Example

$y' = 2yt$ in Leibniz notation becomes $\frac{dy}{dt} = 2yt$

$y' = 2t\frac{y}{1+y} + e^t y'$ becomes $\frac{dy}{dt} = 2t\frac{y}{1+y} + e^t\frac{dy}{dt}$

SEPARABLE EQUATIONS §2.2

A first order ODE is called a <u>separable ODE</u> if it can be rewritten in such a way that the LHS is y' and the RHS is a product of a term containing only y and a term containing only t

<u>Example</u>

$$y' = \underbrace{t}_{\text{contains only } t}\ \underbrace{y^2}_{\text{contains only } y} \qquad y' = \underbrace{(e^t+2)}_{\text{only } t}\underbrace{(y^3+2y)}_{\text{only } y}$$

How to solve. We show this on $y' = ty^2$

1. Check for zeroes of the part containing only y
   Need to solve $y^2 = 0$. Only solution is $y = 0$
   This gives a constant solution $y(t) = 0$
   Let's check: $y'(t) = 0$, $t(y(t))^2 = t \cdot 0^2 = 0$ ✓
2. Rewrite in Leibniz notation, multiply by dt and divide by the part of RHS containing only y
   $\frac{dy}{dt} = ty^2 \rightsquigarrow dy = ty^2\,dt \rightsquigarrow \frac{dy}{y^2} = t\,dt$
   Now y and t are <u>separated</u>

3. Integrate both sides

$$\int \frac{dy}{y^2} = \int t\,dt$$

this is why Leibniz notation is useful

$$-\frac{1}{y} = \frac{t^2}{2} + c$$

4. Solve for y

$$\frac{1}{y} = -\left(\frac{t^2}{2} + c\right)$$

$$y = \frac{-1}{\frac{t^2}{2} + c}$$

Let's check that $y(t) = \dfrac{-1}{\frac{t^2}{2} + c}$ is a solution

LHS $\quad y'(t) = \dfrac{1}{\left(\frac{t^2}{2}+c\right)^2} \cdot t = \dfrac{t}{\left(\frac{t^2}{2}+c\right)^2}$ ✓

RHS $\quad t\,(y(t))^2 = t \cdot \left(\dfrac{-1}{\frac{t^2}{2}+c}\right)^2 = \dfrac{t}{\left(\frac{t^2}{2}+c\right)^2}$

- Let's now solve the initial value problem

$$\begin{cases} y' = t y^2 \\ y(1) = -2 \end{cases}$$

We found that the solutions of $y' = ty^2$ are $y(t) = 0$ and $y(t) = -\dfrac{1}{\frac{t^2}{2}+c}$

$y(t)=0$ is not a solution of the IVP because $0 \neq -2$

Need to find C such that

$$y(1) = -\frac{1}{\frac{1^2}{2}+C} = -2$$

$$\frac{1}{\frac{1}{2}+C} = 2$$

$$\frac{1}{2}+C = \frac{1}{2}$$

$$C = 0$$

So $y(t) = -\frac{1}{\frac{t^2}{2}+0} = -\frac{1}{\frac{t^2}{2}} = -\frac{2}{t^2}$ is the solution

Check

$y'(t) = -2\,\frac{1}{t^3}(-2) = \frac{4}{t^3}$ ✓

$t\,(y(t))^2 = t\left(-\frac{2}{t^2}\right)^2 = t\left(\frac{4}{t^4}\right) = \frac{4}{t^3}$

what is the interval of existence of the solution $y(t) = -\frac{2}{t^2}$ for the IVP?

- What is the solution of the IVP

$$\begin{cases} y' = t y^2 \\ y(1) = 0 \end{cases}$$

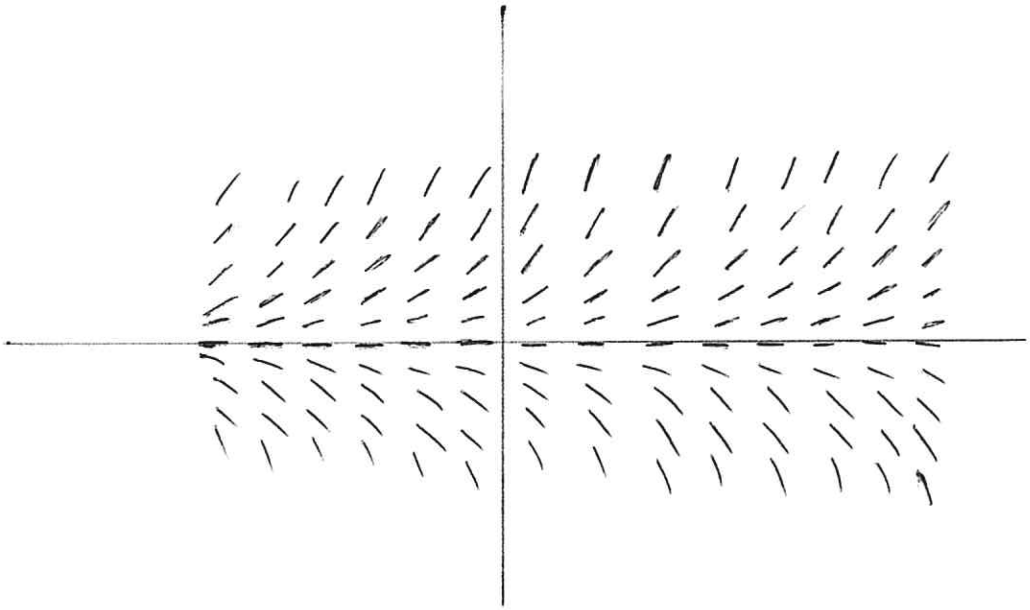
$y(1) = -\frac{1}{\frac{1}{2}+C} = 0$ gives no solution So no solution of the form $y(t) = -\frac{1}{\frac{t^2}{2}+C}$

