Lab. Q. 4. Ans:

We will prove the statement using the principle of Mathematical induction. So,

the base step:

the base step is for $h=0$, i.e., the number of nodes at Height-0 will be $\leq \frac{n}{2^{h+1}} = \frac{n}{2^{0+1}} = \frac{n}{2}$

Now, let's assume the number of nodes at depth H be x. Then, there will be 2^d nodes at any depth $d<H$, because the tree will be complete due to its property.

if x is even

There are $x/2$ nodes at depth H-1 that are parents of depth H nodes, and $2^{H-1} - x/2$ nodes at depth H-1 that are not parents of depth H-nodes. So,

$$\text{total \# of height-0 nodes} = x + 2^{H-1} - x/2$$
$$= 2^{H-1} + x/2$$
$$= (2^H + x)/2$$
$$= \lceil (2^H + x - 1)/2 \rceil$$
$$= \lceil n/2 \rceil$$

If x is odd

Then by similar arguments as in even case,

$$\text{\# of height-0 nodes} = x + 2^{H-1} - (x+1)/2$$
$$= 2^{H-1} + (x-1)/2$$
$$= (2^H + x - 1)/2$$
$$= n/2 = \lceil n/2 \rceil$$

$\therefore$ The base case is correct.

Inductive Step

We assume that the ~~pro~~ statement holds true for height $h-1$, i.e.,

$$\# \text{ of nodes at } h-1 \leq \frac{n}{2^{h-1+1}} = \frac{n}{2^h}$$

Now, let's prove it is also correct for height h, i.e., let n_h be the number of nodes at height h for the tree T.

Let's remove the leaves from tree T, so that it becomes T' and has nodes $n' = n - n_0$

From base case,

$$n_0 = \lceil n/2 \rceil$$

$$\therefore n' = n - n_0 = n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$$

Here, the nodes at height h in T would be at height $h-1$ if the leaves of the tree are removed. Let n'_{h-1} denote # of nodes at $h-1$ in T', then

$$n_h = n'_{h-1}$$

i.e. $$n_h \leq \left\lceil \frac{n'}{2^h} \right\rceil = \left\lceil \lfloor n/2 \rfloor / 2^h \right\rceil$$

$$\leq \left\lceil (n/2) / 2^h \right\rceil$$

$$= \left\lceil \frac{n}{2^{h+1}} \right\rceil$$

$\therefore$ The no. of nodes at height h will have at most $\frac{n}{2^{h+1}}$ nodes.

Proved