Relation among Expansiveness, Shadowing property, and Topological stability on impulsive dynamical system

Jiye Jung
Mathematics, Chungnam university, Daejeon, South Korea

Abstract

This paper is a study of expansiveness, shadowing property, and topological stability on impulsive dynamical system. Especially, we extend a well-known stability property of continuous flow by R.F, Thomas(1982) to impulsive dynamical system.

1 Impulsive dynamical system

[9] Let X be a metric space and $\mathbb{R}_+$ be the set of real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a semidynamical system, if the continuous function $\pi : X \times \mathbb{R}_+ \to X$ satisfies the conditions:

(a) $\pi(x, 0) = x,\ for\ all\ x \in X$

(b) $\pi(\pi(x,t), s) = \pi(x, t+s),\ for\ all\ x \in X\ and\ t, s \in \mathbb{R}_+$

For fixed $t \in \mathbb{R}_+$, we can define $\pi_t : X \to X$ by $\pi_t(x) = \pi(x,t)$, for $x \in X$. And for fixed $x \in X$, we can also define $\pi_x : \mathbb{R}_+ \to X$ as $\pi_x(t) = \pi(x,t)$, for $t \in \mathbb{R}_+$. Easy to know that π_t is a homeomorphism of X and π_x is a continuous. π_x called the trajectory of x. Let $F(x,t) = \{y : \pi(y,t) = x\}$. We denote the *positive orbit of* x by $\pi^+(x) = \{\pi(x,t) : t \in \mathbb{R}_+\}$.

[9] An impulsive dynamical system $(X, \pi; M, I)$ consists of a semdynamical system, (X, π), a non-empty closed subset M of X and a continuous function $I : M \to X$ such that for every $x \in M$, there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \emptyset \ \ and \ \ \pi(x, (0, \epsilon_x)) \cap M = \emptyset$$

Let $M^+(x) = \pi^+(x) \cap M \setminus \{x\}$, $I(x) = x^+$.

Define $\Phi : X \to (0, +\infty]$

$$\Phi(x) = \begin{cases} s & \pi(x,s) \in M, \pi(x,t) \notin M \text{ for } 0 < t < s \\ +\infty & M^+(x) = \emptyset \end{cases}$$

Existance of s is shown in [Lemma 2.1 [4]].

Given $x \in X$ we define a function $\tilde{\pi}_x : [0, s) \to X$ where $s \in \mathbb{R}_+ \cup \{\infty\}$. This function is called *impulsive trajectory* of x and defined inductively as follows. Set $x = x_0$. If $M^+(x_0) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}_+$. If $M^+(x_0) \neq \emptyset$,then there exists a positive $s_0 \in \mathbb{R}_+$ such that $\pi(x_0, s_0) = x_1 \in M$ and $\pi(x_0, t) \notin M$ for $0 < t < s_0$. We define $\tilde{\pi}_x$ on $[0, s_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x, t) & 0 \leq t < s_0 \\ x_1^+ & t = s_0 \end{cases}$$

We continue the above process starting at x_1^+. If $M^+(x_1) = \emptyset$, then $\tilde{\pi}(t) = \pi(x^+, t - s_0)$ for $t \geq s_0$. If else, $M + (x_1^+) \neq \emptyset$, shich implies the existence of an $s_1 > 0$ as before, and we define

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0) & s_0 \leq t < s_0 + s_1 \\ x_2^+ & t = s_0 + s_1 \end{cases}$$

where $x_2 = \pi(x_1^+, s_0 + s_1)$. Repeating this process, if $M^+(x_n^+) = \emptyset$ for n process ends after a finite number of steps. If else, $M^+(x_n^+) \neq \emptyset,\ b = 1, 2, \ldots$, the process continues indefinitely. However $M^+(x_n^+) \neq \emptyset$ for n, we can get

$$\tilde{\pi}_x(t) = \begin{cases} \pi({x_n}^+, t - t_n) & t_n \leq t < t_{n+1} \\ {x_{n+1}}^+ & t = t_{n+1} \end{cases}$$

Where $\pi({x_n}^+, s_n) = x_{n+1}, \ \ I(x_{n+1}) = {x_{n+1}}^+, \ \ t_n = \sum_{i=0}^{n-1} s_i$

Let$(X, \tilde{\pi}; M, I)$ be an impulsive semidynamical system. If $x \in X$, then

(a) $\tilde{\pi}(x, 0) = x$

(b) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$, with $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$

Definition 1.1 ([8], Definition 3.1.) *Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing $x \in X$ is called a section or a λ-section through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that*

(a) $F(L, \lambda) = S$

(b) $F(L, [0, 2\lambda])$ is neighbourhood of x

(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a tube or λ tube and the set L is called a bar. Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subseteq M \cap F(L, [0, 2\lambda])$ is called TC-tube on x. We say that a point $x \in M$ fulfills the Tube Condition, shortly TC, if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. Also we define $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S = M \cap F(L, [0, 2\lambda])$ is called STC-tube on x. We say that a point $x \in M$ fulfills the Strong Tube Condition, shortly STC, if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

Theorem 1.2 ([8], Theorem 3.4.) *If in an impulsive system $(X, \tilde{\pi}; M, I)$ given by a semidynamical system (X, π) each point contained in the impulse set M satisfies TC, then Φ is upper semicontinuous.*

Theorem 1.3 ([8], Theorem 2.7.) *Assume that $(X, \tilde{\pi}; M, I)$ is an impulsive system given by a dynamical system (X, π). Then for any $x \notin M$ the function Φ is lower semicontinuous at x.*

Proof) Let $x \notin M$ and $\Phi(x) = c \in (0, +\infty)$. The function Φ is lower semicontinuous at x if and only if for any sequence $x_n \to x$ with $\Phi(x_n) \to t$ we have $t \geq c$. Suppose to the contrary that there exists a sequence $p_n \to x$ with $\Phi(p_n) \to t < c$. For n large enough we have $p_n \notin M$ as M is closed. Thus $\pi(\Phi(p_n), p_n) \in M = \overline{M}$. On the other hand, $\pi(\Phi(p_n), p_n) \to \pi(t, x)$, so $\pi(t, x) \in M$, which means that $c = \Phi(x) \leq t < c$ and contradiction. So proof finished. □

Theorem 1.4 ([8], Theorem 3.8.) *Assume that $(X, \tilde{\pi}; M, I)$ is an impulsive system given by a semidynamical system (X, π) and no start point of the system belongs to M. Then:*

(1) if each element of M satisfies TC then Φ is continuous at x if and only if $x \notin M$,

(2) Φ is not lower semicontinuous at x if and only if $x \in M$,

(3) if each element of M satisfies STC then $\Phi|_M$ is lower semicontinuous.

2 Definition of expansiveness, shadowing property and topological stability on Impulsive dynamical system

Let $\phi : X \times \mathbb{R} \to X$ be a flow of compact metric space X.

- Impulsive semidynamical system $(X, \tilde{\pi}; M, I)$ is expansive if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $x, y \in X$ satisfy $d(\tilde{\pi}_t(x), \tilde{\pi}_{s(t)}(y)) < \delta$ $\forall t \in \mathbb{R}$ for $x, y \in X$ and a continuous map $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$, then $y \in F(x, (0, \epsilon)) \cup \tilde{\pi}(x, [0, \epsilon))$

- given $\delta, a > 0$ real numbers, a finite $(\delta, a)-$pseudo orbit is a pair of sequences $(\{x_i\},_0^k, \{t_i\}_0^{k-1})$ so that $t_i \geq a$ and $d(\tilde{\pi}_{t_n}(x_n), x_{n+1}) < \delta$ for $k \in \mathbb{N}$. An infinite (δ, a)-pseudo orbit is a pair of doubly infinite sequences $(\{x_i\},_{-\infty}^{\infty}, \{t_i\}_{-\infty}^{\infty})$ so that $t_i \geq a$ and $d(\tilde{\pi}_{t_n}(x_n), x_{n+1}) < \delta$ for all $i \in \mathbb{Z}$.

