importance, as we shall see. However, it is also very useful for obtaining numerous trigonometric identities and for evaluating powers of complex numbers, as the following examples will show.

Example 2.16. Prove the double angle formulas

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \quad \sin(2\theta) = 2\sin\theta\cos\theta.$$

Proof. We apply (2.4.2) in the case $n = 2$ to obtain

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta).$$

However, if we expand the left-hand side we obtain

$$(\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).$$

Equating real and imaginary parts, we immediately obtain the two desired identities. □

Example 2.17. Find the Cartesian representation of $\left(\frac{i+\sqrt{3}}{-1-i}\right)^{123}$.

Solution. We first transform the number $\frac{i+\sqrt{3}}{-1-i}$ into polar coordinates, dealing with numerator and denominator separately:
Since $\left|i+\sqrt{3}\right| = \sqrt{1+3} = 2$, we have

$$i+\sqrt{3} = 2(\tfrac{1}{2}\sqrt{3} + i\tfrac{1}{2}) = 2(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}),$$

where we have used our knowledge of special values of sine and cosine. Similarly, since $|1+i| = \sqrt{1+1} = \sqrt{2}$, we have

$$1+i = \sqrt{2}(\tfrac{1}{2}\sqrt{2} + i\tfrac{1}{2}\sqrt{2}) = \sqrt{2}(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}),$$

having used again some basic facts concerning sine and cosine. Now, by Theorem 2.11 we have

$$\begin{aligned}\frac{i+\sqrt{3}}{1+i} &= \frac{2}{\sqrt{2}} \cdot \frac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}} \\ &= \sqrt{2}\left(\cos(\tfrac{\pi}{6} - \tfrac{\pi}{4}) + i\sin(\tfrac{\pi}{6} - \tfrac{\pi}{4})\right) \\ &= \sqrt{2}\left(\cos(\tfrac{-\pi}{12}) + i\sin(\tfrac{-\pi}{12})\right).\end{aligned}$$

As second part of the solution, we apply De Moivre's formula to this identity, obtaining

$$\begin{aligned}\left(\frac{i+\sqrt{3}}{-1-i}\right)^{123} &= (-1)^{123}\left(\frac{i+\sqrt{3}}{1+i}\right)^{123}\\ &= -(\sqrt{2})^{123}\left(\cos(-\tfrac{123}{12}\pi) + i\sin(-\tfrac{123}{12}\pi)\right).\end{aligned}$$

Now note that $-\frac{123}{12}\pi = -10\pi - \frac{\pi}{4}$, and because of periodicity of sine and cosine with period 2π we have

$$\begin{aligned}\left(\frac{i+\sqrt{3}}{-1-i}\right)^{123} &= -2^{123/2}\left(\cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4})\right)\\ &= -2^{123/2}\left(\tfrac{1}{2}\sqrt{2} + i(-\tfrac{1}{2}\sqrt{2})\right)\\ &= -2^{61}(1-i) = -2^{61} + 2^{61}i.\end{aligned}$$

This is the desired Cartesian representation.

2.4.2 Roots

Just as in the real case the equation $x^2 = a$, for $a > 0$, has two roots $x = \sqrt{a}$ and $x = -\sqrt{a}$, we will see that in the complex case a nonzero complex number a has n different nth roots. We begin with a definition.

Definition 2.18. Let $n \in \mathbb{N}$. The *nth roots* of a nonzero number $a \in \mathbb{C}$ are the solution in $z \in \mathbb{C}$ of the equation $z^n = a$.

The following theorem describes all the solutions. Roughly speaking, what it tells us is that we get one nth root by "reading De Moivre's formula backwards". However, there are more, and we get all n of them by spacing them equally around the circle centered at the origin with radius $\sqrt[n]{|a|}$.

Theorem 2.19. *Let $a = r(\cos\theta + i\sin\theta)$ be a nonzero complex number. The nth roots of a are given by*

$$\sqrt[n]{r}\left(\cos\tfrac{\theta+2\pi j}{n} + i\sin\tfrac{\theta+2\pi j}{n}\right) \qquad (0 \le j \le n-1), \tag{2.4.3}$$

where $\sqrt[n]{r}$ is the (unique) positive nth root of r.

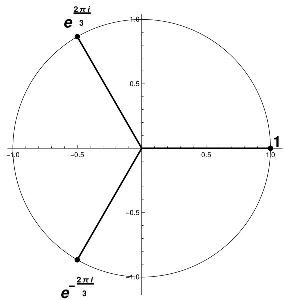
Proof. By De Moivre's formula, the nth powers of the expressions in (2.4.3) are

$$r(\cos(\theta + 2\pi j) + i\sin(\theta + 2\pi j)) = r(\cos\theta + i\sin\theta) = a,$$

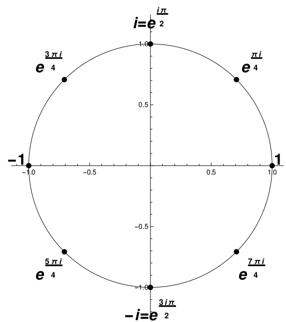
once again by periodicity of sine and cosine. Therefore all the expressions in (2.4.3) are indeed nth roots of a. However, by Theorem 2.7 (a) there cannot be any more. Also, the numbers in (2.4.3) are all different from each other. $\square$

Of particular interest are the nth *roots of unity*, which satisfy $z^n = 1$. They are given by

$$\cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right), \qquad (0 \leq j \leq n-1).$$



cube roots of unity



The eight eighth roots of unity

We can write these as the jth powers of

$$\zeta = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right),$$

for $0 \leq j \leq n-1$. (The Greek letter on the left is read "zeta"). It follows that if z_0 is any nth root of $a \neq 0$, then all nth roots of a are given by $z_0\zeta^j$, for $0 \leq j \leq n-1$. Geometrically, this means that the nth roots of a nonzero complex number a form a regular n-gon, being equally spaced on the circle of radius $\sqrt[n]{|a|}$, centered at the origin. (See Example 2.21 below).

Square roots. By what we have just seen, a nonzero complex number always has two square roots, and they are negative of each other. We distinguish a few cases:

(a) A positive real number x has two square roots $\sqrt{x}$ and $-\sqrt{x}$, where by convention $\sqrt{x}$ always denotes the unique root that is positive (i.e., has positive real part).

(b) A negative real number x has two square roots $\sqrt{x}$ and $-\sqrt{x}$, where by convention $\sqrt{x}$ always denotes the unique root having positive imaginary part. In particular, -1 has roots $i = \sqrt{-1}$ and $-i = -\sqrt{-1}$.

(c) In all other cases, $\sqrt{z}$ is ambiguous, and one must specify which root is meant.

Example 2.20. (a) The square roots of 1 are 1 and -1.

(b) The square roots of -1 are i and $-i$.

(c) The fourth roots of 1 are ± 1 and $\pm i$.

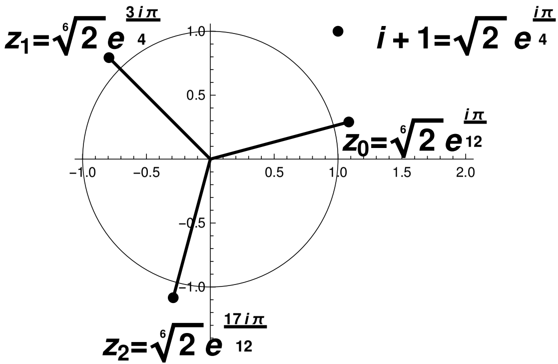
Example 2.21. Find the cube roots of $1+i$.

Solution. Since $|1+i| = \sqrt{2}$, we have

$$1+i = \sqrt{2}\left(\tfrac{1}{2}\sqrt{2} + i\tfrac{1}{2}\sqrt{2}\right) = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right).$$

Theorem 2.19 now gives the three cube roots

$$\sqrt[3]{\sqrt{2}}\left(\cos\tfrac{\pi/4+2\pi j}{3} + i\sin\tfrac{\pi/4+2\pi j}{3}\right), \qquad j = 0, 1, 2.$$



cube roots of 1 + i

More specifically, the roots are

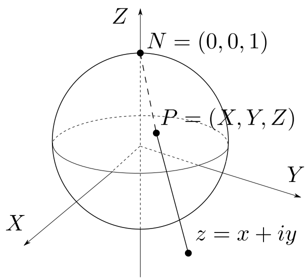
$$z_0 = 2^{1/6}\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right),$$
$$z_1 = 2^{1/6}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right),$$
$$z_2 = 2^{1/6}\left(\cos\tfrac{17\pi}{12} + i\sin\tfrac{17\pi}{12}\right).$$

It is alright to leave the roots in this form, although z_1 could easily be given more explicitly.

2.5 The Point at Infinity

Recall that in single-variable Calculus we deal with the different concepts of ∞ and $-\infty$. Given that in the complex plane we don't just have one straight line that goes through the origin, but infinitely many (one for each angle, or argument), can we then assume that there are infinitely many different "infinities"? It turns out that *there is only one point at infinity*. This fact, which at first sight appears counter-intuitive, can be seen as follows.

Consider a sphere of diameter 1, tangent to the complex plane at the origin; let N be its "north pole".



Draw a line from N to each point z in the complex plane. Then let z

correspond to the point on the sphere where the line cuts the sphere. There is now a one-to-one correspondence between every point on the complex plane and every point on the sphere, except the point N. Now adjoin to the plane a single "ideal" point ∞, to correspond to this one point N, which would make the sphere complete.

The sphere is called the *Riemann sphere*, and the correspondence is called a *stereographic projection*. The complex plane with ∞ adjoined is called the *extended complex plane*, denoted by $\overline{\mathbb{C}}$.

Some conventions about dealing with infinity:

$$\begin{gathered}
a + \infty = \infty + a = \infty \quad (a \in \mathbb{C}), \\
a \cdot \infty = \infty \cdot a = \infty \quad (a \in \overline{\mathbb{C}}, a \neq 0), \\
\frac{a}{0} = \infty \quad (a \in \overline{\mathbb{C}}, a \neq 0), \\
\frac{a}{\infty} = 0 \quad (a \in \mathbb{C}), \\
|\infty| = \infty, \quad \overline{\infty} = \infty.
\end{gathered}$$

More can be said about stereographic projections, but we will leave it at this. Further information can be found, e.g., in the book by Gamelin [3], pp. 11–13.

Chapter 3

Complex Functions

In this chapter the main objects of this course will be introduced, namely functions from $\mathbb{C}$ to $\mathbb{C}$. We will cover many concepts you already met in Calculus or Analysis courses, such as open sets, limits, continuity, derivatives, and power series. We will also study the exponential, sine and cosine functions as functions of complex variables. In this chapter we will see that there are many similarities with the real case, but also some fundamental differences.

As is usually the case in an Analysis course, we begin with a few basic topological concepts.

3.1 Topology of the Complex Plane

The main purpose of this brief section is to define a *domain*, which is a fundamental concept for the remainder of this course.

Definition 3.1. (*a*) For $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, the ε-*neighbourhood* of z_0 is

$$U_\varepsilon(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}.$$

(*b*) Let $D \subseteq \mathbb{C}$ be some set. A point $z_0 \in D$ is called an *interior point* of D if D contains an ε-neighbourhood of z_0, for some $\varepsilon > 0$.

Example 3.2. Let $D := \{z \mid \mathrm{Re}(z) > 0\}$. Then:

- $z_0 = 1 + i$ is an interior point of D;
- $z_0 = i$ is not an interior point of D.

Example 3.3. The set $D := \{z = x + iy \mid -1 < x < 1, y = 0\}$ has no interior points.

Definition 3.4. The set D is called *open* if all points $z \in D$ are interior points of D.

Example 3.5. The set $D := \{z \mid |z| < 1\}$ is open.

Example 3.6. The set $D := \{z \mid \mathrm{Re}(z) \geq 0\}$ is not open since D contains points that are not interior points.

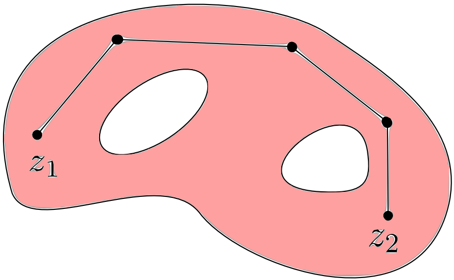
Definition 3.7. A set in which any two points can be joined by a polygonal line without self-intersection is called *connected*.

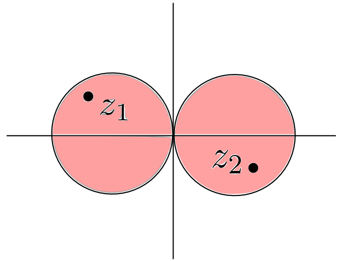
We combine the last two definitions for the following important concept.

Definition 3.8. An open and connected set of points in the complex plane is called a *domain* (sometimes also called a *region*).

Be careful not to confuse this notion of a domain with "the domain of a function". Usually it is clear from the context which concept is meant.

Example 3.9. Both sets shown below are open (if we assume they do not contain their boundaries). The set shown on the left is a domain (it is connected), while the one on the right is not a domain (it is not connected since the origin does not belong to the set).





3.2 Complex Functions

In this very brief section we address the concept of a complex-valued function $f : D \to \mathbb{C}$, where D is some subset of $\mathbb{C}$, at this point not necessarily a domain.

In general, and informally, a *function* assigns to each number z of a set $D \subseteq \mathbb{C}$ one (or more) complex numbers w; in short: $w = f(z)$. If to every $z \in D$ there is only one such w, then f is called *single-valued*.

Example 3.10. $w = z^2$, $w = \operatorname{Re} z$, $w = \overline{z}^3$ are all single-valued functions.

Example 3.11. $w = \arg z$ is not single-valued; in fact, it takes on infinitely many values. However, the principal argument of $z \neq 0$ is a single-valued function.

Example 3.12. $w = \sqrt[n]{z}$, for $n \in \mathbb{N}$ and $n \geq 2$, is an n-valued function.

In what follows, a function is assumed to be single-valued, unless otherwise noted.

Remark. A complex function can be identified with a pair of real functions of two real variables:

$$w = f(z) = u(x, y) + i\,v(x, y) \qquad (z = x + iy).$$

Example 3.13. To write $w = z^2$ in the form $w = u(x, y) + i\,v(x, y)$, we expand

$$w = (x + iy)^2 = x^2 - y^2 + i \cdot 2xy,$$

and therefore

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.$$

3.3 Limits and Continuity

In this section, which we also keep very brief, we define the concepts of limits and continuity, already well known from Calculus. However, in contrast to the usual informal definition of the limit, as done in Calculus, we need a more formal and careful definition, as usually done in Analysis courses.

Definition 3.14. Let f be defined in a neighbourhood of $a \in \mathbb{C}$. Then the number $L \in \mathbb{C}$ is said to be the *limit* of $f(z)$ as z approaches a if, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - a| < \delta \quad \text{implies} \quad |f(z) - L| < \varepsilon.$$

If this is the case, we write

$$\lim_{z \to a} f(z) = L.$$

Remarks. (a) This definition says the following: L is the limit of $f(z)$ if we can make $f(z)$ arbitrarily close to L by taking z sufficiently close to a. The idea of "arbitrarily close" is encoded in the phrase "given an $\varepsilon > 0$" (where "as small as we wish" is implied), and the idea of "sufficiently close" is encoded in the phrase "there exists a $\delta > 0$" (where "as small as necessary" is implied).

(b) It is customary to use the small Greek letters δ (read “delta”) and ε (read “epsilon”) in this setting.

(c) Normally it is not possible to draw a graph of a complex functions since it would be a curve in $\mathbb{C} \times \mathbb{C}$, which has 4 real dimensions. Therefore Definition 3.14 cannot be illustrated with a diagram, as is the case with real-valued functions defined on a subset of $\mathbb{R}$.

We now state, without proof, some properties of the limit that are specific to complex functions.

Theorem 3.15. (*a*) *We have the following equivalences:*

$$\begin{aligned}\lim_{z\to a} f(z) = L &\Leftrightarrow \lim_{z\to a} \overline{f(z)} = \overline{L} \\ &\Leftrightarrow \lim_{z\to a} \operatorname{Re} f(z) = \operatorname{Re} L \quad \text{and} \quad \lim_{z\to a} \operatorname{Im} f(z) = \operatorname{Im} L.\end{aligned}$$

(*b*) *When $L \neq 0$, we have the following equivalence:*

$$\lim_{z\to a} f(z) = L \Leftrightarrow \lim_{z\to a} |f(z)| = |L| \quad \text{and} \quad \lim_{z\to a} \arg f(z) = \arg L.$$

(c) *We have the following equivalence:*

$$\lim_{z\to a} f(z) = 0 \Leftrightarrow \lim_{z\to a} |f(z)| = 0.$$

Remark. Since later in this course we will be dealing with sequences, it is worth mentioning at this point that limits of sequences are defined analogously.

The following properties of limits are identical with the corresponding properties in the real case.

Theorem 3.16. *If* $\lim_{z\to a} f(z) = L$ *and* $\lim_{z\to a} g(z) = K$, *then*

(a) $\lim_{z\to a}(f(z) \pm g(z)) = L \pm K$.

(b) $\lim_{z\to a}(f(z)g(z)) = LK$.

(c) $\lim_{z\to a} \dfrac{f(z)}{g(z)} = \dfrac{L}{K}$ *(for* $K \neq 0$*).*

The concept of continuity is central to all areas of (real and complex) analysis. Here we define it in the same way as we did in Calculus.

Definition 3.17. (*a*) A function $w = f(z)$, defined in a neighbourhood of $a \in \mathbb{C}$, is said to be *continuous at* a if

$$\lim_{z \to a} f(z) = f(a).$$

(*b*) The function $f(z)$ is *continuous in* a domain D if it is continuous at every $a \in G$.

Remark. Continuity of f at a point a is in fact defined by three properties:

- f is defined at a;
- the limit of f as $z \to a$ exists;
- the two numbers are the same.

Each one of these properties may fail, to make f discontinuous at a, as we usually see in Calculus.

The following rules for continuous functions are immediate consequences of the limit rules in Theorems 3.15 and 3.16, with the exception of part (c) which would require a separate proof, similar to the real case. We skip the proofs.

Theorem 3.18. *Let f and g be continuous at $a \in \mathbb{C}$. Then*

(*a*) *$f \pm g$, fg, and f/g $(g(a) \neq 0)$ are continuous at a;*

(*b*) *$\mathrm{Re} f$, $\mathrm{Im} f$, $|f|$, and $\arg f$ $((f(a) \neq 0)$ are continuous at a;*

(*c*) *if g is continuous at a and f is continuous at $g(a)$, then $f(g(a))$ is continuous at a.*

Example 3.19. It is clear that $f(z) = c$ (a constant) and $f(z) = z$ are continuous in $\mathbb{C}$. By using Theorem 3.18 (a) multiple times, this implies that all polynomials are continuous in $\mathbb{C}$. Furthermore, all rational functions (i.e., quotients of two polynomials) are continuous in $\mathbb{C}$ except at the points where the denominator polynomial is zero.

Example 3.20. Again by Theorem 3.18, the function

$$f(z) = \frac{\mathrm{Re}\, z - i\,\overline{z}}{1 - |z|^3}$$

is continuous in $\mathbb{C} \setminus \{z \mid |z| = 1\}$.

3.4 Derivatives

3.4.1 Some Basics

We now define one of the central concepts of this course, namely differentiability and the derivative of a complex function. The definition is identical with that of real-valued functions, and many properties carry over from the real case. Still, we will see that there are some fundamental differences.

Definition 3.21. Let f be a complex function defined on an open set U. We say that f is *differentiable* at $z_0 \in U$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If so, we call this limit the *derivative* of f at z_0, denoted by $f'(z_0)$. If f is differentiable at every point of U, we say that f is *differentiable on* U.

Remark. If f is differentiable on U, we can consider $f'(z)$ a function on U. As in the real case, we also use the *Leibniz notation* $\frac{df(z)}{dz} = f'(z)$.

Example 3.22. Is $f(z) = z^2$ differentiable? Using the definition, we have

$$\begin{aligned}\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{z^2-z_0^2}{z-z_0} &= \lim_{z\to z_0} \frac{(z-z_0)(z+z_0)}{z-z_0} \\ &= \lim_{z\to z_0} (z+z_0) = 2z_0.\end{aligned}$$

Hence f is differentiable at any $z_0 \in \mathbb{C}$, and $f'(z_0) = 2z_0$.

The following theorem is also the exact analogue of the real case.

Theorem 3.23. *If f is differentiable at z_0, then f is continuous at z_0.*

Proof. The main trick in the proof that follows is to multiply and divide by $z - z_0$; we then use Theorem 3.16 (b):

$$\begin{aligned}\lim_{z\to z_0} (f(z) - f(z_0)) &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) \\ &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0} (z-z_0) \\ &= f'(z_0)\cdot 0 = 0.\end{aligned}$$

Hence $\lim_{z\to z_0} f(z) = f(z_0)$, as desired. □

The following differentiation rules are identical with those in the real case. The proofs are also the same, so we skip them.

Theorem 3.24. *If f and g are differentiable at z_0, then so are $f+g$, $f-g$, fg, and f/g (if $g(z_0) \neq 0$). Furthermore,*

(a) $(f \pm g)' = f' \pm g'$;

(b) $(fg)' = fg' + f'g$;

(c) $\left(\frac{f}{g}\right)' = \frac{1}{g^2}(f'g - fg')$. *(for $K \neq 0$).*

Example 3.25. Show that $\frac{d}{dz} z^n = nz^{n-1}$, where $n \in \mathbb{N}$.

Proof. There are two ways of proving this. First, one could use a slightly modified definition of derivative (see a bit later in this section) and the binomial expansion of $(z+h)^n$.

