Design and Analysis of Algorithms

Part II: Sorting and Searching

Lecture 5: Heapsort and Sorting in Linear Time

Yongxin Tong (童咏昕)

School of CSE, Beihang University

yxtong@buaa.edu.cn

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Introduction to Part II

- In Part II, we will illustrate sorting and searching problems using several examples:
  - Quicksort (快速排序)
  - Selection Problem (选择问题)
  - Heapsort and Priority Queues (堆排序与优先队列)
  - Lower Bound for Sorting (基于比较排序的下界)
  - Sorting in Linear Time (线性时间排序)

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Priority Queue: Motivating Example

3 jobs have been submitted to a printer in the order A, B, C. Consider the printing pool at this moment.

Sizes: Job A — 100 pages

Job B — 10 pages

Job C — 1 page

Priority Queue: Motivating Example

3 jobs have been submitted to a printer in the order A, B, C. Consider the printing pool at this moment.

Sizes: Job A — 100 pages

Job B — 10 pages

Job C — 1 page

Average finish time with FIFO service:

$$(100 + 110 + 111) \,/\, 3 \;=\; 107\ time\ units$$

Priority Queue: Motivating Example

3 jobs have been submitted to a printer in the order A, B, C. Consider the printing pool at this moment.

Sizes: Job A — 100 pages

Job B — 10 pages

Job C — 1 page

Average finish time with FIFO service:

$$(100 + 110 + 111)\ /\ 3 \ = \ 107\ time\ units$$

Average finish time for shortest-job-first service:

$$(1 + 11 + 111)\ /\ 3 \ = \ 41\ time\ units$$

Priority Queue: Motivating Example

- The elements in the queue are printing jobs, each with the associated number of pages that serves as its priority
- Processing the shortest job first corresponds to extracting the smallest element from the queue
- Insert new printing jobs as they arrive

Priority Queue: Motivating Example

- The elements in the queue are printing jobs, each with the associated number of pages that serves as its priority
- Processing the shortest job first corresponds to extracting the smallest element from the queue
- Insert new printing jobs as they arrive

A queue capable of supporting two operations: Insert and Extract-Min?

Priority Queue

Priority queue is an abstract data structure that supports two operations

- Insert: inserts the new element into the queue
- Extract-Min: removes and returns the smallest element from the queue

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in O(n) time
  - Extract-Min in

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in O(n) time
  - Extract-Min in O(1) time
- Sorted doubly linked list
  - Insert in

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in O(n) time
  - Extract-Min in O(1) time
- Sorted doubly linked list
  - Insert in O(n) time
  - Extract-Min

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in O(n) time
  - Extract-Min in O(1) time
- Sorted doubly linked list
  - Insert in O(n) time
  - Extract-Min in O(1) time

Possible Implementations

- Unsorted list + a pointer to the smallest element
  - Insert in O(1) time
  - Extract-Min in O(n) time, since it requires a linear scan to find the new minimum
- Sorted array
  - Insert in O(n) time
  - Extract-Min in O(1) time
- Sorted doubly linked list
  - Insert in O(n) time
  - Extract-Min in O(1) time

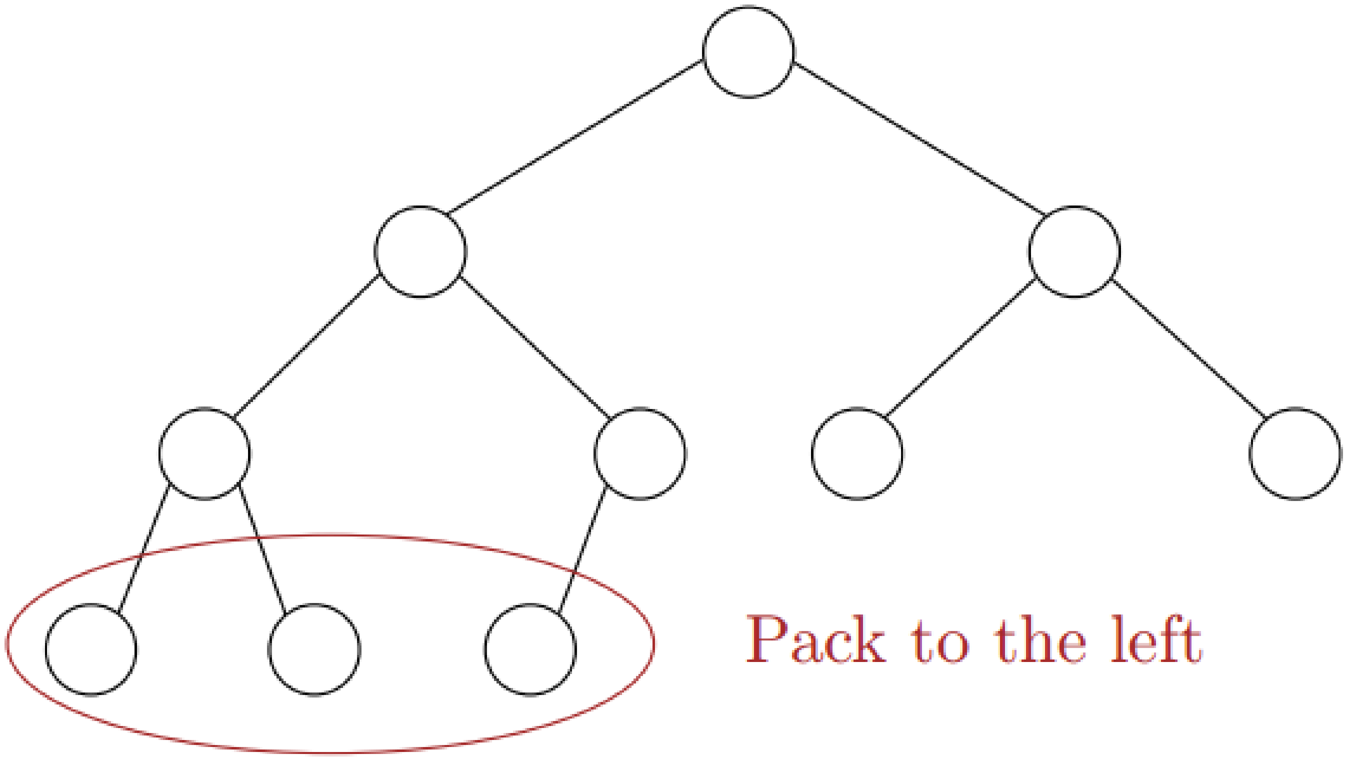
Question

Is there any data structure that supports both these priority queue operations in $O(\log n)$ time?

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

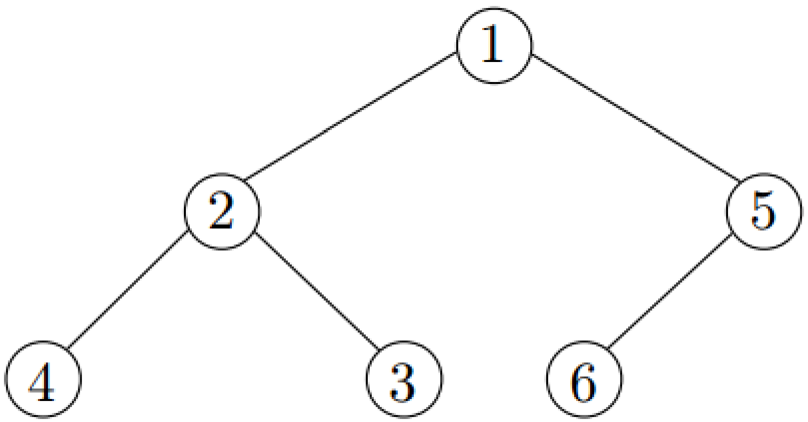
(Binary) Heap



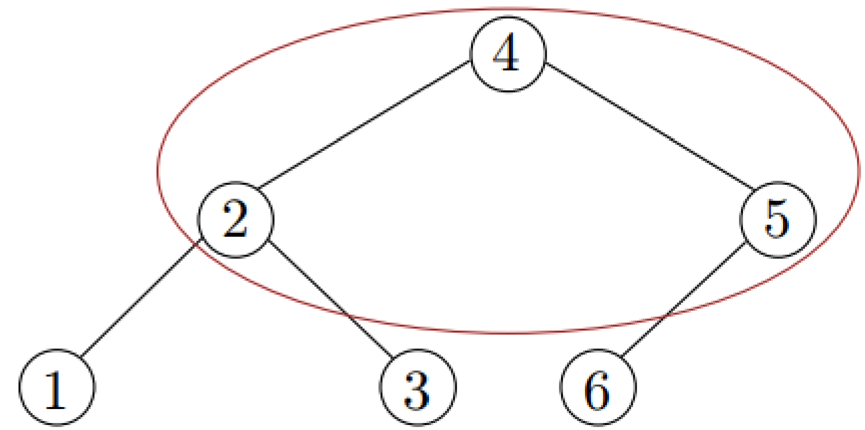
Heaps are "almost complete binary trees"

- All levels are full except possibly the lowest level.
- If the lowest level is not full, then nodes must be packed to the left.

Heap-order Property



A min-heap



Not a heap

Heap-order property (*Min-heap*):

The value of a node is at least the value of its parent.

A[Parent(i)] ≤ A[i]

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are *n* elements in the heap)
  - Insert in O(log *n*) time
  - Extract-Min in O(log *n*) time

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are *n* elements in the heap)
  - Insert in O(log *n*) time
  - Extract-Min in O(log *n*) time

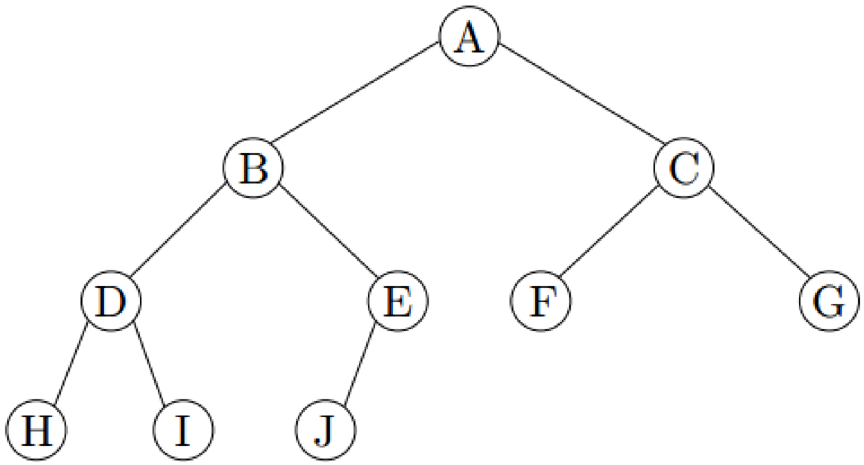
- Structure properties
  - A heap of height *h* has between 2^h to $2^{h+1}-1$ nodes. Thus, an n-element heap has height Θ(log n).

Heap Properties

- If the heap-order property is maintained, heaps support the following operations efficiently (assume there are *n* elements in the heap)
  - Insert in O(log *n*) time
  - Extract-Min in O(log *n*) time

- Structure properties
  - A heap of height *h* has between 2^h to $2^{h+1}-1$ nodes. Thus, an n-element heap has height Θ(log n).
  - The structure is so regular, it can be represented in an array and no links are necessary !

Array Implementation of Heap

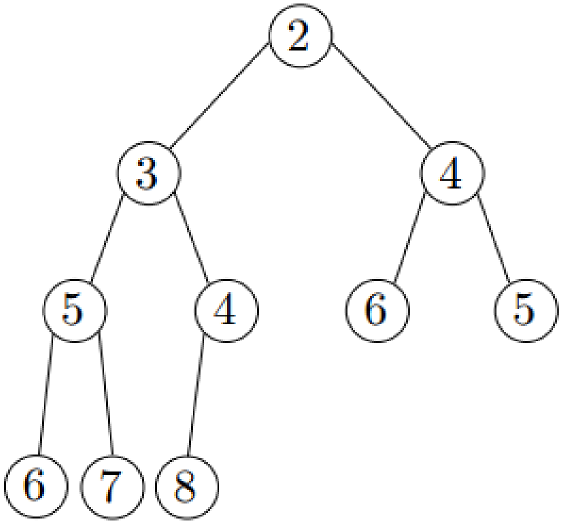


1	2	3	4	5	6	7	8	9	10
A	B	C	D	E	F	G	H	I	J

- The root is in array position 1.
- For any element in array position i,
  - The left child is in position 2i.
  - The right child is in position 2i+1.
  - The parent is in position $\lfloor i/2 \rfloor$.
- We will draw the heaps as trees, with the understanding that an actual implementation will use simple arrays.

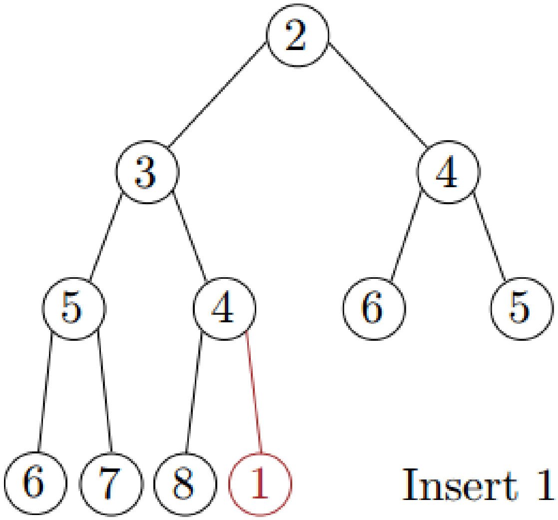
Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



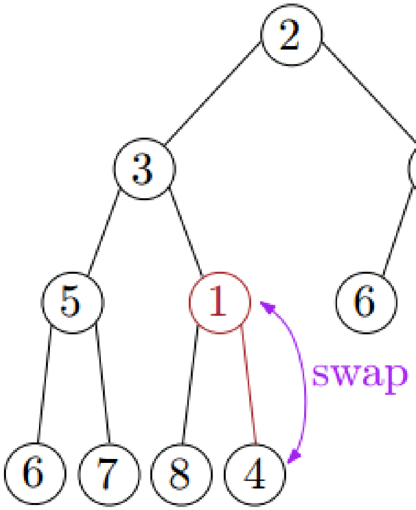
Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



Insertion

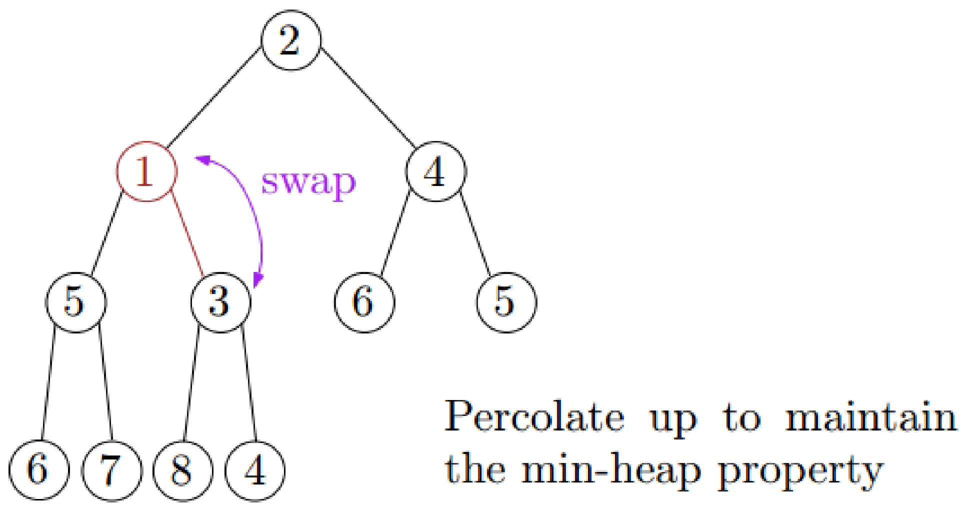
- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



Percolate up to maintain the min-heap property

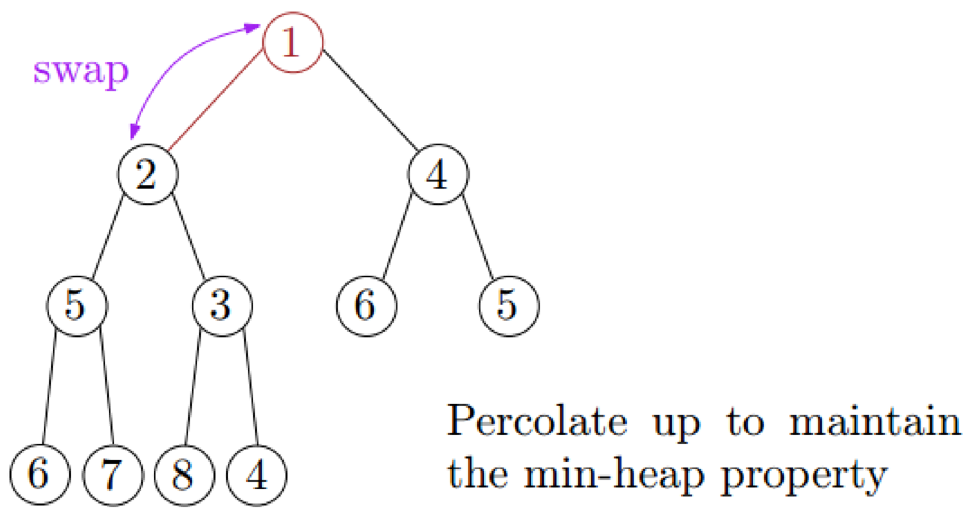
Insertion

- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



Insertion

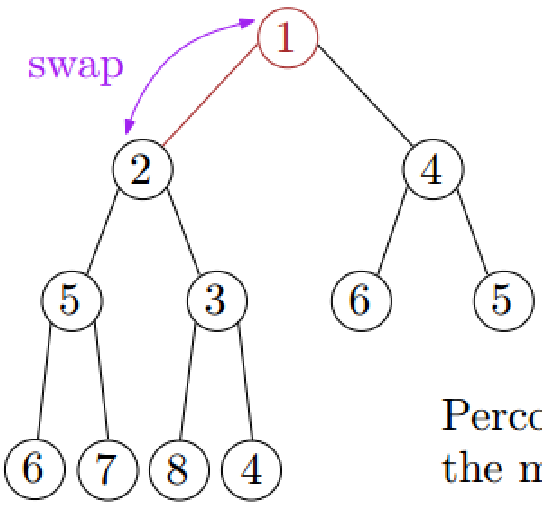
- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



- Correctness: after each swap, the min-heap property is satisfied for the subtree rooted at the new element

Insertion

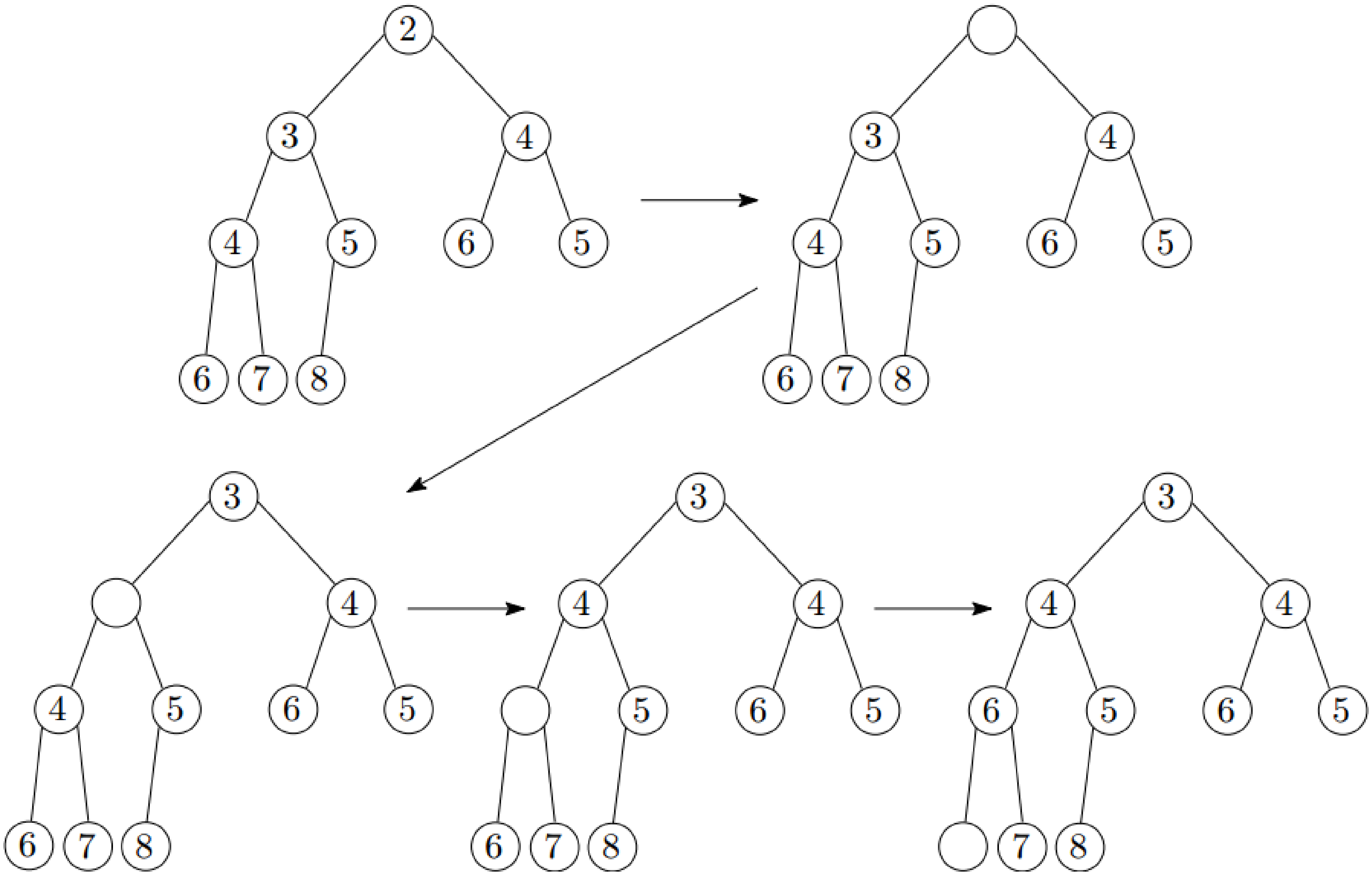
- Add the new element to the next available position at the lowest level
- Restore the min-heap property if violated
  - General strategy is percolate up (or bubble up): if the parent of the element is larger than the element, then interchange the parent with child.



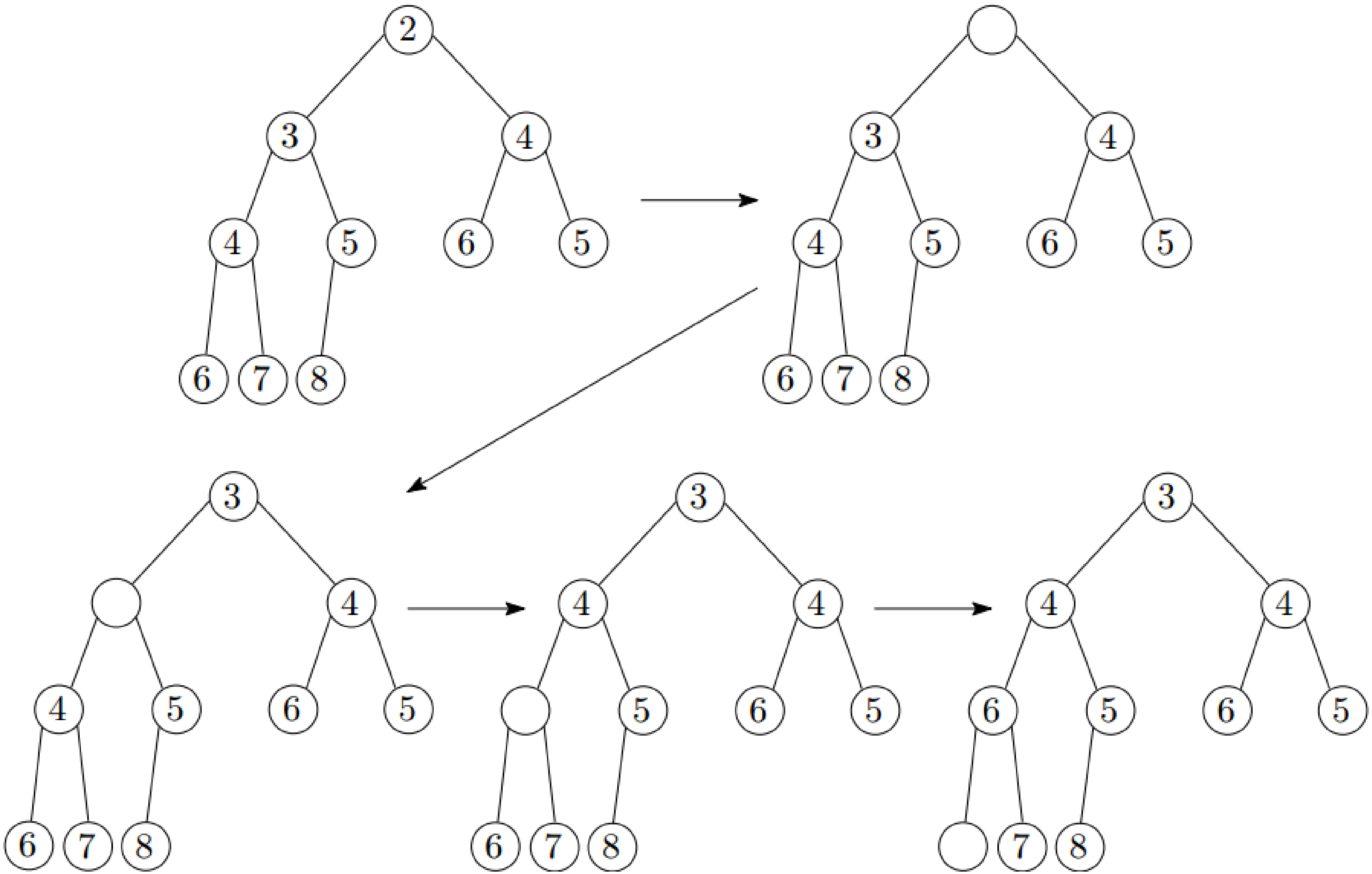
Percolate up to maintain the min-heap property

- Correctness: after each swap, the min-heap property is satisfied for the subtree rooted at the new element
- Time complexity = O(height) = O(log n)

Extract-Min: First Attempt



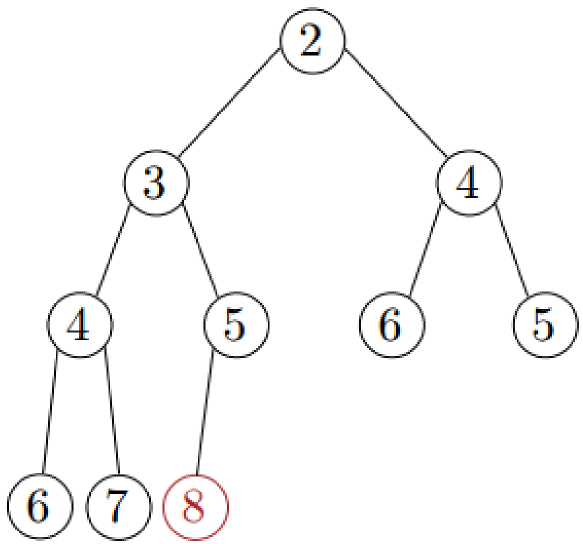
Extract-Min: First Attempt



Min-heap property preserved, but completeness not preserved!

