Uniform Laws of Large Numbers

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1. Motivation
2. A uniform law via Rademacher complexity
3. Upper bounds on the Rademacher complexity

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Empirical cumulative distribution function (ECDF)

- Let X be a random varible with cumulative distribution function (CDF) $F(x) = \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$ and $\{X_i\}_{i=1}^n$ be independent samples which have same distribution with X.

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- A natural estimation of F is the ECDF based on $\{X_i\}_{i=1}^n$, given by

$$\hat{F}_n(x) = n^{-1} \sum_{i=1}^{n} \mathbb{I}_{(-\infty,x]}(X_i) = n^{-1} \sum_{i=1}^{n} \mathbb{I}(X_i \leq x).$$

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- For any fixed $x \in \mathbb{R}$, the strong law of large numbers implies that $\hat{F}_n(x) \to F(x)$ almost surely as $n \to \infty$.
- A natural goal is to strengthen this pointwise convergence to a form of uniform convergence.

The functionals of CDFs

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- In statistical settings, a typical use of the ECDF is to construct estimators of various quantities associated with the (population) CDF.
- Many such estimation problems can be formulated in a terms of functional γ which maps any CDF F to a real number $\gamma(F)$.
- Given a set of samples distributed according to F, the plug-in principle suggests replacing the unknown F by $\hat{F}_n$, thereby obtaining $\gamma(\hat{F}_n)$ as an estimation of $\gamma(F)$.

Expectation functionals

- Given some integrable function g, define the expectation functional γ_g by

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$$\gamma_g(\hat{F}_n) = \int g(x)\mathrm{d}\hat{F}_n(x) = n^{-1}\sum_{i=1}^{n} g(X_i).$$

Quantile functionals

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- It is frequently of interest to test the hypothesis of whether or not a given set of data has been drawn from a known distribution F_0.
- Such tests can be performed using functionals that measure the distance between F and F_0, including sup-norm distance $\|F - F_0\|_\infty$ and Cramér–von Mises criterion based on the functional

$$\gamma(F) = \int_{-\infty}^{\infty} \{F(x) - F_0(x)\}^2 \mathrm{d}F_0(x).$$

The continuity of a functional

- Let F and G be two CDF both defined on $\mathbb{R}$. Define the sup-norm between them by

$$\|G - F\|_\infty = \sup_{x \in \mathbb{R}} \big|G(x) - F(x)\big|.$$

Definition 1 (The continuity of a functional)

Let F and G are two CDFs. We say that the functional γ is continuous at F in the sup-norm if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|G - F\|_\infty \leq \delta$ implies that $\big|\gamma(G) - \gamma(F)\big| \leq \epsilon$.

Glivenko-Cantelli's Theorem

- For any continuous funcional γ, the consistency question for the plug-in estimator $\gamma(\hat{F}_n)$ can be reduced to the issue of whether or not $\|\hat{F}_n - F\|_\infty$ tends to zero.

Theorem 2 (Glivenko-Cantelli's Theorem)

For any CDF F, as $n \to \infty$, the ECDF $\hat{F}_n$ is a strongly consistent estimator of F in the uniform norm, i.e.,

$$\|\hat{F}_n - F\|_\infty = \sup_{x \in \mathbb{R}} \left|\hat{F}_n(x) - F(x)\right| \to 0$$

almost surely.

Uniform laws for more general function classes

- Let $\mathcal{F}$ be a class of integrable real-valued functions with domain $\mathcal{X}$ and X be a random variable with distribution $\mathbb{P}$. Let $\{X_i\}_{i=1}^n$ be independent random variables which have same distribution with X.

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- Define the random variable

$$\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} \left| n^{-1} \sum_{i=1}^{n} f(X_i) - \mathbb{E}\{f(X)\} \right|.$$

Glivenko-Cantelli class

Definition 3 (Glivenko-Cantelli class)

We say that $\mathcal{F}$ is $\mathbb{P}$-Glivenko-Cantelli [or strong $\mathbb{P}$-Glivenko-Cantelli] if $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$ converges to zero in probability [or almost surely].

- When $\mathcal{F} = \{\mathbb{I}_{(-\infty,x]}(\cdot) : x \in \mathbb{R}\}$, one has that

$$\mathbb{E}\{\mathbb{I}_{(-\infty,x]}(X)\} = \mathbb{P}(X \leq x)$$

for fixed x, so that the classical Glivenko-Cantelli theorem is equivalent to a strong uniform law for the class $\mathcal{F}$.

