Solve $Z_1 = Z_2$ for x_1

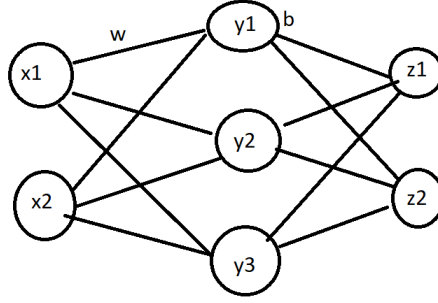Daniel Fishbein

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1 Given:

Figure 1: Given Neural Net

Figure 1 shows the given Neural Net that will be analysed.

x_1, x_2 are the input neurons
y_1, y_2, y_3 are the hidden layers neurons
z_1, z_2 are the output neurons

w denotes a weight
$w_{y_1 -> z_2}$ denotes the weight from y_1 to z_2

b deontes a bias
b_{y_3} deontes the bias associated with neuron y_3

The input to a y_i neuron will be denoted as:
$y_i = \sigma(w_{x_1 -> y_i} * x_1 + w_{x_2 -> y_i} * x_2 + b_{y_i})$

Where $\sigma(x) = \frac{1}{1+e^{-x}} = \frac{1}{1+\exp[-x]}$

The input to a z_i neuron will be denoted as:
$z_i = \sigma(w_{y_1 -> z_i} * y_1 + w_{y_2 -> z_i} * y_2 + w_{y_3 -> z_i} * y_3 + b_{z_i})$

2 Solve $z_1 = z_2$ for x_1

$$z_1 = z_2 \tag{1}$$

$$z_1 = \sigma(w_{y_1->z_1} * y_1 + w_{y_2->z_1} * y_2 + w_{y_3->z_1} * y_3 + b_{z_1}) \tag{2}$$

$$\begin{aligned}
z_1 = \sigma(&w_{y_1->z_1} * \sigma(w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}) \\
&+ w_{y_2->z_1} * \sigma(w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}) \\
&+ w_{y_3->z_1} * \sigma(w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + b_{z_1})
\end{aligned} \tag{3}$$

$$\begin{aligned}
z_2 = \sigma(&w_{y_1->z_2} * \sigma(w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}) \\
&+ w_{y_2->z_2} * \sigma(w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}) \\
&+ w_{y_3->z_2} * \sigma(w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + b_{z_2})
\end{aligned} \tag{4}$$

$$\begin{aligned}
(1 + \exp[-(&w_{y_1->z_1} * \frac{1}{1 + \exp[w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}]} \\
&+ w_{y_2->z_1} * \frac{1}{1 + \exp[w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}]} \\
&+ w_{y_3->z_1} * \frac{1}{1 + \exp[w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}]} \\
&\qquad\qquad\qquad + b_{z_1})])^{-1} \\
&\qquad\qquad = \\
(1 + \exp[-(&w_{y_1->z_2} * \frac{1}{1 + \exp[w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1}]} \\
&+ w_{y_2->z_2} * \frac{1}{1 + \exp[w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2}]} \\
&+ w_{y_3->z_2} * \frac{1}{1 + \exp[w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3}]} \\
&\qquad\qquad\qquad + b_{z_2})])^{-1}
\end{aligned} \tag{5}$$

NOTE: this equation is too big. Lets scope it down.

$$B_1 = -w_{x_1->y_1} * x_1 + w_{x_2->y_1} * x_2 + b_{y_1} \tag{6}$$

$$B_2 = -w_{x_1->y_2} * x_1 + w_{x_2->y_2} * x_2 + b_{y_2} \tag{7}$$

$$B_3 = -w_{x_1->y_3} * x_1 + w_{x_2->y_3} * x_2 + b_{y_3} \tag{8}$$

$$\tag{9}$$

NOTE: Substatuting B in.

