プログラムの最適化手法を用いた Erasure Codingの最適化

上里 友弥 @ Dwango

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Accelerating XOR-Based Erasure Coding using Program Optimization Techniques

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“Accelerating Erasure Coding (EC)” means ...

Optimizing matrix multiplication over two finite fields:

$$\text{(for Standard EC)} \quad A \times B \ \text{ over } \mathbb{F}[2^8],$$
$$\text{(for XOR-based EC)} \quad C \times D \ \text{ over } \mathbb{F}[2].$$

- *Byte* Finite Field $\mathbb{F}[2^8]$ is a field with 256 elements.
- *Bit* Finite Field $\mathbb{F}[2] = \{0, 1\}$ is a field with the two bits.

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- *Byte* F
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What we need to know about $\mathbb{F}[2^8]$ and $\mathbb{F}[2]$.

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- ⋆ 1-byte (8-bits) data can be seen as an element of $\mathbb{F}[2^8]$.
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$\mathbb{F}[2] = \{0, 1\}$ is a field of bits.

- Its addition is XOR $\oplus$.
- Its multiplication is AND $\&$.
- 0 and 1 satisfy the following:

$$x \oplus 0 = 0 \oplus x = x, \quad y \,\&\, 1 = 1 \,\&\, y = y.$$

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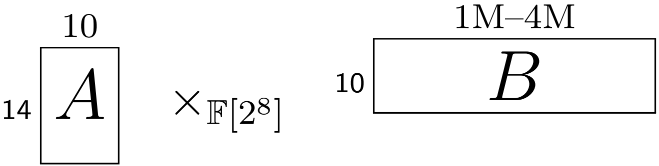

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$$\underbrace{\overbrace{\boxed{A}}^{10}}_{14 \text{ rows}} \times_{\mathbb{F}[2^8]} \overbrace{\boxed{\quad B \quad}}^{\text{1M–4M}} \; (10 \text{ rows})$$

This setting comes from ***erasure coding***.

What are Erasure Coding (EC) and XOR-Based EC

Erasure Coding: Method for Data Redundancy

Example (Building a streaming media server with criteria)

1. We have 14 nodes. Each node has a 20TB disk.
2. We can load data even if nodes ≤ 4 are down.
3. The total capacity of our server = 200TB.
   - $14 \cdot 20 - 200 = 80$TB can be used for data redundancy.

試しに RAID を使ってみる

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RAID-5 データを 13 分割し、parity と呼ばれるものを 1 つ作る。
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RAID-6 データを 12 分割し、parity を 2 つ作る。
- 冗長性はやや良い: 2 台なら壊れても大丈夫
- 空間効率: 12×20TB $= 240$TB の総容量

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For this criteria, we can employ Reed-Solomon EC $\mathbf{RS}(d = 10, p = 4)$.

- d: we can assume d-nodes are living. $(d = 14 - p = 10)$.
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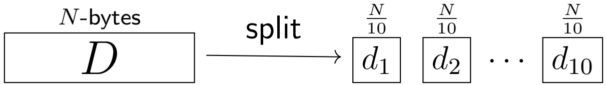
How encoding and decoding are implemented in $\mathbf{RS}(10, 4)$?



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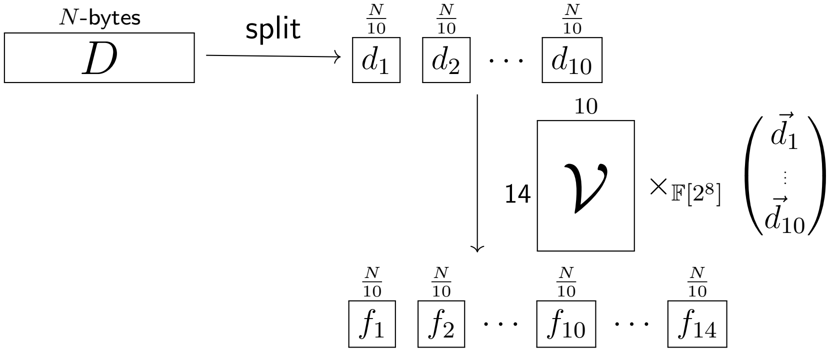

