Solve $Z_1 = Z_2$ for x_1

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1 Given:

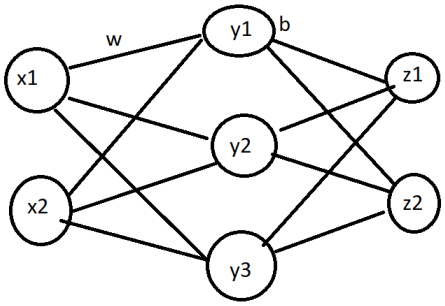

Figure 1: Given Neural Net

Figure 1 shows the given Neural Net that will be analysed.

x_1, x_2 are the input neurons
y_1, y_2, y_3 are the hidden layers neurons
z_1, z_2 are the output neurons

w denotes a weight
$w_{y_1->z_2}$ denotes the weight from y_1 to z_2

b deontes a bias
b_{y_3} deontes the bias associated with neuron y_3

The input to a y_i neuron will be denoted as:
$y_i = \sigma(w_{x_1->y_i} * x_1 + w_{x_2->y_i} * x_2 + b_{y_i})$
Where $\sigma(x) = \frac{1}{1+e^{-x}} = \frac{1}{1+\exp[-x]}$

The input to a z_i neuron will be denoted as:
$z_i = \sigma(w_{y_1->z_i} * y_1 + w_{y_2->z_i} * y_2 + w_{y_3->z_i} * y_3 + b_{z_i})$

2 Solve $z_1 = z_2$ for x_1

$$z_1 = z_2 \tag{1}$$

$$z_1 = \sigma(w_{y_1->z_1} * y_1 + w_{y_2->z_1} * y_2 + w_{y_3->z_1} * y_3 + b_{z_1}) \tag{2}$$

$$\begin{aligned} z_1 = \sigma(w_{y_1->z_1} * \sigma(w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}) \\ + w_{y_2->z_1} * \sigma(w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}) \\ + w_{y_3->z_1} * \sigma(w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}) \\ + b_{z_1}) \end{aligned} \tag{3}$$

$$\begin{aligned} z_2 = \sigma(w_{y_1->z_2} * \sigma(w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}) \\ + w_{y_2->z_2} * \sigma(w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}) \\ + w_{y_3->z_2} * \sigma(w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}) \\ + b_{z_2}) \end{aligned} \tag{4}$$

$$\begin{aligned} (1 + \exp[-(w_{y_1->z_1} * \frac{1}{1 + \exp[w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}]} \\ + w_{y_2->z_1} * \frac{1}{1 + \exp[w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}]} \\ + w_{y_3->z_1} * \frac{1}{1 + \exp[w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}]} \\ + b_{z_1})])^{-1} \\ = \\ (1 + \exp[-(w_{y_1->z_2} * \frac{1}{1 + \exp[w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}]} \\ + w_{y_2->z_2} * \frac{1}{1 + \exp[w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}]} \\ + w_{y_3->z_2} * \frac{1}{1 + \exp[w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}]} \\ + b_{z_2})])^{-1} \end{aligned} \tag{5}$$

NOTE: this equation is too big. Lets scope it down.

$$B_1 = -w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1} \tag{6}$$
$$B_2 = -w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2} \tag{7}$$
$$B_3 = -w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3} \tag{8}$$
$$\tag{9}$$

NOTE: Substatuting B in.

$$(1 + \exp[-(w_{y_1->z_1} * \frac{1}{1 + \exp[B_1]} \tag{10}$$
$$+w_{y_2->z_1} * \frac{1}{1 + \exp[B_2]} \tag{11}$$
$$+w_{y_3->z_1} * \frac{1}{1 + \exp[B_3]} \tag{12}$$
$$+b_{z_1})])^{-1} \tag{13}$$
$$= \tag{14}$$
$$(1 + \exp[-(w_{y_1->z_2} * \frac{1}{1 + \exp[B_1]} \tag{15}$$
$$+w_{y_2->z_2} * \frac{1}{1 + \exp[B_2]} \tag{16}$$
$$+w_{y_3->z_2} * \frac{1}{1 + \exp[B_3]} \tag{17}$$
$$+b_{z_2})])^{-1} \tag{18}$$

