# The property of Chain recurrent set on impulsive semidynamical system (Extension of Conley's theory)

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**Abstract**

In this paper, we extend a well-known result about chain recurrence on compact metric spaces to impulsive dynamical system. Especially we will deal with the theory of Conley(1978) about attractor-repeller decomposition.

## 1 Impulsive dynamical system

[3] Let $X$ be a metric space and $\mathbb{R}_+$ be the set of real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a semidynamical system, if the continuous function $\pi : X \times \mathbb{R}_+ \to X$ satisfies the conditions:

(a) $\pi(x, 0) = x,\ for\ all\ x \in X$

(b) $\pi(\pi(x,t), s) = \pi(x, t+s),\ for\ all\ x \in X\ and\ t, s \in \mathbb{R}_+$

For fixed $t \in \mathbb{R}_+$, we can define $\pi_t : X \to X$ by $\pi_t(x) = \pi(x,t)$, for $x \in X$. And for fixed $x \in X$, we can also define $\pi_x : \mathbb{R}_+ \to X$ as $\pi_x(t) = \pi(x,t)$, for $t \in \mathbb{R}_+$. Easy to know that $\pi_t$ is a homeomorphism of X and $\pi_x$ is a continuous. $\pi_x$ called the trajectory of $x$. Let $F(x,t) = \{y : \pi(y,t) = x\}$. We denote the *positive orbit of* $x$ by $\pi^+(x) = \{\pi(x,t) : t \in \mathbb{R}_+\}$.

[3] An impulsive dynamical system $(X, \tilde{\pi}; M, I)$ consists of a semdynamical system, $(X, \pi)$, a non-empty closed subset $M$ of $X$ and a continuous function $I : M \to X$ such that for every $x \in M$, there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \emptyset\ \ and\ \ \pi(x, (0, \epsilon_x)) \cap M = \emptyset$$

Let $M^+(x) = \pi^+(x) \cap M \setminus \{x\}$, $I(x) = x^+$.

Define $\Phi : X \to (0, +\infty]$

$$\Phi(x) = \begin{cases} s & \pi(x,s) \in M, \pi(x,t) \notin M \text{ for } 0 < t < s \\ +\infty & M^+(x) = \emptyset \end{cases}$$

Existance of $s$ is shown in [Lemma 2.1 [4]].

Given $x \in X$ we define a function $\tilde{\pi}_x : [0, s) \to X$ where $s \in \mathbb{R}_+ \cup \{\infty\}$. This function is called *impulsive trajectory* of $x$ and defined inductively as follows. Set $x = x_0$. If $M^+(x_0) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}_+$. If $M^+(x_0) \neq \emptyset$,then there exists a positive $s_0 \in \mathbb{R}_+$ such that $\pi(x_0, s_0) = x_1 \in M$ and $\pi(x_0, t) \notin M$ for $0 < t < s_0$. We define $\tilde{\pi}_x$ on $[0, s_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x, t) & 0 \leq t < s_0 \\ x_1^+ & t = s_0 \end{cases}$$

We continue the above process starting at $x_1^+$. If $M^+(x_1) = \emptyset$, then $\tilde{\pi}(t) = \pi(x_.^+, t - s_0)$ for $t \geq s_0$. If else, $M + (x_1^+) \neq \emptyset$, shich implies the existence of an $s_1 > 0$ as before, and we define

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0) & s_0 \leq t < s_0 + s_1 \\ x_2^+ & t = s_0 + s_1 \end{cases}$$

where $x_2 = \pi(x_1^+, s_0 + s_1)$. Repeating this process, if $M^+(x_n^+) = \emptyset$ for $n$ process ends after a finite number of steps. If else, $M^+(x_n^+) \neq \emptyset,\ b = 1, 2, \ldots$, the process continues indefinitely. However $M^+(x_n^+) \neq \emptyset$ for $n$, we can get

$$\tilde{\pi}_x(t) = \begin{cases} \pi({x_n}^+, t - t_n) & t_n \leq t < t_{n+1} \\ {x_{n+1}}^+ & t = t_{n+1} \end{cases}$$

Where $\pi({x_n}^+, s_n) = x_{n+1},\ \ I(x_{n+1}) = {x_{n+1}}^+,\ \ t_n = \sum_{i=0}^{n-1} s_i$

