Design and Analysis of Algorithms
Part II: Sorting and Searching
Lecture 4: Quicksort and Selection Problem

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Outline

- Introduction to Part II
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort
- Randomized Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

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Introduction to Part II

- In Part II, we will illustrate sorting and searching problems using several examples:
  - Quicksort (快速排序)
  - Selection Problem (选择问题)
  - Heapsort and Priority Queues (堆排序与优先队列)
  - Lower Bound for Sorting (基于比较排序的下界)
  - Sorting in Linear Time (线性时间排序)

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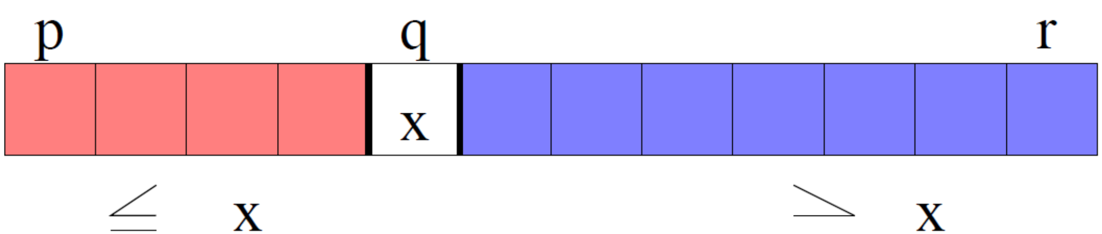
Partition

- Partition
  - Given: An array of numbers
  - Partition: Rearrange the array A[p..r] in place into two (possibly empty) subarrays A[p..q-1] and A[q+1..r] such that

$$A[u] < A[q] < A[v] \ \ for\ any\ p \leq u \leq q-1\ and\ q+1 \leq v \leq r$$



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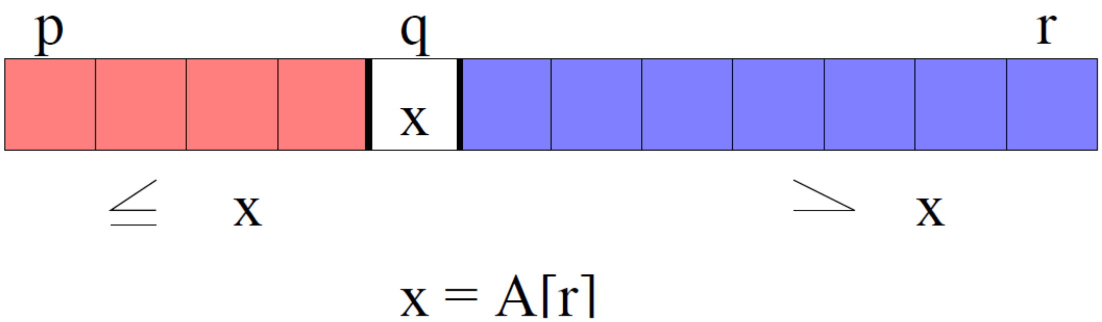

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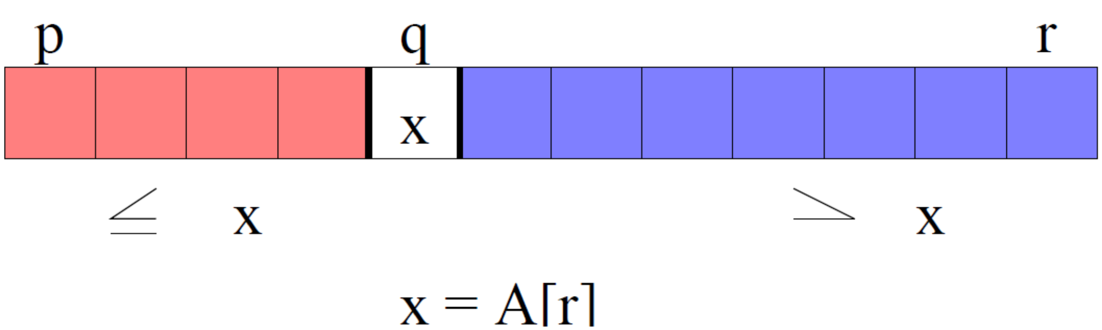

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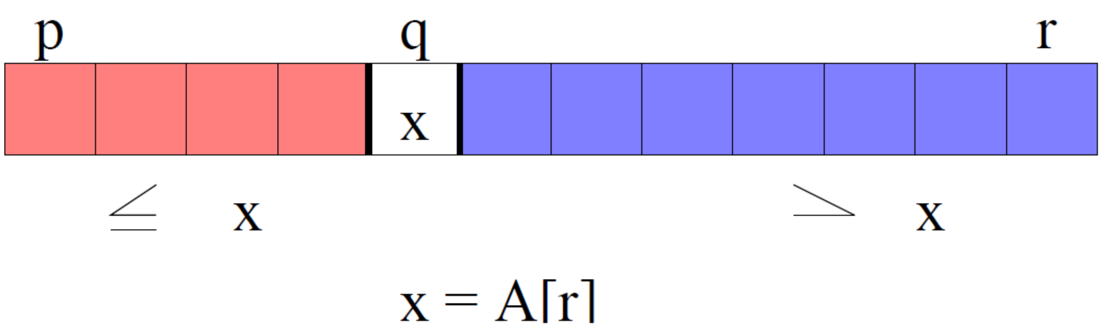

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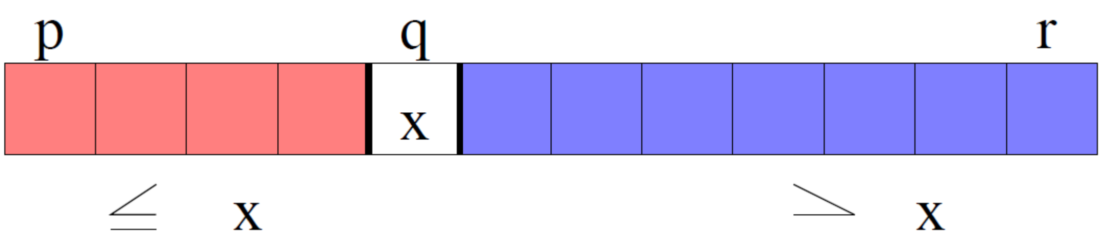

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    - recursively sorting A[ ] and A[ ]

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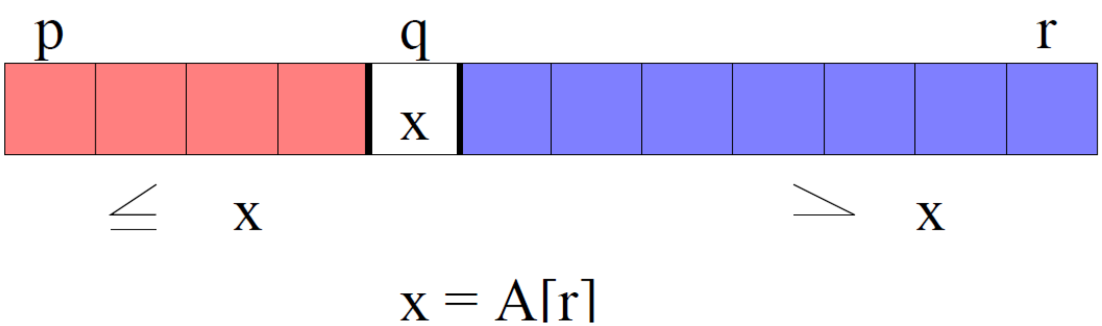

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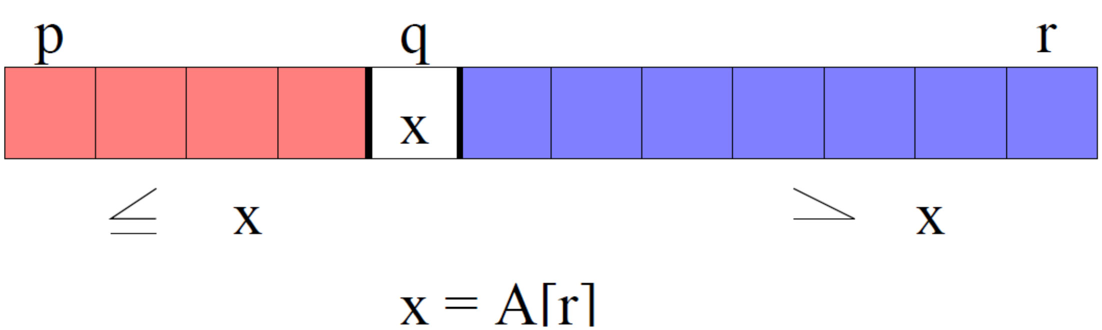

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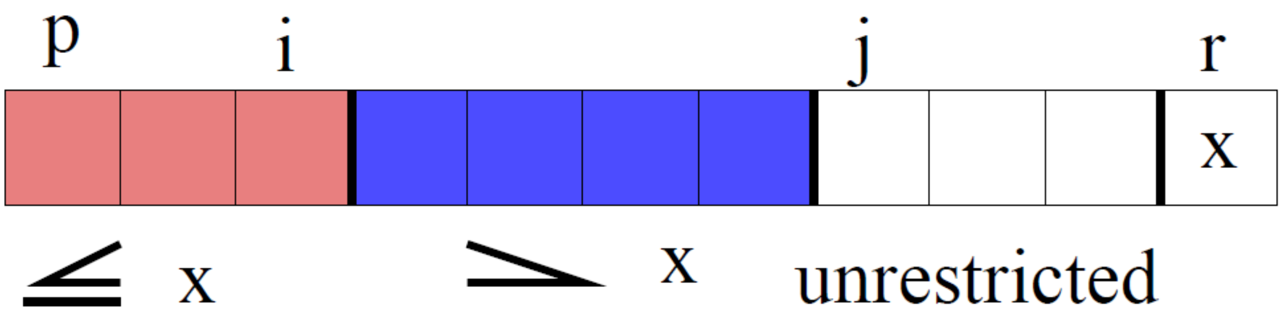
Partition

- The idea of Partition(A, p, r)
  - Use A[r] as the pivot, and grow partition from left to right



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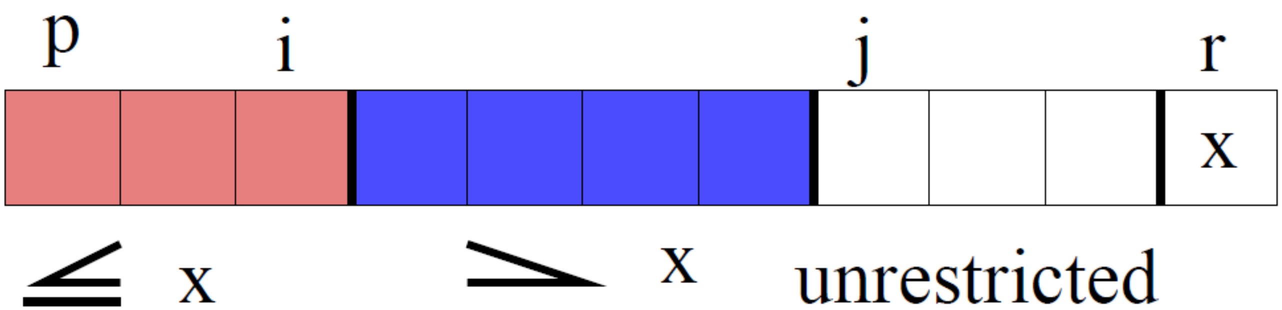

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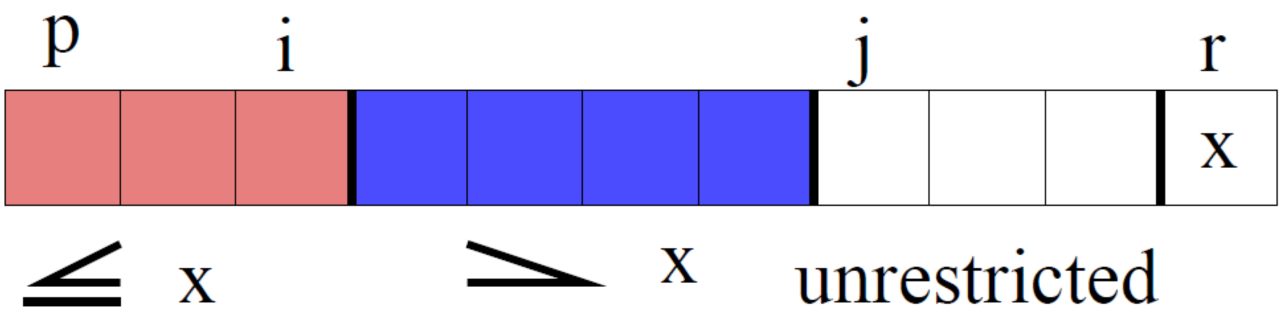

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    At the same time increase i when necessary

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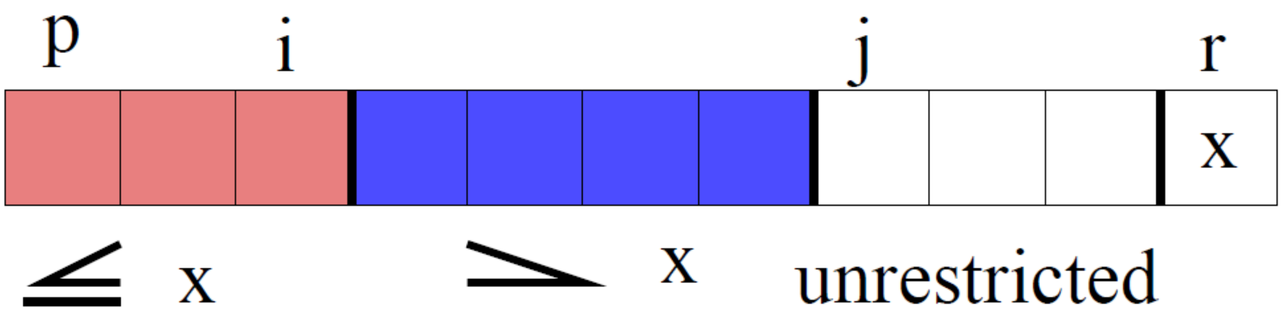

  - Initially (i, j) = (p–1, p)
  - Increase j by 1 each time to find a place for A[j]
    At the same time increase i when necessary
  - Stops when j = r

Partition

- One Iteration of the Procedure Partition
  - Increase j by 1 each time to find a place for A[j]

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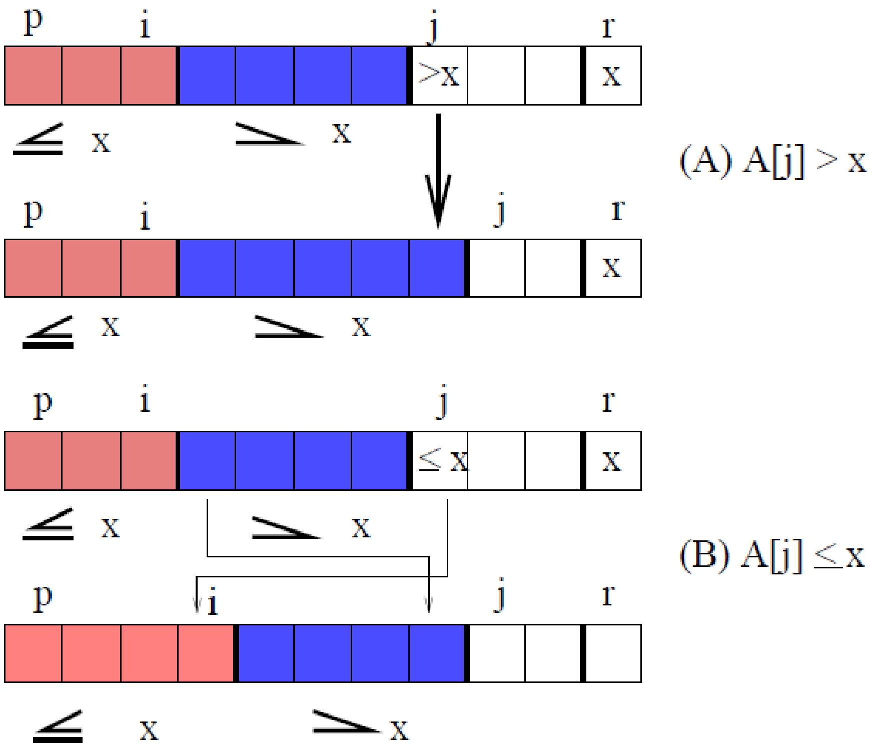

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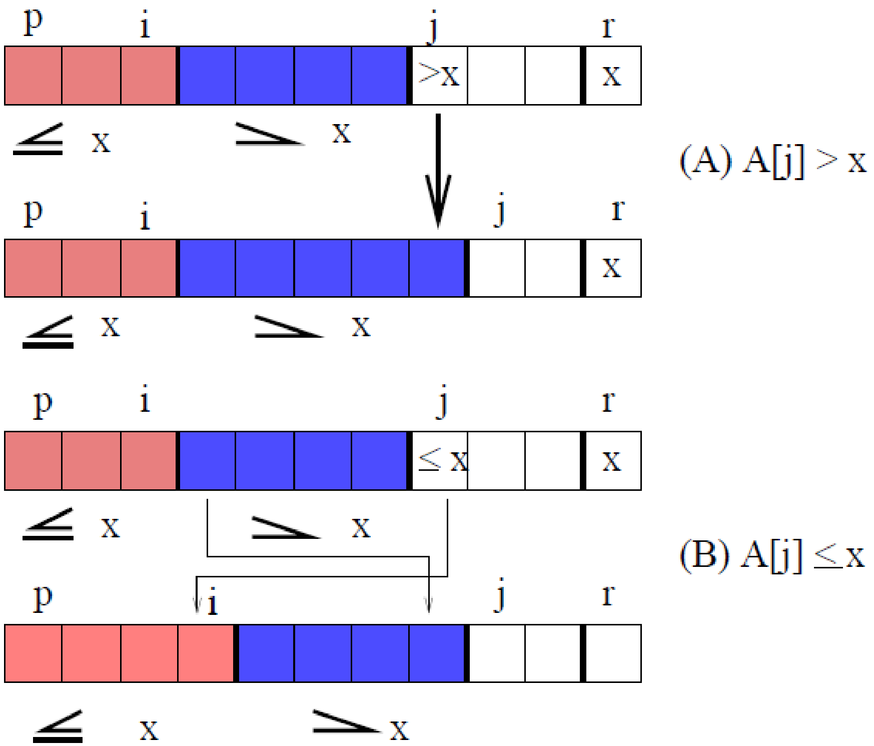