A second possibility is to use the definition of the derivative to show that $\frac{d}{dz} z = 1$ and $\frac{d}{dz} c = 0$ for any constant c, which is very easy (easier than Example 3.22). Then use induction as follows: We already have the induction beginning. Now assume that the derivative holds for some n. Using the

Product Rule (Theorem 3.24 (b)), we then obtain

$$\frac{d}{dz}z^{n+1} = \frac{d}{dz}(z \cdot z^n) = z\frac{d}{dz}z^n + \frac{d}{dz}z \cdot z^n = z \cdot nz^{n-1} + 1 \cdot z^n = (n+1)z^n,$$

where we have used the induction hypothesis. This completes the proof. $\square$

Example 3.26. Is $f(z) = \overline{z}$ differentiable at 0?

Solution. If f were differentiable, then by definition the limit

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{z\to 0} \frac{x - iy}{x + iy}$$

would have to exist. But this means that it would have to be the same regardless of how z approaches 0. We choose the two most convenient ways of approaching 0:

(a) Along the real axis, i.e., $y = 0$. Then

$$\lim_{z\to 0} \frac{x - iy}{x + iy} = \lim_{x\to 0} \frac{x}{x} = 1.$$

(*b*) Along the imaginary axis, i.e., $x = 0$. Then

$$\lim_{z \to 0} \frac{x - iy}{x + iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1.$$

The two are not the same, so the limit as such does not exist, which means that the function $f(z) = \overline{z}$ is *not* differentiable at 0.

Soon we will see that this function is, in fact, not differentiable anywhere on $\mathbb{C}$.

We continue with another important differentiation rule, which is again the same as in the real case. The proof, once again, is the same as in the real case, so we omit it here.

Theorem 3.27 (The Chain Rule)**.** *If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then the composition $f(g(z))$ is differentiable at z_0, and*

$$\frac{d}{dz} f(g(z))\Big|_{z=z_0} = f'(g(z_0))g'(z_0).$$

3.4.2 The Cauchy-Riemann Equations

With the exception of Example 3.26, we haven not seen any difference between the real and the complex case so far. The following basic result will change this possible misconception; we will see that complex differentiability of a function f poses very strict constraints on f.

As we did in Section 3.2, we write f in terms of its real and complex parts:

$$f(z) = u(x,y) + i\,v(x,y),$$

where, as usual, $z = x + iy$. Since u and v are both real-valued functions in the two real variable x, y, we can consider the partial derivatives, as they occur in any second-year calculus course.

Theorem 3.28 (The Cauchy-Riemann Equations)**.** *If f is differentiable at $z = x + iy$, then the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ all exist at (x,y) and satisfy the two equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Proof. We first note that the limit in the definition of the derivative can be slightly rewritten as

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

where h is complex. Now, as we did in Example 3.26, we evaluate this limit separately along the real axis and along the imaginary axis. Since by assumption the limit as such exists, the two evaluations must agree.

(a) For real h we have

$$
\begin{aligned}
f'(z) &= \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\\
&= \lim_{h\to 0}\frac{u(x+h,y)+i\,v(x+h,y)-u(x,y)-i\,v(x,y)}{h}\\
&= \lim_{h\to 0}\frac{u(x+h,y)-u(x,y)}{h}+\lim_{h\to 0}\frac{i(v(x+h,y)-v(x,y))}{h}\\
&= \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.
\end{aligned}
$$

(b) For imaginary h we set $h = ik$, $k \in \mathbb{R}$, and we get

$$
\begin{aligned}
f'(z) &= \lim_{k\to 0}\frac{u(x,y+k)+i\,v(x,y+k)-u(x,y)-i\,v(x,y)}{ik}\\
&= \lim_{k\to 0}\frac{1}{i}\,\frac{u(x,y+k)-u(x,y)}{k}+\lim_{k\to 0}\frac{i}{i}\,\frac{(v(x,y+k)-v(x,y))}{k}
\end{aligned}
$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Finally, equating the real and imaginary parts of the results of (a) and (b), we get the two Cauchy-Riemann equations. □

Example 3.29. Let's revisit the function $f(z) = \overline{z}$ of Example 3.26. Since $f(z) = x - iy$, we have $u(x, y) = x$ and $v(x, y) = -y$, and thus

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1.$$

Obviously, the first Cauchy-Riemann equation is not satisfied, and there is no need to check the second one. Since this is independent of the particular (x, y), we conclude that the function $f(z) = \overline{z}$ is not differentiable anywhere in $\mathbb{C}$.

It is important to realize that the Cauchy-Riemann equations are a *necessary condition* for differentiability; it can be shown by way of examples that the converse is in general *not* true. However, if we impose some extra conditions on the given function f, then the converse is in fact true:

Theorem 3.30. *Suppose that $f(z) = u(x, y) + i\,v(x, y)$ is defined on an open set U and that at some $z_0 \in U$ the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$*

exist, are continuous, and satisfy the Cauchy-Riemann equations, then f is differentiable at z_0.

We skip the proof, which can be found in the textbook. Note that the extra condition is the continuity of all four partial derivatives.

Remark. The two Cauchy-Riemann equations can be written as one single equation

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

To prove this, we use simple differentiation rules and both Cauchy-Riemann equations, to obtain

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial(u+iv)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\left(\frac{-\partial u}{\partial y}\right) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ &= -i\frac{\partial(u+iv)}{\partial y} = -i\frac{\partial f}{\partial y},\end{aligned}$$

as desired.

It is now time for a definition that is closely related to that of differentiability, but is somewhat stronger.

Definition 3.31. A function $f(z)$ is called *analytic at* a point z_0 if it is differentiable in a neighbourhood of z_0 (including z_0 itself).

From now on we will almost exclusively deal with analytic functions. An often used synonymous term is *holomorphic function.* The term *analytic* is actually related to the possibility of writing a function as a power series. We will get to this later in this course.

As mentioned before, the Cauchy-Riemann equations show that differentiability poses strong conditions on a function f and its real and imaginary parts u and v. In particular, u and v are not independent from each other, as the following example shows.

Example 3.32. Find all differentiable function $f(z)$ for which $\operatorname{Re} f(z) = x$, where $z = x + iy$.

Solution. We set, as usual, $f(z) = u(x, y) + i\,v(x, y)$. Then $u(x, y) = x$, and our task is to find the most general function $v(x, y)$ such that u and v satisfy the Cauchy-Riemann equations. We consider the two equations in sequence.

(*a*) Since $\frac{\partial u}{\partial x} = 1$, we also have $\frac{\partial v}{\partial y} = 1$. Taking the antiderivative with respect to y we get

$$v(x,y) = y + c(x),$$

where it is crucial to realize that the constant $c(x)$ is constant with respect to y, but is still a function of x.

(*b*) Since $\frac{\partial u}{\partial y} = 0$, we also have $\frac{\partial v}{\partial x} = 0$, which means $0 + c'(x) = 0$, or (taking the antiderivative with respect to x), $c(x) = c$, a constant. So we have obtained $v(x,y) = y + c$, and thus

$$f(z) = u(x,y) + i\,v(x,y) = x + i(y + c) = z + ic.$$

So the only possible differentiable functions that satisfy $\operatorname{Re} f(z) = x$ are $f(z) = z + ic$, where c is an arbitrary real constant.

Example 3.33. The function $f(z) = |z|$ is not differentiable at any $z \in \mathbb{C}$.

Proof. Since $|z| = \sqrt{x^2 + y^2}$, we have $u(x,y) = \sqrt{x^2 + y^2}$ and $v(x,y) = 0$. This gives the partial derivatives

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0,$$

where the first tho identities are valid when $(x, y) \neq (0,0)$. It is now clear that the Cauchy-Riemann equations do not hold when $(x, y) \neq (0,0)$.
It remains to deal with the case $z = 0$. We use the definition of the derivative and take the limit along the real axis:

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{x \to 0} \frac{|x|}{x}.$$

In Calculus we saw that the limit on the right does not exist. Let's review the argument: Since $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x \leq 0$, we have the two one-sided limits

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1,$$
$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

Since the two don't agree, the limit as such does not exist, and the function is not differentiable at 0. Altogether, $f(z) = |z|$ is nowhere differentiable, as claimed. $\square$

Remark. The functions $\operatorname{Re} z$ and $\operatorname{Im} z$ are also nowhere differentiable (see Assignment 3), as well as $\overline{z}$, as we have seen.

We conclude this section with two important consequences of the Cauchy-Riemann equations. The first one is analogous to what we know about real-valued functions.

Theorem 3.34. *If $f(z)$ is analytic in a domain D and if $f'(z) = 0$ on D, then $f(z)$ is constant on D.*

Proof. We saw in the proof of the Cauchy-Riemann equations that

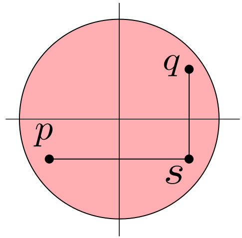
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

So, if $f'(z) = 0$ for all $z \in D$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{for all } z \in D.$$

(a) We first prove the result for an open disk $U_R(0) = \{z \mid |z| < R\}$. We have to show: $f(p) = f(q)$ for arbitrary $p, q \in U_R(0)$.

Fix such $p = a+ib$, $q = c+id$. Then at least one of $s = c+ib$ and $t = a+id$ lies in $U_R(0)$. Without loss of generality suppose that s does; see the image below.



Now the functions $x \mapsto u(x,b), y \mapsto U(c,y)$ are real-valued functions with 0 derivative, and so they are constant. Hence

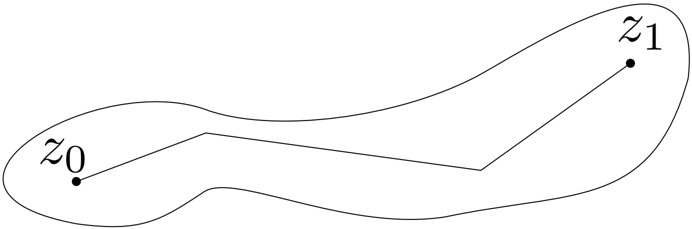
$$u(a,b) = u(c,b), \qquad u(c,b) = u(c,d).$$

Similarly,

$$v(a,b) = v(c,b), \qquad v(c,b) = v(c,d),$$

and so $f(p) = f(s) = f(q)$, which was to be shown.

(b) More generally, this argument works for any circle. Finally, in any domain we can cover the polygonal line connecting two arbitrary points z_0, z_1 (see the illustration below) with a sequence of overlapping circles; this, with the first part of the proof, shows that $f(z_0) = f(z_1)$. The proof is now complete.



□

The following theorem generalizes what we've seen in Example 3.33.

Theorem 3.35. *If $f(z)$ is analytic and real-valued in a domain D, then $f(z)$ is constant on D.*

Proof. Since f is real-valued, we have $v(x, y) = 0$ for all $z = x + iy \in D$. Hence

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

and by the Cauchy-Riemann equations also

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad \text{for all } z = x + iy \in D.$$

The result now follows from the proof of Theorem 3.34. □

3.5 Power Series

The term *analytic* at z_0, defined to mean "differentiable in a neighbourhood of z_0", originally means "the function can be written as a power series around z_0". It is the purpose of this section to review power series, which are usually introduced in first-year Calculus (for real-valued function). The surprising fact that complex-differentiable functions can always be written as power series is one of the main results of complex analysis; this will be proved later in this course. We begin with some background, which is mainly analogous to the real case.

3.5.1 Infinite Series

This may be a good point to review infinite series, as usually studied in a second-semester Calculus course (MATH 1010 at Dalhousie).

Definition 3.36. Let $\{a_k\}_{k\geq 0}$ be a sequence of complex numbers. The series $\sum_{k=0}^{\infty} a_k$ is said to *converge* to the sum S if the sequence of partial sums $\{S_n\}$, where

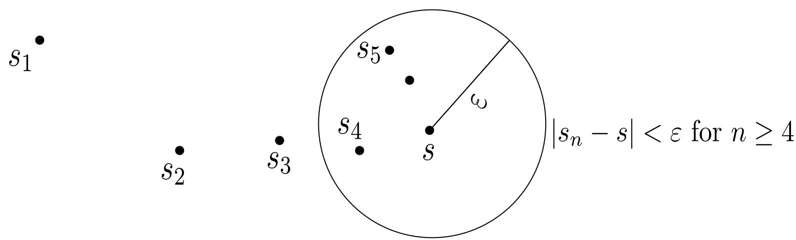
$$S_n := a_0 + a_1 + a_2 + \cdots + a_n,$$

converges to the limit S. In this case we write

$$S = \sum_{k=0}^{\infty} a_k.$$

Remark. While for our purposes an "intuitive idea" of the limit of a sequence $\{S_n\}$ usually suffices, it is worth giving a more exact definition anyway. We say that the sequence $\{S_n\}$ of complex numbers *converges* to (the limit) S if for any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|S_n - S| < \varepsilon \quad \text{for all } n \geq N.$$



In other words: No matter how small a circle we draw around S, eventually all the S_n will lie inside this circle.

Example 3.37. For which z does the series $\sum_{k=0}^{\infty} z^k$ converge, and for which z does it diverge?

Solution. We have the partial sum

$$S_n = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}. \tag{3.5.1}$$

This can be seen by multiplying both sides by the denominator $1 - z$:

$$(1 - z)(1 + z + z^2 + \cdots + z^n) = 1 - z^{n+1},$$

noting that upon expanding the left-hand side most terms cancel, with only two on the right remaining.

Now, if $|z| < 1$, then $z^{n+1} \to 0$ as $n \to \infty$, and therefore $S_n \to \frac{1}{1-z}$ as $n \to \infty$. Consequently, we have the important and well-known series expansion

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z} \qquad (|z| < 1), \tag{3.5.2}$$

an identity you will already know from Calculus in the case of real z. Here, however, as we just saw, (3.5.2) holds for all *complex* z with $|z| < 1$.

On the other hand, if $|z| \geq 1$ and $z \neq 1$, then (3.5.1) shows that the sequence S_n does not converge. Finally, when $z = 1$, we have $S_n = n + 1$, which obviously also diverges. To summarize: we have shown that the series in question converges if and only if $|z| < 1$.

Remarks. (1) As you will remember from Calculus and/or Analysis, the series (3.5.2) is called the *geometric series.*

(2) Adding or subtracting finitely many terms to (or from) a series does not change convergence or divergence of the series. Therefore we often write simply $\sum a_k$ for a series when we are only interested in convergence/divergence and not in its sum (in case it converges).

The following theorem lists some important properties of infinite series. The three parts are completely analogous to the real case. The proofs are also very similar, so we skip them.

Theorem 3.38. *(a) If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$.*

(b) If $\sum |a_k|$ converges, then $\sum a_k$ converges.

(c) *Let* $\sum b_k$ *be a convergent series, where* $b_k \geq 0$ *for all* k*. If for some constant* $c > 0$ *we have* $|a_k| \leq c\, b_k$ *for all* k*, then* $\sum a_k$ *converges.*

Remarks. (1) It is *most important* to realize that the converse of part (a) of the theorem does *not* hold. The standard example is the *harmonic series* $\sum \frac{1}{k}$ which *diverges* (this is often shown in Calculus and/or in 2nd year Analysis), while, obviously, $\frac{1}{k} \to 0$.

(2) Part (c) of this theorem is a variant of what is known as the *comparison test.*

Part (b) of this theorem gives rise to the following definition.

Definition 3.39. The series $\sum a_k$ with complex terms is called *absolutely convergent* if $\sum |a_k|$ converges.

Remarks. (1) We can now restate Theorem 3.38 (b): *Every absolutely convergent series is convergent.*

(2) Related to this, since $\sum |a_k|$ is a series of real positive terms, all known tests for convergence can be applied to establish absolute convergence of $\sum a_k$.

3.5.2 Power Series

Example 3.37 can actually be seen as a series of *functions*, rather than just numbers. A general study of series of functions is very important in some areas of real and complex analysis (for instance, it is usually covered in Dalhousie's MATH 3501/3502). However, here we will restrict ourselves to the most important case, namely that of monomials, as in the case of Example 3.37. The following definition is basic to much of the remainder of this course.

Definition 3.40. A series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

with coefficients $a_k \in \mathbb{C}$ is called a *power series* about $z_0 \in \mathbb{C}$.

Remark. Most often we consider the case $z_0 = 0$. This presents no loss of generality as we can make a change of variable $z \mapsto z - z_0$.

Power series have some very important and special properties:

Theorem 3.41. *Let* $\sum a_k z^k$ *be a power series. Then either*

(a) *the series converges only for* $z = 0$*; or*

(b) *the series converges for all* $z \in \mathbb{C}$*; or*

(c) *there is a number* $R > 0$ *such that the series*

- *converges for all* z *with* $|z| < R$*, and*
- *diverges for all* z *with* $|z| > R$.

Before proving this important result, a few remarks, definitions, and examples are in order.

Remarks. (1) The case (a) of power series that converge only at $z = 0$ is not very interesting, and is usually ignored.

(2) In case (c), the power series may or may not converge for a z with $|z| = R$.

Case (c) of Theorem 3.41 also gives rise to the following definition.

Definition 3.42. The number $R > 0$ in Theorem 3.41 is called the *radius of convergence* of the power series $\sum a_k z^k$, and the open disk $\{z \mid |z| < R\}$ is called the *disk of convergence.*

When the series converges only at $z = 0$, we set $R = 0$, and when it converges for all $z \in \mathbb{C}$, we set $R = \infty$.

Example 3.43. We saw in Example 3.37 that the power series $\sum z^n$ has radius of convergence $R = 1$.

Example 3.44. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^k}$ converges for all $z \in \mathbb{C}$.

Solution. We fix a $z \in \mathbb{C}$ and choose an integer $N > |z|$. Then for all $k > N$ we have

$$\frac{|z|^k}{k^k} < \frac{|z|^k}{N^k} = \left(\frac{|z|}{N}\right)^k . \tag{3.5.3}$$

But by Example 3.37 we know that

$$\sum_{k=0}^{\infty} \left(\frac{|z|}{N}\right)^k \quad \text{converges, since} \quad \frac{|z|}{N} < 1,$$

and by (3.5.3) and the comparison test (Theorem 3.38(c) with $c = 1$),

$$\sum_{k=N+1}^{\infty} \frac{|z|^k}{k^k} \quad \text{converges,}$$

and so

$$\sum_{k=N+1}^{\infty} \frac{z^k}{k^k} \quad \text{converges absolutely,}$$

and therefore it converges, by Theorem 3.38 (b). Finally, since the first finitely many terms don't matter, we see that the given series converges. We did all this for an arbitrary $z \in \mathbb{C}$, so the series converges for all z, and so $R = \infty$.

The proof that now follows is actually quite similar to the solution of this last example.

Proof of Theorem 3.41. First note that the series always converges for $z = 0$.

(1) If (a) does not hold, then it must converge for some $b \neq 0$. By Theorem 3.38 (a), the kth terms of the series must tend to 0, that is,

$$a_k b^k \to 0 \quad \text{as} \quad k \to \infty.$$

So, given any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\left|a_k b^k\right| < \varepsilon$ for all $k \geq N$. Now, for such $k \geq N$ we have

$$\left|a_k z^k\right| = \left|a_k b^k\right| \cdot |z/b|^k < \varepsilon\, |z/b|^k\,. \tag{3.5.4}$$

If $||z| < b$, then $\sum |z/b|^k$ converges (again by Example 3.37), and with (3.5.4) and the comparison test (Theorem 3.38 (c) with $c = \varepsilon$), we see that $\sum \left|a_k z^k\right|$ converges.
As in Example 3.44, the fact that k starts at N does not matter, so $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely, and therefore it converges, for all z with $|z| < |b|$.

(2) If case (b) does not hold, that is, if the series does not converge for all $z \in \mathbb{C}$, then it must diverge for some $z_0 \in \mathbb{C}$. Suppose now that the series converges for some z with $|z| > |z_0|$. But then by part (1) it would also have to converge for z_0, which is a contradiction.

So, altogether, parts (1) and (2) of this proof tell us: If the series is neither of type (a) nor of type (b), then there is a unique $R > 0$ as stated in (c) of the theorem. This completes the proof. $\square$

The most useful aid in computing the radius of convergence is the *ratio test*, applied to the corresponding series of absolute values. We recall from Calculus that this important test can be stated as follows; a proof can be found in most Calculus or Analysis books.

Theorem 3.45 (The Ratio Test). *Let $\sum c_k$ be a series with $c_k > 0$ for all k.*

(*a*) *If $\displaystyle\lim_{k\to\infty} \frac{c_{k+1}}{c_k} = L < 1$, then $\sum c_k$ is convergent.*

(*b*) *If $\displaystyle\lim_{k\to\infty} \frac{c_{k+1}}{c_k} = L > 1$, then $\sum c_k$ is divergent.*

For the sake of completeness it should also be mentioned that the test is inconclusive if $L = 1$. The following examples show how the ratio test can be applied.

Example 3.46. Find the radius of convergence of $\sum k^3 z^k$.

Solution. We apply the ratio test to $\sum \left|k^3 z^k\right|$: For $z \neq 0$,

$$\left|\frac{(k+1)^3 z^{k+1}}{k^3 z^k}\right| = \left(\frac{k+1}{k}\right)^3 |z| = \left(1 + \frac{1}{k}\right)^3 |z| \to |z|,$$

as $k \to \infty$.

- If $|z| < 1$, by Theorem 3.45, the series converges (absolutely).
- If $|z| > 1$, consider a real number r with $1 < r < |z|$. By Theorem 3.45,

$$\left|\frac{(k+1)^3 r^{k+1}}{k^3 r^k}\right| = \left(1 + \frac{1}{k}\right)^3 r \to r > 1 \quad \text{as } k \to \infty,$$

  the series at $z = r$ diverges. By the proof of Theorem 3.41, if $|z| > r$, then the series diverges.

