Concentration Inequalities

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Materials and resources

- Textbook: *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, Martin J. Wainwright. Cambridge. 2019.
- Contents: 2, 4, 5, 6, 7, 8, 12, 13, 15
- Notes and slides: https://jiuzhoumiao.github.io/

Overview

- Tools and techniques:
  1. Concentration inequalities (Chapter 2).
  2. Uniform laws of large number (Chapter 4).
  3. Metric entropy (Chapter 5).
  4. Reproducing kernel Hilbert spaces (Chapter 12).
  5. Minimax lower bounds (Chapter 15).
- Models and estimators:
  1. Random matrices and covariance estimation (Chapter 6).
  2. High dimensional sparse linear models (Chapter 7).
  3. High dimensional principal component analysis (Chapter 8).
  4. Nonparametric least squares (Chapter 13).

Three regimes

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- Non-asymptotics: The pair (n, d), as well as other problem parameters, are viewed as fixed, and high-probability statements are made as a function of them.

From Markov to Chernoff

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$$\mathbb{P}\big\{|X - \mu| \geq t\big\} \leq t^{-2}\mathrm{Var}(X),$$

where $\mu = \mathbb{E}(X)$.

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- If X has a central moment of order k, then Markov's inequality yields a more sharp upper bound $t^{-k}\mathbb{E}\big\{|X - \mu|^k\big\}$.

From Markov to Chernoff

- Let $\varphi(\lambda) = \mathbb{E}\{e^{\lambda(X-\mu)}\}$ be the moment generating function of $X - \mu$. Assume that there is a constant $b > 0$ such that $\varphi(\lambda)$ exists for all $|\lambda| \leq b$.

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- For any $\lambda \in [0, b]$, Markov's inequality can be applied to the random variable $e^{\lambda(X-\mu)}$. i.e., for all $t > 0$,

$$\mathbb{P}(X - \mu \geq t) = \mathbb{P}\{e^{\lambda(X-\mu)} \geq e^{\lambda t}\} \leq e^{-\lambda t}\varphi(\lambda).$$

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- Chernoff's inequality: For all $t > 0$,

$$\log \mathbb{P}(X - \mu \geq t) \leq \inf_{\lambda \in [0,b]} \{-\lambda t + \log \varphi(\lambda)\}.$$

From Markov to Chernoff

Proposition 1 (Gaussian tail bounds)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$, then one has that for all $t > 0$,

(1) *One-sided inequality:*

$$\mathbb{P}(X \geq \mu + t) \leq e^{-\frac{t^2}{2\sigma^2}}.$$

(2) *Two-sided inequality:*

$$\mathbb{P}\big(|X - \mu| \geq t\big) \leq 2e^{-\frac{t^2}{2\sigma^2}}.$$

Sub-Gaussian random variables

Definition 2 (Sub-Gaussian random variable)

A random variable X with mean $\mu = \mathbb{E}(X)$ is sub-Gaussian if there is a positive number σ such that for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left\{e^{\lambda(X-\mu)}\right\} \leq e^{\sigma^2\lambda^2/2}.$$

- $-X$ is sub-Gaussian if and only if X is sub-Gaussian.

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- $-X$ is sub-Gaussian if and only if X is sub-Gaussian.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then X is sub-Gaussian.
- If X_1 and X_2 are independent sub-Gaussian variables with parameters σ_1 and σ_1, then $X_1 + X_2$ is sub-Gaussian with parameter $\sqrt{\sigma_1^2 + \sigma_2^2}$.

Sub-Gaussian tail bounds

Proposition 3 (Sub-Gaussian tail bounds)

Let X be a sub-Gaussian random variable with mean $\mu = \mathbb{E}(X)$ and sub-Gaussian parameter σ, then one has that for all $t > 0$,

$$\mathbb{P}\big(|X - \mu| \geq t\big) \leq 2e^{-\frac{t^2}{2\sigma^2}}.$$

Hoeffding's inequality

Theorem 4 (Hoeffding's inequality)

Let $\{X_i\}_{i=1}^n$ are independent sub-Gaussian random variables with mean $\mu_i = \mathbb{E}(X_i)$ and sub-Gaussian parameter σ_i, then one has that for all $t > 0$,

$$\mathbb{P}\Big\{\sum_{i=1}^{n}(X_i - \mu_i) \geq t\Big\} \leq \exp\Big\{-\frac{t^2}{2\sum_{i=1}^n \sigma_i^2}\Big\}.$$

Rademacher random variables

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- We claim that ε is sub-Gaussian with parameter $\sigma = 1$.
- By taking expectations and using Taylor's expansion

$$\mathbb{E}(e^{\lambda\varepsilon}) = \frac{e^{\lambda} + e^{-\lambda}}{2} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} \leq 1 + \sum_{k=1}^{\infty} \frac{\lambda^{2k}}{2^k k!} = e^{\lambda^2/2}.$$

Bounded random variables

Proposition 5 (Bounded random variables are sub-Gaussian)

Let X be a zero-mean and supported on some interval $[a, b]$, then one has that X is sub-Gaussian with parameter $\sigma = (b-a)/2$.