- A finite(infinite) (δ, a)-pseudo orbit $(\{x_n\}, \{t_n\})$ is ϵ- traced by the orbit $(\tilde{\pi}_t(z))$ if ther exists an increasing homeomorphis $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\pi}_{t - \sum_{i=0}^{n-1} t_i}(x_n)) < \epsilon \ \ whenever \ \ t \geq 0, \ \sum_0^{n-1} t_i \leq t < \sum_0^{n} t_i \ \ n = 0, 1, 2, ...$$

- Impulsive semidynamical system $(X, \tilde{\pi}; M, I)$ is said to have the *pseudo orbit tracing property, POTP*, if for all $\epsilon > 0$ there exists $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit is ϵ-traced by an orbit of $\tilde{\pi}$. This property also called as the shadowing property.

- Impulsive dynamical system $(X, \tilde{\pi}; M, I)$ is said to be topologically stable if for any $\epsilon > 0$ there is $\delta > 0$ such that for every other Impulsive dynamical system (X, ψ, M, I) with $d(\tilde{\phi_t}, \tilde{\psi}_{s(t)}) < \delta$ for all $t \in [0, 1]$, for some continuous map s from $[0, 1]$ to itself. there exists $h : X \to X$ continuous such that $d(h, id) < \epsilon$ and $h(orbit\ of\ \tilde{\psi}) \subset orbit\ of\ \tilde{\phi}$.

3 Properties on impulsive dynamical system

[9] Troughout this section we shall consider an impulsive semidynamical system $(X, \tilde{\pi};; M, I)$, where (X, d) is a metic space. Moreover, we shall assume the following additional hypotheses:

(H1) No initial point in (X, π) velongs to the impulsive set M and each element of M satisfies the condition(STC), consiquently Φ is continuous on $X \setminus M$ and $\Phi|_M$ is continuous on M.

(H2) $M \cap I(M) = \emptyset$

(H3) For each $x \in X$, the motion $\pi(\tilde{x}, t)$ is defined for every $t \geq 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\tilde{\pi}_x$

Proposition 3.1 *If $(X, \tilde{\pi}; M, I)$ be an expansive Impulsive dynamical system, then each fixed point of $\tilde{\pi}$ is an isolated point of X.*

Proof) Since $\tilde{\pi}$ is expansive, there is a constant δ for $\epsilon = 1$ by definition of expansive. If $d(x, y) < \delta$, then putting $s(t) = 0$, $\forall t \in \mathbb{R}$, we have $d(\tilde{\pi}_t(x), \tilde{\pi}_{s(t)}(y)) < \delta\ \forall t \in \mathbb{R}$. By the fact that $\tilde{\pi}_t(x) = x\ \forall t \in \mathbb{R}$ hence $y \in F(x, (0, 1)) \cup \tilde{\pi}(x, [0, 1)) = \{x\}$, so $y = x$.

From now on we shall always assume $\tilde{\pi}$ has no fixed points.

Lemma 3.2 ([6] Corollary 3.9) *Let $\{x_n\}$ be a sequence in X which converges to $x \in X \setminus M$. Then, given $t \geq 0$, there is a sequence $\{\epsilon_n\} \subset \mathbb{R}_+$ such that $\epsilon_n \to 0$ as $n \to \infty$ and $\tilde{\pi}(x_n, t + \epsilon_n) \to \tilde{\pi}(x, t)$.*

Lemma 3.3 ([7] Lemma 2) *Let $(X, \tilde{\pi}; M, I)$ be an impulsive dynamical system and π has no fixed point. There is $T_0 > 0$ such that, for $0 < T < T_0$ there is $\gamma > 0$ such that $d(\tilde{\pi}_T(x), x) > \gamma$ for all $x \in X \setminus M$.*

Proof) Let $x \in X \setminus M$ be given. Let T_0 be the smallest positive number satisfying $\tilde{\pi}_{T_0}(x) = x$. To contrary we suppose that there is $t \in (0, T_0)$ such that for given $n \in \mathbb{N}$ there exists $z_n \in X$ such that

$$d(\tilde{\pi}_t(z_n), z_n) < 1/n$$

for all $n \in \mathbb{N}$. Since X is compact, we may assume that $z_n \to x$, as $n \to \infty$. M is closed set, so we can insist on $z_n \in X \setminus M$ for all $n \in \mathbb{N}$. Think about three cases below.

Case (1) $t < \Phi(x)$.
Let $\epsilon > 0$ be given. There is $N_1 \in \mathbb{N}$ such that $d(z_n, x) < \epsilon/4$ whenever $n > N_1$. Take $N_2 \in \mathbb{N}$ satisfying $1/n < \epsilon/4$ whenever $n > N_2$. By Lemma 3.2 there is a sequence $\{\epsilon_n\} \subset \mathbb{R}_+$ such that

$$\epsilon_n \to 0 \ \text{ as } \ n \to \infty \ \text{ and } \ \tilde{\pi}_{t+\epsilon_n}(z_n) \to \tilde{\pi}_t(x).$$

Hence there is $N_3 \in \mathbb{N}$ such that $d(\tilde{\pi}_{t+\epsilon_n}(z_n), \tilde{\pi}_t(x)) < \epsilon/4$, for all $n > N_3$. Since $\pi(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is continuous for all $x \in X$. Since Φ is continuous, there is $N_4 > 0$ such that

$$|\Phi(z_n) - \Phi(x)| < \frac{\Phi(x) - t}{2},$$

whenever $n > N_4$. Since $\epsilon_n \to 0$ there is $N_5 \in \mathbb{N}$ such that if $n > N_5$ then $\epsilon_n < \frac{\Phi(x)-t}{2}$. Take $N = \max\{N_i : i = 1, \ldots, 5\}$. Then we have

$$\begin{aligned}
d(\tilde{\pi}_t(x), x) &< d(\tilde{\pi}_t(x), \tilde{\pi}_t(z_n)) + d(\tilde{\pi}_t(z_n), z_n) + d(z_n, x) \\
&< d(\tilde{\pi}_t(x), \tilde{\pi}_{t+\epsilon_n}(z_n)) + d(\tilde{\pi}_{t+\epsilon_n}(z_n), \tilde{\pi}_t(z_n)) + d(\tilde{\pi}_t(z_n), z_n) + d(z_n, x) \\
&< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 \\
&< \epsilon
\end{aligned}$$

Case (2) $t > \Phi(x)$.
In this case, there exists $m \in \mathbb{N}$ such that $t = \sum_{i=0}^{m-1} \Phi(x_i^+) + t'$ with $0 \leq t' < \Phi(x_m^+)$. Define $(z_n)_i$ by

$$(z_n)_1 = \pi_{\Phi(z_n)}(z_n) \ \text{ and } \ (z_n)_{i+1} = \pi_{\Phi((z_n)_i^+)}((z_n)_i^+), \quad i = 1, \ldots, m-1$$

Set $t_n = \sum_{i=0}^{m-1} \Phi((z_n)_i^+)$. Since $\Phi(z_n) \to \Phi(x)$, we have

$$(z_n)_1^+ = I((z_n)_1) \to I(x_1) = x_1^+.$$

Now, since $\Phi((z_n)_1^+) \to \Phi(x_1^+)$, because $x_1^+ \notin M$, we get

$$(z_n)_2 = \pi_{\Phi(z_n)_1^+}(z_n)_1^+ \to \pi_{\Phi(x_1^+)}(x_1^+) = x_2.$$

By continuing with this process, we obtain

$$(z_n)_i \to x_i \text{ and } (z_n)_i^+ \to x_i^+, \text{for } \ \text{all } i = 1, \ldots, m.$$