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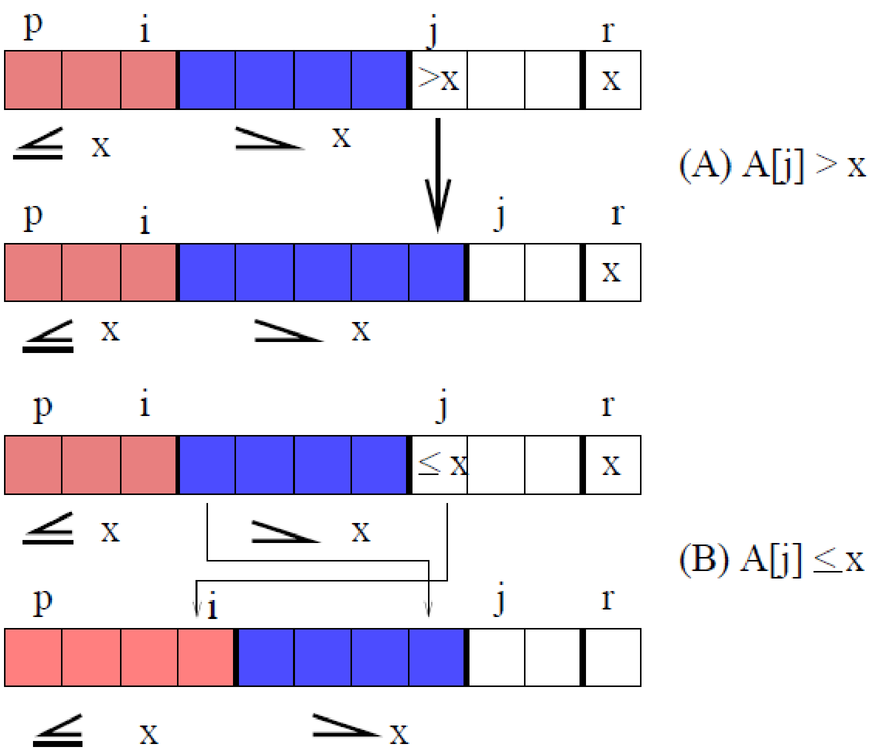

  - Case (A): Only increase j by 1
  - Case (B): i = i + 1

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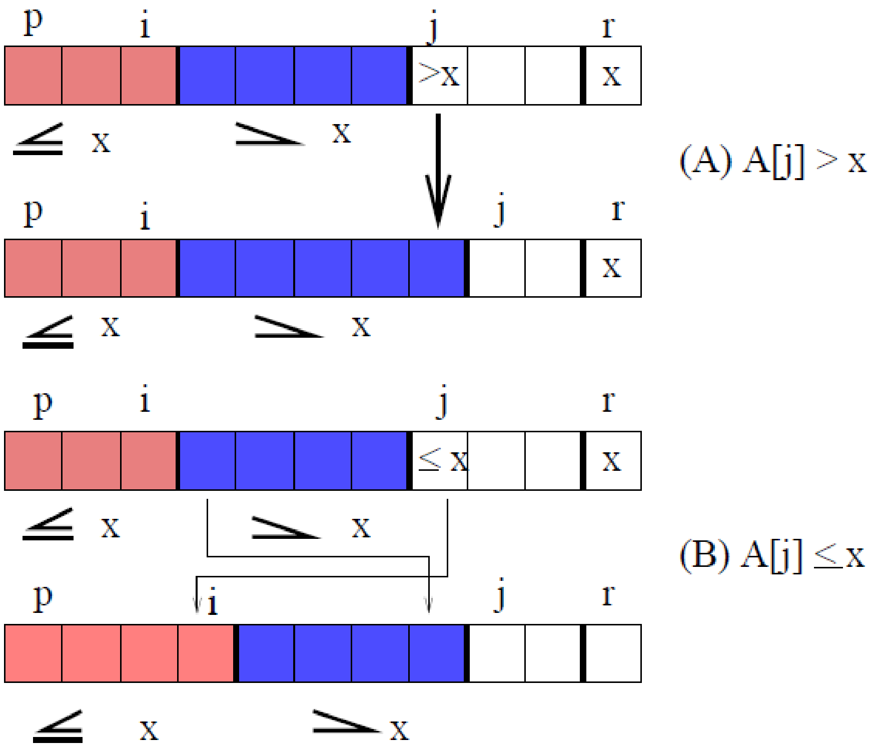

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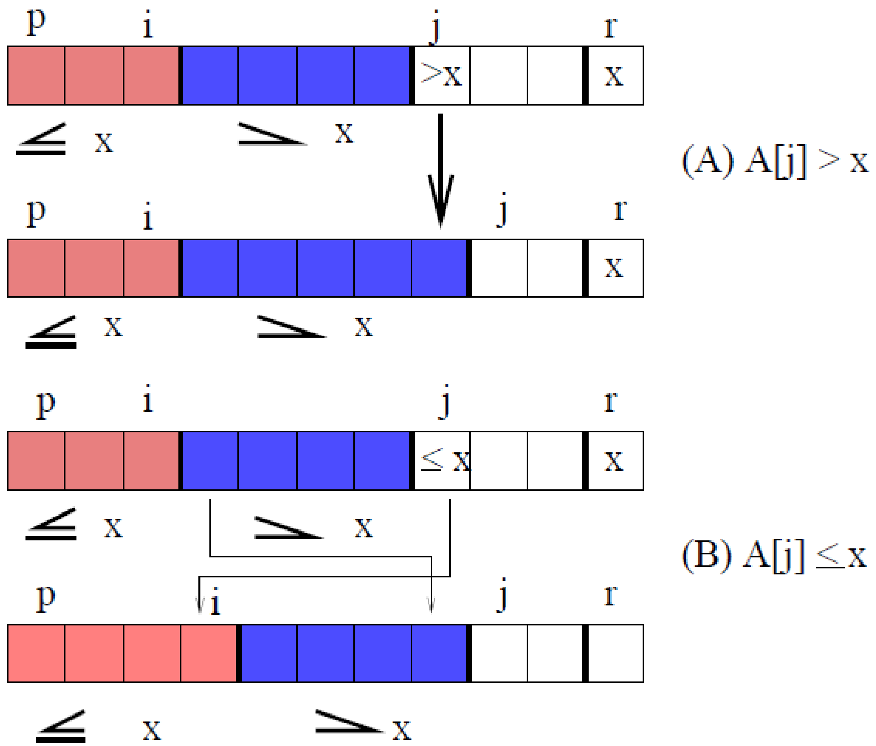

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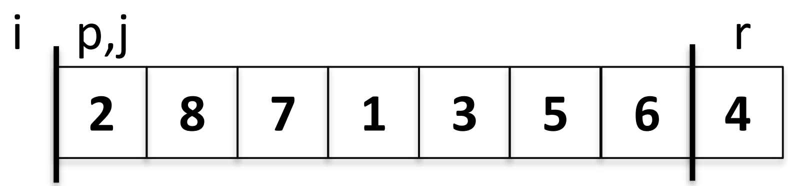
- The Operation of Partition(A, p, r)



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i	p,j							r
	2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

$$A[j] < A[r]$$

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Partition-Example

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$$A[i+1] \leftrightarrow A[r]$$

Partition - Pseudocode

Partition(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p},\boldsymbol{r}$
Output: Index of the pivot after partition
$x \leftarrow A[r]$;$//A[r]$ is the pivot element

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for $j \leftarrow p$ *to* $r - 1$ **do**

end

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Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Index of the pivot after partition

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x ← A[r]; //A[r] is the pivot element
i ← p − 1;
for j ← p to r − 1 do
    if A[j] ≤ x then

    end
end
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    if A[j] ≤ x then
        i ← i + 1;
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- Running time is O(   )

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- Running time is O(r – p)

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- Running time is O(r – p)
  - linear in the length of the array A[p..r]

Quicksort

Quicksort(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
    $q \leftarrow$Partition(A, p, r);
    Quicksort(A,   );
    Quicksort(A,   );
end
return A;

Quicksort

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A Divide-and-Conquer Framework

Quicksort

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n\log n)$.
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$.

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Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[ ] is always used as the pivot *x* to partition the array A[p..r].

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  - In the algorithm Randomized-Partition(A, p, r), we randomly choose an j , p ≤ j ≤ r , and use A[j] as pivot.

Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot x to partition the array A[p..r].
  - In the algorithm Randomized-Partition(A, p, r), we randomly choose an j , p ≤ j ≤ r , and use A[j] as pivot.
  - Idea is that if we choose randomly, then the chance that we get unlucky every time is extremely low.

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

Randomized-Partition(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: A random index in $[p..j]$

Partition(A, p, r);
return j;

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

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Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: A random index in $[p..j]$
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Partition(A, p, r);
return j;

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exchange $A[r]$ and $A[j]$;
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Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
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  Randomized-Quicksort(A,   );
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Randomized-Quicksort(A,p,r)

