Liu's Solutions

1.

(a)Yes. Firstly we can check that $P(X)$ has no roots in $\mathbb{F}_2$, so the only way it could not be irreducible is by being a product of two irreducible polynomials of degree 2. But the only irreducible polynomials of degree 2 in $\mathbb{F}_2[X]$ is X^2+X+1, and P is not its square.

(b) No. We can check that $P(0)=P(2)=P(3)=1, P(1)=3$, so $P(X)$ has no roots in $\mathbb{F}_4$.

(c) No. If it is irreducible, the it will have a root in $\mathbb{F}_{16}$, which is impossible by (d).

(d) Yes. Notice that $\mathbb{F}_8$ is a degree 3 extension of $\mathbb{F}_2$ and 3 is prime to 4, so by the lecture the polynomial remains irreducible in $\mathbb{F}_8$.

(e) Yes. $\mathbb{F}_{16}$ is its splitting field according to the lecture.

(f) No. Notice that X^4+X^3 will always be even if $X \in \mathbb{Z}$, $P(X)$ are all odd number in $\mathbb{F}_{32}$ on integer points between 0 and 31, so $P(X)$ has no roots in $\mathbb{F}_{32}$.

(g) No. Notice that X^4+X^3 will always be even if $X \in \mathbb{Z}$, $P(X)$ are all odd number in $\mathbb{F}_{64}$ on integer points between 0 and 63, so $P(X)$ has no roots in $\mathbb{F}_{64}$.

(h) No. Since $\mathbb{F}_{64}$ contains $\mathbb{F}_4$ and our polynomial is the product of two quadratic factors over $\mathbb{F}_4$, it is not irreducible in $\mathbb{F}_{64}$.

2.

(a) Now denote $f(X)$ that polynomial. Consider $f(X+1)=\frac{(X+1)^p-1}{X}=X^{p-1}+pX^{p-2}+\cdots+p$ Then use Eisenstein's criterion by p to reach our conclusion.

(b) By (a), the polynomial $X^6+X^5+c\cdots+X+1$ is irreducible and has ξ as a root, so it must be the minimal polynomial of ξ. Therefore $[L:\mathbb{Q}]=6$.

(c) We could check the polynomial $(X-\xi)(X-\frac{1}{\xi})=x^2-2\cos\frac{2\pi}{7}x+1$ is irreducible in M and has ξ as a root, so it must be the minimal polynomial of ξ over M. Therefore $[L:M]=2$ and hence $[M,\mathbb{Q}]=6$.

(d) An automorphism f of L must satisfy $f(\xi)=\xi^k(k=1,2,3,4,5,6)$. Moreover, $f(\cos\frac{2\pi}{7})=\frac{\xi^{2k}+1}{\xi^k}=\cos\frac{2k\pi}{7}$, where $k=1,2,3$.

3.

(a) Notice that $\mathbb{Q}(2^{1/3})\cong\mathbb{Q}[t]/(t^3-2)$, we have

$$\mathbb{Q}(\sqrt{2})\otimes_{\mathbb{Q}}\mathbb{Q}(2^{1/3})\cong\mathbb{Q}(\sqrt{2})[t]/(t^3-2)$$

Since t^3-2 is irreducible over $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}(\sqrt{2})[t]/(t^3-2)$ is a field extension of $\mathbb{Q}[\sqrt{2}]$.

(b) Notice that $\mathbb{Q}(2^{1/4}) \cong \mathbb{Q}[t]/(t^4 - 2)$, we have

$$\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(2^{1/4}) \cong \mathbb{Q}(\sqrt{2})[t]/(t^4 - 2)$$

Since $t^4 - 2 = (t^2 - \sqrt{2})(t^2 + \sqrt{2})$ over $\mathbb{Q}[\sqrt{2}]$ and $t^2 - \sqrt{2}, t^2 + \sqrt{2}$ are irreducible over $\mathbb{Q}[\sqrt{2}]$, we have

$$\mathbb{Q}[t]/(t^4 - 2) \cong \mathbb{Q}[t]/(t^2 + \sqrt{2}) \times \mathbb{Q}[t]/(t^2 - \sqrt{2})$$

(c) Notice that $\mathbb{F}_2(\sqrt{T}) \cong \mathbb{F}_2(T)[X]/(X^2 - T)$, we have

$$\mathbb{F}_2(\sqrt{T}) \otimes_{\mathbb{F}_2(T)} \mathbb{F}_2(\sqrt{T}) \cong \mathbb{F}_2(T)[X]/(X^2 - T)$$

Since $X^2 - T = (X - \sqrt{T})^2$ over $\mathbb{F}_2(\sqrt{T})$, our algebra contains nilpotent and in this case it can never be a field or product of fields.
(d) Similarly, we have

$$\mathbb{F}_4(T^{1/3}) \otimes_{\mathbb{F}_4(T)} \mathbb{F}_4(T^{1/3}) \cong \mathbb{F}_4(T^{1/3})[X]/(X^3 - T)$$

The polynomial $X^3 - T$ is split over $\mathbb{F}_4(T^{1/3})$, with roots $T^{1/3}, T^{1/3} \cdot \xi, T^{1/3} \cdot \xi^{-1}$, where ξ is a root of $X^2 + X + 1$ in $\mathbb{F}_4$. Then we have(by chinese remainder theorem):

$$\mathbb{F}_4(T^{1/3}) \otimes_{\mathbb{F}_4(T)} \mathbb{F}_4(T^{1/3}) \cong \mathbb{F}_4(T^{1/3}) \times \mathbb{F}_4(T^{1/3}) \times \mathbb{F}_4(T^{1/3})$$