$$(1 + \exp[-(w_{y_1->z_1} * \frac{1}{1 + \exp[B_1]} \tag{10}$$

$$+w_{y_2->z_1} * \frac{1}{1 + \exp[B_2]} \tag{11}$$

$$+w_{y_3->z_1} * \frac{1}{1 + \exp[B_3]} \tag{12}$$

$$+b_{z_1})])^{-1} \tag{13}$$

$$= \tag{14}$$

$$(1 + \exp[-(w_{y_1->z_2} * \frac{1}{1 + \exp[B_1]} \tag{15}$$

$$+w_{y_2->z_2} * \frac{1}{1 + \exp[B_2]} \tag{16}$$

$$+w_{y_3->z_2} * \frac{1}{1 + \exp[B_3]} \tag{17}$$

$$+b_{z_2})])^{-1} \tag{18}$$

Define A

$$A_1 = w_{y_1->z_1} * \frac{1}{1 + \exp[B_1]} \tag{19}$$

$$+w_{y_2->z_1} * \frac{1}{1 + \exp[B_2]} \tag{20}$$

$$+w_{y_3->z_1} * \frac{1}{1 + \exp[B_3]} \tag{21}$$

$$+b_{z_1} \tag{22}$$

$$A_2 = w_{y_1->z_2} * \frac{1}{1 + \exp[B_1]} \tag{23}$$

$$+w_{y_2->z_2} * \frac{1}{1 + \exp[B_2]} \tag{24}$$

$$+w_{y_3->z_2} * \frac{1}{1 + \exp[B_3]} \tag{25}$$

$$+b_{z_2} \tag{26}$$

3

NOTE: Substatuting A in.

$$(1 + \exp[-(A_1)])^{-1} = (1 + \exp[-(A_2)])^{-1} \tag{27}$$

$$(1 + \exp[-(A_1)]) = (1 + \exp[-(A_2)]) \tag{28}$$

$$1 + \exp[-(A_1)] = 1 + \exp[-(A_2)] \tag{29}$$

$$\exp[-(A_1)] = \exp[-(A_2)] \tag{30}$$

$$(A_1) = (A_2) \tag{31}$$

$$A_1 = A_2 \tag{32}$$

Sub in the values of A_1 and A_2

$$
\begin{gathered}
w_{y_1->z_1} * \frac{1}{1+\exp[B_1]} + w_{y_2->z_1} * \frac{1}{1+\exp[B_2]} + w_{y_3->z_1} * \frac{1}{1+\exp[B_3]} \\
+ b_{z_1} \\
= \\
w_{y_1->z_2} * \frac{1}{1+\exp[B_1]} + w_{y_2->z_2} * \frac{1}{1+\exp[B_2]} + w_{y_3->z_2} * \frac{1}{1+\exp[B_3]} \\
+ b_{z_2}
\end{gathered}
\tag{33}
$$

$$
\begin{gathered}
\frac{w_{y_1->z_1}}{1+\exp[B_1]} + \frac{w_{y_2->z_1}}{1+\exp[B_2]} + \frac{w_{y_3->z_1}}{1+\exp[B_3]} + b_{z_1} \\
= \\
\frac{w_{y_1->z_2}}{1+\exp[B_1]} + \frac{w_{y_2->z_2}}{1+\exp[B_2]} + \frac{w_{y_3->z_2}}{1+\exp[B_3]} + b_{z_2}
\end{gathered}
\tag{34}
$$

Multiply by botoms to create common denominators.

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_2]) * (1 + \exp[B_1]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_3]) * (1 + \exp[B_1]) * (1 + \exp[B_2])} +$$

$$b_{z_1}$$

$$=$$

$$\frac{w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_2]) * (1 + \exp[B_1]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_3]) * (1 + \exp[B_1]) * (1 + \exp[B_2])} +$$

$$b_{z_2}$$

$$\tag{35}$$

REORDRER DENOMINATORS

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$b_{z_1}$$

$$=$$

$$\frac{w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$b_{z_2}$$

$$\tag{36}$$

MOVE b_{z_1}

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])}$$

$$=$$

$$\frac{w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$b_{z_2} - b_{z_1}$$

$$(37)$$

MOVE THE BIG PIECE

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} -$$

$$\frac{w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} -$$

$$\frac{w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} -$$

$$\frac{w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])}$$

$$= b_{z_2} - b_{z_1}$$

$$(38)$$

REORDER FOR SIMMILAR TERMS

6

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} - \frac{w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} - \frac{w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} - \frac{w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])}$$

$$= b_{z_2} - b_{z_1}$$

(39)