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Input: An array A waiting to be sorted,the range of index p,r
Output: Sorted array A
if p < r then
    q ← Randomized-Partition(A, p, r);
    Randomized-Quicksort(A, p, q − 1);
    Randomized-Quicksort(A, q + 1, r);
end
return A;
```

Quicksort - Example

2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

Quicksort - Example

Divide

2	8	7	1	3	5	6	4

Quicksort - Example

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2	8	7	1	3	5	6	4

Quicksort - Example

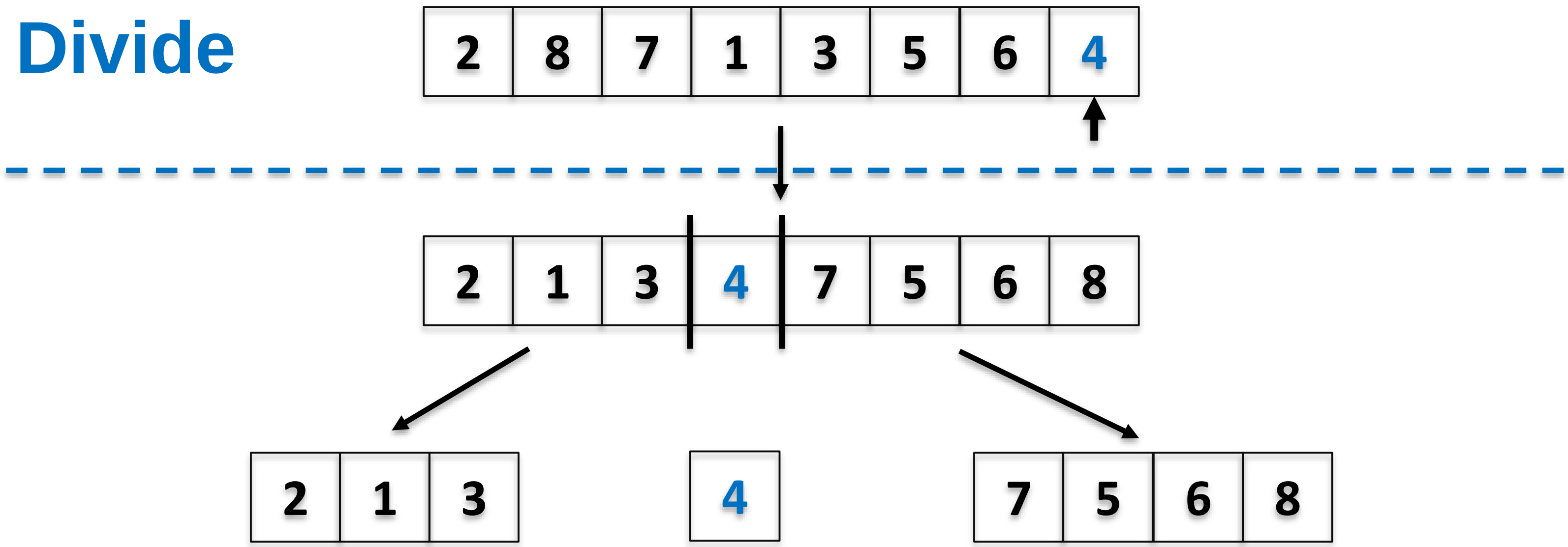
Divide

2	8	7	1	3	5	6	4

2	1	3	4	7	5	6	8

Quicksort - Example

Divide



Quicksort - Example

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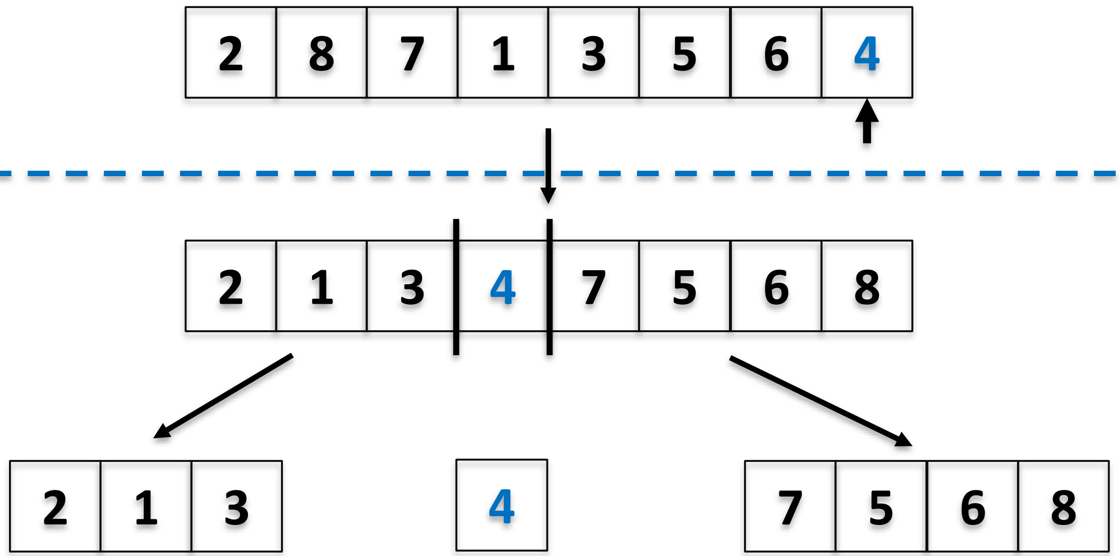

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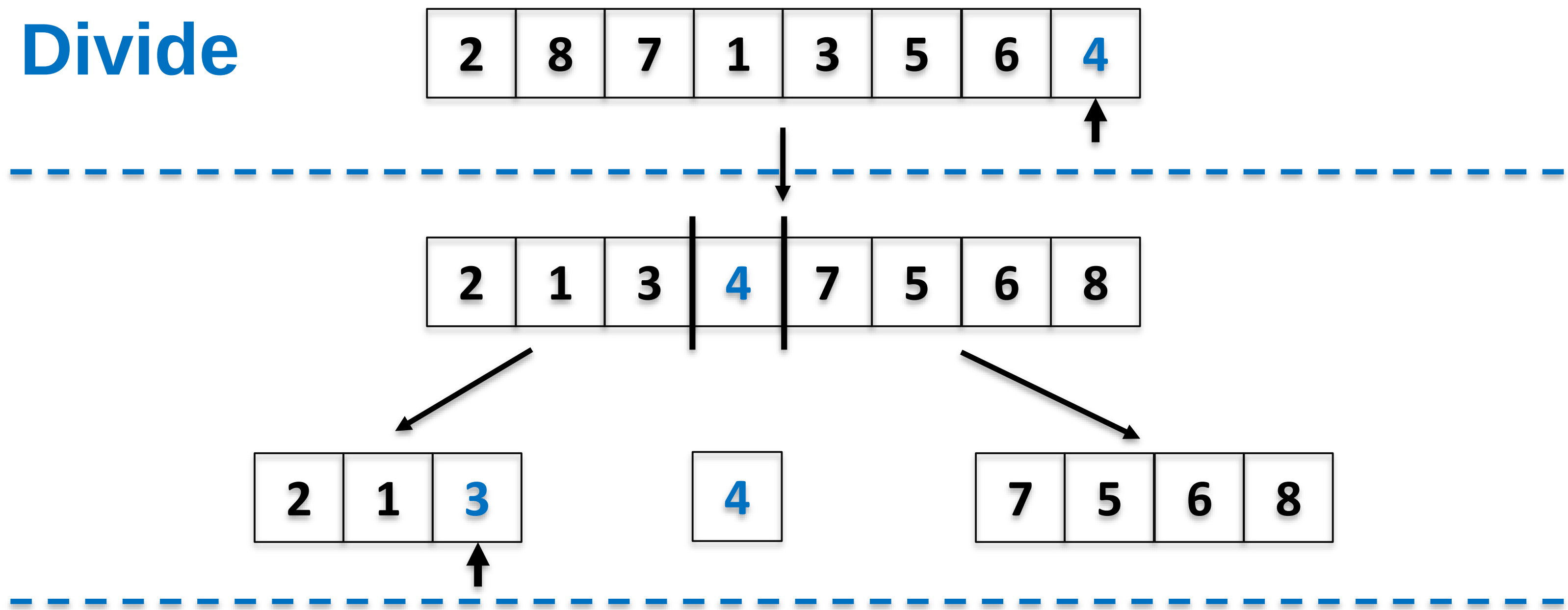

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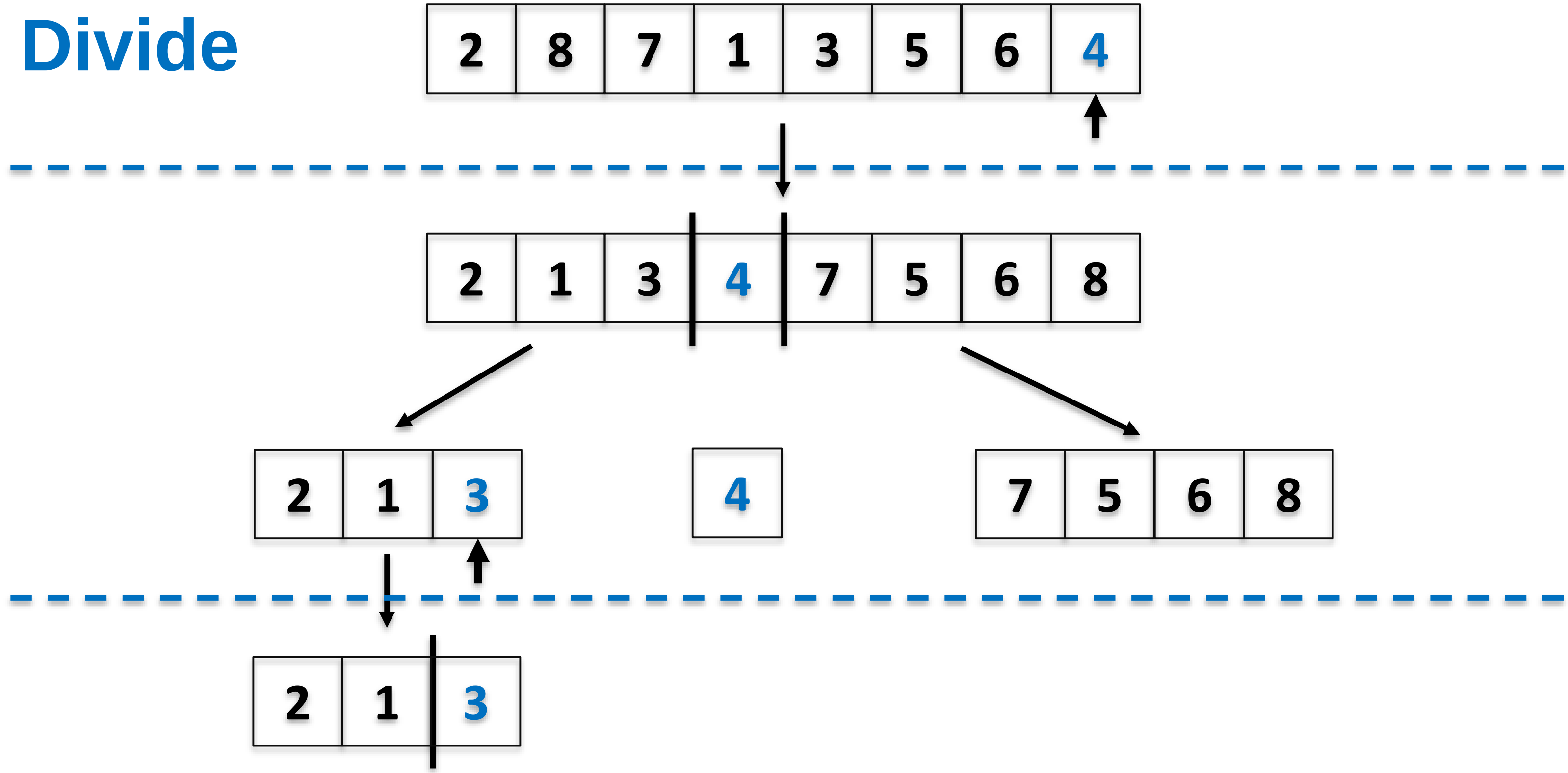

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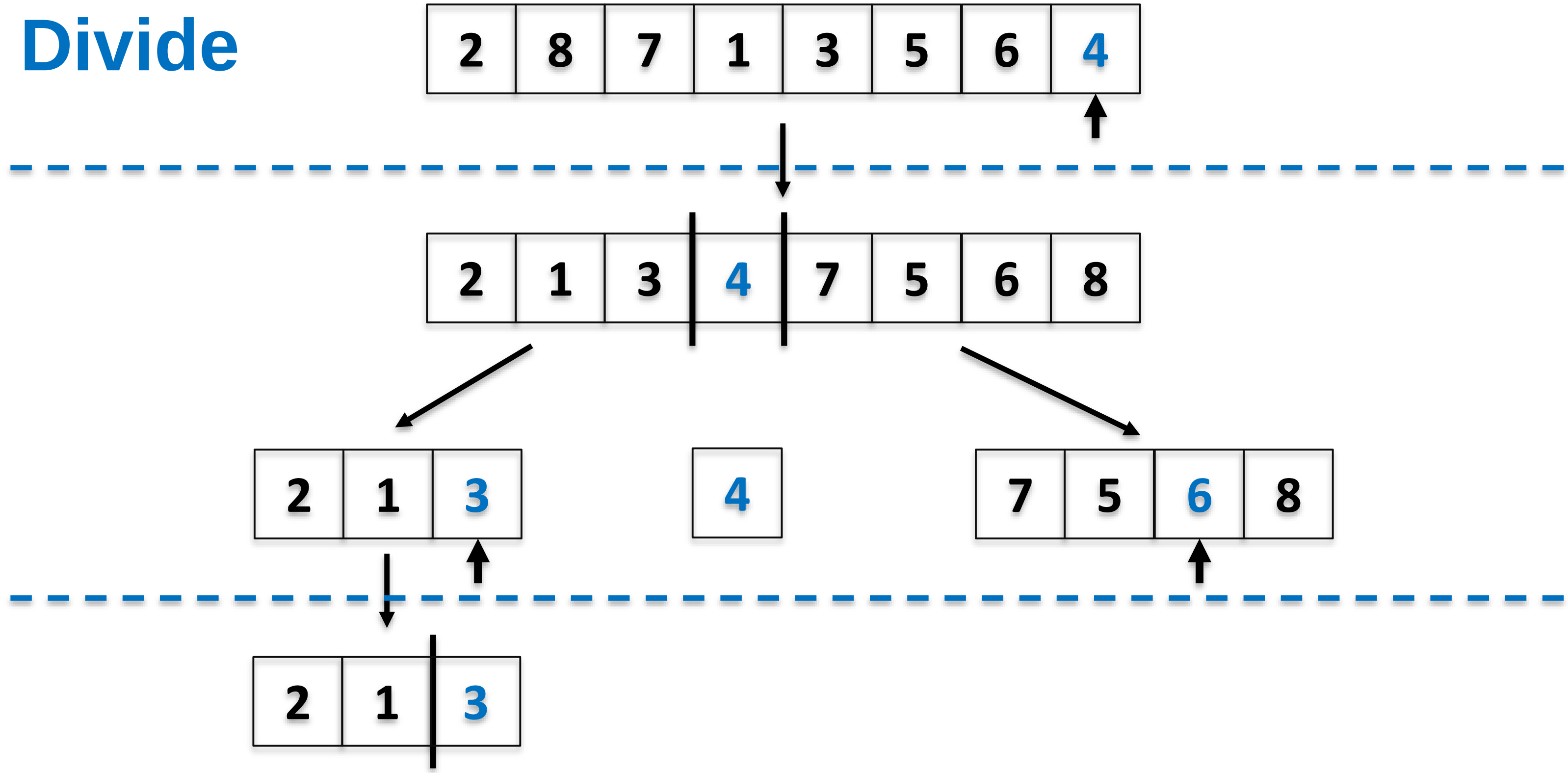

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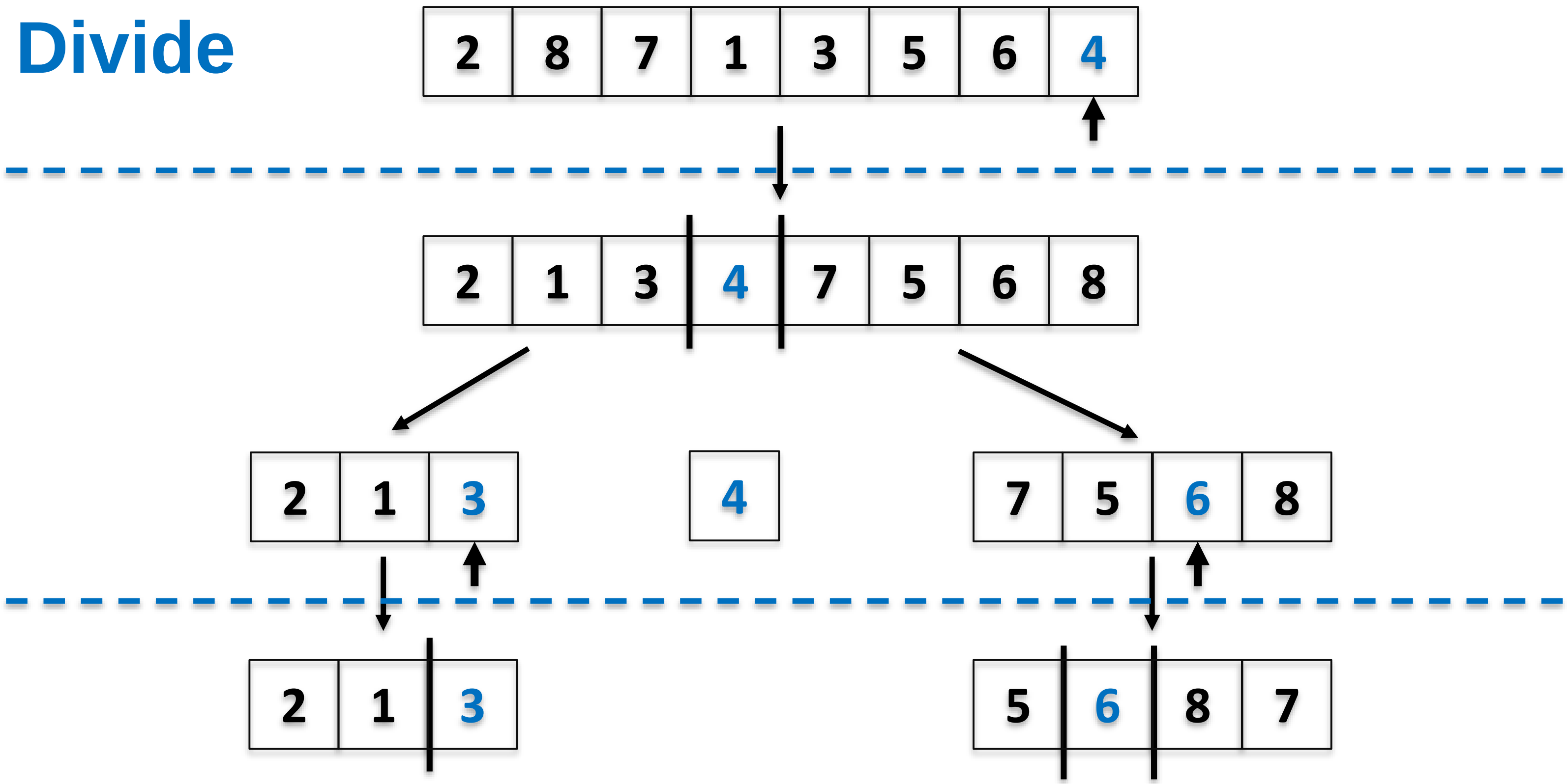

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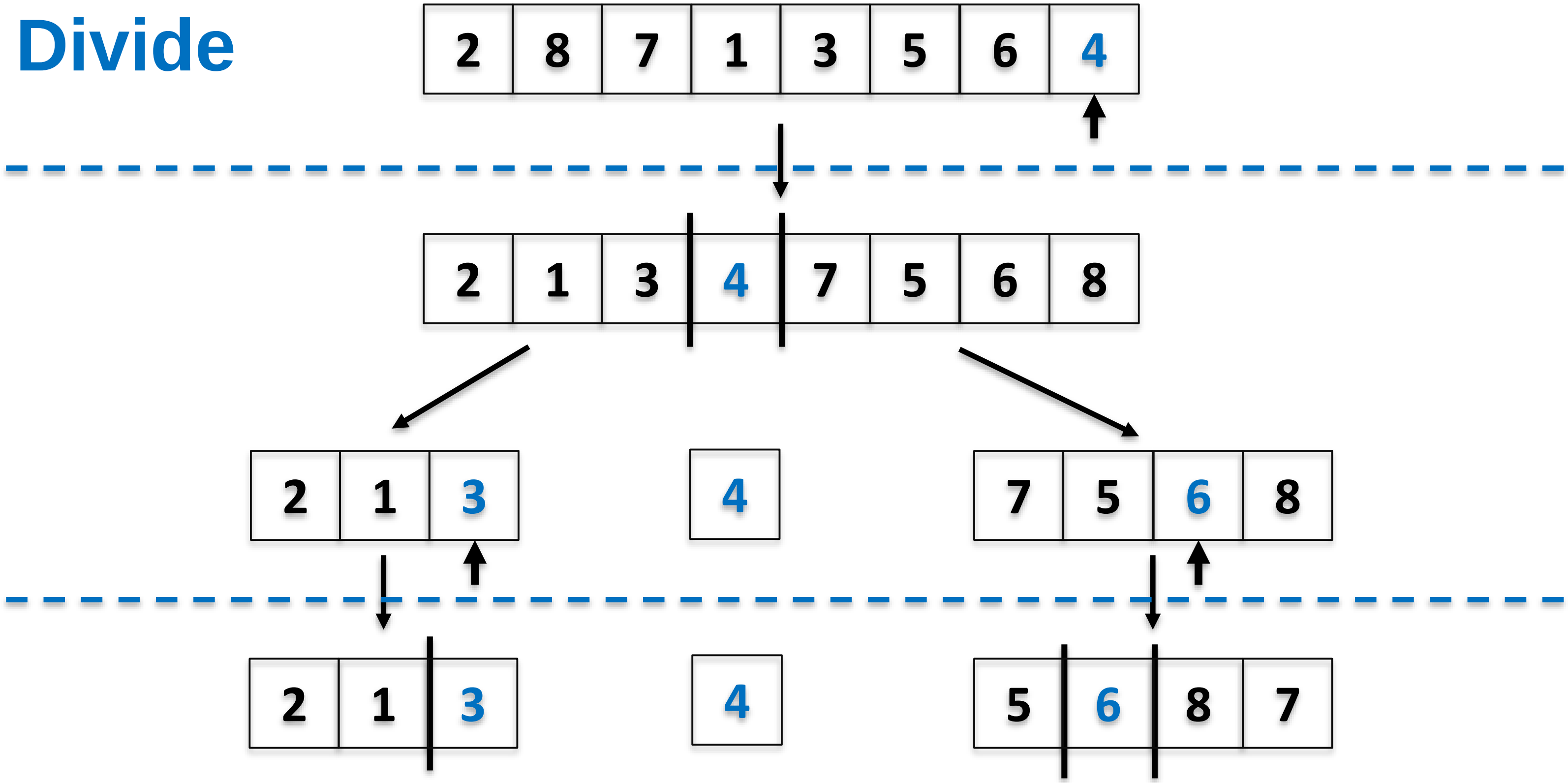

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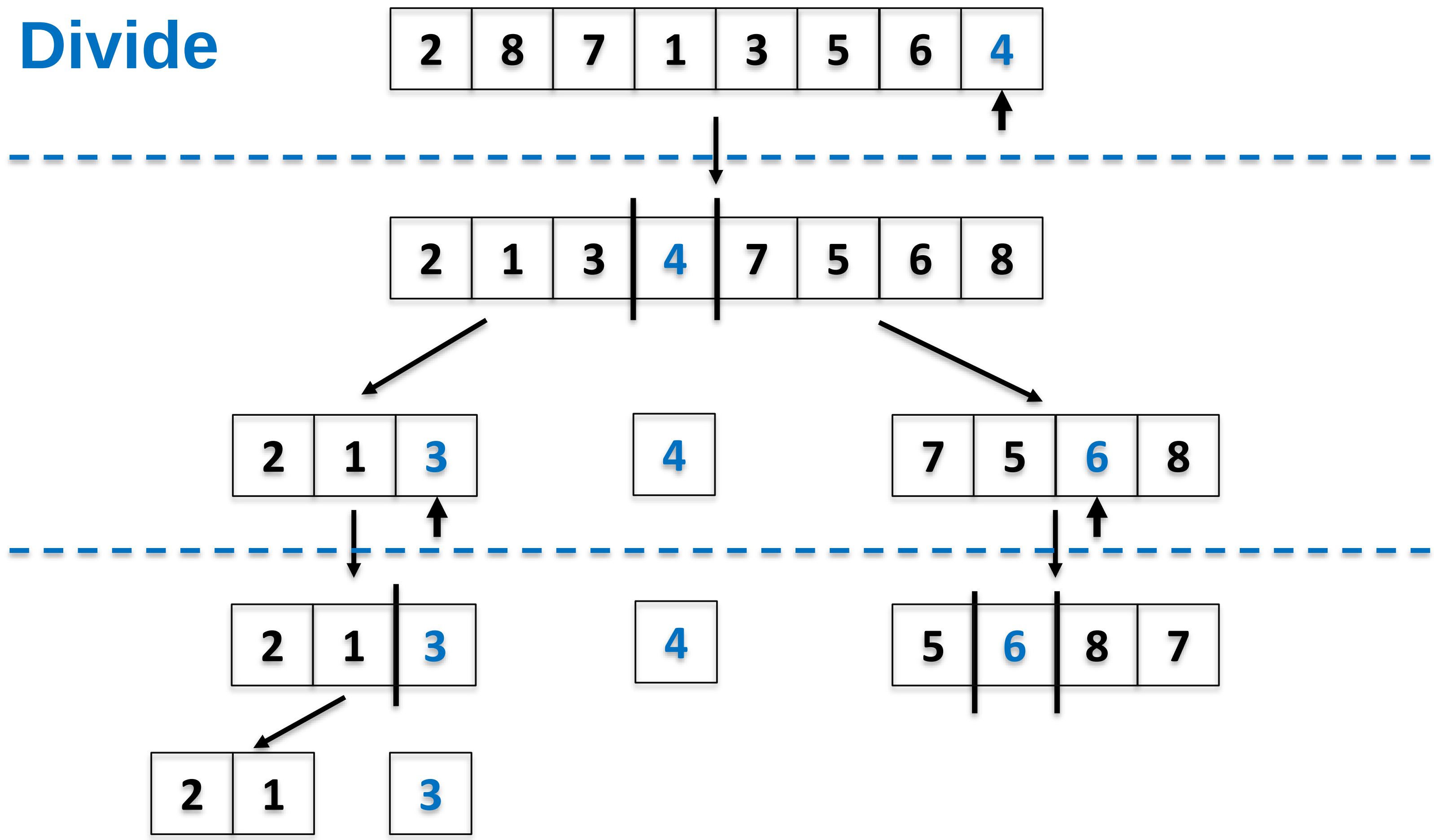

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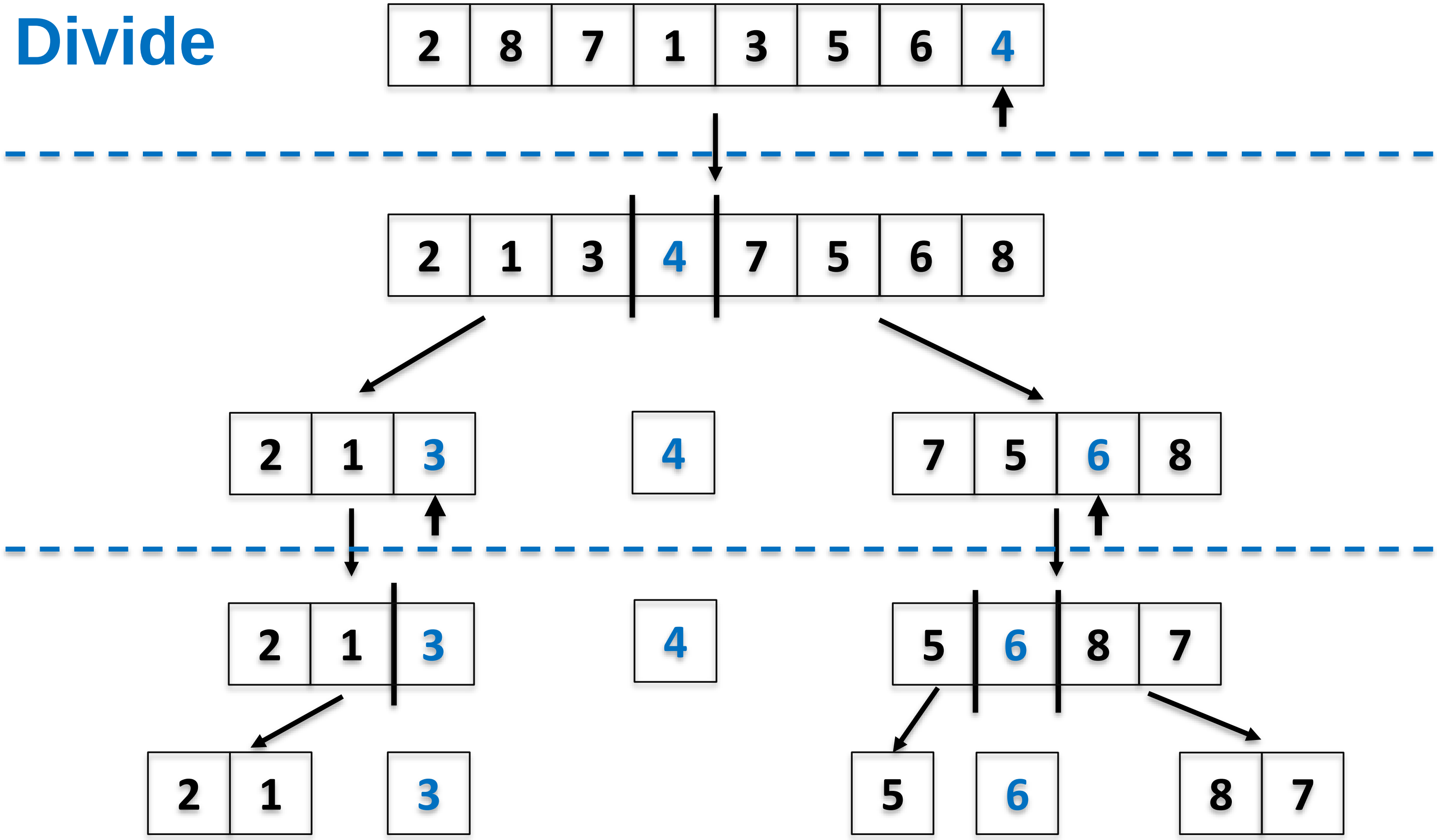

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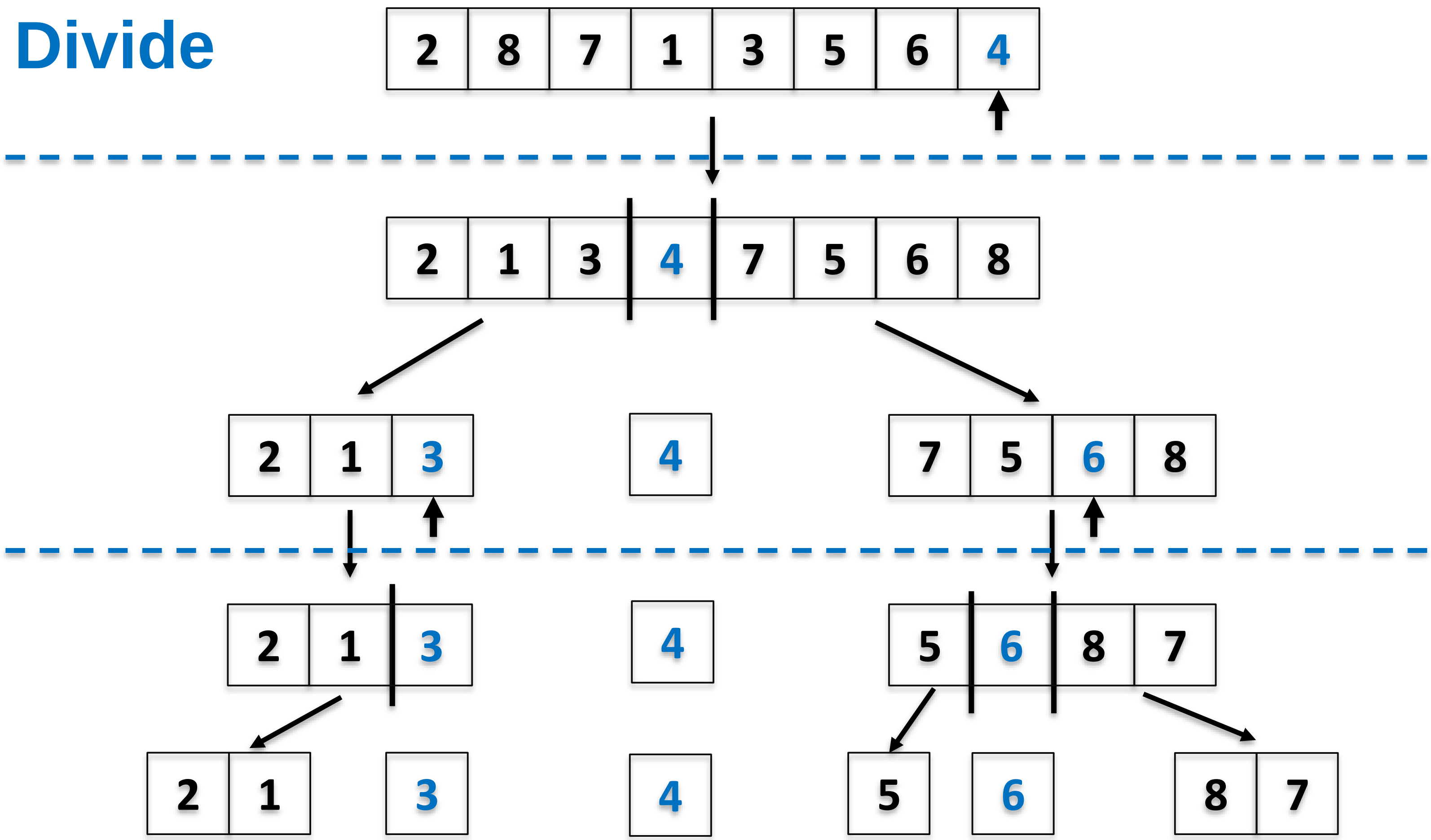

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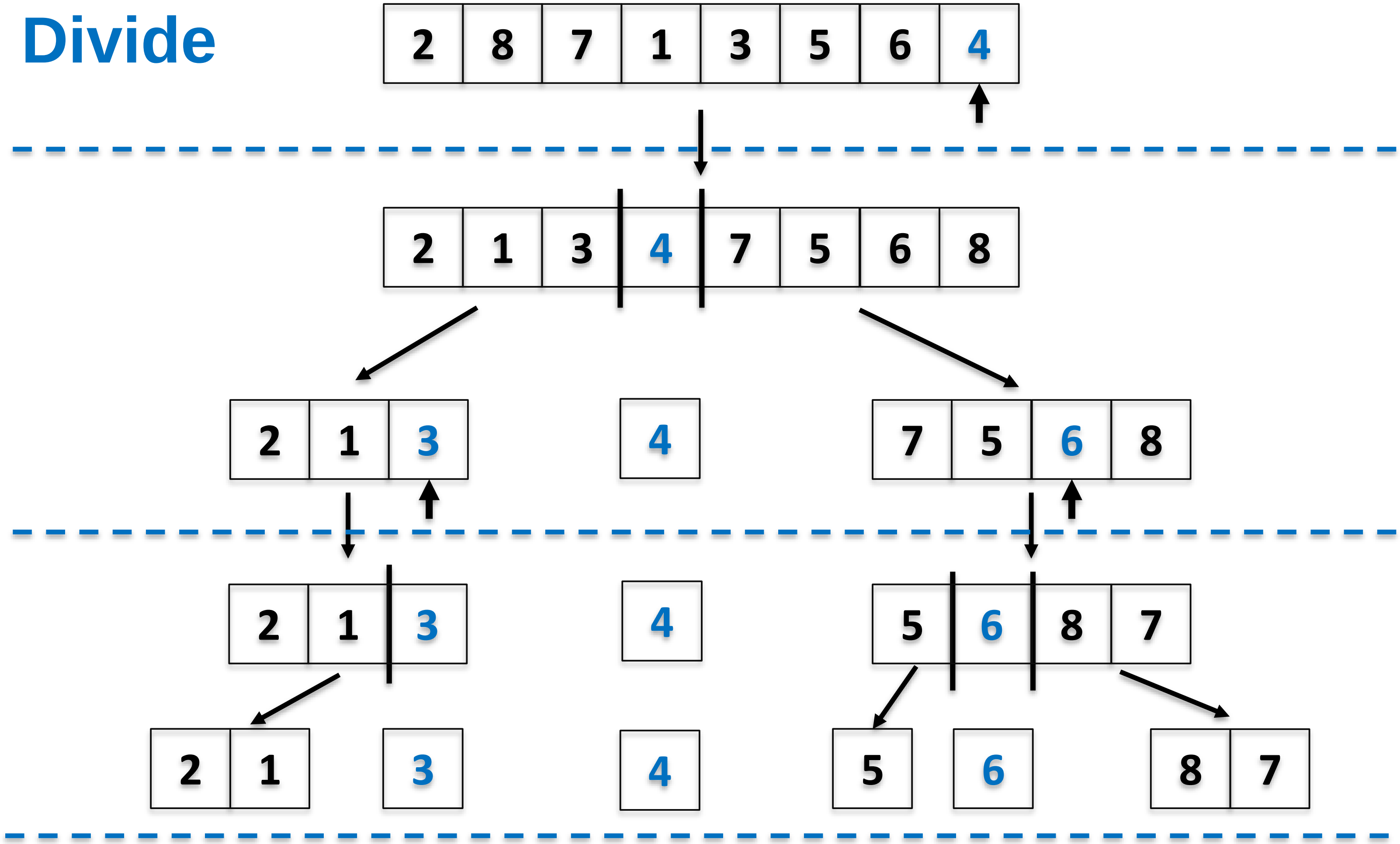

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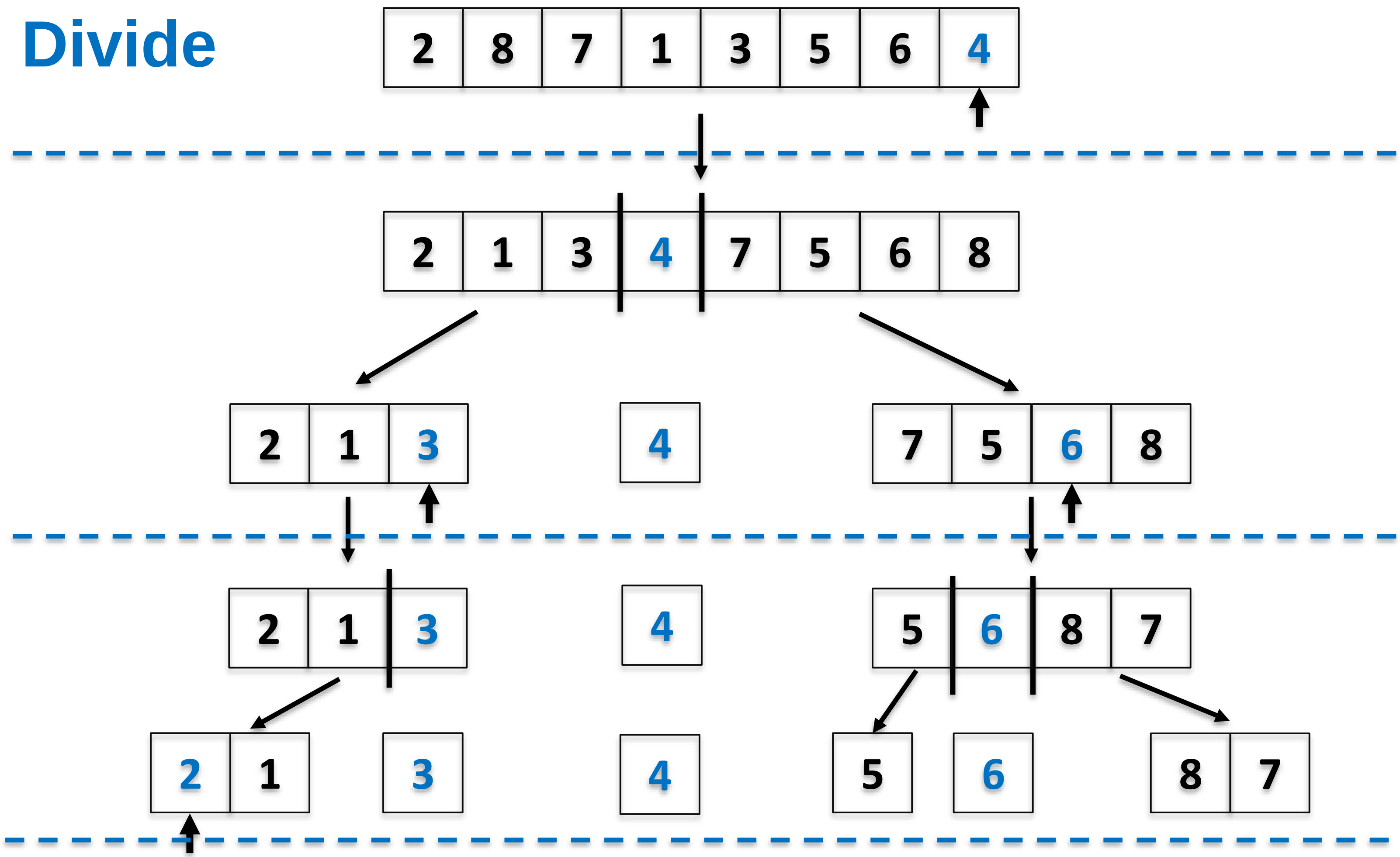