Therefore, the radius of convergence is $R = 1$.

Remark. If we further wish to explore the case $|z| = 1$, at the boundary, we notice that if $|z| = 1$,

$$\lim_{k\to\infty} \left|k^3 z^k\right| = k^3 = \infty,$$

which implies the terms do not converge to 0, so that the series is divergent.

Example 3.47. Find the radius of convergence of $\sum \frac{z^k}{k!}$.

Solution. Apply the ratio test again: For $z \neq 0$,

$$\left|\frac{z^{k+1}}{(k+1)!} \cdot \frac{k!}{z^k}\right| = \frac{|z|}{k+1} \to 0 \quad \text{as} \quad k \to \infty,$$

and we see that the limit is independent of z. So the series converges (absolutely) for any $z \in \mathbb{C}$, and thus $R = \infty$.

Example 3.48. Find the radius of convergence of $\sum_{n=0}^{\infty} (-1)^n 2^n z^{2n+2}$.

Solution. Apply the ratio test again: For $z \neq 0$,

$$\left| \frac{2^{n+1} z^{2n+4}}{2^n z^{2n+2}} \right| = 2 \left| z \right|^2 ,$$

which is already the limit. Hence the series converges (absolutely) when $2 \left| z \right|^2 < 1$, i.e., when $|z| < 1/\sqrt{2}$, and diverges when $2 \left| z \right|^2 > 1$, i.e., when $|z| > 1/\sqrt{2}$. Therefore $R = 1/\sqrt{2}$.

Second Solution. Rewrite the series:

$$\sum_{n=0}^{\infty} (-1)^n 2^n z^{2n+2} = z^2 \sum_{n=0}^{\infty} \left(-2z^2 \right)^n .$$

But this is a geometric series, which converges for $|-2z^2| < 1$, and this means $|z| < 1/\sqrt{2}$, so again $R = 1/\sqrt{2}$.

Example 3.49. Determine for which values of z, the series $\sum_{k=1}^{\infty} \frac{(z-i)^k}{k^3}$ converges.

Solution. Note that we just need to consider the series

$$\sum_{k=1}^{\infty} \frac{z^k}{k^3}, \tag{3.5.5}$$

then the original one can studied by translation. Since

$$\lim_{k\to\infty} \left| \frac{\frac{z^{k+1}}{(k+1)^3}}{\frac{z^k}{k^3}} \right| = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)^3 |z| = |z|,$$

we have the following results.

- If $|z| < 1$, by Theorem 3.45, the series (3.5.5) is absolutely convergent.
- If $|z| > 1$, let r be a real number such that $|z| > r > 1$. Then by

Theorem 3.45 once again, we see

$$\lim_{k\to\infty}\left|\frac{\frac{r^{k+1}}{(k+1)^3}}{\frac{r^k}{k^3}}\right| = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)^3 r = r > 1,$$

which implies when $z = r$, the series (3.5.5) is divergent. Hence, it is divergent for $|z| > r$. Therefore, if $|z| > 1$, series (3.5.5) is divergent.

- If $|z| = 1$, we see

$$\left|\frac{z^k}{k^3}\right| = \frac{1}{k^3}.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^3}$ is a convergent series, by the property of p-series, we see when $|z| = 1$, the series (3.5.5) is absolutely convergent, by the comparison test.

Now, finally, by translation, we see for the series $\sum_{k=1}^{\infty}\frac{(z-i)^k}{k^3}$,

- it is absolutely convergent if $|z - i| \le 1$;

- and it is divergent if $|z - i| > 1$.

3.5.3 Power Series as Functions

A power series with radius of convergence $R > 0$ can be considered as defining a function on $D = U_R(0)$, so we write

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (|z| < R).$$

Our aim is to show that $f(z)$ is complex differentiable (or analytic) in D. We begin with a lemma.

Lemma 3.50. *The power series* $\sum a_k z^k$ *and* $\sum k a_k z^{k-1}$ *have the same radius of convergence.*

Proof. Two different directions need to be proved.

(1) Suppose that $\sum a_k z^k$ has radius of convergence $R > 0$. We want to show that the same is true for $\sum k a_k z^{k-1}$. We fix a z with $0 < |z| < R$ and

choose an r such that $|z| < r < R$. Then

$$\left|ka_k z^{k-1}\right| = \frac{k}{|z|}\left(\frac{|z|}{r}\right)^k \left|a_k r^k\right|.$$

By the ratio test (similar to Example 3.46),

$$\sum k\left(\frac{|z|}{r}\right)^k \qquad \text{converges.}$$

Hence by Theorem 3.38 (a), there is an $M > 0$ such that

$$k\left(\frac{|z|}{r}\right)^k \le M \qquad \text{for all } k,$$

so that

$$\left|ka_k z^{k-1}\right| \le \frac{M}{|z|}\left|a_k r^k\right|.$$

The result now follows.

(2) The opposite direction, i.e., assuming that $\sum ka_k z^{k-1}$ has radius of convergence $R > 0$, is similar. The details are left as an exercise.

$\square$

With this lemma we can now obtain the main result of this section.

Theorem 3.51. *Suppose that* $\sum a_k z^k$ *has radius of convergence* $R > 0$, *and define*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (|z| < R).$$

Then f *is analytic on* $U_R(0)$, *and*

$$f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad (|z| < R).$$

Proof. By Lemma 3.50 we know that

$$g(z) := \sum_{k=1}^{\infty} k a_k z^{k-1} \qquad (|z| < R)$$

is well-defined. We want to show: $f'(z)$ exists, and $f'(z) = g(z)$.

To do so, we choose a $z \in U_R(0)$ and $h \in \mathbb{C}$ small enough so that $z+h \in U_R(0)$. Then consider

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| = \left|\sum_{k=1}^{\infty} a_k \left(\frac{(z+h)^k - z^k}{h} - kz^{k-1}\right)\right|. \tag{3.5.6}$$

Now we use the binomial expansion

$$(z+h)^k = z^k + kz^{k-1}h + \sum_{j=2}^{k} \binom{k}{j} z^{k-j} h^j,$$

so that

$$\frac{(z+h)^k - z^k}{h} - kz^{k-1} = \sum_{j=2}^{k} \binom{k}{j} z^{k-j} h^{j-1}.$$

Also, using a binomial identity, we have for $j \geq 2$,

$$\binom{k}{j} = \frac{k}{j}\binom{k-1}{j-1} = \frac{k(k-1)}{j(j-1)}\binom{k-2}{j-2} \leq \frac{k(k-1)}{2}\binom{k-2}{j-2},$$

so that with (3.5.6) we have, first using the triangle inequality,

$$\begin{aligned}
\left|\frac{f(z+h)-f(z)}{h}-g(z)\right| &\leq \sum_{k=1}^{\infty}|a_k|\sum_{j=2}^{k}\binom{k}{j}|z|^{k-j}|h|^{j-1} \\
&\leq |h|\sum_{k=1}^{\infty}\frac{k(k-1)}{2}|a_k|\sum_{j=2}^{k}\binom{k-2}{j-2}|z|^{k-j}|h|^{j-2} \\
&\leq |h|\sum_{k=1}^{\infty}\frac{k(k-1)}{2}|a_k|\sum_{j=0}^{k-2}\binom{k-2}{j}|z|^{k-2-j}|h|^{j} \\
&= |h|\sum_{k=1}^{\infty}\frac{k(k-1)}{2}|a_k|(|z|+|h|)^{k-2}.
\end{aligned}$$

Note that in the last step we have again used a binomial expansion. Now choose an r such that $|z| < r < R$. By Lemma 3.50, the series

$$\sum_{k=1}^{\infty} k(k-1)|a_k|r^{k-2}$$

converges, with sum $K > 0$, say. So for $|h| < r - |z|$,

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq \frac{1}{2} K |h|.$$

Finally, as $h \to 0$, we have

$$\frac{f(z+h) - f(z)}{h} \to g(z),$$

which is what we wanted to show. □

By iterating Theorem 3.51 and evaluating the nth derivative at 0, we immediately get the following consequence.

Corollary 3.52. *Suppose that* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *has radius of convergence* $R > 0$. *Then* f *has derivatives of all orders at 0, and*

$$f^{(n)}(0) = n! a_n \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

Remarks. (1) This corollary shows that the coefficients in a power series expansion are unique.

(2) We showed in this section: Every power series with positive radius of convergence is analytic (at 0). The remarkable fact (proved later in this course) is that, conversely, every function that is analytic at 0 has a power series expansion about 0 with positive radius of convergence.

3.6 Elementary Functions

So far we have only dealt with some very simple polynomials or rational functions, followed by some generalities in the last section. In this section we will meet the complex versions of the most important elementary functions, namely the exponential function, trigonometric and hyperbolic functions, and the logarithmic function.

3.6.1 The Exponential Function, Sine and Cosine

We begin by defining the most important function of all.

Definition 3.53. We define the *exponential function* e^z by the power series

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad (z \in \mathbb{C}).$$

Recall from Example 3.47 that this series has radius of convergence $R = \infty$. We now list some further properties.

Theorem 3.54. *(a) e^z is analytic in $\mathbb{C}$, and*

$$\frac{d}{dz} e^z = e^z \qquad (z \in \mathbb{C}).$$

(b) For all $z, w \in \mathbb{C}$ we have

$$e^{z+w} = e^z e^w.$$

(c) $e^z \neq 0$ for any $z \in \mathbb{C}$; $e^z > 0$ if $z \in \mathbb{R}$.

(d) $|e^z| = e^{\operatorname{Re} z}$ for all $z \in \mathbb{C}$; $\left|e^{iz}\right| = 1$ for $z \in \mathbb{R}$.

Proof. (a) It follows from Theorem 3.51; note that

$$\frac{d}{dz} e^z = \frac{d}{dz} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=1}^{\infty} \frac{k}{k!} z^{k-1} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

where in the second-last step we shifted the summation by 1.

(*b*) Fix $z_0 \in \mathbb{C}$ and consider $f(z) := e^z e^{z_0 - z}$. Differentiate, using product rule and chain rule:

$$f'(z) = e^z e^{z_0 - z} + e^z e^{z_0 - z}(-1) = 0.$$

So by Theorem 3.34, $f(z)$ is constant in $U_R(0)$ for any R, and so in all of $\mathbb{C}$. Therefore $f(z) = f(0)$, i.e., $e^z e^{z_0 - z} = e^{z_0}$. Finally, setting $z_0 = z + w$, we get the desired identity.

(*c*) The first statement follows from $e^z e^{-z} = e^0 = 1$; the second statement follows from this and the fact that $e^z > 0$ for $z \in \mathbb{R}$ and $z \geq 0$, which follows from the definition of e^z.

(*d*) By Theorem 2.6 (5), by the definition of e^z, and by part (*b*) we have

$$|e^z|^2 = e^z \overline{e^z} = e^z e^{\overline{z}} = e^{z + \overline{z}} = e^{2\mathrm{Re} z} = \left(e^{\mathrm{Re} z}\right)^2,$$

and thus

$$|e^z| = e^{\mathrm{Re} z},$$

as required. Next, for $z \in \mathbb{R}$ we have $\mathrm{Re}(iz) = 0$, and so, by what we just showed, $\left|e^{iz}\right| = e^0 = 1$.

□

The following important definitions will already be familiar, in the real case, as Taylor (or Maclaurin) series from Calculus.

Definition 3.55. For all $z \in \mathbb{C}$ we define

$$\cos z := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \ldots$$

$$\sin z := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \ldots$$

The next theorem shows that the complex cosine and sine functions have the expected properties and that, restricted to the reals, they are identical with the real sine and cosine functions.

Theorem 3.56. *(a) The functions* $\cos z$ *and* $\sin z$ *are analytic in* $\mathbb{C}$.

(b) $\frac{d}{dz} \cos z = -sin z,$ $\quad \frac{d}{dz} \sin z = \cos z.$

Proof. (a)follows from Theorem 3.51. (b) is left as an exercise (also related to Theorem 3.51). □

Combining Definitions 3.53 and 3.55, we get the following most important identity.

Theorem 3.57. *For all $z \in \mathbb{C}$ we have*

$$e^{iz} = \cos z + i \sin z.$$

Proof. Recall from Calculus and/or Analysis: Because all series involved are absolutely convergent, rearrangement of the terms is allowed. Now, by Definitions 3.53 and 3.55 we have

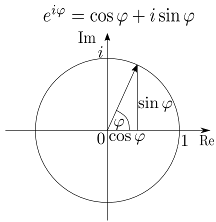
$$\begin{aligned} e^{iz} &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \dots \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) \\ &= \cos z + i \sin z, \end{aligned}$$

and we are done. □

Consequences.

(*a*) If $\varphi \in \mathbb{R}$, we have

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \qquad \text{and} \qquad \left|e^{i\varphi}\right| = 1.$$



Therefore, from now on we will usually write $e^{i\varphi}$ for a complex number on the unit circle with argument φ.

(*b*) If we set $z = \pi$ in Euler's formula, we get $e^{i\pi} = -1$, or

$$\boxed{e^{i\pi} + 1 = 0.}$$

This is probably *the* most remarkable identity in all of mathematics, deserving of its own frame. It combines the five most important numbers in mathematics (0, 1, i, e, and π), and it involves addition, multiplication and exponentiation.

(c) The polar representation of $z \in \mathbb{C}$ can now be written in the form

$$z = r\,e^{i\theta}, \qquad (r, \theta \in \mathbb{R}, r \geq 0.)$$

(d) De Moivre's formula now has the easy form

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

(e) The nth roots of $z = re^{i\theta}$ can be written in the form

$$z^{1/n} = r^{1/n} e^{i(\theta+2j\pi)/n}, \qquad j = 0, 1, \ldots, n-1.$$

As illustrations, see the three figures in Subsection 2.4.2

(f) With Euler's formula we can obtain easy proofs for many trigonometric identities, as shown in the following example.

Example 3.58. From the identity $e^{i(\theta+\varphi)} = e^{i\theta}e^{i\varphi}$ we get

$$\begin{aligned}\cos(\theta+\varphi) + i\sin(\theta+\varphi) &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)\\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)\end{aligned}$$

Equating real and imaginary parts, we get

$$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi,$$
$$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi.$$

(Note: The Greek letters θ and φ are read "theta" and "phi", respectively.)

Do all the "standard properties" of e^z, $\cos z$ and $\sin z$ carry over from the real case? Many do, as we've seen above, but $|\cos z|$ and $|\sin z|$ are, in general, no longer ≤ 1.

Example 3.59. We evaluate $\cos i$, using Definition 3.55:

$$\cos i = \sum_{k=0}^{\infty}(-1)^k \frac{i^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^k}{(2k)!} = \sum_{k=0}^{\infty} \frac{1}{(2k)!}$$
$$= 1 + \frac{1}{2} + \frac{1}{24} + \cdots > 1.$$

The following identities are also most important and useful.

Theorem 3.60 (Euler's Formulas for Sine and Cosine)**.** *For all $z \in \mathbb{C}$ we have*

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

Proof. Euler's formula (Theorem 3.60) gives

$$e^{iz} = \cos z + i \sin z, \qquad e^{-iz} = \cos z - i \sin z.$$

Adding, respectively subtracting, we get

$$e^{iz} + e^{-iz} = 2 \cos z, \qquad e^{iz} - e^{-iz} = 2i \sin z,$$

and the desired identities follow immediately. □

Example 3.61. With $z = i$ in Theorem 3.60, we get

$$\cos i = \frac{1}{2}\left(e^{-1} + e\right) \simeq 1.54308.$$

(Compare this with Example 3.59).

Other trigonometric functions.
$\tan z$, $\cot z$, $\sec z$ and $\csc z$ can be defined exactly as in the real case; for instance,

$$\tan z = \frac{\sin z}{\cos z}.$$

Hyperbolic functions.

The *hyperbolic cosine*, $\cosh z$, and *hyperbolic sine*, $\sinh z$, are defined as below, which is the same definition (in terms of the exponential function) as in the real case. These functions can be found in most Calculus textbooks, but they are usually not covered in MATH 1000/1010. With Euler's formulas for sine and cosine (Theorem 3.60), these functions can also be written in terms of $\sin z$ and $\cos z$:

$$\begin{aligned} \cosh z &= \frac{1}{2}\left(e^z + e^{-z}\right) = \cos(iz), \\ \sinh z &= \frac{1}{2}\left(e^z - e^{-z}\right) = -i\sin(iz). \end{aligned}$$

3.6.2 The Zeros of Sine and Cosine

We know that the real exponential function always satisfies $e^x > 0$, and in Theorem 3.54 (c) we proved that also the complex function e^z is never 0 anywhere in $\mathbb{C}$. How about $\cos z$ and $\sin z$? Do they have complex zeros in addition to their known real zeros?

Theorem 3.62. *For $z \in \mathbb{C}$ we have*

- $\cos z = 0$ *if and only if* $z = (k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$;

- $\sin z = 0$ *if and only if* $z = k\pi,\ k \in \mathbb{Z}$.

Proof. By Euler's formulas (Theorem 3.57), $\cos z$ and $\sin z$ cannot be 0 unless $\left|e^{iz}\right| = 1$. But we have, for $z = x + iy$,

$$\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = e^{-y}\left|e^{ix}\right| = e^{-y},$$

so $\left|e^{iz}\right| = 1$ if and only if $y = 0$, i.e., $z \in \mathbb{R}$. But for real z, we know that the zeros are as stated in the theorem. □

3.6.3 The Complex Logarithm

Recall the logarithm in the real case: To every $x > 0$ there is a unique t satisfying $e^t = x$:

$$e^t = x \quad \Longleftrightarrow \quad t = \log_e x = \log x \quad (= \ln x).$$

In the complex case: Given $z \in \mathbb{C}$ (with certain restrictions?), find a $w \in \mathbb{C}$ such that $e^w = z$.

Problem. e^w is not a one-to-one function, so we cannot expect the complex logarithm to be a single-valued function. Let $z \in \mathbb{C}$, $z \neq 0$, and set

$$z = e^w = e^{u+iv} \qquad (u, v \in \mathbb{R}).$$

Then

$$|z| = \left|e^u e^{iv}\right| = |e^u| \cdot \left|e^{iv}\right| = e^u,$$

so that

$$u = \log |z| \qquad \text{and} \qquad \arg z = v + 2k\pi \quad (k \in \mathbb{Z}).$$

This gives rise to the following definition.

Definition 3.63. If $z \in \mathbb{C}$ and $z \neq 0$, we define $\log z$ to be any of the values

$$\begin{aligned} \log z &= \log |z| + i \arg z \\ &= \log |z| + i \operatorname{Arg} z + 2k\pi i \quad (k \in \mathbb{Z}), \end{aligned}$$

where $\operatorname{Arg} z$ is the principal value of the argument.

Remark. Most of the known properties of the logarithm also hold in the complex case.

Example 3.64. We have the following evaluations:

(*a*) $\log 3 = \log |3| + i \arg 3 = (1.098\ldots) + 2k\pi i$,

(*b*) $\log(-1) = \log |-1| + i \arg(-1) = \pi i + 2k\pi i = (2k+1)\pi i$,

(c) $\log(1+i) = \log|1+i| + i \arg(1+i) = \frac{1}{2}\log 2 + (2k + \frac{1}{4})\pi i,$

where in each case k is an arbitrary integer.

Chapter 4

Integration

The centerpiece of this chapter, and indeed of the whole course, will be *Cauchy's Theorem* which can be stated as follows: *If f is analytic inside and on a closed curve γ, then*

$$\int_\gamma f(z)dz = 0.$$

Of course, at this point none of this makes sense. So, before we can even discuss and then prove this fundamental result, we need to address a few basic questions, among them: What exactly is a closed curve? What is a complex integral? And how do we define such an integral over a curve?

4.1 Integrals

Parts of this section will be quite similar to what you saw in second-year Calculus. This is therefore a good time to review line integrals. We begin with a discussion of curves.

4.1.1 Curves

Definition 4.1. (*a*) A *curve* γ is a continuous complex-valued function $z = \gamma(t)$ defined for t in some interval $[a, b] \in \mathbb{R}$.

(*b*) The curve γ is called *simple* if $\gamma(t_1) \neq \gamma(t_2)$ whenever $a \leq t_1 < t_2 < b$.

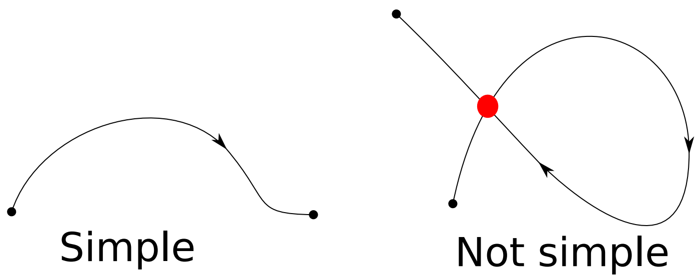
(*c*) The curve γ is called *closed* if $\gamma(a) = \gamma(b)$.

(*d*) A curve that is not closed is called an *arc*; in this case, $\gamma(a)$ and $\gamma(b)$ are called the *initial*, resp. the *terminal point* (collectively the *endpoints*) of γ.

(*e*) A *contour* is either a closed curve or an arc.

(*f*) The range of the function $\gamma(t), t \in [a, b]$, is called the *trace* of γ, denoted by γ^*.

(*g*) A simple closed curve is called a *Jordan curve*.



Remarks. (*a*) The small Greek letter γ is read “gamma”.

(*b*) Jordan curves are named after Camille Jordan (1838–1922), a well-known French mathematician.

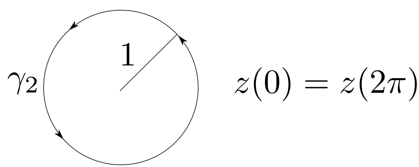
(*c*) If we write $\gamma(t) = x(t) + i\,y(t)$, then $\gamma(t)$ is continuous if and only if both $x(t)$ and $y(t)$ are continuous.