Failure of uniform law

- Let $\mathcal{S}$ be the class of all subsets S of $[0,1]$ such that the subset S has a finite number of elements. Consider the function class

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- This class includes any distribution that has a density with respect to Lebesgue measure. Then $\mathbb{P}(S) = 0$ for all $S \in \mathcal{S}$.

Failure of uniform law

- However, for any positive integer n, the discrete set $\{X_1, \ldots, X_n\}$ belongs to $\mathcal{S}$, which implies that

$$\mathbb{P}\big[\{X_1, \ldots, X_n\}\big] = 1$$

and

$$\sup_{S \in \mathcal{S}} \big|\mathbb{P}_n(S) - \mathbb{P}(S)\big| = 1.$$

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- $\mathcal{F}_S$ is not $\mathbb{P}$-Glivenko–Cantelli.

Empirical risk minimization

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- Let $\{X_i\}_{i=1}^n$ be i.i.d. samples lying in some space $\mathcal{X}$ which are drawn according to $\mathbb{P}_{\theta^*}$, where $\theta^* \in \Theta$ is fixed and unknown.
- Let $L : \mathcal{X} \times \Theta \to \mathbb{R}$ be a loss function. The quantity

$$R(\theta) = \mathbb{E}_{X \sim \mathbb{P}_{\theta^*}} \{L(X, \theta)\}$$

is called as population risk.

Empirical risk minimization

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  is called as empirical risk.
- A standard decision-theoretic approach to estimating θ^* is based on minimizing the empirical risk $\hat{R}_n(\theta)$ over $\Theta_0 \subseteq \Theta$, thereby obtaining an estimator $\hat{\theta}$.

Maximum likelihood

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- Consider the loss function

$$L(x, \theta) = \log\left\{p_{\theta^*}(x)/p_\theta(x)\right\}.$$

The term $p_{\theta^*}(x)$ has no effect on the minimization over θ.

Maximum likelihood

- The maximum likelihood estimation is obtained by minimizing

$$\hat{\theta} = \arg\min_{\theta \in \Theta_0 \subseteq \Theta} \underbrace{\left[n^{-1} \sum_{i=1}^{n} \log \left\{ p_{\theta^*}(X_i) / p_\theta(X_i) \right\} \right]}_{\hat{R}_n(\theta)}$$

$$= \arg\min_{\theta \in \Theta_0 \subseteq \Theta} \left[n^{-1} \sum_{i=1}^{n} \log \left\{ 1 / p_\theta(X_i) \right\} \right].$$

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- The population risk is given by

$$R(\theta) = \mathbb{E}_{X\sim\mathbb{P}_{\theta^*}}\left[\log\left\{p_{\theta^*}(x)/p_\theta(x)\right\}\right],$$

known as the Kullback-Leibler divergence between p_{θ^*} and p_θ.

Bound the excess risk

- The statistical question is how to bound the excess risk

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- With this notation, the excess risk can be decomposed as

$$\mathrm{ER}(\theta) = \underbrace{R(\hat{\theta}) - \hat{R}_n(\hat{\theta})}_{T_{n1}} + \underbrace{\hat{R}_n(\hat{\theta}) - \hat{R}_n(\theta_0)}_{T_{n2}} + \underbrace{\hat{R}_n(\theta_0) - R(\theta_0)}_{T_{n3}}.$$

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- Obviously, $T_{n2} \leq 0$.

Bound the excess risk

- Recall that

$$T_{n1} = \mathbb{E}_{X\sim\mathbb{P}_{\theta^*}}\{L(X,\hat{\theta})\} - n^{-1}\sum_{i=1}^{n} L(X_i,\hat{\theta}),$$

$$T_{n3} = n^{-1}\sum_{i=1}^{n} L(X_i,\theta_0) - \mathbb{E}_{X\sim\mathbb{P}_{\theta^*}}\{L(X,\theta_0)\}.$$

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$$T_{n3} = n^{-1}\sum_{i=1}^{n} L(X_i,\theta_0) - \mathbb{E}_{X\sim\mathbb{P}_{\theta^*}}\{L(X,\theta_0)\}.$$

- Define the function class

$$\mathcal{L}(\Theta_0) = \{x \mapsto L(x,\theta) : \theta \in \Theta_0\},$$

then $T_{n1} + T_{n3}$ is bounded by $2\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{L}(\Theta_0)}$.