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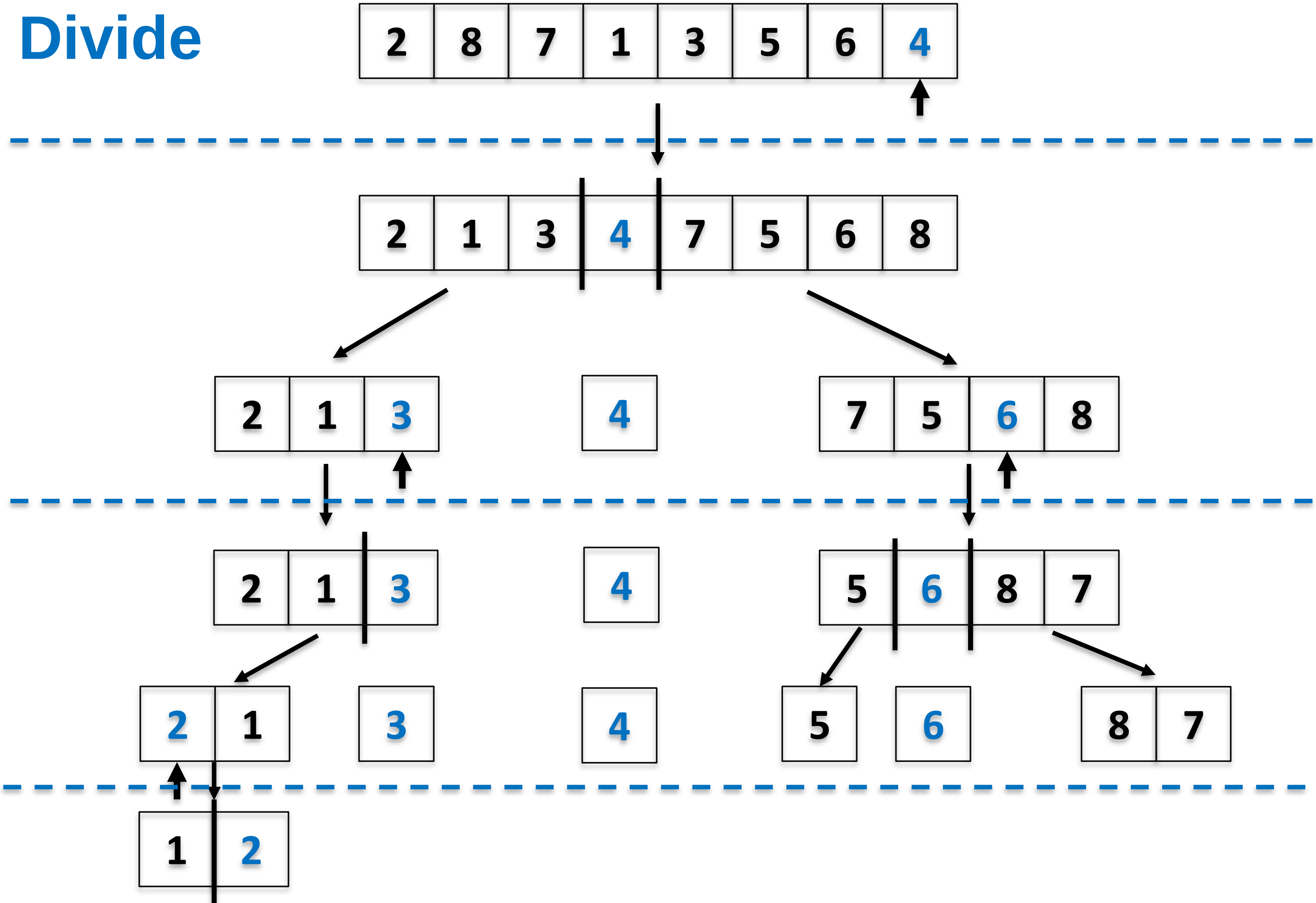

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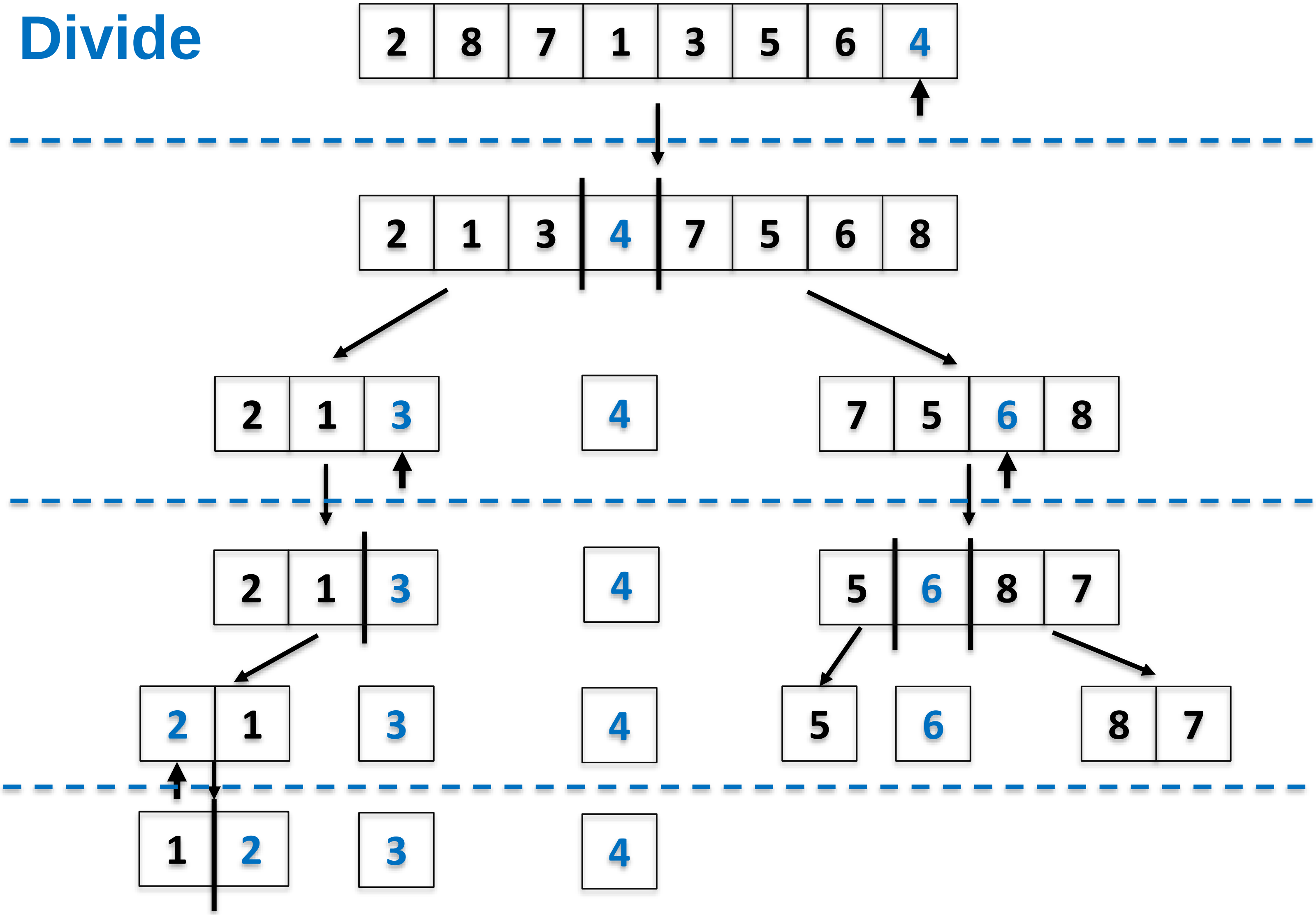

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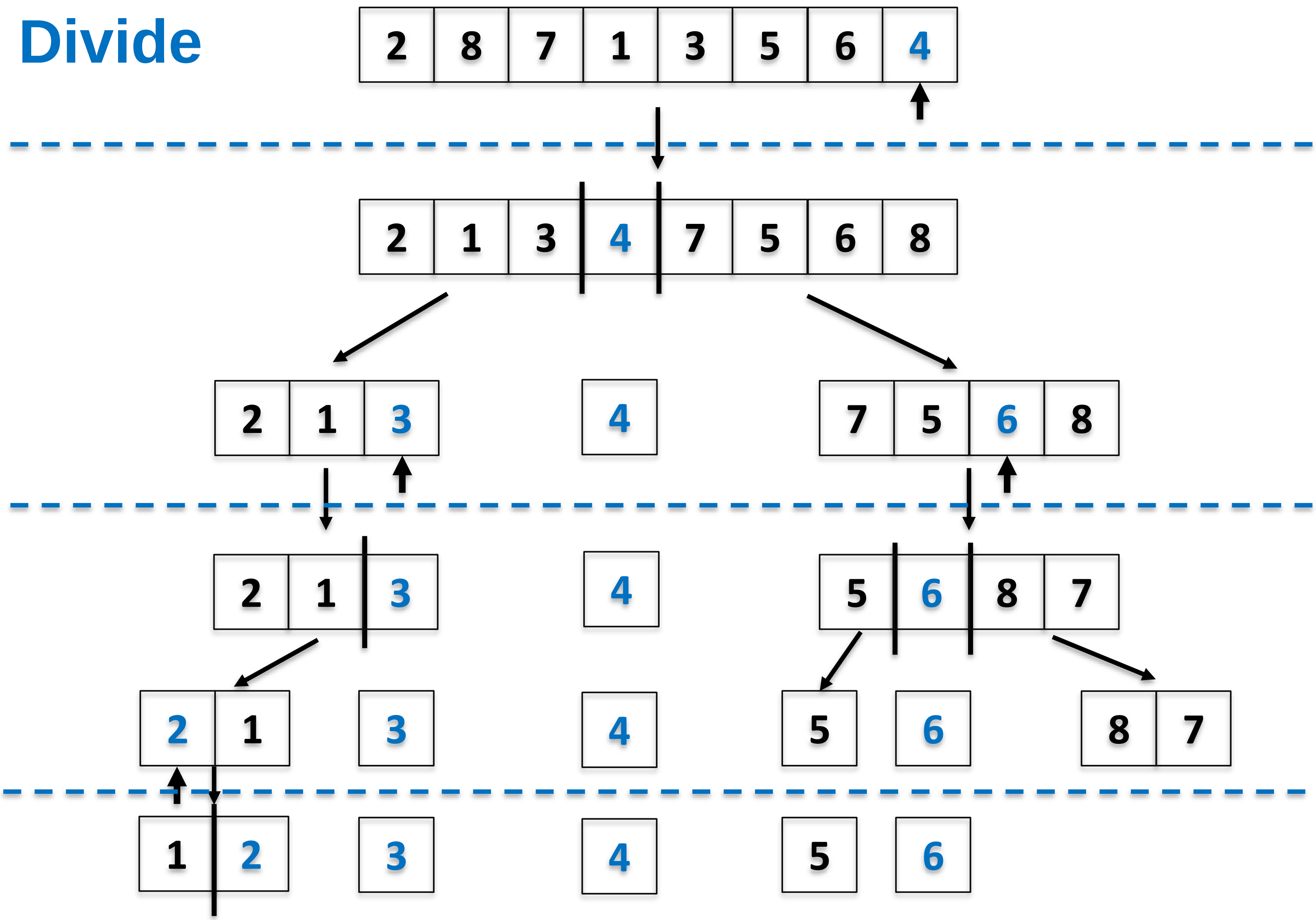

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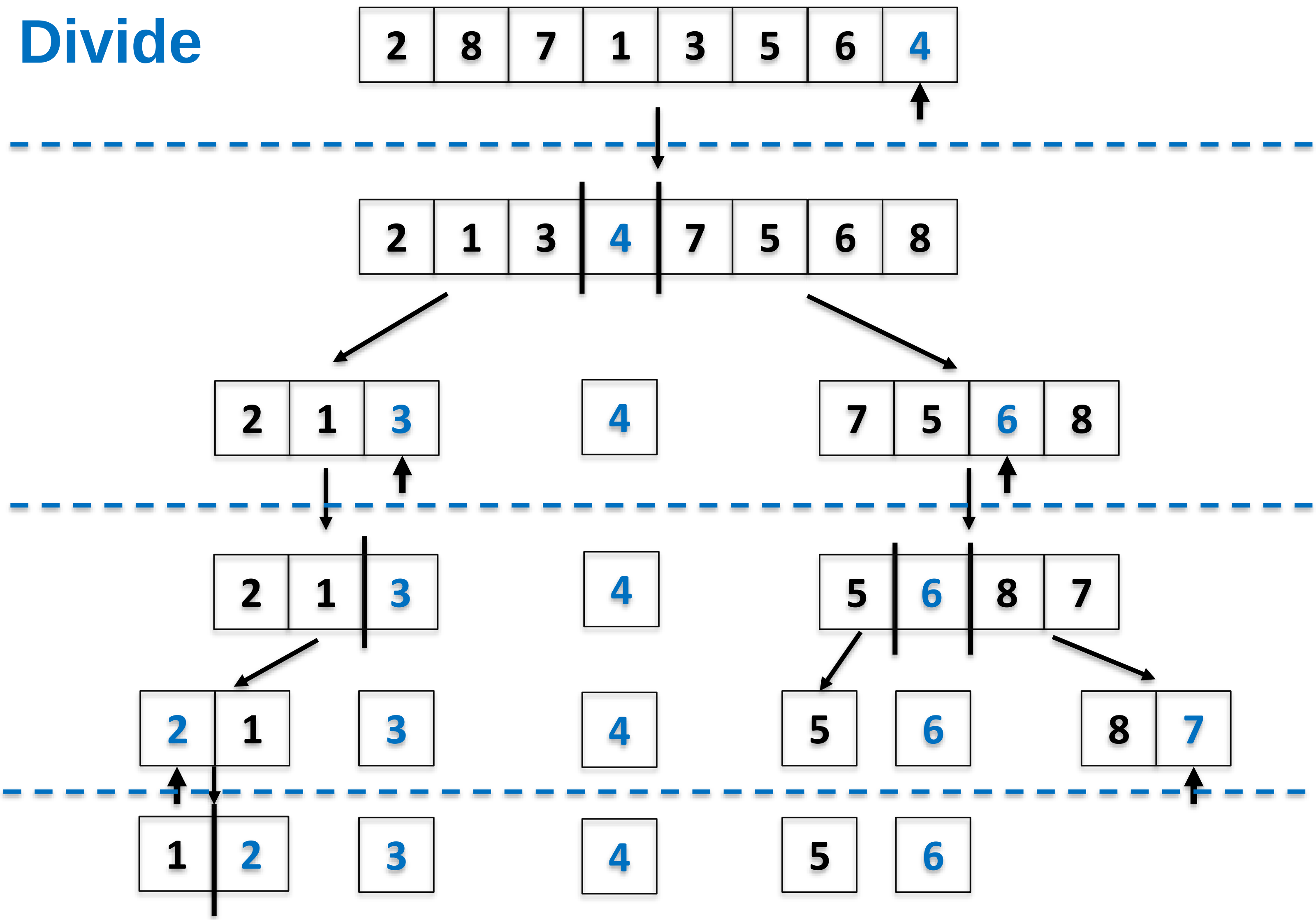

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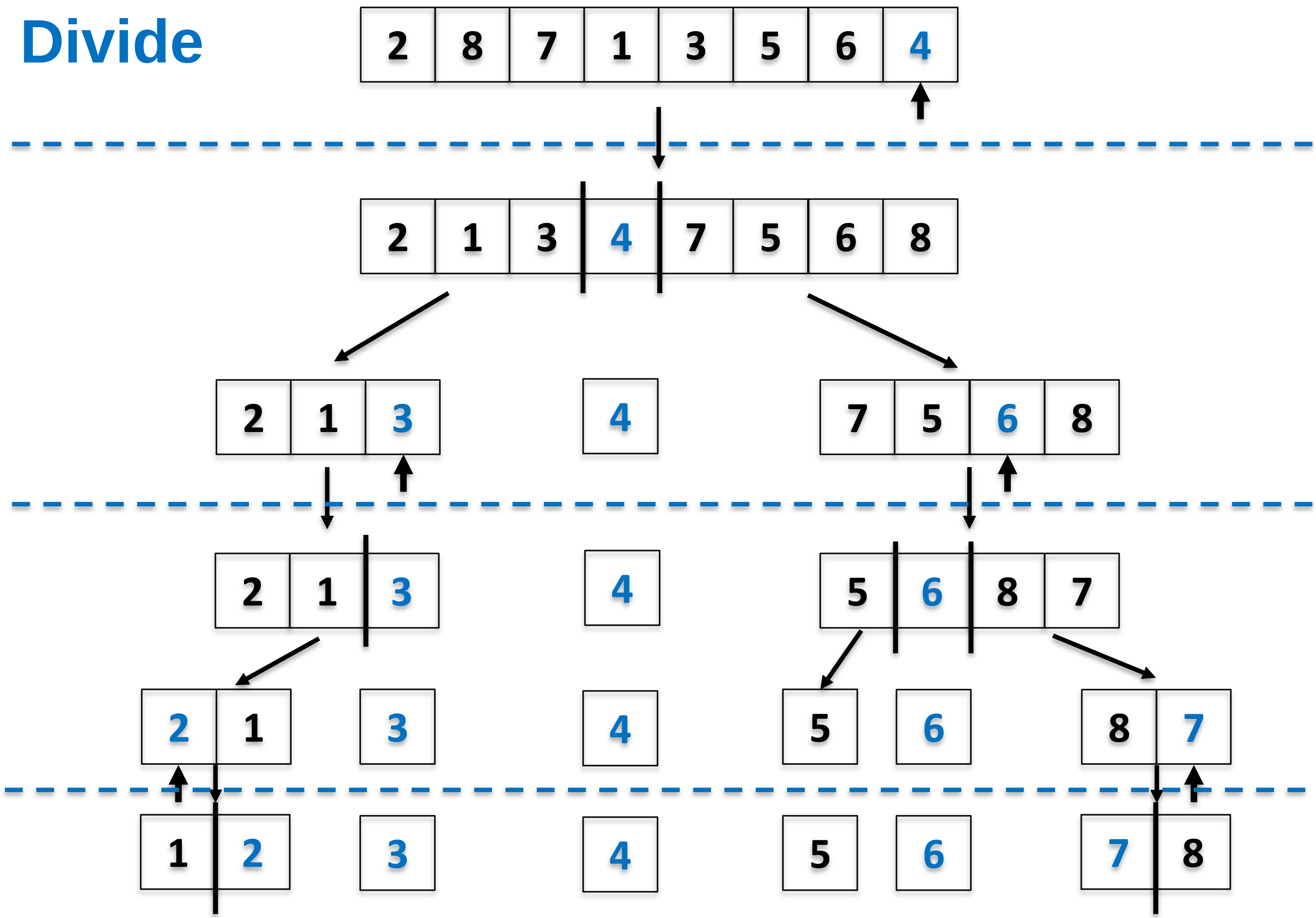

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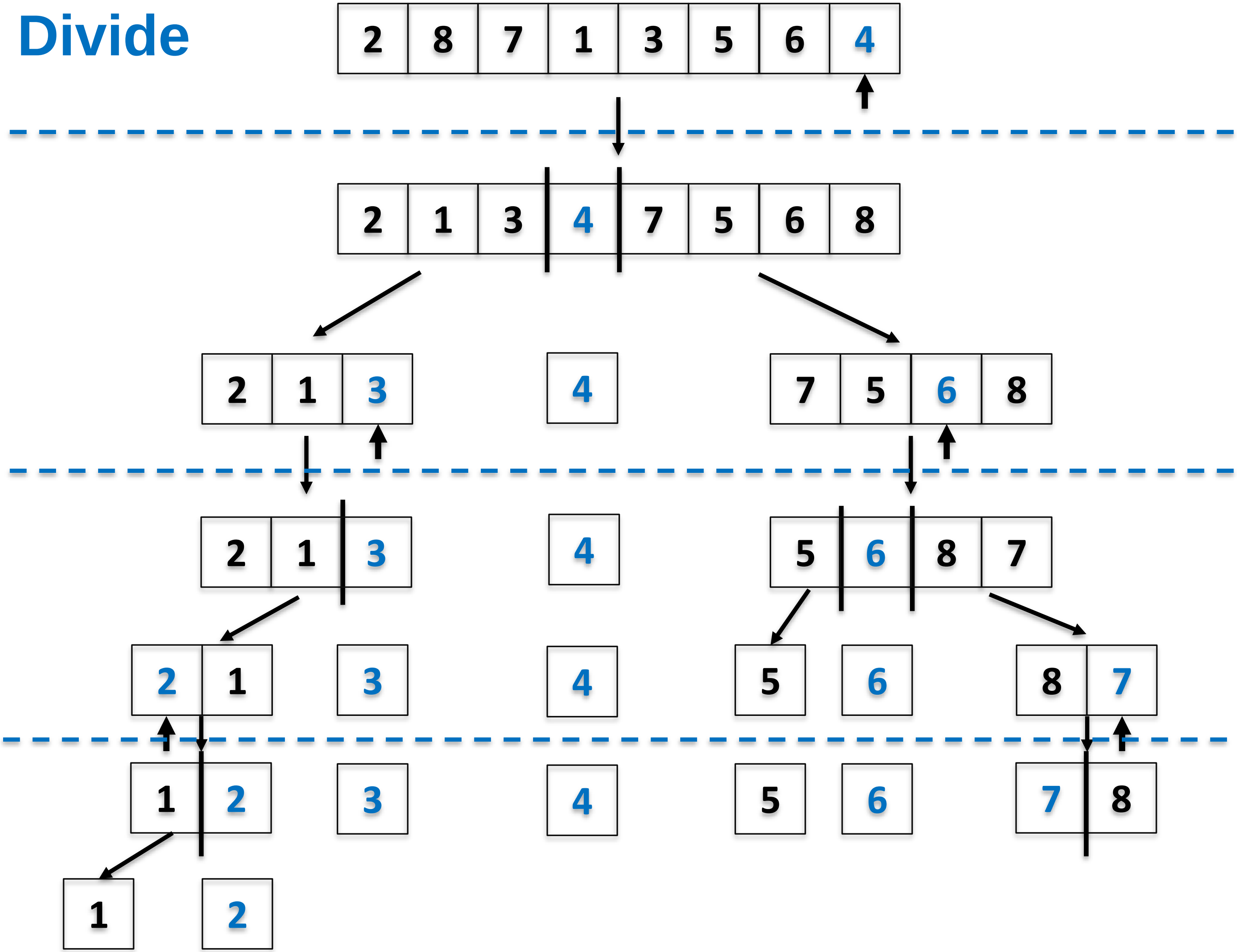

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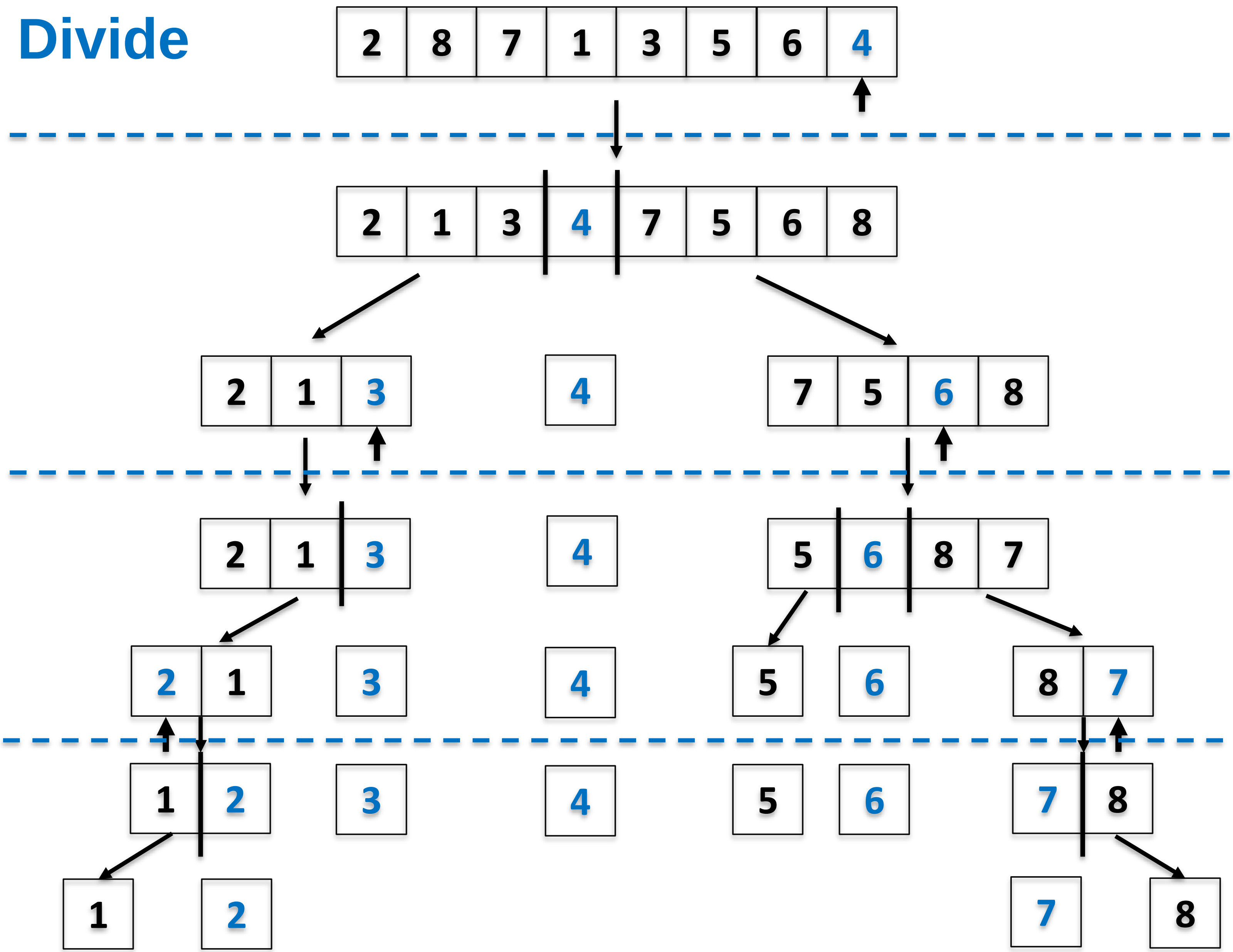

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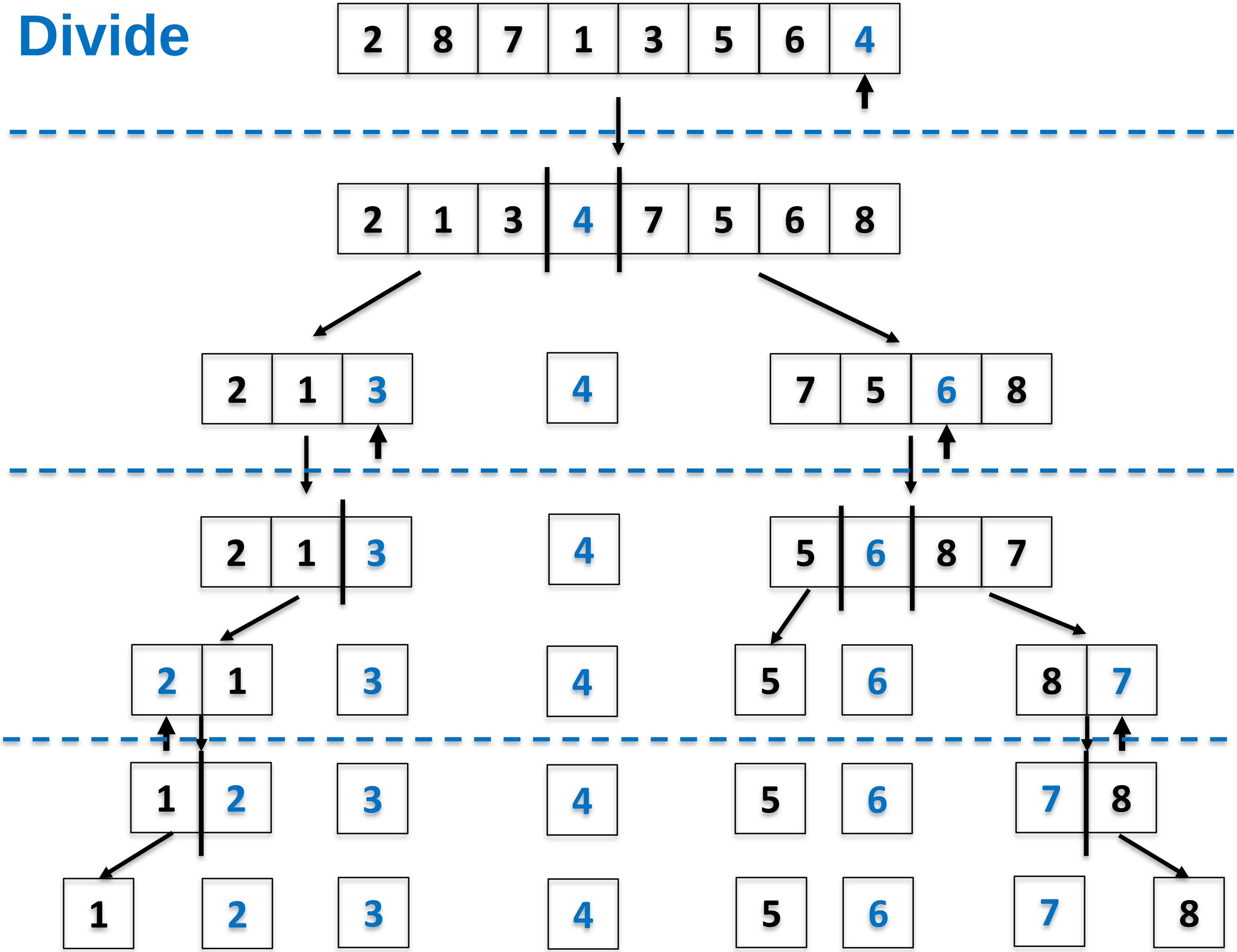


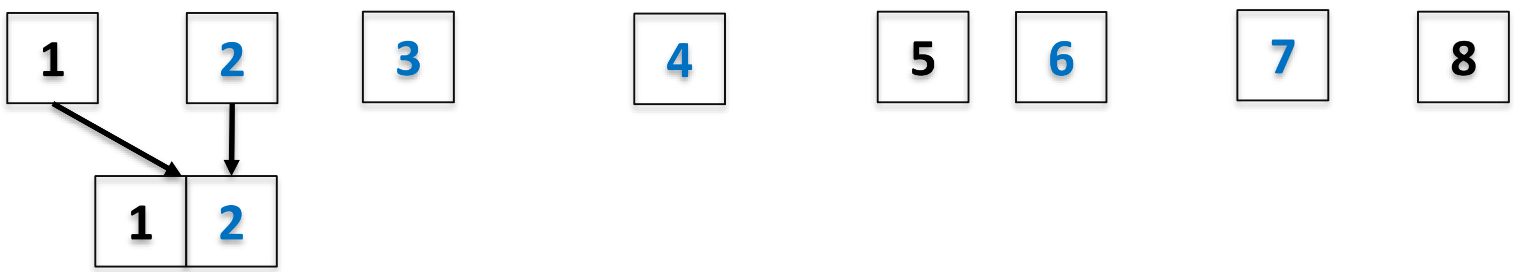
Quicksort - Example

Conquer

1	2	3	4	5	6	7	8

Quicksort - Example

Conquer



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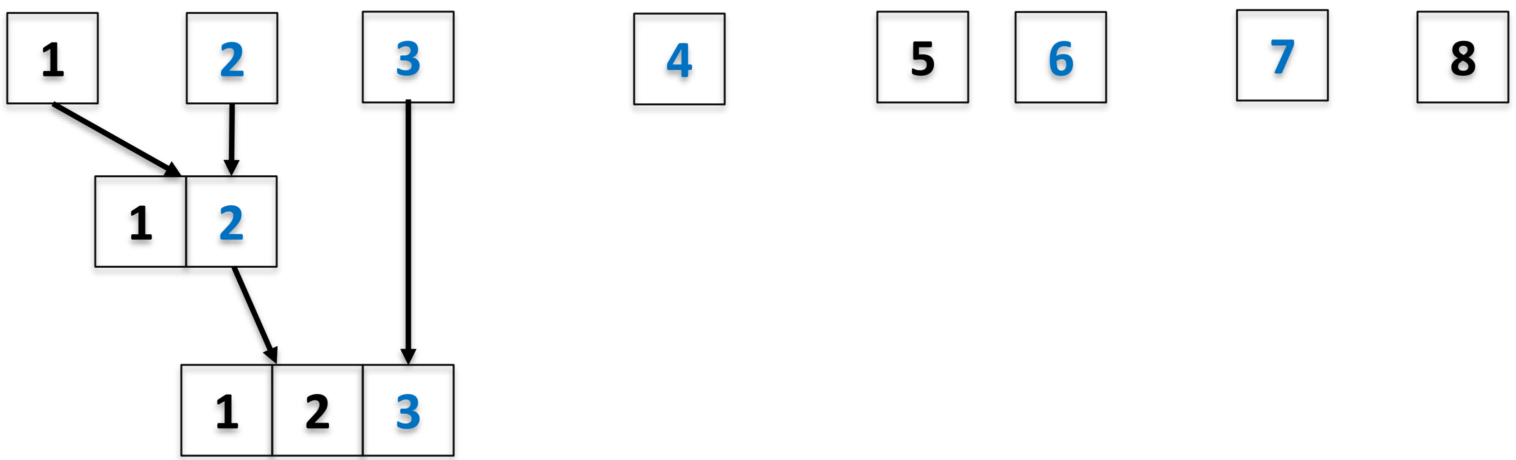

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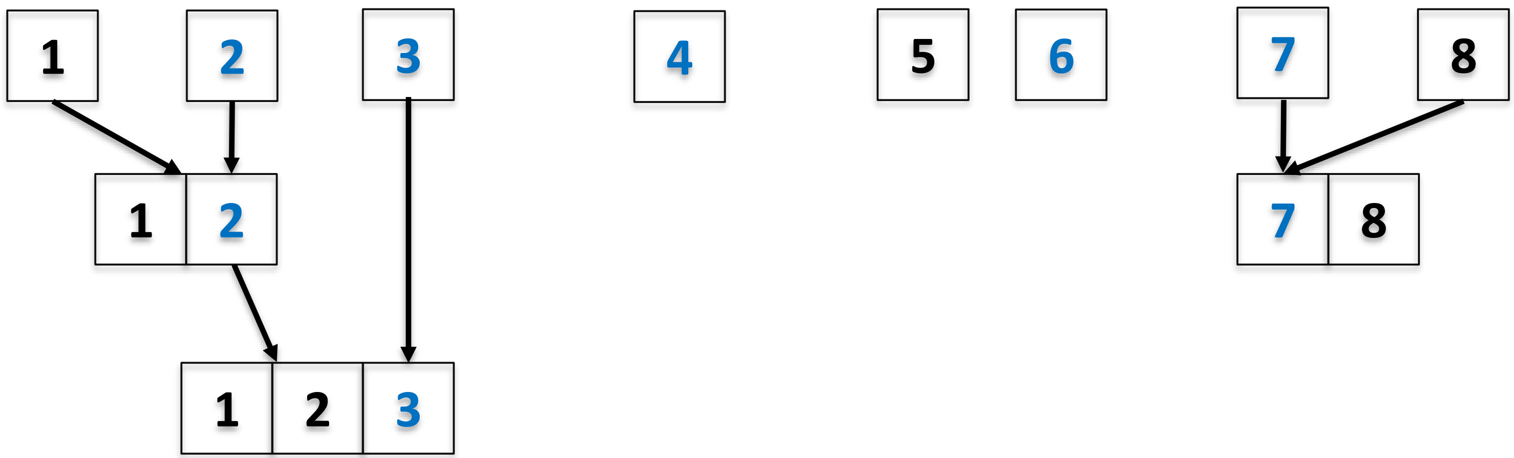

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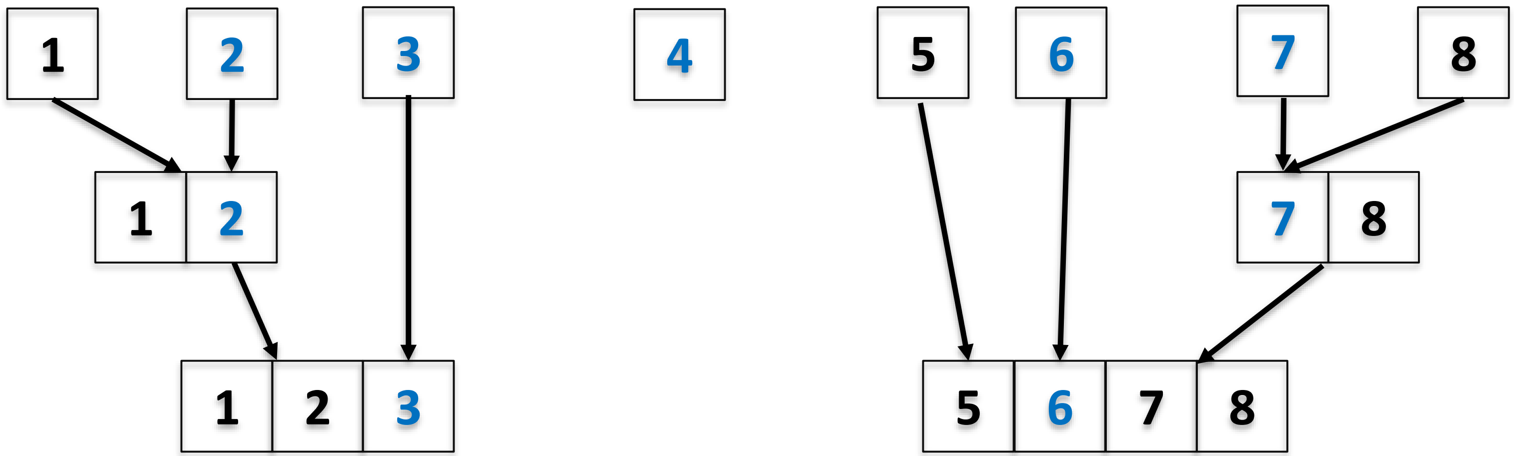

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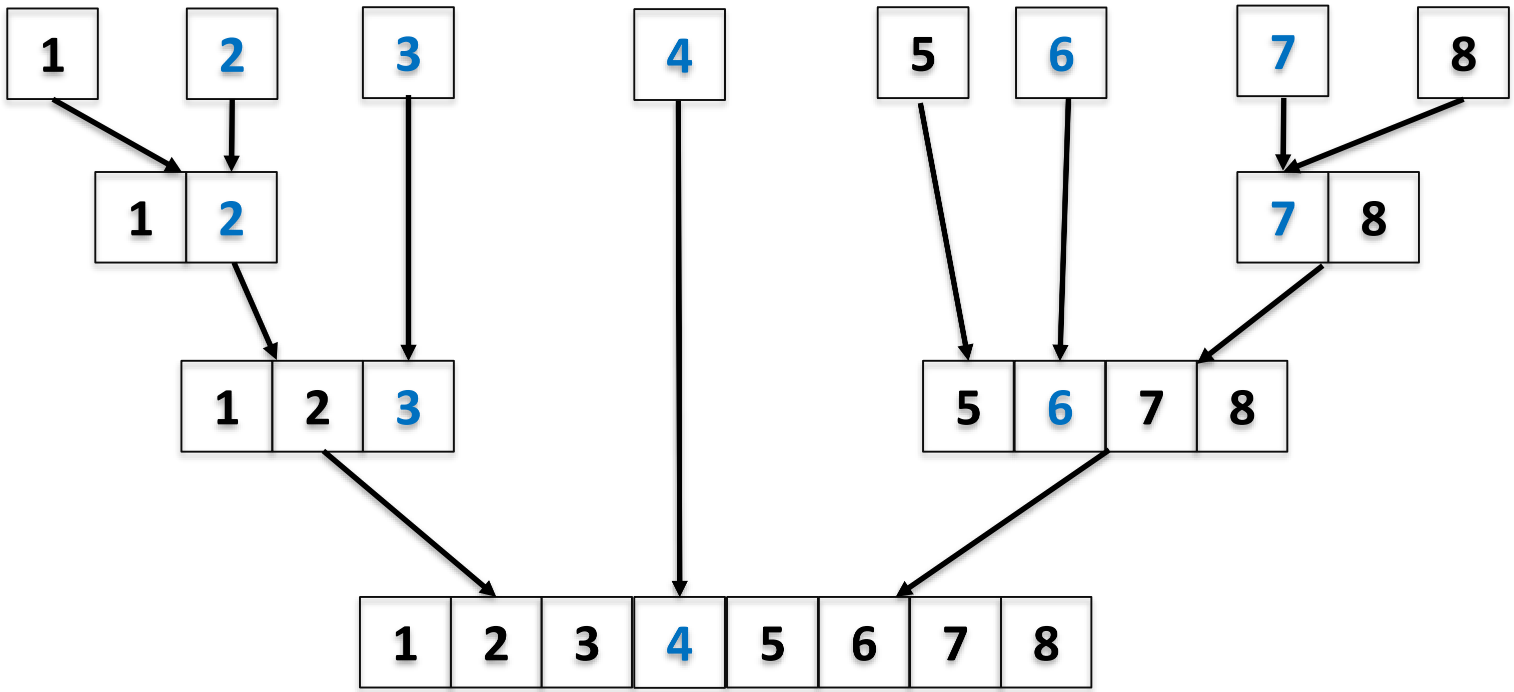

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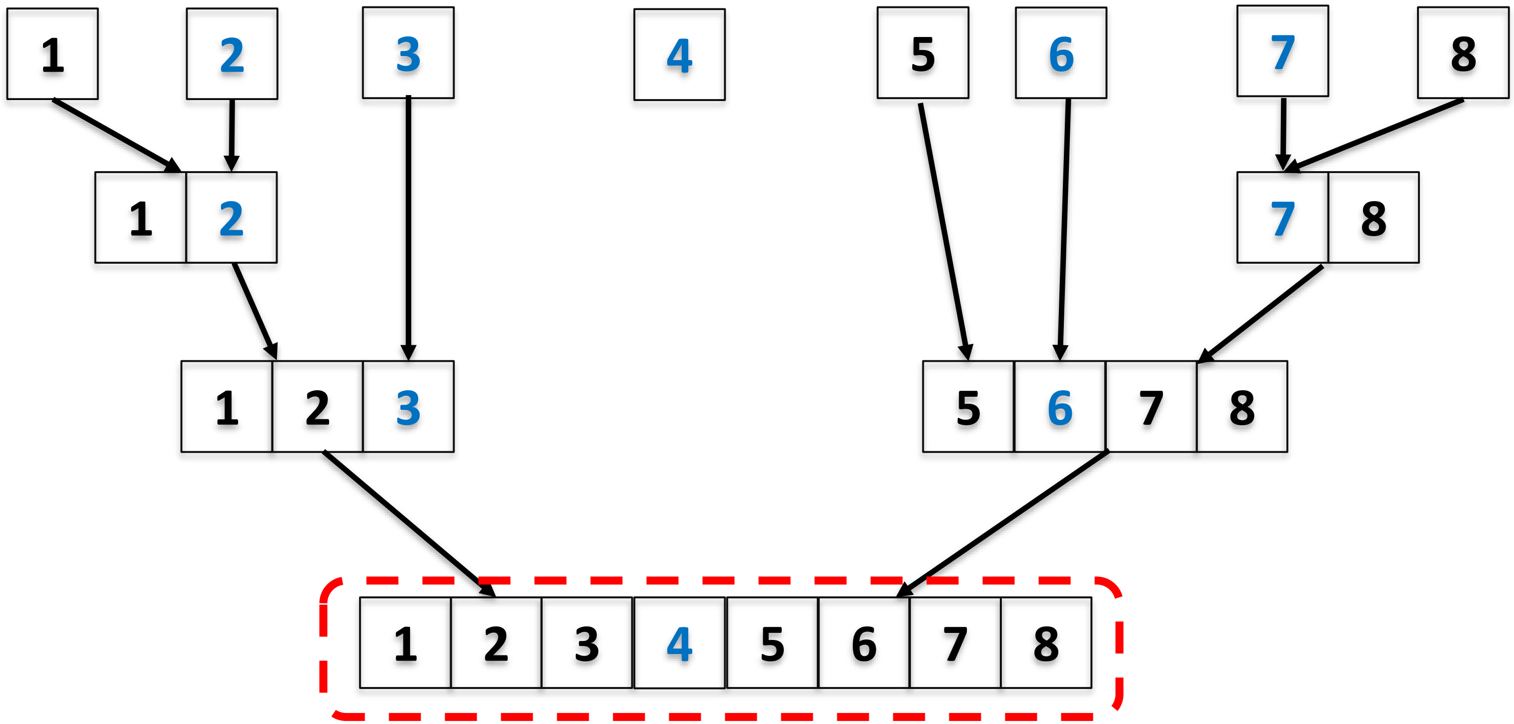

Outline

- Introduction to Part II
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort
- Randomized Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

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  - New: Indicator variables.
    —— Simple and elegant, but needs practice to master.