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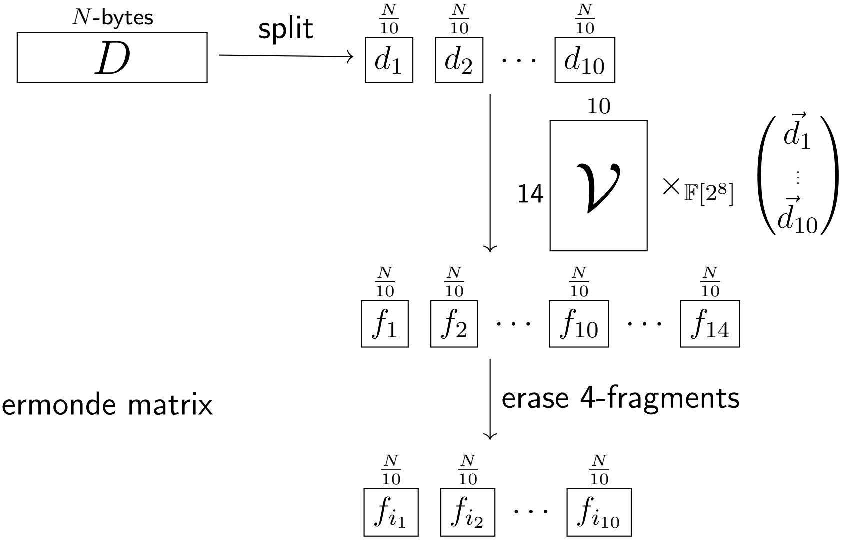

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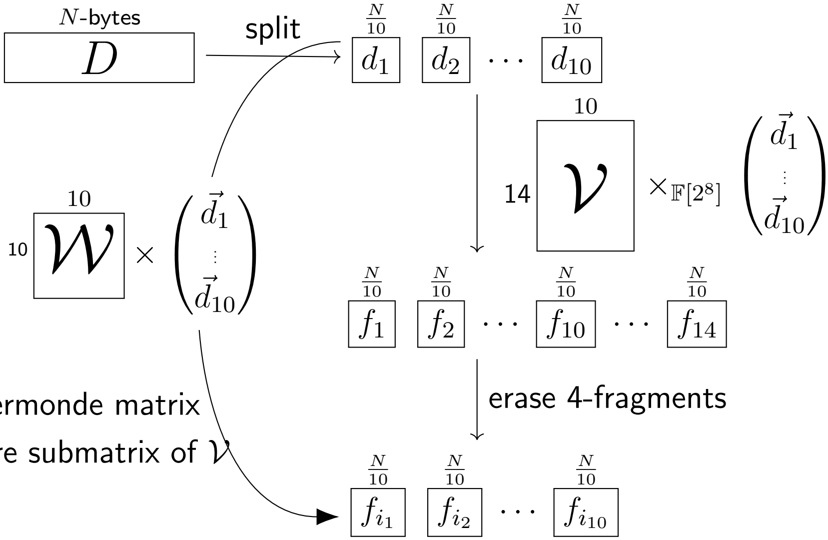

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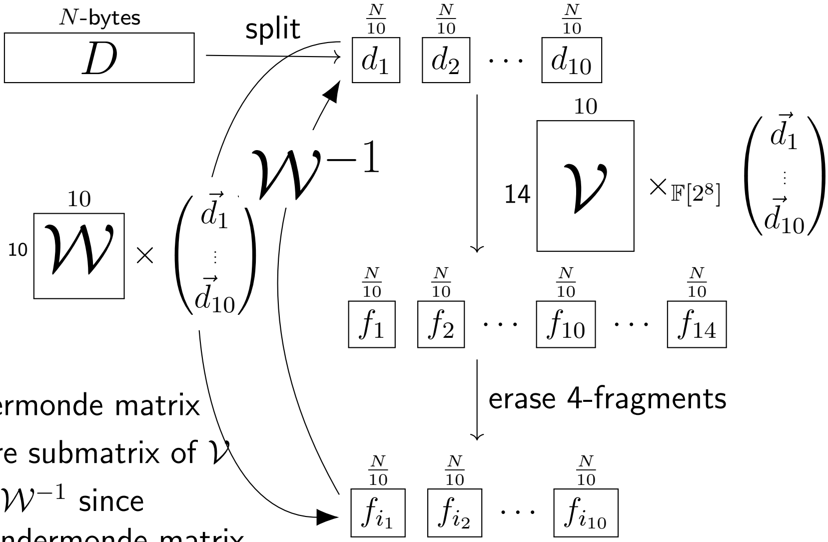