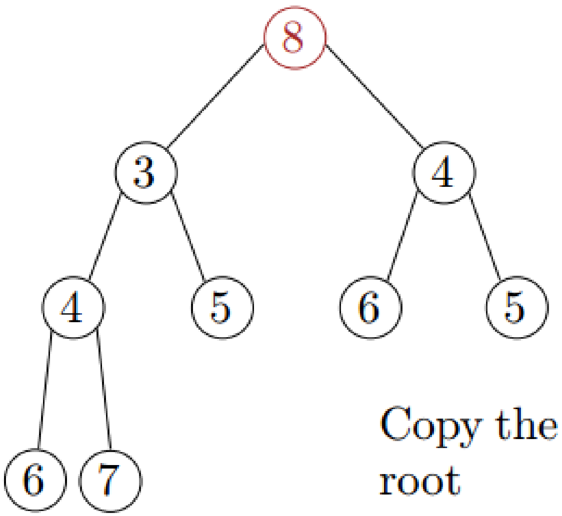
Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



Extract-Min

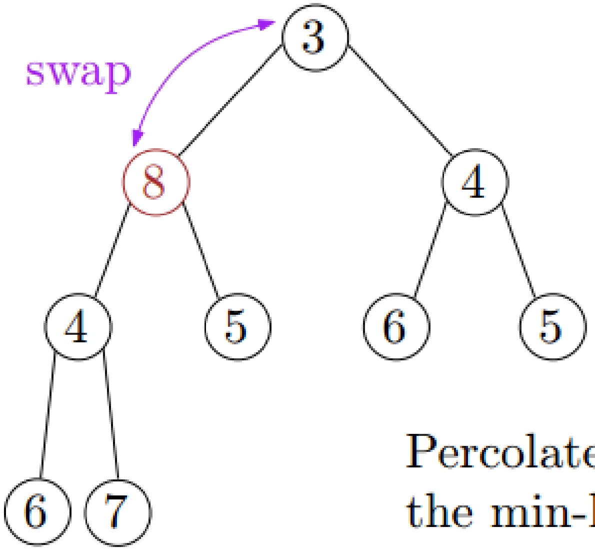
- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



Copy the last element to the root

Extract-Min

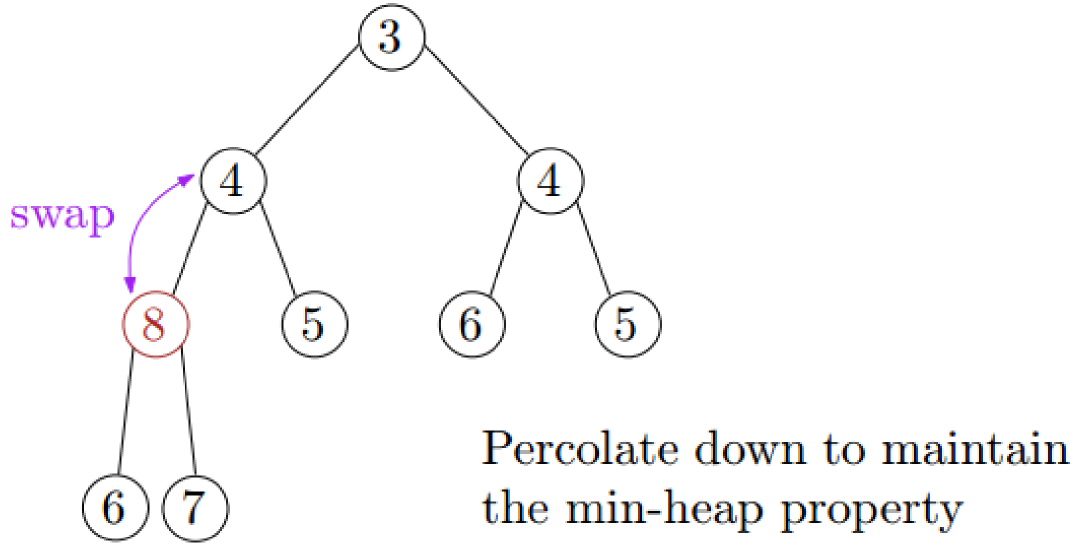
- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



Percolate down to maintain the min-heap property

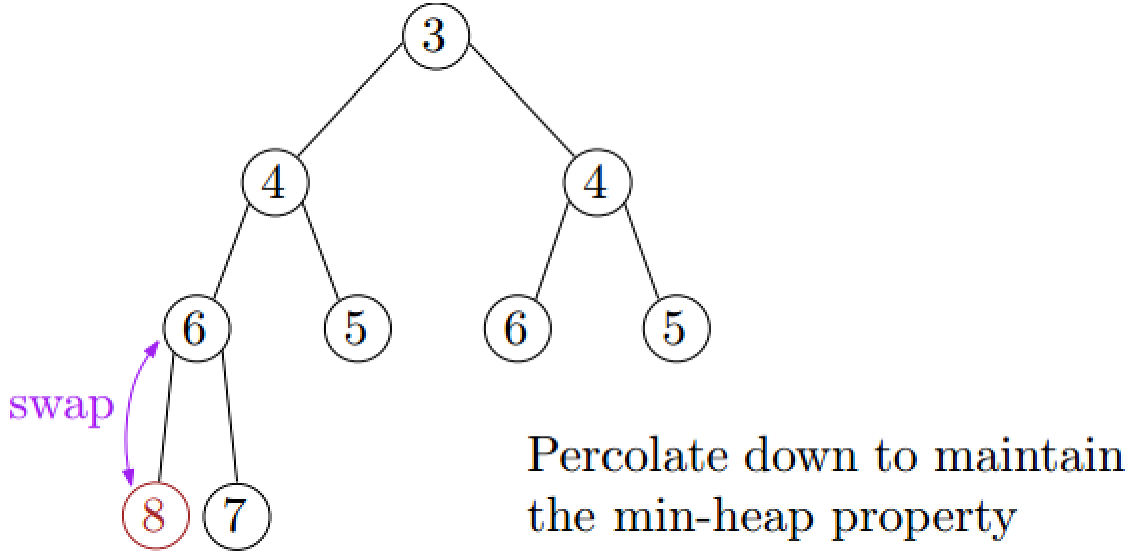
Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



Extract-Min

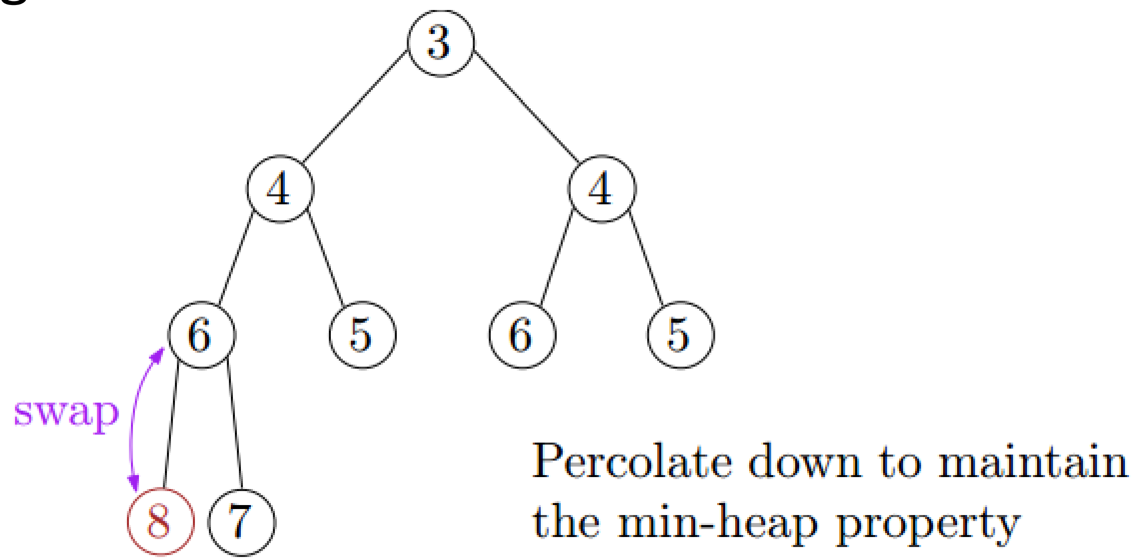
- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



- Correctness: after each swap, the min-heap property is satisfied for all nodes except the node containing the element (with respect to its children)

Extract-Min

- Copy the last element to the root (i.e., overwrite the minimum element stored there)
- Restore the min-heap property by percolate down (or bubble down): if the element is larger than either of its children, then interchange it with the smaller of its children.



- Correctness: after each swap, the min-heap property is satisfied for all nodes except the node containing the element (with respect to its children)
- Time complexity = O(height) = O(log n)

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap
  - insert n elements one by one
    → O(n log n)
    (there is a more efficient way, check CLRS 6.3 if interested)

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap
  - insert n elements one by one
    $\rightarrow$ O(n log n)
    (there is a more efficient way, check CLRS 6.3 if interested)

- Perform n Extract-Min operations
  - the elements are extracted in sorted order

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap
  - insert n elements one by one
    → O(n log n)
    (there is a more efficient way, check CLRS 6.3 if interested)

- Perform n Extract-Min operations
  - the elements are extracted in sorted order
  - each Extract-Min operation takes O(log n) time
    → O(n log n)

Heapsort

- Build a binary heap of n elements
  - the minimum element is at the top of the heap
  - insert n elements one by one
    → O(n log n)
    (there is a more efficient way, check CLRS 6.3 if interested)

- Perform n Extract-Min operations
  - the elements are extracted in sorted order
  - each Extract-Min operation takes O(log n) time
    → O(n log n)

- Total time complexity: O(n log n)

Heapsort - Example

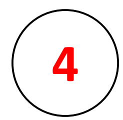
- Build a binary heap of n elements

4	**1**	**3**	**2**	**16**	**9**	**10**	**14**	**8**	**7**

Heapsort - Example

- Build a binary heap of n elements

4	1	3	2	16	9	10	14	8	7



Heapsort - Example

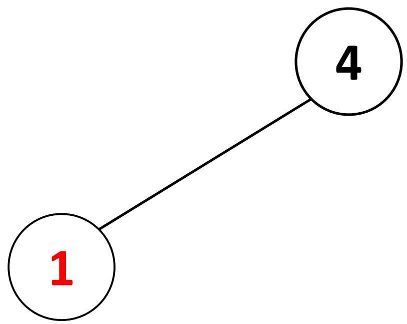
- Build a binary heap of n elements

4	1	3	2	16	9	10	14	8	7

Heapsort - Example

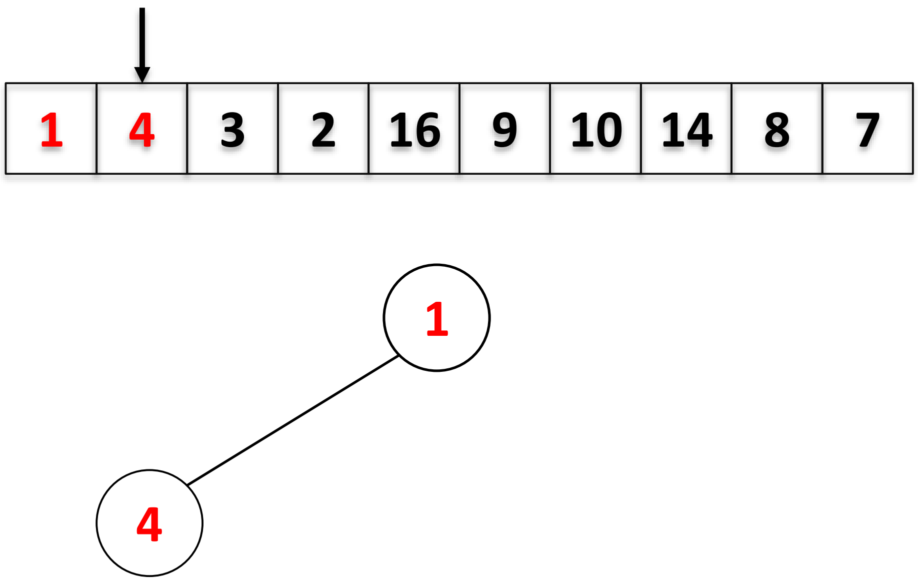
- Build a binary heap of n elements

4	1	3	2	16	9	10	14	8	7



Heapsort - Example

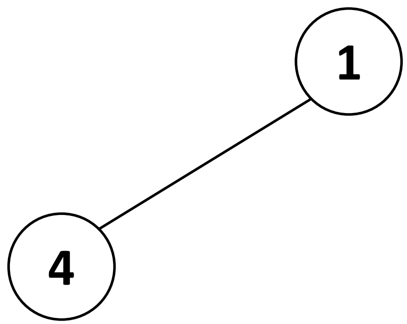
- Build a binary heap of n elements



Heapsort - Example

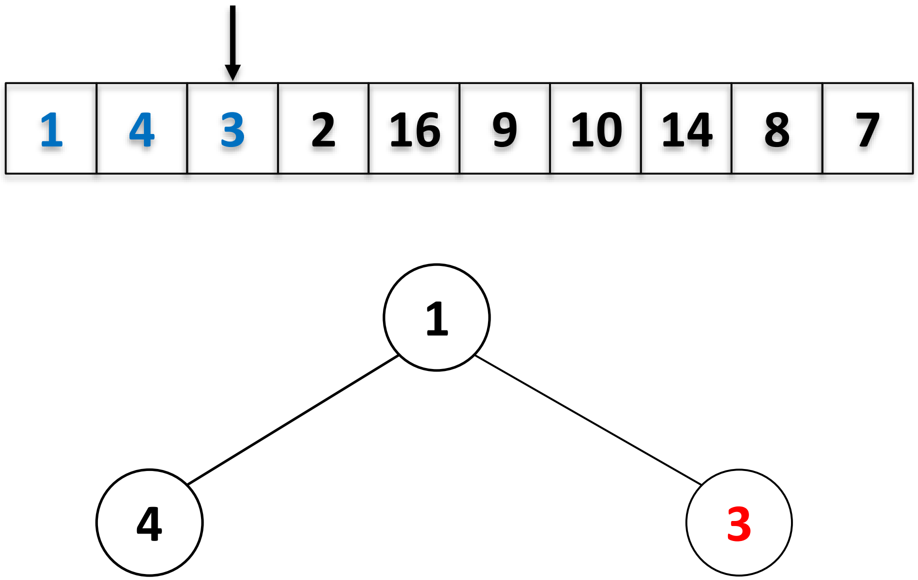
- Build a binary heap of n elements

1	4	3	2	16	9	10	14	8	7



Heapsort - Example

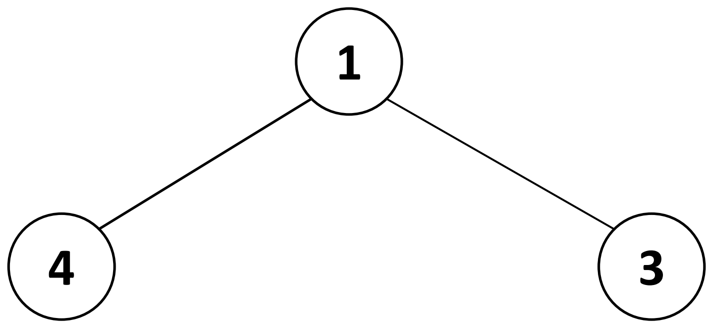
- Build a binary heap of n elements



Heapsort - Example

- Build a binary heap of n elements

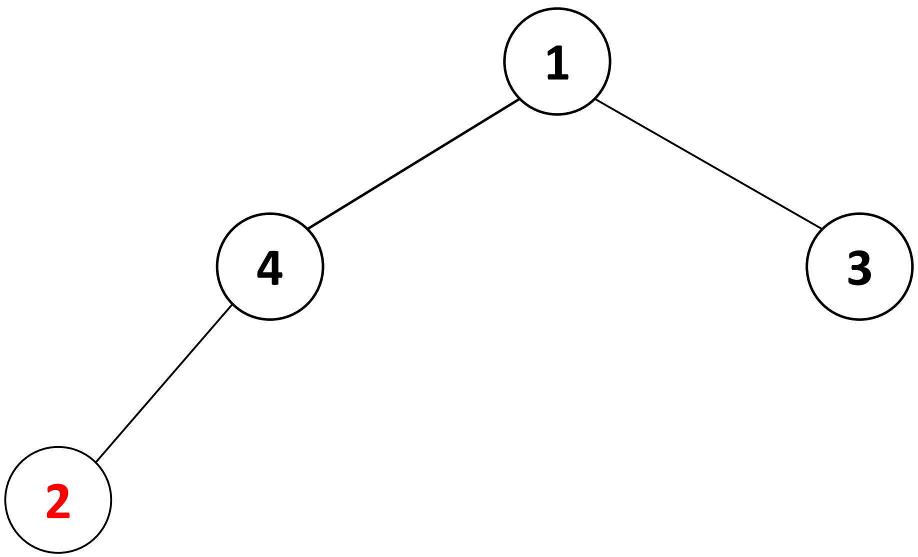
1	4	3	2	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

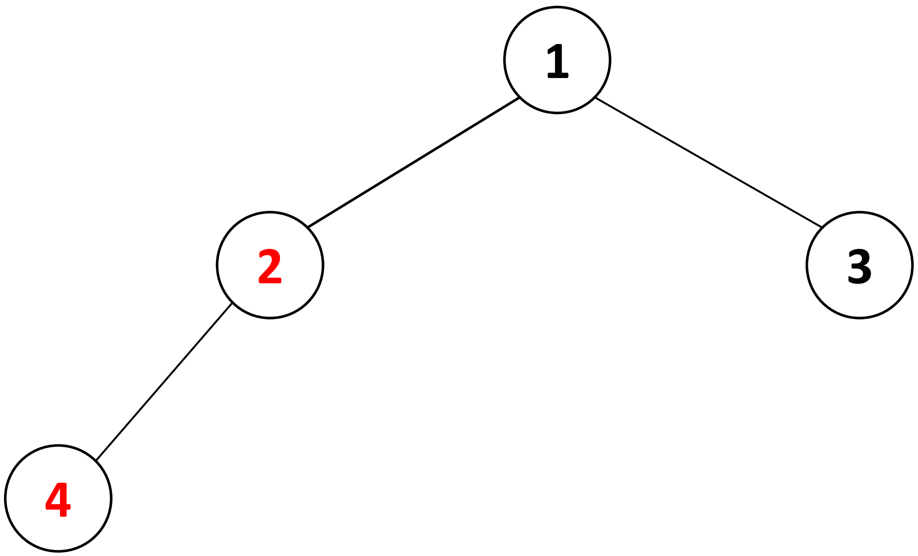
1	**4**	**3**	**2**	**16**	**9**	**10**	**14**	**8**	**7**



Heapsort - Example

- Build a binary heap of n elements

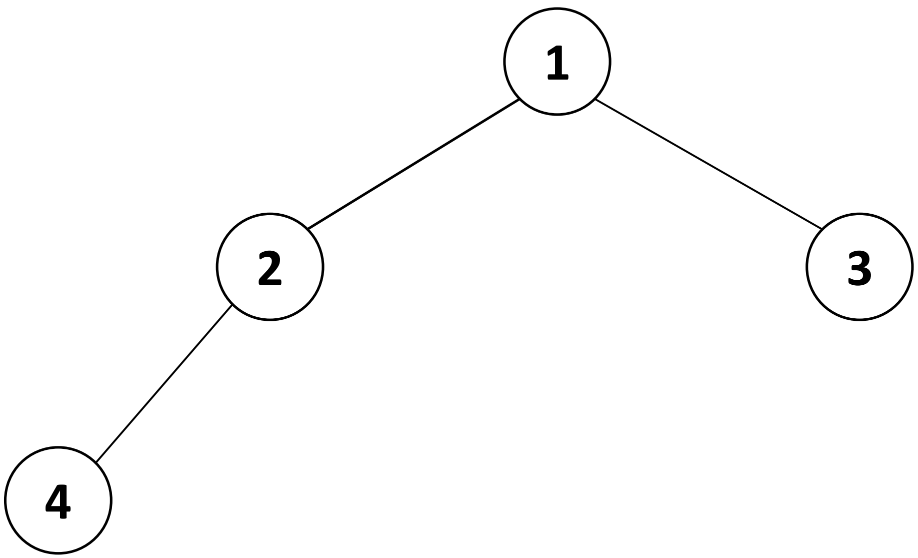
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

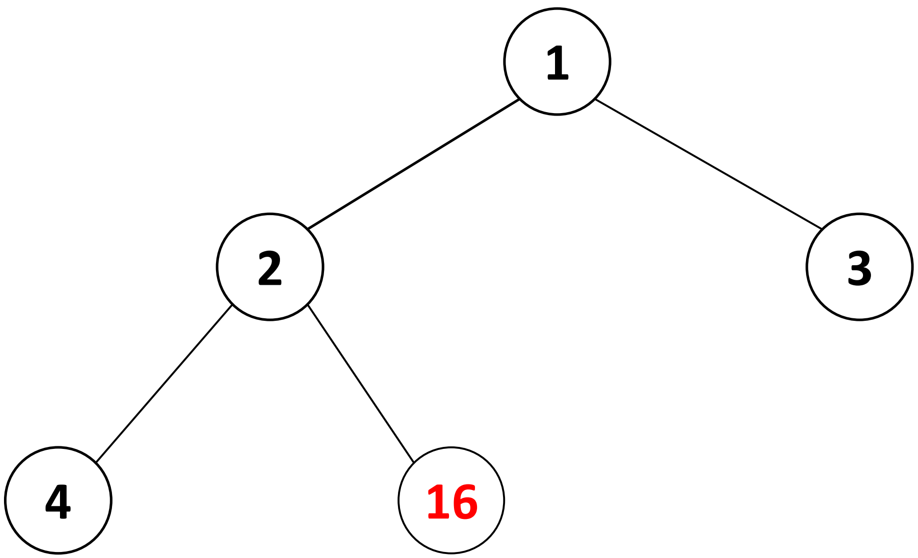
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

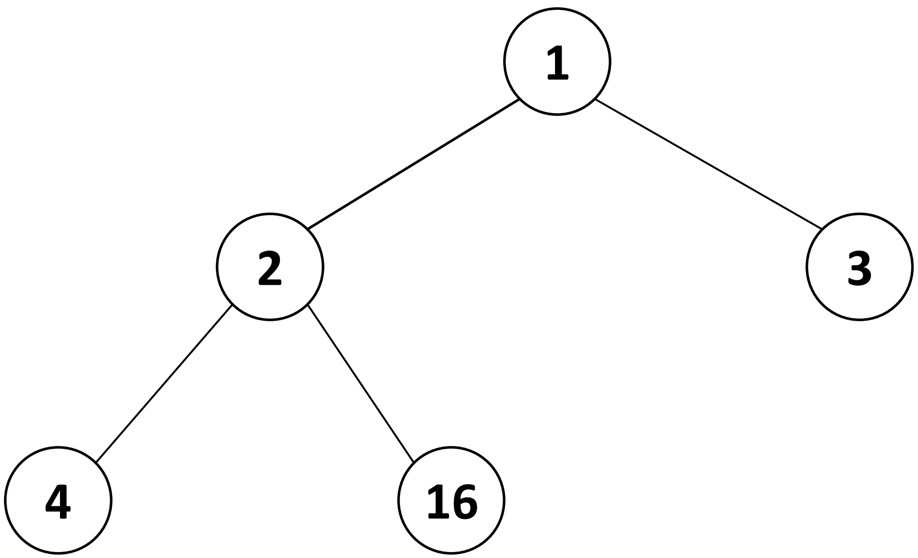
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