- I will provide the proofs for both cases that $\sigma = b - a$ and $\sigma = (b-a)/2$.

Sub-Exponential random variables

Definition 6 (Sub-Exponential random variable)

A random variable X with mean $\mu = \mathbb{E}(X)$ is sub-Exponential if there are non-negative parameters (ν, α) such that for all $|\lambda| < \alpha^{-1}$,

$$\mathbb{E}\left\{e^{\lambda(X-\mu)}\right\} \leq e^{\nu^2\lambda^2/2}.$$

- Sub-Gaussian variable (with parameter σ) is sub-Exponential $(\nu, \alpha) = (\sigma, 0)$, $1/0 = \infty$.

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- Sub-Gaussian variable (with parameter σ) is sub-Exponential $(\nu, \alpha) = (\sigma, 0)$, $1/0 = \infty$.
- However, the converse statement is not true.

Sub-Exponential but not sub-Gaussian

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- For $\lambda < 1/2$, one has that

$$\mathbb{E}\{e^{\lambda(X-1)}\} = \frac{e^{-\lambda}}{\sqrt{1-2\lambda}},$$

but for $\lambda \geq 1/2$, the moment generating function is infinite, which reveals that X is not sub-Gaussian.

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but for $\lambda \geq 1/2$, the moment generating function is infinite, which reveals that X is not sub-Gaussian.

- Following some calculus, one has that for $|\lambda| < 1/4$,

$$\frac{e^{-\lambda}}{\sqrt{1-2\lambda}} \leq e^{2\lambda^2} = e^{4\lambda^2/2},$$

which shows that X is sub-Exponential with parameters $(\nu, \alpha) = (2, 4)$.

Sub-Exponential tail bounds

Proposition 7 (Sub-Exponential tail bounds)

Let $\{X_i\}_{i=1}^n$ be independent sub-Exponential random variables with mean $\mu_i = \mathbb{E}(X_i)$ and sub-Exponential parameters (ν_i, α_i), then one has that $\sum_{i=1}^n (X_i - \mu_i)$ is sub-Exponential with parameters (ν_*, α_*), where $\nu_* = \left(\sum_{i=1}^n \nu_i^2\right)^{1/2}$, $\alpha_* = \max_{1\le i\le n} \alpha_i$, and

$$\mathbb{P}\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \ge t\right\} \le \begin{cases} 2e^{-\frac{t^2}{2\nu_*^2}} & , 0 < t \le \frac{\nu_*^2}{\alpha} \\ 2e^{-\frac{t}{2\alpha_*}} & , t > \frac{\nu_*^2}{\alpha_*} \end{cases}.$$

Chi-square random variables

- Let $\{Z_i\}_{i=1}^n$ be independent $\mathcal{N}(0,1)$ random variables. We have shown that Z_i^2 is sub-Expnential with parameters $(\nu, \alpha) = (2, 4)$ for each i.

Chi-square random variables

- Let $\{Z_i\}_{i=1}^n$ be independent $\mathcal{N}(0,1)$ random variables. We have shown that Z_i^2 is sub-Expnential with parameters $(\nu, \alpha) = (2, 4)$ for each i.
- By using Proposition 7, one has that $\sum_{i=1}^n (Z_i^2 - 1)$ is sub-Expnential with parameters $(\nu_*, \alpha_*) = (2\sqrt{n}, 4)$ and

$$\mathbb{P}\left\{\left|n^{-1}\sum_{i=1}^n Z_i^2 - 1\right| \geq t\right\} = \mathbb{P}\left\{\left|\sum_{i=1}^n (Z_i^2 - 1)\right| \geq nt\right\}$$
$$\leq \begin{cases} 2e^{-\frac{nt^2}{8}} & , 0 < t \leq 1 \\ 2e^{-\frac{nt}{8}} & , t > 1 \end{cases}.$$

Johnson–Lindenstrauss embedding

- Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$, $N \geq 2$ be given distinct vectors, with each vector lying in $\mathbb{R}^d$. If d is large, then it might be expensive to store and manipulate the dataset.