1 Motivation

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3 Upper bounds on the Rademacher complexity

Rademacher complexity of the function class

- Let $\mathcal{F}$ be a function class. For any collection $x_1^n = \{x_1, \ldots, x_n\}$, consider the subset of $\mathbb{R}^n$ given by

$$\mathcal{F}(x_1^n) = \left\{ \big(f(x_1), \ldots, f(x_n)\big)^\mathsf{T} : f \in \mathcal{F} \right\}.$$

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$$\mathcal{F}(x_1^n) = \Big\{\big(f(x_1), \ldots, f(x_n)\big)^\top : f \in \mathcal{F}\Big\}.$$

- Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)^\top$ where $\{\varepsilon_i\}_{i=1}^n$ are i.i.d. Rademacher random variables. The Rademacher complexity of $\mathcal{F}(x_1^n)/n$ is given by

$$\mathcal{R}(\mathcal{F}(x_1^n)/n) = \mathbb{E}_{\boldsymbol{\varepsilon}}\left\{\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^n \varepsilon_i f(x_i)\right|\right\},$$

where $\mathcal{F}(x_1^n)/n$ denotes the set with elements $\big(f(x_1)/n, \ldots, f(x_n)/n\big)^\top$ for $f \in \mathcal{F}$.

Rademacher complexity of the function class

- Let $\mathbf{X} = (X_1, \ldots, X_n)^\mathsf{T}$ where $\{X_i\}_{i=1}^n$ are i.i.d. random variables. The quantity $\mathcal{R}(\mathcal{F}(X_1^n)/n)$ is still a random variable.

Rademacher complexity of the function class

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- The quantity

$$\mathcal{R}_n(\mathcal{F}) = \mathbb{E}_{\mathbf{X}}\{\mathcal{R}(\mathcal{F}(X_1^n)/n)\} = \mathbb{E}_{\mathbf{X},\boldsymbol{\varepsilon}}\left\{\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^n \varepsilon_i f(X_i)\right|\right\}$$

is called as Rademacher complexity of the function class $\mathcal{F}$.

Upper bound of $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$

Theorem 4 (Upper bound of $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$)

Let $\mathcal{F}$ be a function class which satisfies that $\|f\|_\infty \le b$ for each $f \in \mathcal{F}$. For any positive integer n and $t > 0$, one has that

$$\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}} \le 2\mathcal{R}_n(\mathcal{F}) + t$$

with $\mathbb{P}$-probability at least $1 - e^{-\frac{nt^2}{2b^2}}$. Consequently, as long as $\mathcal{R}_n(\mathcal{F}) = o(1)$, one has that $\mathcal{F}$ is $\mathbb{P}$-Glivenko-Cantelli.

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- Upper bound on mean: Show that

$$\mathbb{E}\big(\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}\big) \leq 2\mathcal{R}_n(\mathcal{F}).$$

Necessary conditions with Rademacher complexity

- The proof of Theorem 4 illustrates an important technique known as symmetrization, which relates the random variable $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$ to its symmetrized version

$$\|S_n\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} \left| n^{-1} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right|.$$

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- It is natural to wonder whether much was lost in moving from the variable to its symmetrized version.
- Denote

$$\check{\mathcal{F}} = \left\{ f - \mathbb{E}_X \{ f(X) \} : f \in \mathcal{F} \right\},$$

where X is a random variable from $\mathbb{P}$.

Necessary conditions with Rademacher complexity

Proposition 5 (Sandwich results on $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$ and $\|S_n\|_{\mathcal{F}}$)

For any convex and non-decreasing function $\phi : \mathbb{R} \to \mathbb{R}$, one has that

$$\mathbb{E}_{\mathbf{X},\varepsilon}\left\{\phi\left(\|S_n\|_{\check{\mathcal{F}}}/2\right)\right\} \leq \mathbb{E}_{\mathbf{X}}\left\{\phi\left(\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}\right)\right\} \leq \mathbb{E}_{\mathbf{X},\varepsilon}\left\{\phi\left(2\|S_n\|_{\mathcal{F}}\right)\right\}.$$

- When $\phi(t) = t$, Proposition 5 implies that

$$\mathbb{E}_{\mathbf{X},\varepsilon}\{\|S_n\|_{\check{\mathcal{F}}}\}/2 \leq \mathbb{E}_{\mathbf{X}}\{\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}\} \leq 2\mathbb{E}_{\mathbf{X},\varepsilon}\{\|S_n\|_{\mathcal{F}}\}.$$

Lower bound of $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$

Theorem 6 (Lower bound of $\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}}$)

Under the assumption of Theorem 4, for any positive integer n and $t > 0$, one has that

$$\|\mathbb{P}_n - \mathbb{P}\|_{\mathcal{F}} \geq 2^{-1}\mathcal{R}_n(\mathcal{F}) - 2^{-1}n^{-1/2}\sup_{f\in\mathcal{F}}\Big|\mathbb{E}\{f(X)\}\Big| - t$$

with $\mathbb{P}$-probability at least $1 - e^{-\frac{nt^2}{2b^2}}$.