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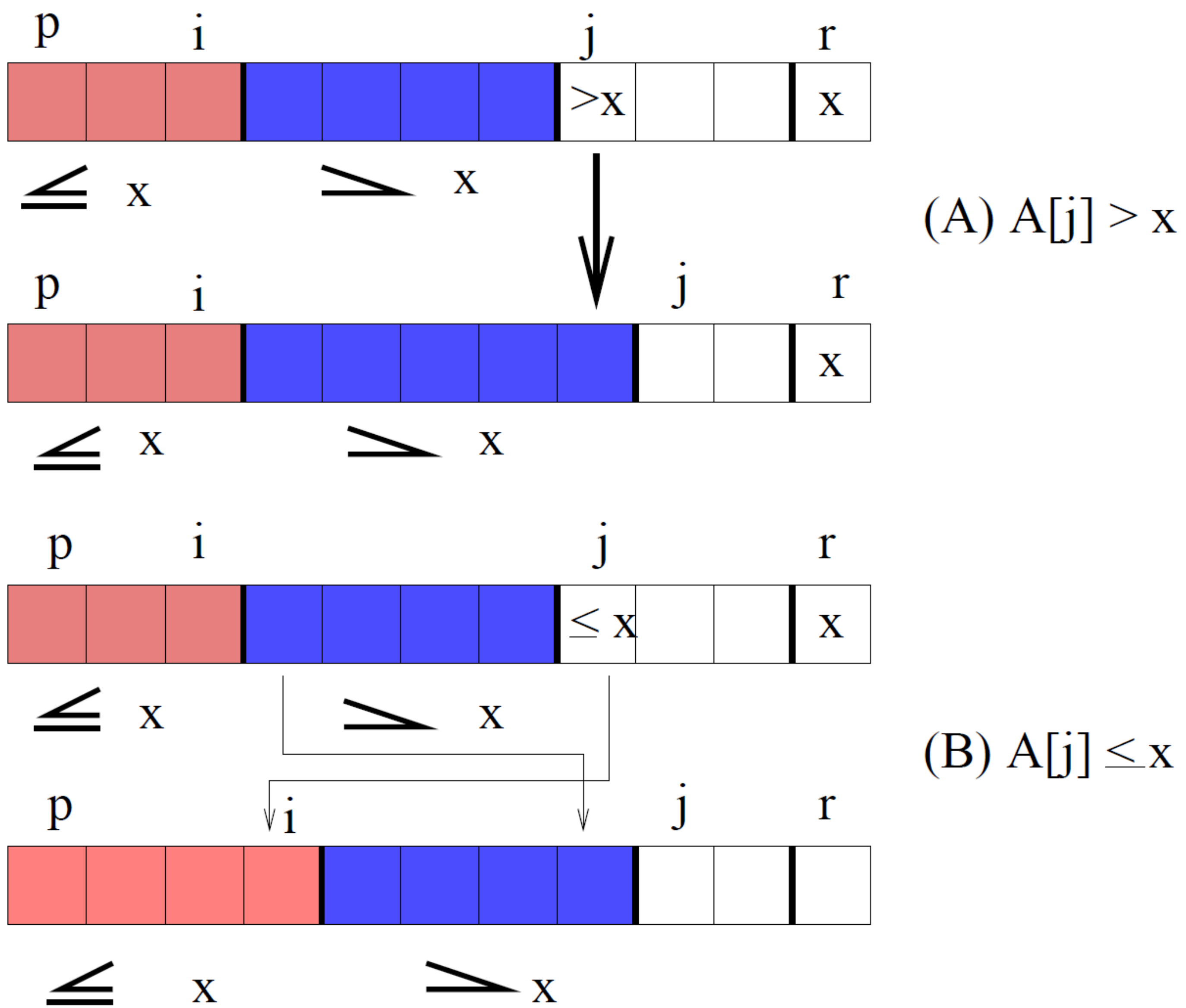
Expected Case



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Expected Running Time

- Expected Running Time of Randomized-Quicksort:
  - Let $z_1 < z_2 < \cdots < z_n$ be the n elements in sorted order
  - X: total number of comparisons performed in all calls to randomized-partition
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For $1 \leq i \leq j \leq n$, let $Z_{ij} = \{z_i , z_{i+1}, \ldots , z_j\}$

- remember $z_i < z_{i+1} < \cdots < z_j$

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- If the pivot is either z_i or z_j
  - z_i and z_j will be compared
- If the pivot is any element in Z_{ij} other than z_i or z_j
  - z_i and z_j are not compared with each other in all randomized-partition calls

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \text{ is compared with } z_j\}$$

$$=$$

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Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

Hence, the expected number of comparisons is $O(n \log n)$, which is the expected running time of Randomized-Quicksort

Outline

- Introduction to Part II
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort
- Randomized Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Linear Time Selection

Definition (Selection Problem)

Given a sequence of numbers $\langle a_1, \ldots, a_n \rangle$, and an integer i, $1 \leq i \leq n$, find the ith smallest element. When $i = \lceil n/2 \rceil$, it is called the median problem.

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Example

Given $\langle 1, 8, 23, 10, 19, 33, 100 \rangle$, the 4th smallest element is 19.

