

Solve  $Z_1 = Z_2$  for  $x_1$

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April 22, 2023

## 1 Given:

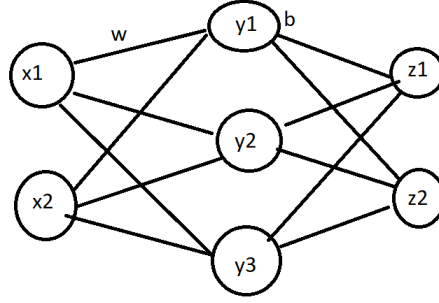


Figure 1: Given Neural Net

Figure 1 shows the given Neural Net that will be analysed.

$x_1, x_2$  are the input neurons

$[x]$  is the columnised vector notation of all "x" neurons of dimensions  $1 \times x$

$y_1, y_2, y_3$  are the hidden layers neurons

$[y]$  is the columnised vector notation of all "y" neurons of dimensions  $1 \times y$

$z_1, z_2$  are the output neurons

$[z]$  is the columnised vector notation of all "z" neurons of dimensions  $1 \times z$

$w$  denotes a weight

$w_{y_1 \rightarrow z_2}$  denotes the weight from  $y_1$  to  $z_2$

$[w_1]$  is the columnised vector notation of all " $w_1$ " weights of dimensions  $y \times x$

$$[w_1] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \dots & w_{1x} \\ \vdots & & & & \\ w_{y1} & w_{y2} & w_{y3} & \dots & w_{yx} \end{bmatrix}$$

$[w_2]$  is the columnised vector notation of all " $w_2$ " weights of dimensions  $z \times y$

$$[w_2] = \begin{bmatrix} w_{11}w_{12}w_{13}\dots w_{1y} \\ \vdots \\ w_{ij} \\ \vdots \\ w_{z1}w_{z2}w_{z3}\dots w_{zy} \end{bmatrix}$$

$b$  deontes a bias

$b_{y_3}$  deontes the bias associated with neuron  $y_3$

$[b_1]$  is the columnised vector notation of all " $b_1$ " biases of dimentions  $1 \times y$

$$[b_1] = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{1y} \end{bmatrix}$$

$[b_2]$  is the columnised vector notation of all " $b_2$ " biases of dimentions  $1 \times z$

$$[b_2] = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{1z} \end{bmatrix}$$

The input to a  $y_i$  neuron will be denoted as:

$$y_i = \sigma(w_{x_1 \rightarrow y_i} * x_1 + w_{x_2 \rightarrow y_i} * x_2 + b_{y_i})$$

OR in vector notation:

$$[y] = \sigma([x][w_1] + [b_1])$$

$$\text{Where } \sigma(x) = \frac{1}{1+e^{-x}} = \frac{1}{1+\exp[-x]}$$

The input to a  $z_i$  neuron will be denoted as:

$$z_i = \sigma(w_{y_1 \rightarrow z_i} * y_1 + w_{y_2 \rightarrow z_i} * y_2 + w_{y_3 \rightarrow z_i} * y_3 + b_{z_i})$$

OR in vector notation:

$$[z] = \sigma([y][w_2] + [b_2])$$

## 2 Solve $z_1 = z_2$ for $x_1$

$$z_1 = z_2 \tag{1}$$

NOTE: Observe  $I$  as the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

NOTE: Observe  $[x]I$

$$[x]I = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

NOTE: Observe:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

Denote:

$$[I_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Denote:

$$[I_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can rewrite:  $z_1 = z_2$  as:

$$[z][I_1] = [z][I_2] \quad (3)$$

move everything to one side

$$[z][I_1] - [z][I_2] = 0 \quad (4)$$

factor  $[z]$

$$[z]([I_1] - [I_2]) = 0 \quad (5)$$

expand  $[I_1]$  and  $[I_2]$

$$[z] \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 0 \quad (6)$$

simplify

$$[z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad (7)$$

Denote:

$$[I_{1-2}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (8)$$

return to matrix notation

$$[z][I_{1-2}] = 0 \quad (9)$$

expand  $[z]$

$$[z] = \sigma([y][w_2] + [b_2]) \quad (10)$$

$$[z] = \sigma(\sigma([x][w_1] + [b_1])[w_2] + [b_2]) \quad (11)$$

substatute eq.11 into eq.9

$$\sigma(\sigma([x][w_1] + [b_1])[w_2] + [b_2])[I_{1-2}] = 0 \quad (12)$$

expand out matrices

$$\theta(\theta(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} w_{y11} & w_{y12} & w_{y13} \\ w_{y21} & w_{y22} & w_{y23} \\ w_{y31} & w_{y32} & w_{y33} \end{bmatrix} + \begin{bmatrix} b_{y1} \\ b_{y2} \\ b_{y3} \end{bmatrix}) \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

NOTE: will need the transpos of  $[x]$

$$\theta(\theta(\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} w_{y11} & w_{y12} & w_{y13} \\ w_{y21} & w_{y22} & w_{y23} \\ w_{y31} & w_{y32} & w_{y33} \end{bmatrix} + \begin{bmatrix} b_{y1} \\ b_{y2} \\ b_{y3} \end{bmatrix}) \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

simplify

$$\theta(\theta(\begin{bmatrix} x_1 * w_{y11} + x_2 * w_{y21} + x_3 * w_{y31} \\ x_1 * w_{y12} + x_2 * w_{y22} + x_3 * w_{y32} \\ x_1 * w_{y13} + x_2 * w_{y23} + x_3 * w_{y33} \end{bmatrix} + \begin{bmatrix} b_{y1} \\ b_{y2} \\ b_{y3} \end{bmatrix}) \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\theta(\theta(\begin{bmatrix} x_1 * w_{y11} + x_2 * w_{y21} + x_3 * w_{y31} + b_{y1} \\ x_1 * w_{y12} + x_2 * w_{y22} + x_3 * w_{y32} + b_{y2} \\ x_1 * w_{y13} + x_2 * w_{y23} + x_3 * w_{y33} + b_{y3} \end{bmatrix}) \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\theta(\begin{bmatrix} \theta(x_1 * w_{y11} + x_2 * w_{y21} + x_3 * w_{y31} + b_{y1}) \\ \theta(x_1 * w_{y12} + x_2 * w_{y22} + x_3 * w_{y32} + b_{y2}) \\ \theta(x_1 * w_{y13} + x_2 * w_{y23} + x_3 * w_{y33} + b_{y3}) \end{bmatrix} \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

define:

$$A_1 = \theta(x_1 * w_{y11} + x_2 * w_{y21} + x_3 * w_{y31} + b_{y1}) A_2 = \theta(x_1 * w_{y12} + x_2 * w_{y22} + x_3 * w_{y32} + b_{y2}) \quad (13)$$

$$\theta(\begin{bmatrix} A_1 \\ A_2 \\ \theta(x_1 * w_{y13} + x_2 * w_{y23} + x_3 * w_{y33} + b_{y3}) \end{bmatrix} \begin{bmatrix} w_{z11} & w_{z12} & w_{z13} \\ w_{z21} & w_{z22} & w_{z23} \\ w_{z31} & w_{z32} & w_{z33} \end{bmatrix} + \begin{bmatrix} b_{z1} \\ b_{z2} \\ b_{z3} \end{bmatrix}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

### **3 Conclusion**

Not much of a paper, but it's a start.