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Classes with polynomial discrimination

- Recall the notation

$$\mathcal{F}(x_1^n) = \left\{ \left(f(x_1), \ldots, f(x_n) \right)^{\mathsf{T}} : f \in \mathcal{F} \right\}.$$

For a given collection of points $x_1^n = \{x_1, \ldots, x_n\}$, the "size" of $\mathcal{F}(x_1^n)$ provides a sample-dependent measure of the complexity of $\mathcal{F}$.

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- Consider that $\mathcal{F}(x_1^n)$ contains only a finite number of vectors for all sample sizes, so that its "size" can be measured via its cardinality.
- If $\mathcal{F}$ consists of a family of binary-valued functions, then $\mathcal{F}(x_1^n)$ can contain at most 2^n elements. Of interest to us are function classes for which this cardinality grows only as a polynomial function of n.

Classes with polynomial discrimination

Definition 7 (Polynomial discrimination)

Let $\mathcal{F}$ be a class consisting a family of binary-valued functions on $\mathcal{X}$. We say that $\mathcal{F}$ has polynomial discrimination of order $\nu \geq 1$ if for each positive integer n and collection $x_1^n = \{x_1, \ldots, x_n\}$ of n points in $\mathcal{X}$, $\mathcal{F}(x_1^n)$ has cardinality upper bounded as

$$\operatorname{Card}\big(\mathcal{F}(x_1^n)\big) \leq (n+1)^{\nu}.$$

- The significance of this property is that it provides a straightforward approach to controlling the Rademacher complexity.

Upper bound of the Rademacher complexity

Proposition 8 (Upper bound of $\mathcal{R}(\mathcal{F}(x_1^n)/n)$)

Suppose that $\mathcal{F}$ has polynomial discrimination of order ν. Then for all positive integers n and any collection of points $x_1^n = \{x_1, \ldots, x_n\}$, one has that

$$\underbrace{\mathbb{E}_{\varepsilon}\left\{\sup_{f\in\mathcal{F}}\left|n^{-1}\sum_{i=1}^{n}\varepsilon_i f(x_i)\right|\right\}}_{\mathcal{R}(\mathcal{F}(x_1^n)/n)} \leq 4D(x_1^n)\sqrt{\nu\log(n+1)/n},$$

where $D(x_1^n) = n^{-1/2}\sup_{f\in\mathcal{F}}\sqrt{\sum_{i=1}^{n} f^2(x_i)}$.

Upper bound of the Rademacher complexity

- When the function class is b uniformly bounded, then one has that $D(x_1^n)$ is bounded by b uniformly for all points x_1^n, which implies that

$$\mathcal{R}_n(\mathcal{F}) = \mathbb{E}_{\mathbf{X}}\{\mathcal{R}(\mathcal{F}(X_1^n)/n)\} \le 4b\sqrt{\nu \log(n+1)/n}.$$

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$$\mathcal{R}_n(\mathcal{F}) = \mathbb{E}_{\mathbf{X}}\{\mathcal{R}(\mathcal{F}(X_1^n)/n)\} \leq 4b\sqrt{\nu \log{(n+1)}/n}.$$

- As discussed previously, the classical Glivenko-Cantelli law is based on indicator functions of $(-\infty, t]$, which are uniformly bounded by $b = 1$.

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- We will apply Proposition 8 and Theorem 4 to give a version proof of Theorem 2.

Classical Glivenko-Cantelli's Theorem

Theorem 9 (Classical Glivenko-Cantelli)

Let $F(x) = \mathbb{P}(X \leq x)$ be the CDF of a random variable X and $\hat{F}_n(x)$ be the ECDF based on n i.i.d. samples $\{X_i\}_{i=1}^n$ from $\mathbb{P}$. Then one has that for all $t > 0$

$$\|\hat{F}_n - F\|_\infty \leq 8\sqrt{\log(1+n)/n} + t$$

with $\mathbb{P}$-probability at least $1 - e^{-nt^2/2}$, which implies that as $n \to \infty$, $\|\hat{F}_n - F\|_\infty \to 0$ almost surely.

