Design and Analysis of Algorithms
Tutorial 1

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Asymptotic notations

Asymptotic upper bound

Definition (big-Oh)

$f(n) = O(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \leq c \cdot g(n)$ for $n \geq n_0$

Asymptotic lower bound

Definition (big-Omega)

$f(n) = \Omega(g(n))$: There exists constant $c > 0$ and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.

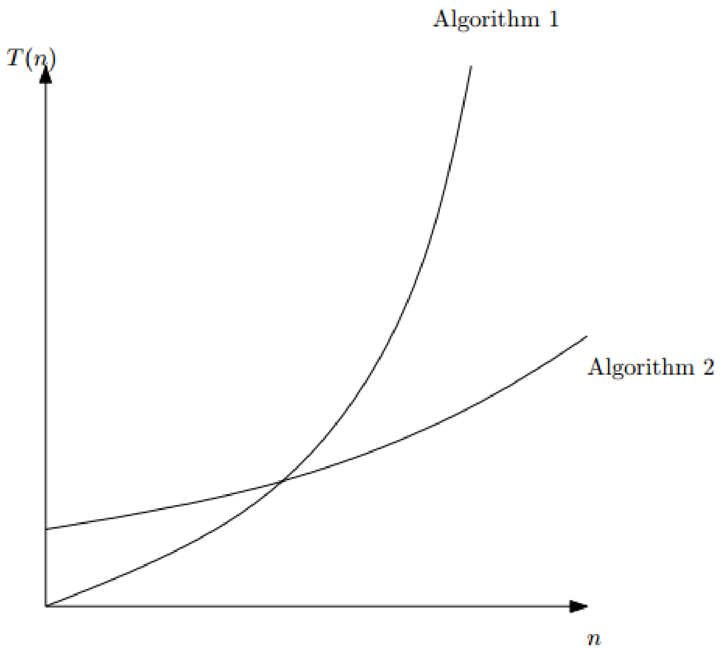
Asymptotic tight bound

Definition (big-Theta)

$f(n) = \Theta(g(n))$: $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Comparing time complexity

Example:



Algorithm 2 is clearly superior

- T(n) for Algorithm 1 is $O(n^3)$
- T(n) for Algorithm 2 is $O(n^2)$
- Since n^3 grows much more rapidly, we expect Algorithm 1 to take much more time than Algorithm 2 when n increases

Some Basic mathematic background on exponentials

For all real a ≠ 0, m and n, we have the following identities:

$$\mathrm{a}^0 = 1$$

$$a^1 = a$$

$$a^{-1} = \frac{1}{a}$$

$$(a^m)^n = (a^n)^m = a^{mn}$$

$$a^m a^n = a^{m+n}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Some Basic mathematic background on logarithms

For all real a > 0, b > 0, c > 0, and n:

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(\frac{1}{a}) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b n} = n^{\log_b a}$$

Question 1

For each of the following statement, answer whether the statement is true or false.

(a) $1000n + nlogn = O(nlogn)$

(b) $n^2 + nlog(n^3) = O\left(nlog(n^3)\right)$

(c) $n^3 = \Omega(n)$

(d) $n^2 + n = \Omega(n^3)$

(e) $n^3 = O(n^{10})$

(f) $n^3 + 1000n^{2.9} = \Theta(n^3)$

(g) $n^3 - n^2 = \Theta(n)$

Solution 1

(a) True.

(b) False.

(c) True.

(d) False.

(e) True.

(f) True.

(g) False.

Question 2

For each pair of expressions (A,B) below, indicate whether A is O, Ω, or Θ of B. Note that zero, one, or more of these relations may hold for a given pair; list all correct ones. Justify your answers.

