MATH 392 Notes

An ordinary differential equation (ODE) is an equation involving functions and derivatives

Examples

- $y' = t^2$
  What is a solution for this equation? A function $y(t)$ such that $y'(t) = t^2$. That is, a function whose derivative is t^2. For example $y(t) = \frac{t^3}{3}$
  However, it is not the unique solution. Indeed, $y(t) = \frac{t^3}{3} + C$
  is a solution for every real number C.
  So there are infinitely many solutions
  This is the same as computing the indefinite integral
  $\int t^2 dt$

- $y' = \sin(t)$
  The solutions are found by computing $\int \sin(t)\, dt$
  So all the solutions are $y(t) = -\cos(t) + C$

- $y' = y$
  A solution of this equation is a function $y(t)$ which coincides with its derivative
  One of them is $y(t) = e^t$. Are there other such functions?
  All the solutions are of the form $y(t) = Ce^t$ where C is a real number. There are infinitely many solutions.

- $y' = 2y$

  A solution of this equation is a function $y(t)$ whose derivative is equal to twice the function itself. For example $y(t) = e^{2t}$. All the solutions are of the form $y(t) = Ce^{2t}$ for C a real number. Infinite solutions.

- $y' = y^2$

  In this case it's hard to guess a solution.
  It turns out that $y(t) = -\frac{1}{t}$ is a solution. Let's check this

  LHS $y'(t) = \frac{1}{t^2}$ both sides coincide ✓

  RHS $(y(t))^2 = \left(-\frac{1}{t}\right)^2 = \frac{1}{t^2}$

  But also $y(t) = \frac{1}{C-t}$ is a solution for C real number

  Let's check

  LHS $y'(t) = -\frac{1}{(C-t)^2} \cdot (-1) = \frac{1}{(C-t)^2}$ coincide ✓

  RHS $(y(t))^2 = \left(\frac{1}{C-t}\right)^2 = \frac{1}{(C-t)^2}$

  We'll see how to solve this kind of equations later

- There are much more crazy equations like

  $y' \cos(t) = (y^2 + e^t)\, t^2$

  for which finding a solution might be really hard

When the highest derivative is the first one, they are called <u>first order</u> ODE

Later we will also study higher order ODE. In particular we will solve some <u>second order</u> ODE which involve also the second derivative

Example

- $y'' = -y$

We can guess some solutions like

$y(t) = 0, \quad y(t) = \sin(t), \quad y(t) = \cos(t)$

But also $y(t) = \sin(t) + \cos(t)$ is a solution,

also $y(t) = 5\cos(t)$ is a solution

More generally sum of solutions is a solution and a multiple by a constant of a solution is a solution

It turns out that all its solutions are of the form

$C_1 \cos(t) + C_2 \sin(t)$ for C_1, C_2 real numbers.

Indeed, if $y(t) = C_1 \cos(t) + C_2 \sin(t)$

$y'(t) = -C_1 \sin(t) + C_2 \cos(t)$

$y''(t) = -C_1 \cos(t) - C_2 \sin(t) = -\left(C_1 \cos(t) + C_2 \sin(t)\right) = -y(t)$ ✓

- But there are also third order ODE and so on. For example

$y''' + t^2 y'' + e^{(y')} = \sin(t)$

they are in general very hard to solve

What about partial derivatives?

(Ask how many of them know what is a partial derivative)

We obtain what are called <u>Partial Differential Equations</u> (PDE). For example

- <u>Laplace equation</u>: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

  whose solution needs to be a function $f(x,y,z)$ of threee variables x, y, z

- <u>Heat equation</u> to model how the distribution of heat evolves over time

- Equations modelling how the vibrations of a membrane evolves over time

PDE are very useful but we are not interested in them in this class.

(End of Overview)

★ Some definitions:

- An <u>ordinary differential equation</u> (ODE) is an equation involving an unknow function y of a <u>single</u> variable t (called independent variable) together with its derivatives

- A <u>solution</u> of an ODE is a <u>differentiable</u> function $y(t)$ that, once plugged into the equation makes the LHS equal to the RHS of the equation for all the t in the interval where $y(t)$ is defined

For example

$y(t) = t+1$ is a solution of the equation $y' = y - t$

Indeed, the left-hand side is $y'(t) = 1$ — same ✓

the right hand side is $y(t) - t = t+1-t = 1$

Recall that there might be many solutions of an equation
for example, for every real number C

$y(t) = Ce^{2t}$ is a solution of $y' = 2y$

Indeed, LHS $y'(t) = 2Ce^{2t}$ ✓

RHS $2y(t) = 2Ce^{2t}$

A first order ODE is said to be in <u>normal form</u> if the left hand side is y' and the RHS doesn't contain y'. For example $y' = 2y + t$

We saw that there are many solutions in general. If we want to get a single solution we have to add additional constraints.
For example we may require that the solution has a particular value at some point.
These kinds of problems are called <u>Initial Value Problems</u>

Example

$$\begin{cases} y' = 2y \\ y(0) = 3 \end{cases}$$

We saw before that the solutions of $y' = 2y$ are $y(t) = Ce^{2t}$

We need to require $y(0) = 3$, i.e. $Ce^{2\cdot 0} = 3$

$$Ce^{0} = 3$$

$$C = 3$$

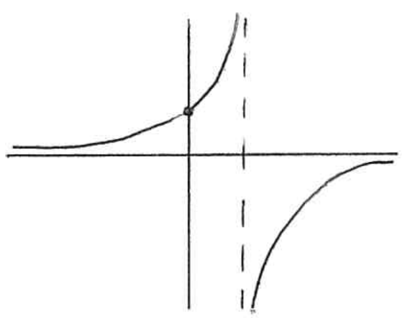
So $y(t) = 3e^{2t}$ is the unique solution of the IVP

The <u>interval of existence</u> of a solution is the largest interval over which the solution can be defined

Example: Consider the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$ it has solution $y(t) = \frac{1}{1-t}$ (check as HW)

what is the largest interval on which it is still a solution?



We need 0 to be in the interval.
Since $y(t)$ is not defined at $t=1$, the interval of existence is $(-\infty, 1)$

Using variables other than y and t:

Examples:

$y' = x + y$ whose solutions are the functions
$y(x) = -1 - x + Ce^x$ (Check)
here x is the independent variable
(it's just notation)

$s' = \sqrt{r}$ whose solutions are $s(r) = \frac{2}{3} r^{3/2} + C$ (check)
here r is the independent variable and the function is denoted by s

<u>Leibniz notation</u>:

For the equations we're about to solve, it is useful to write $\frac{dy}{dt}$ instead of y'