Vapnik–Chervonenkis (VC) dimension

Definition 10 (Shattering and VC dimension)

Given a class $\mathcal{F}$ of binary-valued functions, we say that the set $x_1^n = \{x_1, \ldots, x_n\}$ is shattered by $\mathcal{F}$ if $\mathrm{Card}\big(\mathcal{F}(x_1^n)\big) = 2^n$. The VC dimension $\nu(\mathcal{F})$ is the largest integer n for which there is some collection x_1^n of n points that is shattered by $\mathcal{F}$.

- When $\nu(\mathcal{F}) < \infty$, $\mathcal{F}$ is said to be a VC class.

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- When $\nu(\mathcal{F}) < \infty$, $\mathcal{F}$ is said to be a VC class.
- When $\mathcal{F}$ is consisted by indicator functions $\mathbb{I}_S(\cdot)$ for $S \in \mathcal{S}$, we use $\mathcal{S}(x_1^n)$ and $\nu(\mathcal{S})$ to denote $\mathcal{F}(x_1^n)$ and $\nu(\mathcal{F})$ respectively.

Intervals in $\mathbb{R}$

- Consider the class of all indicator functions for left-sided half-intervals on the real line, i.e., the class

$$\mathcal{S}_1 = \big\{(-\infty, a] : a \in \mathbb{R}\big\}.$$

We have shown that for any collection $x_1^n = \{x_1, \ldots, x_n\}$, $\mathrm{Card}\big(\mathcal{S}_1(x_1^n)\big) \leq n+1$.

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  We have shown that for any collection $x_1^n = \{x_1, \ldots, x_n\}$, $\mathrm{Card}\big(\mathcal{S}_1(x_1^n)\big) \leq n + 1$.

- For any single point x_1, the collection $\{x_1\}$ can be be picked out by the class $\mathcal{S}_1$. But given two distinct points $x_1 < x_2$, it is impossible to find a left-sided interval that contains x_2 but not x_1. Therefore, we conclude that $\nu(\mathcal{S}_1) = 1$.

Intervals in $\mathbb{R}$

- Consider the class of all indicator functions for two-sided intervals on the real line, i.e., the class

$$\mathcal{S}_2 = \big\{(a, b] : a, b \in \mathbb{R}, a < b\big\}.$$

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$$\mathcal{S}_2 = \{(a, b] : a, b \in \mathbb{R}, a < b\}.$$

- The class $\mathcal{S}_2$ can shatter any two-point set. But given three distinct points $x_1 < x_2 < x_3$, it cannot pick out the subset $\{x_1, x_3\}$, which implies that $\nu(\mathcal{S}_2) = 2$.

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- The class $\mathcal{S}_2$ can shatter any two-point set. But given three distinct points $x_1 < x_2 < x_3$, it cannot pick out the subset $\{x_1, x_3\}$, which implies that $\nu(\mathcal{S}_2) = 2$.
- Note that any collection of n distinct points $x_1 < \cdots < x_n$ divides up the real line into $n + 1$ intervals. Thus, any set of the form $(a, b]$ can be specified by choosing one of $n + 1$ intervals for a and a second interval for b, which implies that this class has polynomial discrimination with $\nu = 2$.

Connection between VC dimension and polynomial discrimination

Theorem 11 (Vapnik-Chervonenkis, Sauer and Shelah)

Consider a set class $\mathcal{S}$ with $\nu(\mathcal{S}) < \infty$. Then for any collection of points $x_1^n = \{x_1, \ldots, x_n\}$ with $n \geq \nu(\mathcal{S})$, one has that

$$\mathrm{Card}\big(\mathcal{S}(x_1^n)\big) \leq \sum_{i=0}^{\nu(\mathcal{S})} \binom{n}{i} \leq (n+1)^{\nu(\mathcal{S})}.$$

Operations on VC classes

Proposition 12 (Operations on VC classes)

Let $\mathcal{S}$ and $\mathcal{T}$ be set classes, each with finite VC dimensions $\nu(\mathcal{S})$ and $\nu(\mathcal{T})$ respectively. Then each of the following set classes also have finite VC dimension:

(1) $\mathcal{S}^c = \{S^c : S \in \mathcal{S}\}$.