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- Let $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$, $N \geq 2$ be given distinct vectors, with each vector lying in $\mathbb{R}^d$. If d is large, then it might be expensive to store and manipulate the dataset.
- The idea of dimensionality reduction is to construct a mapping $F : \mathbb{R}^d \to \mathbb{R}^m$, with the projected dimension m substan tially smaller than d, that preserves some "essential" features of the dataset.

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- The idea of dimensionality reduction is to construct a mapping $F : \mathbb{R}^d \to \mathbb{R}^m$, with the projected dimension m substan tially smaller than d, that preserves some "essential" features of the dataset.
- We consider that F can preserve pairwise distances, or equivalently norms and inner products. i.e., for some $\delta \in (0, 1)$, F satisfies that for all pairs $\boldsymbol{u}_i \neq \boldsymbol{u}_j$,

$$(1-\delta)\|\boldsymbol{u}_i - \boldsymbol{u}_j\|_2^2 \leq \left\|F(\boldsymbol{u}_i) - F(\boldsymbol{u}_j)\right\|_2^2 \leq (1+\delta)\|\boldsymbol{u}_i - \boldsymbol{u}_j\|_2^2.$$

Johnson–Lindenstrauss embedding

Theorem 8 (Johnson-Lindenstrauss embedding)

For N vectors $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$, $N \geq 2$ and some $\delta \in (0,1)$, there is a mapping $F : \mathbb{R}^d \to \mathbb{R}^m$ which satisfies that

$$(1-\delta)\|\boldsymbol{u}_i - \boldsymbol{u}_j\|_2^2 \leq \left\|F(\boldsymbol{u}_i) - F(\boldsymbol{u}_j)\right\|_2^2 \leq (1+\delta)\|\boldsymbol{u}_i - \boldsymbol{u}_j\|_2^2,$$

for all pairs $\boldsymbol{u}_i \neq \boldsymbol{u}_j$ with probability at least $1 - N^2 e^{-m\delta^2/8}$.

Bernstein's condition

Definition 9 (Bernstein's condition)

Given a random variable X with mean $\mu = \mathbb{E}(X)$ and variance $\sigma^2 = \mathbb{E}(X^2) - \mu^2$, we say that Bernstein's condition with parameter b holds if for $k = 2, 3, \ldots,$

$$\left|\mathbb{E}\{(X-\mu)^k\}\right| \leq 2^{-1} k! \sigma^2 b^{k-2}.$$

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- One sufficient condition for Bernstein's condition to hold is that X be bounded.
- In particular, if $|X - \mu| \leq b$, then it is straightforward to verify that Bernstein's condition holds.

Bernstein's inequality

Theorem 10 (Bernstein's inequality)

Let $\{X_i\}_{i=1}^n$ are independent random variables with mean $\mu_i = \mathbb{E}(X_i)$ and variance $\sigma_i^2 = \mathbb{E}(X_i^2) - \mu_i^2$. If X_i satisfies Bernstein's condition with parameter b for $1 \le i \le n$, one has that:

(1) *For all $|\lambda| < 1/b$,*

$$\mathbb{E}\left\{e^{\lambda \sum_{i=1}^n (X_i - \mu_i)}\right\} \le \exp\left\{\frac{2^{-1}\lambda^2 \sum_{i=1}^n \sigma_i^2}{1 - |\lambda| b}\right\}.$$

(2) *For all $t > 0$,*

$$\mathbb{P}\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \ge t\right\} \le 2\exp\left\{-\frac{t^2}{2(bt + \sum_{i=1}^n \sigma_i^2)}\right\}.$$

Motivations

- Let $\{X_k\}_{k=1}^n$ be i.i.d. random variables and $f : \mathbb{R}^n \to \mathbb{R}$. We want to derive the bounds of

$$\mathbb{P}\left[\left|f(X_1, \ldots, X_n) - \mathbb{E}\{f(X_1, \ldots, X_n)\}\right| \geq t\right]$$

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for all $t > 0$.

- Denote $Y_0 = \mathbb{E}\big\{f(X_1,\ldots,X_n)\big\}$, $Y_n = f(X_1,\ldots,X_n)$ and

$$Y_k = \mathbb{E}\big\{f(X_1,\ldots,X_n)\big|X_1,\ldots,X_k\big\}$$

for $k = 1,\ldots,n-1$.