Let$(X, \tilde{\pi}; M, I)$ be an impulsive semidynamical system. If $x \in X$, then

(a) $\tilde{\pi}(x, 0) = x$

(b) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$, with $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$

**Definition 1.1 ([2], Definition 3.1.)** *Let $(X, \pi)$ be a semidynamical system. Any closed set $S \subset X$ containing $x \in X$ is called a section or a $\lambda$-section through $x$, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that*

*(a) $F(L, \lambda) = S$*

*(b) $F(L, [0, 2\lambda])$ is neighbourhood of $x$*

*(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.*

The set $F(L, [0, 2\lambda])$ is called a tube or $\lambda$ tube and the set $L$ is called a bar. Any tube $F(L, [0, 2\lambda])$ given by a section $S$ through $x \in X$ such that $S \subseteq M \cap F(L, [0, 2\lambda])$ is called TC-tube on $x$. We say that a point $x \in M$ fulfills the Tube Condition, shortly TC, if there exists a TC-tube $F(L, [0, 2\lambda])$ through $x$. Also we define $F(L, [0, 2\lambda])$ given by a section $S$ through $x \in X$ such that $S = M \cap F(L, [0, 2\lambda])$ is called STC-tube on $x$. We say that a point $x \in M$ fulfills the Strong Tube Condition, shortly STC, if there exists a STC-tube $F(L, [0, 2\lambda])$ through $x$.

**Theorem 1.2 ([2], Theorem 3.4.)** *If in an impulsive system $(X, \tilde{\pi}; M, I)$ given by a semidynamical system $(X, \pi)$ each point contained in the impulse set $M$ satisfies TC, then $\Phi$ is upper semicontinuous.*

**Theorem 1.3 ([2], Theorem 2.7.)** *Assume that $(X, \tilde{\pi}; M, I)$ is an impulsive system given by a dynamical system $(X, \pi)$. Then for any $x \notin M$ the function $\Phi$ is lower semicontinuous at $x$.*

Proof) Let $x \notin M$ and $\Phi(x) = c \in (0, +\infty)$. The function $\Phi$ is lower semicontinuous at $x$ if and only if for any sequence $x_n \to x$ with $\Phi(x_n) \to t$ we have $t \geq c$. Suppose to the contrary that there exists a sequence $p_n \to x$ with $\Phi(p_n) \to t < c$. For $n$ large enough we have $p_n \notin M$ as $M$ is closed. Thus $\pi(\Phi(p_n), p_n) \in M = \overline{M}$. On the other hand, $\pi(\Phi(p_n), p_n) \to \pi(t, x)$, so $\pi(t, x) \in M$, which means that $c = \Phi(x) \leq t < c$ and contradiction. So proof finished. □

**Theorem 1.4 ([2], Theorem 3.8.)** *Assume that $(X, \tilde{\pi}; M, I)$ is an impulsive system given by a semidynamical system $(X, \pi)$ and no start point of the system belongs to $M$. Then:*

*(1) if each element of $M$ satisfies TC then $\Phi$ is continuous at $x$ if and only if $x \notin M$,*

*(2) $\Phi$ is not lower semicontinuous at $x$ if and only if $x \in M$,*

*(3) if each element of $M$ satisfies STC then $\Phi|_M$ is lower semicontinuous.*

## 2 Attractor on Impulsive dynamical system

Let $\phi : X \times \mathbb{R} \to X$ be a flow of compact metric space X.

