Appendix

The proof of Theorem 2 relies on Lemmas 1 and 2. The first lemma shows that the standard deviation of power flow related to customer i is at least as much as σ_i. Therefore, by specifying σ_i, the DSO attains the desired degree of randomization.

Lemma 1. *If OPF mechanism (4) returns optimal solution, then σ_ℓ is the lower bound on Std$[\tilde{f}_\ell^p]$.*

Proof. Consider a single flow perturbation with $\xi_\ell \sim \mathcal{N}(0, \sigma_\ell^2)$ and $\xi_j = 0,\ \forall j \in \mathcal{L}\backslash\ell$. The standards deviation of active power flow (3b) in optimum finds as

$$\text{Std}\Big[\overset{\star}{f}{}_\ell^p - \Big[\overset{\star}{\rho}{}_\ell^p + \sum_{j\in\mathcal{D}_\ell}\overset{\star}{\rho}{}_j^p\Big]\xi\Big] = \text{Std}\Big[\Big[\overset{\star}{\rho}{}_\ell^p + \sum_{j\in\mathcal{D}_\ell}\overset{\star}{\rho}{}_j^p\Big]\xi\Big] = \text{Std}\Big[\sum_{j\in\mathcal{D}_\ell}\overset{\star}{\alpha}_{j\ell}\xi_\ell\Big] \overset{(2b)}{=} \text{Std}\Big[\xi_\ell\Big] = \sigma_\ell, \tag{9}$$

where the second to the last equality follows from balancing conditions (2b). As for any pair $(\ell, j) \in \mathcal{L}$ the covariance matrix returns $\Sigma_{\ell,j} = 0$, σ_ℓ is a lower bound on Std$[\tilde{f}_\ell^p]$ in the optimum for any additional perturbation in the network. □

Remark 1. *The result of Lemma 1 holds independently from the choice of objective function and is solely driven by the feasibility conditions.*

The second lemma shows that $\beta_i \geqslant \Delta_i^\beta$, i.e., if σ_i is parameterized by β_i, then σ_i is also parameterized by sensitivity Δ_i^β.

Lemma 2. *Let D and D' be two adjacent datasets differing in at most one load d_i^p by at most $\beta_i > 0$. Then,*

$$\Delta_i^\beta = \max_{\ell\in\mathcal{L}} \|\mathcal{M}(D)_{|f_\ell^p} - \mathcal{M}(D')_{|f_\ell^p}\|_2 \leqslant \beta_i$$

where the notation $\mathcal{M}(\cdot)_{|f_\ell^p}$ denotes the value of the optimal active power flow on line ℓ returned by the computation $\mathcal{M}(\cdot)$.

Proof. Let $\overset{\star}{f}{}_\ell^p$ be the optimal solution for the active power flow in line ℓ obtained on input dataset $D = (d_1^p, \dots, d_n^p)$. From OPF equation (1c), it can be written as

$$\overset{\star}{f}{}_\ell^p = d_\ell^p - \overset{\star}{g}{}_\ell^p + \sum_{i\in\mathcal{D}_\ell}(d_i^p - \overset{\star}{g}{}_i^p),$$

which expresses the flow as a function of the downstream loads and the optimal DER dispatch. A change in the active load d_ℓ^p translates into a change of power flow as

$$\frac{\partial \overset{\star}{f}{}_\ell^p}{\partial d_\ell^p} = \underbrace{\frac{\partial d_\ell^p}{\partial d_\ell^p}}_{1} - \frac{\partial \overset{\star}{g}{}_\ell^p}{\partial d_\ell^p} + \sum_{i\in\mathcal{D}_\ell}\Big(\underbrace{\frac{\partial d_i^p}{\partial d_\ell^p}}_{0} - \frac{\partial \overset{\star}{g}{}_i^p}{\partial d_\ell^p}\Big) = 1 - \frac{\partial \overset{\star}{g}{}_\ell^p}{\partial d_\ell^p} - \sum_{i\in\mathcal{D}_\ell}\frac{\partial \overset{\star}{g}{}_i^p}{\partial d_\ell^p}, \tag{10}$$

where the last two terms are always non-negative due to convexity of model (1). The value of (10) attains maximum when

$$\overset{\star}{g}{}_k^p = \overline{g}_k^p \mapsto \frac{\partial \overset{\star}{g}{}_k^p}{\partial d_\ell^p} = 0, \ \ \forall k \in \{\ell\} \cup \mathcal{D}_\ell. \tag{11}$$

Therefore, by combining (10) with (11) we obtain the maximal change of power flows as

$$\frac{\partial \overset{\star}{f}{}_\ell^p}{\partial d_\ell^p} = 1.$$

Since the dataset adjacency relation considers loads d_ℓ^p that differ by at most β_ℓ, it suffices to multiply the above by β_ℓ to attain the result. It finds similarly that for a β_i change of any load $i \in \mathcal{N}$, all network flows change by at most β_i. □

Proof of Theorem 2. Consider a customer at non-root node i. Mechanism $\tilde{\mathcal{M}}$ induces a perturbation on the active power flow f_i^p by a random variable $\xi_i \sim \mathcal{N}(0, \sigma_i^2)$. The randomized active power flow f_i^p is then given as follows:

$$\tilde{f}_i^p = \overset{\star}{f}{}_i^p - [\overset{\star}{\rho}{}_i^p + \sum_{j\in\mathcal{D}_i}\overset{\star}{\rho}{}_j^p]\xi$$

where $\star$ denotes optimal solution for optimization variables. For privacy parameters (ε, δ), the mechanism specifies

$$\sigma_i \geqslant \beta_i\sqrt{2\ln(1.25/\delta)}/\varepsilon, \ \forall i \in \mathcal{L}.$$

As per Lemma 1, we know that σ_i is the lower bound on the standard deviation of power flow f_ℓ^p. From Lemma 2 we also know that the sensitivity Δ_i^β of power flow in line i to load d_i^p is upper-bounded by β_i, so we have

$$\text{Std}[\tilde{f}_i^p] \geqslant \sigma_i \geqslant \Delta_i^\beta\sqrt{2\ln(1.25/\delta)}/\varepsilon.$$

Since the randomized power flow follow is now given by a Normal distribution with the standard deviation Std$[\tilde{f}_i^p]$ as above, by Theorem 1, mechanism $\tilde{\mathcal{M}}$ satisfies (ε, δ)-differential privacy for each grid customer up to adjacency parameter β. □