1. Aggregate Analysis

We have a sequence of n-operations. And, the cost function is defined as -

$$c_i = \begin{cases} i, & \text{for } i = 2^k \ (k = 0, 1, 2, \cdots) \\ 1, & \text{otherwise} \end{cases}$$

Now,

Aggregate cost of n-operations

$= \text{Cost of operations for } i = 2^k + \text{cost of operations for } i \neq 2^k$

$$= \sum_{j=0}^{\lfloor \log_2 n \rfloor} 2^j + \left[n - \{ \lfloor \log n \rfloor + 1 \} \right] * 1$$

$$= \frac{2^{\lfloor \log_2 n \rfloor} \cdot 2 - 1}{2 - 1} + n - \lfloor \log n \rfloor - 1$$

$$= 2n - 1 + n - \lfloor \log_2 n \rfloor - 1$$

$< 3n$

Thus, amortized cost per operation $= \dfrac{\text{Agg. cost}}{\# \text{operations}}$

$< \dfrac{3n}{n} = 3$

$\therefore$ By aggregate analysis, the amortized cost per operation is $O(1)$.

1. Amortized Analysis (Accounting Method)

Here,

the actual cost of i^{th} operation is

$$c_i = \begin{cases} i, & \text{for } i = 2^k \text{ any } k = 0,1,\dots \\ 1 & \text{otherwise.} \end{cases}$$

We define amortized cost as follows:

$$\hat{c}_i = \begin{cases} 2, & \text{for } i = 2^k \text{ any } k = 0,1,2,\dots \\ 3 & \text{otherwise.} \end{cases}$$

Now,

- for an operation, i, with $i = 2^k$, it uses all previously accumulated credits plus one unit of its own amortized cost to pay its true cost. It then assigns remaining one unit as credit.

- for an operation, i, with $i \neq 2^k$, it uses one unit to pay its actual cost and assign remaining two as credit.

This covers the actual cost of the operation. Now, we need to show that there will be enough credit to pay the cost for 2^k operations.

- for 1st operation ($i = 1$), it is clear.
- Now suppose $j > 0$ and after 2^{j-1}th operation, there is one unit of credit.

Betn operations 2^{j-1} and 2^j, there are $2^{j-1} - 1$ operations & all are not-power-of-2. So, total credit

$$= 1 + 2\cdot(2^{j-1} - 1)$$
$$= 2^j - 1, \text{ before } 2^{j\,th} \text{ operation}$$

$\therefore$ This accumulated credit along with the one of its own is enough to cover its true cost. Therefore, the total amortized cost per operation is an upper bound for the total actual cost.