- Impulsive semidynamical system $(X, \tilde{\pi}; M, I)$ is expansive if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $x, y \in X$ satisfy $d(\tilde{\pi}_t(x), \tilde{\pi}_{s(t)}(y)) < \delta$ $\forall t \in \mathbb{R}$ for $x, y \in X$ and a continuous map $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$, then $y \in F(x, (0, \epsilon)) \cup \tilde{\pi}(x, [0, \epsilon))$
- given $\delta, a > 0$ real numbers, a finite $(\delta, a)-$pseudo orbit is a pair of sequences $(\{x_i\}_0^k, \{t_i\}_0^{k-1})$ so that $t_i \geq a$ and $d(\tilde{\pi}_{t_n}(x_n), x_{n+1}) < \delta$ for $k \in \mathbb{N}$. An infinite $(\delta, a)$-pseudo orbit is a pair of doubly infinite sequences $(\{x_i\}_{-\infty}^{\infty}, \{t_i\}_{-\infty}^{\infty})$ so that $t_i \geq a$ and $d(\tilde{\pi}_{t_n}(x_n), x_{n+1}) < \delta$ for all $i \in \mathbb{Z}$.
- A finite(infinite) $(\delta, a)$-pseudo orbit $(\{x_n\}, \{t_n\})$ is $\epsilon$- traced by the orbit $(\tilde{\pi}_t(z))$ if ther exists an increasing homeomorphis $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\pi}_{t - \sum_{i=0}^{n-1} t_i}(x_n)) < \epsilon \ \textit{whenever} \ t \geq 0, \ \sum_0^{n-1} t_i \leq t < \sum_0^n t_i \ \ n = 0, 1, 2, ...$$

- Impulsive semidynamical system $(X, \tilde{\pi}; M, I)$ is said to have the *pseudo orbit tracing property, POTP*, if for all $\epsilon > 0$ there exists $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit is $\epsilon$-traced by an orbit of $\tilde{\pi}$.

- Impulsive dynamical system $(X, \tilde{\pi}; M, I)$ is said to be topologically stable if for any $\epsilon > 0$ there is $\delta > 0$ such that for every other Impulsive dynamical system $(X, \psi, M, I)$ with $d(\tilde{\phi}_t, \tilde{\psi}_{s(t)}) < \delta$ for all $t \in [0, 1]$, for some continuous map $s$ from $[0, 1]$ to itself. there exists $h : X \to X$ continuous such that $d(h, id) < \epsilon$ and $h(orbit\ of\ \tilde{\psi}) \subset orbit\ of\ \tilde{\phi}$.

- Let $\mathcal{T}$ be a family of subset from $X$. An impulsive semidynamical system $(X, \pi; M, I)$ is called $\mathcal{T}$- dissipative if there exist a bounded set $A \subset X \setminus M$ such that for every $\epsilon > 0$ and $\in \mathcal{T}$ there exist $l(\epsilon, V) > 0$ such that $\tilde{\pi}(V, t) \subset B(A, \epsilon)$ for all $t \geq l(\epsilon, V)$. The minimal set of $A$ which satisfies above condition called an attractor for the family $\mathcal{T}$.

- Let $(X, \tilde{\pi}; M, I)$ is $\mathcal{T}$- dissipative and $A$ is an attractor for the family $\mathcal{T}$. $U = \tilde{\pi}_{[t,\infty)}(\bigcup_{V \in \mathbb{T}} V)$ is called a preattractor if $U$ is a neighborhood of $A$ and there is $T > 0$ such that $\overline{\tilde{\pi}_{[T,\infty)}(U)} \subset U$.

- Given a subset $U \subset X$ we define the impulsive $\omega$ - limit of $U$ as the set

$$\tilde{\omega}(U) = \bigcup_{t \geq 0} \overline{O_t^+(U)}$$

**Proposition 2.1** *We have $\tilde{\omega}(U) = \{x \in X|$ there exist wequences$\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $t_n \to \infty$ such that $\tilde{\pi}_{t_n}(x_n) \to x$ as $n \to \infty\}$ and $\tilde{\omega}(U)$ is closed for every subset $U \subset X$*

Proof) [1] Lemma 3.2.□

**Proposition 2.2** *Let $(X, \tilde{\pi}; M, I)$ is $\mathcal{T}$- dissipative and set $A$ is an attractor for the family $\mathcal{T}$ and $U$ is a preattroactor for the family $\mathcal{T}$. Then $\overline{A} = \bigcap_{t \geq 0} \overline{\tilde{\pi}_{[t,\infty)}(U)}$*