but $y(t)=0$ is a solution!

Geometric meaning of a differential equation and its solutions

A direction field is the result of attaching to every point of the plane a small slanted line segment

Example If the slope of the segent at the point (t,y) is given by y you get

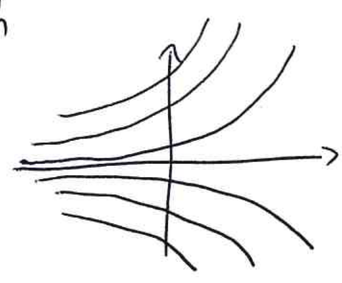


Given a first order eq in normal form $y' = f(t,y)$ we can associate the direction field ~~whose slope~~ such that the slope of the segment attached to ~~(t,y)~~ the point (t,y) has slope $f(t,y)$

The example above is the direction field associated to the equation $y' = y$

Then it turns out that $y(t)$ is a solution to $y' = f(t,y)$ if and only if the graph of $y(t)$ is tangent to the segent attached to every point it passes through

In the example above:



For pictures of direction fields see section 2.1 of the book

See: DESMOS.COM/CALCULATOR/KGSYCYWBUG

for an interactive tool

It's default setting is the direction field for $y' = ty^2$ that we solved before

Back to separable equations:

Solve the IVP: $\quad y' = \dfrac{e^x}{1+y}, \quad y(0) = -4$

now x denotes the independent variable

Check for zeroes in the y part of the RHS i.e. $\frac{1}{1+y}$. There are none

Leibniz notation:

$$\frac{dy}{dx} = \frac{e^x}{1+y}$$

Separate variables: $\quad (1+y)\,dy = e^x\,dx$

Integrate: $\quad \int (1+y)\,dy = \int e^x\,dx$

$$y + \frac{y^2}{2} = e^x + c$$

Solve for y: $\quad \dfrac{y^2}{2} + y - (e^x + c) = 0$

$$y^2 + 2y - 2(e^x + c) = 0$$

$$y = \frac{-2 \pm \sqrt{4 + 8(e^x + c)}}{2}$$

$$y = -1 \pm \sqrt{1 + 2(e^x + c)}$$

So the solutions are

$$y(x) = -1 + \sqrt{1 + 2(e^x + c)} \quad \text{and} \quad y(x) = -1 - \sqrt{1 + 2(e^x + c)}$$

~~[illegible]~~

~~[illegible]~~

Find the solution such that $y(0)=-4$

By plugging 0 in place of x we get

$y(0)=-1+\sqrt{3+2c}$ and $y(0)=-1-\sqrt{3+2c}$

Since we want $y(0)=-4$, we look at the second formula (because $\sqrt{3+2c}$ is always positive)

$$y(0)=-1-\sqrt{3+2c}=-4$$

$$3=\sqrt{3+2c}$$

$$9=3+2c$$

$$6=2c$$

$$c=3$$

So the solution is $y(x)=-1-\sqrt{1+2(e^x+3)}=-1-\sqrt{7+2e^x}$

What if we want to solve

$$y'=\frac{e^x}{1+y}, \quad y(0)=1 \quad ?$$

Then we look at the first formula

$$y(0)=-1+\sqrt{3+2c}=1$$

$$\sqrt{3+2c}=2$$

$$3+2c=4$$

$$2c=1$$

$$c=\frac{1}{2}$$

So the solution is $y(x)=-1+\sqrt{1+2(e^x+\frac{1}{2})}=-1+\sqrt{2+2e^x}$

<u>Solve the IVPs:</u>

$$x'=\frac{2tx}{1+x} \quad \text{with } x(0)=1 \quad \text{or} \quad x(0)=-2 \quad \text{or} \quad x(0)=0$$

This is a separable equation because $x' = 2t \frac{x}{1+x}$

Check zeroes of the x part of the RHS: $x=0$ is a zero

therefore $x(t)=0$ is a constant solution!