COMBINE LIKE TERMS

$$\frac{w_{y_1->z_1} * (1 + \exp[B_2]) * (1 + \exp[B_3]) - w_{y_1->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_2->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_3]) - w_{y_2->z_2} * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{w_{y_3->z_1} * (1 + \exp[B_1]) * (1 + \exp[B_2]) - w_{y_3->z_2} * (1 + \exp[B_2]) * (1 + \exp[B_1])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])}$$

$$= b_{z_2} - b_{z_1}$$

(40)

fACTOR COMMON NUMERATORS

$$\frac{(w_{y_1->z_1} - w_{y_1->z_2}) * (1 + \exp[B_2]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{(w_{y_2->z_1} - w_{y_2->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_3])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])} +$$

$$\frac{(w_{y_3->z_1} - w_{y_3->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_2])}{(1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3])}$$

$$= b_{z_2} - b_{z_1}$$

(41)

SANITY CHECK

$$aAB - bAB = (a - b)AB \qquad (42)$$

MULTIPLY BY COMMON DENOMINATOR

$$(w_{y_1->z_1} - w_{y_1->z_2}) * (1 + \exp[B_2]) * (1 + \exp[B_3]) +$$
$$(w_{y_2->z_1} - w_{y_2->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_3]) +$$
$$(w_{y_3->z_1} - w_{y_3->z_2}) * (1 + \exp[B_1]) * (1 + \exp[B_2])$$
$$= (b_{z_2} - b_{z_1}) * ((1 + \exp[B_1]) * (1 + \exp[B_2]) * (1 + \exp[B_3]))$$

$$(43)$$

The next step is to multiply this out and emilimate terms. I do not want to do that rn :/

GO BACK TO EQ.34

$$\frac{w_{y_1->z_1}}{1+\exp[B_1]} + \frac{w_{y_2->z_1}}{1+\exp[B_2]} + \frac{w_{y_3->z_1}}{1+\exp[B_3]} + b_{z_1}$$
$$=$$
$$\frac{w_{y_1->z_2}}{1+\exp[B_1]} + \frac{w_{y_2->z_2}}{1+\exp[B_2]} + \frac{w_{y_3->z_2}}{1+\exp[B_3]} + b_{z_2}$$

(44)

SUBRTRACT b

$$\frac{w_{y_1->z_1}}{1+\exp[B_1]} + \frac{w_{y_2->z_1}}{1+\exp[B_2]} + \frac{w_{y_3->z_1}}{1+\exp[B_3]}$$
$$=$$
$$\frac{w_{y_1->z_2}}{1+\exp[B_1]} + \frac{w_{y_2->z_2}}{1+\exp[B_2]} + \frac{w_{y_3->z_2}}{1+\exp[B_3]} + b_{z_2} - b_{z_1}$$

(45)

SUBRTRACT REST OF THE STUFF

$$\frac{w_{y_1->z_1}}{1+\exp[B_1]} - \frac{w_{y_1->z_2}}{1+\exp[B_1]} + \frac{w_{y_2->z_1}}{1+\exp[B_2]} - \frac{w_{y_2->z_2}}{1+\exp[B_2]} + \frac{w_{y_3->z_1}}{1+\exp[B_3]} - \frac{w_{y_3->z_2}}{1+\exp[B_3]}$$
$$=$$
$$b_{z_2} - b_{z_1}$$

(46)

COMBINE LIKE TERMS

$$\frac{w_{y_1->z_1} - w_{y_1->z_2}}{1+\exp[B_1]} + \frac{w_{y_2->z_1} - w_{y_2->z_2}}{1+\exp[B_2]} + \frac{w_{y_3->z_1} - w_{y_3->z_2}}{1+\exp[B_3]}$$
$$=$$
$$b_{z_2} - b_{z_1}$$

(47)

This will got to EQ.43. This is a dead end.

3 Conclusion

Not much of a paper, but it's a start.