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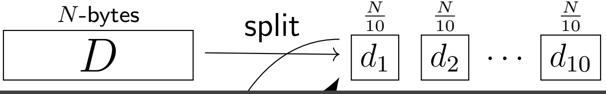
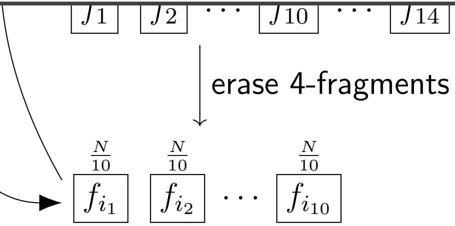


How large is N in a real application?

In my company, D is a short video whose size is 10MB–40MB:

- The size of 10 secs videos of 1080p & 30fps $\sim 12\mathrm{MB}$.
- The size of 5 secs videos of 4K & 30fps $\sim 35\mathrm{MB}$.



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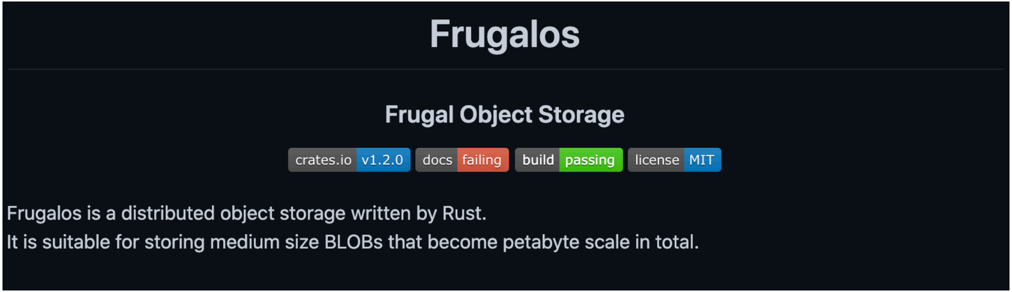
自己紹介

- 筑波大学の SCORE 研出身の博士です。博論はオートマトンのお話。
- いまはドワンゴで働いています。
- もう少し具体的には: Frugalos という 分散オブジェクトストレージを作っていて・改良していて・基礎研究しています。

  https://github.com/frugalos/frugalos



Optimizing $\mathcal{V} \times_{\mathbb{F}[2^8]} D$

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XOR-based EC is one way to vanish $\cdot$ of $\mathbb{F}[2^8]$.

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- もちろんフェルマーの小定理を用いても良いです:

$$\forall a. (a \bmod p \neq 0) \implies a^{p-1} = 1 \pmod{p}$$

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$\mathbb{F}[2^8] = \{0, 1, 2, \ldots, 255\}$ の場合はどうか?

- $x \cdot y \pmod{256}$ で乗算を定義すると、一般に逆元が存在しないです。
- 例: $2 \cdot y = 1 \pmod{256}$ とする y がない（2 の積逆元がない）。

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- 加算は $x + y \pmod 2$ で、これは実質 bit-XOR
- 乗算は $x \cdot y \pmod 2$ で、これは実質 bit-AND

$\mathbb{F}[2^8] = \{0, 1, 2, \ldots, 255\}$ の場合はどうか?

- $x \cdot y \pmod{256}$ で乗算を定義すると、一般に逆元が存在しないです。
- 例: $2 \cdot y = 1 \pmod{256}$ とする y がない（2 の積逆元がない）。

この問題を克服するために

$$k_7 x^7 + k_6 x^6 + \cdots + k_1 x + k_0 \quad (k_i \in \mathbb{F}[2])$$

という多項式全体（濃度は 2^8）を使えるというのが大きいです。

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7 次多項式の積は 7 次を超えてしまうので、
素数っぽい振る舞いをする 8 次多項式（＝既約多項式）で割ります。
原始多項式というフェルマーの小定理っぽいのを満たすものもあります。

XOR-based EC: From $\mathbb{F}[2^8]$ to BitMatrix ($\mathbb{F}[2]$-Matrix)

- 1-byte and 8-bits are isomorphic: $x \in \mathbb{F}[2^8] \;\cong\; \widetilde{x} \in \begin{array}{cc} & 1 \\ 8 & \boxed{\mathbb{F}[2]} \end{array}$.