Thus $\sum_{i=0}^{m-1} \Phi((z_n)_i^+) \to \sum_{i=0}^{m-1} \Phi(x_i^+)$. If we define $t'_n = t - \sum_{i=0}^{m-1} \Phi((z_n)_i^+)$ then $t'_n \to t'$. Let $\epsilon > 0$ be given. There is $N_1 \in \mathbb{N}$ such that $d(z_n, x) < \epsilon/4$ whenever $n > N_1$. Take $N_2 \in \mathbb{N}$ satisfying $1/n < \epsilon/4$ whenever $n > N_2$. By Lemma 3.2, there is a sequence $\{\epsilon_n\} \subset \mathbb{R}_+$ such that

$$\epsilon_n \to 0 \ \text{ as } \ n \to \infty \ \text{ and } \ \tilde{\pi}_{t+\epsilon_n}(z_n) \to \tilde{\pi}_t(x).$$

Hence there is $N_3 \in \mathbb{N}$ such that $d(\tilde{\pi}_{t+\epsilon_n}(z_n), \tilde{\pi}_t(x)) < \epsilon/4$, for all $n > N_3$. Since $\pi(\centerdot, x) : \mathbb{R} \to \mathbb{R}$ is continuous for all $x \in X$. Since Φ is continuous, there is $N_4 > 0$ such that

$$\left| \sum_{i=0}^{m} \Phi((z_n)_i^+) - \sum_{i=0}^{m} \Phi(x_i^+) \right| < \frac{\sum_{i=0}^{m} \Phi(x_i^+) - t}{2}$$

whenever $n > N_4$. Since $\epsilon_n \to 0$ there is $N_5 \in \mathbb{N}$ such that

$$\epsilon_n < \frac{\sum_{i=0}^{m} \Phi(x_i^+) - t}{2}$$

whenever $n > N_5$. Take $N = \max\{N_i : i = 1, \ldots, 5\}$. Then

$$\epsilon_n < \frac{\sum_{i=0}^{m} \Phi(x_i^+) - t}{2} < \frac{\sum_{i=0}^{m} \Phi((z_n)_i^+) - t}{2} < \sum_{i=0}^{m} \Phi((z_n)_i^+) - t$$

and

$$\begin{aligned}
d(\tilde{\pi}_t(x), x) &< d(\tilde{\pi}_t(x), \tilde{\pi}_t(z_n)) + d(\tilde{\pi}_t(z_n), z_n) + d(z_n, x) \\
&< d(\tilde{\pi}_t(z_n), \tilde{\pi}_{t+\epsilon_n}(z_n)) + d(\tilde{\pi}_{t+\epsilon_n}(z_n), \tilde{\pi}_t(x)) + d(\tilde{\pi}_t(z_n), z_n) + d(z_n, x) \\
&< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 \\
&< \epsilon
\end{aligned}$$

So that $\tilde{\pi}_t(x) = x$ and contradicting the choice of T_0.

Case (3) $t = \Phi(x)$.
Since $z_n \to x$ and $d(\tilde{\pi}_t(z_n), z_n) \to 0$ as $n \to \infty$, we get $d(x, \tilde{\pi}_t(z_n)) \to 0$ as $n \to \infty$. Also, from the continuity of Φ, we obtain $\Phi(z_n) \to t$ as $n \to \infty$. Then we have $d(\tilde{\pi}_t(z_n), M) \to 0$ as $n \to \infty$. So

$$d(x, M) \leq d(x, \tilde{\pi}_t(z_n)) + d(\tilde{\pi}_t(z_n), M).$$

Letting $n \to \infty$, we get $d(x, M) = 0$. This is a contradiction to $x \in X \setminus M$.

Lemma 3.4 *Let (X, π, M, I) be an expansive impulsive dynamical system such that $I(M) \cap M = \emptyset$ where π is continuous flow without fixed point. For given $\epsilon > 0$ there is r, if $(t_i), (u_i)$ be a sequence of $\mathbb{R}_+$ satisfying $t_i \to \infty$ as $i \to \infty$, $u_0 = t_0 = 0$, $0 < t_{i+1} - t_i \leq r$, $0 < u_{i+1} - u_i \leq r$, $d(\tilde{\pi}_{t_i}(x), \tilde{\pi}_{u_i}(y)) < r$ for $x, y \in X$ then $y \in F(x, (0, \epsilon)) \cup \tilde{\pi}(x, [0, \epsilon))$.*

Proof) Let $\epsilon > 0$ be given. Since $I(M)$ is compact and $I(M) \cap M = \emptyset$, by [6] Remark 1.6 (2), there exists $\xi > 0$ such that $\Phi(z) \geq \xi$ for all $z \in I(M)$. Choose r so that $0 < r < \xi$ and

$$r + 2 \sup_{z \in X |u| \leq r} d(z, \pi_u(z)) < \delta$$

where δ corresponds to ϵ by expansiveness. Let $(t_i), (u_i)$and $x, y \in X$ satisfy the hypotheses. Define $s : \mathbb{R} \to \mathbb{R}$ by $s(t_i) = u_i$. Since $r < \xi$, neither $\tilde{\pi}(x, (t_i, t_{i+1})) \cap I(M)$ nor

$\tilde{\pi}(y,(u_i,u_{i+1})) \cap I(M)$ can not have more than one point. So we can think the three case below.

Case1) If $\tilde{\pi}(x,(t_i,t_{i+1})) \cap I(M) = \emptyset$ and $\tilde{\pi}(y,(u_i,u_{i+1})) \cap I(M) = \emptyset$, then extending s linearly on each interval $[t_i,t_{i+1})$. If $t \in [t_i,t_{i+1})$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_i}(x)) + d(\tilde{\pi}_{t_i}(x),\tilde{\pi}_{u_i}(y)) + d(\tilde{\pi}_{u_i}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

Case2) If $\tilde{\pi}(x,(t_i,t_{i+1})) \cap I(M) \neq \emptyset$ and $\tilde{\pi}(y,(u_i,u_{i+1})) \cap I(M) = \emptyset$ then extending s linearly on each interval $[t_i,t_{i+1}]$. If $t \in [t_i,t')$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_i}(x)) + d(\tilde{\pi}_{t_i}(x),\tilde{\pi}_{u_i}(y)) + d(\tilde{\pi}_{u_i}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

If $t \in [t',t_{i+1})$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_{i+1}}(x)) + d(\tilde{\pi}_{t_{i+1}}(x),\tilde{\pi}_{u_{i+1}}(y)) + d(\tilde{\pi}_{u_{i+1}}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

Case3) If $\tilde{\pi}(x,(t_i,t_{i+1})) \cap I(M) = \emptyset$ and $\tilde{\pi}(y,(u_i,u_{i+1})) \cap I(M) \neq \emptyset$ then extending s linearly on each interval $[t_i,t_{i+1}]$. If $t \in [t_i,s^{-1}(u'))$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_i}(x)) + d(\tilde{\pi}_{t_i}(x),\tilde{\pi}_{u_i}(y)) + d(\tilde{\pi}_{u_i}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

If $t \in [s^{-1}(u'),t_{i+1})$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_{i+1}}(x)) + d(\tilde{\pi}_{t_{i+1}}(x),\tilde{\pi}_{u_{i+1}}(y)) + d(\tilde{\pi}_{u_{i+1}}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

Case4) If $\tilde{\pi}(x,(t_i,t_{i+1})) \cap I(M) \neq \emptyset$ and $\tilde{\pi}(y,(u_i,u_{i+1})) \cap I(M) \neq \emptyset$, then we set $s(t') = u'$ and e extending s linearly on each interval $[t_i,t')$ and $[t',t_{i+1})$. If $t \in [t_i,t')$, then

$$\begin{aligned} d(\tilde{\pi}_t(x),\tilde{\pi}_{s(t)}(y)) &\leq d(\tilde{\pi}_t(x),\tilde{\pi}_{t_i}(x)) + d(\tilde{\pi}_{t_i}(x),\tilde{\pi}_{u_i}(y)) + d(\tilde{\pi}_{u_i}(y),\tilde{\pi}_{s(t)}(y)) \\ &\leq r + 2 \sup_{z\in X|u|\leq r} d(z,\tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