Leibniz and separate: $\frac{dx}{dt} = 2t \frac{x}{1+x}$

$$\frac{1+x}{x}\, dx = 2t\, dt$$

Integrate $\int \frac{1+x}{x}\, dx = \int 2t\, dt$

$$\int \left(\frac{1}{x}+1\right) dx = \int 2t\, dt$$

$$\ln|x| + x = t^2 + c$$

Solve for x: there is a problem, we have to keep the solution in this implicit form

Solution for $x(0)=1$

$$1 + \ln|1| = 0^2 + C$$

$$1 = C$$

So $x(t)$ defined by $\ln|x| + x = t^2 + 1$

Solution for $x(0)=0$

we found $x(t)=0$ before!

Solution for $x(0)=-2$

$$\ln|-2| - 2 = 0^2 + c$$

$$\ln 2 - 2 = c$$

So $x(t)$ defined by $\ln|x| + x = t^2 + \ln 2 - 2$

For pictures of solutions see Figure 5 page 34 of the book

LINEAR EQUATIONS §2.4

A first order <u>linear</u> equation is one of the form

$$x' = a(t)x + f(t) \quad \text{where } a(t), f(t) \text{ are functions of } t$$

Example

$$x' = 2x + 3$$
$$x' = \sin(t)\,x + \cos(t)$$
$$y' = e^{2t}y + \cos(t)$$
$$y' = (3t+2)\,y + t^2 - 1$$

are all linear

Homogeneous linear equations

When $f(t) = 0$ a linear equation is called <u>homogeneous</u>
So they are equations of the form

$$x' = a(t)x$$

But this is a separable equation that we already know how to solve!

$$x' = a(t)x$$

$$\frac{dx}{dt} = a(t)x$$

$$\frac{dx}{x} = a(t)\,dt$$

$$\int \frac{dx}{x} = \int a(t)\,dt + C$$

$$\ln|x| = \int a(t)\,dt + C$$

$$|x| = e^{\int a(t)\,dt + C}$$

$$|x| = e^c e^{\int a(t)dt}$$

the constant e^c is always positive so we can replace it with the constant A that we allow to be positive, zero or negative so we can get rid of the absolute value and obtain the zero solution

$$x = A e^{\int a(t)dt}$$

<u>Example</u> $x' = \sin(t) x$

in this case $a(t) = \sin(t)$ and $\int a(t)dt = \int \sin(t) dt = -\cos(t)$

In this case we can forget the constant C when computing $\int a(t)dt$ because A already takes care of that

Then $x(t) = A e^{\int a(t)dt} = A e^{-\cos(t)}$

<u>Inhomogeneous linear equations</u>

This is the general case $x' = a(t)x + f(t)$

Before stating the methods to solve them, let's see how to solve an example

<u>Example:</u> $x' = x + e^{-t}$

the <u>integrating factor</u> of the equation $x' = a(t)x + f(t)$

is $u(t) = e^{-\int a(t)dt}$ (this is like the solution of the homogeneous equation with a $-$ in the exponent)

in this case it is $u(t) = e^{-\int a(t)dt} = e^{-\int 1 dt} = e^{-t}$

(we still don't care about constants in this step)

Bring the term involving x to the LHS

$$x' - x = e^{-t}$$

Multiply both sides by the integrating factor $u(t) = e^{-t}$

$$e^{-t}(x' - x) = e^{-2t}$$

the LHS turns out to be the derivative of $e^{-t}x$
then the equation becomes

$$\left[e^{-t}x\right]' = e^{-2t}$$

Integrate both sides w.r.t. t to get

$$e^{-t}x = \int e^{-2t}\,dt$$

$$e^{-t}x = -\frac{1}{2}e^{-2t} + C$$

Solve for x by multiplying both sides by e^{t}

$$x = -\frac{1}{2}e^{-t} + Ce^{t}$$

This is the solution

Summary of the method

Given the equation $x' = ax + f$

1. Rewrite it as $x' - ax = f$
2. Multiply by the integrating factor $u(t) = e^{-\int a(t)dt}$
   so the equation becomes $u(x' - ax) = uf$
   which is the same as $(ux)' = uf$
3. Integrate $u(t)x(t) = \int u(t)f(t)\,dt + C$
4. Solve for $x(t)$

Example $x' = x\sin(t) + 2te^{-\cos(t)}$

Let's rewrite it as

$$x' = \sin(t)x + 2te^{-\cos(t)}$$

Bring the term involving x to the LHS

$$x' - \sin(t)x = 2te^{-\cos(t)}$$

Find the integrating factor

$$u(t) = e^{-\int \sin(t)\,dt} = e^{\cos(t)}$$

Multiply both sides by u

$$e^{\cos(t)}(x' - \sin(t)x) = 2t$$

$$(e^{\cos(t)}x)' = 2t$$

Integrate

$$e^{\cos(t)}x = \int 2t\,dt = t^2 + C$$

Multiply by $e^{-\cos(t)}$ to solve for x

$$x(t) = e^{-\cos(t)}t^2 + e^{-\cos(t)}C = (t^2+C)e^{-\cos(t)}$$

Find the solution that satisfies $x(0)=1$

$$x(0) = (0^2+C)e^{-\cos(0)}$$

$$1 = Ce^{-1}$$

$$C = e$$

the solution is $x(t) = (t^2+e)e^{-\cos(t)}$