Proof) If $x \in A$ then for given $n \in \mathbb{N}$ there exist $l_n \in [0, \infty)]$ such that if $t \geq l_n$ then $\tilde{\pi}_t(U) \subset B(A, 1/n)$. If we define $t_n = 1/n$ and pick $x_n \in \tilde{\pi}_{l_n}(U)$. Then $A \subset \tilde{\omega}(U)$. Since $\tilde{\omega}(U)$ is closed set, $\overline{A} \subset \tilde{\omega}(U)$. Now suppose that $x \in A \subset \tilde{\omega}(U)$ but $x \notin \overline{A}$. Define $\epsilon = d(\overline{A}, x)$. Then $x \in A \subset \tilde{\omega}(U) \subset \overline{\tilde{\pi}_{[t,\infty)}(\bigcup_{V \in \mathbb{T}} V))} \subset \overline{B(A, \epsilon/2)} \subset \overline{B(\overline{A}, \epsilon)} \subset \{x\}^c$. This is contradiction. Hence $\tilde{\omega}(U) \subset \overline{A}$.□

**Proposition 2.3** *Let $(X, \tilde{\pi}; M, I)$ is $\mathcal{T}$- dissipative and $U$ is a preattroactor for the family $\mathcal{T}$. Then there is a $T \in \mathbb{R}_+$ such that if $t \geq T$ then $\overline{\tilde{\pi}_{[T,\infty)}(U)} \subset U$.*

- Let $F(x, t) = \{y : \pi(y, t) = x\}$, $U = \bigcup_{V \in \mathcal{T}} O^+(V)$. We define the set $O^-(U)$ such that

$$O^-(U) = \bigcup_{t \geq 0} F(U, t)$$

- the set $\mathbb{T}$ denoted by the set of all family $\mathcal{T}$ that $U = \bigcup_{V\in\mathcal{T}} O^+(V)$ is a preattractor determines $A$. Define $B(A)$, the basin of $A$, such that

$$B(A) = \bigcup_{\mathcal{T}\in\mathbb{T}} O^-(U)$$

- Given a map $\epsilon \in \mathcal{P}$ and a positive constant $T$, a finite sequence $(x_0, t_0), (x_1, t_1), \ldots, (x_n, t_n)$ in $X \times (0, \infty)$ is called an $(\epsilon, T)$ - chain for $\tilde{\pi}$ if $d(\tilde{\pi}(x_i), x_{i+1}) < B(\tilde{\pi}(x_i), \epsilon(\tilde{\pi}(x_i))), i = 0, 1, \ldots, n-1$. As above, $n$ is the length of the chain and the chain is said to go from $x_0$ to $x_n$.

- A point $p$ is chain recurrent of for each choice of $T > 0$ and $\epsilon \in \mathcal{P}$ there is an $(\epsilon, T)$-chain of length at least 1 from $p$ to $p$. The chain recurrent set of $\tilde{\pi}$ is denoted $\mathcal{CR}(\tilde{\pi})$

# 3 Main theorem

**Lemma 3.1** *If $K$, $L$ are disjoint closed subsets of $X$, then there is a positive, continuous function $\epsilon$ sith the property that $B_{\epsilon(x)}(x) \cap L = \emptyset$*

**Lemma 3.2** *Suppose that $(X, d)$ and $(Y, \rho)$ are metric spaces and that $g : X \to Y$ and $\epsilon : Y \to (0, \infty)$ are continuous. Then there is a continuous map $\delta : X \to (0, \infty)$ such that $\rho(g(x), g(y)) < \epsilon(g(x))$ whenever $d(x, y) < \delta(x)$.*

**Lemma 3.3** *If $p \in X \setminus M$ is chain recurrent for $\tilde{\pi}$ and $\tau \geq 0$, then $\tilde{\pi}_\tau(p)$ is also chain recurrent for $\tilde{\pi}$.*