If $t \in [t', t_{i+1})$

$$\begin{aligned} d(\tilde{\pi}_t(x), \tilde{\pi}_{s(t)}(y)) &\le d(\tilde{\pi}_t(x), \tilde{\pi}_{t_{i+1}}(x)) + d(\tilde{\pi}_{t_{i+1}}(x), \tilde{\pi}_{u_{i+1}}(y)) + d(\tilde{\pi}_{u_{i+1}}(y), \tilde{\pi}_{s(t)}(y)) \\ &\le r + 2 \sup_{z \in X |u| \le r} d(z, \tilde{\pi}_u(z)) \\ &< \delta. \end{aligned}$$

4 Main theorem

Theorem 4.1 *Let $(X, \tilde{\pi}; M, I)$ be an expansive impulsive dynamical system without fixed point and $I(M) \cap M = \emptyset$. If (X, π, M, I) has P.O.T.P then topologically stable.*

Proof) For given $\epsilon > 0$, we take T_0 as in Lemma 3.3 and take r as in Lemma 3.4. Whitout loss of generality, we can assume that $0 < \epsilon < T_0/2$ and $0 < r < \epsilon$. Using Lemma 3.3, there is $\gamma > 0$ so that $d(\tilde{\pi}_r(y), y) \ge \gamma$ for all $y \in X$. From definition of expansiveness we also take $\epsilon' < min\{\gamma, r\}$ and if $d(\tilde{\pi}_{s(t)}(x), \tilde{\pi}_t(y)) \le \epsilon'$ for all $x, y \in X$ and a continuous map $s : \mathbb{R}_+ \to \mathbb{R}_+$ with $s(0) = 0$, then $y = \tilde{\pi}_t(x)$ with $|t| \le r$. Also choose $0 < \delta < \epsilon'/12$ so that every $(\delta, 1)$-pseudo orbit is $\epsilon'/12$-traced by an orbit of $\tilde{\pi}$. Assume (X, ψ, M, I) be another impulsive dynamical system with $d(\tilde{\pi}_t, \tilde{\psi}_t) < \delta$ for all $t \in [0, 1]$. Note that $d(\tilde{\pi}_t, \tilde{\psi}_t) = \sup_{x \in X} d(\tilde{\pi}_t(x), \tilde{\psi}_t(x))$. Now we fix a point $y \in X$. Since $d(\tilde{\pi}_1 \tilde{\psi}_n(y), \tilde{\psi}_{n+1}(y)) < \delta$ for all $n \in \mathbb{Z}$, therefore $(\tilde{\psi}_n(y), t_n = 1)$ is a $(\delta, 1)$-pseudo orbit for $\tilde{\pi}$. Using the definition of shadowing property, there exists $z \in X$ and $\alpha : \mathbb{R} \to \mathbb{R}$ increasing homeomorphism with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\pi}_{t-n}(\tilde{\psi}_n(y)) < \epsilon'/12 \quad \text{whenever} \quad n \le t < n+1, \quad n = 0, 1, 2, \cdots.$$

For the case when $n \le t < n+1$,

$$\begin{aligned} d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\psi}_t(y)) &\le d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\pi}_{t-n}(\tilde{\psi}_n(y))) + d(\tilde{\pi}_{t-n}(\tilde{\psi}_n(y)), \tilde{\psi}_{t-n}(\tilde{\psi}_n(y))) \\ &< \epsilon'/12 + \epsilon'/12 = \epsilon'/6. \end{aligned}$$

So this means that we proved here for an orbit $(\psi_t(y))$ there exists a point $z \in X$ and increasing homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\psi}_t(y)) < \epsilon'/6 \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Assume $z' \in X$ and $\alpha' : \mathbb{R} \to \mathbb{R}$ increasing homeomorphism with $\alpha'(0) = 0$ and such that we have also

$$d(\tilde{\pi}_{\alpha'(t)}(z'), \tilde{\psi}_t(y)) < \epsilon'/6 \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Then $d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\pi}_{\alpha'(t)}(z')) < \epsilon'/3 \quad for\ all\ t \in \mathbb{R}_+$. Using expansiveness and the way we choose ϵ' implies $z' = \pi_t(z)$ with $|t| \le r < \epsilon$. It means that we proved every orbit of ψ is $\epsilon'/6$-traced by a unique orbit of π. Now for $y \in X$ define the set A_y as follows
$A_y = \{x \in X :$ for every η, $T > 0$, there exists a homeomorphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ such that $d(\tilde{\pi}_{\alpha(t)}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta$ for all $t \in [0, T]\}$. We know that $(\psi_t(y))$ is uniquely traced by an orbit of π say $(\pi_t(z))$ we want to show

(1) $A_y \subseteq (\pi_t(z))$ and time diameter of $A_y < \epsilon$,

(2) A_y is a closed set in Y.

To prove (1) As we know $d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\psi}_t(y)) < \epsilon'/6$ for all $t \in \mathbb{R}_+$ so $z \in A_y$. Assume $\{\eta_i\}, \{T_i\}$ are sequences of positive real numbers such that $\eta \to 0$, $T_i \to \infty$. If $x \in A_y$, there are $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ homeomorphisms with $\alpha_i(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha_i(t)}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta_i \quad \text{for} \quad t \in [0, T_i]$$

Using $d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\psi}_t(y)) \leq \epsilon'/6$ for all $t \in \mathbb{R}_+$ we have

$$d(\tilde{\pi}_{\alpha_i(t)}(x), \tilde{\pi}_{\alpha(t)}(z)) < \epsilon'/3 + \eta \quad \text{for } t \in [0, T_i]$$

Since $\eta_i \to 0$, WLOG assume $\eta_i < \epsilon'/6$ for all i. Take $T_i' = \alpha(T_i)$. It is clear that $T_i' \to \infty$, so

$$d(\tilde{\pi}_{\alpha_i\alpha^{-1}(u)}(x), \tilde{\pi}_u(z)) < \epsilon'/3 + \eta_i \quad \text{for} \quad \text{all} \quad u \in [0, T_i'].$$

Denote $\gamma_i = \alpha_i\alpha^{-1}$ it is clear $\gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing homeomorphism with $\gamma_i(0) = 0$ for all i. By continuity of γ_i choose $0 < s_i < n$ such that $|u - u'| < s_i$ implies $|\gamma_i(u) - \gamma_i(u')| < r$. Now,

$$\begin{aligned} d(\tilde{\pi}_{\gamma_{i+1}(u)-\gamma_i(u)}\tilde{\pi}_{\gamma_i(u)}(x), \tilde{\pi}_{\gamma_i(u)}(x)) &= d(\tilde{\pi}_{\gamma_{i+1}(u)}(x), \tilde{\pi}_{\gamma_i(u)}(x)) \\ &\leq d(\tilde{\pi}_{\gamma_{i+1}(u)}(x), \tilde{\pi}_u(z)) + d(\tilde{\pi}_u(z), \tilde{\pi}_{\gamma_i(u)}(x)) \\ &< \epsilon'/3 + \eta_{i+1} + \epsilon'/3 + \eta_i < \epsilon' \end{aligned}$$

for all $u \in [0, T_i']$. Let $\gamma_{i+1}(u) - \gamma_i(u) = \Gamma(u)$ then $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is homeomorphism and $\Gamma(0) = 0$.