(2) $\mathcal{S} \sqcup \mathcal{T} = \{S \cup T : S \in \mathcal{S}, T \in \mathcal{T}\}$.

(3) $\mathcal{S} \sqcap \mathcal{T} = \{S \cap T : S \in \mathcal{S}, T \in \mathcal{T}\}$.

Vector space structure

Definition 13 (Subgraph)

Let $g : \mathcal{X} \to \mathbb{R}$ be a function, the subset of $\mathcal{X}$

$$S_g = \{x \in \mathcal{X} : g(x) \leq 0\}$$

is called as the subgraph of g at level zero. Let $\mathcal{G}$ be a function class, the collection of subsets

$$\mathcal{S}(\mathcal{G}) = \{S_g : g \in \mathcal{G}\}$$

is called as the subgraph class of $\mathcal{G}$.

Vector space structure

Proposition 14 (Finite-dimensional vector spaces)

Let $\mathcal{G}$ be a vector space of functions $g : \mathbb{R}^d \to \mathbb{R}$ with dimension $\dim(\mathcal{G}) < \infty$. Then the subgraph class $\mathcal{S}(\mathcal{G})$ has VC dimension at most $\dim(\mathcal{G})$.

Linear functions in $\mathbb{R}^d$

- For a pair $(\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R}$, define $f_{\boldsymbol{a},b}(\boldsymbol{x}) = \langle \boldsymbol{a}, \boldsymbol{x} \rangle + b$ and consider the family

$$\mathcal{L}^d = \{ f_{\boldsymbol{a},b} : (\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R} \}.$$

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- The associated subgraph class $\mathcal{S}(\mathcal{L}^d)$ corresponds to the collection of all half-spaces of the form

$$H_{\boldsymbol{a},b} = \{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{a}, \boldsymbol{x} \rangle + b \leq 0 \}.$$

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$$H_{\boldsymbol{a},b} = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{a}, \boldsymbol{x} \rangle + b \leq 0 \right\}.$$

- $\mathcal{L}^d$ forms a vector space of dimension $d+1$, one has that $\mathcal{S}(\mathcal{L}^d)$ has VC dimension at most $d+1$.

Spheres in $\mathbb{R}^d$

- Consider the sphere

$$S_{\boldsymbol{a},b} = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{a}\|_2 \leq b, (\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R}_+\}$$

and let $\mathcal{S}^d$ be the collection of all such spheres.

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$$S_{\boldsymbol{a},b} = \left\{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x} - \boldsymbol{a}\|_2 \le b, (\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R}_+\right\}$$

and let $\mathcal{S}^d$ be the collection of all such spheres.

- Define

$$f_{\boldsymbol{a},b}(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^2 - 2\langle \boldsymbol{a}, \boldsymbol{x}\rangle + \|\boldsymbol{a}\|_2^2 - b^2,$$

then one has that

$$S_{\boldsymbol{a},b} = \left\{\boldsymbol{x} \in \mathbb{R}^d : f_{\boldsymbol{a},b}(\boldsymbol{x}) \le 0\right\},$$

so that the sphere $S_{\boldsymbol{a},b}$ is a subgraph of the function $f_{\boldsymbol{a},b}$.

Spheres in $\mathbb{R}^d$

- Define a feature map $\phi : \mathbb{R}^d \to \mathbb{R}^{d+2}$ via

$$\phi(\boldsymbol{x}) = \left(1, x_1, \ldots, x_d, \|x\|_2^2\right)^{\mathsf{T}}$$

and then consider functions of the form $g_{\boldsymbol{c}}(\boldsymbol{x}) = \langle \boldsymbol{c}, \boldsymbol{x} \rangle$, $\boldsymbol{c} \in \mathbb{R}^{d+2}$.

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- The family of functions $\{g_{\boldsymbol{c}} : \boldsymbol{c} \in \mathbb{R}^{d+2}\}$ is a vector space of dimension $d+2$ and contains the function class $\left\{f_{\boldsymbol{a},b} : (\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R}_+\right\}$.

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- The family of functions $\{g_{\boldsymbol{c}} : \boldsymbol{c} \in \mathbb{R}^{d+2}\}$ is a vector space of dimension $d+2$ and contains the function class $\left\{f_{\boldsymbol{a},b} : (\boldsymbol{a}, b) \in \mathbb{R}^d \times \mathbb{R}_+\right\}$.
- By applying Proposition 14 to this larger vector space, one has that $\nu(\mathcal{S}^d) \leq d+2$.

Thank You