(*d*) Similarly for differentiability of $\gamma(t)$, and we define the derivative by

$$\gamma'(t) = x'(t) + i\,y'(t).$$

(*f*) We think of a curve as being "oriented from $\gamma(a)$ to $\gamma(b)$". We can reverse the orientation by considering $\gamma(a + b - t), t \in [a, b]$.

Example 4.2. Let $z = \gamma(t) = e^{it}, 0 \leq t \leq 2\pi$. This is a simple closed curve, and the trace is the unit circle: $\gamma^* = \{z \mid |z| = 1\}$.

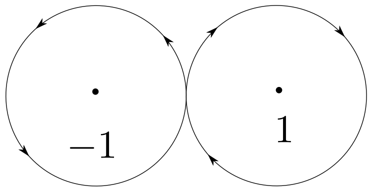


Example 4.3. Let $z = \gamma(t) = e^{it}, 0 \le t \le 4\pi$. This is the "unit circle traversed twice". It is a closed curve, but is not simple; the trace is the unit circle again.

Example 4.4. The curve defined by

$$z = \gamma(\theta) = \begin{cases} -1 + e^{i\theta}, & 0 \le \theta \le 2\pi, \\ 1 + e^{i(\pi - \theta)}, & 2\pi \le \theta \le 4\pi, \end{cases}$$

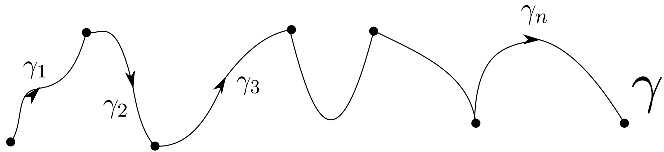
traverses a "double circle". This curve is again closed, but not simple.



Many of the curves we will be dealing with in this course require an additional property, defined as follows.

Definition 4.5. (*a*) A curve γ, defined on $[a, b]$, is *smooth* if $\gamma(t)$ is differentiable and $\gamma'(t)$ is continuous on $[a, b]$, where the limits and derivatives at a and b are taken to be one-sided.

(*b*) A curve γ is *piecewise smooth* if it is composed of a finite number of smooth curves, the end of one coinciding with the beginning of the next.



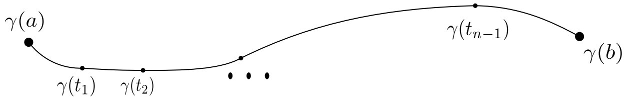
4.1.2 Contour Integration

The definition of a complex contour integral is very similar to that of a line integral in $\mathbb{R}^2$ (as met in 2nd-year Calculus), and is analogous to the usual Riemann integral on an interval of $\mathbb{R}$.

Let γ be a piecewise smooth curve defined by $\gamma(t), t \in [a, b]$. Consider a partition $a = t_0 < t_1 < \cdots < t_n = b$, and the Riemann sum

$$S_n(f, \gamma) := \sum_{j=1}^{n} f(\gamma(t_j))\,(\gamma(t_j) - \gamma(t_{j-1}))\,,$$

where f is a continuous function on the trace γ^* of γ.



We make the partitions finer and finer, and in the limit we set

$$\int_a^b f(\gamma(t))\gamma'(t)dt = \lim_{n\to\infty} S_n(f,\gamma).$$

Here the integrand is a complex-valued function, say $g : [a,b] \to \mathbb{C}$, and the integral is interpreted as follows: If $g(t) = u(t) + i\,v(t)$, then

$$\int_a^b g(t)dt = \int_a^b (u(t) + i\,v(t))dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

where the two integrals on the right are usual real integrals. We are now ready for the main definition of this section.

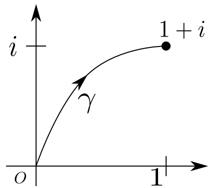
Definition 4.6. Let γ be a piecewise smooth curve with parameter interval $[a,b]$, and let $f : \gamma^* \to \mathbb{C}$ be continuous. Then the *integral of f along γ* (or *around* γ if f is closed) is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{4.1.1}$$

Remarks. (1) Since $f(\gamma(t))\gamma'(t)$ is a piecewise continuous function (i.e., it has piecewise continuous real and imaginary parts), the integral exists.

(2) The motivation for the notation on the left of (4.1.1) is similar to integration by substitution: Let $z = \gamma(t)$; then $dz = \gamma'(t)dt$.

Example 4.7. Let $f(z) = z^2$ and $\gamma(t) = t^2 + it, 0 \le t \le 1$.



Then

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt$$
$$= \int_0^1 (t^2 + it)^2(2t + i)dt$$

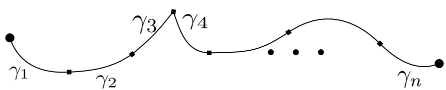
$$= \int_0^1 (t^4 + 2it^3 - t^2)(2t + i)dt$$
$$= \int_0^1 (2t^5 - 4t^3)dt + i\int_0^1 (5t^4 - t^2)dt$$
$$= \left[\tfrac{1}{3}t^6 - t^4\right]_0^1 + i\left[t^5 - \tfrac{1}{3}t^3\right]_0^1 = -\tfrac{2}{3} + \tfrac{2}{3}i.$$

Complex (contour) integrals have many important properties in common with the usual Riemann integral. Here is a first such property; more will follow later. The proof is similar to the real case.

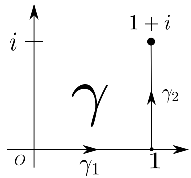
Theorem 4.8 (Additivity of Contour Integrals)**.** *If the piecewise smooth curve γ is made up of the pieces $\gamma_1, \gamma_2, \ldots, \gamma_n$, then*

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

γ



Example 4.9. Evaluate $\int_\gamma \overline{z}dz$, where γ consists of γ_1 and γ_2, as shown below.



Solution. We parametrize γ_1 and γ_2:

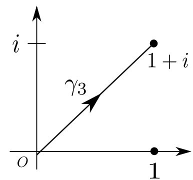
$$\gamma_1(t) = t, \quad 0 \le t \le 1; \quad \gamma_1'(t) = 1.$$
$$\gamma_2(t) = 1 + it, \quad 0 \le t \le 1; \quad \gamma_2'(t) = i.$$

With Theorem 4.8 and Definition 4.6 we have

$$\begin{aligned}\int_\gamma \overline{z}dz &= \int_{\gamma_1} \overline{z}dz + \int_{\gamma_2} \overline{z}dz \\ &= \int_0^1 \overline{\gamma_1(t)}\gamma_1'(t)dt + \int_0^1 \overline{\gamma_2(t)}\gamma_2'(t)dt \\ &= \int_0^1 tdt + \int_0^1 (1-it)idt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 tdt + i\int_0^1 dt + \int_0^1 tdt = 2\int_0^1 tdt + i\int_0^1 dt \\
&= 2\left[\tfrac{1}{2}t^2\right]_0^1 + i\,[t]_0^1 = 1 + i.
\end{aligned}
$$

Example 4.10. Evaluate $\int_{\gamma_3} \overline{z}dz$, where γ_3 is the straight line connecting 0 with $1+i$.



Solution. We parametrize

$$\gamma_3(t) = (1+i)t, \quad 0 \le t \le 1; \quad \gamma_3'(t) = 1 + i.$$

Then by Definition 4.6 we have

$$\int_{\gamma_3} \overline{z}dz = \int_0^1 (1-i)t(1+i)dt = \int_0^1 2tdt = \left[t^2\right]_0^1 = 1.$$

Remark. Note that the paths in Examples 4.9 and 4.10 both connect the origin with $1+i$. Although the integrands are the same, the integrals have different values. See, however, the results of Q.6 on Assignment 5.

4.2 The Fundamental Theorem of Calculus

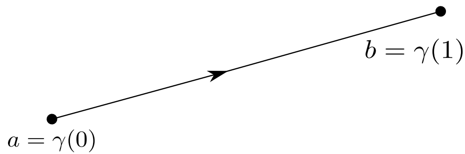
In this section we are going to prove the complex contour integral analogue of the Fundamental Theorem of Calculus (FTC). The original FTC is an essential result in first-year Calculus, but here it will be one step towards obtaining Cauchy's Theorem. In the process we will take a closer look at parametrizations, deal with arc lengths, and prove further properties of the integral.

4.2.1 Parametrization

In Examples 4.9 and 4.10 we already used parametrizations of the curves in question. Here are another two, somewhat more general, examples.

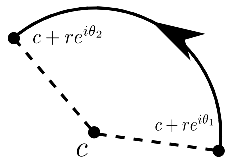
Example 4.11. A straight line from $a \in \mathbb{C}$ to $b \in \mathbb{C}$:

$$\gamma(t) = (1-t)a + tb = a + t(b-a), \quad t \in [0,1].$$



Example 4.12. A circular segment (or circle) of radius r and center $c \in \mathbb{C}$:

$$\gamma(t) = c + re^{it}, \quad t \in [\theta_1, \theta_2].$$



It is important to note that a curve can be parametrized in many different ways. For instance, in Example 4.11 we could take

$$\gamma(t) = a + \frac{1}{4}t^2(b-a), \quad t \in [0,2].$$

An essential question is now: Does the integral defined in Definition 4.6 depend on the particular parametrization? In Examples 4.9 and 4.10 we didn't even consider this question and simply chose particular parametrizations. We will now see that we were justified in doing so. We begin with a definition.

Definition 4.13. Let $t = u(s)$ be a real-valued function with piecewise continuous derivative and increasing in the interval $[c, d]$. If $\gamma(t)$, $a \leq t \leq b$, defines a curve, then

$$\gamma(u(s)), \quad c \leq s \leq d, \quad \text{with } u(c) = a,\ u(d) = b,$$

is also a curve whose trace covers the same points in the same order. We say that these two curves are *equivalent*. This is also called a *reparametrization*.

Theorem 4.14. (*a*) *Let γ and $\tilde{\gamma}$ be equivalent, i.e., $\tilde{\gamma} = \gamma \circ u$, where u maps the parameter interval $[c, d]$ of $\tilde{\gamma}$ onto $[a, b]$ and is piecewise continuously differentiable and increasing. Then*

$$\int_{\tilde{\gamma}} f(z)dz = \int_{\gamma} f(z)dz,$$

where γ and f are as in Definition 4.6.

(*b*) *Let* $-\gamma$ *denote the curve* γ *with opposite orientation. Then*

$$\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz.$$

Proof. (*a*) In view of Theorem 4.8 we may assume that u is continuously differentiable and that γ is smooth (i.e., both without the qualifier "piecewise"). Then $\tilde{\gamma}$ is also smooth. By Definition 4.6 and the chain rule we then have

$$\begin{aligned}\int_{\tilde{\gamma}} f(z)dz &= \int_c^d f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds \\ &= \int_c^d f(\gamma(u(s)))\gamma'(u(s))u'(s)ds \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt \quad (\text{where } t = u(s)) \\ &= \int_{\gamma} f(z)dz,\end{aligned}$$

where in the last step we have used Definition 4.6 again, and in the second-last step the substitution $t = u(s)$ (so that $dt = u'(s)ds$.)

(*b*) This is left as an exercise. □

Remark. This theorem shows that the integral depends only on γ^* and on the direction in which it is traced, but not on the parametrization.

Example 4.15. Compute $\int_\gamma (z-a)^n dz$, where $n \in \mathbb{Z}$ and γ is the circle $|z-a| = r$, traversed once in the positive (i.e., counterclockwise) direction.

Solution. We use the standard parametrization $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$. Then $\gamma'(t) = ire^{it}$, and

$$\int_\gamma (z-a)^n dz = \int_0^{2\pi} (re^{it})^n ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt.$$

When $n = -1$, then

$$\int_0^{2\pi} e^{i(n+1)t} dt = \int_0^{2\pi} dt = 2\pi,$$

while for $n \neq -1$ we have

$$\int_0^{2\pi} e^{i(n+1)t} dt = \left[\frac{1}{i(n+1)} e^{i(n+1)t} \right]_0^{2\pi} = 0.$$

So, altogether,

$$\int_\gamma (z-a)^n dz = \begin{cases} 0 & \text{for } n \neq -1, \\ 2\pi i & \text{for } n = -1. \end{cases}$$

This integral will be most important for later on.

4.2.2 Arc Length

Recall from Calculus: The length of a curve, parametrized as $(x(t), y(t))$, $a \le t \le b$, in the real xy plane is given by

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

The figure below indicates how this formula is obtained from the finite sum of small segments, the limit of which gives the integral.

The situation in the complex plane is very similar: Let $\gamma(t) = x(t) + iy(t)$, $a \leq t \leq b$, be a path in $\mathbb{C}$. Then $\gamma'(t) = x'(t) + iy'(t)$, and the length of γ will be as follows.

Definition 4.16. The *length* of the piecewise smooth curve γ with parameter interval $[a, b]$ is

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

This is often written as

$$\int_\gamma ds \qquad \text{or} \qquad \int_\gamma |dz|.$$

Example 4.17. Find the length of the circle described by $\gamma(t) = z_0 + re^{it}$. $t \in [0, 2\pi]$.

Solution. By Definition 4.16 we have

$$\int_\gamma |dz| = \int_0^{2\pi} |\gamma'(t)|\, dt = \int_0^{2\pi} \left|ire^{it}\right| dt = \int_0^{2\pi} r dt = 2\pi r.$$

Theorem 4.18 (The Fundamental Theorem of Calculus)**.** *Let γ be a path with parameter interval $[a,b]$, let F be defined on an open set containing γ^*, and suppose that $F'(z)$ exists and is continuous on γ^*. Then*

$$\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is a closed curve, then

$$\int_\gamma F'(z)dz = 0.$$

Proof. As before, without loss of generality we may assume that γ is smooth. Then $F \circ \gamma$ is differentiable on $[a,b]$, and by the Chain Rule,

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t).$$

Then, by definition of the integral,

$$\begin{aligned} \int_\gamma F'(z)dz &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b (F \circ \gamma)'(t)dt \end{aligned}$$

$$\begin{aligned}
&= \int_a^b \mathrm{Re}(F\circ\gamma)'(t)dt + i\int_a^b \mathrm{Im}(F\circ\gamma)'(t)dt \\
&= (\mathrm{Re}(F\circ\gamma)(t) + i\mathrm{Im}(F\circ\gamma)(t))\big|_a^b \\
&= F(\gamma(b)) - F(\gamma(a)),
\end{aligned}$$

where in the second-last step we have applied the usual Fundamental Theorem of Calculus. □

Remarks. (1) This theorem is sometimes called the Fundamental Theorem of Contour Integration.

(2) It is just a step on the way to Cauchy's Theorem, and is not as important as in the real case.

Example 4.19. Find $\int_\gamma z^2 dz$, where γ is any path from $z_0 = 0$ to $z_1 = 1+i$.

Solution. The function $F(z) = \frac{1}{3}z^3$ is an antiderivative of $f(z) = z^2$. Therefore, by Theorem 4.18,

$$\int_\gamma z^2 dz = \frac{1}{3}z_1^3 - \frac{1}{3}z_0^3 = \frac{1}{3}(1+i)^3 = \frac{1}{3}(1+3i-3-i) = -\frac{2}{3} + \frac{2}{3}i.$$

Note that this is consistent with Example 4.7.

4.2.3 Integral Inequalities

For what follows, we require some inequalities that are analogues to corresponding inequalities which we met in first-year Calculus for real integrals. First, recall the "triangle inequality for integrals":

$$\left|\int_a^b g(x)dx\right| \le \int_a^b |g(x)|dx,$$

where $g(x)$ is piecewise continuous on $[a, b]$. In analogy we have:

Theorem 4.20 (Estimation Theorem)**.** *Let γ be a path with parameter interval $[a, b]$, and $f(z)$ be continuous on γ^*. Then*

$$\left|\int_\gamma f(z)dz\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|dt. \tag{4.2.1}$$

Proof. Using the definition of the integral, we get

$$\left|\int_\gamma f(z)dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right|$$

$$
\begin{aligned}
&= e^{i\theta} \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b e^{i\theta} f(\gamma(t))\gamma'(t)dt,
\end{aligned}
$$

for some $\theta \in \mathbb{R}$. Taking the real part on both sides, we get

$$
\begin{aligned}
\left|\int_\gamma f(z)dz\right| &= \int_a^b \operatorname{Re}\left[e^{i\theta} f(\gamma(t))\gamma'(t)\right] dt \\
&= \left|\int_a^b \operatorname{Re}\left[e^{i\theta} f(\gamma(t))\gamma'(t)\right] dt\right| \\
&\le \int_a^b \left|\operatorname{Re}\left[e^{i\theta} f(\gamma(t))\gamma'(t)\right]\right| dt,
\end{aligned}
$$

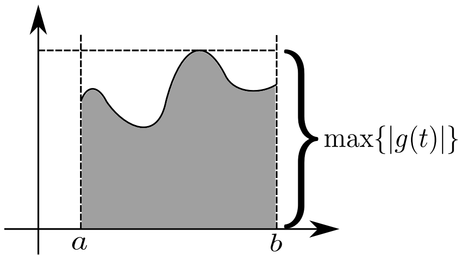
where we have used (4.2.1). Now we use the fact that $|\operatorname{Re} w| \le |w|$ for any $w \in \mathbb{C}$ and that $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$. So, finally,

$$
\left|\int_\gamma f(z)dz\right| \le \int_a^b |e^{i\theta}| \cdot |f(\gamma(t))\gamma'(t)|dt = \int_a^b |f(\gamma(t))\gamma'(t)|dt,
$$

which was to be shown. $\square$

As a consequence we obtain an analogue of the following important inequality from Calculus: If $g(x)$ is continuous on the interval $[a, b]$, then

$$\left|\int_a^b g(t)dt\right| \leq (b-a) \cdot \max_{a \leq t \leq b}\{|g(t)|\}.$$



Corollary 4.21. *With γ and f as in Theorem 4.20, we have*

$$\left|\int_\gamma f(z)dz\right| \leq \text{length}(\gamma) \cdot \max_{z \in \gamma^*}\{|f(z)|\}.$$

Proof. By Theorem 4.20 we have

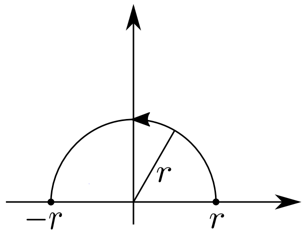
$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \int_a^b |f(\gamma(t))\gamma'(t)|dt \\ &\leq \int_a^b \max_{z\in\gamma^*}\{|f(z)|\}\cdot|\gamma'(t)|dt \\ &= \max_{z\in\gamma^*}\{|f(z)|\}\int_a^b |\gamma'(t)|dt.\end{aligned}$$

But by Definition 4.16 the last integral gives the length of γ, which completes the proof. $\square$

Example 4.22. Find an upper bound for the modulus of

$$\int_\gamma \frac{dz}{z^4+1},$$

where γ is the upper semicircle with radius $r \neq 1$, from $z = r$ to $z = -r$.



Solution. We know that length$(\gamma) = \pi r$. Now note that the smallest possible value of $|z^4+1|$ is r^4-1 when $r>1$, and $1-r^4$ when $r<1$, and thus $|r^4-1|$ for all $r \neq 1$. Hence

$$\max_{z\in\gamma^*}\left\{\left|\frac{1}{z^4+1}\right|\right\} = \frac{1}{|r^4-1|},$$

and by Corollary 4.21 we have

$$\left|\int_\gamma \frac{dz}{z^4+1}\right| \leq \frac{\pi r}{|r^4-1|} \qquad (r \neq 1).$$

As an important application of Corollary 4.21 we will prove a theorem that allows us to interchange the order of an integral and an infinite series.

Theorem 4.23. *Suppose that γ is a path, that $U(z), u_0(z), u_1(z), \ldots$, are continuous on γ^*, and that for each $z \in \gamma^*$, $\sum_{k=0}^{\infty} u_k(z)$ converges with*

$$U(z) = \sum_{k=0}^{\infty} u_k(z).$$

Suppose there exist constants M_k such that $\sum M_k$ converges and $|u_k(z)| \leq M_k$ for all $z \in \gamma^*$. Then

$$\sum_{k=0}^{\infty} \int_{\gamma} u_k(z)dz = \int_{\gamma} \sum_{k=0}^{\infty} u_k(z)dz = \int_{\gamma} U(z)dz.$$

Proof. The function $U(z)$ and all the partial sums $\sum_{k=0}^{n} u_k(z)$ are continuous on γ^*; hence they can be integrated there. Also, by the Comparison Test (Theorem 3.41 (c)), $\sum |u_k(z)|$ converges. Now, with Corollary 4.21 we get

$$\begin{aligned}\left| \int_{\gamma} U(z)dz - \sum_{k=0}^{n} \int_{\gamma} u_k(z)dz \right| &= \left| \int_{\gamma} \left(U(z) - \sum_{k=0}^{n} u_k(z) \right) dz \right| \\ &= \left| \int_{\gamma} \left(\sum_{k=n+1}^{\infty} u_k(z) \right) dz \right|\end{aligned}$$

$$\begin{aligned}
&\leq \text{length}(\gamma) \cdot \max_{z \in \gamma^*} \left\{ \left| \sum_{k=n+1}^{\infty} u_k(z) \right| \right\} \\
&\leq \text{length}(\gamma) \cdot \max_{z \in \gamma^*} \left\{ \sum_{k=n+1}^{\infty} |u_k(z)| \right\} \\
&\leq \text{length}(\gamma) \cdot \sum_{k=n+1}^{\infty} M_k.
\end{aligned}$$

But this last sum approaches 0 as $n \to \infty$ since by assumption the series $\sum M_k$ converges. This means that

$$\sum_{k=0}^{n} \int_\gamma u_k(z)dz \to \int_\gamma U(z)dz \qquad \text{as} \quad n \to \infty. \qquad \square$$

4.3 Simply Connected Domains

It is the purpose of this brief section to introduce a concept that is important for the following section, and indeed for much of the remainder of this course.