Since $\epsilon' < \gamma$, $\Gamma(u) \neq r$ for all $u \in [0, T_i']$. Suppose that there is $s \in [0, T_i']$ such that $|\gamma_{i+1}(s) - \gamma_i(s)| > r$. Then there is $k \in [0, T_i']$ such that $|\gamma_{i+1}(k) = \gamma_i(k)| = r$ by the continuity of Γ. But this is a contradiction. Hence we have $|\gamma_{i+1}(u) - \gamma_i(u)| < r$ for all $u \in [0, T_i']$.

So we can take $0 < \xi_i < r$ such that $u \leq T_i' < u'$ and $|u' - u| < \xi_i$ imply $|\gamma_{i+1}(u') - \gamma_i(u)| < |\gamma_{i+1}(u') - \gamma_{i+1}(u)| + |\gamma_{i+1}(u) - \gamma_i(u)| < r$. Fix i and define a sequence (u_j) of real numbers such that $u_0 = 0, u_j < u_{j+1},\ u_j \to \infty$ as $j \to \infty$, and $u_{j+1} - u_j < \min\{\xi_i, s_i\}$ if $u_j \in [T_i', T_{i+1}']$ for $j \geq 0$. It follows that if we take $t_j = \gamma_i(u_j)$ if $u_j \in [T_i', T_{i+1}]$ and so on that $t_{j+1} - t_j < r$ for all $j \in \{0\} \cup \mathbb{N}$ and $t_0 = u_0 = 0$ and

$$d(\tilde{\pi}_{t_j}(x), \tilde{\pi}_{u_j}(z)) < \epsilon' < r \quad \text{for all } j \in \{0\} \cup \mathbb{N}$$

Using Lemma 3.5 we have that $x = \tilde{\pi}_t(z)$ with $|t| < \epsilon$ and this proves (1).

Now to prove (2) because of (1) we only need to show that A_y is closed in the orbit $(\tilde{\pi}_t(z))$ with relative topology. Let z' be a limit point of A_y and assume $z' \in (\tilde{\pi}_t(z))$. Take $\{z_i\}$ be a sequence in A_y such that $z_i \to z'$. Given $\eta, T > 0$ there are $\alpha_i : \mathbb{R} \to \mathbb{R}$ homeomorphisms with $\alpha_i(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta/2 \text{ for all } t \in [0, T],\ \text{all } i \in \mathbb{N}$$

There is ξ such that if $|t-t'|<\xi$ then $d(\pi_{t'}(z'),\pi_t(z'))<\eta/2$. Since all z_i and z' are in the same orbit, WLOG we can suppose $\tilde{\pi}_{\Phi(z_i)}(z_i)=\tilde{\pi}_{\Phi(z')}(z')$ and $\Phi(z_i)\geq\Phi(z')$ or $\Phi(z_i)<\Phi(z')$ for all $i\in\mathbb{N}$.
Case1) $\Phi(z_i)\geq\Phi(z')$ for all $i\in\mathbb{N}$.
Let $\Phi(z')+\tau_i=\Phi(z_i)$ then $\tilde{\pi}_{\tau_i}(z_i)=z'$. There is sufficiently large $i\in\mathbb{N}$ such that $\tau_i<\min\{\xi,\Phi(z')\}$. We can pick the $t_i>0$ such that $\alpha_i(t_i)=2\tau_i$. Define $\beta:\mathbb{R}_+\to\mathbb{R}_+$ by

$$\beta(t)=\begin{cases}\frac{\tau_i}{t_i}t & 0\leq t<t_i\\ \alpha_i(t)-\tau_i & t\geq t_i\end{cases}$$

Then β is a homeomorphism and $\beta(0)=0$.

Let $0\leq t<t_i$. Then $\left|\frac{\tau_i}{t_i}t-\{\alpha_i(t)-\tau_i\}\right|\leq\tau_i<\xi$. Since $\tau_i<\alpha_i(t)$, we can write down

$$\tilde{\pi}_{\alpha_i(t)}(z_i)=\tilde{\pi}_{\alpha_i(t)-\tau_i}(z').$$

Since $\frac{\tau_i}{t_i}t<\tau_i<\Phi(z')$, $\alpha_i(t)-\tau_i<\tau_i<\Phi(z')$. So we know that $\pi_{[\tilde{\pi}_{\frac{\tau_i}{t_i}t}(z'),\tilde{\pi}_{\alpha_i(t)-\tau_i}(z')]}\cap M=\emptyset$. So we can get

$$\begin{aligned}d(\tilde{\pi}_{\beta(t)}(z'),\tilde{\psi}_t(y))&\leq d(\tilde{\pi}_{\beta(t)}(z'),\tilde{\pi}_{\alpha_i(t)}(z_i))+d(\tilde{\pi}_{\alpha_i(t)}(z_i),\tilde{\psi}_t(y))\\&=d(\tilde{\pi}_{\frac{\tau_i}{t_i}t}(z'),\tilde{\pi}_{\alpha_i(t)-\tau_i}(z'))+\epsilon'/6+\eta/2\\&\leq\eta+\epsilon'/6.\end{aligned}$$

If $t\geq t_i$ we have,

$$\begin{aligned}d(\tilde{\pi}_{\beta(t)}(z'),\tilde{\psi}_t(y))&=d(\tilde{\pi}_{\alpha_i(t)-\tau_i}(z'),\tilde{\psi}_t(y))\\&=d(\tilde{\pi}_{\alpha_i(t)}(z_i),\tilde{\psi}_t(y))\\&\leq\eta/2+\epsilon'/6\leq\eta+\epsilon'/6.\end{aligned}$$

Case2) $\Phi(z_i)<\Phi(z')$ for all $i\in\mathbb{N}$.
Let $\Phi(z_i)+\tau_i=\Phi(z')$ then $\tilde{\pi}_{\tau_i}(z')=z_i$. there is sufficiently big $i\in\mathbb{N}$ such that $2\tau_i<\min\{\Phi(z_i),\xi\}$.We can pick the $t_i>0$ such that $\alpha_i(t_i)=\Phi(\tau_i)$. Define $\beta:\mathbb{R}_+\to\mathbb{R}_+$

$$\beta(t)=\begin{cases}\frac{2\tau_i}{t_i}t & 0\leq t<t_i\\ \alpha_i(t)+\tau_i & t\geq t_i\end{cases}$$

Then β is homeomorphism and $\beta(0)=0$. If $0\leq t<t_i$ then $|\frac{2\tau_i}{t_i}t-\{\alpha_i(t)+\tau_i\}|\leq2\tau_i<\xi$. If $0\leq t<t_i$

$$\begin{aligned}d(\tilde{\pi}_{\beta(t)}(z'),\tilde{\psi}_t(y))&\leq d(\tilde{\pi}_{\beta(t)}(z'),\tilde{\pi}_{\alpha_i(t)}(z_i))+d(\tilde{\pi}_{\alpha_i(t)}(z_i),\tilde{\psi}_t(y))\\&=d(\tilde{\pi}_{\frac{2\tau_i}{t_i}t}(z'),\tilde{\pi}_{\alpha_i(t)+`\tau_i}(z'))+\epsilon'/6+\eta/2\\&\leq\eta+\epsilon'/6.\end{aligned}$$

If $t \geq t_i$ we have,

$$\begin{aligned} d(\tilde{\pi}_{\beta(t)}(z'), \tilde{\psi}_t(y)) &= d(\tilde{\pi}_{\alpha_i(t)+\tau_i}(z'), \tilde{\psi}_t(y)) \\ &= d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\psi}_t(y)) \\ &\leq \eta/2 + \epsilon'/6 \leq \eta + \epsilon'/6. \end{aligned}$$

