Ungraded assignment 1 for Introduction to Galois theory

March 12, 2016

1. Let $F(x)/G(x) \in K(X)$ be a rational function over a field K. Show that the extension $K(X)/K(F/G)$ is algebraic and compute its degree.

2. Let L/K be an algebraic extension and $\varphi : L \to L$ a K-algebra homomorphism. Show that φ is always an isomorphism. Give a counterexample when L/K is not algebraic.

3. Let m, n be square-free integers with $m \neq n$ and $m, n \neq 1$. Show that $\mathbb{Q}(\sqrt{m}, \sqrt{n}) = \mathbb{Q}(\sqrt{m} + \sqrt{n})$. Find the degree of $\sqrt{m} + \sqrt{n}$ over $\mathbb{Q}$ and compute its minimial polynomial.

4. Show $x^4 + 1$ is reducible modulo every prime p, but irreducible over $\mathbb{Z}$. Hint: $p = 2$ is easy and for $p \neq 2$, consider the group of units in ${\mathbb{F}_p}^2$.

5. Show $x^p - x - 1$ is irreducible over $\mathbb{F}_p$. Hint: Show it is has no root in $\mathbb{F}_p$ and show that if α is a root in some extension, then all other roots of the form $\alpha + a$ for $a \in \mathbb{F}_p$.

6. What is the degree of the splitting field of $x^5 - 7$ over $\mathbb{Q}$?

7. What is the degree of the splitting field of $x^6 + x^3 + 1$ over $\mathbb{F}_p$ for $p \equiv 1 \bmod 9$, $p \equiv 2 \bmod 9$, $p \equiv 7 \bmod 9$.