Recall (Definition 3.4) that a *domain* is an open and connected set of points in $\mathbb{C}$, where *connected* means that any two points in the set can be joined by a polygonal line that lies in the set (Definition 3.3). We now need to consider a special case of a domain.

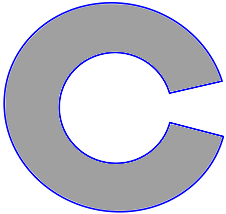
Definition 4.24. Let $\gamma(t)$, $a \leq t \leq b$, be a closed curve in a domain D. We say that γ is *deformable to a point* if there are closed curves $\gamma_s(t)$, $a \leq t \leq b$, $0 \leq s \leq 1$ in D such that $\gamma_s(t)$ depends continuously on both s and t, and $\gamma_0 = \gamma$, while $\gamma_1(t) = z_1$ identically for some $z_1 \in D$.

The following defines the main concept of this section.

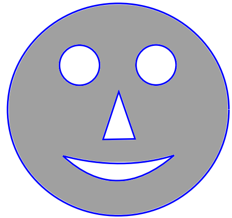
Definition 4.25. A domain $D \subseteq \mathbb{C}$ is *simply connected* if every closed curve in D can be deformed to a point.

Example 4.26. (*a*) The open disk $D := \{z : |z| < 1\}$ is a simply connected domain, while the set $D' := D \setminus \{0\}$ is a domain, but is not simply connected.

(*b*) Two further examples are shown here:



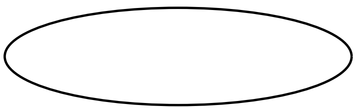
Simply Connected



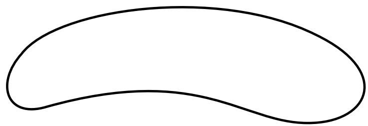
Not Simply Connected

The following concept, very important in its own right, provides a large class of simply connected domains.

Definition 4.27. If a set $S \subseteq \mathbb{C}$ is such that any two points in S can be joined by a straight line that lies in S, then S is called *convex*.



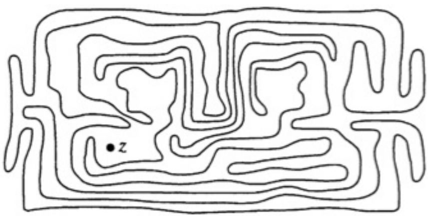
Convex



Not Convex

It is clear from the definition that a convex domain is simply connected.

Finally in this section, to give an alternate definition for simple connectedness. I cite a famous theorem which is also interesting and important in its own right. We have an intuitive idea of what the "inside" and the "outside" of a simple closed curve should be. However, is it *really* clear that there always is an inside and an outside? For instance, is the point z below in the inside or the outside of the curve?



Theorem 4.28 (The Jordan Curve Theorem). *The trace γ^* of a simple closed curve γ separates $\mathbb{C}$ into two disjoint regions, one of which is bounded (the **inside** of γ) and one of which is unbounded (the **outside** of γ).*

We skip the proof which in the general case is quite difficult. But it now makes sense to rephrase Definition 4.25 as follows:

The domain D is simply connected if the interior of any simple closed curve in D lies completely in D.

4.4 Cauchy's Theorem

In this section, central to the whole course, we will state and prove the theorem of Cauchy which is fundamental to much of complex analysis. Our aim is to prove that

$$\int_\gamma f(z)dz = 0$$

if f is analytic inside and on a simple closed curve γ. Recall that the Fundamental Theorem of Calculus already gives us something similar, namely $\int_\gamma F'(z)dz = 0$. This means that it remains to prove that an analytic function f has an antiderivative F. We begin with a special case.

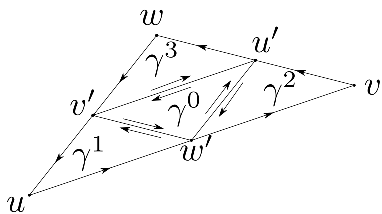
Lemma 4.29 (Cauchy's Theorem for a Triangle)**.** *If f is analytic inside and on a triangle γ, then $\int_\gamma f(z)dz = 0$.*

The idea of the proof is as follows: By the Fundamental Theorem of Calculus we know that

$$\int_\gamma p(z)dz = 0 \qquad \text{for any polynomial} \quad p(z).$$

Now, an analytic function has a good "linear approximation" in the neighbourhood of a given point. We therefore divide the triangle of the lemma into many small subtriangles, and use the facts just mentioned.

Proof of Lemma 4.29. Let the triangle of the lemma be given by the corners u, v, w (in that order), and suppose it is traversed in the positive direction. Let u', v', w' be the midpoints lying opposite u, v, w. Consider the four subtriangles $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, where γ^0 lies in the middle, and $\gamma^1, \gamma^2, \gamma^3$ contain the points u, v, w, respectively.



Then by additivity and Theorem 4.14 (b) we have

$$I := \int_{\gamma} f(z)dz = \sum_{k=0}^{3} \int_{\gamma^k} f(z)dz.$$

For at least one k we then have

$$\left| \int_{\gamma^k} f(z)dz \right| \geq \frac{1}{4}|I|.$$

Rename this γ^k as γ_1, and repeat the process with γ_1 in place of γ. Continuing like this, we get a sequence of triangles $\gamma_0 = \gamma,\ \gamma_1,\ \gamma_2, \ldots$ with the properties that for all $n \geq 0$,

(i) $\Delta_{n+1} \subseteq \Delta_n$, where $\Delta_n = \gamma_n^* \cup$ (interior of γ_n);

(ii) $\mathrm{length}(\gamma_n) = 2^{-n}L$, where $L = \mathrm{length}(\gamma)$;

(iii) $\left| \int_{\gamma_n} f(z)dz \right| \geq 4^{-n}|I|$.

Now consider the intersection of all the triangular regions Δ_n, $n = 0, 1, 2, \ldots$; it contains some point z_0, which must then belong to all Δ_n. Fix an $\varepsilon > 0$. Since f is differentiable at z_0, there is an $r > 0$ such that

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \varepsilon|z - z_0| \tag{4.4.1}$$

for all $z \in U_r(z_0)$. Let N be such that $\Delta_N \subseteq U_r(z_0)$. Then by (ii),

$$|z - z_0| \leq 2^{-N}L \qquad \text{for all} \quad z \in \Delta_N, \tag{4.4.2}$$

and by the Fundamental Theorem of Calculus we have

$$\int_{\gamma_N} [f(z_0) + (z - z_0)f'(z_0)]dz = 0. \tag{4.4.3}$$

Now (4.4.3) gives

$$\left|\int_{\gamma_n} f(z)dz\right| = \left|\int_{\gamma_n} [f(z) - f(z_0) - (z - z_0)f'(z_0)]\, dz\right|,$$

and if we set $A(z) := f(z) - f(z_0) - (z - z_0)f'(z_0)$, then by Corollary 4.21 we have

$$\begin{aligned}\left|\int_{\gamma_N} f(z)dz\right| &\le \max_{z\in\gamma_N^*}\{|A(z)|\}\cdot \mathrm{length}(\gamma_N)\\ &\le \varepsilon 2^{-N}L\cdot \mathrm{length}(\gamma_N) && \text{(by(4.4.1), (4.4.2))}\\ &\le \varepsilon 2^{-N}L\cdot 2^{-N}L = \varepsilon(2^{-N}L)^2 && \text{(by (ii)).}\end{aligned}$$

On the other hand, by (iii) we have

$$|I| \le 4^N \left|\int_{\gamma_N} f(z)dz\right| \le 4^N\varepsilon(2^{-N}L)^2 = \varepsilon L^2.$$

But L is a constant, and ε was chosen arbitrarily small. This means that $|I| = 0$, and thus $I = 0$. $\square$

Recall from first-year Calculus: *If f is continuous on $[a,b] \subseteq \mathbb{R}$, then F defined by*

$$F(x) = \int_a^x f(t)dt, \qquad a \le x \le b,$$

is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$.

This is usually known as Part I of the Fundamental Theorem of Calculus. To state and prove a complex analogue of this, we introduce the following:

Notation. By $[a, z] \subseteq \mathbb{C}$ we mean the straight line segment connecting $a \in \mathbb{C}$ and $z \in \mathbb{C}$.

Lemma 4.30. *Let D be a convex domain, and $f : D \to \mathbb{C}$ continuous such that $\int_\gamma f(z)dz = 0$ for any triangle γ with $\gamma^* \subseteq D$. Let $a \in D$ be fixed, and define*

$$F(z) := \int_{[a,z]} f(w)dw.$$

Then F is analytic in D, and $F' = f$.

Proof. Fix $z \in D$ and let r be such that $U_r(z) \subseteq D$; then $|h| < r$ implies $z + h \in D$. We claim that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z). \tag{4.4.4}$$

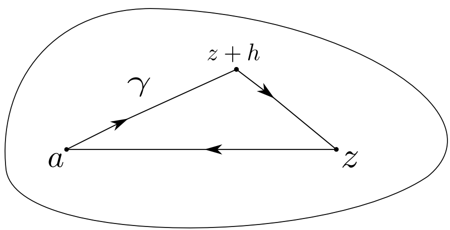
This would prove the lemma.

To prove (4.4.4), we use the definition of $F(z)$:

$$F(z+h) - F(z) = \int_{[a,z+h]} f(w)dw - \int_{[a,z]} f(w)dw$$
$$= \int_{[a,z+h]} f(w)dw + \int_{[z,a]} f(w)dw. \tag{4.4.5}$$

Now, since the segments $[a, z+h]$, $[z, a]$ and $[z+h, z]$ all lie in D (by convexity), we have by hypothesis,

$$\int_{[z,a]} f(w)dw + \int_{[a,z+h]} f(w)dw + \int_{[z+h,z]} f(w)dw = \int_{\gamma} f(w)dw = 0,$$



and therefore with (4.4.5),

$$F(z+h)-F(z) = -\int_{[z+h,z]} f(w)dw = \int_{[z,z+h]} f(w)dw. \tag{4.4.6}$$

Also,

$$\int_{[z,z+h]} f(z)dw = f(z)\int_{[z,z+h]} 1\cdot dw = \int_0^1 h\,dt = h\cdot f(z), \tag{4.4.7}$$

where we have used the parametrization $w(t) = z + th$, and thus $w'(t) = h$. Now, combining (4.4.6) with (4.4.7),

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \frac{1}{|h|}\left|\int_{[z,z+h]} [f(w)-f(z)]dw\right| \\ &\le \frac{1}{|h|}|h| \max_{w\in[z,z+h]} |f(w)-f(z)| \to 0\end{aligned}$$

as $h \to 0$, by continuity of f at z. But this proves the claim (4.4.4), and we are done. $\square$

Combining these last two lemmas, we get

Lemma 4.31 (The Antiderivative Theorem)**.** *If D is a convex domain and f is analytic in D, then there is a function F, analytic in D, such that $F' = f$.*

Proof. By Lemma 4.29 we have $\int_\gamma f(z)dz = 0$ for every triangle γ in D. Lemma 4.30 then gives an F, analytic in D, with $F' = f$, as required. $\square$

This, in turn, leads to another lemma that brings us closer yet to Cauchy's theorem.

Lemma 4.32 (Cauchy's Theorem for Convex Domains)**.** *If D is a convex domain and f is analytic in D, then $\int_\gamma f(z)dz = 0$ for every closed path γ in D.*

Proof. By Lemma 4.31, f has an antiderivative F, i.e., $f = F'$. But by the Fundamental Theorem of Calculus, $\int_\gamma F'(z)dz = 0$. $\square$

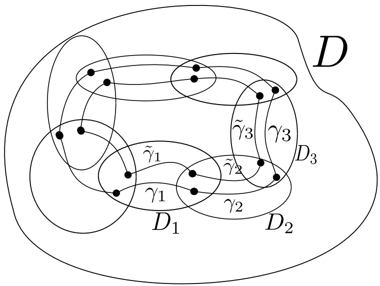
For a more general version of Cauchy's theorem we need the following definition.

Definition 4.33. Let $D \subseteq \mathbb{C}$ be non-empty and open, and let $\gamma, \tilde{\gamma}$ be closed paths in D.

(*a*) $\tilde{\gamma}$ is said to be obtained by an *elementary deformation* from γ if there are open convex subsets $D_1, D_2, \ldots, D_N$ of D such that γ (respectively $\tilde{\gamma}$) can be written as the join of paths $\gamma_1, \gamma_2, \ldots, \gamma_N$ (resp. $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_N$) such that

$$\gamma_j^*, \tilde{\gamma}_j^* \subseteq D_j \qquad \text{for} \quad j = 1, 2, \ldots N.$$

(*b*) Two closed paths γ and $\tilde{\gamma}$ are called *homotopic* in D if $\tilde{\gamma}$ can be obtained from γ by finitely many elementary deformations.



Remark. The *constant path* mentioned in Definition 4.24 is sometimes also called a *null path* in D; another characterization is $\gamma^* = \{z_1\}$, where $z_1 \in D$.

With Definition 4.33 (*b*) we can now give a somewhat more precise version of Definition 4.25: A domain D is *simply connected* if every closed path in D is homotopic to a null path in D.

Theorem 4.34 (The Deformation Theorem)**.** *Let f be analytic in an open set D, and $\gamma, \tilde{\gamma}$ homotopic paths in D. Then*

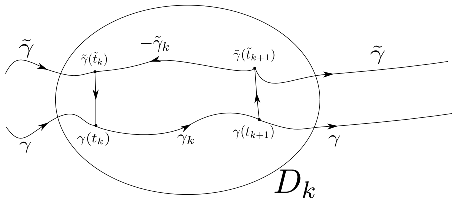
$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz.$$

Proof. Without loss of generality we may assume that $\tilde{\gamma}$ is obtained from γ by an elementary deformation. For a fixed k, $1 \leq k \leq N$, let

$$\begin{array}{lll} \gamma_k & \text{be given by} & \gamma(t), t_k \leq t \leq t_{k+1}, \\ \tilde{\gamma}_k & \text{be given by} & \tilde{\gamma}(t), \tilde{t}_k \leq t \leq \tilde{t}_{k+1}. \end{array}$$

Now consider the closed path Γ_k, made up of

$$\gamma_k, \quad [\gamma(t_{k+1}), \tilde{\gamma}(\tilde{t}_{k+1})], \quad -\tilde{\gamma}_k, \quad [\tilde{\gamma}(\tilde{t}_k, \gamma(t_k)].$$



The closed path Γ_k lies in the convex domain D_k, and so by Lemma 4.32,

$$\int_{\Gamma_k} f(z)dz = 0, \qquad k = 1, 2, \ldots, N.$$

Finally,

$$\begin{aligned}\int_{\gamma} f(z)dz - \int_{\tilde{\gamma}} f(z)dz &= \sum_{k=1}^{N} \left(\int_{\gamma_k} f(z)dz - \int_{\tilde{\gamma}_k} f(z)dz \right) \\ &= \sum_{k=1}^{N} \int_{\Gamma_k} f(z)dz = 0,\end{aligned}$$

where the middle equality comes from the fact that the integrals along the straight line segment of adjacent Γ_k cancel each other. □

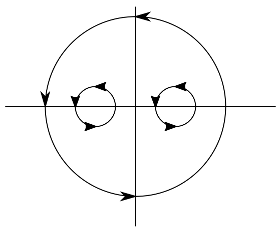
For the following example, and for later, we introduce the following

Notation. We denote by $\gamma(c;r)$ the circle of radius $r > 0$, centered at $c \in \mathbb{C}$, and traversed once in the positive direction.

Example 4.35. Evaluate $\displaystyle\int_{\gamma(0;1)} \left(\frac{1}{2z-1} - \frac{1}{2z+1}\right) dz.$

Solution. We split the integral into two and use the deformation theorem:

$$I = \int_{\gamma(0;1)} \frac{dz}{2z-1} - \int_{\gamma(0;1)} \frac{dz}{2z+1} = \frac{1}{2}\int_{\gamma(\frac{1}{2};\frac{1}{4})} \frac{dz}{z-\frac{1}{2}} - \frac{1}{2}\int_{\gamma(-\frac{1}{2};\frac{1}{4})} \frac{dz}{z+\frac{1}{2}}.$$



But these last two integrals are instances of Example 4.15, with $n = -1$; hence we have

$$I = \frac{1}{2} \cdot 2\pi i - \frac{1}{2} \cdot 2\pi i = 0.$$

We are now ready to state and prove the most central result of this course, if not of complex analysis as a whole.

Theorem 4.36 (Cauchy's Theorem)**.** *Let f be analytic in a simply connected domain D. Then*

$$\int_\gamma f(z)dz = 0$$

for every closed path γ in D.

Proof. Since D is simply connected, γ is homotopic to a null path $\tilde{\gamma}$. Hence, by the Deformation Theorem (Theorem 4.34),

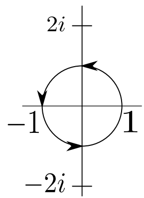
$$\int_\gamma f(z)dz = \int_{\tilde{\gamma}} f(z)dz = 0,$$

where the last equality follows from $\tilde{\gamma}'(t) = 0$ since $\tilde{\gamma}(t)$ is a constant function. $\square$

Example 4.37. We have, by Cauchy's Theorem,

$$\int_{\gamma(0;1)} \frac{e^{iz^2}}{4+z^2} dz = 0,$$

since the integrand is analytic inside and on $\gamma(0;1)$. (Note that if fails to be analytic only when $4+z^2=0$, i.e., for $z=\pm 2i$.)



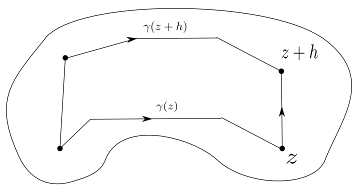
Remark. The integral in Example 4.35 *cannot* be evaluated by Cauchy's Theorem: The integrand is *not* analytic everywhere inside $\gamma(0;1)$.

Finally in this section, we prove a more general version of Lemma 4.31.

Theorem 4.38 (The Antiderivative Theorem – General version)**.** *Let D be a simply connected domain and f analytic on D. Then there is a function F, analytic on D, such that $F' = f$.*

Proof. As in the proof of Lemma 4.30, we choose appropriate complex numbers a, z and h. Since we now lack convexity, we can no longer define $F(z)$ as an integral over the line segment $[a, z]$. However, since D is connected (being a domain), we denote by $\gamma(z)$ a polygonal path going from a to z, and define

$$F(z) := \int_{\gamma(z)} f(w)dw.$$



Then

$$F(z+h) - F(z) = \int_{\gamma(z+h)} f(w)dw - \int_{\gamma(z)} f(w)dw = \int_{[z,z+h]} f(w)dw,$$

where the last equality is due to Cauchy's Theorem, applied to the closed path $\gamma(z)$, $[z, z+h]$, $-\gamma(z+h)$. Now proceed exactly as in the proof of Lemma 4.30 $\square$

Almost everything in this course that comes after this section will be a direct or indirect consequence of Cauchy's fundamental theorem.

Chapter 5

Consequence of Cauchy's Theorem

In this chapter we will derive a number of important and interesting consequences of Cauchy's Theorem, and we will see that analyticity poses very strong restrictions on a function. We will also return to power series, for a fresh look with what we've learned in Chapter 4.

5.1 Cauchy's Integral Formulas

The first important consequence of our work in Chapter 4 says that the value of an analytic function anywhere inside a closed curve is completely determined by the values of the function *on* the curve. This is as important as it is surprising.

We begin by generalizing the concept of a circle being traversed in the positive direction.

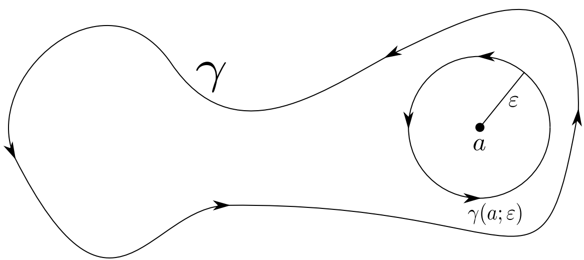
Definition 5.1. A simple closed curve γ is said to be *positively oriented* if the parametrization is chosen such that the curve is traversed counterclockwise, i.e., the inside of the curve is always on the left as one walks around the curve.

Theorem 5.2 (Cauchy's Integral Formula). *Let f be analytic inside and on a positively oriented simple closed curve γ. Then*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw \qquad \text{for } a \text{ inside } \gamma.$$

Proof. If a is in the interior of γ, then there is an $\varepsilon > 0$ such that $U_\varepsilon(a)$ is also in the interior. Then by the Deformation Theorem,

$$\int_\gamma \frac{f(w)}{w-a}\,dw = \int_{\gamma(a;\varepsilon)} \frac{f(w)}{w-a}\,dw.$$



But also,

$$\int_{\gamma(a;\varepsilon)} \frac{f(a)}{w-a}dw = f(a)\int_{\gamma(a;\varepsilon)} \frac{1}{w-a}dw = f(a)\cdot 2\pi i,$$

by Example 4.15 in Chapter 4. So, by using the standard parametrization $w = a + \varepsilon e^{it}$, we get

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}dw - f(a)\right| &= \left|\frac{1}{2\pi i}\int_{\gamma(a;\varepsilon)} \frac{f(w)-f(a)}{w-a}dw\right| \\ &= \left|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+\varepsilon e^{it})-f(a)}{\varepsilon e^{it}} i\varepsilon e^{it}dt\right| \\ &\leq \frac{1}{2\pi}\cdot 2\pi \cdot \max_{t\in[0,2\pi]}\left\{\left|f(a+\varepsilon e^{it})-f(a)\right|\right\},\end{aligned}$$

but this approaches 0 as $\varepsilon \to 0$ since f, being analytic, is also continuous at a. This means that the left-hand side above has to be 0, which completes the proof. □

Remark. One may wonder what can be said if the point a lies *outside* of γ. It turns out that in this case we have

$$\int_\gamma \frac{f(w)}{w-a}\,dw = 0.$$

Indeed, $\frac{f(w)}{w-a}$ (as a function of w, with a fixed) is analytic inside and on γ. Hence, by Cauchy's Theorem, the integral is 0.