So $z' \in A_y$ and A_y is closed.
In order to be able to define a function h on X we need to select a certain point in the set A_y in this way. We denote the largest limit point of A_y to $L(A_y)$."A point $x \in A_y$ is called a largest limit of $A_y(l.l.A_y)$ if and only if $x = \tilde{\pi}_w(x')$ with $w \geq 0$ for all $x' \in A_y$". This point $x = l.l.A_y$ is the unique point in A_y with this property. Define $h : X \to X$ by

$$h(y) = l.l.A_y \quad \text{for} \quad \text{all} \quad y \in X.$$

By uniqueness of $l.l.A_y$ this function is well defined, from definition of A_y, if $x \in A_y$ we have for every $\eta, T > 0$, there is a homeomorphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta \text{ for all } t \in [0, T]$$

If we take $\eta < \epsilon'/2$, then at $t = 0$, $d(x, y) < \epsilon' < \epsilon$. And this means that $d(y, l.l.A_y) < \epsilon$ so that $d(h, I) < \epsilon$. Now to show $h(\text{orbit of } \tilde{\psi}) \subseteq (\text{orbit of } \tilde{\pi})$. Let $y \in X$, as we show above the orbit $(\tilde{\psi}_t(y))$ is $\epsilon'/6 - \textit{traced}$ by a nuique orbit of $\tilde{\pi}$ say $(\tilde{\pi}_t(z))$ (i.e., $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ increasing homeomorphism with $\alpha(0) = 0$ such that $d(\tilde{\pi}_{\alpha(t)}(z), \tilde{\psi}_t(y)) \leq \epsilon'/6$ for all $t \in \mathbb{R}_+$.) Let $\tilde{\psi}_u(y)$ is point of the orbit $(\tilde{\psi}_t(y))$. Obviously $d(\tilde{\pi}_{\alpha(t+u)-\alpha(u)}\tilde{\pi}_{\alpha(u)}(z), \tilde{\psi}_t\tilde{\psi}_u(y))) \leq \epsilon'/6$ for all $t \in \mathbb{R}_+$. Now if we take $\gamma(t) = \alpha(t+u) - \alpha(u)$, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing homeomorphism with $\gamma(0) = 0$ (i.e., a reparametrization for the orbit $\tilde{\pi}_t(z)$). Therefore $\tilde{\pi}_{\alpha(u)}(z) \in A_{\tilde{\psi}_u(y)}$. So $h(\tilde{\psi}_u(y)) = l.l.A_{\tilde{\psi}_u(y)} \in A_{\tilde{\psi}_u(y)} \subset (\tilde{\pi}_t(z^*))$ for some $z^* \in X$. And this means that $h(\text{orbit of } \tilde{\psi}) \subseteq (\text{orbit of } \tilde{\pi})$.
To show h is continuous, let $\eta, T > 0, y \in X$ and define $A_{y,\eta,T} = \{x$: there exists a homeomerphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ such that $d(\tilde{\pi}_{\alpha(t)}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta$ for all $t \in [0, T]\}$.
Properties of such sets are

(a) $0 < \eta_2 \leq \eta_1 \Rightarrow A_{u,\eta_2,T} \subseteq A_{y,\eta_1,T}$for all $y \in X, T \in \mathbb{R}_+$

(b) $0 < T_1 \leq T_2 \Rightarrow A_{Y,\eta,T_2} \subseteq A_{y,\eta,T_1}$ for all $y \in X, T \in \mathbb{R}_+$

(c) $0 < \eta_2 \leq \eta_1$ and $0 < T_1 \leq T_2 \Rightarrow A_{y,\eta_2,T_2} \subseteq A_{y,\eta_1,T_1}$ for all $y \in X$

(d) if $\{\eta_i\}, \{T_i\}$ are sequences of positive real numbers with $\eta_i \to 0$ and $T_i \to \infty$ as $i \to \infty$, then $A_y = \bigcap_i A_{y,\eta_i,T_i}$

Lemma 4.2 *For every $\lambda > 0, y \in X \setminus M$ there are $\eta, T > 0$ such that $d(x, A_y) < \lambda$ for all $x \in A_{y,\eta,T}$*

Proof) Let $\{\eta_i\}, \{T_i\}$ be sequences in $\mathbb{R}_+$ satisfying $\eta_i \to 0, T_i \to \infty$ as $i \to \infty$. Assume that $\eta_i < \epsilon'/6$ for all $i \in \mathbb{N}$. Let $\{z_i\}$ be a sequence of points with $z_i \to x$ as $i \to \infty$, where

$z_i \in A_{y,\eta_i,T_i}$ for each $i \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, there is a homeomorphisms $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha_i(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta_i \text{ for all } \ t \in [0, T_i], i \in \mathbb{N}$$

Since $z_i \to x$, one can take sequences $\{\beta_i\}, \{w_i\}$ in $\mathbb{R}_+$ with $\beta_i \to 0, w_i \to \infty$ as $i \to \infty$ such that

$$d(\tilde{\pi}_t(z_i), \tilde{\pi}_t(x)) < \beta_i \ \text{ for } t \in [0, w_i].$$

Since $y \in X \setminus M$ and $d(z_i, y) < \epsilon'/6 + \eta_i$, there is $N \in \mathbb{N}$ such that $z_i \in X \setminus M$ whenever $i > N$. So $\Phi((z_i)_n^+) \to \Phi(x_n^+)$. Define $\gamma_i : \mathbb{R}_+ \to \mathbb{R}_+$ by $\gamma_i\left(\sum\limits_{n=0}^{k} \Phi((z_i)_n^+)\right) = \sum\limits_{n=0}^{k} \Phi(x_n^+)$ for $t = \sum\limits_{n=0}^{k} \Phi((z_i)_n^+)$ otherwise γ_i is defined linearly for $t \neq \sum\limits_{n=0}^{k} \Phi((z_i)_n^+)$. Then we have $\gamma_i(0) = 0$, $\gamma_i \to id$ as $i \to \infty$ and $d(\tilde{\pi}_t(z_i), \tilde{\pi}_{\gamma_i(t)}(x)) < \beta_i$ for $t \in [0, w_i]$. We first show that $\gamma_i \to id$ as $i \to \infty$. Indeed, since for each $n \in \mathbb{N}$, $\Phi((z_i)_n^+) \to \Phi(x_n^+)$ as $i \to \infty$ we get $\sum\limits_{n=0}^{k} \Phi((z_i)_n^+) \to \sum\limits_{n=0}^{k} \Phi(x_n^+)$ as $i \to \infty$. Since γ_i is defined linearly for $t \neq \sum\limits_{n=0}^{k} \Phi((z_i)_n^+)$, $\gamma_i(t) = \frac{b_{k+1} - b_k}{a_{k+1} - a_k}(t - a_k) + b_k$ where $a_k = \sum\limits_{n=0}^{k} \Phi((z_i)_n^+)$, $b_k = \sum\limits_{n=0}^{k} \Phi(x_n^+)$. Note that $d(\gamma_i, id) = \sup_{l \in \mathbb{R}_+} d(\gamma_i(l), l)$. From the fact that $a_k \to b_k$ and $a_{k+1} \to b_{k+1}$ as $i \to \infty$, we get

$$\begin{aligned}
\lim_{i\to\infty} d(\gamma_i, id) &= \lim_{i\to\infty} d\left(\frac{b_{k+1} - b_k}{a_{k+1} - a_k}(l - a_k) + b_k, l\right) \\
&= \lim_{i\to\infty} d(l, l) \\
&= 0
\end{aligned}$$

Obviously $\gamma_i \to id$ as $i \to \infty$.

Now we show that $d(\tilde{\pi}_t(z_i), \tilde{\pi}_{\gamma_i(t)}(x)) < \beta_i$ for $t \in [0, w_i]$.

Case1) $t = \sum\limits_{n=0}^{k} \Phi((z_i)_n^+)$ for some $k \in \mathbb{N} \cup \{0\}$.