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$$\forall x, y \in \mathbb{F}[2^8].\; \begin{cases} x + y = \mathcal{B}^{-1}(\mathcal{B}(x) + \mathcal{B}(y)), \\ x \cdot y = \mathcal{B}^{-1}(\mathcal{B}(x) \times \mathcal{B}(y)) \end{cases}$$

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Prop: Emulate $\mathcal{W}^{-1} \times (\mathcal{W} \times D) = D$ in the $\mathbb{F}[2]$ world

$$\mathcal{B}(\mathcal{W}^{-1}) \overset{\mathbb{F}[2]}{\times} (\mathcal{B}(\mathcal{W}) \overset{\mathbb{F}[2]}{\times} \widetilde{D}) = \mathcal{B}(\mathcal{W}^{-1} \times \mathcal{W}) \times \widetilde{D} = \widetilde{D}.$$

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$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \times_{\mathbb{F}[2^8]} \begin{pmatrix} d_1 & \cdots \\ d_2 & \cdots \end{pmatrix} = \begin{pmatrix} x_1 \cdot d_1 + x_2 \cdot d_2 & \cdots \\ x_3 \cdot d_1 + x_4 \cdot d_4 & \cdots \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times_{\mathbb{F}[2]} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \\ \vec{x}_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vec{x}_1 \oplus \vec{x}_2 \oplus \vec{x}_4 \oplus \cdots \\ \vec{x}_3 \oplus \vec{x}_4 \oplus \cdots \\ \vec{x}_2 \oplus \vec{x}_3 \oplus \cdots \\ \vec{x}_1 \oplus \vec{x}_4 \oplus \cdots \\ \vdots \end{pmatrix}$$

$\oplus$ is byte-array XOR.

Comparing MM over $\mathbb{F}[2^8]$ and MM over $\mathbb{F}[2]$ for Encoding

Trade-off in Matrix Multiplication	$\mathbf{RS}(10,4)$ by $\mathbb{F}[2^8]$	$\mathbf{RS}(10,4)$ by $\mathbb{F}[2]$
Number of Core Operation	$\mathcal{V}$: 14 $\overset{10}{\boxed{\mathbb{F}[2^8]}}$	$\mathcal{B}(\mathcal{V})$: 112 $\overset{80}{\boxed{\mathbb{F}[2]}}$
Speed of Core Operation	$+$ of $\mathbb{F}[2^8]$ is fast $\cdot$ of $\mathbb{F}[2^8]$ is slow	bytevec-XOR $\oplus$ is fast (SIMDable)

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Encoding Throughput Comparison (on Intel CPU):

GB/s	ISA-L♣ $\mathbb{F}[2^8]$	State-of-the-art♠ $\mathbb{F}[2]$	
RS(10, 4)	6.79	4.94	
RS(10, 3)	6.78	6.15	
RS(9, 3)	7.31	6.17	

♣ ISA-L: Intel's EC library `https://github.com/intel/isa-l`

♠ T. Zhou & C. Tian. 2020. *Fast Erasure Coding for Data Storage: A Comprehensive Study of the Acceleration Techniques.*

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GB/s	ISA-L♣ $\mathbb{F}[2^8]$	State-of-the-art♠ $\mathbb{F}[2]$	Ours(New!) $\mathbb{F}[2]$
RS(10, 4)	6.79	4.94	8.92
RS(10, 3)	6.78	6.15	11.78
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Our Contribution:

Optimizing Bitmatrix Multiplication

as

Program Optimization Problem

MM over $\mathbb{F}[2]$ = Running Straight Line Program

We identify bitmatrix multiplication as *straight line program* (SLP):

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \times_{\mathbb{F}[2]} \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \\ \vec{d} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{array}{l} P(a, b, c, d) \\ \hline v_1 \leftarrow a \oplus b; \\ v_2 \leftarrow a \oplus b \oplus c; \\ v_3 \leftarrow b \oplus c \oplus d; \\ \textsf{return}(v_1, v_2, v_3) \end{array}$$

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$$\begin{aligned} [\![P]\!] &= \mathsf{return}(v_1, v_2, v_3) \\ &= \langle a \oplus b, \\ &\quad\ a \oplus b \oplus c, \\ &\quad\ b \oplus c \oplus d \rangle \end{aligned}$$

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★ "Bitmatrix as SLP" is not a new idea (See. Boyar+ 2008)

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★ "Bitmatrix as SLP" is not a new idea (See. Boyar+ 2008)

- SLP only allow assignments with one kind *binary* operator $\oplus$.
- SLP do not have functions, `if`-branchings, and `while`-loop, etc.