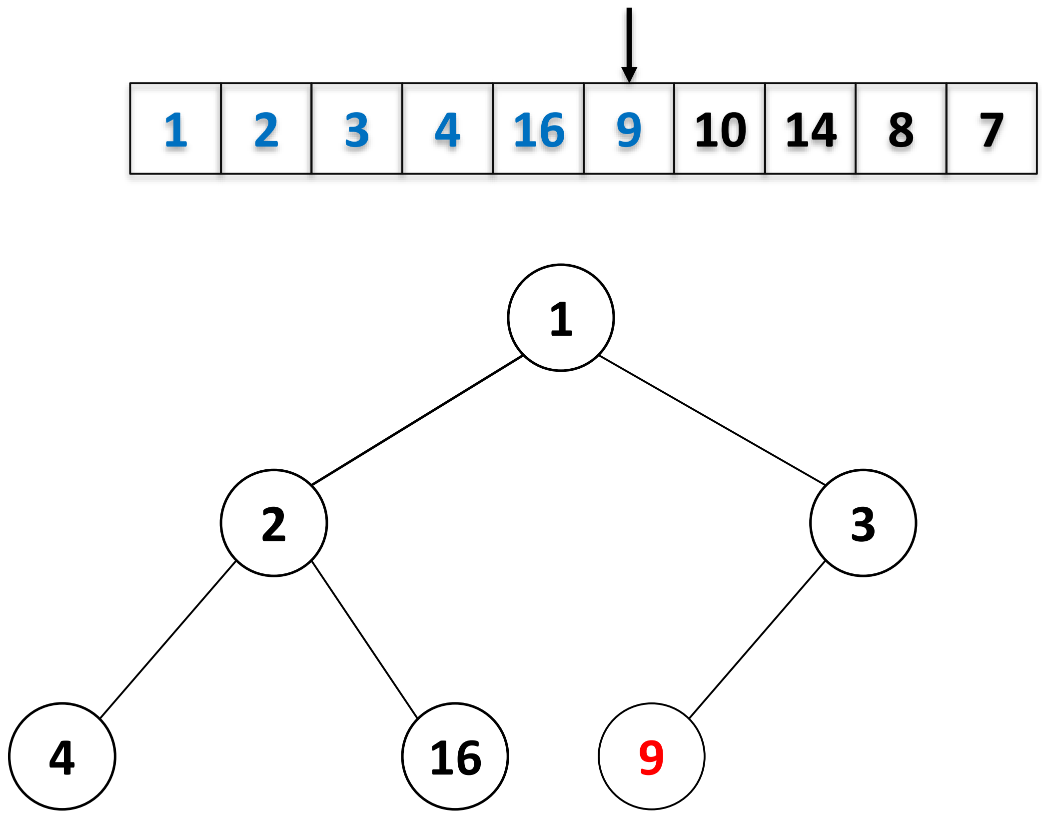
- Build a binary heap of n elements

1	2	3	4	16	9	10	14	8	7



Heapsort - Example

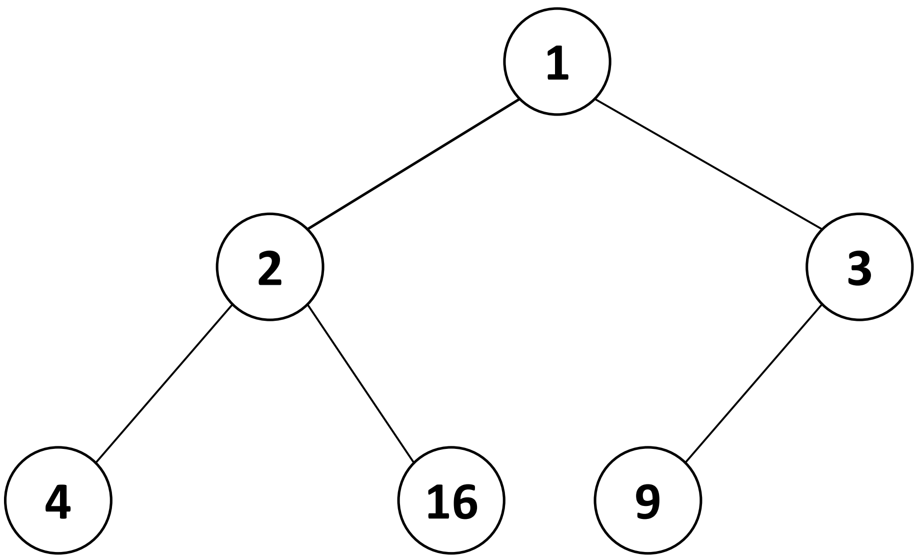
- Build a binary heap of n elements



Heapsort - Example

- Build a binary heap of n elements

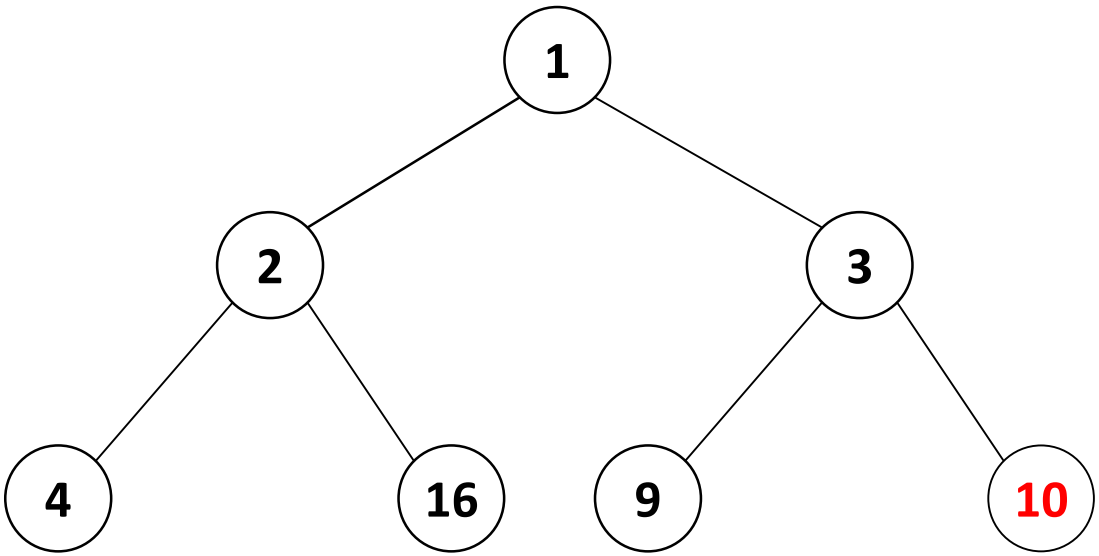
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

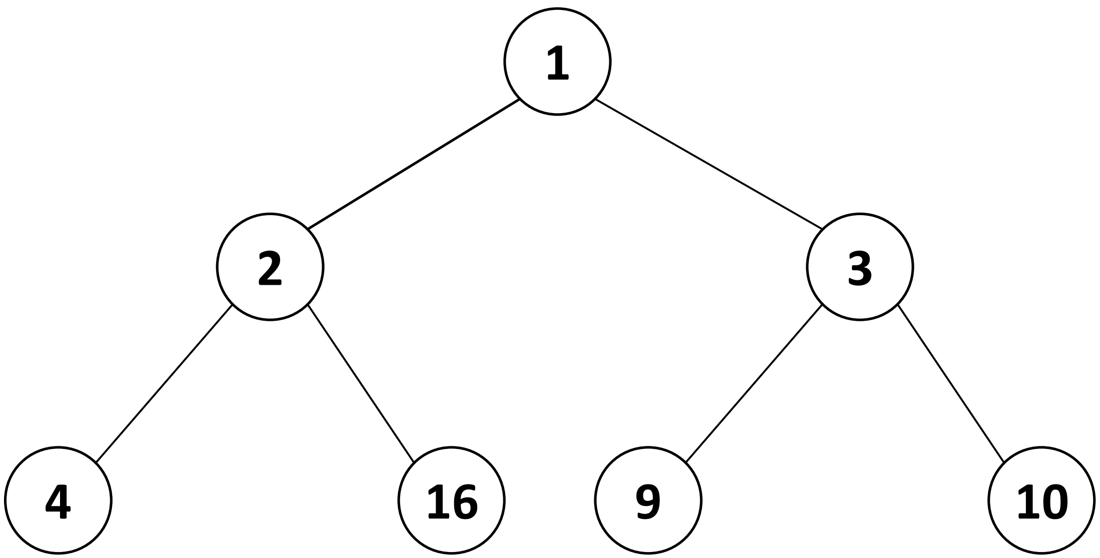
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

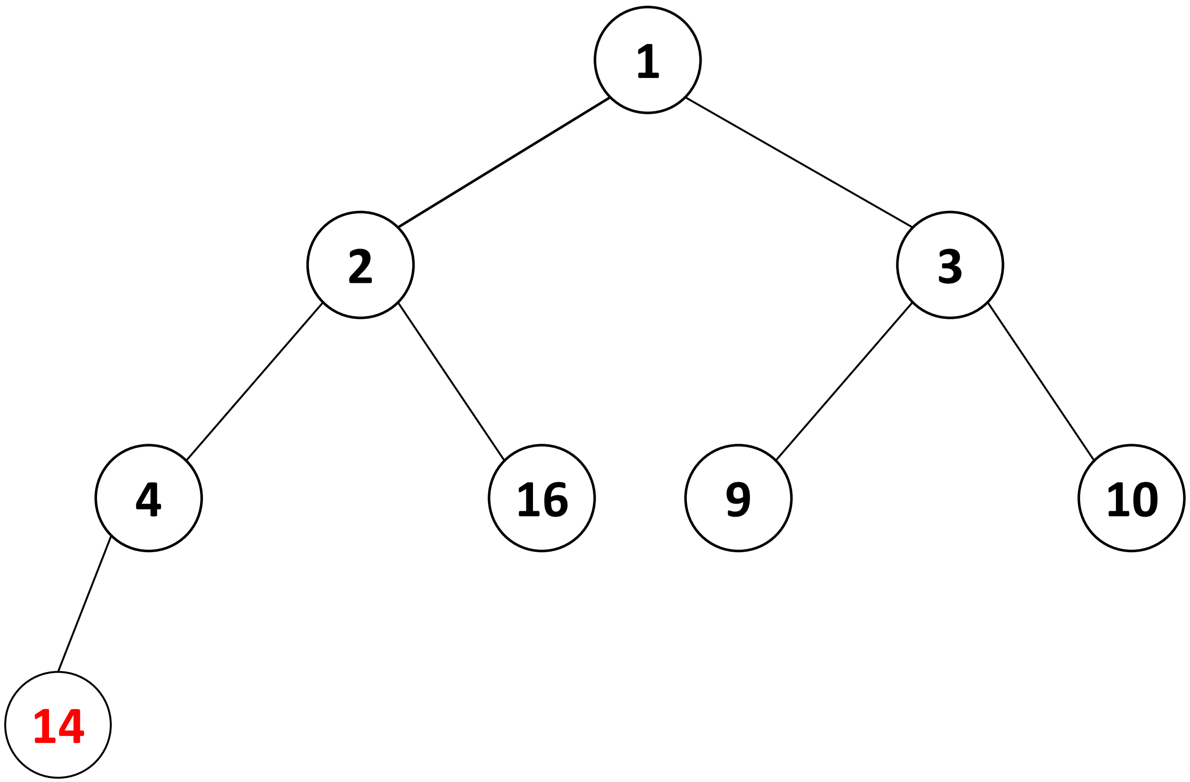
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

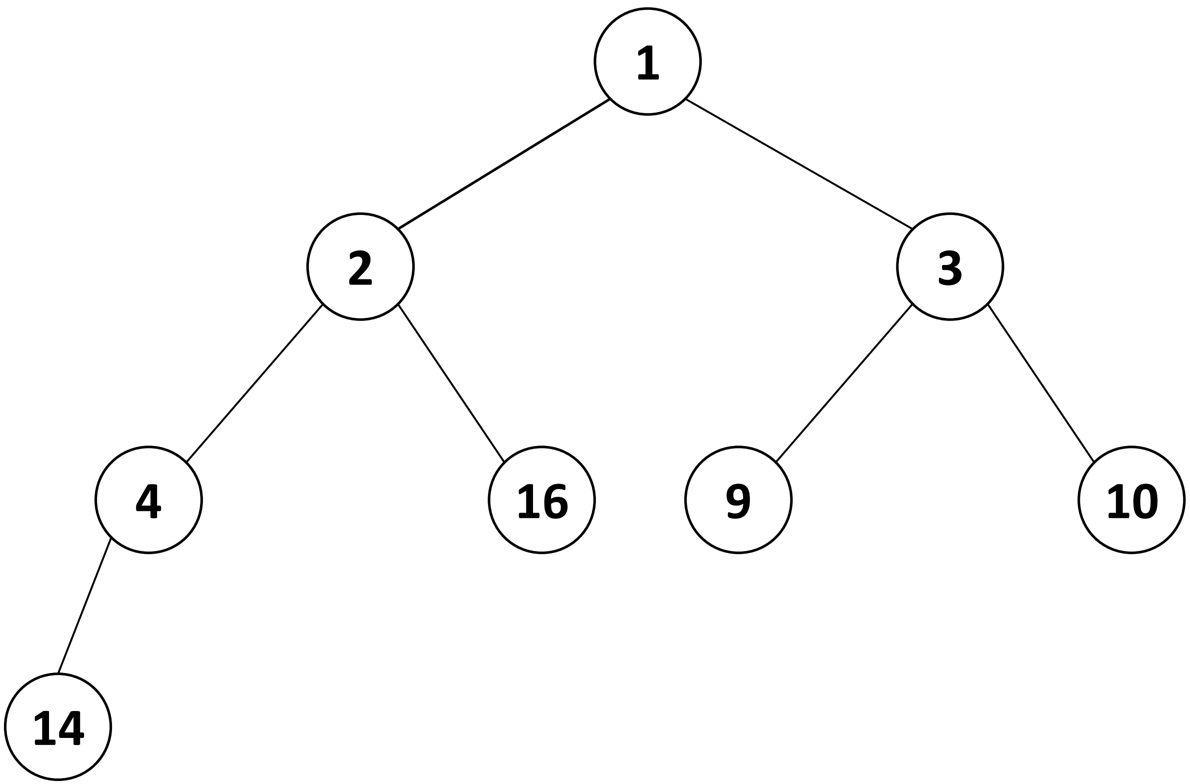
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

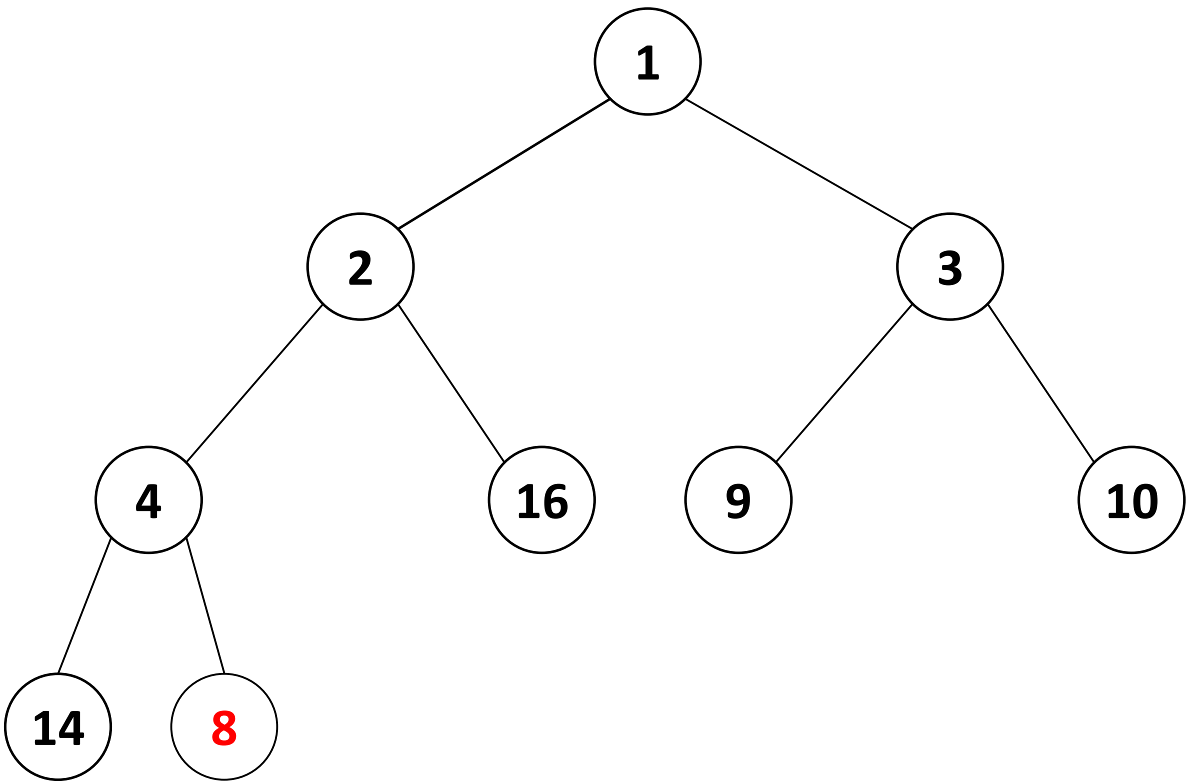
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

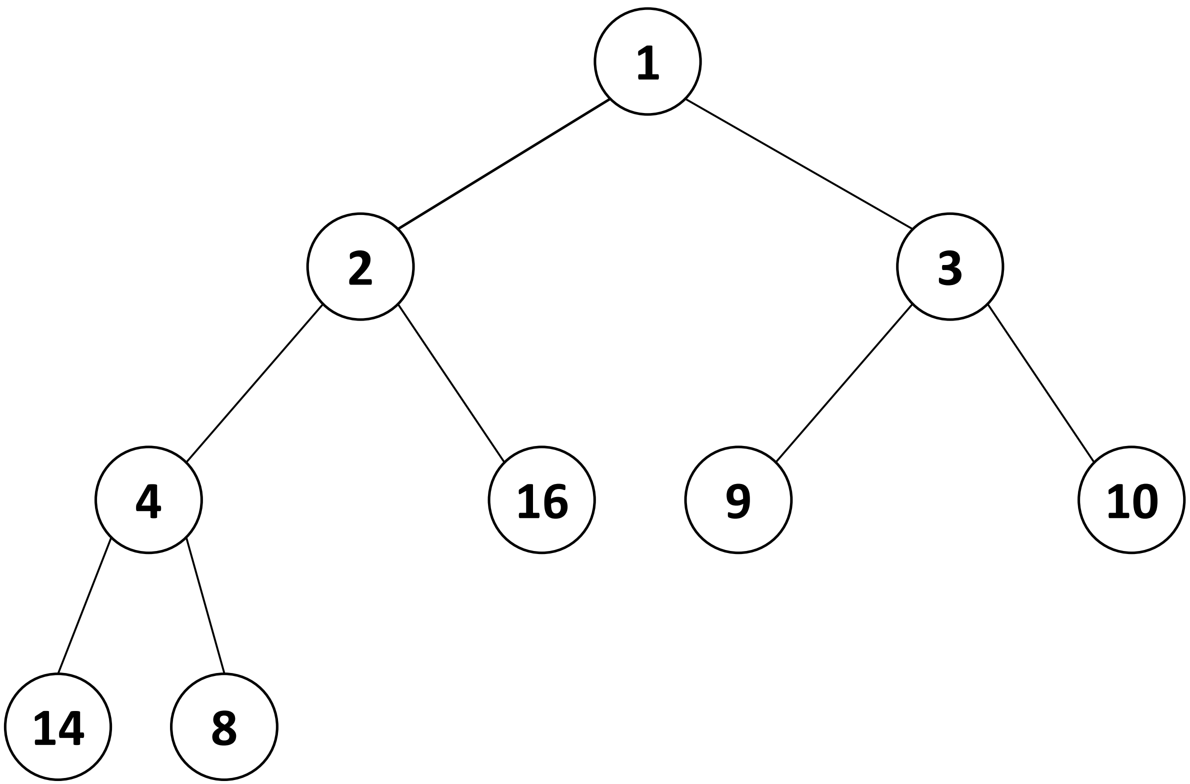
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

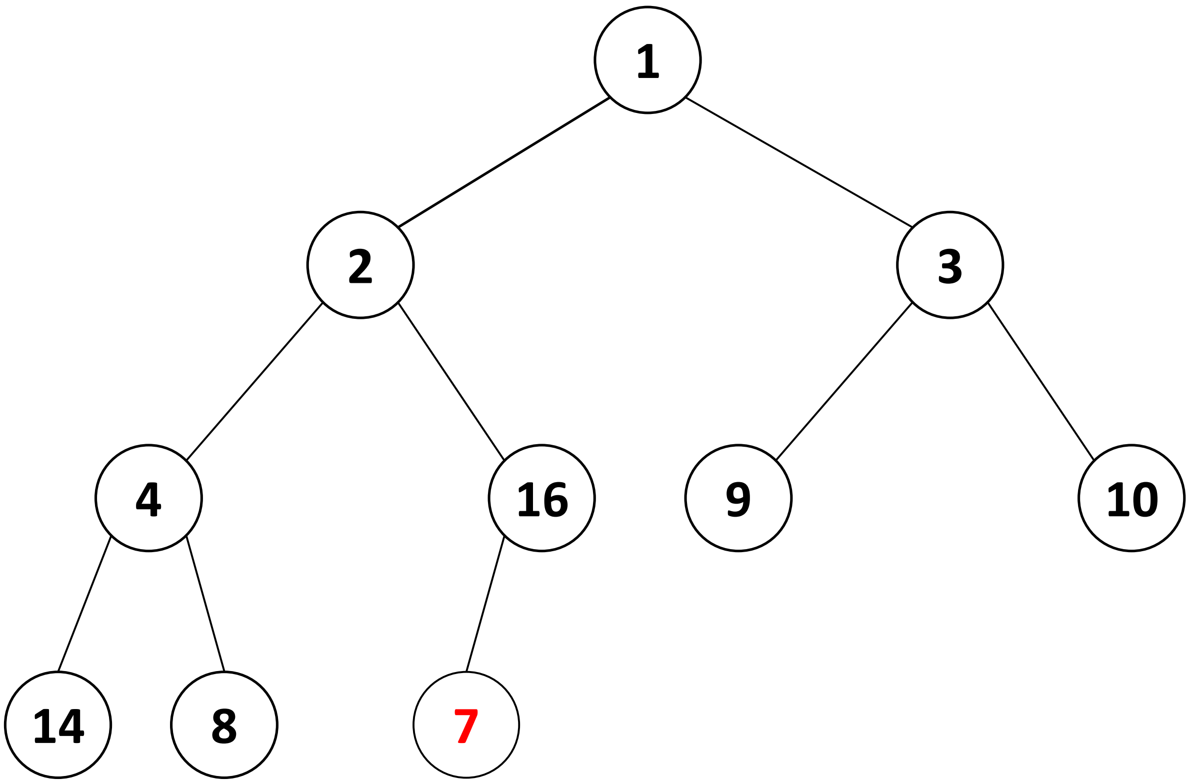
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