(a) $A = n^3 + nlogn; B = n^3 + n^2logn$

(b) $A = log\sqrt{n}; B = \sqrt{logn}$

(c) $A = nlog_3n; B = nlog_4n$

(d) $A = 2^n; B = 2^{\frac{n}{2}}$

(e) $A = \log(2^n)\,; B = \log(3^n)$

Solution 2

	A	Relation:	B
(a)	$n^3+nlogn$	Ω, Θ, O	$n^3 + n^2 logn$
(b)	$log\sqrt{n}$	Ω	$\sqrt{logn}$
(c)	$nlog_3 n$	Ω, Θ, O	$nlog_4 n$
(d)	2^n	Ω	$2^{\frac{n}{2}}$
(e)	$\log(2^n)$	Ω, Θ, O	$\log(3^n)$

Solution 2: Step by step

Notes:

(a) Both are $\Theta(n^3)$, the lower order terms can be ignored. Note that if A(n) = Θ(B(n)), then automatically A(n) = O(B(n)) and A(n) = Ω(B(n)).

(b) After simplifying, A is (1/2) log n, and B is $\sqrt{logn}$. Substituting m = log n, we can see ratio A/B grows as $\frac{m}{2\sqrt{m}} = \frac{\sqrt{m}}{2}$ which tends to infinity as n (and hence m) tends to infinity, i.e., A(n) = Ω(B(n)).

(c) Log base conversion only introduces a constant factor.

(d) The ratio is $\frac{2^n}{2^{\frac{n}{2}}} = (2)^{\frac{n}{2}}$which goes to infinity in the limit.

(e) After simplifying these are n log 2 and n log 3, both of which are Θ(n).

Question 3

Suppose $T_1(n) = O(f(n))$ and $T_2(n) = O(f(n))$. Which of the following are true? Justify your answers.

(a) $T_1(n) + T_2(n) = O\big(f(n)\big)$

(b) $\frac{T_1(n)}{T_2(n)} = O(1)$

(c) $T_1(n) = O(T_2(n))$

Solution 3

(a) True. Since $T_1(n) = O(f(n))$ and $T_2(n) = O(f(n))$, it follows from the definition that there exist constants $c_1, c_2 > 0$ and positive integers n_1, n_2 such that $T_1(n) \le c_1 f(n)$ for $n \ge n_1$ and $T_2(n) \le c_2 f(n)$ for $n \ge n_2$. This implies that, $T_1(n) + T_2(n) \le (c1+c2)f(n)$ for $n \ge \max(n_1, n_2)$. Thus, $T1(n) + T2(n) = O(f(n))$.

(b) False. For a counterexample to the claim, let

$T_1(n) = n^2$, $T_2(n) = n$, $f(n) = n^2$. Then $T_1(n) = O(f(n))$ and $T_2(n) = O(f(n))$ but $\frac{\mathrm{T}_1(n)}{T_2(n)} = n \neq O(1)$.

(c) False. We can use the same counterexample as in part (b). Note that $\mathrm{T}_1(n) \neq O(T_2(n))$

Question 4

Let $f(n)$ and $g(n)$ be non-negative functions. Using the basic definition of Θ-notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

Solution 4

For any value of n, max(f(n), g(n)) is either equal to f(n) or equal to g(n). Therefore, for all n,

$$\max\big(f(n), g(n)\big) \leq f(n) + g(n).$$

Using c = 1 and n_0 = 1 in the big-oh definition, it follows that

$$\max\big(f(n), g(n)\big) = O\big(f(n) + g(n)\big)$$

Also, for all n,

$$\max\big(f(n), g(n)\big) \geq f(n)$$

And

$$\max\big(f(n), g(n)\big) \geq g(n)$$

Adding we have

$$2 \times \max\big(f(n), g(n)\big) \geq f(n) + g(n)$$

Therefore,

$$\max\big(f(n), g(n)\big) \geq 1/2(f(n) + g(n))$$

Solution 4

Using c = 1/2 and n_0 = 1 in the Omega definition, it follows that

$$\max\big(f(n), g(n)\big) = \Omega\big(f(n) + g(n)\big)$$

Since max(f(n), g(n)) = O(f(n) + g(n)) and

max(f(n), g(n)) = Ω(f(n) + g(n)), it implies that

max(f(n), g(n)) = Θ(f(n) + g(n)).