Proof) Given $\epsilon \in \mathcal{P}$ and $T > 0$, define $T' = T + \tau$. Let $c = \epsilon(\tilde{\pi}_\tau(p))/2$ and choose a small neighborhood $N$ of $\tilde{\pi}_\tau(p)$ with the property that if $y \in N$ then $\epsilon(y) > c$ and $d(y, \tilde{\pi}_\tau(p)) < c$. Let $\eta_n(x) = \min\{\epsilon(x), \frac{1}{n}\}$. Then there is a $(\eta_n, T')-$chain $(x_i, t'_i)$, $0 \leq i \leq k_n$ which start and end at $p$. For convenience, denote $\tilde{\pi}_{t'_{k_n-1}}(x_{k_n-1}) = x^n$. Then $x^n \to p$ as $n \to \infty$. Since $p \in X \setminus M$, $\tau \geq 0$ by [1] Corollary 3.9, there is a sequence $\{\epsilon_n\}$ such that $\epsilon_n \geq 0$, $\epsilon_n \to 0$ as $n \to \infty$ and $\tilde{\pi}_{\tau+\epsilon_n}(x^n) \to \tilde{\pi}_\tau(p)$. So there is $K \in \mathbb{N}$ such that if $K \leq n$ then $\tilde{\pi}_{\tau+\epsilon_n}(x^n) = \tilde{\pi}_{t'_{n-1}+\tau+\epsilon_n}(x_{n-1}) \in N$. So we pick $(\eta_K, T')-$chain $(x_j, t'_j), 0 \leq j \leq k$ and define $(\epsilon, T)-$chain $(y_j, t_j)$ as follows. Let $y_0 = y_k = \tilde{\pi}_\tau(p)$, $t_0 = t'_0 - \tau$, and $t_{k-1} = t'_{k-1} + \tau + \epsilon_n$. Otherwise let $y_j = x_j$ and $t_j = t'_j$. Clearly, each $t_j \geq T$ and $\tilde{\pi}_{t_0}(y_0) = \tilde{\pi}_{t'_0-\tau}(\tilde{\pi}_\tau(p)) = \tilde{\pi}_{t'_0}(p)$ and $\tilde{\pi}_{t_j}(y_j) = \tilde{\pi}_{t'_j}(x_j)$ for $j = 1, \ldots, k-1$. So we have

$$d(\tilde{\pi}_{t_j}(y_j), y_{j+1}) = d(\tilde{\pi}_{t'_j}(x_j), x_{j+1}) < \epsilon(\tilde{\pi}_{t'_j}(y_j)).$$

And the choice of $K$ ensures that $\tilde{\pi}_{t'_{k-1}+\tau+\epsilon_n}(x_{k-1}) = \tilde{\pi}_{t_{k-1}}(y_{k-1}) \in N$. So we obtain $\epsilon(\tilde{\pi}_{t_{k-1}}(y_{k-1})) > c$ and $d(\tilde{\pi}_{t_{k-1}}(y_{k-1}), \tilde{\pi}_\tau(p)) < c$. Then we get that

$$d(\tilde{\pi}_\tau(p), \tilde{\pi}_{t_{k-1}}(y_{k-1})) < c < \epsilon(\tilde{\pi}_{t_{k-1}}(y_{k-1})).$$

Hence $(y_j, t_j)$ is an $(\epsilon, T)-$ chain. $\square$

**Proposition 3.4** *Let $(X, \tilde{\pi}; M, I)$ is $\mathcal{T}$- dissipative and set $A$ is an attractor for the family $\mathcal{T}$. Let the set $U$ is a neighborhood of $A$ preattroactor for the family $\mathcal{T}$. If $p$ is chain recurrent point of $\tilde{\pi}$ that is contained in $O^-(U)$, then $p \in A$.*

Proof) First consider the special case where $pinU$.

Proof) First consider the special case where $p \in U$. Since $U$ is a preattractor, there is a time $T'$ with $\overline{\tilde{\pi}_{[T',\infty)}(U)} \subset U$. Take $\epsilon$ satisfyng $\overline{\tilde{\pi}_{[T',\infty)}(U)} \subset B(A, \epsilon)$