Motivations

- Then one can rewrite $f(X_1, \ldots, X_n) - \mathbb{E}\{f(X_1, \ldots, X_n)\}$ by

$$Y_n - Y_0 = \sum_{k=1}^{n} (Y_k - Y_{k-1}) = \sum_{k=1}^{n} D_k.$$

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- Then one can rewrite $f(X_1, \ldots, X_n) - \mathbb{E}\{f(X_1, \ldots, X_n)\}$ by

$$Y_n - Y_0 = \sum_{k=1}^{n} (Y_k - Y_{k-1}) = \sum_{k=1}^{n} D_k.$$

- Here, $\{Y_k\}_{k=1}^n$ is a particular example of a martingale sequence, whereas $\{D_k\}_{k=1}^n$ is an example of a martingale difference sequence.

Martingale

Definition 11 (Filtration)

Let $\{\mathcal{A}_k\}_{k=1}^{\infty}$ be a sequence of σ-fields. We say that $\{\mathcal{A}_k\}_{k=1}^{\infty}$ is a filtration if $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$ for all $k \geq 1$. For a sequence of random variables $\{Y_k\}_{k=1}^{\infty}$, we say that $\{Y_k\}_{k=1}^{\infty}$ is adapted to the filtration $\{\mathcal{A}_k\}_{k=1}^{\infty}$ if Y_k is measurable with respect to $\mathcal{A}_k$ for all $k \geq 1$.

Definition 12 (Martingale)

Let $\{Y_k\}_{k=1}^{\infty}$ be a sequence of random variables adapted to the filtration $\{\mathcal{A}_k\}_{k=1}^{\infty}$. We say that $(Y_k, \mathcal{A}_k)_{k=1}^{\infty}$ is a martingale if for all $k \geq 1$, $\mathbb{E}\big(|Y_k|\big) < \infty$ and almost surely

$$\mathbb{E}(Y_{k+1}|\mathcal{A}_k) = Y_k.$$

Two special cases

- If $\mathcal{A}_k = \sigma(Y_1, \ldots, Y_k)$ in Definition 12, we say simply that $\{Y_k\}_{k=1}^{\infty}$ is a martingale.

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- If $\mathcal{A}_k = \sigma(Y_1, \ldots, Y_k)$ in Definition 12, we say simply that $\{Y_k\}_{k=1}^{\infty}$ is a martingale.
- Let $\{X_k\}_{k=1}^{\infty}$ be another sequence of random variables. If $\mathcal{A}_k = \sigma(X_1, \ldots, X_k)$ in Definition 12, we say that $\{Y_k\}_{k=1}^{\infty}$ is a martingale with respect to $\{X_k\}_{k=1}^{\infty}$.

Partial sums as martingales

- Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with mean zero. Denote $S_k = \sum_{i=1}^{k} X_i$ and $\mathcal{A}_k = \sigma(X_1, \ldots, X_k)$.

Partial sums as martingales

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- One has that for all $k \geq 1$,

$$\mathbb{E}\big(|S_k|\big) \leq \sum_{i=1}^{k} \mathbb{E}\big(|X_k|\big) < \infty,$$

and almost surely

$$\begin{aligned}\mathbb{E}(S_{k+1}|\mathcal{A}_k) &= \mathbb{E}(S_k + X_{k+1}|X_1, \ldots, X_k) \\ &= S_k + \mathbb{E}(X_{k+1}|X_1, \ldots, X_k) \\ &= S_k.\end{aligned}$$

Doob construction

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- Let $\{X_k\}_{k=1}^n$ be i.i.d. random variables and $f : \mathbb{R}^n \to \mathbb{R}$.
- For short hand, denote $X_1^k = (X_1, \ldots, X_k)$ for $1 \le k \le n$. Denote $Y_0 = \mathbb{E}\{f(X_1^n)\}$, $Y_n = f(X_1^n)$ and $Y_k = \mathbb{E}\{f(X_1^n) \big| X_1^k\}$ for $k = 1, \ldots, n-1$.