Then $\gamma_i(t) = \sum\limits_{n=0}^{k} \Phi(x_n^+)$. Since $(z_i)_{k+1} \to x_{k+1}$ as $i \to \infty$, there is N such that if $i > N$ then $d((z_i)_{k+1}^+, x_{k+1}^+) < \beta_i$. So we get that

$$\begin{aligned}
d\left(\tilde{\pi}_t(z_i), \tilde{\pi}_{\gamma_i(\alpha_i(t))}(x)\right) &= d\left(\tilde{\pi}_{\sum_{n=0}^{k} \Phi((z_i)_n^+)}(z_i), \tilde{\pi}_{\sum_{n=0}^{k} \Phi(x_n^+)}(x)\right) \\
&= d\left((z_i)_{k+1}^+, x_{k+1}^+\right) < \beta_i
\end{aligned}$$

for $t \in [0, w_i]$ and $i > N$.

Case2) $\sum\limits_{n=0}^{k} \Phi((z_i)_n^+) < t < \sum\limits_{n=0}^{k+1} \Phi((z_i)_n^+)$ for some $k \in \mathbb{N} \cup \{0\}$.

In this case, $\sum\limits_{n=0}^{k} \Phi(x_n^+) < \gamma_i(t) < \sum\limits_{n=0}^{k+1} \Phi(x_n^+)$. Let $t = \sum\limits_{n=0}^{k} \Phi((z_i)_n^+) + t_i$ and $\gamma_i(t) = \sum\limits_{n=0}^{k} \Phi(x_n^+) +$

p_i. Since $\sum_{n=0}^{k}\Phi((z_i)_n^+) \to \sum_{n=0}^{k}\Phi(x_n^+)$ and $\gamma_i \to id$ as $i \to \infty$ we have $t_i \to p_i$, i.e., $|t_i - p_i| \to 0$ as $i \to \infty$. By continuity of π_t, $\pi(\cdot, z)$ and compactness of X, there is N' such that if $i > N'$ then $d(\pi_{t_i}(p)), \pi_{p_i}(p)) < \beta_i/2$ for all $p \in X$ and $d(\pi_t(z_i)_k^+), \pi_t(x_k^+))$ for all $t \in \mathbb{R}_+$. Hence

$$\begin{aligned} d(\tilde{\pi}_t(z_i), \tilde{\pi}_{\gamma_i(\alpha_i(t))}(x)) &= d(\pi_{t_i}((z_i)_k^+), \pi_{p_i}(x_k^+)) \\ &\le d(\pi_{t_i}((z_i)_k^+), \pi_{p_i}((z_i)_k^+)) + d(\pi_{p_i}((z_i)_k^+), \pi_{p_i}(x_k^+)) \\ &< \beta_i/2 + \beta_i/2 = \beta_i \end{aligned}$$

So we have

$$d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\pi}_{\gamma_i(\alpha_i(t))}(x)) < \epsilon'/6 + \eta_i \text{ for all } t \in [0, \alpha_i^{-1}(w_i)],$$

and $\alpha_i^{-1}(w_i) \to \infty$ as $i \to \infty$. So one can assume that there are sequence of positive real numbers $\{\beta_i\}, \{\nu_i\}$ with $\beta_i \to 0, \nu_i \to \infty$ such that
$d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\pi}_{\gamma_i(t)}(x)) < \beta_i$ for all $t \in [0, \nu_i]$. Then we have

$$\begin{aligned} d(\tilde{\pi}_{\gamma_i(\alpha_i(t))}(x), \tilde{\psi}_t(y)) &\le d(\tilde{\pi}_{\gamma_i(\alpha_i(t))}(x)), \tilde{\pi}_{\alpha_i(t)}(z_i)) + d(\tilde{\pi}_{\alpha_i(t)}(z_i), \tilde{\psi}_t(y)) \\ &< \beta_i + \epsilon'/6 + \eta_i, \quad \text{for all } \ t \in [0, k_i] \end{aligned}$$

where $k_i = \min\{\nu_i, T_i\}$. Hence $x \in A_{y,\eta_i+\beta_i,k_i}$ for all i. Since $\eta_i + \beta_i \to 0, k_i \to \infty$ as $i \to \infty$, $x \in A_y$. Suppose that $d(z_i, A_y) \ge \lambda$ for all i, then $d(x, A_y) \ge \lambda$. This is a contradiction and finishes the proof of the lemma. □

Lemma 4.3 *For every $\lambda > 0$ there are $\eta, T > 0$ such that for every $y \in X$, $d(x, A_y) < \lambda$ for all $x \in A_{y,\eta,T}$.*

Proof) For every point $y \in X$ let U_y be a neighbourhood of y such that if $y' \in U_y$ there is a homeomorphism $\beta_{y'} : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\beta_{y'}(0) = 0$ and

$$d(\tilde{\psi}_{\beta_{y'}(t)}(y), \tilde{\psi}_t(y')) < \eta_y/2 \text{ for all } \ t \in [0, T_y]$$

where η_y, T_y are taken as in lemma 3.7(i.e., $d(x, A_y) < \lambda/2$ for all $x \in A_{y,\eta_y,T_y}$). Let x be any point in $A_{y',\eta_y/2,\beta_{y'}^{-1}(T_y)}$. There exists $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ homeomorphism with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha(t)}(x), \tilde{\psi}_t(y')) < \epsilon'/6 + \eta_y/2 \text{ for } \ t \in [0, \beta_{y'}^{-1}(T_y)]$$

Hence

$$d(\tilde{\pi}_{\alpha(t)}(x), \tilde{\psi}_{\beta_{y'}(t)}(y)) < \epsilon'/6 + \eta_y \text{ for } t \in [0, \tilde{T}_y],$$

where $\tilde{T}_y = min\{T_y, \beta_{y'}^{-1}(T_y)\}$ It consequence that

$$d(\tilde{\pi}_{\alpha(\beta_{y'}^{-1}(t))}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta_y \text{ for } t \in [0, \beta_{y'}(\tilde{T}_y)].$$

Since $\beta_{y'}(\tilde{T}_y) = min\{\beta_{y'}(T_y), T_y\}$, $A_{y',\eta_y/2,\beta_{y'}^{-1}(T_y)} \subseteq A_{y,\eta_y,\beta_{y'}(\tilde{T}_y)} \subseteq A_{y,\eta_y,T_y}$ for all $y' \in U_y$. By compactness of X, there are points $y_1, y_2, \ldots, y_k$ with an open cover $U_1, U_2, \ldots, U_k$ and $\eta_1, \eta_2, \ldots, \eta_k, T_1, T_2, \ldots T_k$ takens as above. Let $\eta = \min_{1\le i\le k}\{\eta_i\}, T = \max_{1\le i\le k}\{T_i\}$. Now if y' be any point in X, there exists $j, 1 \leq j \leq k$ such that $y' \in U_j$. But $A_{y',\eta/2,\beta^{-1}_{y'}(T_j)} \subseteq A_{y_j,\eta_j,T_j}$, therefore if we take $z \in A_{y',\eta/2,\beta^{-1}_{y'}(T_j)}$ we will have $d(z, A_{y_j}) < \lambda/2$. Since $A_{y'} \subseteq A_{y',\eta/2,\beta^{-1}_{y'}(T_j)} \subseteq A_{y_j,\eta,T}$, we have $d(A_{y'}, A_{y_j}) < \lambda/2$. Hence $d(z, A_{y'}) \leq d(z, A_{y_j}) + d(A_{y_j}, A_{y'}) < \lambda$.□

Lemma 4.4 *Let $\{y_i\}, \{z_i\}$ be sequences of points in X. Then if $z_i \in A_{y_i}$ for all i, $z_i \to z$, $y_i \to y$, then $z \in A_y$.*