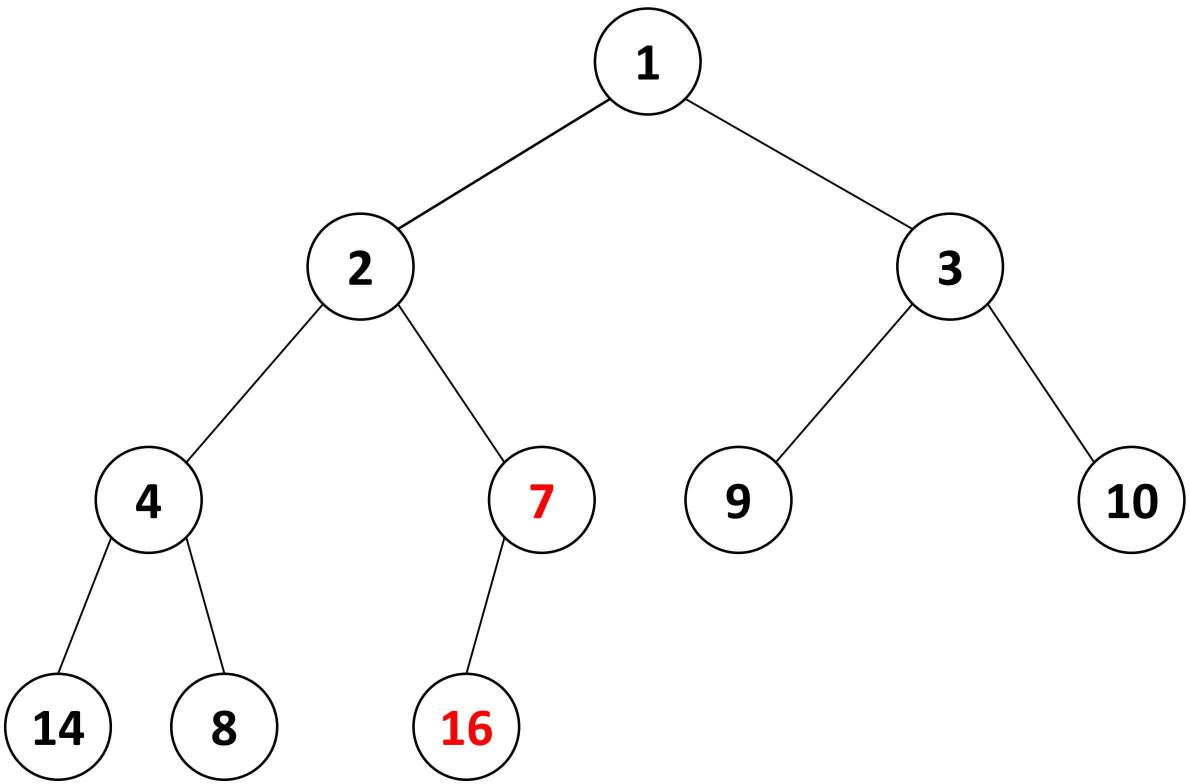
1	2	3	4	16	9	10	14	8	7



Heapsort - Example

- Build a binary heap of n elements

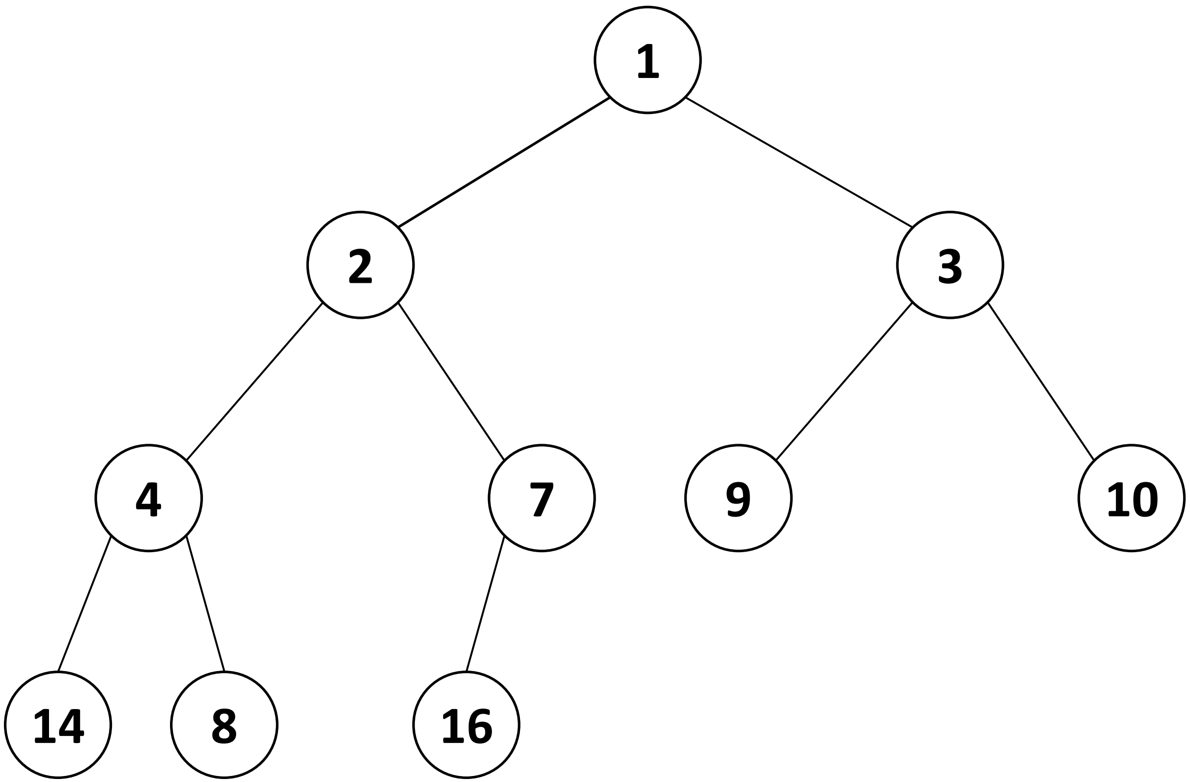
1	2	3	4	7	9	10	14	8	16



Heapsort - Example

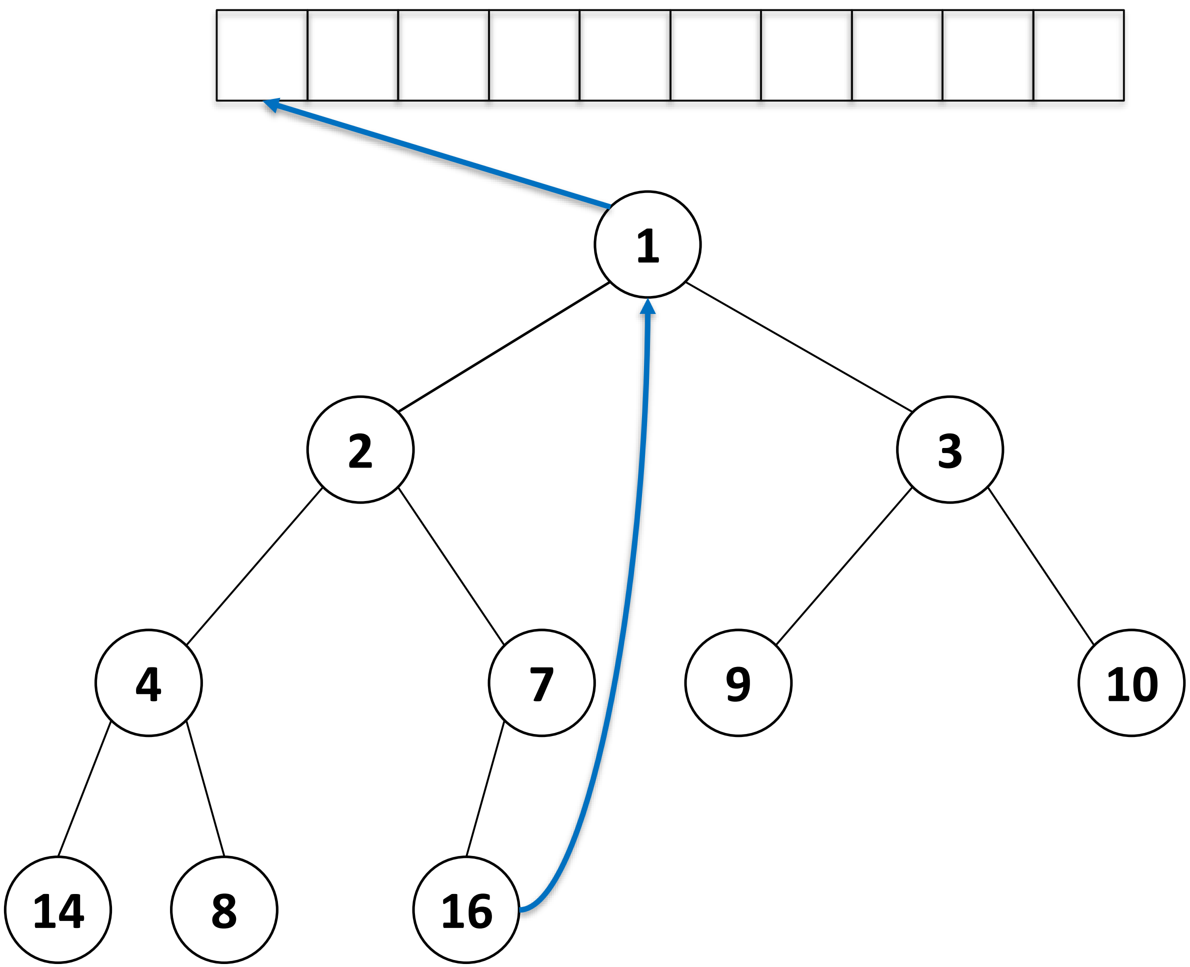
- Build a binary heap of n elements

1	2	3	4	7	9	10	14	8	16



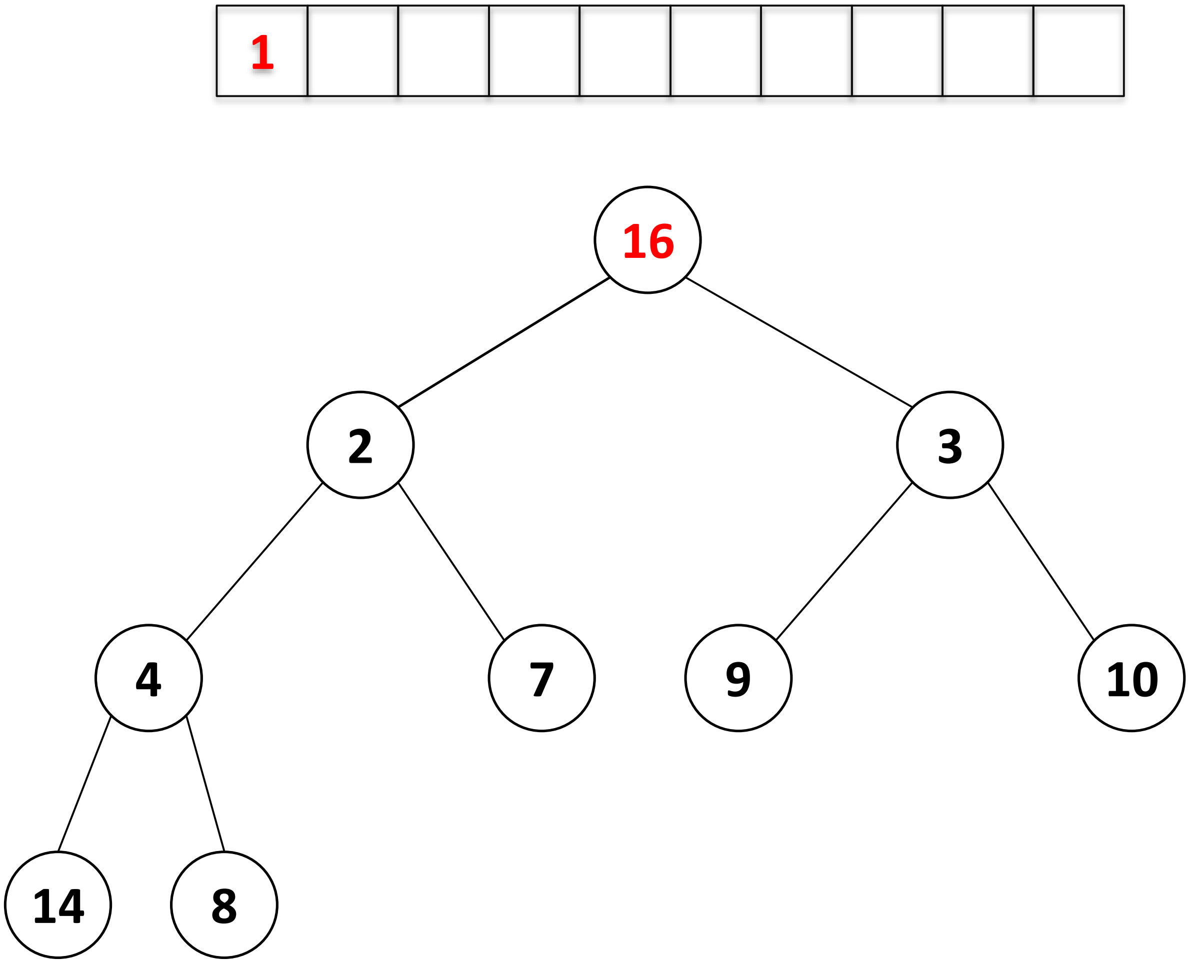
Heapsort - Example

- Perform n Extract-Min operations



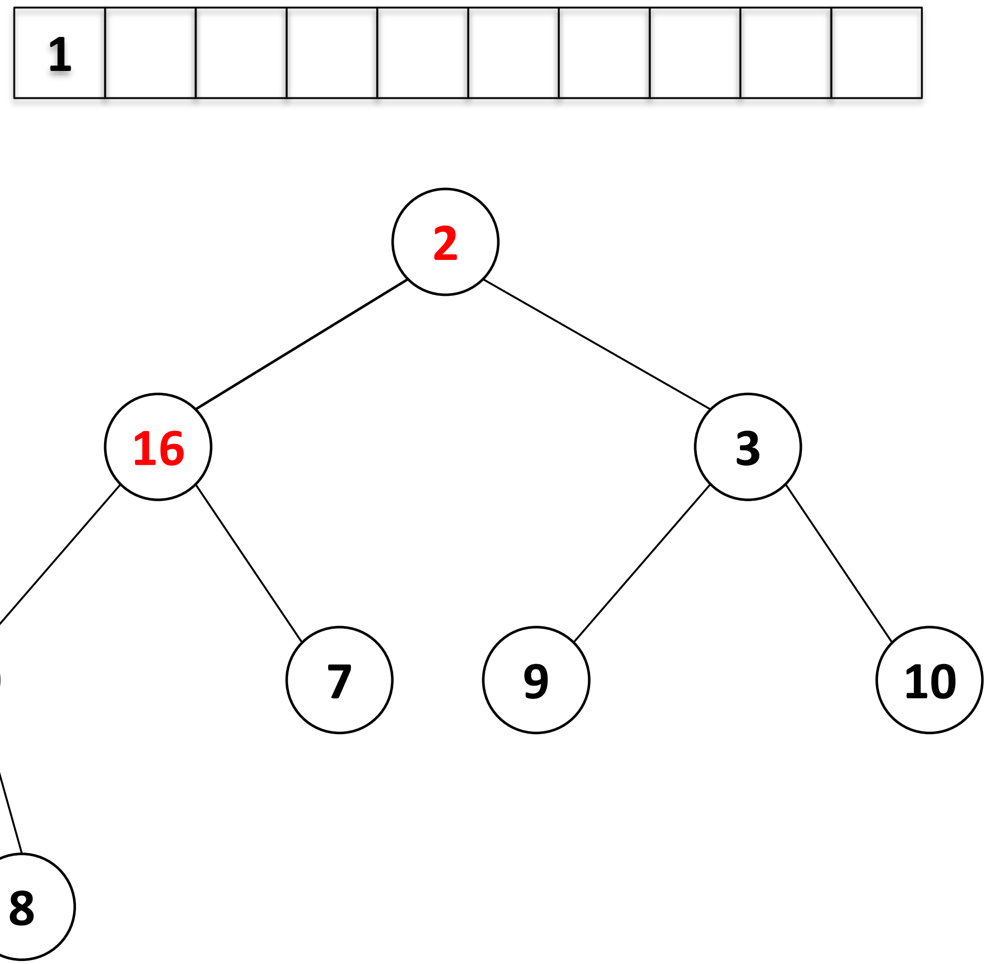
Heapsort - Example

- Perform n Extract-Min operations



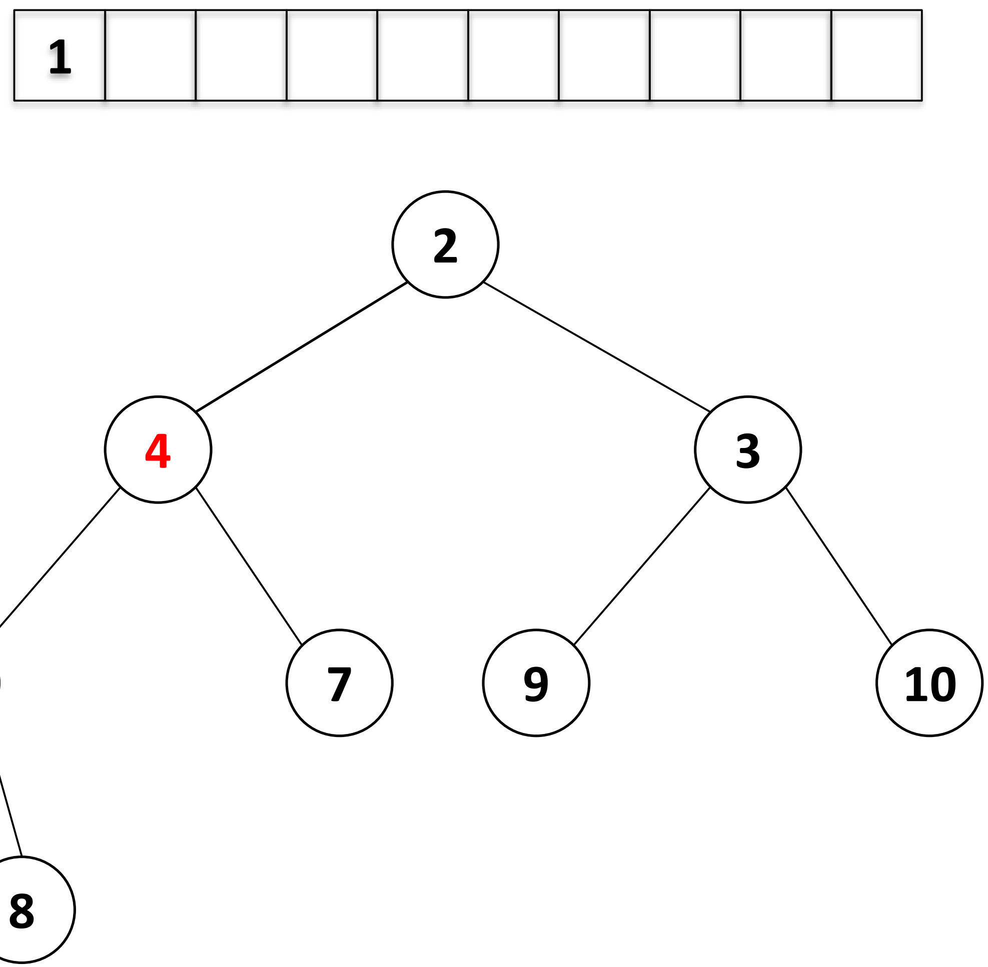
Heapsort - Example

- Perform n Extract-Min operations



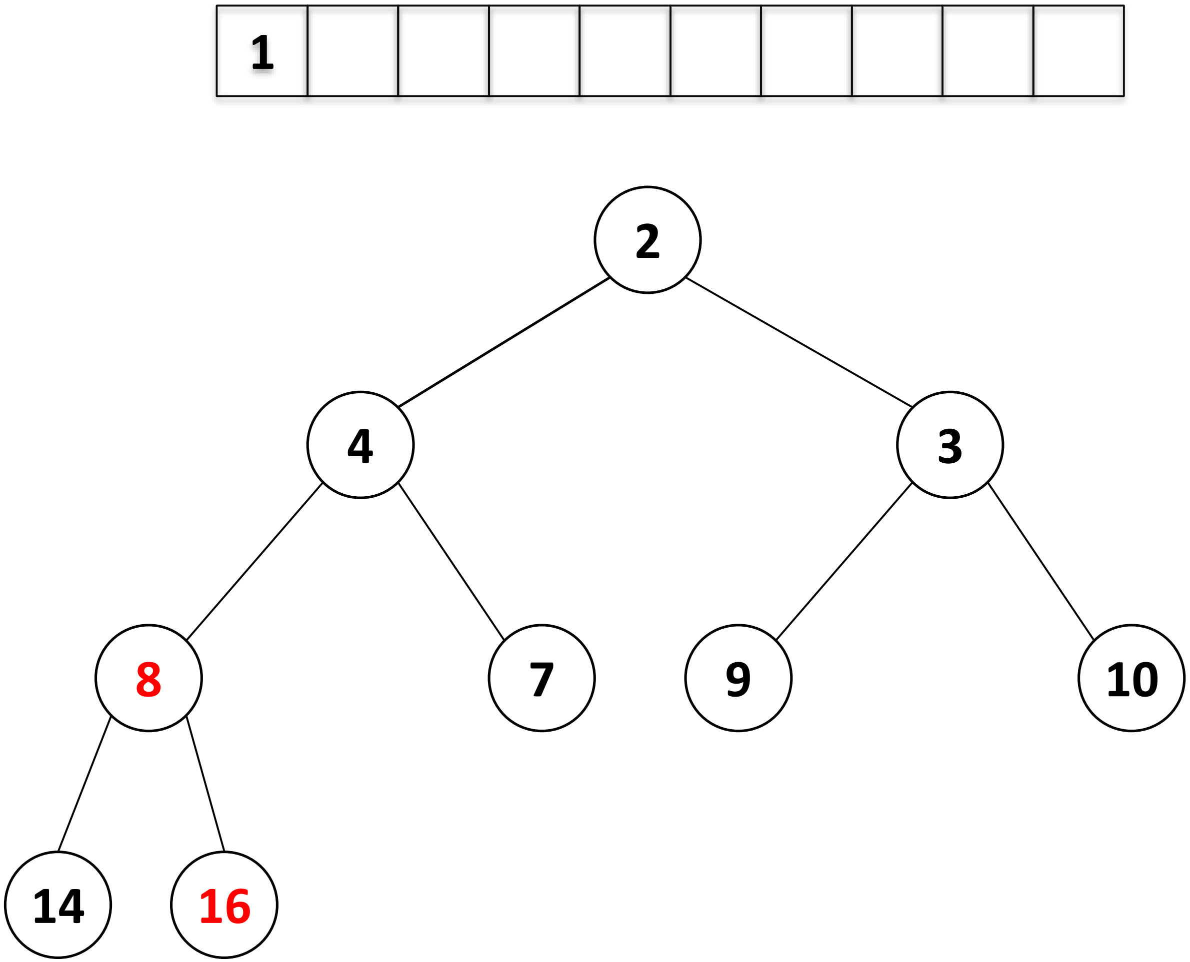
Heapsort - Example

- Perform n Extract-Min operations



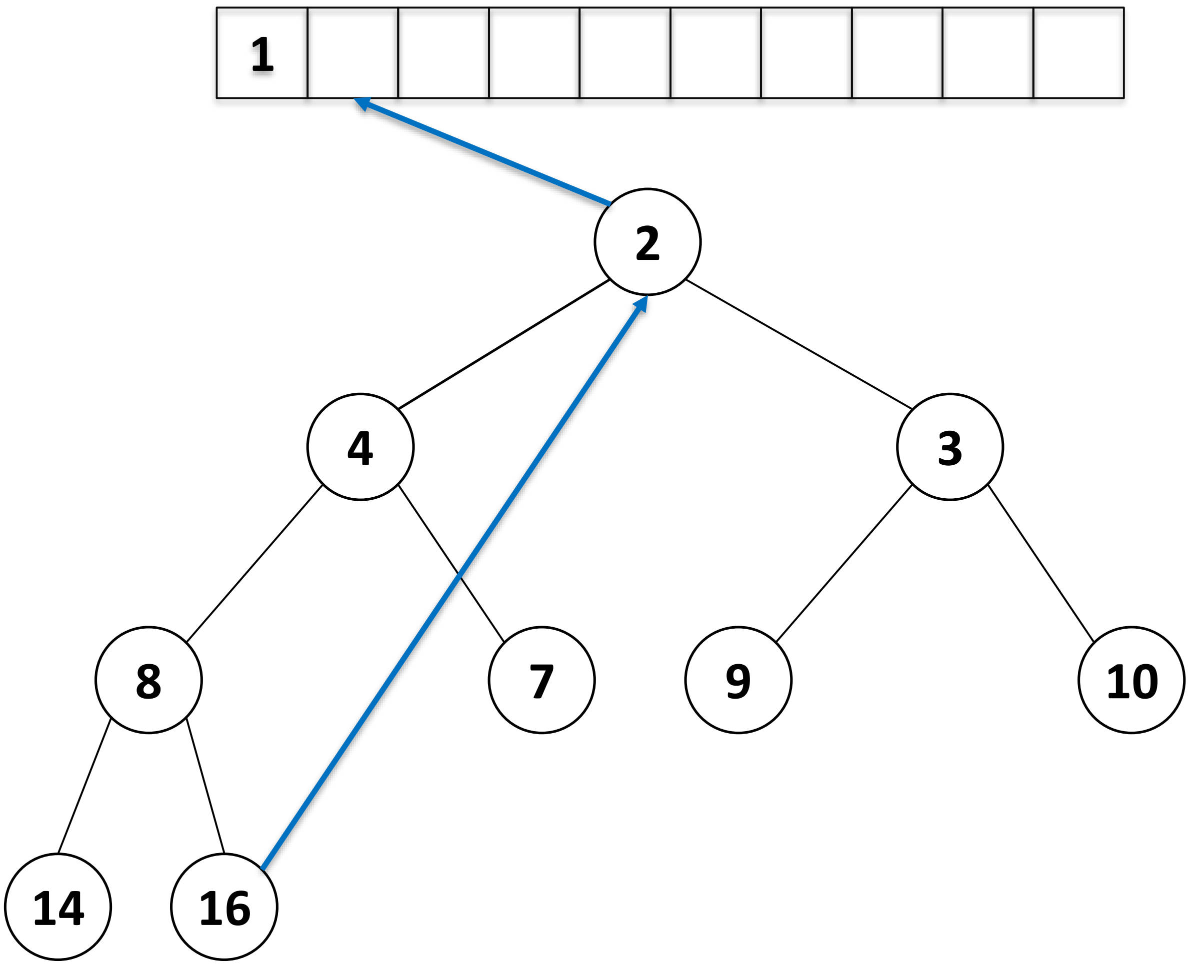
Heapsort - Example

- Perform n Extract-Min operations



Heapsort - Example

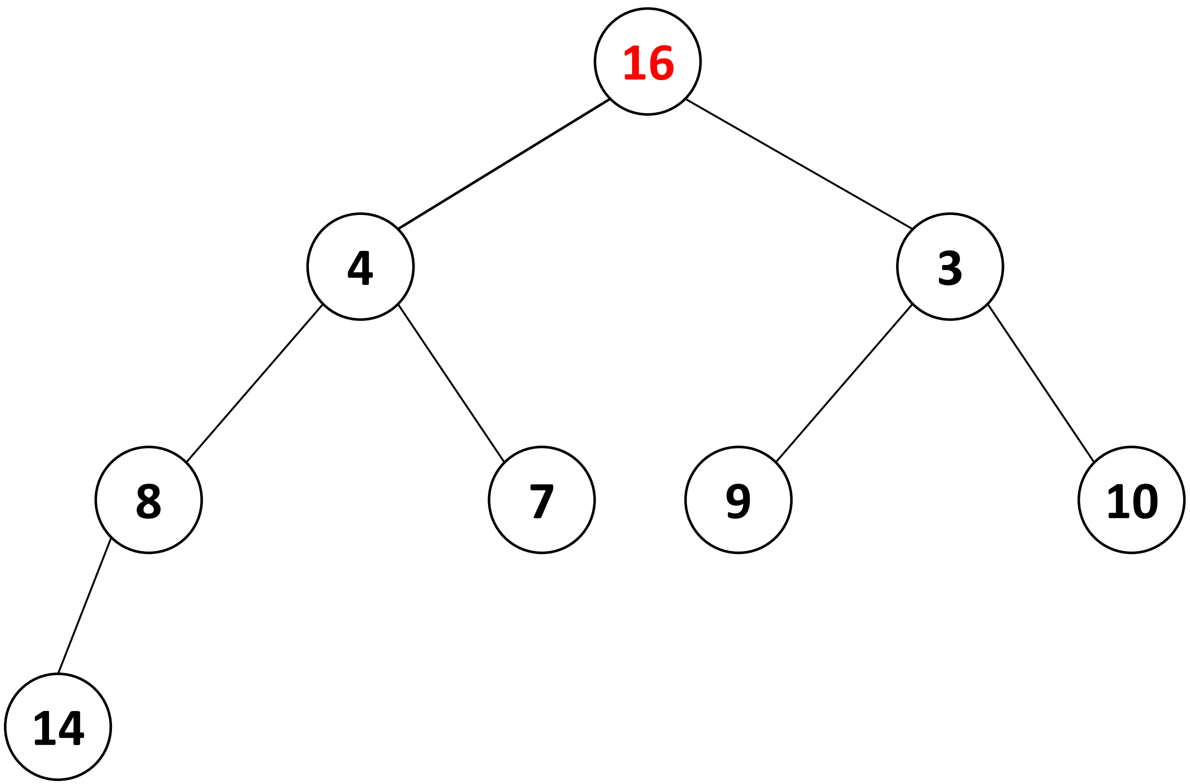
- Perform n Extract-Min operations



Heapsort - Example

- Perform n Extract-Min operations

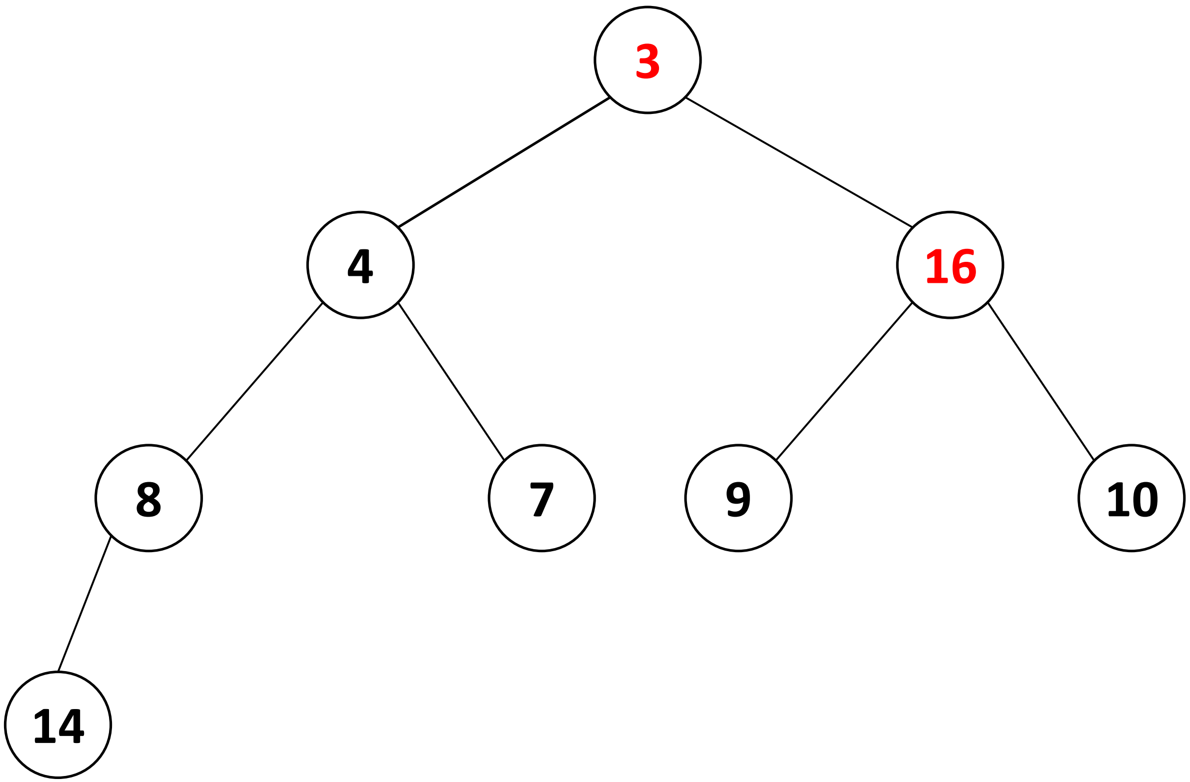
1	2								



Heapsort - Example

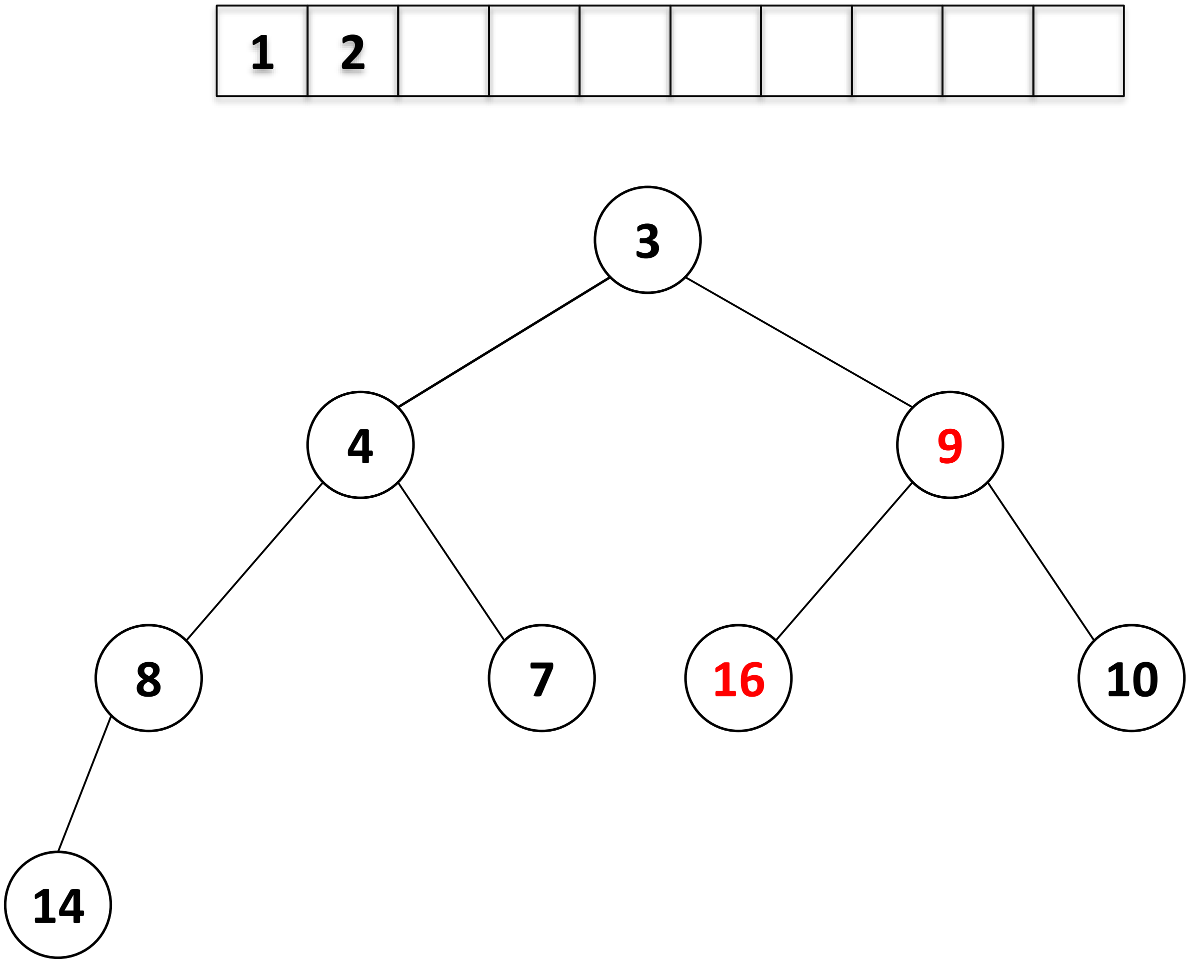
- Perform n Extract-Min operations

1	2								



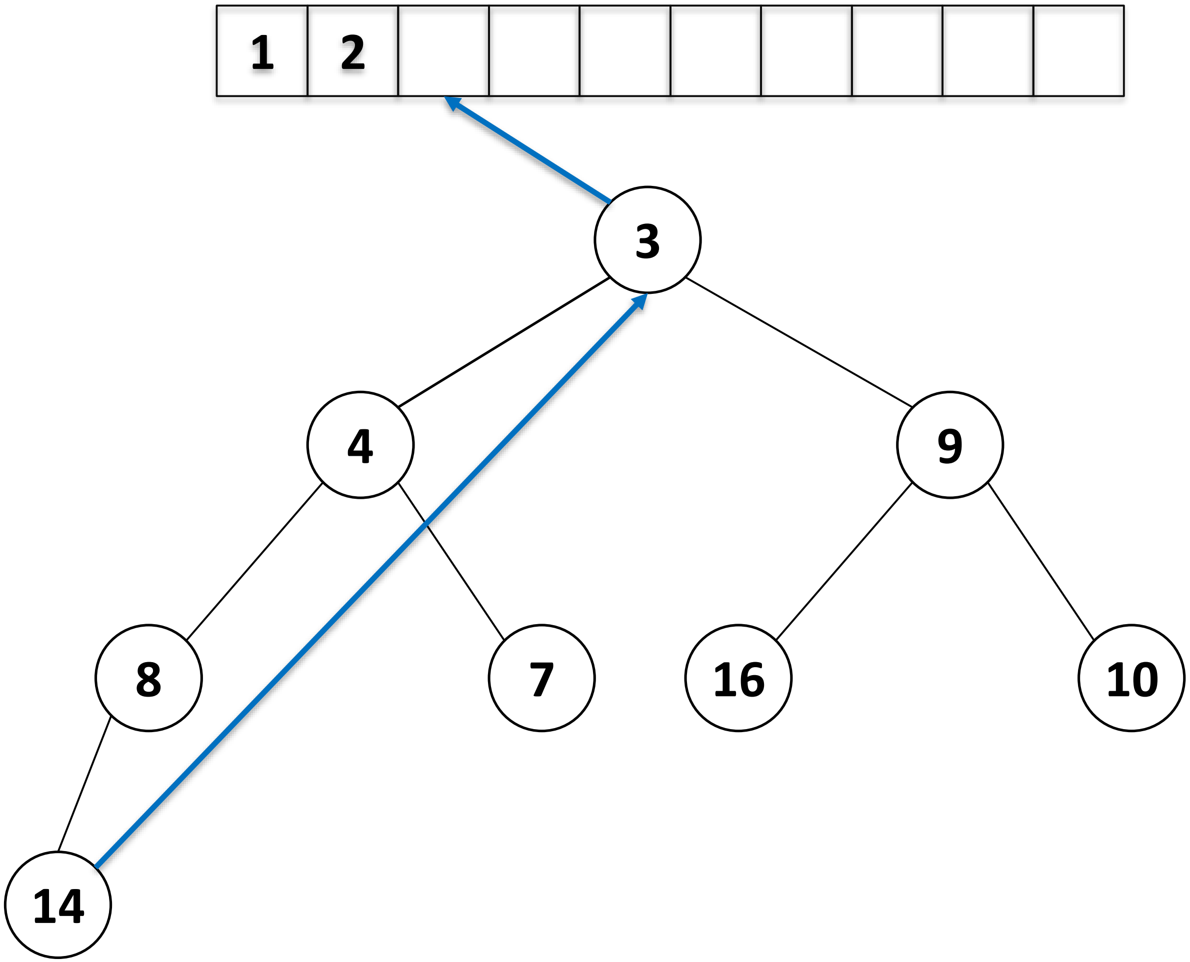
Heapsort - Example

- Perform n Extract-Min operations



Heapsort - Example

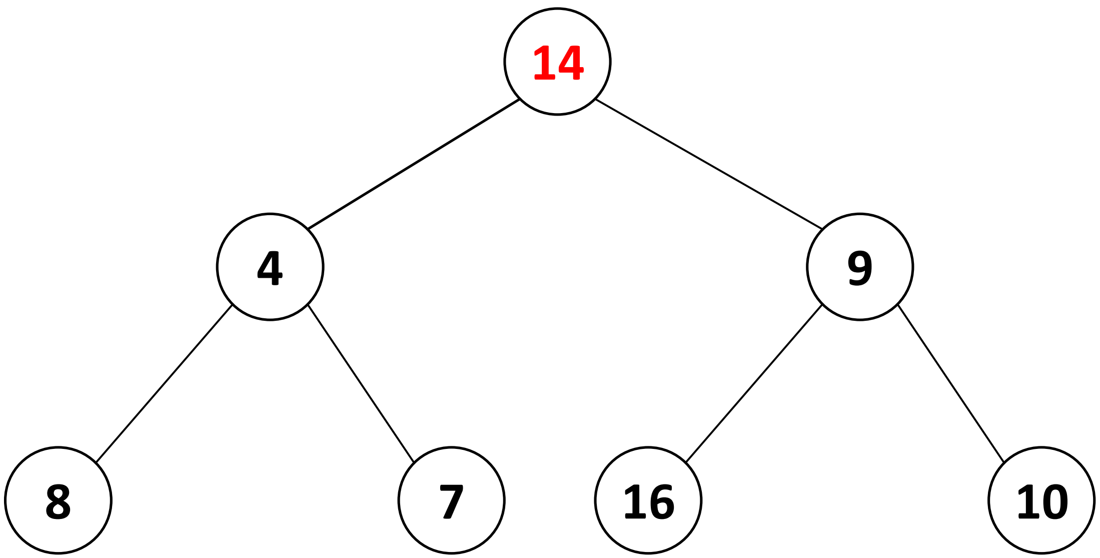
- Perform n Extract-Min operations



Heapsort - Example

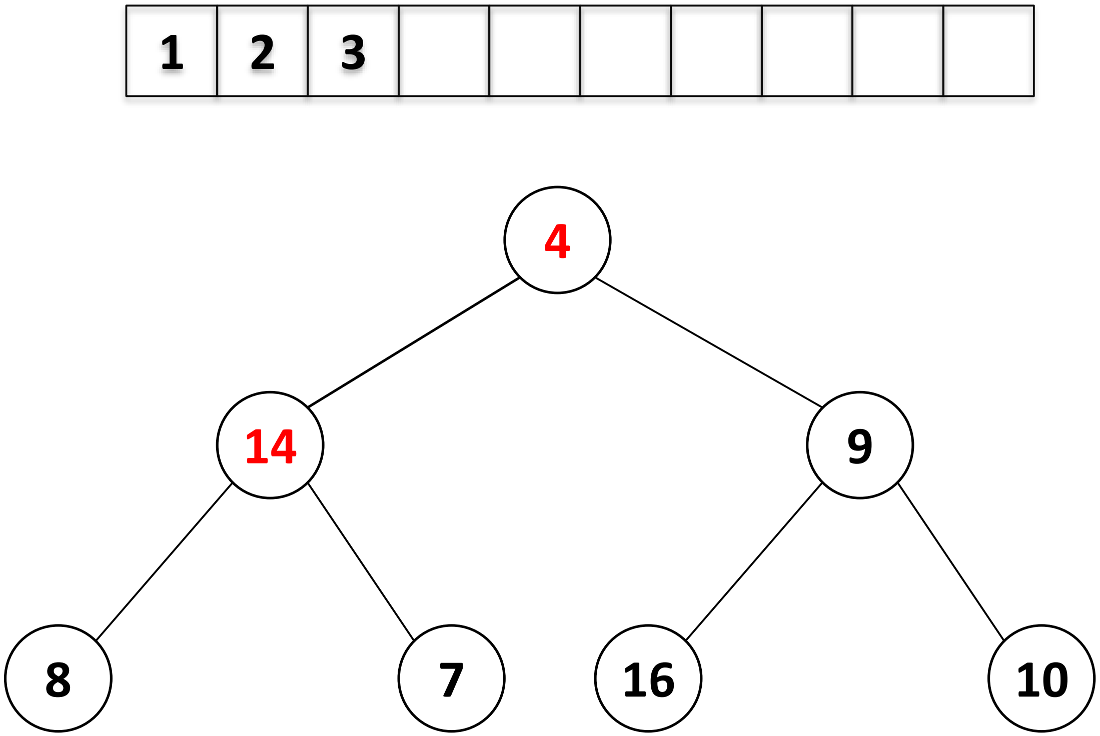
- Perform n Extract-Min operations

1	2	3							



Heapsort - Example

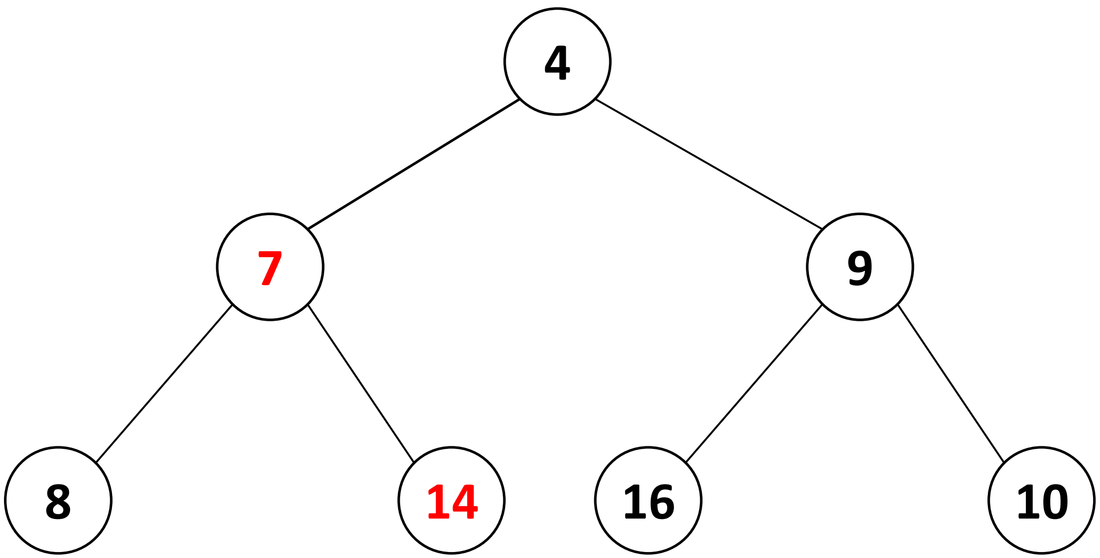
- Perform n Extract-Min operations



Heapsort - Example

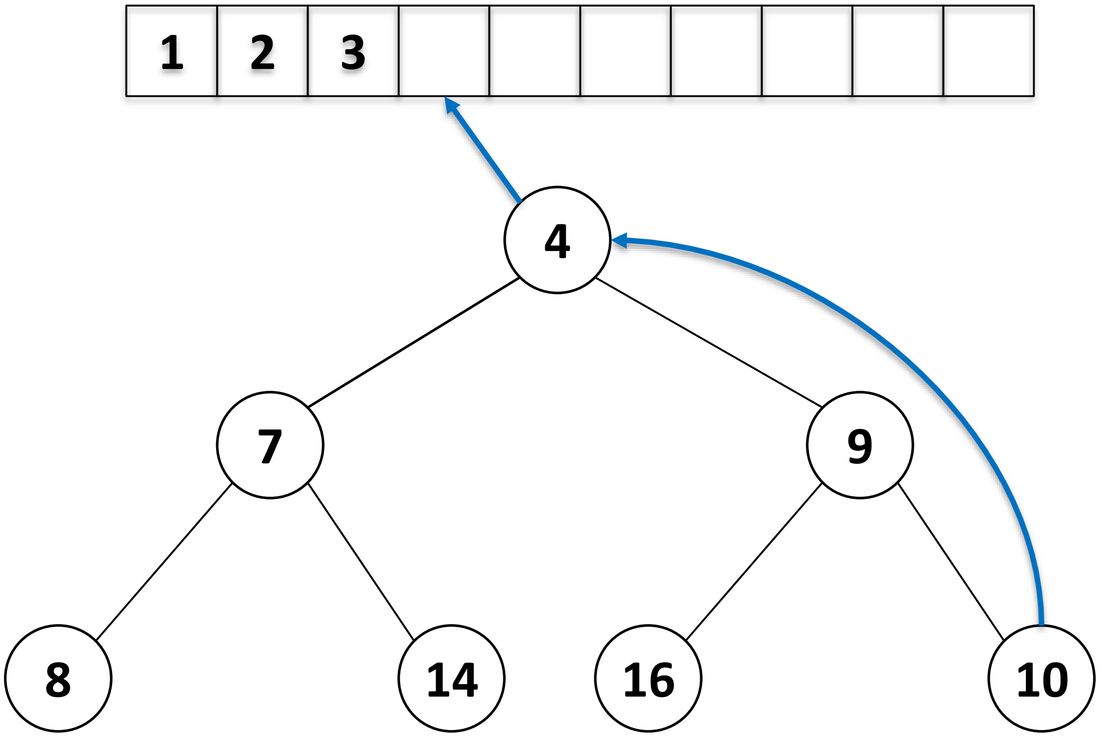
- Perform n Extract-Min operations

1	2	3							



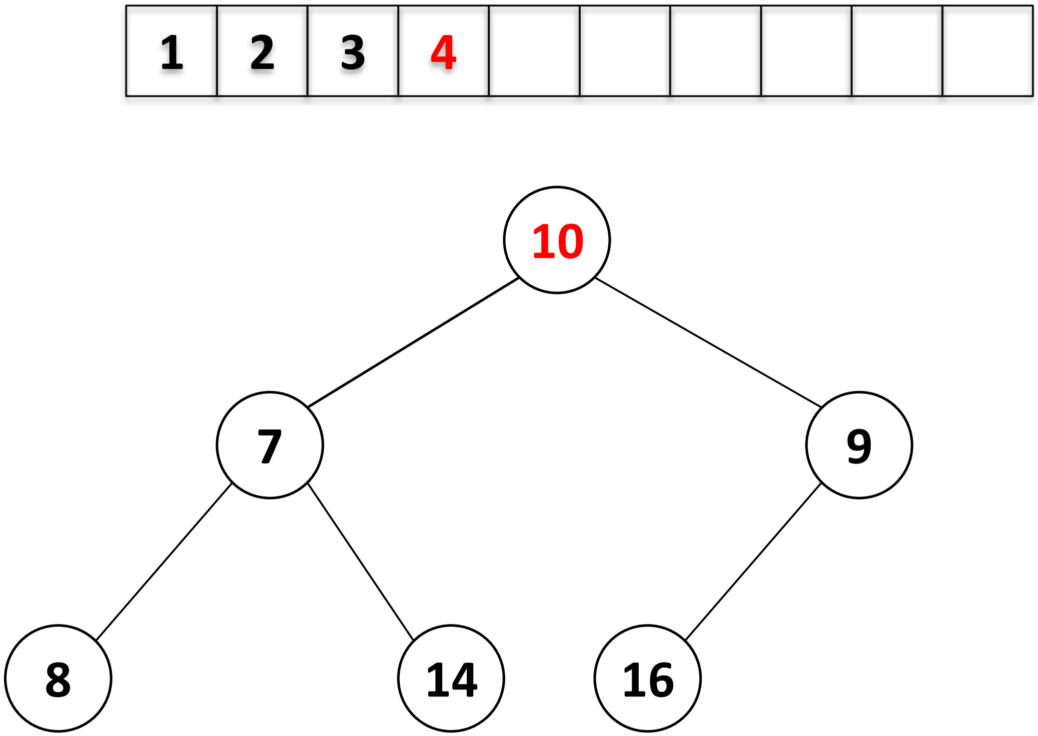
Heapsort - Example

- Perform n Extract-Min operations



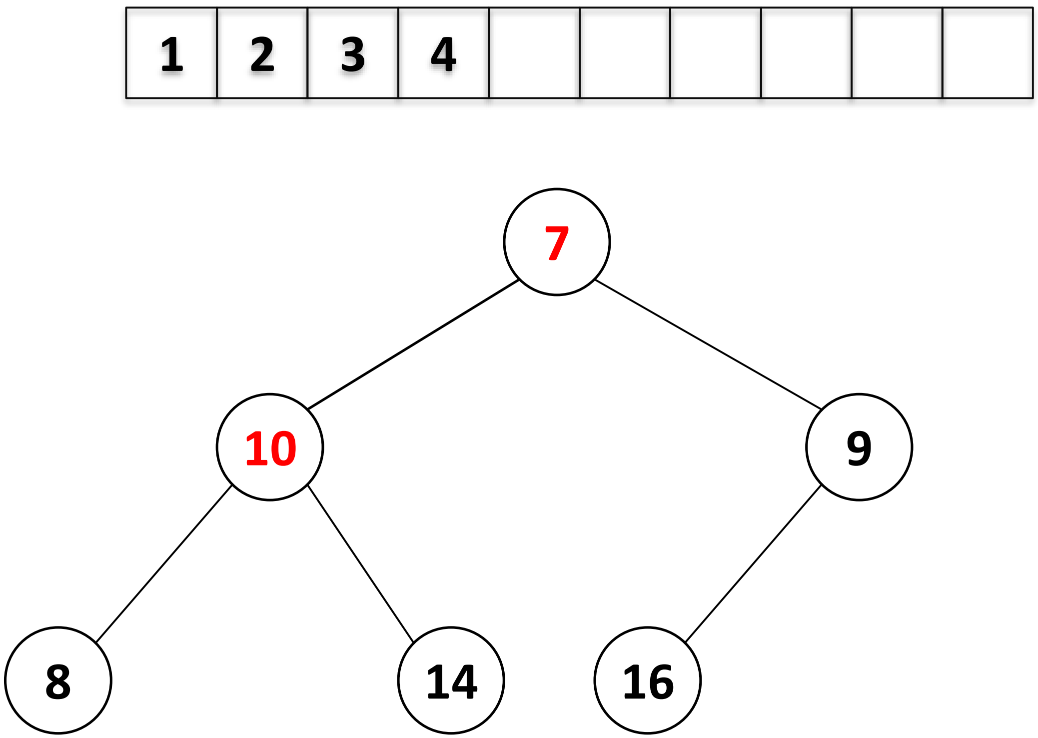
Heapsort - Example

- Perform n Extract-Min operations



Heapsort - Example

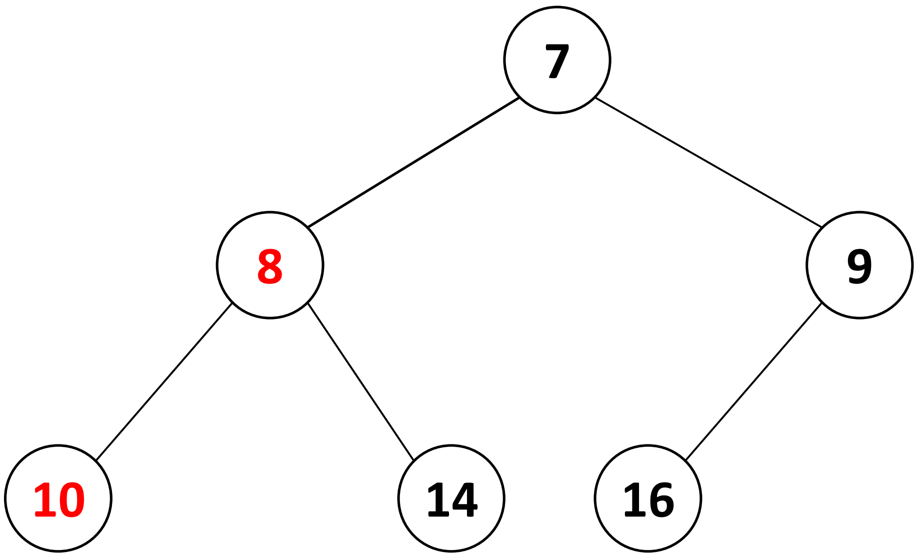