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- Suppose that $\mathbb{E}\{|f(X_1^n)|\} < \infty$, then one has that for all $k \ge 1$,

$$\mathbb{E}(|Y_k|) \le \mathbb{E}\left[\mathbb{E}\{|f(X_1^n)||X_1^k\}\right] = \mathbb{E}\{|f(X_1^n)|\} < \infty,$$

and almost surely

$$\begin{aligned}\mathbb{E}(Y_{k+1}|X_1^k) &= \mathbb{E}\left[\mathbb{E}\{f(X_1^n)|X_1^{k+1}\}\middle|X_1^k\right] \\ &= \mathbb{E}\{f(X_1^n)|X_1^k\} \\ &= Y_k.\end{aligned}$$

Likelihood ratio

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- For each $k \geq 1$, denote $Y_k = \prod_{i=1}^{k} \{g(X_i)/f(X_i)\}$.

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- For each $k \geq 1$, denote $Y_k = \prod_{i=1}^{k} \{g(X_i)/f(X_i)\}$.
- Note that for each $i \geq 1$, one has that

$$\mathbb{E}\{g(X_i)/f(X_i)\} = \int_{-\infty}^{\infty} \{g(x)/f(x)\} f(x) \mathrm{d}x = 1,$$

one has that for each $k \geq 1$, $\mathbb{E}(|Y_k|) < \infty$ and almost surely

$$\mathbb{E}(Y_{k+1}|X_1^k) = Y_k \mathbb{E}\{g(X_{k+1})/f(X_{k+1})\} = Y_k.$$

Martingale difference sequence

Definition 13 (Martingale difference sequence)

Let $\{D_k\}_{k=1}^{\infty}$ be a sequence of random variables adapted to the filtration $\{\mathcal{A}_k\}_{k=1}^{\infty}$. We say that $(D_k, \mathcal{A}_k)_{k=1}^{\infty}$ is a martingale difference sequence if for all $k \geq 1$, $\mathbb{E}\big(|D_k|\big) < \infty$ and almost surely

$$\mathbb{E}(D_{k+1}|\mathcal{A}_k) = 0.$$

- Let $(Y_k, \mathcal{A}_k)_{k=0}^{\infty}$ be a martingale. One can construct a martingale difference sequence $(D_k, \mathcal{A}_k)_{k=1}^{\infty}$ by setting $D_k = Y_k - Y_{k-1}$ for $k \geq 1$.

Concentration inequalities for martingale difference sequences

Theorem 14 (Sub-exponential bounds for martingale difference sequences)

Let $(D_k, \mathcal{A}_k)_{k=1}^{\infty}$ be a martingale difference sequence, and suppose that $\mathbb{E}(e^{\lambda D_k}|\mathcal{A}_{k-1}) \leq e^{\nu_k^2\lambda^2/2}$ almost surely for $|\lambda| < \alpha_k^{-1}$. Then one has that:

(1) *$\sum_{k=1}^n D_k$ is sub-exponential with parameters (ν_*, α_*), which is the same as the notation given in Proposition 7.*

(2) *The sum satisfies the concentration inequality*

$$\mathbb{P}\left\{\left|\sum_{k=1}^n D_k\right| \geq t\right\} \leq \begin{cases} 2e^{-\frac{t^2}{2\nu_*^2}} & , 0 < t \leq \frac{\nu_*^2}{\alpha} \\ 2e^{-\frac{t}{2\alpha_*}} & , t > \frac{\nu_*^2}{\alpha_*} \end{cases}.$$

Concentration inequalities for martingale difference sequences

Theorem 15 (Azuma–Hoeffding)

Let $(D_k, \mathcal{A}_k)_{k=1}^{\infty}$ be a martingale difference sequence for which there are constants $(a_k, b_k)_{k=1}^{n}$ such that $D_k \in [a_k, b_k]$ almost surely for all $k = 1, \ldots, n$. Then one has that for all $t > 0$,

$$\mathbb{P}\left\{\left|\sum_{k=1}^{n} D_k\right| \geq t\right\} \leq 2\exp\left\{-\frac{2t^2}{\sum_{i=1}^{n}(b_k - a_k)^2}\right\}.$$

Bounded differences inequality

Definition 16 (Bounded differences property)

Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$ that their i-th element are x_i and x_i' respectively. Define $\boldsymbol{x}^{\sim k} \in \mathbb{R}^n$ via

$$x_i^{\sim k} = \begin{cases} x_i, & i \neq k \\ x_k', & i = k \end{cases}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f satisfies the bounded difference property with parameters $(L_1, \ldots, L_n)$ if for each $k = 1, \ldots, n$,

$$\left| f(\boldsymbol{x}) - f(\boldsymbol{x}^{\sim k}) \right| \leq L_k$$

for all $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{R}^n$.