Proof) Take $\{\lambda_i\}$ to be a sequence of positive real number such that $\lambda_i \to 0$. Using lemma 5.9, there are $\eta_i, T_i > 0$ such that $d(x, A_y) < \lambda_i$ for all $x \in A_{y,\eta_i,T_i}$ and for every $y \in Y$. Since $y_i \to y$ there is a subsequences $\{y_{j_i}\}$ of $\{y_i\}$ and homermorphism $\beta_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\beta_i(0) = 0$ such that

$$d(\tilde{\psi}_{\beta_i(t)}(y), \tilde{\psi}_t(y_{j_i})) < \eta_i/2 \text{ for } t \in [0, T_i], \text{ all } i.$$

But $z_{j_i} \in A_{y_{j_i}}$, therefore there is homeomorphisms $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha_i(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha_i(t)}(z_{j_i}), \tilde{\psi}_t(y_{j_i})) < \epsilon'/6 + \eta_i/2 \text{ for } t \in [0, T_i]$$

So

$$\begin{aligned} d(\tilde{\pi}_{\alpha_i(t)}(z_{j_i}), \tilde{\psi}_{\beta_i(t)}(y)) &\leq d(\tilde{\pi}_{\alpha_i(t)}(z_{j_i}), \tilde{\psi}_t(y_{j_i})) + d(\tilde{\psi}_t(y_{j_i}), \tilde{\psi}_{\beta_i(t)}(y)) \\ &< \epsilon'/6 + \eta_i \text{ for } t \in [0, T_i]. \end{aligned}$$

Hence

$$d(\tilde{\pi}_{\alpha_i(\beta_i^{-1}(t))}(z_{j_i}), \tilde{\psi}_(y)) < \epsilon'/6 + \eta_i \text{ for } t \in [0, \beta_i(T_i)].$$

It means that $z_{j_i} \in A_{y,\eta_i,\beta_i(T_i)}$ for all i. WOLG we can suppose $\eta_i \to 0,\ T_i \to \infty$. Then one can take subsequence $\{z_{j_{i_k}}\}$ for $\{z_{j_i}\}$ such that $z_{j_{i_k}} \in A_{y,\eta_i,\beta_{j_{i_k}}(T_{j_{i_k}})}$. By Lemma 3.7, we have $d(z_{j_{i_k}}, A_y) < \lambda_{j_{i_k}}$ for all i. But $z_{j_{i_k}} \to z$, $\lambda_{j_{i_k}} \to 0$ therefore $d(z, A_y) = 0$. A_y is closed set, hence $z \in A_y$.□

Now, we have to show that h is continuous. Assume $\{y_i\}$ is sequence of X and converse to y. Claim $\lim_{i\to\infty} h(y_i) = \lim_{i\to\infty} l.l.A_{y_i} = l.l.A_y$. Define $z_i = l.l.A_{y_i}$. Since X is compact, WLOG $z_i \to z'$ for some $z' \in X$. By using Lemma 3.8, $z' \in A_y$. For given $x \in A_y$ and $\{\lambda_i\}$, sequence of $\mathbb{R}_+$ with converge to 0, we can choose $\eta_i, T_i > 0$ as in lemma5.9. (i.e., $d(x, A_y) < \lambda$ for all $x \in A_{y,\eta,T}$.) Since $y_i \to y$, there exists a subsequence $\{y_{k_i}\}$ such that

$$d(\tilde{\psi}_t(y_{k_i}), \tilde{\psi}_{\beta_{k_i}(t)}(y)) < \eta_i/2 \ \text{ for } t \in [0, T_i]$$

Since $x \in A_y$, there exists a homeomorphism $\alpha_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ such that

$$d(\tilde{\pi}_{\alpha_i(t)}(x), \tilde{\psi}_t(y)) < \epsilon'/6 + \eta_i/2 \ \text{ for } t \in [0, \beta_{k_i}^{-1}(T_i)].$$

Therefore

$$d(\tilde{\pi}_{\alpha_i(t)}(x), \tilde{\psi}_{\beta_{k_i}(t)}(y_{k_i})) < \epsilon'/6 + \eta_i/2 \ \text{ for } t \in [0, \tilde{T}_i],$$

where $\tilde{T}_i = \min\{\beta_{k_i}^{-1}(T_i), T_i\}$. So

$$d(\tilde{\pi}_{\alpha_i(\beta_{k_i}^{-1}(t)(t)}(x), \tilde{\psi}_t(y_{k_i})) < \epsilon'/6 + \eta_i/2 \text{ for } t \in [0, \beta_{k_i}(\tilde{T}_i)].$$

Since $\beta_{j_{i_k}}(\tilde{T}_i) \leq \beta_{j_{i_k}}(\beta_{j_{i_k}}^{-1}(T_i)) = T_i$, we get $x \in A_{y_{k_i}, \eta_i, \beta_{j_{i_k}}(\tilde{T}_i)} \subset A_{y_{k_i}, \eta_i, T_i}$. By Lemma 3.8, $d(x, A_{y_{k_i}}) < \lambda_i$ for all $i \in \mathbb{N}$. Because $A_{y_{k_i}}$ is closed, we can choose $x_{k_i} \in A_{y_{k_i}}$ such that $d(x, x_{k_i}) = d(x, A_{y_{k_i}}) = \lambda_i$. Obviously $x_{k_i} \to x$. Since $z_{k_i} = l.l.A_{y_{k_i}}$, there is a sequence $\{w_{k_i}\}$ of $\mathbb{R}_+$ such that $z_{k_i} = \tilde{\pi}_{w_{k_i}} x_{k_i}$. Let $w_{k_i} \to w > 0$. Then we get

$$\begin{aligned} z' &= \lim_{i \to \infty} z_{k_i} \\ &= \lim_{i \to \infty} \tilde{\pi}_{w_{k_i}}(x_{k_i}) \\ &= \tilde{\pi}_w(x) \end{aligned}$$

Hence $z' = \tilde{\pi}_w(x)$. This means that $z' = l.l.A_y$. By the uniqueness of $l.l.A_y$, so $z = z'$. So we showed $\lim_{i \to \infty} h(y_i) = h(y)$. This means that h is continuous and complete the proof of Theorem 3.6 $\square$

References

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems: Recent Advances*, North-Holland Mathematical Library, vol. 52, North-Holland, Amsterdam. 1994.

[2] A. Arbieto and E. Rego, *On The Entropy of Continuous Flows With Uniformly Expansive Points*, arXiv. (2019)

[3] A. Artigue, *Positive expansive flows*, Topology Appl., **165** (2014), 121-132.

[4] E.M. Bonotto and M. Federson, *Topological conjugation and asymptotic stability in impulsive semidynamical systems*, J. Math. Anal. Appl., **326** (2007), no. 2, 869-–881.

[5] E.M. Bonotto, D.P. Demuner, and G.M. Souto, *Weak Topological Conjugacy Via Character of Recurrence on Impulsive Dynamical Systems*, Bull. Braz. Math. Soc., **50** (2019), no. 2, 399—417.

[6] E.M. Bonotto, M.C. Bortolan, A.N. Carvalho, and R. Czaja, *Global attractors for impulsive dynamical systems-a precompact approach*, J. Differential Equations, **259** (2015), no. 7, 2602—2625.

[7] R. Bowen and P. Walters, *Expansive one-parameter flows*, J. Differential Equations, **12** (1972), 180—193.

[8] K. Ciesielski, *On semicontinuity in impulsive dynamical systems*, Bull. Pol. Acad. Sci. Math., **52** (2004), no. 1, 71-–80.

[9] Saroop K. Kaul, *On impulsive semidynamical systems II. Recursive properties*. Nonlinear Anal., **16** (1991), no. 7-8, 635-–645.

[10] R.F. Thomas, *Stability properties of one-parameter flows*, Proc. London Math. Soc., **45** (1982), no. 3, 479—505.