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Question

How do you solve this problem?

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Can we do better?

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Question

Can we do better?

Answer: YES, but we need to recall Partition(A,p,r) used in Quicksort!

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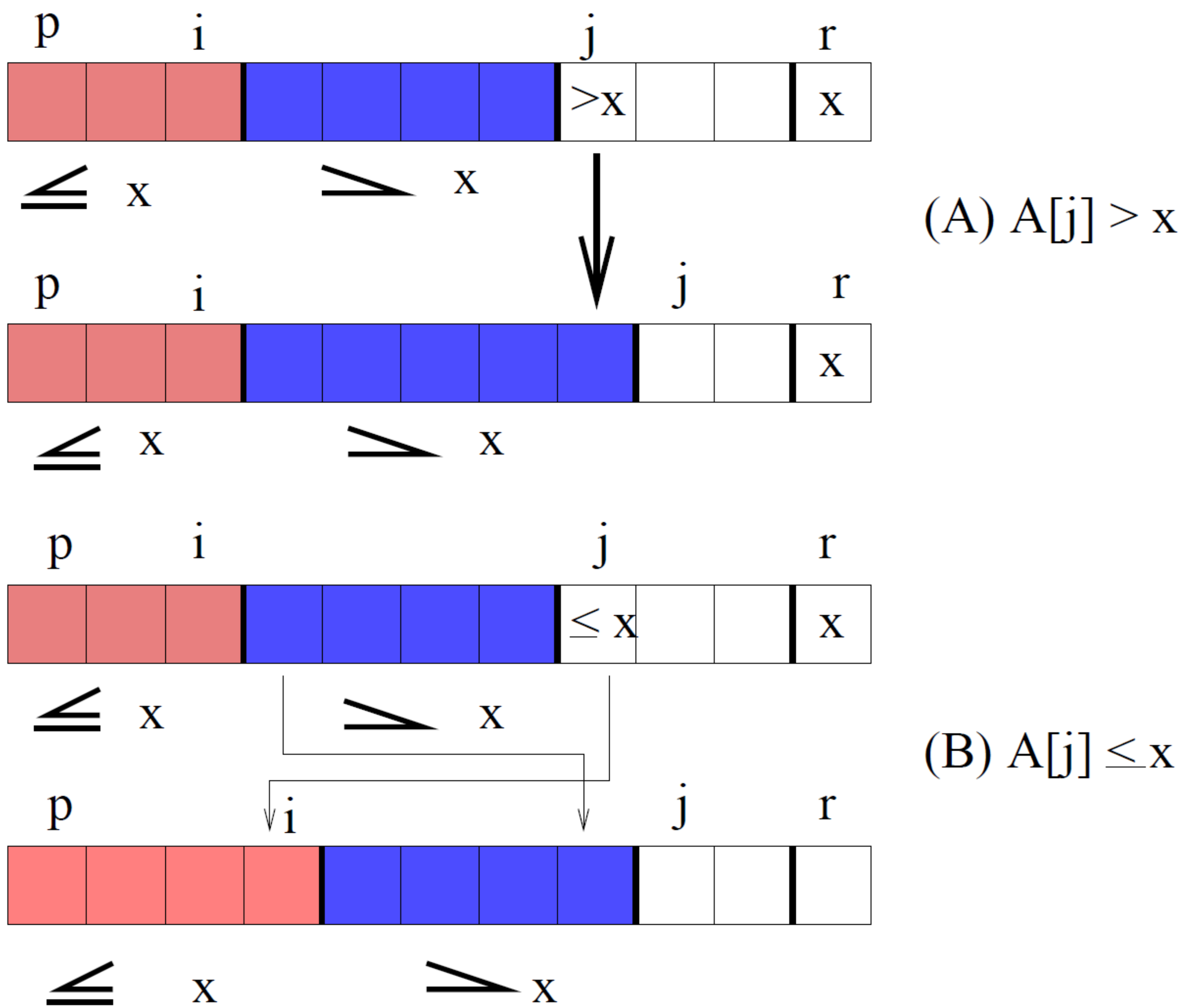
Review of Randomized-Partition (A,p,r)



Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Problem: Select the *i*th smallest element in A[p..r], where
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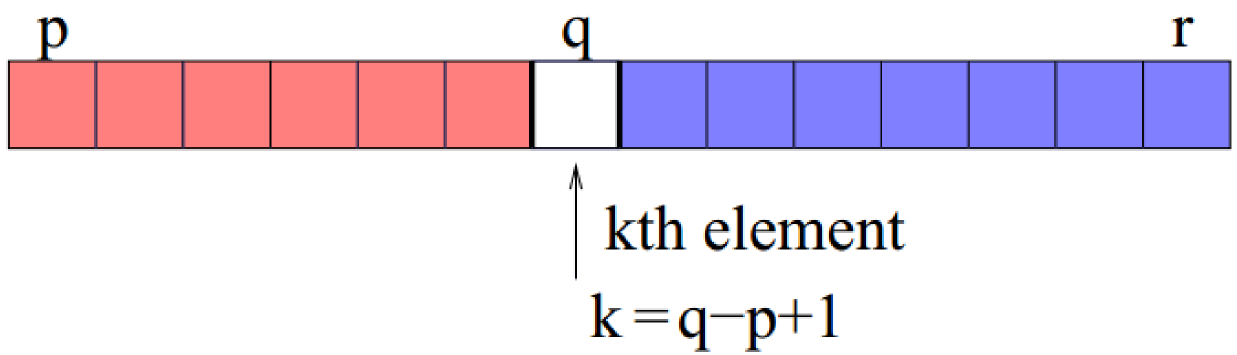

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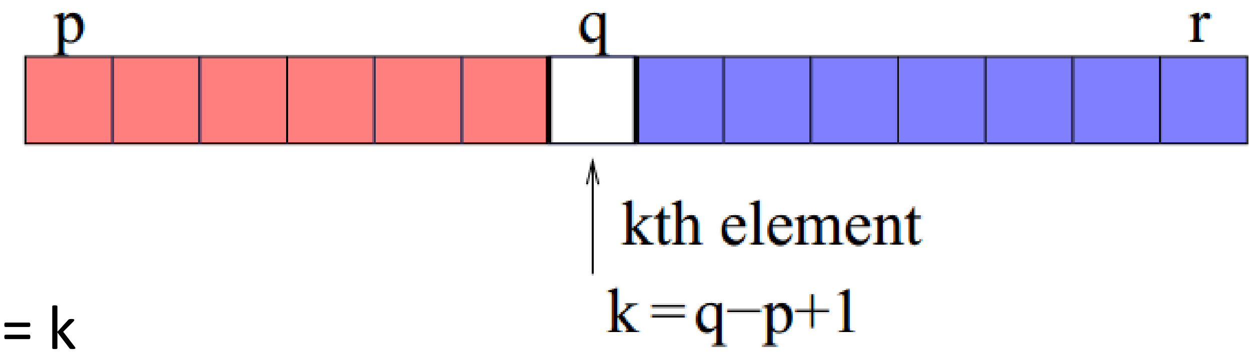

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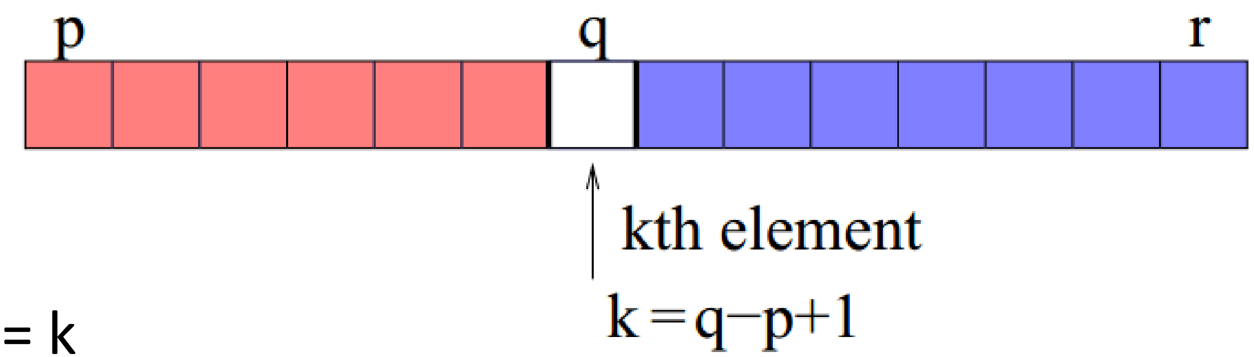

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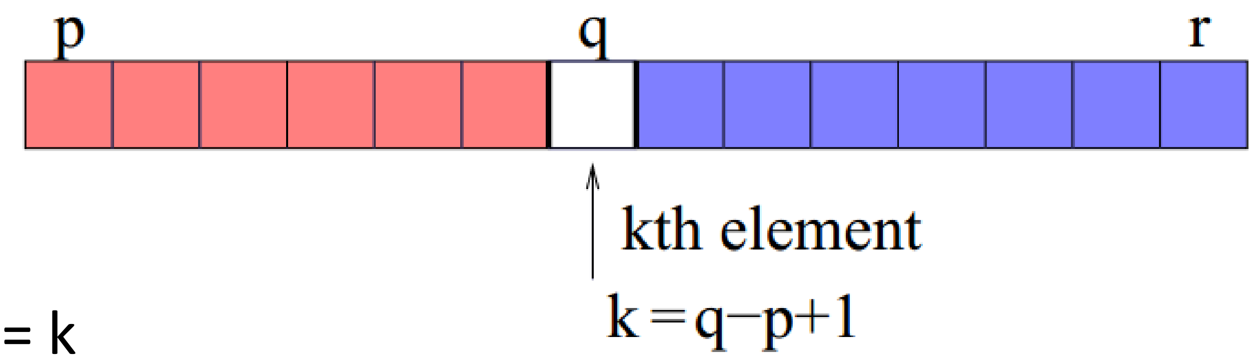

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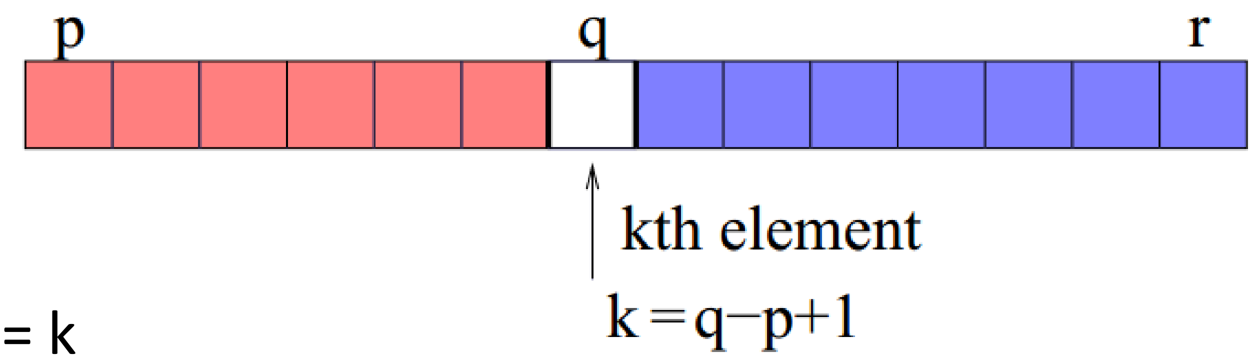

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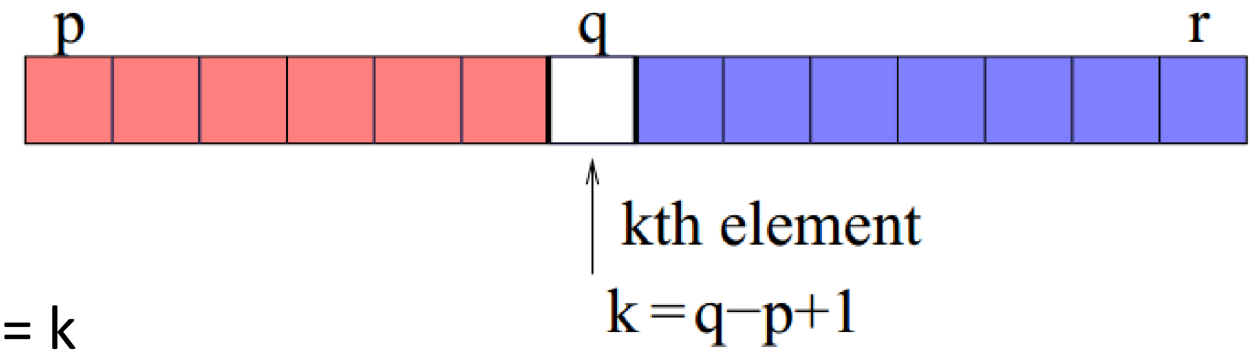

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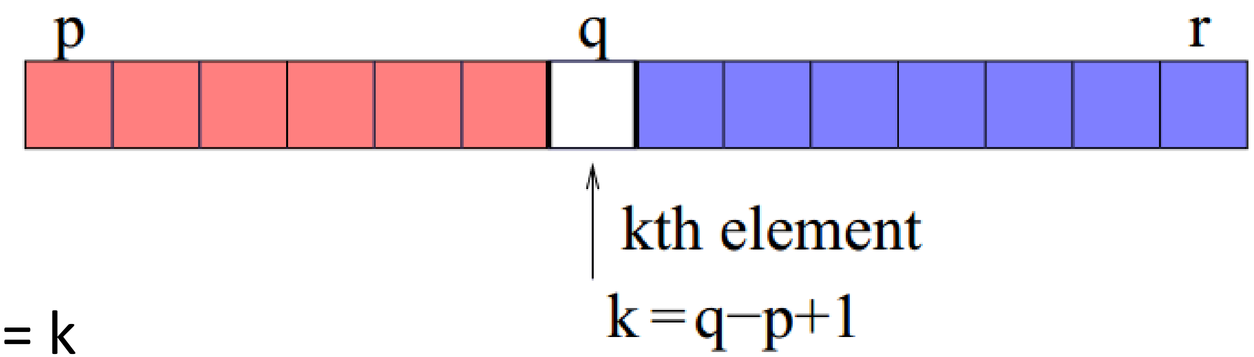

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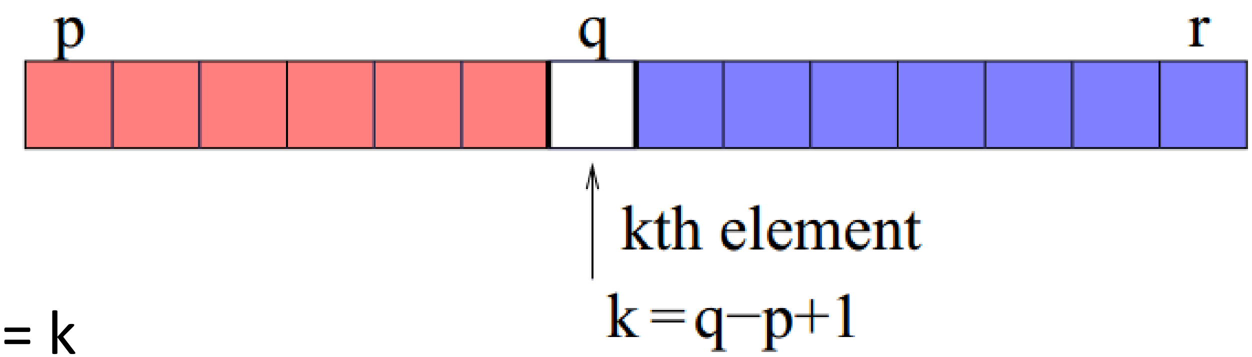

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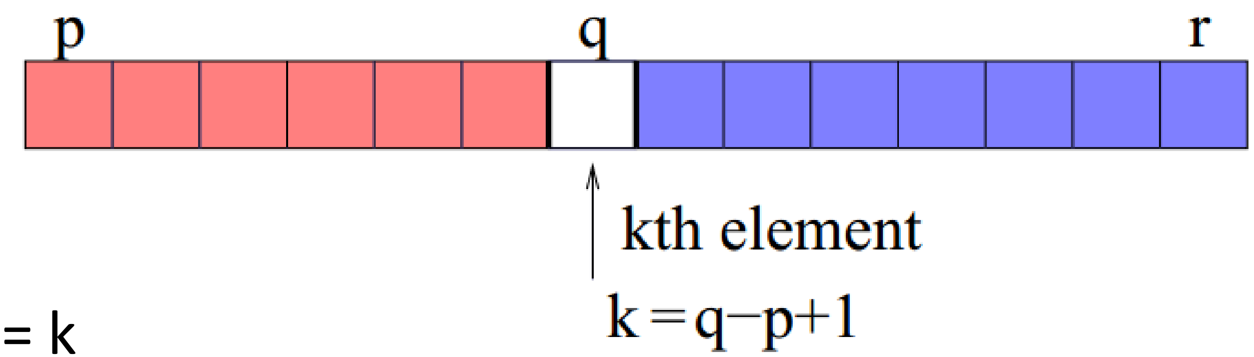

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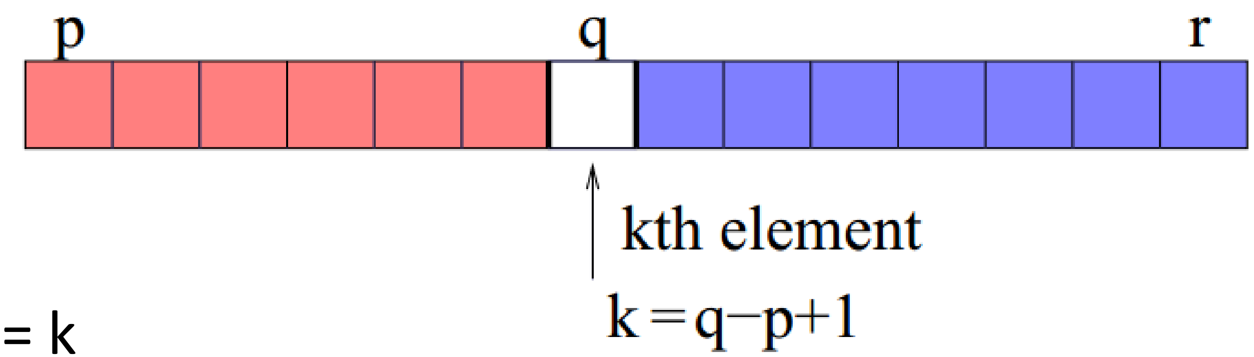
    in A[q+1..r]

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Problem: Select the *i*th smallest element in A[p..r], where $1 \le i \le r-p+1$

Solution: Apply Randomized-Partition(A, p, r), getting



- $i = k$
  - pivot is the solution
- $i < k$
  - the *i*th smallest element in A[p..r] must be the *i*th smallest element in A[p..q−1]
- $i > k$
  - the *i*th smallest element in A[p..r] must be the (*i* − *k*)th smallest element in A[q+1..r]

If necessary, recursively call the same procedure to the subarray

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

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Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

if p *is equal to* r **then**
  | **return** $A[p]$;
end

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

if p *is equal to* r **then**

  return $A[p]$;

end

$q \leftarrow$Randomized-Partition(A, p, r);

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select
Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$
if p *is equal to* r **then**
  return $A[p]$;
end
$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
if $i \leftarrow k$ **then**

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

if p *is equal to* r **then**
| **return** $A[p]$;
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$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
if $i \leftarrow k$ **then**
| **return** $A[q]$;//The pivot is the answer
end

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select
Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$
if p *is equal to* r **then**
| **return** $A[p]$;
end
$q \leftarrow$Randomized-Partition(A, p, r);
$k \leftarrow q - p + 1$;
if $i \leftarrow k$ **then**
| **return** $A[q]$;//The pivot is the answer
end
else if $i < k$ **then**
| **return** Randomized-Select$(A, p, q - 1, i)$;
end

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

