Solve $Z_1 = Z_2$ for x_1

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1 Given:

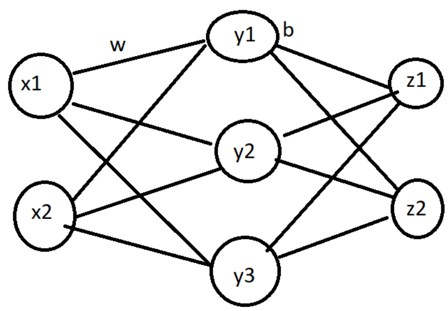

Figure 1: Given Neural Net

Figure 1 shows the given Neural Net that will be analysed.

x_1, x_2 are the input neurons
$[x]$ is the columnised vector notation of all "x" neurons of dimentions 1Xx
y_1, y_2, y_3 are the hidden layers neurons
$[y]$ is the columnised vector notation of all "y" neurons of dimentions 1Xy
z_1, z_2 are the output neurons
$[z]$ is the columnised vector notation of all "z" neurons of dimentions 1Xz
w denotes a weight
$w_{y_1->z_2}$ denotes the weight from y_1 to z_2
$[w_1]$ is the columnised vector notation of all "w_1" weights of dimentions yXx

$$[w_1] = [\begin{matrix} w_{11}w_{12}w_{13}...w_{1x} \\ \vdots \\ w_{ij} \\ \vdots \\ w_{y1}w_{y2}w_{y3}...w_{yx} \end{matrix}]$$

$[w_2]$ is the columnised vector notation of all "w_2" weights of dimentions zXy

$$[w_2] = [\begin{matrix} w_{11}w_{12}w_{13}...w_{1y} \\ : \\ w_{ij} \\ : \\ w_{z1}w_{z2}w_{z3}...w_{zy} \end{matrix}]$$

b deontes a bias
b_{y_3} deontes the bias associated with neuron y_3
$[b_1]$ is the columnised vector notation of all "b_1" biases of dimentions 1Xy

$$[b_1] = [\begin{matrix} b_{11} \\ b_{21} \\ : \\ b_{1y} \end{matrix}]$$

$[b_2]$ is the columnised vector notation of all "b_2" biases of dimentions 1Xz

$$[b_2] = [\begin{matrix} b_{11} \\ b_{21} \\ : \\ b_{1z} \end{matrix}]$$

The input to a y_i neuron will be denoted as:
$y_i = \sigma(w_{x_1->y_i} * x_1 + w_{x_2->y_i} * x_2 + b_{y_i})$
OR in vector notation:
$[y] = \sigma([x][w_1] + [b_1])$
Where $\sigma(x) = \frac{1}{1+e^{-x}} = \frac{1}{1+\exp[-x]}$

The input to a z_i neuron will be denoted as:
$z_i = \sigma(w_{y_1->z_i} * y_1 + w_{y_2->z_i} * y_2 + w_{y_3->z_i} * y_3 + b_{z_i})$
OR in vector notation:
$[z] = \sigma([y][w_2] + [b_2])$

2 Solve $z_1 = z_2$ for x_1

$$z_1 = z_2 \tag{1}$$

NOTE: Observe I as the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

NOTE: Observe $[x]I$

$$[x]I = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

NOTE: Observe:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

Denote:

$$[I_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Denote:

$$[I_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we can rewrite: $z_1 = z_2$ as:

$$[z][I_1] = [z][I_2] \tag{3}$$

move everything to one side

$$[z][I_1] - [z][I_2] = 0 \tag{4}$$

factor $[z]$

$$[z]([I_1] - [I_2]) = 0 \tag{5}$$

expand $[I_1]$ and $[I_2]$

$$[z](\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = 0 \tag{6}$$

simplify

$$[z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \tag{7}$$

Denote:

$$[I_{1-2}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

return to matrix notation

$$[z][I_{1-2}] = 0 \tag{9}$$

expand $[z]$

$$[z] = \sigma([y][w_2] + [b_2]) \tag{10}$$

$$[z] = \sigma(\sigma([x][w_1] + [b_1])[w_2] + [b_2]) \tag{11}$$

substatute eq.11 into eq.9

$$\sigma(\sigma([x][w_1] + [b_1])[w_2] + [b_2])[I_{1-2}] = 0 \tag{12}$$

Go back to equ.3

$$[z][I_1] = [z][I_2] \tag{13}$$

expand Z