Choose a function $\delta \in P$ with the property that if $x \in U$ and $t \geq T'$, then the $\delta(x)$ ball centered at $\tilde{\pi}_t(x)$ is disjoint from $B(A, \epsilon)^c$. WLOG we can assume that $\delta$ is bounded above by 1.Take $T = \max\{\Sigma_{i=0}^{n-1} s_i, T'\}$. Suppose that $t \geq T$, $(\delta, t)$- chain $(p_i, t_i), p_0 = p = p_n$ is starts at a point of $U$. Then $p_0 \in U$, $\tilde{\pi}_{t_0}(p_0) \in \tilde{\pi}_{[T',\infty)}(U)$, $d(\tilde{\pi}_{T_0}(p_0), p_1) < \delta(\tilde{\pi}_{T_0}(x_0)) \leq 1$, $p_1 \in B(\tilde{\pi}_{T_0}(x_0), \delta(\tilde{\pi}_{T_0}(x_0))) \in U$, $\tilde{\pi}_{T_1}(x_1) \in \tilde{\pi}_{[t,\infty)}(U)$. By induction we can show that $p$, the end point of the chain, is in $U$ and within 1 of $\tilde{\pi}_{[t,\infty)}(U)$. Using the same argument with $\delta$ replaced by $\delta/k$, $(k = 2, 3, \dots)$ shows that $d(p, \tilde{\pi}_{[t,\infty)}(U)) \leq 1/k$ and so $p \in \overline{\tilde{\pi}_{[t,\infty)}(U)}$. This inclusion holds for all large $t$, we conclude that $p \in \overline{A}$.
In the general case we have a chain recurrent point $p$. There is $t_1$ such that $\tilde{\pi}_{t_1}(p) \in U$ for $t \geq t_1$. Since $\tilde{\pi}_{t_1}(p) \in U$ is also chain recurrent point, $\tilde{\pi}_{t_1}(p) \in \overline{A}$. Take $\epsilon = d(A, U^c)$, $l > 0$ such that $\tilde{\pi}_t(U) \subset B(A, \epsilon/2) \subset X \setminus U$ for all $t \geq l$. Let $T = max\{t_1, l\}$. If we suppose that $p \in X \setminus \overline{A}$ then any $(T, \epsilon/2)$ - chain from $p$ to $p$ can not be exist. □

**Proposition 3.5** *Suppose that $q$ is not chain recurrent for $\tilde{\pi}$. Then there is a family $\mathcal{T}$ such that $q \in O^-(U) - A$ where $U$ and $A$ is a preattractor and an attractor for the family $\mathcal{T}$, respectively.*

Proof) Select $T > 0$ and a function $\epsilon \in \mathcal{P}$ with the property that no $(\epsilon, T)$-chain beginning at $q$ can end at $q$. Define

$$U = \{x \in X | \text{there is an} (\epsilon, T) - \text{chain beginning at q and ending at } x\}$$

$U$ is open follows from the strict inequality $d(\tilde{\pi}_{t_j}(x_j), x_{j+1}) < \epsilon(\tilde{\pi}_{t_j}(x_j))$ in the definition of an $(\epsilon, T)$-chain. Show that $t > T$, $\overline{\tilde{\pi}_{[t,\infty)}(U)} \subset U$. Let $\gamma \in \overline{\tilde{\pi}_t(U)}$ and $\gamma' \in B(\gamma, \epsilon) \cap \overline{\tilde{\pi}_t(U)}$. For $\gamma'' \in B(\gamma, \epsilon)$, $\{F(\gamma, t)\}, \gamma'' \mid t\}$ is a $(\epsilon, T)$ - chain from $F(\gamma, t)$ to $\gamma''$. Using the transitivity it follows that $\gamma'' \in U$ and $\overline{\tilde{\pi}_{[t,\infty)}(U)} \subset U$. Define $\mathcal{T} = \{U\}$ and $A$ be a attractor for the family $\mathcal{T}$. Then $A \subset U$ by above fact. Suppose that $p \in \overline{A}$. For given $\delta > 0$ there is $t > 0$ such that $B(p, \delta) \cap \tilde{\pi}_t(U) \neq \emptyset$. If $t \geq T$ then $\tilde{\pi}_t(p) \in U$. So $B(p, \epsilon) \cap U \neq \emptyset$. Since $p \notin U$ then $p \in U'$. This is contrary to $A \subset U$. □

**Theorem 3.6** *If $(X, d)$ is a compact metric space and $(X, \tilde{\pi}, M, I)$ is an impulsive dynamical system, then $\mathcal{CR}(\tilde{\pi})$ is the complement of the union of the sets $B(A) - A$ as $A$ caries over the collection of attractors of $\tilde{\pi}$ :*

$$X - \mathcal{CR}(\tilde{\pi}) = \bigcup [B(A) - A]$$

Proof) It is directly gained by Proposition 3.5 , [5], and [6].□

## References

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