- Perform n Extract-Min operations



Heapsort - Example

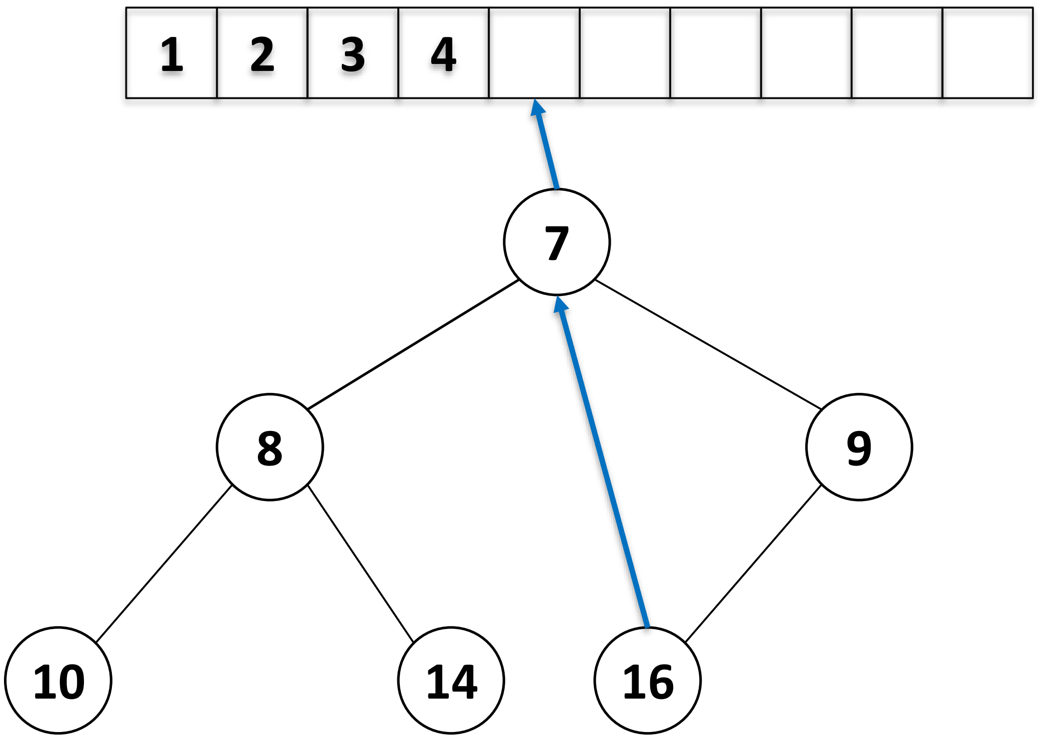
- Perform n Extract-Min operations

1	2	3	4						



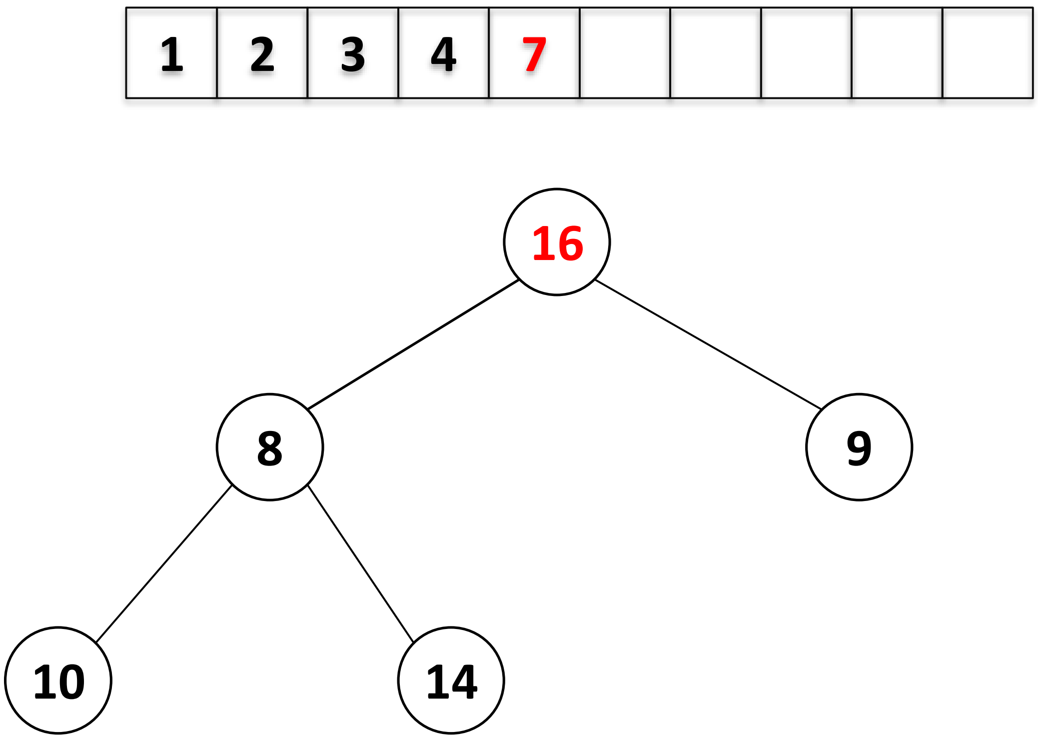
Heapsort - Example

- Perform n Extract-Min operations



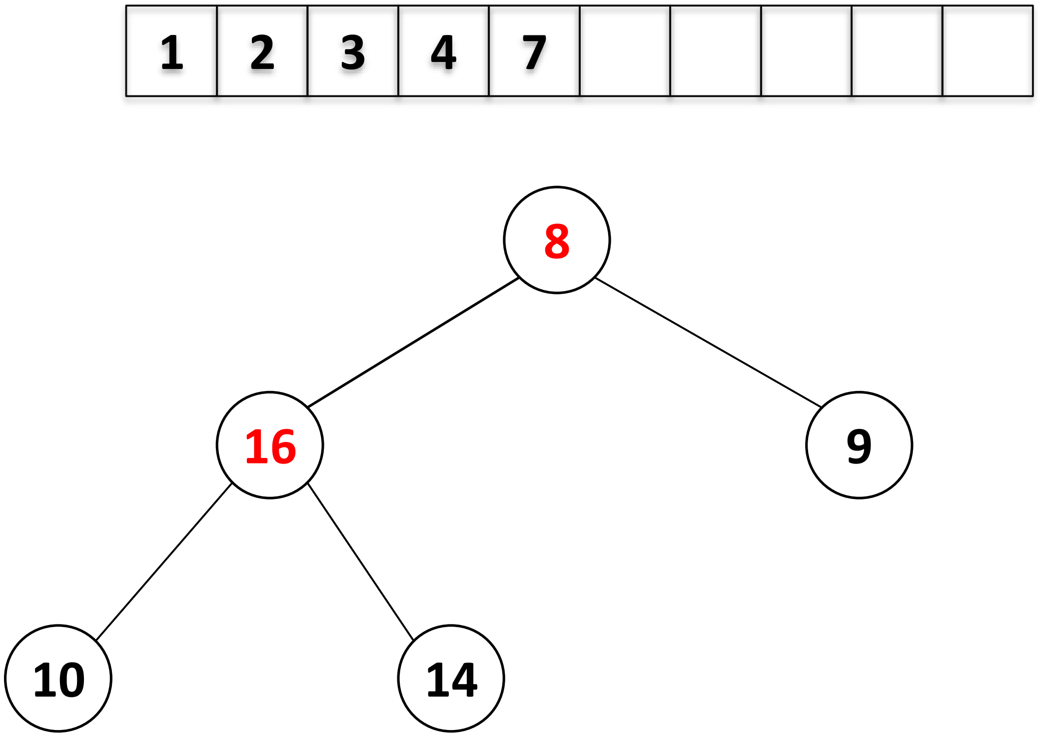
Heapsort - Example

- Perform n Extract-Min operations



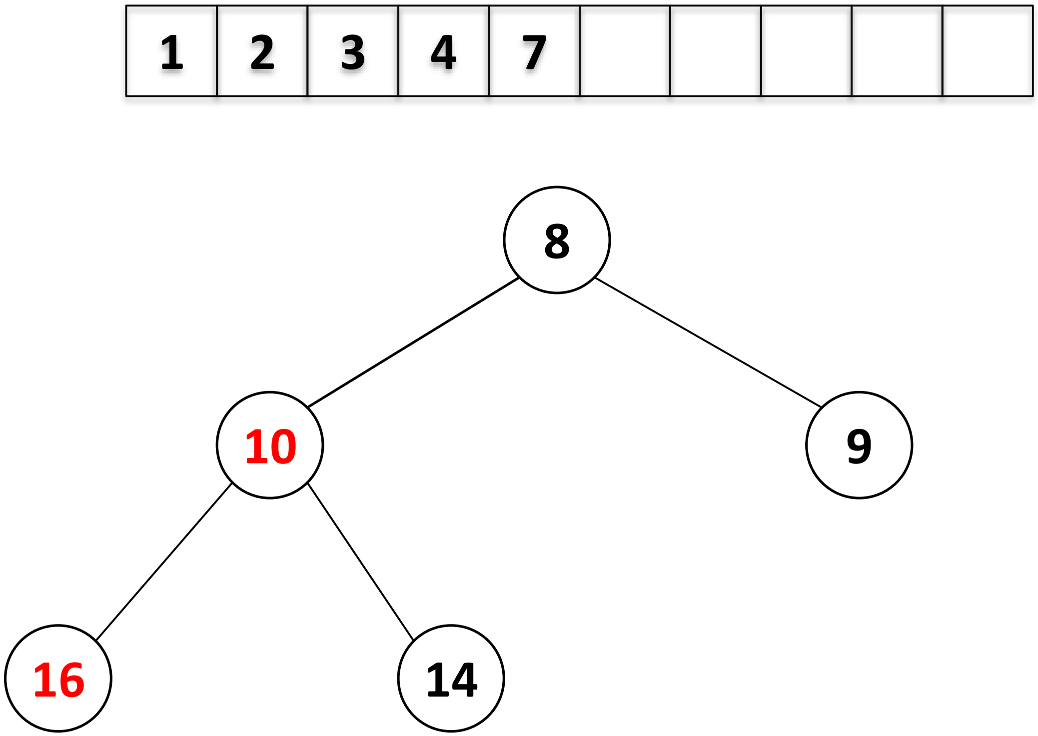
Heapsort - Example

- Perform n Extract-Min operations



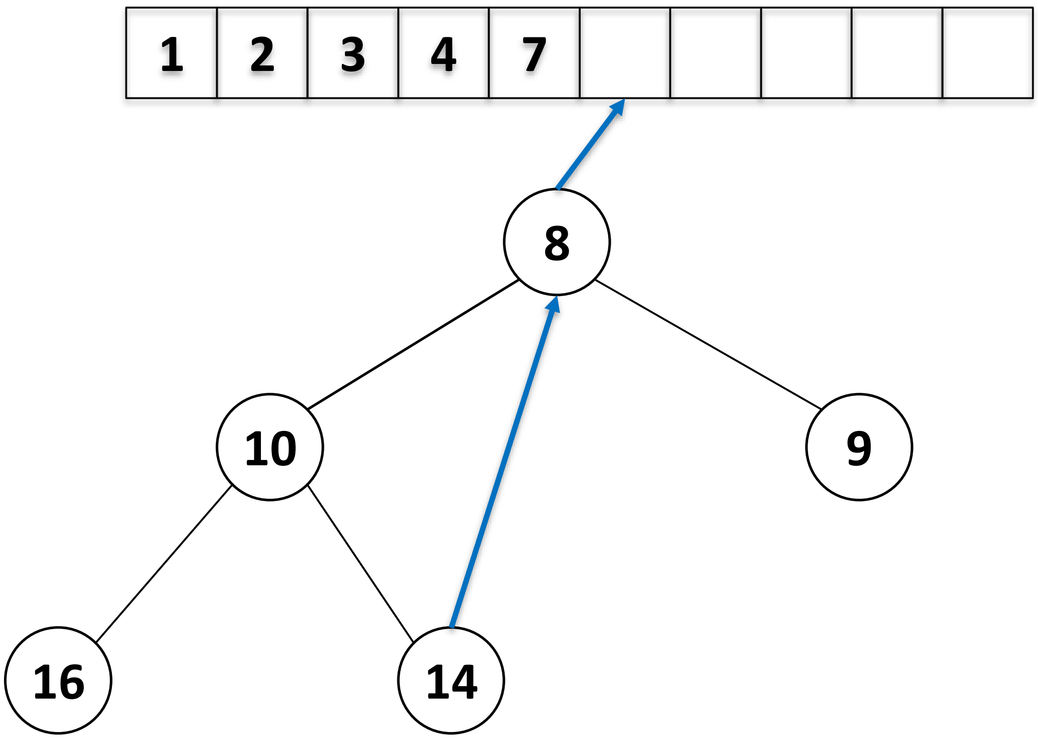
Heapsort - Example

- Perform n Extract-Min operations



Heapsort - Example

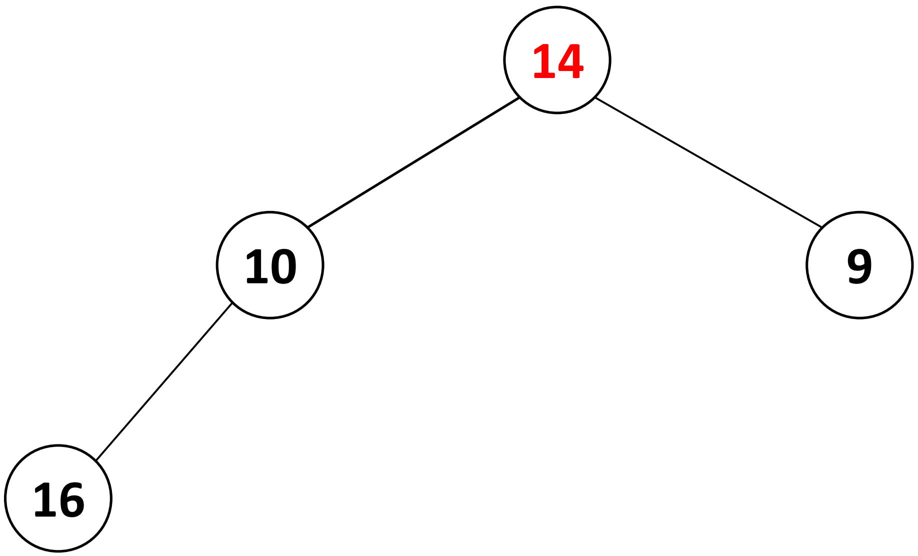
- Perform n Extract-Min operations



Heapsort - Example

- Perform n Extract-Min operations

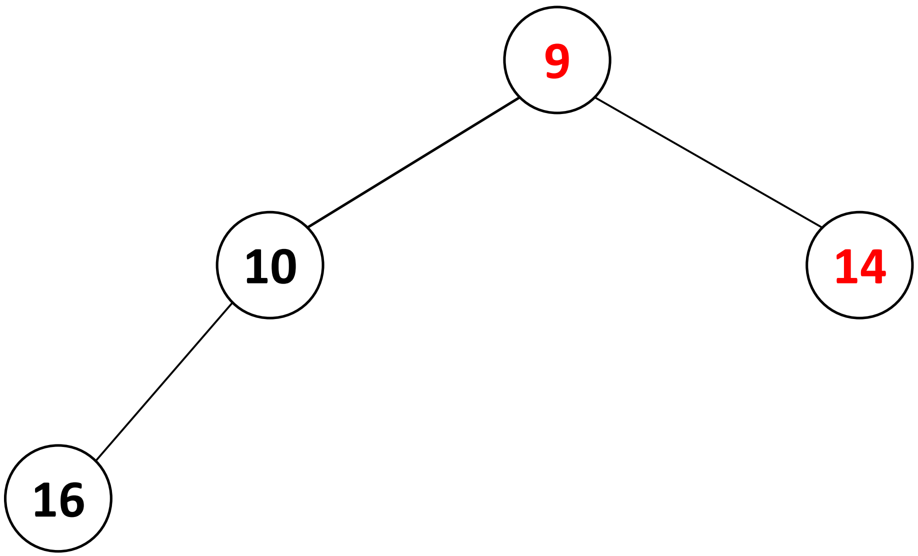
1	2	3	4	7	8				



Heapsort - Example

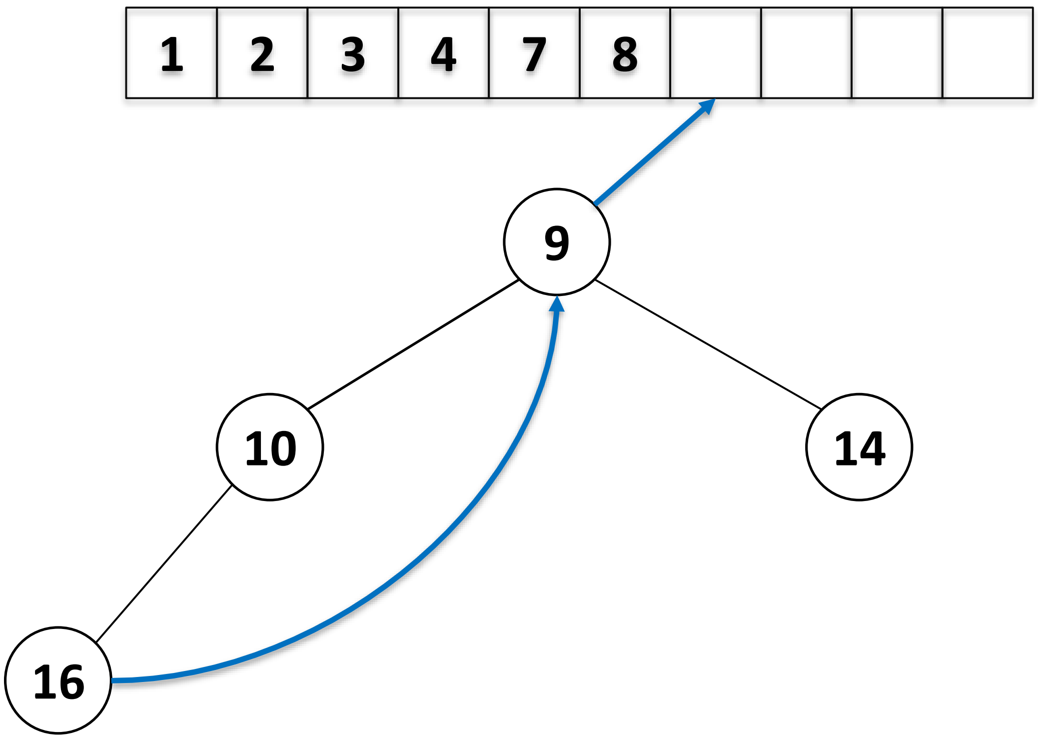
- Perform n Extract-Min operations

1	2	3	4	7	8				



Heapsort - Example

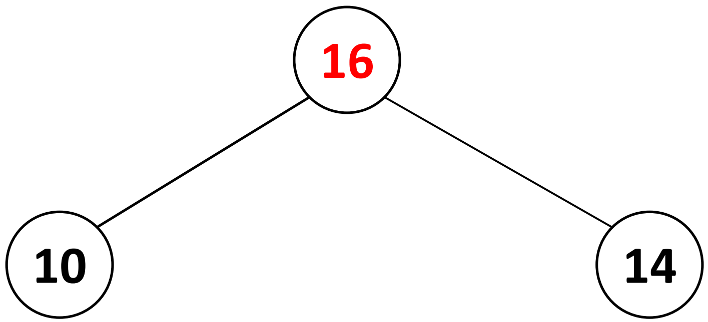
- Perform n Extract-Min operations



Heapsort - Example

- Perform n Extract-Min operations

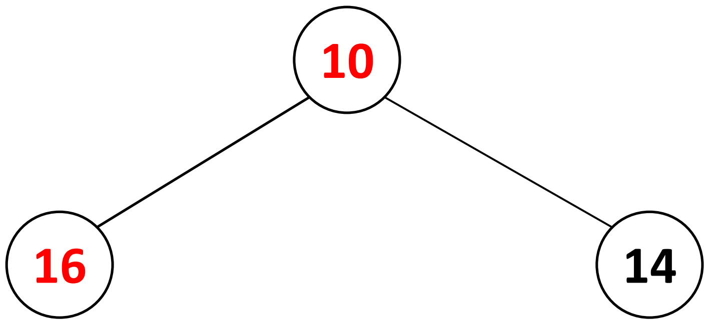
1	2	3	4	7	8	9			



Heapsort - Example

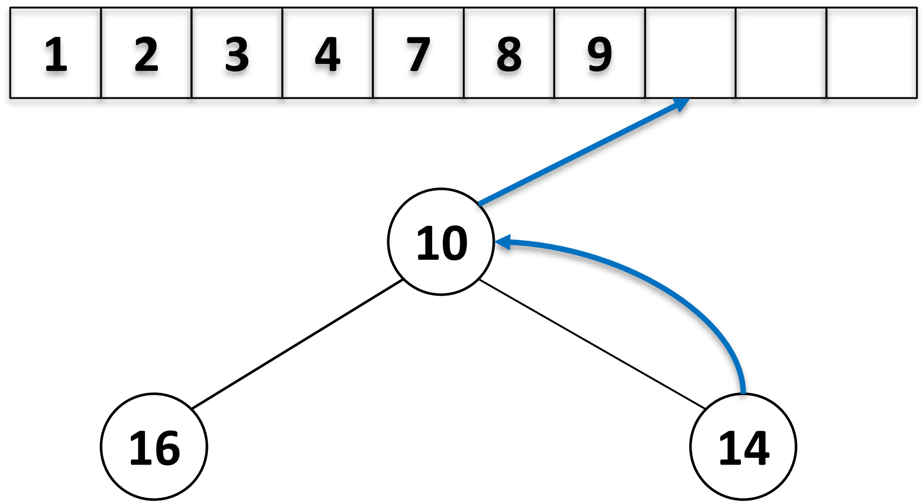
- Perform n Extract-Min operations

1	2	3	4	7	8	9			



Heapsort - Example

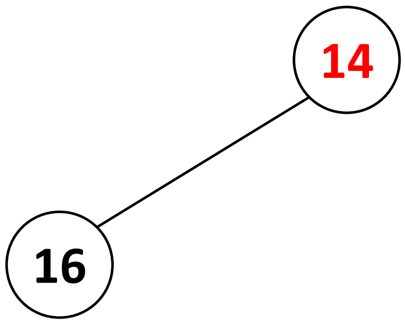
- Perform n Extract-Min operations



Heapsort - Example

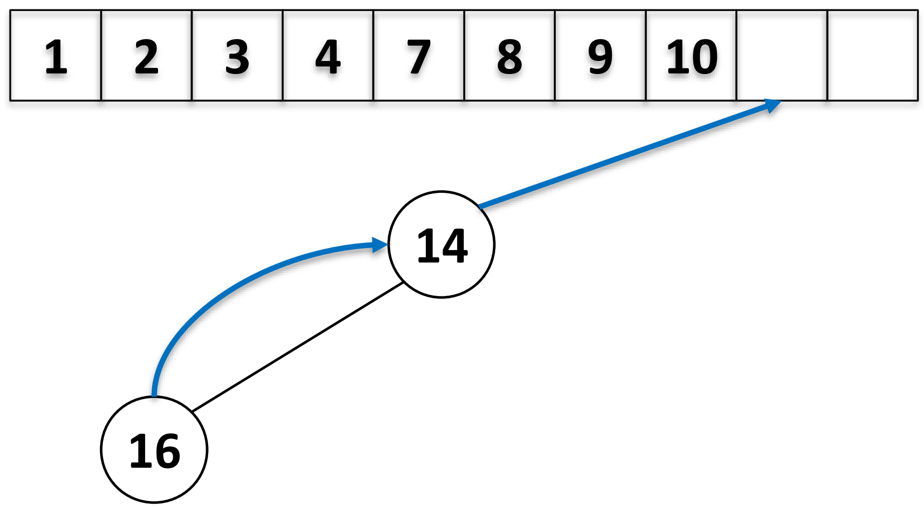
- Perform n Extract-Min operations

1	2	3	4	7	8	9	10		



Heapsort - Example

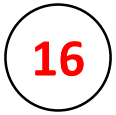
- Perform n Extract-Min operations



Heapsort - Example

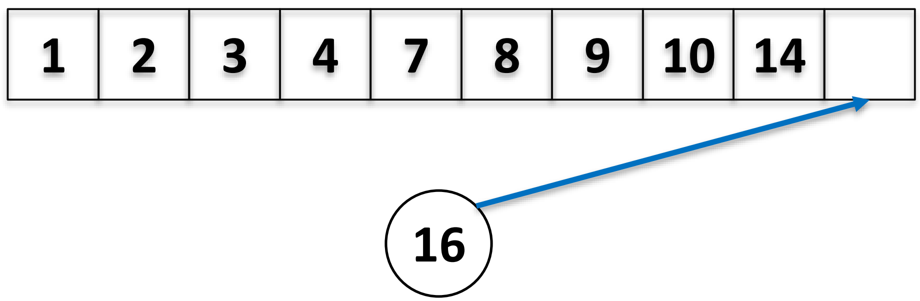
- Perform n Extract-Min operations

1	2	3	4	7	8	9	10	14	



Heapsort - Example

- Perform n Extract-Min operations



Heapsort - Example

- Perform n Extract-Min operations

1	**2**	**3**	**4**	**7**	**8**	**9**	**10**	**14**	**16**

Heapsort - Example

- Perform n Extract-Min operations

1	2	3	4	7	8	9	10	14	16

Summary