```
Input: An array A,the range of index p,r,the ith smallest element that
       we want to select
Output: The ith smallest element A[i]
if p is equal to r then
 | return A[p];
end
q ←Randomized-Partition(A, p, r);
k ← q − p + 1;
if i ← k then
 | return A[q];//The pivot is the answer
end
else if i < k then
 | return Randomized-Select(A, p, q − 1, i);
end
else
 | return Randomized-Select(A, q + 1, r, i − k);
end
```

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

```
Input: An array A,the range of index p,r,the ith smallest element that
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 |  return Randomized-Select(A, q + 1, r, i − k);
end
```

To find the ith smallest element in A[1..n], call
Randomized-Select(   )

Randomized-Select(A,p,r,i), 1≤i≤r−p+1

Randomized-Select(A,p,r,i)

Input: An array $\boldsymbol{A}$,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$,the $\boldsymbol{i}$th smallest element that we want to select

Output: The ith smallest element $\boldsymbol{A}[\boldsymbol{i}]$

```
if p is equal to r then
|  return A[p];
end
q ←Randomized-Partition(A, p, r);
k ← q − p + 1;
if i ← k then
|  return A[q];//The pivot is the answer
end
else if i < k then
|  return Randomized-Select(A, p, q − 1, i);
end
else
|  return Randomized-Select(A, q + 1, r, i − k);
end
```

To find the ith smallest element in A[1..n], call Randomized-Select(A, 1, n, i)

Randomized Selection - Example

- Find the 8th smallest element of the following list of numbers:
  - 8 25 2 14 3 20 15 13 12 11 7 5 9 10 16 18 1 23 26 21

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p																			r
8	25	2	14	3	20	15	13	12	11	7	5	9	10	16	18	1	23	26	21

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i*-*k*)th smallest element in A[q+1..r]

p																			r
8	25	2	14	3	20	15	13	12	11	7	5	9	10	16	18	1	23	26	21

$$i = 8, p = 1, r = 20$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p																			r
8	**25**	**2**	**14**	**3**	**20**	**15**	**13**	**12**	**11**	**7**	**5**	**9**	**10**	**16**	**18**	**1**	**23**	**26**	**21**
					↑														

$$i = 8, p = 1, r = 20$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p															q				r
8	**2**	**14**	**3**	**15**	**13**	**12**	**11**	**7**	**5**	**9**	**10**	**16**	**18**	**1**	**20**	**25**	**23**	**26**	**21**

$$i = 8, p = 1, r = 20$$
$$q = 16, \mathrm{k} = 16$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p															q				r
8	2	14	3	15	13	12	11	7	5	9	10	16	18	1	20	25	23	26	21

$$i = 8, p = 1, r = 20$$
$$q = 16, \mathrm{k} = 16$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i-k*)th smallest element in A[q+1..r]

p														r					
8	2	14	3	15	13	12	11	7	5	9	10	16	18	1	20	25	23	26	21

$$i = 8, p = 1, r = 15$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i*-*k*)th smallest element in A[q+1..r]

p														r					
8	2	14	3	15	13	12	11	7	5	9	10	16	18	1	20	25	23	26	21

$$i = 8, p = 1, r = 15$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
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p						**q**								**r**					
8	**2**	**3**	**7**	**5**	**1**	**9**	**11**	**14**	**15**	**13**	**10**	**16**	**18**	**12**	**20**	**25**	**23**	**26**	**21**

$$i = 8, p = 1, r = 15$$
$$q = 7, k = 7$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
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p						q								r					
8	2	3	7	5	1	9	11	14	15	13	10	16	18	12	20	25	23	26	21

$$i = 8, p = 1, r = 15$$
$$q = 7, k = 7$$

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							p							r					
8	2	3	7	5	1	9	**11**	**14**	**15**	**13**	**10**	**16**	**18**	**12**	20	25	23	26	21

$$i = 1, p = 8, r = 15$$

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							p							r					
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							p			q				r					
8	2	3	7	5	1	9	**11**	**12**	**10**	**13**	**15**	**14**	**18**	**16**	20	25	23	26	21

$$i = 1, p = 8, r = 15$$
$$q = 11, k = 4$$

Randomized Selection - Example

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							p			q				r					
8	2	3	7	5	1	9	**11**	**12**	**10**	**13**	**15**	**14**	**18**	**16**	20	25	23	26	21

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							p		r										
8	2	3	7	5	1	9	**11**	**12**	**10**	13	15	14	18	16	20	25	23	26	21

$$i = 1, p = 8, r = 10$$

Randomized Selection - Example

- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
  - i = k : pivot is the solution
  - i < k : the *i*th smallest element in A[p..q-1]
  - i > k : the (*i*-*k*)th smallest element in A[q+1..r]

							p		r										
8	2	3	7	5	1	9	**11**	**12**	**10**	13	15	14	18	16	20	25	23	26	21

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- Select the *i*th smallest element in A[p..r], pivot is A[q], k = q-p+1.
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8	2	3	7	5	1	9	10	12	11	13	15	14	18	16	20	25	23	26	21

$$i = 1, p = 8, r = 10$$
$$q = 8, k = 1$$

Randomized Selection - Example

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							p,q		r										
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							p,q		**r**										
8	2	3	7	5	1	9	**10**	**12**	**11**	13	15	14	18	16	20	25	23	26	21

$$i = 1, p = 8, r = 10$$
$$q = 8, k = 1$$

10 is the 8th smallest element of the array.

Outline

- Introduction to Part II
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort
- Randomized Selection Problem
  - Problem Definition
  - First solution: Selection by sorting
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Running Time of Randomized-Select(A,1,n,i)

- We let the running time on an input array A[p..r] of n elements be a random variable denoted by $T(n)$.

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- For k = 1,2,…,n, we define indicator random variables X_k where

$$X_k = I\{the\ subarray\ A[p..q] has\ exactly\ k\ elements\}$$

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$$E[X_k] = 1/n$$

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- When we choose $A[q]$ as the pivot element, we will either
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  - recurse on the subarray $A[p..q-1]$
  - recurse on the subarray $A[q+1..r]$

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- To obtain an upper bound, we assume that the *i*th element is always on the side of the partition with the greater number of elements.

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  - recurse on the subarray $A[q+1..r]$
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- When $X_k = 1$, the two subarrays on which we might recurse have sizes $k-1$ and $n-k$.Hence,

Running Time of Randomized-Select(A,1,n,i)

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- To obtain an upper bound, we assume that the *i*th element is always on the side of the partition with the greater number of elements.
- When $X_k = 1$, the two subarrays on which we might recurse have sizes $k-1$ and $n-k$.Hence,

$$T(n) \leq \sum_{k=1}^{n} X_k \cdot \big(T(\max(k-1, n-k)) + O(n)\big)$$

Running Time of Randomized-Select(A,1,n,i)

Taking expected values, we have

$$E[T(n)] \leq E\left[\sum_{k=1}^{n} X_k \cdot \big(T(\max(k-1, n-k)) + O(n)\big)\right]$$

Running Time of Randomized-Select(A,1,n,i)

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$$= \sum_{k=1}^{n} E[X_k \cdot T(\max(k-1, n-k))] + O(n)$$

Running Time of Randomized-Select(A,1,n,i)

Taking expected values, we have

$$E[T(n)] \leq E\left[\sum_{k=1}^{n} X_k \cdot \big(T(\max(k-1, n-k)) + O(n)\big)\right]$$

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Linearity of expectation

Running Time of Randomized-Select(A,1,n,i)

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$$= \sum_{k=1}^{n} E[X_k] \cdot E[T(\max(k-1, n-k))] + O(n)$$

Independence between X_k and $T(max(k-1, n-k))$

Running Time of Randomized-Select(A,1,n,i)

Taking expected values, we have

$$
\begin{aligned}
E[T(n)] &\leq E\left[\sum_{k=1}^{n} X_k \cdot \big(T(\max(k-1, n-k)) + O(n)\big)\right] \\
&= \sum_{k=1}^{n} E[X_k \cdot T(\max(k-1, n-k))] + O(n) \\
&= \sum_{k=1}^{n} E[X_k] \cdot E[T(\max(k-1, n-k))] + O(n) \\
&= \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max(k-1, n-k))] + O(n)
\end{aligned}
$$

Running Time of Randomized-Select(A,1,n,i)

Taking expected values, we have

$$E[T(n)] \le E\left[\sum_{k=1}^{n} X_k \cdot \big(T(\max(k-1, n-k)) + O(n)\big)\right]$$

$$= \sum_{k=1}^{n} E[X_k \cdot T(\max(k-1, n-k))] + O(n)$$

$$= \sum_{k=1}^{n} E[X_k] \cdot E[T(\max(k-1, n-k))] + O(n)$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max(k-1, n-k))] + O(n)$$

$$\boldsymbol{E[X_k] = 1/n}$$

Running Time of Randomized-Select(A,1,n,i)

- Consider the expression $\max(k-1, n-k)$,

$$\max(k-1, n-k) = \begin{cases} k-1 \ \ if \ k > \lceil n/2 \rceil \\ n-k \ if \ k \leq \lceil n/2 \rceil \end{cases}$$

Running Time of Randomized-Select(A,1,n,i)

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- Guess:

$$T(n) \leq c\, n, \qquad for\ all\ n$$

for some constant *c* to be figured out later.

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Running Time of Randomized-Select(A,1,n,i)

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&= \frac{2c}{n}\left(\frac{n^2-n}{2} - \frac{n^2/4 - 3n/2 + 2}{2}\right) + an \\
&= \frac{c}{n}\left(\frac{3n^2}{4} + \frac{n}{2} - 2\right) + an \\
&= c\left(\frac{3n}{4} + \frac{1}{2} - \frac{2}{n}\right) + an
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&\le \frac{3cn}{4} + \frac{c}{2} + an \\
&= cn - \left(\frac{cn}{4} - \frac{c}{2} - an\right)
\end{aligned}
$$

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- In order to complete the proof, we need to show that for sufficiently large n, this last expression is at most $c \cdot n$ or equivalently,

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- If $\frac{c}{4} - a > 0$, we have

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Induction basis:

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So if we choose $c = \max\{12, T(1), T(2)/2\}$, then the entire proof works.

Running Time of Randomized-Select(A,1,n,i)

Worst Case:

$$T(n) = n - 1 + T(n-1), T(n) = O(n^2)$$

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Expected running time much better than worst case!

Randomized Quicksort vs Randomized Selection

Question

Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

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Randomized Quicksort vs Randomized Selection

Question

Why does Randomized Selection take $O(n)$ time while Randomized Quicksort takes $O(n \log n)$ time?

Answer:

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- Randomize Quicksort needs to work on both of the two subproblems.

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- The running time is the expected running time for any given input, expectation is with respect to the random choices made by the algorithm internally.


dank u
ju faleminderit
Tack
谢谢
Tak
mulţumesc
Asante
kiitos
Gracias
Salamat!
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
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