Define A

$$A_1 = w_{y_1->z_1} * \frac{1}{1 + \exp[B_1]} \tag{19}$$
$$+w_{y_2->z_1} * \frac{1}{1 + \exp[B_2]} \tag{20}$$
$$+w_{y_3->z_1} * \frac{1}{1 + \exp[B_3]} \tag{21}$$
$$+b_{z_1} \tag{22}$$

$$A_2 = w_{y_1->z_2} * \frac{1}{1 + \exp[B_1]} \tag{23}$$
$$+w_{y_2->z_2} * \frac{1}{1 + \exp[B_2]} \tag{24}$$
$$+w_{y_3->z_2} * \frac{1}{1 + \exp[B_3]} \tag{25}$$
$$+b_{z_2} \tag{26}$$

NOTE: Substatuting A in.

$$(1 + \exp[-(A_1)])^{-1} = (1 + \exp[-(A_2)])^{-1} \tag{27}$$

$$(1 + \exp[-(A_1)]) = (1 + \exp[-(A_2)]) \tag{28}$$

$$1 + \exp[-(A_1)] = 1 + \exp[-(A_2)] \tag{29}$$

$$\exp[-(A_1)] = \exp[-(A_2)] \tag{30}$$

$$(A_1) = (A_2) \tag{31}$$

$$A_1 = A_2 \tag{32}$$

Sub in the values of A_1 and A_2

$$\begin{gathered} w_{y_1->z_1} * \frac{1}{1 + \exp[B_1]} + w_{y_2->z_1} * \frac{1}{1 + \exp[B_2]} + w_{y_3->z_1} * \frac{1}{1 + \exp[B_3]} \\ + b_{z_1} \\ = \\ w_{y_1->z_2} * \frac{1}{1 + \exp[B_1]} + w_{y_2->z_2} * \frac{1}{1 + \exp[B_2]} + w_{y_3->z_2} * \frac{1}{1 + \exp[B_3]} \\ + b_{z_2} \end{gathered} \tag{33}$$

$$\begin{gathered} \frac{w_{y_1->z_1}}{1 + \exp[B_1]} + \frac{w_{y_2->z_1}}{1 + \exp[B_2]} + \frac{w_{y_3->z_1}}{1 + \exp[B_3]} + b_{z_1} \\ = \\ \frac{w_{y_1->z_2}}{1 + \exp[B_1]} + \frac{w_{y_2->z_2}}{1 + \exp[B_2]} + \frac{w_{y_3->z_2}}{1 + \exp[B_3]} + b_{z_2} \end{gathered} \tag{34}$$

Multiply by botoms to create common denominators.

$$\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_2]) * (1+\exp[B_1]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_3]) * (1+\exp[B_1]) * (1+\exp[B_2])} +$$
$$b_{z_1}$$
$$=$$
$$\frac{w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_2]) * (1+\exp[B_1]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_3]) * (1+\exp[B_1]) * (1+\exp[B_2])} +$$
$$b_{z_2} \tag{35}$$

REORDRER DENOMINATORS

$$\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$b_{z_1}$$
$$=$$
$$\frac{w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$b_{z_2} \tag{36}$$

MOVE b_{z_1}

$$\begin{gathered}
\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} \\
= \\
\frac{w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
b_{z_2} - b_{z_1}
\end{gathered} \tag{37}$$

MOVE THE BIG PIECE

$$\begin{gathered}
\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} + \\
\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \\
\frac{w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \\
\frac{w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \\
\frac{w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} \\
= b_{z_2} - b_{z_1}
\end{gathered} \tag{38}$$

REORDER FOR SIMMILAR TERMS

$$\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \frac{w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \frac{w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} - \frac{w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])}$$
$$= b_{z_2} - b_{z_1} \tag{39}$$