- Priority queue is an abstract data structure that supports two operations: Insert and Extract-Min.

Summary

- Priority queue is an abstract data structure that supports two operations: Insert and Extract-Min.
- If priority queues are implemented using heaps, then these two operations are supported in O(log n) time.

Summary

- Priority queue is an abstract data structure that supports two operations: Insert and Extract-Min.

- If priority queues are implemented using heaps, then these two operations are supported in O(log n) time.

- Heapsort takes O(n log n) time, which is as efficient as merge sort and quicksort.

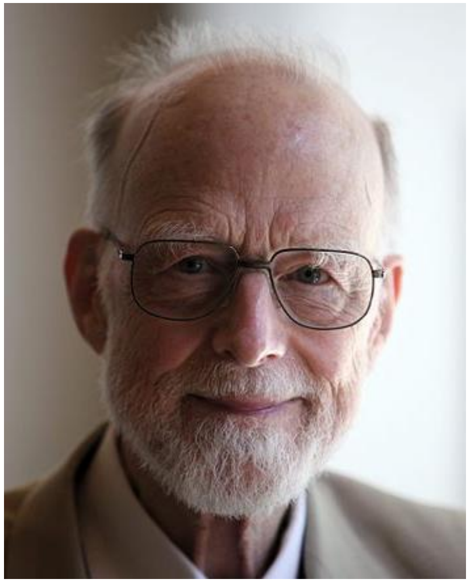
Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Review of Classical Sorting Algorithms

John von Neumann

Merge Sort Algorithm was invented in 1945



Tony Hoare

Quicksort Algorithm was invented in 1959

J. W. J. Williams

Heapsort Algorithm was invented in 1964

Which algorithm is the best in practice?

Objective

- All sorting algorithms seen so far are based on comparing elements
  - E.g., insertion sort, merge sort, and heapsort

Objective

- All sorting algorithms seen so far are based on comparing elements
  - E.g., insertion sort, merge sort, and heapsort
- Insertion sort has worst-case running time $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Objective

- All sorting algorithms seen so far are based on comparing elements
  - E.g., insertion sort, merge sort, and heapsort
- Insertion sort has worst-case running time $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Question

Can we do better?

Objective

- All sorting algorithms seen so far are based on comparing elements
  - E.g., insertion sort, merge sort, and heapsort
- Insertion sort has worst-case running time $\Theta(n^2)$, while the others have worst-case running time $\Theta(n \log n)$

Question

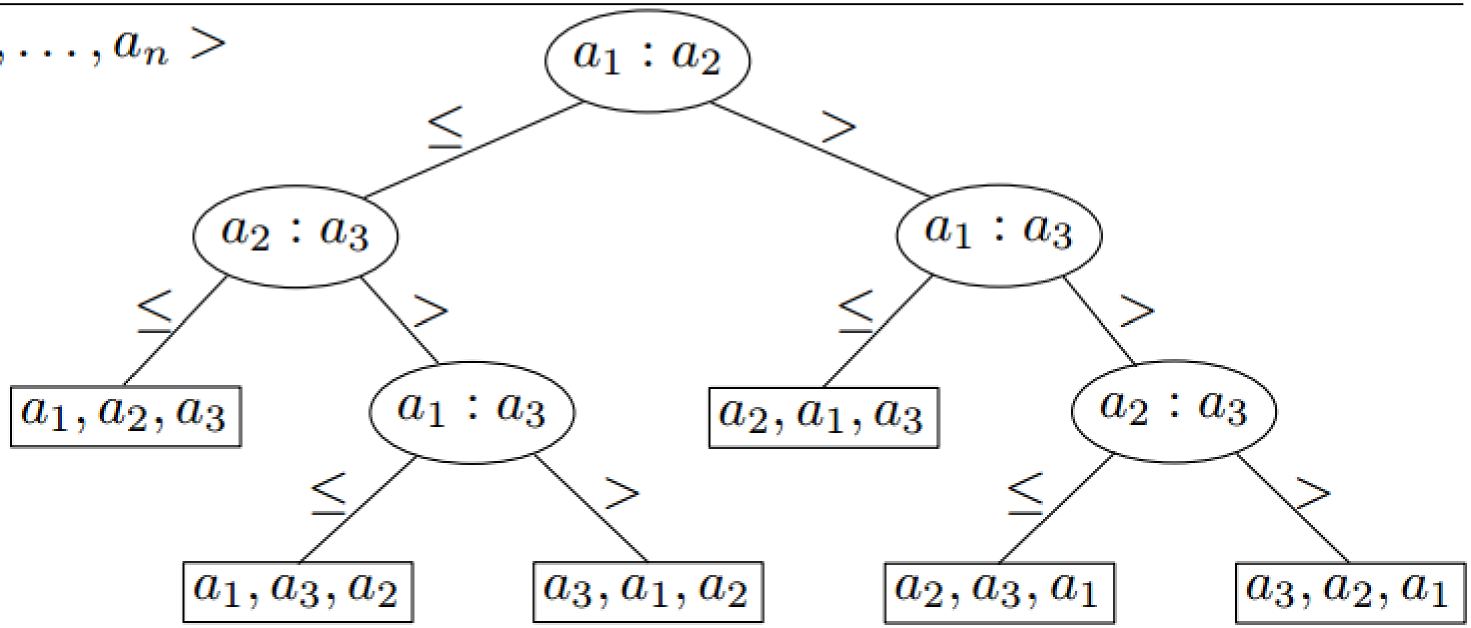
Can we do better?