Bounded differences inequality

Theorem 17 (Bounded differences inequality)

Let $\mathbf{X} = (X_1, \ldots, X_n)^\mathsf{T}$ be a random vector with independent components and $f : \mathbb{R}^n \to \mathbb{R}$ which satisfies the bounded difference property with parameters $(L_1, \ldots, L_n)$. Then one has that for all $t > 0$,

$$\mathbb{P}\left\{\left|f(\mathbf{X}) - \mathbb{E}\{f(\mathbf{X})\}\right| \geq t\right\} \leq 2\exp\left\{-\frac{2t^2}{\sum_{k=1}^n L_k^2}\right\}.$$

Classical Hoeffding from bounded differences

- Let $\{X_i\}_{i=1}^n$ be independent bounded random variables on $[a, b]$. Consider $f(\boldsymbol{x}) = \sum_{i=1}^n (x_i - \mu_i)$, where $\mu_i = \mathbb{E}(X_i)$.

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- Then one has that for all $t > 0$,

$$\mathbb{P}\left\{\left|\sum_{i=1}^n (X_i - \mu_i)\right| \geq t\right\} = \mathbb{P}\left\{\left|f(\mathbf{X}) - \mathbb{E}\{f(\mathbf{X})\}\right| \geq t\right\}$$
$$\leq 2\exp\left\{-\frac{2t^2}{n(b-a)^2}\right\}.$$

U-statistics

- Let $\{X_i\}_{i=1}^n$ be i.i.d. random variables and $g : \mathbb{R}^2 \to \mathbb{R}$ be a symmetric function of its arguments.

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is called as a pairwise U-statistic, which is an unbiased estimator of $\mathbb{E}\{g(X_1, X_2)\}$.

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- Assume that $\|g\|_\infty \le b$ for some $b > 0$ and let

$$f(\boldsymbol{x}) = \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} g(x_i, x_j).$$

U-statistics

- One has that

$$\begin{aligned}\left|f(\boldsymbol{x})-f(\boldsymbol{x}^{\sim k})\right| &\leq \binom{n}{2}^{-1}\sum_{i\neq j}\left|g(x_i,x_j)-g(x_i,x_j')\right| \\ &\leq \binom{n}{2}^{-1}(n-1)\times 2b \\ &= 4bn^{-1}.\end{aligned}$$

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- Then one has that for all $t>0$,

$$\mathbb{P}\Big\{\left|U_n-\mathbb{E}(U_n)\right|\geq t\Big\}\leq 2e^{-nt^2/(8b^2)}.$$

Rademacher complexity

- Let $\{\varepsilon_k\}_{k=1}^n$ be independent Rademacher random variables and $\mathcal{A} \subseteq \mathbb{R}^n$ be a collection of vectors.

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- For $\boldsymbol{a} = (a_1, \ldots, a_n)^\mathsf{T} \in \mathcal{A}$, define

$$Z(\mathcal{A}) = \sup_{\boldsymbol{a} \in \mathcal{A}} \sum_{k=1}^n a_k \varepsilon_k = \sup_{\boldsymbol{a} \in \mathcal{A}} \langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle.$$

The quantity $\mathcal{R}(\mathcal{A}) = \mathbb{E}\big\{Z(\mathcal{A})\big\}$ is called as the Rademacher complexity of the collection $\mathcal{A}$.

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  The quantity $\mathcal{R}(\mathcal{A}) = \mathbb{E}\big\{Z(\mathcal{A})\big\}$ is called as the Rademacher complexity of the collection $\mathcal{A}$.

- Let $f(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{a} \in \mathcal{A}} \langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle$. Since $f(\boldsymbol{\varepsilon}^{\sim k}) \geq \langle \boldsymbol{a}, \boldsymbol{\varepsilon}^{\sim k} \rangle$ for any $\boldsymbol{a} \in \mathcal{A}$, one has that

$$\langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle - f(\boldsymbol{\varepsilon}^{\sim k}) \leq \langle \boldsymbol{a}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\sim k} \rangle \leq a_k(\varepsilon_k - \varepsilon_k') \leq 2|a_k|.$$

Rademacher complexity

- Taking the supremum over $\mathcal{A}$ on both sides, one has that

$$f(\boldsymbol{\varepsilon}) - f(\boldsymbol{\varepsilon}^{\sim k}) \leq 2 \sup_{a \in \mathcal{A}} |a_k|$$

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- By symmetry, one can conclude that f satisfies the bounded difference inequality in coordinate k with parameter $2 \sup_{a \in \mathcal{A}} |a_k|$.
- Hence, $Z(\mathcal{A})$ is sub-Gaussian with parameter $2\sqrt{\sum_{k=1}^{n} \sup_{a \in \mathcal{A}} a_k^2}$.