XOR Optimization: Reducing XORs

Optimization Metric $\#_{\oplus}(\_)$: the number of XORs.

$P \quad \#_{\oplus} = 8$		Q
$v_1 \leftarrow a \oplus b;$ $v_2 \leftarrow a \oplus b \oplus c;$ $v_3 \leftarrow a \oplus b \oplus c \oplus d;$ $v_4 \leftarrow b \oplus c \oplus d;$ return(v_1, v_2, v_3, v_4)	$\Rightarrow$	

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$v_2 \leftarrow a \oplus b \oplus c;$		$v_2 \leftarrow v_1 \oplus c;$
$v_3 \leftarrow a \oplus b \oplus c \oplus d;$	$\Longrightarrow$	$v_3 \leftarrow v_2 \oplus d;$
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XOR Optimization: Reducing XORs

Optimization Metric $\#_{\oplus}(\_)$: the number of XORs.

$P \quad \#_{\oplus} = 8$		$Q \quad \#_{\oplus} = 4$
$v_1 \leftarrow a \oplus b;$		$v_1 \leftarrow a \oplus b;$
$v_2 \leftarrow a \oplus b \oplus c;$		$v_2 \leftarrow v_1 \oplus c;$
$v_3 \leftarrow a \oplus b \oplus c \oplus d;$	$\Longrightarrow$	$v_3 \leftarrow v_2 \oplus d;$
$v_4 \leftarrow b \oplus c \oplus d;$		$v_4 \leftarrow v_3 \oplus a;$
		$\because (a \oplus b \oplus c \oplus d) \oplus a = b \oplus c \oplus d.$
return(v_1, v_2, v_3, v_4)		return(v_1, v_2, v_3, v_4)

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$v_2 \leftarrow a \oplus b \oplus c;$		$v_2 \leftarrow v_1 \oplus c;$
$v_3 \leftarrow a \oplus b \oplus c \oplus d;$	$\Longrightarrow$	$v_3 \leftarrow v_2 \oplus d;$
$v_4 \leftarrow b \oplus c \oplus d;$		$v_4 \leftarrow v_3 \oplus a;$
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return(v_1, v_2, v_3, v_4)		return(v_1, v_2, v_3, v_4)

- P and Q are equivalent: $[\![P]\!] = [\![Q]\!]$.
- Intuitively, Q ($\#_{\oplus}(Q) = 4$) runs faster than P ($\#_{\oplus}(P) = 8$).

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$v_2 \leftarrow a \oplus b \oplus c;$		$v_2 \leftarrow v_1 \oplus c;$
$v_3 \leftarrow a \oplus b \oplus c \oplus d;$	$\Rightarrow$	$v_3 \leftarrow v_2 \oplus d;$
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  Question. For a given SLP P,
  can we quickly find the most efficient equivalent SLP Q?

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$P \quad \#_{\oplus} = 8$		$Q \quad \#_{\oplus} = 4$
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$v_2 \leftarrow a \oplus b \oplus c;$		$v_2 \leftarrow v_1 \oplus c;$
$v_3 \leftarrow a \oplus b \oplus c \oplus d;$	$\Longrightarrow$	$v_3 \leftarrow v_2 \oplus d;$
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Theorem (Boyar+ 2013)

Unless **P = NP**, *for a given SLP* P, *in polynomial time,*
we cannot find Q *such that* $[\![P]\!] = [\![Q]\!]$ *and minimizes* $\#_{\oplus}(Q)$.