Goal

We will prove that any comparison-based sorting algorithm has a worst-case running time $\Omega(n \log n)$.

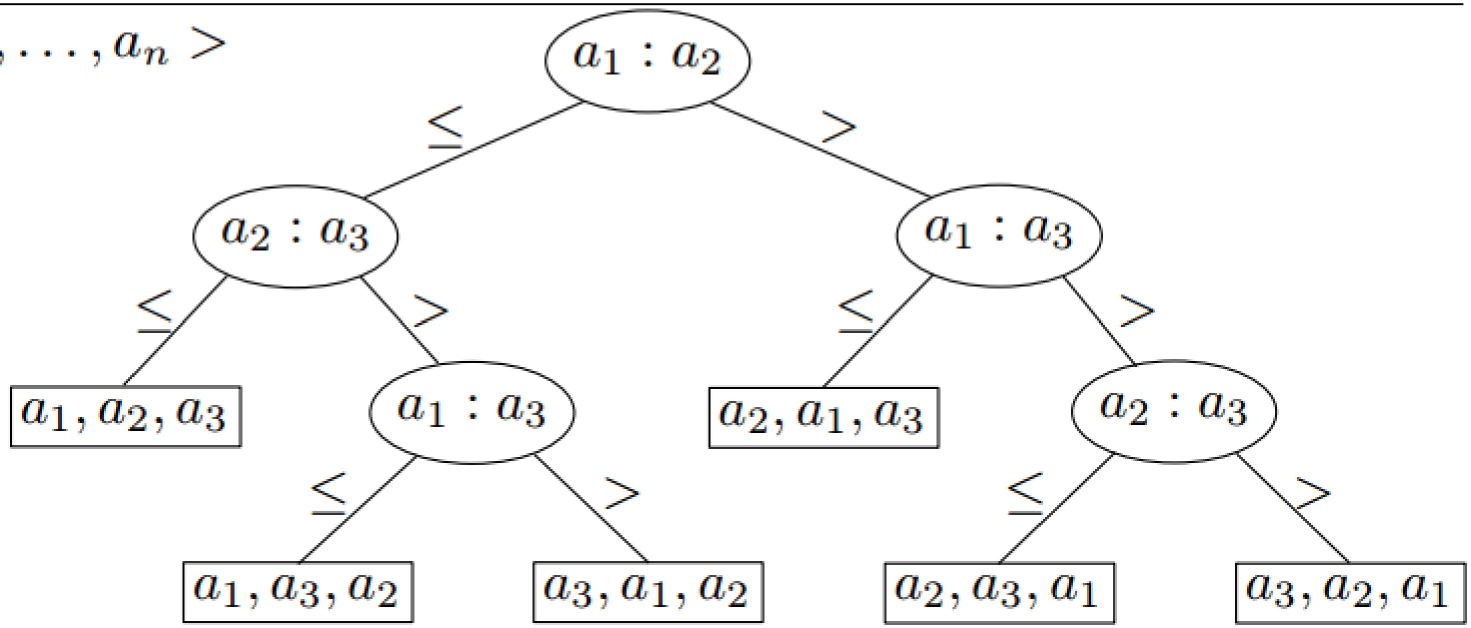
Decision-tree Example

Sort $< a_1, a_2, \ldots, a_n >$



Decision-tree Example

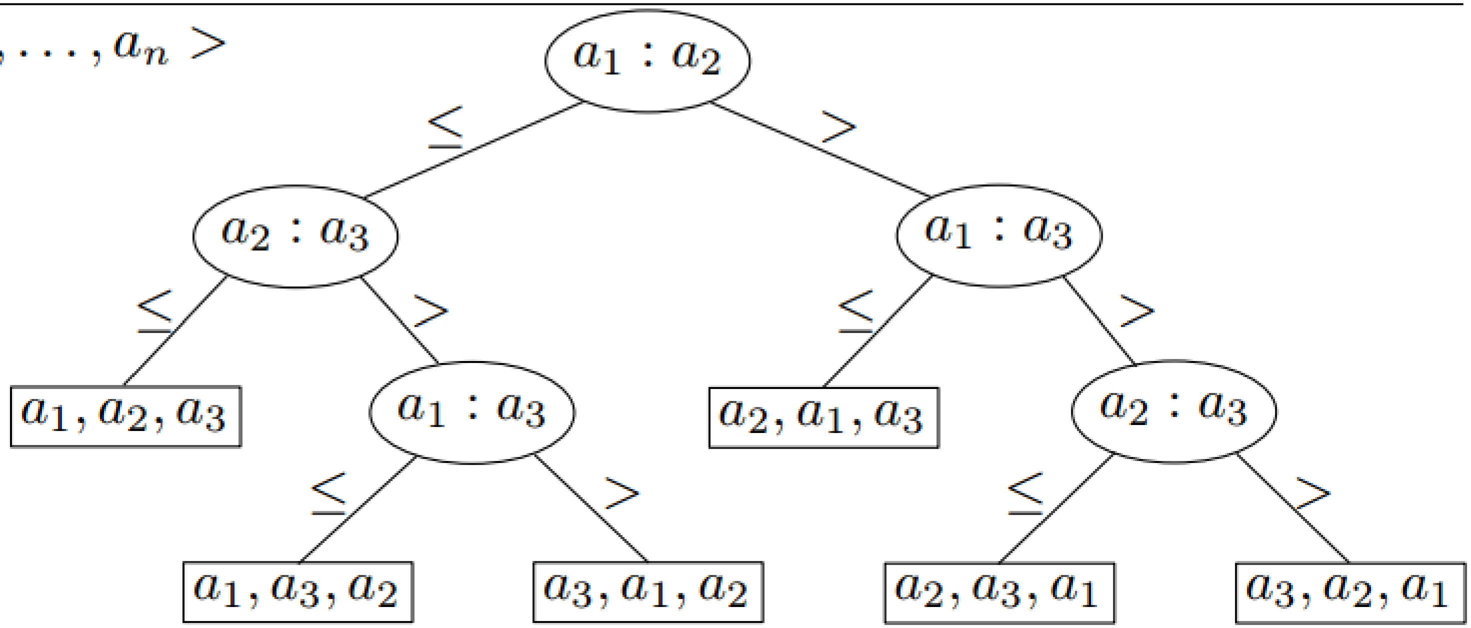
Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}

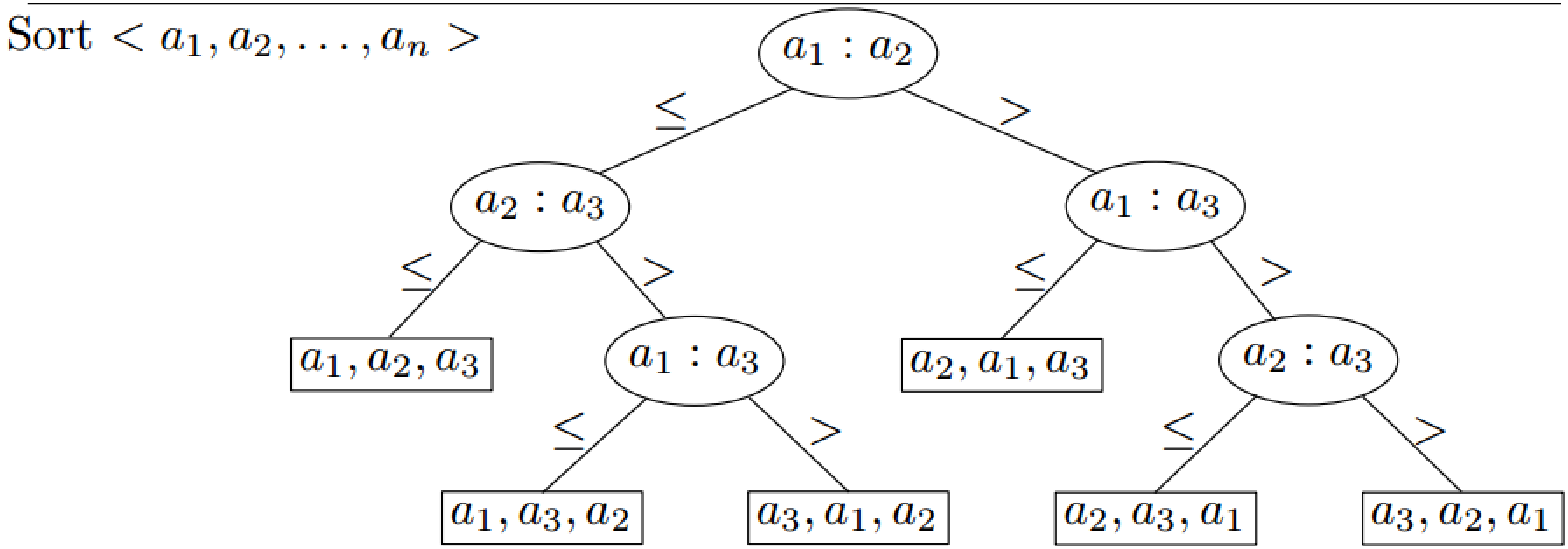
Decision-tree Example

Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$

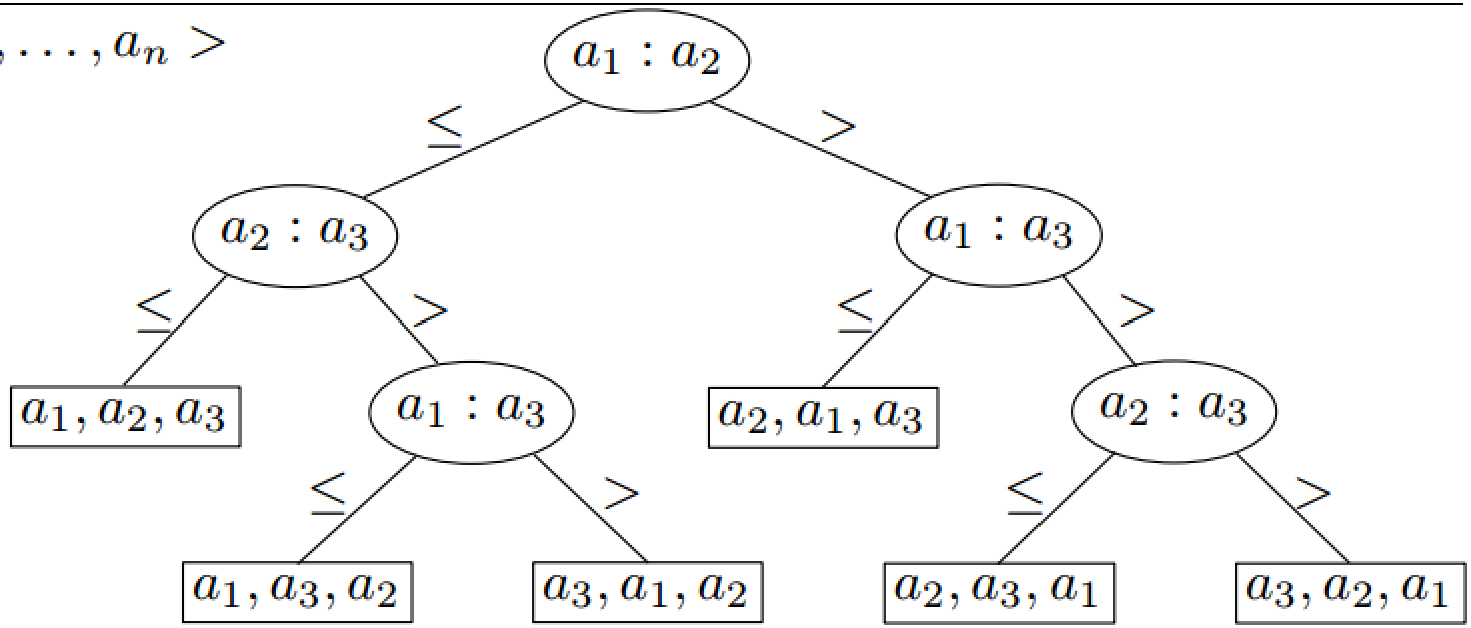
Decision-tree Example



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$

Decision-tree Example

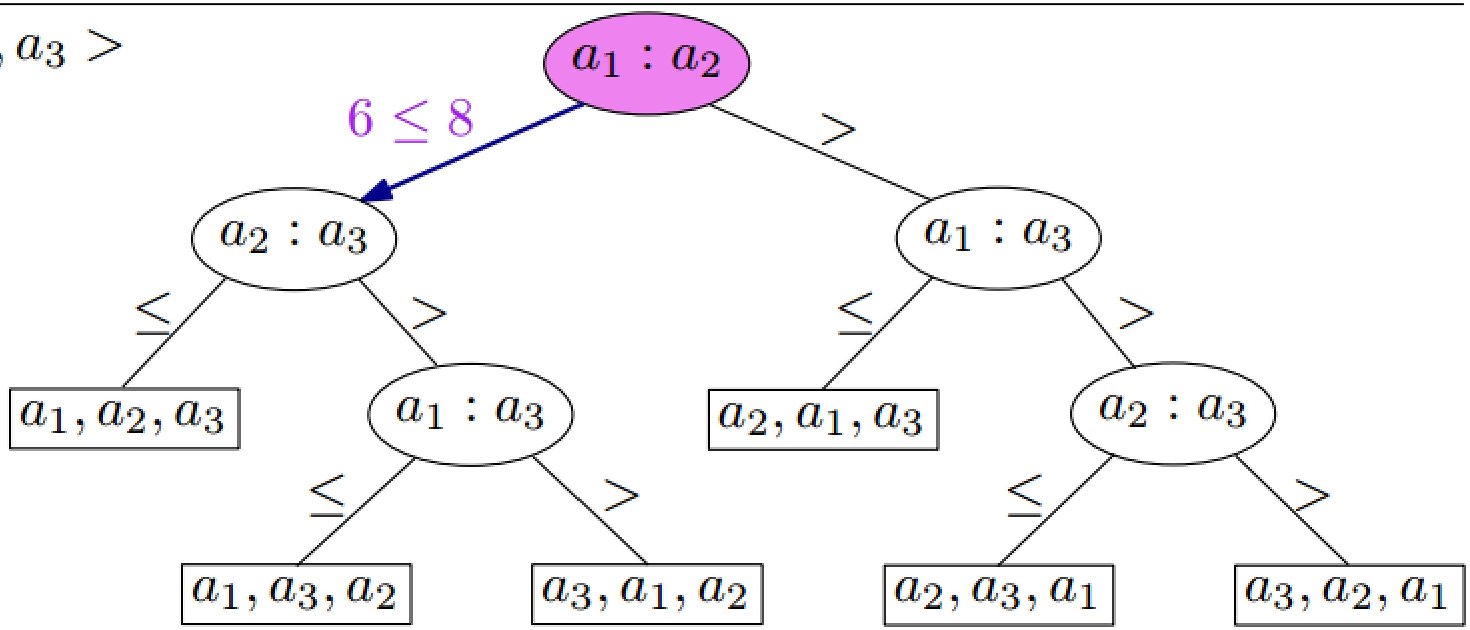
Sort $< a_1, a_2, \ldots, a_n >$



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

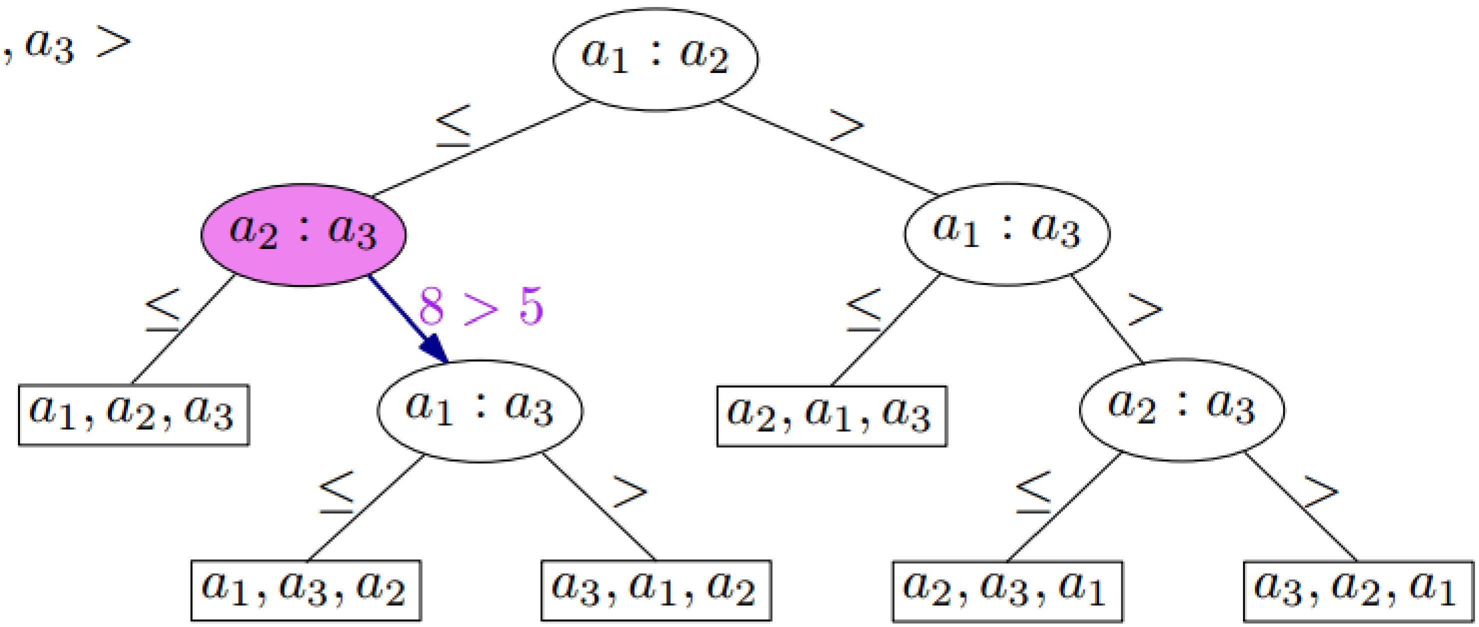
Sort $< a_1, a_2, a_3 >$ $=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

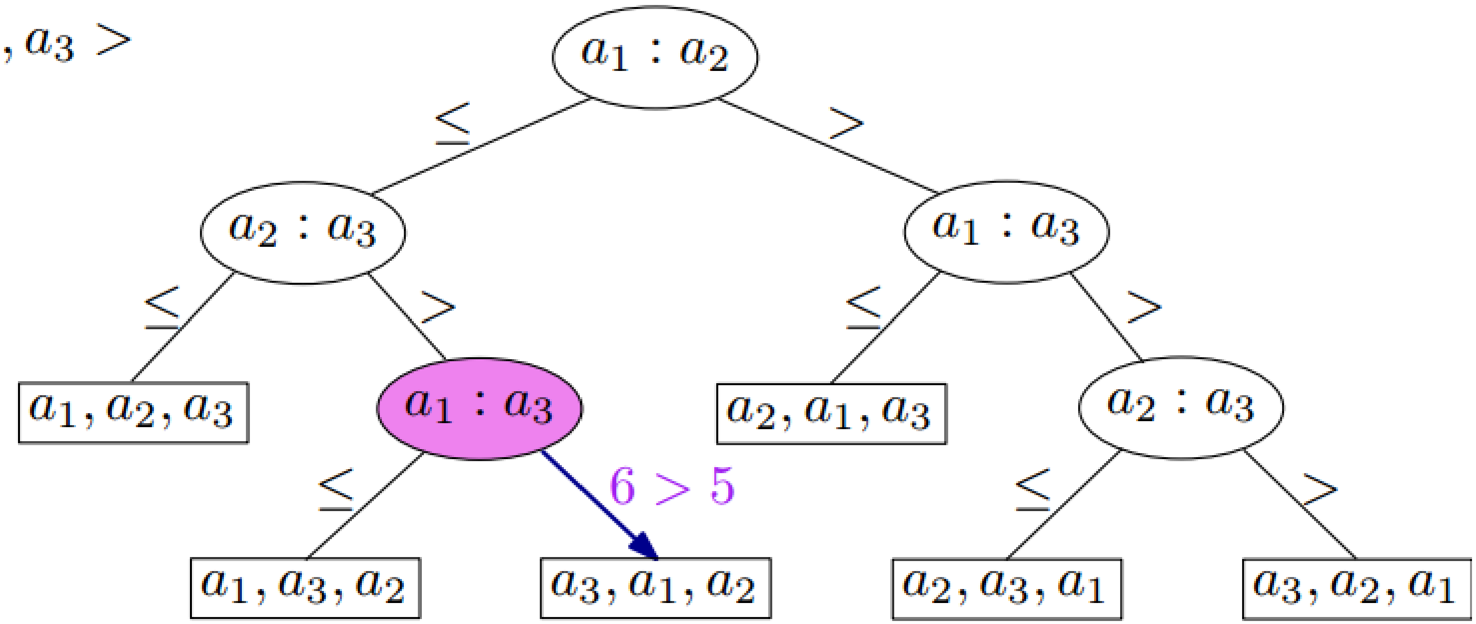
Sort $< a_1, a_2, a_3 >$ $=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

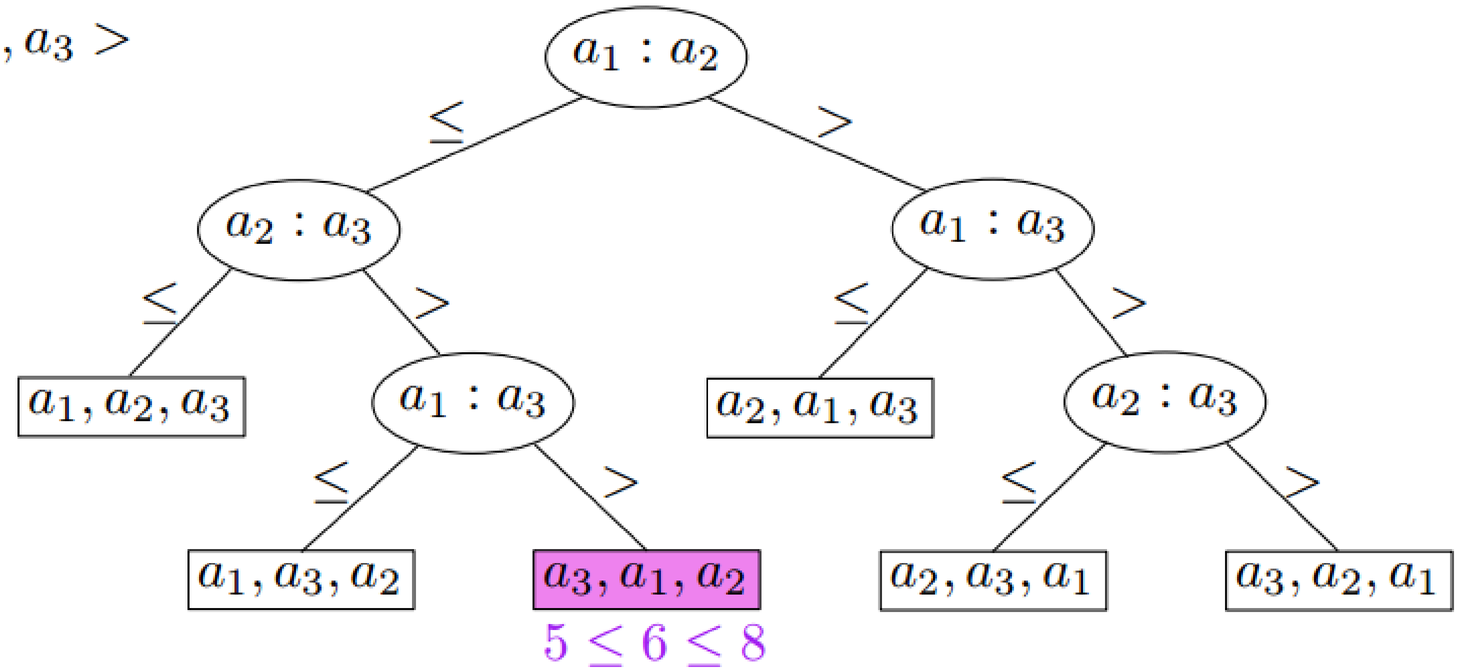
Sort $< a_1, a_2, a_3 >$ $=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Example

Sort $< a_1, a_2, a_3 >$
$=< 6, 8, 5 >$:



- Each internal node is labeled $a_i : a_j$ for {1,2,…,n}
  - The left subtree shows subsequent comparisons if $a_i \leq a_j$
  - The right subtree shows subsequent comparisons if $a_i > a_j$
- Each leaf corresponds to an input ordering

Decision-tree Model

A decision tree can model the execution of any comparison-based sorting algorithm

Decision-tree Model

A decision tree can model the execution of any comparison-based sorting algorithm

- One tree for each input size n

Decision-tree Model

A decision tree can model the execution of any comparison-based sorting algorithm

- One tree for each input size n
- Worst-case running time = height of tree

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ *comparisons.*

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since there are $n!$ possible orderings.

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since there are $n!$ possible orderings.
- A binary tree of height h has at most 2^h leaves

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since there are $n!$ possible orderings.
- A binary tree of height h has at most 2^h leaves
- Thus, $n! \leq 2^h$
  $\Rightarrow h \geq \log n! = \Omega(n \log n)$ (proved in previous lecture)

$\square$

Lower Bound for Sorting

Theorem

Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons.

Proof.

- A decision tree to sort n elements must have at least $n!$ leaves, since there are $n!$ possible orderings.
- A binary tree of height h has at most 2^h leaves
- Thus, $n! \leq 2^h$
  $\Rightarrow h \geq \log n! = \Omega(n \log n)$ (proved in previous lecture)

$\square$

Corollary

Heapsort and merge sort are asymptotically optimal comparison-based sorting algorithms.

Can we do better?

Are there sorting algorithms which are not based on comparisons? Do they beat the $\Omega(n \log n)$ lower bound?

Can we do better?

Are there sorting algorithms which are not based on comparisons? Do they beat the $\Omega(n \log n)$ lower bound?

- Counting sort (计数排序)
- Radix sort (基数排序)

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Main Ideas

- Counting sort determines, for each input element x, the number of elements less than x.

Main Ideas

- Counting sort determines, for each input element *x*, the number of elements less than *x*.

- It uses this information to place element *x* directly into its position in the output array.

Main Ideas

- Counting sort determines, for each input element *x*, the number of elements less than *x*.

- It uses this information to place element *x* directly into its position in the output array.
  - For example, if 17 elements are less than *x*, then *x* belongs in output position 18.

