Matrix Representations for Bernoulli and Euler Polynomials

Lin JIU (Joint work with Diane Yahui Shi)

Department of Mathematics and Statistics, Dalhousie University
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Objects

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n	0	1	2	3	4
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
$B_n(x)$	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$

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$$B_n = (-1)^n n! \det\begin{pmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \cdots & \frac{1}{2!} \end{pmatrix}$$

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$$R := \begin{pmatrix} x-\frac{1}{2} & \omega_1 & 0 & 0 & \cdots & \cdots \\ 1 & x-\frac{1}{2} & \omega_2 & 0 & \cdots & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \cdots \\ 0 & 0 & 1 & \ddots & \omega_n & \cdots \\ 0 & 0 & 0 & \ddots & x-\frac{1}{2} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \text{where } \omega_n = -\frac{n^4}{4(2n+1)(2n-1)}.$$

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For exmaple,

$$R_4 := \begin{pmatrix} x-1/2 & -1/12 & 0 & 0 \\ 1 & x-1/2 & -4/15 & 0 \\ 0 & 1 & x-1/2 & -81/140 \\ 0 & 0 & 1 & x-1/2 \end{pmatrix}$$

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If letting $\omega_n = -n^2/4$, we could similarly obain $E_n(x)$.

Main Result

Theorem [L. J and D. Shi]

Define $M_{n,k}$ by $M_{0,0} = 1$ and for $n > 0$

$$M_{n+1,k} = M_{n,k-1} + \left(x - \frac{1}{2}\right) M_{n,k} + \omega_{k+1} M_{n,k+1},$$

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P_n satisfies a three-term recurrence: for some sequences $(s_n)_{n \geq 0}$ and $(t_n)_{n \geq 1}$,

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Theorem

$$\sum_{n=0}^{\infty} m_n x^n = \cfrac{m_0}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - \cdots}}}.$$

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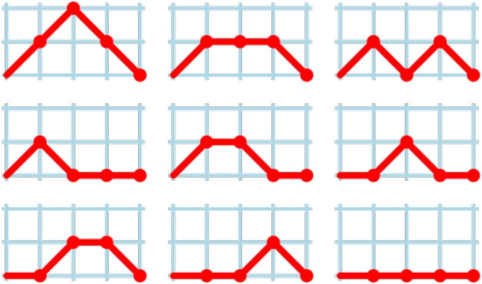

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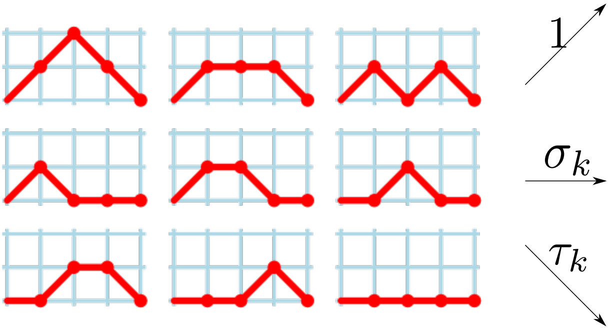

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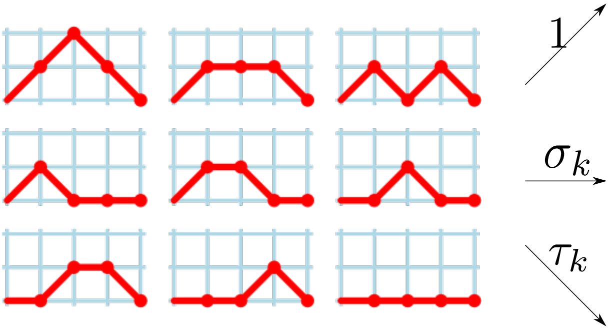

$$\sum_{n=0}^{\infty} M_{n,0} z^n = \cfrac{1}{1 - s_0 z - \cfrac{t_1 z^2}{1 - s_1 z - \cfrac{t_2 z^2}{\cdots}}}$$

General Identification

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For random variable X with moments m_n and monic orthogonal polynomials P_n, satisfying recurrence

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we define generalized Motzkin numbers $M_{n,k}$ by letting $(\sigma_k, \tau_k) = (s_k, t_k)$. If further assuming $m_0 = 1$, we have

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Theorem (J. Touchard)

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Lemma[L. J and D. Shi]

	Moments	Monic Orthogonal Polynomials
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$X + c$	$\sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$	$Q_{n+1}(y) = (y - s_n - c)Q_n(y) - t_n Q_{n-1}(y)$
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Recall

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_n x^{n-k}.$$

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For Bernoulli polynomials $B_n(x)$, the corresponding orthgonal polynomials $\theta_n(y)$ satisfy

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Thus, we define the generalized Motzkin numbers by

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which implies $M_{n,0} = B_n(x)$. The matrix presentation follows from the lattice path interpretation.

Euler Polynoimials

Euler numbers $(E_n)_{n=0}^{\infty}$ and Euler polynomials $(E_n(x))_{n=0}^{\infty}$

$$\operatorname{sech}(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

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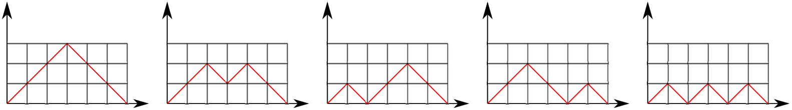

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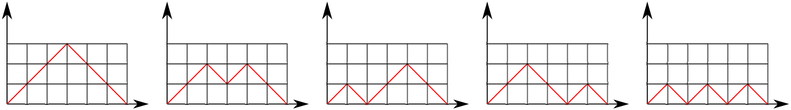

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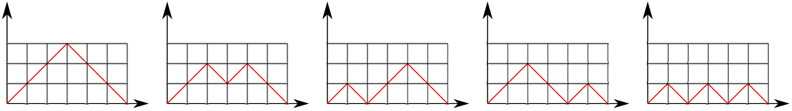

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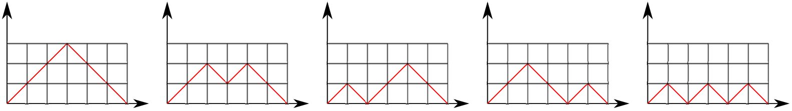

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$$(R_4)^4$$

$$\left(\begin{array}{rrr}
x^4-2x^3+x^2-\frac{1}{30} & -\frac{1}{3}x^3+\frac{1}{2}x^2-\frac{2}{15}x-\frac{1}{135} & \frac{2}{15}x^2 \\
4x^3-6x^2+\frac{8}{5}x+\frac{4}{45} & x^4-2x^3-\frac{3}{5}x^2+\frac{52}{45}x-\frac{37}{3780} & -\frac{16}{15}x^3+\frac{14}{15}x^2 \\
6x^2-\frac{13}{3}x-\frac{89}{1008} & 4x^3-\frac{7}{2}x^2-\frac{635}{252}x+\frac{1673}{1728} & x^4-\frac{1}{3}x^3-\frac{845}{168}x^2+ \\
4x-\frac{19}{12} & 6x^2-\frac{13}{3}x-\frac{89}{1008} & 4x^3-\frac{7}{2}x^2
\end{array}\right.$$

End

Thank you