In real analysis we learn that under certain conditions it is possible to interchange the order of differentiation and integration (i.e., "differentiate under the integral sign"). This can in fact be done with Cauchy's Integral Formula. As is usual, we denote the nth derivative of $f(z)$ by $f^{(n)}(z)$, with $f^{(0)}(z) = f(z)$ and $f^{(1)}(z) = f'(z)$.

Theorem 5.3 (Cauchy's Integral Formula for Derivatives)**.** *Let f be analytic inside and on a positively oriented simple closed curve γ. Then*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw \qquad \text{for } a \text{ inside } \gamma.$$

This formula can be proved by induction on n, where the base case, $n = 0$, is Cauchy's Integral Formula. We skip the details which are similar to the proof of Theorem 5.2.

Corollary 5.4. *If f is analytic in an open set D, then it has derivatives of all orders in D.*

Proof. We apply Theorem 5.3 with $a \in D$ and $\gamma = \gamma(a; r)$, where $r > 0$ is small enough so that $\gamma(a; r) \subseteq D$. $\square$

Application. Cauchy's integral formulas can be used to evaluate certain contour integrals.

Example 5.5. Evaluate $\displaystyle\int_{\gamma(i;1)} \frac{z^2}{z^2+1} dz$.

Solution. Note that the denominator of the integrand has the two zeros i and $-i$, one of which lies inside $\gamma(i;1)$. We therefore rewrite

$$\frac{z^2}{z^2+1} = \frac{1}{z-i} \cdot \frac{z^2}{z+i}$$

and apply Theorem 5.2 with $f(w) = \frac{w^2}{w+i}$ (which is analytic inside and on $\gamma(i;1)$) and $a = i$. Then

$$\int_{\gamma(i;1)} \frac{z^2}{z^2+1} dz = \int_{\gamma(i;1)} \frac{1}{w-i} \frac{w^2}{w+i} dw = 2\pi i f(i) = 2\pi i \frac{-1}{2i} = -\pi.$$

Example 5.6. Evaluate $\displaystyle\int_{\gamma(0;1)} e^w w^{-3} dw.$

Solution. Apply Theorem 5.3 with $f(w) = e^w$, $a = 0$, and $n = 2$. Then

$$\int_{\gamma(0;1)} \frac{e^w}{w^3} dw = \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2} e^0 = \pi i.$$

5.2 Liouville's Theorem

In this brief section we will derive another famous and important theorem, this time from Cauchy's Integral Formula. As a consequence we obtain one of the most important theorems in all of mathematics. We begin with a definition.

Definition 5.7. A function that is analytic in the whole complex plane is called *entire*.

Example 5.8. All polynomials, as well as $e^z, \sin z, \cos z, \sinh z$ and $\cosh z$ are all entire functions. $\frac{1}{z}$ and $\tan z$ are not entire.

Theorem 5.9 (Liouville's Theorem). *The only bounded entire functions are constants.*

Proof. Since f is bounded, there is an $M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We need to show: For any pair $a, b \in \mathbb{C}$ we have $f(a) = f(b)$.

To do so, fix a and b, and choose an $R \in \mathbb{R}$ such that $R \geq 2\max\{|a|, |b|\}$. Then for a $z \in \mathbb{C}$ with $|z| = R$ we have

$$|z-a| \geq |z| - |a| \geq R - \frac{R}{2} = \frac{R}{2},$$

and similarly $|z-b| \geq \frac{R}{2}$. Now use Cauchy's Integral Formula:

$$\begin{aligned} f(a) - f(b) &= \frac{1}{2\pi i}\int_{\gamma(0;R)} f(w)\left(\frac{1}{w-a} - \frac{1}{w-b}\right) dw \\ &= \frac{1}{2\pi i}\int_{\gamma(0;R)} \frac{f(w)(a-b)}{(w-a)(w-b)} dw. \end{aligned}$$

So, using the Estimation Theorem, we get

$$|f(a) - f(b)| \leq \frac{1}{2\pi}(2\pi R)\frac{M|a-b|}{\frac{R}{2}\cdot\frac{R}{2}} = \frac{4M|a-b|}{R},$$

which approaches 0 as $R \to \infty$. Hence $f(a) = f(b)$. □

The following is a most important application.

Theorem 5.10 (The Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a non-constant polynomial with complex coefficients. Then there is a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. To obtain a contradiction, we suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Since $|p(z)| \to \infty$ as $|z| \to \infty$, there is an $R \in \mathbb{R}$ such that

$$\frac{1}{|p(z)|} < 1 \qquad \text{for all} \quad |z| > R.$$

The complement of the set $\{z : |z| > R\}$ is $\{z : |z| \leq R\}$, which is compact (i.e., closed and bounded), and therefore $1/p(z)$, as a continuous function, is bounded on this set. So $1/p(z)$ is bounded in all of $\mathbb{C}$. But it's also entire, and so by Liouville's Theorem it must be constant. Hence $p(z)$ is also constant, which is a contradiction. Therefore our assumption was false, which completes the proof. □

Remarks. (1) Liouville's Theorem is named after the French mathematician Joseph Liouville (1809–1882).

(2) Liouville's Theorem can be generalized: If $f(z)$ is an entire function with

$$|f(z)| \leq M|z|^k \qquad (k \geq 0 \ \text{ an integer}),$$

then f is a polynomial of degree at most k. (A proof will be given later.)

5.3 Power Series

In Section 3.5 we saw that a convergent power series defines an analytic function in its disk of convergence. We will now see that the converse is also true. Parts of this important result will look familiar from Calculus.

Theorem 5.11 (Taylor's Theorem). *Let f be analytic in $U_R(a)$ for some $R > 0$. Then there are unique constants $c_0, c_1, c_2, \ldots$ such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad (z \in U_R(a)).$$

The constants c_n are given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!},$$

where $\gamma = \gamma(a; r)$, $0 < r < R$, *or any contour in* $U_R(a)$ *homotopic to* $\gamma(a; r)$.

Remarks. (1) This important theorem supplements Corollary 5.4 by saying that a function that is once differentiable in a neighbourhood of a point is not only arbitrarily often differentiable there, but also has a power series expansion about that point. This is a very strong statement, especially since in real analysis there are functions that have continuous derivatives of any order at a given point, but still cannot be written as a power series about that point.

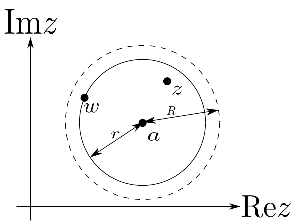
(2) Taylor's theorem also explains why a function that is complex differentiable in a neighbourhood of a point is called analytic. This term originally (including in real analysis) means "can be expanded in a power series".

(3) Taylor's theorem is named after the English mathematician Brook Taylor (1685–1731).

Proof of Theorem 5.11. Fix a $z \in U_R(a)$ and an r such that $|z - a| < r < R$.

Then by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma(a;r)} \frac{f(w)}{w-z} dw.$$



Now we use the following trick:

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} \cdot \frac{1}{1-\frac{z-a}{w-a}}.$$

Since $|z-a| < |w-a|$, we have $\left|\frac{z-a}{w-a}\right| < 1$, and we have the geometric series

$$\frac{1}{w-z} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n,$$

and therefore

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma(a;r)}\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}f(w)dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma(a;r)}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n \\ &= \sum_{n=0}^{\infty}\frac{f^{(k)}(a)}{n!}(z-a)^n \end{aligned}$$

by Cauchy's integral formula for derivatives, provided the interchange of integral and infinite series is allowed.

To see that it is, we use Theorem 4.23: First note that $f(w)$ is continuous on the compact set $\gamma(a;r)^*$, so there is an M such that $|f(w)| \leq M$ for all $w \in \gamma(a;r)^*$. Then

$$\left|\frac{(z-a)^n}{(w-a)^{n+1}}f(w)\right| \leq \frac{|z-a|^n}{r^{n+1}}M = \frac{M}{r}\left(\frac{|z-a|}{r}\right)^n.$$

We denote the right-most term by M_n, and note that $\frac{|z-a|}{r} < 1$ since $|z-a| < r$.

Hence $\sum M_n$ is a converget series, and the interchange of $\sum$ and $\int$ was a legal operation. $\square$

The following generalization of Liouville's theorem is a first application of Taylor's theorem.

Theorem 5.12 (Generalized Liouville's Theorem)**.** *Let f be entire and suppose there are positive constants M, K, and k, with $k \in \mathbb{N}$, such that*

$$|f(z)| \leq M|z|^k \quad \text{for all} \quad |z| \geq K.$$

Then f is a polynomial of degree $\leq k$.

Proof. By Taylor's theorem, the expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is valid in $U_R(0)$ for any $R > 0$, and

$$c_n = \frac{1}{2\pi i} \int_{\gamma(0;R)} \frac{f(w)}{w^{n+1}} dw.$$

Now, for $R \geq K$,

$$|c_n| = \frac{1}{2\pi} \left| \int_{\gamma(0;R)} \frac{f(w)}{w^{n+1}} dw \right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \max_{|w|=R} \left\{ \left| \frac{f(w)}{w^{n+1}} \right| \right\}$$

$$\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M \cdot R^k}{R^{n+1}} = M \cdot R^{k-n}.$$

Since R can be chosen arbitrarily large, we have $c_n = 0$ whenever $n > k$, which means that f is a polynomial of degree $\leq k$. $\square$

Multiplication of power series Given two power series with positive radii of convergence, we form their product:

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n.$$

This gives rise to two questions:
(1) What are the coefficients c_n? (2) Is the product convergent?

Without any consideration of convergence we can ("formally") expand the product on the left and equate coefficients of z^n, thus obtaining

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

A sum of this type is known as a convolution sum, and the resulting product expansion is called a *Cauchy product*. We will now see that all this is in fact legitimate.

Theorem 5.13. *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ and $g(z) := \sum_{n=0}^{\infty} b_n z^n$ be power series with radii of convergence R_1 and R_2, respectively. Then the power series*

$$h(z) := \sum_{n=0}^{\infty} c_n z^n, \quad \textit{with} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

has radius of convergence at least $R := \min\{R_1, R_2\}$, and $h(z) = f(z)g(z)$ for $|z| < R$.

Proof. By definition of R, the functions f and g are analytic in $U_R(0)$, and by Taylor's theorem we have $a_n = f^{(n)}(0)/n!$, $b_n = g^{(n)}(0)/n!$. We know that fg is also analytic in $U_R(0)$ and thus has a power series expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n.$$

By Taylor's theorem and Leibniz's formula (generalization of the product rule) we now get

$$\begin{aligned} c_n = \frac{(fg)^{(n)}(0)}{n!} &= \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} f^{(j)}(0)g^{(n-j)}(0) \\ &= \sum_{j=0}^{n}\frac{1}{n!}\cdot\frac{n!}{j!(n-j)!}f^{(j)}(0)g^{(n-j)}(0) \\ &= \sum_{j=0}^{n}\frac{f^{(j)}(0)}{j!}\frac{g^{(n-j)}(0)}{(n-j)!} = \sum_{j=0}^{n} a_j b_{n-j}, \end{aligned}$$

which completes the proof. □

5.4 The Identity Theorem

It is a consequence of Taylor's theorem that a function that is analytic on a disc $U_r(a)$ is uniquely determined by $f(a)$, $f'(a), \ldots, f^{(n)}(a), \ldots$. Furthermore, by the definition of the derivative we have the following fact: If the derivative exists, we obtain the same value no matter how h approaches 0. So, let h approach 0 along a straight line segment ending at a. This means that the values of $f(z)$ for z along the line segment $[a,p]$ for $p \neq a$ completely determine all the derivatives $f^{(n)}(a)$. This in turn means:

If $f(z) = g(z)$ for $z \in [a,p]$, then $f(z) = g(z)$ in $U_r(a)$. Or:
If $f(z) = 0$ for $z \in [a,p]$, then $f \equiv 0$ in $U_r(a)$.

This is already a strong and surprising statement, but we can do even better. First we need some definitions and notations.

Definition 5.14. Let f be analytic on $U_r(a)$ for some $r > 0$.

(*a*) The point $a \in \mathbb{C}$ is called a *zero* of f if $f(a) = 0$.

(*b*) A zero a of f is called *isolated* if there is an $\varepsilon > 0$ such that $U_\epsilon(a) \setminus \{a\}$ contains no zero of f.

Notation. If f is defined on a set D, we denote by $Z(f)$ the set of zeros of f in D.

Theorem 5.15 (The Identity Theorem)**.** *Let D be a domain and f anlytic in D. If $Z(f)$ has a limit point in D, then $f \equiv 0$ in D.*

This theorem can be rephrased as follows: *The zeros of an analytic function in a domain are isolated unless the function is identically zero.*

Proof of Theorem 5.15. (1) We begin by proving a special case. Let $a \in D$ and suppose that $f(a) = 0$. Let $r > 0$ be such that $U_r(a) \subseteq D$. Then by Taylor's Theorem we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad z \in U_r(a).$$

Then either all $c_n = 0$, in which case $f \equiv 0$ in $U_r(a)$; or, if this is not the case, there is a smallest $m > 0$ such that $c_m \neq 0$. We then write

$$f(z) = (z-a)^m \sum_{n=0}^{\infty} c_{n+m} (z-a)^n,$$

and denote the new series by $g(z)$. It has radius of convergence $\geq r$, so g is analytic in $U_r(a)$, and in particular g is continuous at a. Now, since $g(a) \neq 0$, we have $g(z) \neq 0$ for $z \in U_\varepsilon(a)$, for some $\varepsilon > 0$. But also $(z-a)^m \neq 0$ for $z \neq a$. So $f(z) \neq 0$ for $z \in U_\varepsilon(a) \setminus \{a\}$, i.e., a is not a limit point of $Z(f)$. This proves the theorem for the special domain $U_r(a)$.

(2) To prove the general case, we define the set

$$E := \{z \in \mathbb{C} \mid z \text{ is a limit point of } Z(f)\}.$$

Claim. (i) $E \subseteq Z(f)$; (ii) both E and $D \setminus E$ are open.
Note: Statement (ii) implies:

– either $E = D$, which means that $f \equiv 0$ in D,
– or $E = \emptyset$, which means that $Z(f)$ has no limit point in D.

Proof of the Claim. (i) To obtain a contradiction, suppose that $a \in E \setminus Z(f)$. Then for every n there is an $a_n \in U_{1/n}(a) \setminus \{a\}$ such that $f(a_n) = 0$. But f is continuous, so $f(a) = 0$, which is a contradiction.

(ii) Let $a \in E$. By part (1), $f \equiv 0$ in $U_r(a)$ for some r. But then $U_r(a) \subseteq E$, so E is open.

Now let $a \in D \setminus E$. So a is not a limit point of $Z(f)$, i.e., there is an $r > 0$ such that $f(z) \neq 0$ for $z \in U_r(a)$. Hence

$$U_r(a) \subseteq D \setminus Z(f) \subseteq D \setminus E$$

(since $E \subseteq Z(f)$ by (i)); so $D \setminus E$ is open, and the proof is complete. □

Corollary 5.16. *Let D be a domain and f analytic on D. If $f \equiv 0$ in some $U_r(a) \subseteq D$, then $f \equiv 0$ in D.*

The following theorem is equivalent to the Identity Theorem:

Theorem 5.17 (The Uniqueness Theorem). *Let D be a domain and f, g anlytic in D. If $f(z) = g(z)$ for all $z \in S$, where $S \subseteq D$ is a set which has a limit point in D, then $f \equiv g$ in D.*

Proof. Apply the Identity theorem with $f - g$ in place of f. Then we get $f - g \equiv 0$, which was to be shown. □

Example 5.18. Suppose that f is entire and $f(\frac{1}{n}) = \sin(\frac{1}{n})$ for $n = 1, 2, 3, \ldots$. Since the set $S = \{\frac{1}{n} \mid n = 1, 2, \ldots\}$ has the limit point 0 in $D = \mathbb{C}$, the Uniqueness theorem tells us that $f(z) = \sin z$ for all $z \in \mathbb{C}$.

Example 5.19. Let $D = \mathbb{C} \setminus \{0\}$ and $S = \{\frac{1}{n\pi} \mid n = 1, 2, \ldots\}$. Then the functions $f(z) = \sin(\frac{1}{z})$ and $g \equiv 0$ are analytic in D and have the same values for $z \in S$, namely 0. But f and g are obviously not identical! What is wrong here?

Remark. Example 5.19 shows: The condition in Theorems 5.15 and 5.17 that the limit point lie in D is actually needed.

The Uniqueness theorem can be used to justify that identities for real functions are also valid in the complex case (if the functions in question are defined in a domain in $\mathbb{C}$).

Example 5.20. $f(z) := \sin^2 z + \cos^2 z - 1$ is an entire function, and we know (Pythagoras!) that $f(z) = 0$ for $z \in \mathbb{R}$. But the real line has limit points (all of them are!). So $f \equiv 0$ in $\mathbb{C}$, which means that the identity $\sin^2 z + \cos^2 z = 1$ holds for all $z \in \mathbb{C}$.

5.5 The Maximum Modulus Principle

Let f be analytic in a domain D. Can $|f|$ have a (local) maximum or minimum in the interior of D? In this section we will prove the somewhat surpising fact that this cannot happen.

Theorem 5.21. *Let f be analytic in $U_R(a)$, where $a \in \mathbb{C}$ and $R > 0$. If $|f(z)| \leq |f(a)|$ for all $z \in U_R(a)$, then f is constant.*

Proof. Fix an r, $0 < r < R$, and use Cauchy's integral formula:

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma(a;r)} \frac{f(w)}{w-a} dw \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta}) d\theta. \end{aligned}$$

Using the “triangle inequality” and the Estimation Theorem, our assumption $|f(z)| \leq |f(a)|$ then gives

$$\begin{aligned} f(a) &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(a+re^{i\theta})\right| d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, d\theta \\ &\leq \frac{1}{2\pi} 2\pi |f(a)| = |f(a)|, \end{aligned}$$

and so

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(a+re^{i\theta})\right| d\theta = |f(a)| = \frac{1}{2\pi}\int_0^{2\pi} |f(a)|\, d\theta,$$

or

$$\int_0^{2\pi} \left(|f(a)| - |f(a+re^{i\theta})|\right) d\theta = 0.$$

Note that the integrand is continuous and ≥ 0, so it must be equal to 0. This holds for any $r < R$ so $|f(z)| = |f(a)|$ for all $z \in U_R(a)$. But then $f(z)$ is constant, as a consequence of the Cauchy-Riemann equations. $\square$

From Theorem 5.21 we obtain the following important result.

Theorem 5.22 (The Maximum Modulus Theorem)**.** *Let D be a bounded domain, and let f be analytic in D and continuous on the closure $\overline{D}$ of D (i.e., the union of D and its boundary). Then $|f|$ attains its maximum on the boundary of D.*

Proof. Since $\overline{D}$ is compact, the continuous function $|f|$ attains its maximum at some point $a \in \overline{D}$. To obtain a contradiction, assume that $a \in D$. But then, by Theorem 5.21, f is constant in some disc $U_R(a) \subseteq D$, so by the Identity Theorem, f is constant on D. By continuity, f is constant on $\overline{D}$, so the maximum is attained on $\overline{D}$, a contradiciton. □

5.6 Morera's Theorem

We finish this chapter with a partial converse of Cauchy's Theorem.

Theorem 5.23 (Morerea's Theorem)**.** *Let f be continuous in an open set D, and $\int_\gamma f(z)dz = 0$ for all triangles γ in D. Then f is analytic in D.*

Proof. Let $a \in D$ and r be such that $U_r(a) \subseteq D$. Since $U_r(a)$ is convex, we can apply Lemma 4.30: There is a function F analytic in $U_r(a)$, such that

$F' = f$. But then by Corollary 5.4, f is also analytic in $U_r(a)$. Since a was arbitrary, f is analytic in all of D. $\square$

Chapter 6

Laurent Series and Singularities

Recall: Points where $f(z)$ "fail to be analytic" played an important role in evaluating integrals by use of Cauchy's Integral Formulas. It is the goal of this chapter to study and classify such points, many of which have been of the form

$$\frac{f(z)}{(z-a)^k} \qquad \text{(e.g., } k=1\text{)},$$

where $f(z)$ is analytic (at a).

6.1 Laurent Series

As a main tool of studying such points (which we will later call *singularities*) we generalize the concept of a Taylor series to negative powers.

Theorem 6.1. *Let $A := \{z \mid R < |z-a| < S\}$, with $0 \leq R < S \leq \infty$, and let f be analytic in A. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad z \in A,$$

where

$$c_n := \frac{1}{2\pi i} \int_{\gamma(a;r)} \frac{f(w)}{(w-a)^{n+1}} dw, \qquad R < r < S.$$

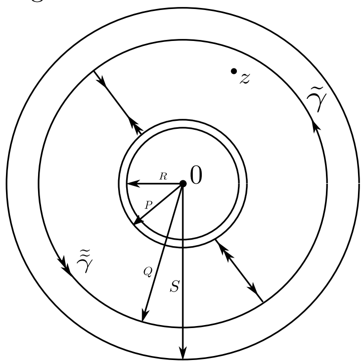
Remark. How do we interpret a "doubly infinite" series $\sum_{n=-\infty}^{\infty} a_n$?