Lipschitz functions

Definition 18 (Lipschitz functions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $L > 0$. We say that f is L-Lipschitz with respect to the Euclidean norm $\|\cdot\|_2$ if

$$\left|f(\boldsymbol{x}) - f(\boldsymbol{y})\right| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|_2$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Lipschitz functions of Gaussian variables

Theorem 19 (Concentration properties of Lipschitz functions of Gaussian variables)

Let $\mathbf{X} = (X_1, \ldots, X_n)^\mathsf{T}$ be a vector of i.i.d. standard Gaussian variables and $f : \mathbb{R}_n \to \mathbb{R}$ be L-Lipschitz with respect to the Euclidean norm. Then $f(\mathbf{X}) - \mathbb{E}\{f(\mathbf{X})\}$ is sub-Gaussian with parameter L and for all $t > 0$,

$$\mathbb{P}\left\{\left|f(\mathbf{X}) - \mathbb{E}\{f(\mathbf{X})\}\right| \geq t\right\} \leq 2\exp\left\{-\frac{t^2}{2L^2}\right\}.$$

- I will prove a weaker version that the sub-Gaussian parameter is $\pi L/2$ when f is both Lipschitz and differentiable.

Gaussian complexity

- Let $\{\varepsilon_k\}_{k=1}^n$ be independent $\mathcal{N}(0,1)$ random variables and $\mathcal{A} \subseteq \mathbb{R}^n$ be a collection of vectors.

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$$Z(\mathcal{A}) = \sup_{\boldsymbol{a} \in \mathcal{A}} \sum_{k=1}^n a_k \varepsilon_k = \sup_{\boldsymbol{a} \in \mathcal{A}} \langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle.$$

The quantity $\mathcal{R}(\mathcal{A}) = \mathbb{E}\big\{Z(\mathcal{A})\big\}$ is called as the Gaussian complexity of the collection $\mathcal{A}$.

Gaussian complexity

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  The quantity $\mathcal{R}(\mathcal{A}) = \mathbb{E}\big\{Z(\mathcal{A})\big\}$ is called as the Gaussian complexity of the collection $\mathcal{A}$.

- Let $f(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{a} \in \mathcal{A}} \langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle$. One has that

$$\langle \boldsymbol{a}, \boldsymbol{\varepsilon} \rangle = \langle \boldsymbol{a}, \boldsymbol{\varepsilon}' \rangle + \langle \boldsymbol{a}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}' \rangle \leq f(\boldsymbol{\varepsilon}') + \Big(\sup_{\boldsymbol{a} \in \mathcal{A}} \|\boldsymbol{a}\|_2\Big) \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\|_2.$$

Gaussian complexity

- Taking the supremum over $\mathcal{A}$ on both sides, one has that

$$f(\boldsymbol{\varepsilon}) - f(\boldsymbol{\varepsilon}') \leq \left(\sup_{\boldsymbol{a}\in\mathcal{A}} \|\boldsymbol{a}\|_2\right) \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\|_2.$$

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$$f(\boldsymbol{\varepsilon}) - f(\boldsymbol{\varepsilon}') \leq \left(\sup_{\boldsymbol{a}\in\mathcal{A}} \|\boldsymbol{a}\|_2\right) \|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'\|_2.$$

- By symmetry, one has that f is a $\sup_{\boldsymbol{a}\in\mathcal{A}} \|\boldsymbol{a}\|_2$-Lipschitz function. Then $Z(\mathcal{A})$ is sub-Gaussian with parameter $\sup_{\boldsymbol{a}\in\mathcal{A}} \|\boldsymbol{a}\|_2$.

Gaussian chaos variables

- Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix with $\mathbf{A} = (a_{ij})_{n\times n}$ and $\mathbf{X}, \mathbf{Y}$ be independent $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ random vectors with $\mathbf{X} = (X_1, \ldots, X_n)^{\mathsf{T}}$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)^{\mathsf{T}}$. Define

$$Z_n = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} X_i Y_j = \mathbf{X}^{\mathsf{T}}\mathbf{A}\mathbf{Y},$$

called as a (decoupled) Gaussian chaos.