COMBINE LIKE TERMS

$$\frac{w_{y_1->z_1} * (1+\exp[B_2]) * (1+\exp[B_3]) - w_{y_1->z_2} * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_2->z_1} * (1+\exp[B_1]) * (1+\exp[B_3]) - w_{y_2->z_2} * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{w_{y_3->z_1} * (1+\exp[B_1]) * (1+\exp[B_2]) - w_{y_3->z_2} * (1+\exp[B_2]) * (1+\exp[B_1])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])}$$
$$= b_{z_2} - b_{z_1} \tag{40}$$

fACTOR COMMON NUMERATORS

$$\frac{(w_{y_1->z_1} - w_{y_1->z_2}) * (1+\exp[B_2]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{(w_{y_2->z_1} - w_{y_2->z_2}) * (1+\exp[B_1]) * (1+\exp[B_3])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])} +$$
$$\frac{(w_{y_3->z_1} - w_{y_3->z_2}) * (1+\exp[B_1]) * (1+\exp[B_2])}{(1+\exp[B_1]) * (1+\exp[B_2]) * (1+\exp[B_3])}$$
$$= b_{z_2} - b_{z_1} \tag{41}$$

SANITY CHECK

$$aAB - bAB = (a-b)AB \tag{42}$$

MULTIPLY BY COMMON DENOMINATOR

$$
\begin{aligned}
&(w_{y_1->z_1} - w_{y_1->z_2}) * (1 + \exp[B_2]) * (1 + \exp[B_3]) + \\
&(w_{y_2->z_1} - w_{y_2->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_3]) + \\
&(w_{y_3->z_1} - w_{y_3->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_2]) \\
&= (b_{z_2} - b_{z_1}) * ((1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3]))
\end{aligned}
\tag{43}
$$

The next step is to multiply this out and emilimate terms. I do not want to do that rn :/

GO BACK TO EQ.34

$$\begin{gathered}\frac{w_{y_1->z_1}}{1+\exp[B_1]}+\frac{w_{y_2->z_1}}{1+\exp[B_2]}+\frac{w_{y_3->z_1}}{1+\exp[B_3]}+b_{z_1}\\=\\\frac{w_{y_1->z_2}}{1+\exp[B_1]}+\frac{w_{y_2->z_2}}{1+\exp[B_2]}+\frac{w_{y_3->z_2}}{1+\exp[B_3]}+b_{z_2}\end{gathered} \tag{44}$$

SUBRTRACT b

$$\begin{gathered}\frac{w_{y_1->z_1}}{1+\exp[B_1]}+\frac{w_{y_2->z_1}}{1+\exp[B_2]}+\frac{w_{y_3->z_1}}{1+\exp[B_3]}\\=\\\frac{w_{y_1->z_2}}{1+\exp[B_1]}+\frac{w_{y_2->z_2}}{1+\exp[B_2]}+\frac{w_{y_3->z_2}}{1+\exp[B_3]}+b_{z_2}-b_{z_1}\end{gathered} \tag{45}$$

SUBRTRACT REST OF THE STUFF

$$\begin{gathered}\frac{w_{y_1->z_1}}{1+\exp[B_1]}-\frac{w_{y_1->z_2}}{1+\exp[B_1]}+\frac{w_{y_2->z_1}}{1+\exp[B_2]}-\frac{w_{y_2->z_2}}{1+\exp[B_2]}+\frac{w_{y_3->z_1}}{1+\exp[B_3]}-\frac{w_{y_3->z_2}}{1+\exp[B_3]}\\=\\b_{z_2}-b_{z_1}\end{gathered} \tag{46}$$

COMBINE LIKE TERMS

$$\begin{gathered}\frac{w_{y_1->z_1}-w_{y_1->z_2}}{1+\exp[B_1]}+\frac{w_{y_2->z_1}-w_{y_2->z_2}}{1+\exp[B_2]}+\frac{w_{y_3->z_1}-w_{y_3->z_2}}{1+\exp[B_3]}\\=\\b_{z_2}-b_{z_1}\end{gathered} \tag{47}$$

This will got to EQ.43. This is a dead end.
continue wiht eq.43

3 Conclusion

Not much of a paper, but it's a start.