If the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ both converge and have sums s_1 and s_2, respectively, then we say that $\sum_{n=-\infty}^{\infty} a_n$ converges, with sum $s = s_1 + s_2$.

Proof of Theorem 6.1. Fix $z \in A$ and choose P and Q such that

$$R < P < |z| < Q < S,$$

and let $\tilde{\gamma}, \tilde{\tilde{\gamma}}$ be the following contours:



Then by Cauchy's Integral Formula we have

$$f(z) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f(w)}{w-z} dw,$$

and by Cauchy's Theorem,

$$\frac{1}{2\pi i} \int_{\tilde{\tilde{\gamma}}} \frac{f(w)}{w-z} dw = 0.$$

If we add the two integrals, the portions along the straight lines cancel, and we get

$$f(z) = \frac{1}{2\pi i} \int_{\gamma(0;Q)} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \int_{\gamma(0;P)} \frac{f(w)}{w-z} dw.$$

Now use the geometric series expansions

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} \qquad (|z| < |w|),$$

$$\frac{1}{w-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{w}{z}} = \sum_{m=0}^{\infty} \frac{-w^m}{z^{m+1}} \qquad (|z| > |w|),$$

and so

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\gamma(0;Q)} \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}} f(w)dw - \frac{1}{2\pi i}\int_{\gamma(0;P)} \sum_{m=0}^{\infty} \frac{-w^m}{z^{m+1}} f(w)dw \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma(0;Q)} \frac{f(w)}{w^{n+1}}dw\right) z^n + \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma(0;P)} f(w)w^m dw\right) z^{-m-1},
\end{aligned}
$$

where we have used Theorem 4.23, with the fact that $|z/w| < 1$ for $w \in \gamma(0;Q)$, and $|w/z| < 1$ for $w \in \gamma(0;P)$.

Finally, note that $f(w)w^m$ is analytic in A for all $m \in \mathbb{Z}$. Use the deformation theorem and set $n = -m-1$ for $m \geq 0$; then we get

$$
f(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma(0;r)} \frac{f(w)}{w^{n+1}}dw\right) z^n,
$$

which completes the proof. □

Definition 6.2. The expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ is called the *Laurent expansion* (or *Laurent series*) of f in the annulus A.

We will now see that a Laurent expansion, just like a Taylor expansion, is unique.

Theorem 6.3. *Let f be analytic in $A = \{z \mid R < |z-a| < S\}$, and suppose that f has the Lauent expansion of Theorem 6.1. If*

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n \qquad (z \in A)$$

is another expansion, then $b_n = c_n$ for all $n \in \mathbb{Z}$.

Proof. Without loss of generality we may again assume that $a = 0$, and we let r be such that $R < r < S$. Then by Theorem 6.1 we have

$$\begin{aligned} 2\pi i c_n &= \int_{\gamma(0;r)} \frac{f(w)}{w^{n+1}} dw = \int_{\gamma(0;r)} \left(\sum_{k=-\infty}^{\infty} b_k w^k \right) \frac{1}{w^{n+1}} dw \\ &= \int_{\gamma(0;r)} \sum_{k=-\infty}^{\infty} b_k w^{k-n-1} dw \\ &= \sum_{k=-\infty}^{\infty} b_k \int_{\gamma(0;r)} w^{k-n-1} dw, \end{aligned}$$

where we have once again used Theorem 4.23. Recall from Example 4.15: the integral in the last expression is $2\pi i$ when $n = k$, and is 0 otherwise. This means that in the last series only the term $n = k$ remains, and we have $2\pi i c_n = 2\pi i b_n$, which was to be shown. □

Remark. This uniqueness theorem means that we can determine the Laurent expansion by whichever means we wish. In particular, we can use known Taylor expansions of related functions.

Example 6.4. Find the Laurent series for

$$f(z) = \frac{z^2 - 2z + 3}{z - 2}, \quad \text{in the region } |z-1| > 1.$$

Solution. We have to write both numerator and denominator in terms of powers of $z - 1$. (i) Numerator:

$$z^2 - 2z + 3 = (z-1)^2 + 0 \cdot (z-1) + 2.$$

(ii) Denominator: Set up a geometric series:

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1} \cdot \frac{1}{1 - \frac{1}{z-1}},$$

and so

$$\frac{1}{z-2} = \frac{1}{z-1}\sum_{j=0}^{\infty}\frac{1}{(z-1)^j} = \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \dots$$

Combining (i) and (ii) we then get

$$\begin{aligned}\frac{z^2-2z+3}{z-2} &= ((z-1)^2+2)\left(\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \dots\right)\\ &= \left((z-1)+1+\frac{1}{z-1} + \frac{1}{(z-1)^2} + \dots\right)\\ &\quad + \left(\frac{2}{z-1} + \frac{2}{(z-1)^2} + \dots\right)\\ &= (z-1)+1+\sum_{j=1}^{\infty}\frac{3}{(z-1)^j}.\end{aligned}$$

Example 6.5. For the function $f(z) := \frac{1}{(z-1)(z-2)}$, find the Laurent series in the regions
(a) $|z| < 1$; (b) $1 < |z| < 2$; (c) $|z| > 2$; (d) $0 < |z-1| < 1$ (e) $|z-1| > 1$.

Solution. We begin with the partial fraction expansion

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

(a) For $|z| < 1$ we set up the following geometric series expansions:

$$\frac{1}{z-2} = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = -\sum_{j=0}^{\infty} \frac{z^j}{2^{j+1}}, \tag{6.1.1}$$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{j=0}^{\infty} z^j,$$

and so we get

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^{j+1}}\right) z^j = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \dots$$

(b) For $1 < |z| < 2$, the identity (6.1.1) is still valid (as the series converges), but we need to rewrite

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \sum_{j=0}^{\infty} \frac{1}{z^j} = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}}, \tag{6.1.2}$$

and so

$$\begin{aligned}\frac{1}{(z-1)(z-2)} &= -\sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}} - \sum_{j=0}^{\infty}\frac{1}{z^{j+1}} \\ &= \cdots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots\end{aligned}$$

(c) For $|z| > 2$, the identity (6.1.2) is still valid, but we must rewrite

$$\frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{j=0}^{\infty}\left(\frac{2}{z}\right)^j = \sum_{j=0}^{\infty}\frac{2^j}{z^{j+1}},$$

and so

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty}\frac{2^j-1}{z^{j+1}} = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \ldots$$

(d) For $0 < |z-1| < 1$, we see that

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n\,.$$

Therefore,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{n=0}^{\infty} (z-1)^n - \frac{1}{z-1} = -\sum_{n=-1}^{\infty} (z-1)^n .$$

(e) For $|z-1| > 1$,

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1} \cdot \frac{1}{1-\frac{1}{z-1}} = \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+1}} = \frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots$$

Therefore,

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+2}} .$$

Example 6.6. Expand $e^{1/z}$ in a Laurent series about $z = 0$.

Solution. We know from the definition of the exponential function that

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots$$

for all $w \in \mathbb{C}$. If $z \neq 0$, we set $w = 1/z$, and obtain

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

Note that in Examples 6.5 (c) and 6.6 we have only nonpositive powers of z. This is in fact true in general, as the following theorem shows:

Theorem 6.7. *Let f be analytic in $\{z : |z| > R\}$ and suppose it is bounded there. Then f has the Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{0} c_n z^n \qquad (|z| > R).$$

Proof. Let $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, and suppose that $|f(z)| \leq M$. For $n > 0$ we have, by Theorem 6.1,

$$\begin{aligned} |c_n| &= \left| \frac{1}{2\pi i} \int_{\gamma(0;r)} \frac{f(w)}{w^{n+1}} dw \right| \qquad \text{(for any } r > R) \\ &\leq \frac{1}{2\pi} \cdot 2\pi r \max_{|w|=r} \left\{ \left| \frac{f(w)}{w^{n+1}} \right| \right\} \leq \frac{M}{r^n}. \end{aligned}$$

But r can be chosen arbitrarily large, so $|c_n| = 0$ whenever $n \geq 1$, which proves the theorem. $\square$

6.2 Singularities and Zeros

Using Laurent series, we will now classify and study the "points where f fails to be analytic". We begin with some definitions.

Definition 6.8. Let $f : D \to \mathbb{C}$ for some set $D \subseteq \mathbb{C}$.

(a) The point $a \in D$ is a *regular point* if f is analytic at a. (Recall: This means f is analytic in a neighbourhood $U_r(a)$ of a).

(b) The point $a \in D$ is a *singularity* of f if it is a limit point of regular points but is not itself regular.

(c) If a is a singularity of f and f is analytic in $U_r(a) \setminus \{a\}$ for some $r > 0$, then a is an *isolated singularity*.

(d) If a is a singularity and f is not analytic in $U_r(a) \setminus \{a\}$ for any $r > 0$, then a is a *non-isolated essential singularity*.

We can now use the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad z \in A := \{z : 0 < |z-a| < R\} \tag{6.2.1}$$

to classify isolated singularities.

Definition 6.9. Suppose that f has an isolated singularity at a, and that f has the (unique) Laurent expansion (6.2.1) around a. Then a is said to be

(a) a *removable singularity* if $c_n = 0$ for all $n < 0$;

(b) a *pole of order* m ($m \geq 1$) if $c_{-m} \neq 0$ but $c_n = 0$ for all $n < -m$. A pole of order 1 is called a *simple pole*.

(c) an (isolated) *essential singularity* if $c_n \neq 0$ for infinitely many negative integers n.

Remarks. (1) The definition above makes sense because of the uniqueness of Laurent expansions.

(2) If for $z \in A = \{z : 0 < |z-a| < R\}$ we write

$$f(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n + \sum_{n=0}^{\infty} c_n (z-a)^n,$$

then the first part of this series is called the **principal part** of the Laurent series.

Example 6.10. (a) The function

$$f(z) = (z+2)^{-3}$$

has a triple pole (or pole of order 3) at $z = -2$.

(b) The function

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - + \dots$$

has a double pole (or pole of order 2) at $z = 0$.

(*c*) The function $(1 - \cos z)/z^2$ is not defined at $z = 0$, while it is analytic everywhere else. However, we have

$$\frac{1 - \cos z}{z^2} = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - + \ldots,$$

so it has a removable singularity at $z = 0$.

(*d*) We saw before that

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n},$$

therefore this function has an essential singularity at $z = 0$.

(*e*) The function $f(z) = \csc(1/z)$ has singularities at $1/k\pi$ for all $k \in \mathbb{Z}$. The point $z = 0$ is not an isolated singularity since $f(z)$ is not analytic in any punctured disc $U_r(0) \setminus \{0\}$, for any $r > 0$.

Definition 6.11. A point $a \in \mathbb{C}$ is called a *zero of order* m of the function f if f is analytic at a and $f, f', \ldots, f^{(m-1)}$ vanish at a, while $f^{(m)}(a) \neq 0$. Zeros of orders $1, 2, 3$ are called simple, double, and triple zeros, respectively.

Remark. If f has a zero of order m at a, then the Taylor series for f around a takes the form

$$\begin{aligned} f(z) &= c_m(z-a)^m + c_{m+1}(z-a)^{m+1} + c_{m+2}(z-a)^{m+2} + \dots \\ &= (z-a)^m \left(c_m + c_{m+1}(z-a) + c_{m+2}(z-a)^2 + \dots\right), \end{aligned}$$

where $c_m \neq 0$. Also recall that $c_n = f^{(n)}(a)/n!$ for all $n \geq 0$.

The following is an important characterization of zeros and poles.

Theorem 6.12. (*a*) *Let f be analytic in $U_r(a)$ for some $r > 0$. Then f has a zero of order m at a if and only if*

$$\lim_{z\to a} \frac{f(z)}{(z-a)^m} = C \qquad (C \textit{ is a non-zero constant.})$$

(*b*) *Let f be analytic in $U_r(a) \setminus \{a\}$ for some $r > 0$. Then f has a pole of order m at a if and only if*

$$\lim_{z\to a} (z-a)^m f(z) = D \qquad (D \textit{ is a non-zero constant.})$$

Proof. (a) The remark above indicates what is at play here. We skip the details.

(b) Two directions need to be shown. (i) "$\Rightarrow$": Suppose that a is a pole of order m. Then by definition,

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n, \qquad c_{-m} \neq 0,$$

for $z \in U_r(a) \setminus \{a\}$. Then

$$(z-a)^m f(z) = c_{-m} + c_{-m+1}(z-a) + c_{-m+2}(z-a)^2 + \dots,$$

which defines a function that is analytic in $U_r(a)$, so it is continuous at a, and we have

$$\lim_{z \to a} (z-a)^m f(z) = c_{-m} =: D \neq 0.$$

(ii) "$\Leftarrow$": We use Laurent's theorem, namely

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a;s)} \frac{f(w)}{(w-a)^{n+1}} dw, \qquad 0 < s < r. \tag{6.2.2}$$

We now claim that $c_n = 0$ for $n < -m$, while $c_{-m} \neq 0$.

To prove this claim, we rewrite the given limit, using the "$\varepsilon - \delta$ definition": For every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|(w-a)^m f(w) - D| < \varepsilon \quad \text{whenever} \quad 0 < |w-a| < \delta.$$

Choose s such that $0 < s < \min\{\delta, r\}$. Then $|w-a| = s$ implies

$$|(w-a)^m f(w)| \leq |D| + \varepsilon,$$

or

$$\left|(w-a)^{-n-1} f(w)\right| \leq (|D| + \varepsilon) s^{-n-m-1}.$$

We use this to estimate the integral (6.2.2):

$$|c_n| \leq \frac{1}{2\pi} \cdot 2\pi s \cdot (|D| + \varepsilon) s^{-n-m-1} = (|D| + \varepsilon) s^{-n-m}.$$

Hence $c_n = 0$ for $n < -m$ because $|c_n| \to 0$ as $s \to 0$. So

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n,$$

and

$$D = \lim_{z \to a} (z-a)^m f(z) = c_{-m} \neq 0,$$

as claimed. This completes the proof. $\square$

By considering the "duality" between parts (a) and (b) of this theorem, we immediately see that the following is true.

Corollary 6.13. *Let f be analytic in $U_r(a)$ for some $a \in \mathbb{C}$ and $r > 0$. Then f has a zero of order m at a if and only if $1/f$ has a pole of order m at a.*

Zeros and poles can also cancel each other out, as the next consequence of Theorem 6.12 shows.

Corollary 6.14. *Suppose that f has a pole of order m at $a \in \mathbb{C}$.*

(a) *If g is analytic in $U_r(a)$ for some $r > 0$ then fg has*

- (i) *a pole of order $m-n$ at a if g has a zero of order n at a ($0 \leq n < m$, where "order 0" means no zero at a);*
- (ii) *a removable singularity (and in fact a zero of order $n-m$) at a if g has a zero of order $n \geq m$ at a.*

(b) *If g has a pole of order n at a, then fg has a pole of order $m+n$ at a.*

Example 6.15. Characterize the singularities of $\dfrac{1}{z\sin z}$.

Solution. With the aim of using Corollary 6.14, we consider the zeros of $z\sin z$. We have seen earlier that all the zeros are given by $z = n\pi$, for $n \in \mathbb{Z}$. Now

$$\frac{d}{dz}(z\sin z) = \sin z + z\cos z \begin{cases} = 0 & \text{for } z = 0, \\ \neq 0 & \text{for } z = n\pi, n \neq 0. \end{cases}$$

Therefore we consider further

$$\frac{d^2}{dz^2}(z\sin z) = 2\cos z - z\sin z \neq 0 \quad \text{for} \quad z = 0.$$

So $z\sin z$ has simple zeros at $z = n\pi, n \in \mathbb{Z}, n \neq 0$, and a double zero at $z = 0$. Therefore, by Corollary 6.14, $\dfrac{1}{z\sin z}$ has simple poles at $z = n\pi, n \in \mathbb{Z}, n \neq 0$, and a double pole at $z = 0$.

Remark. How does $f(z)$ behave "near" an isolated singularity a? To answer this, we distinguish between three cases:

(a) A removable singularity: By definition we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad z \in U_r(a) \setminus \{a\},$$

so $f(z) \to c_0$ as $z \to a$. Note that f becomes analytic in all of $U_r(a)$ by setting $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for all $z \in U_r(a)$.

(b) A pole: By Corollary 6.14 we have $|f(z)| \to \infty$ as $z \to a$.

(c) An essential singularity: It can be shown (Theorem of Casorati and Weierstrass) that, given any $w \in \mathbb{C}$, there is a sequence $\{a_n\}$ with $a_n \to a$, such that $f(a_n) \to w$.

In fact, much more was shown by Picard: In any punctured disc $U_r(a) \setminus \{a\}$, every complex value, with possibly one exception, is assumed infinitely often by f. (Example: For $e^{1/z}$, which has an essential singularity at $z = 0$, this exception is the value 0).

We conclude this chapter with a definition:

Definition 6.16. A function that is analytic in a domain D, with the possible exception of poles, is called *meromorphic* in D.

Note that sums and products of meromorphic functions are meromorphic. Quotients of meromorphic functions are also meromorphic, as long as the denominator function is not identically 0.

Chapter 7

Residues

In this final chapter of the course we apply much of what we have learned about analytic functions and about singularities to evaluate *real* improper integrals, many of which cannot (or only with difficulty) be evaluated by other methods. An important concept in helping us achieve this is that of a *residue*.

7.1 Cauchy's Residue Theorem

Recall the important "standard example" (Example 4.15):

$$\int_{\gamma(a;r)} (z-a)^n dz = \begin{cases} 0 & \text{for } \ n \neq -1, \\ 2\pi i & \text{for } \ n = -1. \end{cases} \tag{7.1.1}$$

This seems to suggest that the term $c_{-1}/(z-a)$ in the Laurent expansion of a function plays a special role. This is in fact the case:

Lemma 7.1. *Let f be analytic inside and on a positively oriented contour γ except at a pole a inside γ, and let*

$$f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$$

be the Laurent expansion of f around a. Then

$$\int_{\gamma} f(z) dz = 2\pi i c_{-1}.$$

Proof. Let r be such that the closure of $U_r(a)$ lies in the interior of γ. Then by the Deformation Theorem,

$$\begin{aligned}\int_\gamma f(z)dz &= \int_{\gamma(a,r)} f(z)dz \\ &= \int_{\gamma(a,r)} \left(\sum_{n=-m}^{\infty} c_n(z-a)^n\right) dz \\ &= \sum_{n=-m}^{\infty} c_n \int_{\gamma(a,r)} (z-a)^n dz = c_{-1}(2\pi i),\end{aligned}$$

where we have used Theorem 4.23 (to interchange the order of the integral and the infinite series), and the identity (7.1.1). $\square$

This lemma gives rise to the following important definition.

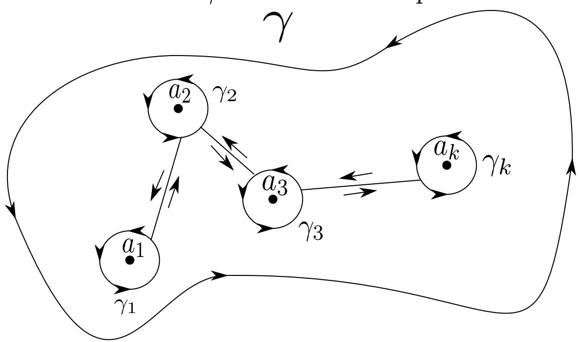
Definition 7.2. Let f be analytic in $U_r(a) \setminus \{a\}$, and suppose it has a pole at a. The *residue* of f at a, $\mathrm{Res}[f;a]$, is the coefficient c_{-1} of $1/(z-a)$ in the Laurent expansion of f about a.

The following is the main theorem of this chapter.

Theorem 7.3 (Cauchy's Residue Theorem)**.** *Let f be analytic inside and on a positively oriented contour γ except for a finite number of poles at $a_1, \ldots, a_m$ in the interior of γ. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{m} \mathrm{Res}[f; a_k].$$

Proof. Let $r_1, \ldots, r_m > 0$ be sufficiently small so that the circles $\gamma(a_k, r_k)$, $k = 1, \ldots, m$, lie in the interior of γ and don't overlap.



Then by the Deformation Theorem we have

$$\int_\gamma f(z)dz = \sum_{k=1}^{m} \int_{\gamma(a_k,r_k)} f(z)dz = \sum_{k=1}^{m} 2\pi i \mathrm{Res}[f; a_k],$$

where in the second identity we have used Lemma 7.1 and the definition of a residue. This completes the proof.

Finding Residues

1. First, suppose that a is a simple pole. Then

$$f(z) = \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots,$$

and

$$(z-a)f(z) = c_{-1} + c_0(z-a) + c_1(z-a)^2 + c_2(z-a)^3 + \ldots,$$

so that

$$\lim_{z\to a}(z-a)f(z) = c_{-1} = \mathrm{Res}[f; a].$$

Example 7.4. Let $\dfrac{e^z}{z(z+1)}$. It has simple poles at $z = 0$ and at $z = -1$. Then

$$\operatorname{Res}[f;0] = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{e^z}{z+1} = 1.$$
$$\operatorname{Res}[f;-1] = \lim_{z\to -1} (z+1)f(z) = \lim_{z\to -1} \frac{e^z}{z} = -\frac{1}{e}.$$

Now, if γ is a positively oriented simple closed curve that contains 0 and -1 in its interior, then we have

$$\int_\gamma \frac{e^z}{z(z+1)}dz = 2\pi i(\operatorname{Res}[f;0] + \operatorname{Res}[f;-1]) = 2\pi i(1 - \tfrac{1}{e}).$$

2. Now suppose that f has a pole of order $m \geq 2$ at a. The Laurent expansion of f around a is

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \cdots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \ldots$$

Upon multiplying by $(z-a)^m$ we get the Taylor series

$$\begin{aligned}(z-a)^m f(z) = c_{-m} + \cdots + c_{-2}(z-a)^{m-2} + c_{-1}(z-a)^{m-1} \\ + c_0(z-a)^m + c_1(z-a)^{m+1} + \ldots\end{aligned}$$

Differentiating this $m-1$ times will then bring the desired residue c_{-1} into the constant coefficient:

$$\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) = (m-1)!c_{-1} + \frac{m!}{1!}c_0(z-a) + \frac{(m+1)!}{2!}c_1(z-a)^2 + \ldots$$

Finally, taking the limit as $z \to a$, we get the following evaluation theorem for residues.