Counting Sort

Counting-Sort(A,B,k)

```
Input: A[1...n] where A[j] ∈ {1, 2, ..., k}
Output: B[1...n],sorted
let C[1...k] be a new array;
for i ← 1 to k do
 |  C[i] ← 0;
end
for j ← 1 to n do
 |  C[A[j]] ← C[A[j]] + 1;//C[i] = |{key = i}|
end
for i ← 2 to k do
 |  C[i] ← C[i] + C[i − 1];//C[i] = |{key ≤ i}|
end
for j ← n to 1 do
 |  B[C[A[j]]] ← A[j];
 |  C[A[j]] ← C[A[j]] − 1;
end
return B;
```

Counting Sort

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
  $C[i] \leftarrow 0;$
end
for $j \leftarrow 1$ *to* n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
  $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
  $B[C[A[j]]] \leftarrow A[j];$
  $C[A[j]] \leftarrow C[A[j]] - 1;$
end
return B;

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C				

B					

Counting Sort

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
  $C[i] \leftarrow 0$;
end
for $j \leftarrow 1$ *to* n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
  $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
  $B[C[A[j]]] \leftarrow A[j]$;
  $C[A[j]] \leftarrow C[A[j]] - 1$;
end
return B;

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	0	0	0

B					

for $i \leftarrow 1$ to k **do**
| $C[i] \leftarrow 0;$
end

Counting Sort

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
 | $C[i] \leftarrow 0;$
end
for $j \leftarrow 1$ *to* n **do**
 | $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
 | $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
 | $B[C[A[j]]] \leftarrow A[j];$
 | $C[A[j]] \leftarrow C[A[j]] - 1;$
end
return B;

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	0	0	1

B					

for $j \leftarrow 1$ *to* n **do**
$\quad | \quad C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	0	1	0	1

B					

for $j \leftarrow 1$ *to* n **do**
$\quad C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	1	0	1

B					

for $j \leftarrow 1$ *to* n **do**

$\quad | \quad C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	1	0	2

B					

for $j \leftarrow 1$ *to* n **do**

$\quad | \quad C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	2	0	2

B					

for $j \leftarrow 1$ *to* n **do**

| $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$

end

Counting Sort

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
  $C[i] \leftarrow 0;$
end
for $j \leftarrow 1$ *to* n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1; // C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
  $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
  $B[C[A[j]]] \leftarrow A[j];$
  $C[A[j]] \leftarrow C[A[j]] - 1;$
end
return B;

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2
B					

	1	2	3	4
C	1	2	0	2
C'	1	3	0	2

for $i \leftarrow 2$ *to* k **do**

$\quad C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2

	1	2	3	4
C	1	2	0	2

B					

C'	1	3	3	2

for $i \leftarrow 2$ *to* k **do**

$\quad C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \le i\}|$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2
B					

	1	2	3	4
C	1	2	0	2
C'	1	3	3	5

for $i \leftarrow 2$ *to* k **do**

| $C[i] \leftarrow C[i] + C[i-1]; // C[i] = |\{key \leq i\}|$

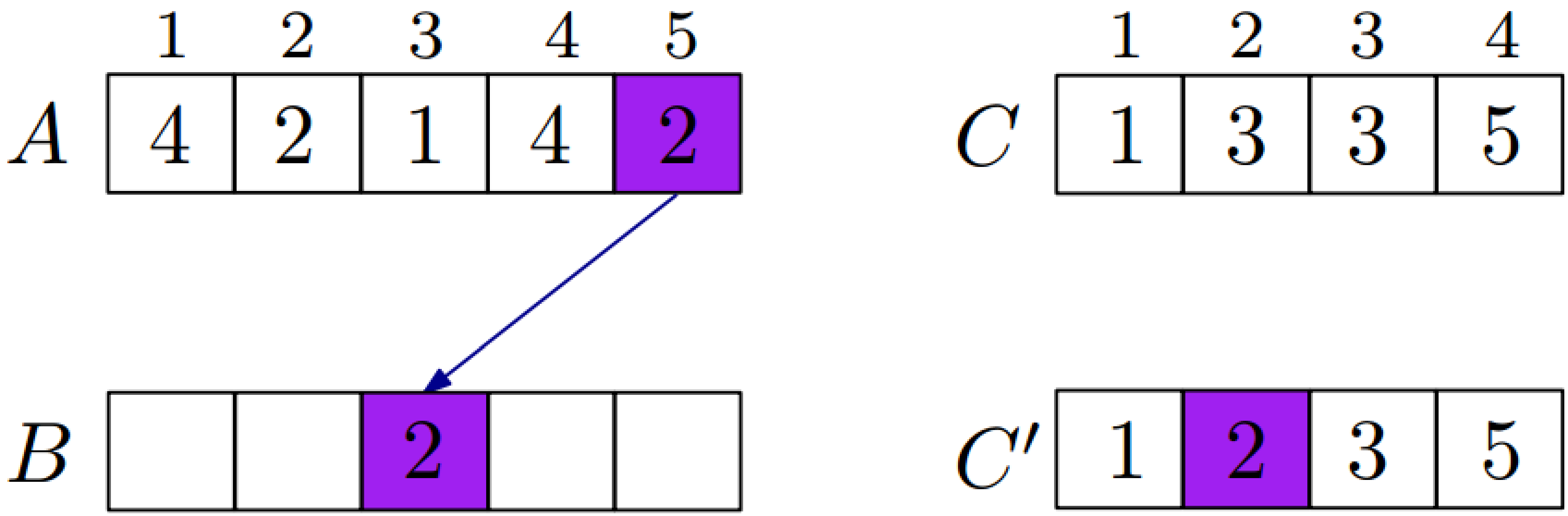
end

Counting Sort

Counting-Sort(A,B,k)

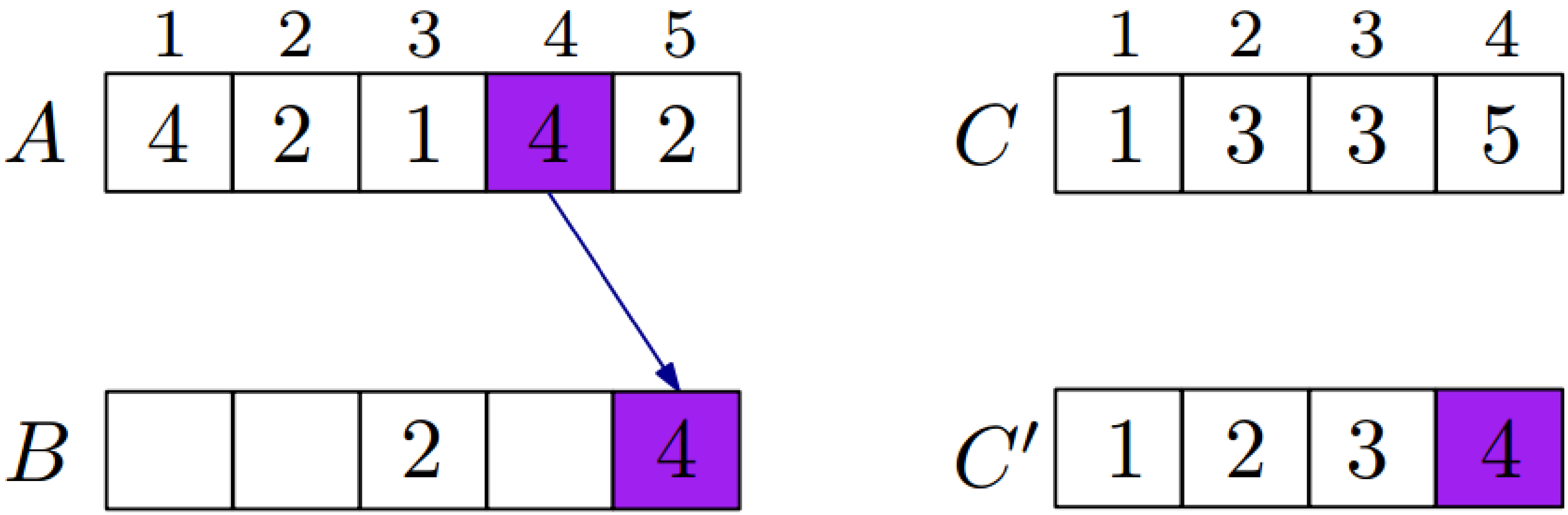
Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
  | $C[i] \leftarrow 0;$
end
for $j \leftarrow 1$ *to* n **do**
  | $C[A[j]] \leftarrow C[A[j]] + 1;//C[i] = |\{key = i\}|$
end
for $i \leftarrow 2$ *to* k **do**
  | $C[i] \leftarrow C[i] + C[i-1];//C[i] = |\{key \leq i\}|$
end
for $j \leftarrow n$ *to* 1 **do**
  | $B[C[A[j]]] \leftarrow A[j];$
  | $C[A[j]] \leftarrow C[A[j]] - 1;$
end
return B;

Example: Counting Sort



$$
\begin{array}{l}
\textbf{for } j \leftarrow n \ \textit{to} \ 1 \textbf{ do} \\
\quad B[C[A[j]]] \leftarrow A[j]; \\
\quad C[A[j]] \leftarrow C[A[j]] - 1; \\
\textbf{end}
\end{array}
$$

Example: Counting Sort



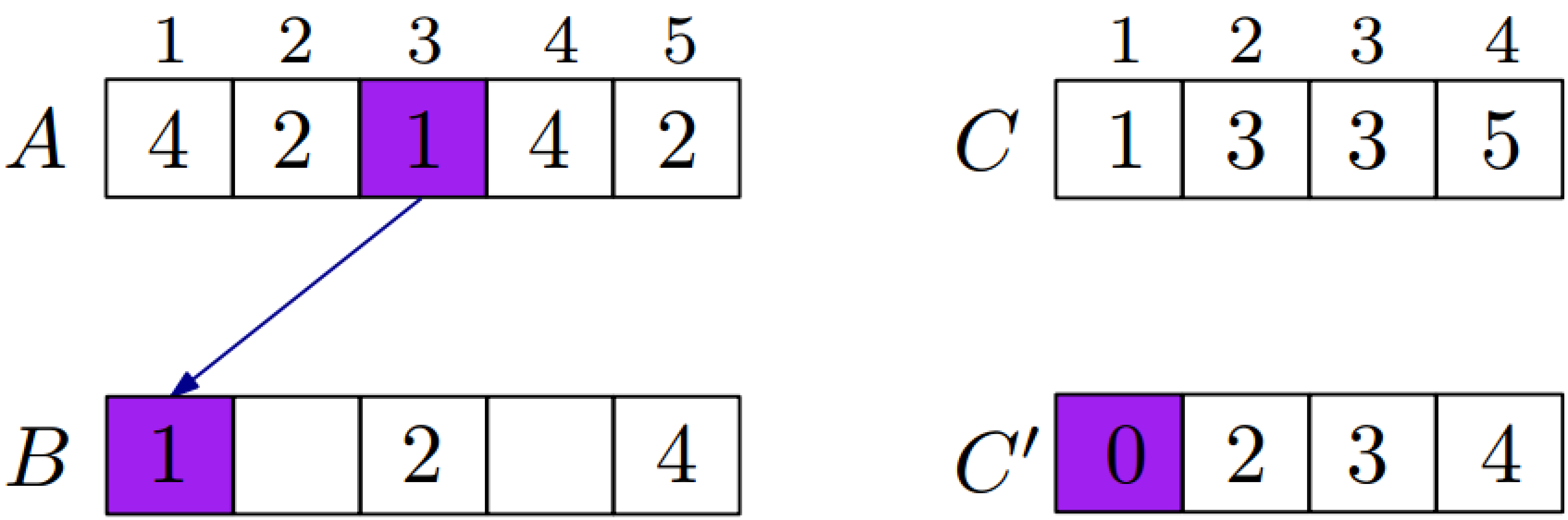
for $j \leftarrow n$ *to* 1 **do**

$\quad B[C[A[j]]] \leftarrow A[j];$

$\quad C[A[j]] \leftarrow C[A[j]] - 1;$

end

Example: Counting Sort



for $j \leftarrow n$ *to* 1 **do**

$B[C[A[j]]] \leftarrow A[j];$

$C[A[j]] \leftarrow C[A[j]] - 1;$

end

Example: Counting Sort

	1	2	3	4	5
A	4	2	1	4	2
B	1	2	2		4

	1	2	3	4
C	1	3	3	5
C'	0	1	3	4

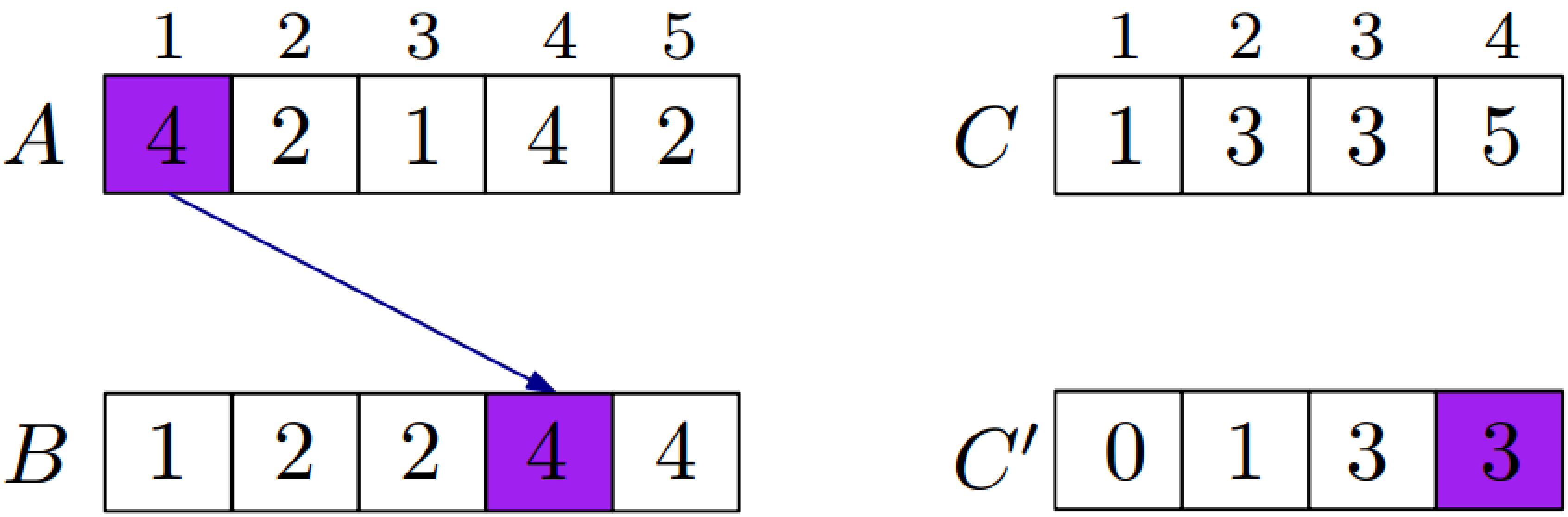
for $j \leftarrow n$ *to* 1 **do**

$B[C[A[j]]] \leftarrow A[j];$

$C[A[j]] \leftarrow C[A[j]] - 1;$

end

Example: Counting Sort



$$
\begin{array}{l}
\textbf{for } j \leftarrow n \textit{ to } 1 \textbf{ do} \\
\quad B[C[A[j]]] \leftarrow A[j]; \\
\quad C[A[j]] \leftarrow C[A[j]] - 1; \\
\textbf{end}
\end{array}
$$

Analysis

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
| $C[i] \leftarrow 0;//O(k)$
end

return B;

Analysis

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
| $C[i] \leftarrow 0;//O(k)$
end
for $j \leftarrow 1$ *to* n **do**
| $C[A[j]] \leftarrow C[A[j]] + 1;//O(n)$
end

return B;

Analysis

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
| $C[i] \leftarrow 0; // O(k)$
end
for $j \leftarrow 1$ *to* n **do**
| $C[A[j]] \leftarrow C[A[j]] + 1; // O(n)$
end
for $i \leftarrow 2$ *to* k **do**
| $C[i] \leftarrow C[i] + C[i-1]; // O(k)$
end

return B;

Analysis

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
| $C[i] \leftarrow 0;//O(k)$
end
for $j \leftarrow 1$ *to* n **do**
| $C[A[j]] \leftarrow C[A[j]] + 1;//O(n)$
end
for $i \leftarrow 2$ *to* k **do**
| $C[i] \leftarrow C[i] + C[i-1];//O(k)$
end
for $j \leftarrow n$ *to* 1 **do**
| $B[C[A[j]]] \leftarrow A[j];$
| $C[A[j]] \leftarrow C[A[j]] - 1;//O(n)$
end
return B;

Analysis

Counting-Sort(A,B,k)

Input: $\boldsymbol{A}[1...n]$ where $A[j] \in \{1, 2, ..., k\}$
Output: $\boldsymbol{B}[1...n]$,sorted
let $C[1...k]$ be a new array;
for $i \leftarrow 1$ *to* k **do**
  $C[i] \leftarrow 0;//O(k)$
end
for $j \leftarrow 1$ *to* n **do**
  $C[A[j]] \leftarrow C[A[j]] + 1;//O(n)$
end
for $i \leftarrow 2$ *to* k **do**
  $C[i] \leftarrow C[i] + C[i-1];//O(k)$
end
for $j \leftarrow n$ *to* 1 **do**
  $B[C[A[j]]] \leftarrow A[j];$
  $C[A[j]] \leftarrow C[A[j]] - 1;//O(n)$
end
return B;

Total: $O(n + k)$

Running Time

If k = O(n), then counting sort takes O(n) time.

- But didn't we prove that sorting must take Ω(n log n) time?

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.

Running Time

If $k = O(n)$, then counting sort takes $O(n)$ time.

- But didn't we prove that sorting must take $\Omega(n \log n)$ time?
- No, actually we proved that any comparison-based sorting algorithm takes $\Omega(n \log n)$ time.
- Note that counting sort is not a comparison-based sorting algorithm.

Running Time

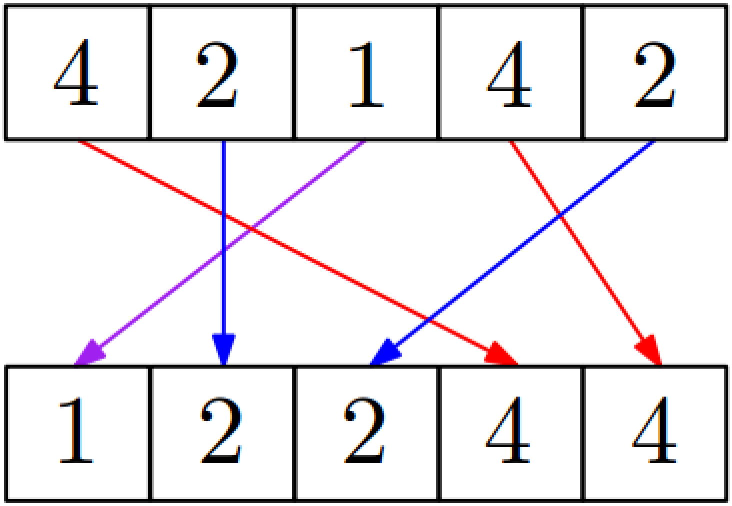
If k = O(n), then counting sort takes O(n) time.

- But didn't we prove that sorting must take Ω(n log n) time?
- No, actually we proved that any comparison-based sorting algorithm takes Ω(n log n) time.
- Note that counting sort is not a comparison-based sorting algorithm.
- In fact, it makes no comparison at all!

Stable Sorting

Counting sort is a stable sort

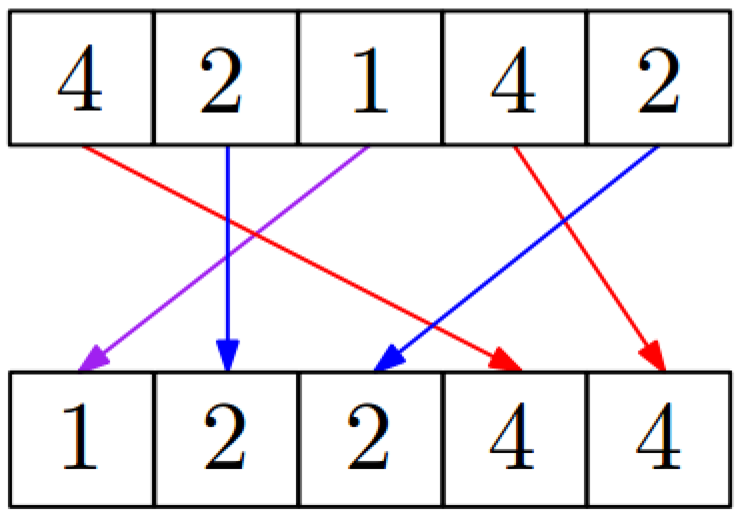
- it preserves the input order among equal elements.



Stable Sorting

Counting sort is a stable sort

- it preserves the input order among equal elements.



Exercise

What other sorts have this property?

Outline

- Introduction to Part II
- Heapsort Problem
  - Priority Queues
  - (Binary) Heap
  - Heapsort
- Lower Bound for Sorting
- Sorting in Linear Time
  - Counting Sort
  - Radix Sort

Radix Sort

- Sort on least significant digit first using stable sort

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
5	4	5	7
3	6	5	7
5	8	3	9
3	4	3	6
2	7	2	0
5	3	5	5

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
5	4	5	7
3	6	5	7
5	8	3	9
3	4	3	6
2	7	2	0
5	3	5	5

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	3	2	9
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	3	2	9
5	8	3	9

Preserves the input order among equal elements

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	3	2	9
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
2	3	2	9
3	4	3	6
5	8	3	9
5	3	5	5
5	4	5	7
3	6	5	7

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
2	3	2	9
3	4	3	6
5	8	3	9
5	3	5	5
5	4	5	7
3	6	5	7

Radix Sort

- Sort on least significant digit first using stable sort

2	7	2	0
2	3	2	9
3	4	3	6
5	8	3	9
5	3	5	5
5	4	5	7
3	6	5	7

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	7	2	0
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	7	2	0
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
5	3	5	5
3	4	3	6
5	4	5	7
3	6	5	7
2	7	2	0
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
2	7	2	0
3	4	3	6
3	6	5	7
5	3	5	5
5	4	5	7
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2	3	2	9
2	7	2	0
3	4	3	6
3	6	5	7
5	3	5	5
5	4	5	7
5	8	3	9

Radix Sort

- Sort on least significant digit first using stable sort

2 3 2 9
2 7 2 0
3 4 3 6
3 6 5 7
5 3 5 5
5 4 5 7
5 8 3 9

Radix Sort: Correctness

Induction on digit position

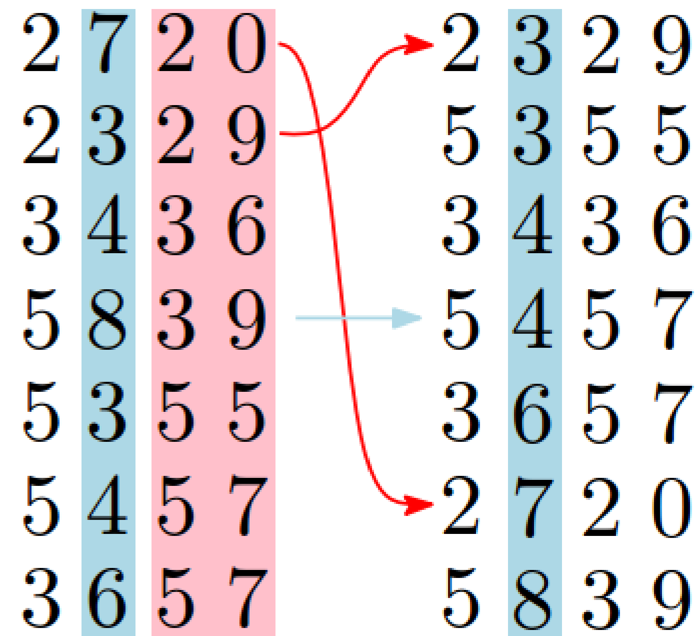
- Assume that the numbers are sorted by their low-order i–1 digits
- Sort on digit i

2 7 2 0		2 3 2 9
2 3 2 9		5 3 5 5
3 4 3 6		3 4 3 6
5 8 3 9	→	5 4 5 7
5 3 5 5		3 6 5 7
5 4 5 7		2 7 2 0
3 6 5 7		5 8 3 9

Radix Sort: Correctness

Induction on digit position

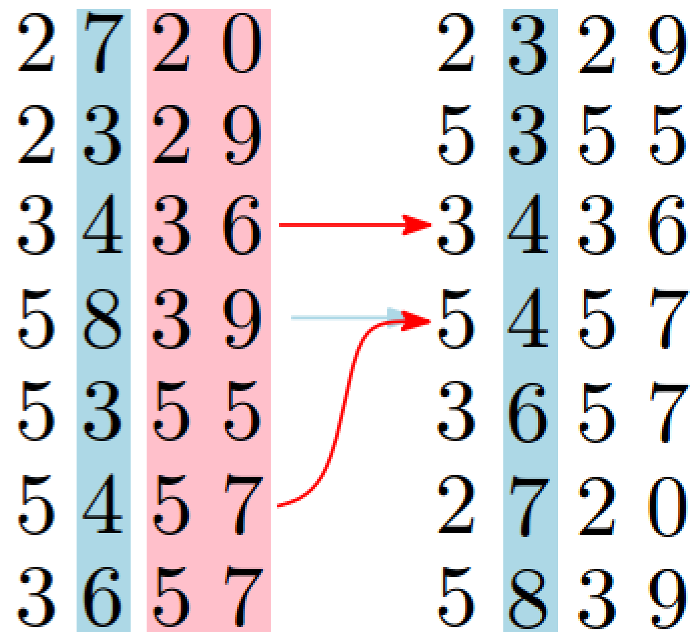
- Assume that the numbers are sorted by their low-order i–1 digits
- Sort on digit i
  - Two numbers that differ on digit i are correctly sorted by their low-order i digits



Radix Sort: Correctness

Induction on digit position

- Assume that the numbers are sorted by their low-order i–1 digits
- Sort on digit i
  - Two numbers that differ on digit i are correctly sorted by their low-order i digits
  - Two numbers equal on digit I are put in the same order as the input → correctly sorted by their low-order i digits



Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:

Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:

Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- b log n bits for each number

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:

Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- b log n bits for each number
- each number can be viewed as having O(b) digits of log n bits each

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:

Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- b log n bits for each number
- each number can be viewed as having O(b) digits of log n bits each
- running time is $O(d(n + k)) = O(b(n + 2^{\log n})) = O(bn)$

Radix Sort: Running Time & Application

Lemma

Given n d-digit numbers in which each digit can take on up to k possible values, radix sort correctly sorts these numbers in $O(d(n+k))$ time if the stable sort it uses takes $O(n+k)$ time.

Application:

Sorting numbers in the range from 0 to $n^b - 1$, where b is a constant

- b log n bits for each number
- each number can be viewed as having O(b) digits of log n bits each
- running time is $O(d(n + k)) = O(b(n + 2^{\log n})) = O(bn)$
- since b is a constant, the running time is O(n)


dank u
ju faleminderit
Tack
谢谢
Tak
mulţumesc
Asante
kiitos
Gracias
Salamat!
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
gam