An alternative way to solve the inhomogeneous equations

Example $y' = -2y + 3$

Take a particular solution y_h of the associated homogeneous equation $y_h' = -2y_h$

which is $y_h = e^{\int -2\,dt} = e^{-2t}$

Let $y = v y_h = v e^{-2t}$

then the equation becomes

$$(ve^{-2t})' = -2ve^{-2t} + 3$$

$$v'e^{-2t} \cancel{- 2ve^{-2t}} = \cancel{-2ve^{-2t}} + 3$$

$$v'e^{-2t} = 3$$

$$v' = 3e^{2t}$$

Integrate both sides

$$v = \int 3e^{2t} = \frac{3}{2}e^{2t} + C$$

So $y = v y_h = \left(\frac{3}{2}e^{2t} + C\right)e^{-2t} = \frac{3}{2} + Ce^{-2t}$

This method is called <u>variation of parameters</u>

<u>Summary of the method</u>

Given an equation $y' = ay + f$

1. Find a particular solution of the associated homogeneous eq $y_h' = ay_h$ which is $y_h(t) = e^{\int a(t)dt}$
2. Substitute $y = vy_h$ into $y' = ay + f$ to find v
   it turns out that you always get the equation $v' = \frac{f}{y_h}$
3. Write down the general solution $y(t) = v(t)\, y_h(t)$

Choose the method you prefer, they are both OK

Example Use both methods to solve $x' = x\tan(t) + \sin(t)$

First method $x' - x\tan(t) = \sin(t)$

integrating factor

$$u(t) = e^{-\int \tan(t)\,dt} = e^{-(-\ln(\cos(t)))} = e^{\ln(\cos(t))} = \cos(t)$$

$$\cos(t)\left(x' - x\tan(t)\right) = \cos(t)\sin(t)$$

$$\left(\cos(t)x\right)' = \cos(t)\sin(t)$$

$$\cos(t)x = \int \cos(t)\sin(t)\,dt = -\frac{\cos^2(t)}{2} + c$$

then $x(t) = -\dfrac{\cos(t)}{2} + \dfrac{c}{\cos(t)}$

Variation of parameters method

$$x_h = e^{\int \tan(t)} = e^{-\ln(\cos(t))} = \frac{1}{e^{\ln(\cos(t))}} = \frac{1}{\cos(t)}$$

Let $x = v x_h$ and replace into $x' = x\tan(t) + \sin(t)$

$$\left(\frac{v}{\cos(t)}\right)' = \frac{v}{\cos(t)}\tan(t) + \sin(t)$$

$$\frac{v'\cos(t) + v\sin(t)}{\cos^2(t)} = \frac{v\sin(t)}{\cos^2(t)} + \sin(t)$$

$$\frac{v'\cos(t)}{\cos^2(t)} = \sin(t)$$

$$v' = \cos(t)\sin(t)$$

$$v(t) = \int \sin(t)\cos(t) = -\frac{\cos^2(t)}{2} + c$$

$$x(t) = v(t)\,x_h(t) = \left(-\frac{\cos^2(t)}{2} + c\right) \cdot \frac{1}{\cos(t)} = -\frac{\cos(t)}{2} + \frac{c}{\cos(t)}$$

Same solutions as before

To finish, let's notice a property of linear equations

<u>Theorem</u> Let $y' = a(t)y + f(t)$ a linear equation
with a particular ~~[illegible]~~ solution $y_p(t)$
and whose associated homogeneous equation
$y' = a(t)y$ has solution $y_h(t)$
then all solutions of the equation $y' = a(t)y + f(t)$
are of the form $y(t) = y_p(t) + A\,y_h(t)$ with A constant

APPLICATIONS OF SEPARABLE & LINEAR EQUATIONS

<u>MODELING POPULATION GROWTH</u> (§ 3.1)

~~[illegible]~~

Malthusian model

We assume that the growth of the population is proportional to the population. Thus we have the equation $P' = rP$

The Malthusian model is quite primitive and assumes infinite availability of resources and so we assume r to be constant.

r is called the <u>reproductive rate</u>

$P' = rP$ is a separable equation

Let P_0 be the population at time $t=0$, i.e. $P(0)=P_0$

The general solution of $P' = rP$ is $P(t) = Ce^{rt}$ for C constant

Since we want $P(0) = P_0$ we get

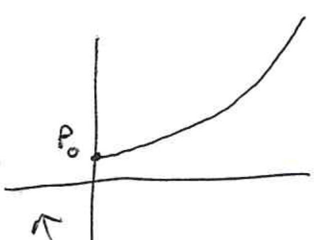
$$P_0 = P(0) = Ce^{r0} = C$$

So the solution is $P(t) = P_0 e^{rt}$

The population behaves differently depending on the sign of the reproductive rate r

If r is positive, $r>0$

then $P(t) = P_0 e^{rt}$ increases (P_0 cannot be negative)

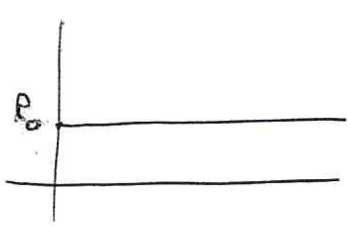


$$\lim_{t\to\infty} P(t) = \infty$$

we are not interested in negative times because we are only interested in predicting the future population

If $r=0$

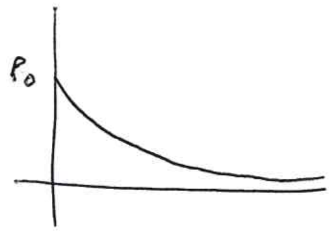
then $P(t) = P_0 e^{0t} = P_0$ is constant



$$\lim_{t\to\infty} P(t) = P_0$$

If r is negative, $r<0$

then $P(t) = P_0 e^{rt}$ decreases



$$\lim_{t\to\infty} P(t) = 0$$

Logistic equation

Let now denote by r the natural reproductive rate (we assume $r>0$ now) which is the reproductive rate assuming infinite resources. In this model we are not assuming infinite resources anymore, let K be the maximum population that is sustainable with the resources available, K is called the carrying capacity (it is never negative)

Then the constant of proportionality connecting P' and P now depends on P. If P is smaller than K then it is positive and the population increases, if P is greater than K then it is negative and the population decreases.

take $r\left(1-\frac{P}{K}\right)$ to be the constant of proportionality

then the equation modeling the population is

$$P' = r\left(1-\frac{P}{K}\right)P$$ this is called the logistic equation

this is still a separable equation, if we denote by P_0 the population at time $t=0$ then its solution is

$$P(t) = \frac{KP_0}{P_0 + (K-P_0)e^{-rt}}$$