Theorem 7.5. *If f has a pole of order m at a, then*

$$\operatorname{Res}[f;a] = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right).$$

We see that the earlier limit formula for a simple pole is a special case of this.

Example 7.6. Compute the residues of the singularities of

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}.$$

Solution. By considering the zeros of $1/f$ we see that f has a pole of order 2 at $z = 0$ and a pole of order 3 at $z = \pi$. First, using Theorem 7.5 with $a = 0$ and $m = 2$, we get

$$\begin{aligned} \operatorname{Res}[f;0] &= \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left(z^2 \frac{\cos z}{z^2(z-\pi)^3}\right) \\ &= \lim_{z\to 0} \frac{d}{dz}\left(\frac{\cos z}{(z-\pi)^3}\right) \\ &= \lim_{z\to 0} \frac{-(z-\pi)\sin z - 3\cos z}{(z-\pi)^4} = \frac{-3}{\pi^4}. \end{aligned}$$

Second, using Theorem 7.5 with $a = \pi$ and $m = 3$, we get

$$\begin{aligned} \operatorname{Res}[f;\pi] &= \frac{1}{2!} \lim_{z\to \pi} \frac{d^2}{dz^2}\left((z-\pi)^3 \frac{\cos z}{z^2(z-\pi)^3}\right) \\ &= \lim_{z\to \pi} \frac{d^2}{dz^2}\left(\frac{\cos z}{z^2}\right) \end{aligned}$$

$$= \frac{1}{2} \lim_{z \to \pi} \frac{(6 - z^2)\cos z + 4z \sin z}{z^4} = \frac{-(6 - \pi^2)}{2\pi^4}.$$

Applying Cauchy's Residue Theorem, we can now use the results of Example 7.6 to evaluate certain integrals.

Example 7.7. Evaluate

$$I = \int_\gamma \frac{\cos z}{z^2(z - \pi)^3} dz,$$

where (a) $\gamma = \gamma(0; 3)$; (b) $\gamma = \gamma(3/2; 2)$.

Solution. (a) We note that only the pole $a = 0$ lies in the interior of γ. Hence Cauchy's Residue Theorem gives

$$I = 2\pi i \text{Res}[f; 0] = 2\pi i \frac{-3}{\pi^4} = \frac{-6i}{\pi^3}.$$

(b) In this case both poles lie in the interior of γ, and therefore we get

$$I = 2\pi i(\text{Res}[f; 0] + \text{Res}[f; \pi]) = 2\pi i \left(\frac{-3}{\pi^4} + \frac{-(6 - \pi^2)}{2\pi^4} \right) = \frac{\pi^2 - 12}{\pi^3} i.$$

Another application of Cauchy's Residue Theorem is the counting of zeros and poles of a function.

Theorem 7.8 (Argument Principle). *Let f be analytic inside and on a positively oriented simple closed curve γ, except for P poles inside γ. Suppose that f is nonzero on γ and has N zeros inside γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P,$$

where the poles and zeros are counted with their multiplicities.

Example 7.9. Let $f(z) = \frac{z^3}{z^2-1}$ and $\gamma = \gamma(0;2)$. Now, $D = U_2(0)$ is bounded by γ. In D, $f(z)$ has

- a zero of order 3 at $z_0 = 0 \Rightarrow N = 3 \cdot 1 = 3$;
- a simple pole at $z_1 = 1$ and a simple pole at $z_2 = -1 \Rightarrow P = 1 + 1 = 2$.

Also, all $z_0, z_1, z_2 \in D$. On one hand,

$$f(z) = \frac{z^3}{z^2-1} \Rightarrow \frac{f'(z)}{f(z)} = \frac{z^2-3}{z(z+1)(z-1)} = \frac{3}{z} - \frac{1}{z+1} - \frac{1}{z-1}.$$

Therefore,

$$\begin{aligned}
\int_\gamma \frac{f'(z)}{f(z)}dz &= \int_{\gamma(0;\frac{1}{4})} \frac{f'(z)}{f(z)}dz + \int_{\gamma(1;\frac{1}{4})} \frac{f'(z)}{f(z)}dz + \int_{\gamma(-1;\frac{1}{4})} \frac{f'(z)}{f(z)}dz \\
&= \int_{\gamma(0;\frac{1}{4})} \frac{3}{z}dz - \int_{\gamma(1;\frac{1}{4})} \frac{1}{z-1}dz - \int_{\gamma(-1;\frac{1}{4})} \frac{1}{z+1}dz \\
\text{Example 4.15} &= 3 \cdot 2\pi i - 2\pi i - 2\pi i \\
&= 2\pi i \left(\underbrace{3}_{N} - \underbrace{(1+1)}_{P} \right) \\
&= 2\pi i.
\end{aligned}$$

Proof of Theorem 7.8. f'/f is analytic inside and on γ, except for the zeros and poles of f inside γ.

(i) Suppose that a is a zero of order m of f. Then there is a function g, analytic in $U_r(a)$ for some $r > 0$ and nonzero in $U_r(a)$, such that

$$f(z) = (z-a)^m g(z), \qquad z \in U_r(a). \tag{7.1.2}$$

Then

$$f'(z) = m(z-a)^{m-1}g(z) + (z-a)^m g'(z),$$

and upon dividing this by (7.1.2),

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}.$$

Now g'/g is analytic in $U_r(a)$, and thus f'/f has a simple pole with residue m at a.

(ii) Now suppose that a is a pole of order n of f. Then we use exactly the same argument as in (i), with m replaced by $-n$, and we see that f'/f has a simple pole at a with residue $-n$.

Finally, use the Residue Theorem; N is the sum of all the m given by the zeros a, and P is the sum of all the n given by the poles a. □

Example 7.10 (Summary)**.** Evaluate

$$I = \int_{\gamma(0;2)} \frac{2z-1}{z(z-1)} dz.$$

Solution. For the integrand, we analyze its poles:

- the numerator is linear, with one simple zero at $z = 1/2$;
- the denominator is quadratic, with simple zeros at $z = 0$ and $z = 1$.

Thus, the integrand has simple poles at 0 and 1, both lie inside the circle $\gamma(0;2)$.
(I) Residue: Let $f(z) = (2z-1)/(z(z-1))$. We first compute that

$$\operatorname{Res}[f;0] = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{2z-1}{z-1} = 1,$$
$$\operatorname{Res}[f;1] = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{2z-1}{z} = 1.$$

Thus, by residue theorem,

$$I == 2\pi i \left(\operatorname{Res}[f;0] + \operatorname{Res}[f;1]\right) = 4\pi i.$$

(II) Partial fraction decomposition: Since

$$\frac{2z-1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{(A+B)z - A}{z(z-1)} \Rightarrow A = B = 1,$$

we see

$$I = \int_{\gamma(0;2)} \left(\frac{1}{z} + \frac{1}{z-1} \right) dz = \int_{\gamma(0;2)} \frac{1}{z} dz + \int_{\gamma(0;2)} \frac{1}{z-1} dz.$$

By deformation theorem,

$$I = \int_{\gamma(0;1)} \frac{1}{z} dz + \int_{\gamma(1;1)} \frac{1}{z-1} dz.$$

Now, we have two methods:

- by Example 4.15, we see directly,

$$\int_{\gamma(0;1)} \frac{1}{z} dz = 2\pi i = \int_{\gamma(1;1)} \frac{1}{z-1} dz \Rightarrow I = 4\pi i;$$

- or by considering the constant function $f(z) = 1$ and apply the Cauchy's integral formulas to see

$$\int_{\gamma(0;1)} \frac{1}{z} dz = 2\pi i f(0) = 2\pi i;$$
$$\int_{\gamma(1;1)} \frac{1}{z-1} dz = 2\pi i f(1) = 2\pi i,$$

so that

$$I = 4\pi i.$$

(III) Argument Principle: Notice that if letting $f(z) := z(z-1) = z^2 - z$,

$$I = \int_{\gamma(0;2)} \frac{f'(z)}{f(z)} dz.$$

Now, $f(z)$ is a polynomial, therefore entire, so that it has no poles. Also as mentioned above, $f(z)$ has two simple zeros inside $\gamma(0;2)$. By argument principle,

$$I = \int_{\gamma(0;2)} \frac{f'(z)}{f(z)} dz = 2\pi i\,(2-0) = 4\pi i.$$

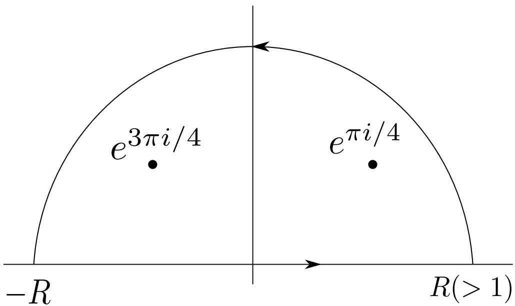
7.2 Applications: Improper Real Integrals

We have seen that the Residue Theorem is very useful in efficiently evaluating integrals of functions that have a finite number of poles inside the closed contour of integration. In this section we will see that this method can actually be used to evaluate improper *real* integrals, some of which may be difficult or impossible to do with other methods. We will study a number of examples.

Example 7.11. Evaluate

$$I := \int_0^\infty \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_0^R \frac{1}{1+x^4}\,dx.$$

Solution. The idea here is to consider the interval $\gamma_1 := [-R, R]$ as part of a single closed curve $\gamma_1 + \gamma_2$, where γ_2 is the semicircle "sitting on top" of γ_1.



Then we have

$$\int_{\gamma_1} \frac{dz}{1+z^4} + \int_{\gamma_2} \frac{dz}{1+z^4} = \int_{\gamma_1+\gamma_2} \frac{dz}{1+z^4}.$$

The plan is as follows:

- The first integral will approach $2I$ as $R \to \infty$.
- We will try to estimate the second integral as $R \to \infty$.
- The third integral can be evaluated using the Residue Theorem.

We now carry this out in detail.
1. We saw in Example 4.22, following the Estimation Theorem (Corollary 4.21), that

$$\left| \int_{\gamma_2} \frac{dz}{1+z^4} \right| \le \frac{\pi R}{|R^4-1|} \to 0 \qquad \text{as } R \to \infty,$$

and therefore,

$$\int_{\gamma_2} \frac{dz}{1+z^4} \to 0 \qquad \text{as } R \to \infty.$$

2. We evaluate the third integral using the Residue Theorem. It is clear that

$$f(z) := \frac{1}{1+z^4}$$

has four simple poles, namely the 4th roots of -1. Two of them, $(\pm 1 + i)/\sqrt{2}$, lie in the interior of $\gamma_1 + \gamma_2$, as long as $R > 1$. It is more convenient to write these poles in polar form: $e^{\pi i/4}, e^{3\pi i/4}$.

In this case the best way to evaluate the limits involved in finding the residues is by using L'Hospital's Rule:

$$\begin{aligned}
\operatorname{Res}[f; e^{\pi i/4}] &= \lim_{z \to e^{\pi i/4}} (z - e^{\pi i/4}) \frac{1}{1+z^4} = \lim_{z \to e^{\pi i/4}} \frac{1}{4z^3} \\
&= \frac{1}{4e^{3\pi i/4}} = \frac{1}{4} e^{-3\pi i/4} = \frac{1}{4\sqrt{2}}(-1-i); \\
\operatorname{Res}[f; e^{3\pi i/4}] &= \lim_{z \to e^{3\pi i/4}} (z - e^{3\pi i/4}) \frac{1}{1+z^4} = \lim_{z \to e^{3\pi i/4}} \frac{1}{4z^3} \\
&= \frac{1}{4e^{9\pi i/4}} = \frac{1}{4} e^{-\pi i/4} = \frac{1}{4\sqrt{2}}(1-i).
\end{aligned}$$

Therefore, by the Residue Theorem,

$$\int_{\gamma_1+\gamma_2} \frac{dz}{1+z^4} = 2\pi i \left(\frac{1}{4\sqrt{2}}(-1-i) + \frac{1}{4\sqrt{2}}(1-i) \right) = 2\pi i \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

So, finally, $2I = \pi/\sqrt{2}$, and so

$$\int_0^\infty \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

The next two examples involve trigonometric functions. In general, for integrals of the type

$$\int_{-\infty}^\infty R(x) \cos x dx \quad \text{or} \quad \int_{-\infty}^\infty R(x) \sin x dx,$$

where $R(x)$ is a rational function, we apply the Residue Theorem to the function $f(z) = R(z)e^{iz}$, and take real and imaginary parts.

Example 7.12. Evaluate

$$\int_{-\infty}^\infty \frac{\cos x}{x^2+1} dx$$

Solution. We set

$$f(z) := \frac{e^{iz}}{z^2+1}$$

and use the same contour $\gamma = \gamma_1 + \gamma_2$ as in the previous example. We note that (for $R > 1$) the only pole in the interior of γ is i. We therefore compute

$$\operatorname{Res}[f; i] = \lim_{z \to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \lim_{z \to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i},$$

and so, by the Residue Theorem,

$$\int_\gamma \frac{e^{iz}}{z^2+1}dz = 2\pi i\frac{e^{-1}}{2i} = \frac{\pi}{e}$$

(for $R > 1$). Next, we use the Estimation Theorem:

$$\left|\int_{\gamma_2} \frac{e^{iz}}{z^2+1}dz\right| \leq \pi R \max_{z \in \gamma_2^*}\left\{\left|\frac{e^{iz}}{z^2+1}\right|\right\}.$$

To determine the "max" expression above, we first use the triangle inequality and get

$$\left|z^2+1\right| \geq R^2 - 1 \qquad \text{for} \quad z \in \gamma_2^*,\ R > 1.$$

Second, using the polar form $z = R(\cos\theta + i\sin\theta)$, we get

$$\left|e^{iz}\right| = \left|e^{iR(\cos\theta + i\sin\theta)}\right| = \left|e^{iR\cos\theta}\right| \cdot \left|e^{-iR\sin\theta}\right|.$$

Now, since the exponent $iR\cos\theta$ is purely imaginary, we have $|e^{iR\cos\theta}| = 1$. To estimate the last factor, note that $0 \leq \theta \leq \pi$, and therefore $\sin\theta \geq 0$, which implies

$$\left|e^{iz}\right| = 1 \cdot \left|e^{-iR\sin\theta}\right| \leq 1.$$

Thus we get

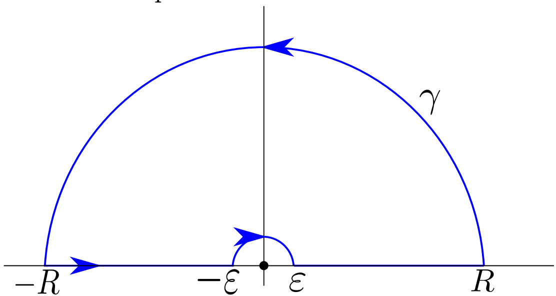
$$\left|\int_{\gamma_2} \frac{e^{iz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1} \to 0 \quad \text{as} \quad R \to \infty.$$

So finally,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}dx &= \lim_{R\to\infty} \operatorname{Re}\left(\int_{\gamma_1} \frac{\cos z + i\sin z}{z^2+1}dz\right) \\ &= \lim_{R\to\infty} \operatorname{Re}\left(\int_{\gamma} \dots - \int_{\gamma_2} \dots\right) = \frac{\pi}{e}.\end{aligned}$$

Example 7.13. Evaluate
$$\int_0^\infty \frac{\sin x}{x} dx.$$

Solution. Following the general rule from before, we consider $f(z) := e^{iz}/z$. Note that f has a pole at 0, which means that we have to modify the contour used in the previous two examples as follows:



In particular, we go around the origin in a small semicircle. We denote by

γ_1: a semicircle of radius ε,

γ_2: a semicircle of radius R,

both positively oriented, and the idea will be to let $R \to \infty$ and $\varepsilon \to 0$. Now, together with $-\gamma_1$ and γ_2 the real intervals $[-R, -\varepsilon]$ and $[\varepsilon, R]$ form a simple closed curve γ, and it is clear that the function f is analytic in the interior and on γ. So, by Cauchy's Theorem we have

$$\int_{-R}^{\varepsilon} f(x)dx - \int_{\gamma_1} f(z)dz + \int_{\varepsilon}^{R} f(x)dx + \int_{\gamma_2} f(z)dz = 0. \tag{7.2.1}$$

1. First we estimate the integral $\int_{\gamma_2} f(z)dz$; this time we begin by parametrizing $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Then $z' = iRe^{i\theta}$, and we have

$$\int_{\gamma_2} \frac{e^{iz}}{z} dz = \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta = i \int_0^{\pi} e^{iR(\cos\theta + i\sin\theta)} d\theta.$$

Using the "triangle inequality for integrals" we get

$$\begin{aligned}\left|\int_{\gamma_2}\frac{e^{iz}}{z}dz\right| &\leq \int_0^\pi \left|e^{iR(\cos\theta+i\sin\theta)}\right| d\theta\\ &= \int_0^\pi \left|e^{iR\cos\theta}\right| e^{-R\sin\theta}d\theta\\ &= \int_0^\pi e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta.\end{aligned}$$

To estimate this last integral, we note that

$$\sin\theta \geq \frac{2}{\pi}\theta \qquad \text{for} \quad 0\leq\theta\leq\frac{\pi}{2},$$

which can be seen by considering the graph of $\sin\theta$. With this, we get

$$\begin{aligned}\left|\int_{\gamma_2} \frac{e^{iz}}{z} dz\right| &\leq 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta \\ &= 2\left[\frac{1}{-2R/\pi} e^{-2R\theta/\pi}\right]_0^{\pi/2} \\ &= \frac{\pi}{R}\left(1-e^{-R}\right) \to 0 \quad \text{as} \quad R\to\infty.\end{aligned}$$

2. Before we can estimate the integral along the small semicircle γ_1, we need a lemma that is something like an interesting generalization of Lemma 7.1.

Lemma 7.14. *Let f be analytic in $U_r(a)\setminus\{a\}$, and suppose that f has a simple pole at a. Define γ_ε by*

$$\gamma_\varepsilon(\theta) = a + \varepsilon e^{i\theta}, \quad \theta\in[\theta_1,\theta_2], \quad 0<\varepsilon<r, \quad 0\leq\theta_1<\theta_2\leq 2\pi.$$

Then

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z)dz = (\theta_2-\theta_1)i\cdot \operatorname{Res}[f;a]$$

Proof. Since a is a simple pole, we have

$$\mathrm{Res}[f;a] = \lim_{z\to a}(z-a)f(z).$$

For simplicity of notation, we set $b := \mathrm{Res}[f;a]$. Now let

$$g(z) := (z-a)f(z) - b.$$

Then by the definition of continuity, given $\overline{\varepsilon} > 0$, there is a $\delta > 0$ such that $|g(z)| < \overline{\varepsilon}$ whenever $0 < |z-a| < \delta$. Let $0 < \varepsilon < \min\{r, \delta\}$. We parametrize

$$z = \gamma_\varepsilon(\theta); \quad \text{then} \quad \gamma_\varepsilon'(\theta) = \varepsilon i e^{i\theta} = i(z-a),$$

and get

$$\begin{aligned}\left|\int_{\gamma_\varepsilon} f(z)dz - ib(\theta_2-\theta_1)\right| &= \left|\int_{\theta_1}^{\theta_2} \left(f(\gamma_\varepsilon(\theta))\gamma_\varepsilon'(\theta) - ib\right) d\theta\right| \\ &= \left|i\int_{\theta_1}^{\theta_2} \left(f(\gamma_\varepsilon(\theta))\left(\gamma_\varepsilon(\theta)-a\right) - b\right) d\theta\right| \\ &= \left|\int_{\theta_1}^{\theta_2} g(\gamma_\varepsilon(\theta))d\theta\right|\end{aligned}$$

$$\begin{aligned} &\leq (\theta_2 - \theta_1) \cdot \max\{|g(\gamma_\varepsilon(\theta))|\} \\ &< \overline{\varepsilon}(\theta_2 - \theta_1). \end{aligned}$$

Since $\overline{\varepsilon}$ was chosen arbitrarily small, this proves the lemma.

3. Now we return to the example, estimating the integral along γ_1. Using Lemma 7.14, we have

$$\lim_{\varepsilon \to 0} \int_{\gamma_1} \frac{e^{iz}}{z} dz = (\pi - 0)i \cdot \operatorname{Res}[e^{iz}/z; 0] = \pi i,$$

since

$$\operatorname{Res}[e^{iz}/z; 0] = \lim_{z \to 0} \left(z \cdot \frac{e^{iz}}{z} \right) = \lim_{z \to 0} e^{iz} = 1.$$

Putting everyting together in (7.2.1): We let jointly $R \to \infty$ and $\varepsilon \to 0$ in the limits below, obtaining

$$\begin{aligned} \pi i &= \lim \left(\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx \right) \\ &= \lim \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x} dx \end{aligned}$$

$$= \lim 2i \int_{\varepsilon}^{R} \frac{\sin x}{x} dx.$$

But this means that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2},$$

which is the desired integral.

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