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  called as a (decoupled) Gaussian chaos.
- One has that $\mathbb{E}(Z_n) = 0$, so it is natural to seek a tail bound on Z_n.

Gaussian chaos variables

- Condition on $\mathbf{X}$, one has that Z_n is a mean zero Gaussian random variable with variance $\mathbf{X}^{\mathsf{T}}\mathbf{A}^2\mathbf{X} = \|\mathbf{A}\mathbf{X}\|_2^2$. Then one has that for all $\delta > 0$,

$$\mathbb{P}\big(|Z_n| \geq \delta \big| \mathbf{X}\big) \leq 2\exp\left\{-\frac{\delta^2}{2\|\mathbf{A}\mathbf{X}\|_2^2}\right\}.$$

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$$\mathbb{P}\big(|Z_n| \geq \delta \big| \mathbf{X}\big) \leq 2\exp\Big\{-\frac{\delta^2}{2\|\mathbf{A}\mathbf{X}\|_2^2}\Big\}.$$

- Let $f(\boldsymbol{x}) = \|\mathbf{A}\boldsymbol{x}\|_2$. One can easily show that f is a $\|\mathbf{A}\|_{\mathrm{op}}$-Lipschitz function, where

$$\|\mathbf{A}\|_{\mathrm{op}} = \max_{\|\boldsymbol{x}\|_2=1} \|\mathbf{A}\boldsymbol{x}\|_2.$$

Gaussian chaos variables

- By Jensen's inequality, one has that

$$\mathbb{E}\big(\|\mathbf{AX}\|_2\big) \leq \sqrt{\mathbb{E}\big(\|\mathbf{AX}\|_2^2\big)} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2} = \|\mathbf{A}\|_{\mathrm{F}}.$$

Gaussian chaos variables

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- Then for all $t > 0$, one has that

$$\begin{aligned}\mathbb{P}\big(\|\mathbf{AX}\|_2 \geq \|\mathbf{A}\|_{\mathrm{F}} + t\big) &\leq \mathbb{P}\Big\{\|\mathbf{AX}\|_2 \geq \mathbb{E}\big(\|\mathbf{AX}\|_2\big) + t\Big\} \\ &\leq 2\exp\left\{-\frac{t^2}{2\|\mathbf{A}\|_{\mathrm{op}}^2}\right\}.\end{aligned}$$

Gaussian chaos variables

- By taking $t^2 = \delta\|\mathbf{A}\|_{\mathrm{op}}$ and note that

$$\left(\|\mathbf{A}\|_{\mathrm{F}} + t\right)^2 \leq 2\|\mathbf{A}\|_{\mathrm{F}}^2 + 2t^2,$$

one has that

$$\mathbb{P}\left(\|\mathbf{A}\mathbf{X}\|_2^2 \geq 2\|\mathbf{A}\|_{\mathrm{F}}^2 + 2\delta\|\mathbf{A}\|_{\mathrm{op}}\right) \leq 2\exp\left\{-\frac{\delta}{2\|\mathbf{A}\|_{\mathrm{op}}}\right\}.$$

Gaussian chaos variables

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$$\big(\|\mathbf{A}\|_{\mathrm{F}} + t\big)^2 \leq 2\|\mathbf{A}\|_{\mathrm{F}}^2 + 2t^2,$$

one has that

$$\mathbb{P}\big(\|\mathbf{AX}\|_2^2 \geq 2\|\mathbf{A}\|_{\mathrm{F}}^2 + 2\delta\|\mathbf{A}\|_{\mathrm{op}}\big) \leq 2\exp\left\{-\frac{\delta}{2\|\mathbf{A}\|_{\mathrm{op}}}\right\}.$$

- Finally, one has that

$$\begin{aligned}\mathbb{P}\big(|Z_n| \geq \delta\big) &= \mathbb{E}\Big\{\mathbb{P}\big(|Z_n| \geq \delta\big|\mathbf{X}\big)\Big\} \\ &\leq 2\exp\left\{-\frac{\delta^2}{4\|\mathbf{A}\|_{\mathrm{F}}^2 + 4\delta\|\mathbf{A}\|_{\mathrm{op}}}\right\} + 2\exp\left\{-\frac{\delta}{2\|\mathbf{A}\|_{\mathrm{op}}}\right\} \\ &\leq 4\exp\left\{-\frac{\delta^2}{4\|\mathbf{A}\|_{\mathrm{F}}^2 + 4\delta\|\mathbf{A}\|_{\mathrm{op}}}\right\}.\end{aligned}$$

Thank You