XOR 最適化の NP 完全性についてもうちょっと

Vertex Cover Problem という古典的な NP 完全問題を使います。

$$\mathcal{G}: \begin{tikzcd} A \arrow[r, dash, "e_1"] & B \arrow[d, dash, "e_3"] \arrow[dl, dash, "e_4"'] \\ C \arrow[r, dash, "e_2"] & D \end{tikzcd}$$

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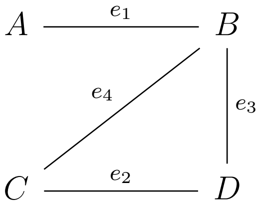
$\mathcal{G}$:



このグラフ $\mathcal{G}$ については、

- 頂点セット $\{B, C\}$ で、全ての辺をカバーできます
- 他の頂点セット $\{B, D\}$ でも、カバーできます

XOR 最適化の NP 完全性についてもうちょっと

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グラフ $\mathcal{G}$ から、次の SLP $P_{\mathcal{G}}$ を作ります:

$$P_{\mathcal{G}}: \quad \begin{aligned} e_1 &\leftarrow p \oplus A \oplus B; \\ e_2 &\leftarrow p \oplus C \oplus D; \\ e_3 &\leftarrow p \oplus B \oplus D; \\ e_4 &\leftarrow p \oplus B \oplus C; \end{aligned}$$

これを XOR 最適化すると、最小頂点セットが実は現れます。

XOR 最適化の NP 完全性についてもうちょっと

$$
\mathcal{G}:\quad
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\end{array}
\quad \Rrightarrow \quad
P^{\mathsf{opt}} : \begin{array}{l}
p_B \leftarrow p \oplus B; \\
e_1 \leftarrow p_B \oplus A; \\
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e_4 \leftarrow p_B \oplus C; \\
p_C \leftarrow p \oplus C; \\
e_2 \leftarrow p_C \oplus D;
\end{array}
$$

これを XOR 最適化すると、最小頂点セットが実は現れます。　実際に示しているのは、いつでも右のような形にできる、という正規化補題です。

Our Heuristic: Grammar Compression Algorithm RePair

Originally, RePair is an algorithm to compress context-free grammars.
We use it identifying SLPs as commutative CFGs.

- Larsson & Moffat. 1999. *Offline dictionary-based compression*
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RePair = **Re**peat Pair. The key operation is Pair:

$$\begin{array}{l} v_1 \leftarrow a \oplus b; \\ v_2 \leftarrow a \oplus b \oplus c; \\ v_3 \leftarrow a \oplus b \oplus c \oplus d; \\ v_4 \leftarrow b \oplus c \oplus d; \\ \qquad\qquad \#_\oplus = 8 \end{array} \xRightarrow{\textsc{Pair}(a,\, c)} \begin{array}{l} t_1 \leftarrow a \oplus c; \\ \hline v_1 \leftarrow a \oplus b; \\ v_2 \leftarrow t_1 \oplus b; \\ v_3 \leftarrow t_1 \oplus b \oplus d; \\ v_4 \leftarrow b \oplus c \oplus d; \end{array} \quad \#_\oplus = 7$$

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\hfill \#_{\oplus} = 8
\end{array}
\xRightarrow{\textsc{Pair}(a,\, c)}
\begin{array}{l}
t_1 \leftarrow a \oplus c; \\
\hline
v_1 \leftarrow a \oplus b; \\
v_2 \leftarrow t_1 \oplus b; \\
v_3 \leftarrow t_1 \oplus b \oplus d; \\
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\end{array}
\quad \#_{\oplus} = 7
$$

How do we choose a pair of terms to do pairing?

Our Heuristic: Grammar Compression Algorithm RePair

Originally, RePair is an algorithm to compress context-free grammars.
We use it identifying SLPs as commutative CFGs.

- Larsson & Moffat. 1999. *Offline dictionary-based compression*
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How do we choose a pair of terms to do pairing? *Greedy*.

$$\begin{array}{l} v_1 \leftarrow a \oplus b; \\ v_2 \leftarrow a \oplus b \oplus c; \\ v_3 \leftarrow a \oplus b \oplus c \oplus d; \\ v_4 \leftarrow b \oplus c \oplus d; \end{array} \quad \xRightarrow{\textsc{Pair}(a,\, b)} \quad \begin{array}{l} t_1 \leftarrow a \oplus b; \\ \hline v_2 \leftarrow t_1 \oplus c; \\ v_3 \leftarrow t_1 \oplus c \oplus d; \\ v_4 \leftarrow b \oplus c \oplus d; \end{array} \quad \#_\oplus = 6$$

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v_4 \leftarrow b \oplus c \oplus d;
\end{array}
\xRightarrow{\textsc{Pair}(t_1, c)}
\begin{array}{l}
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t_2 \leftarrow t_1 \oplus c; \\
\hline
v_3 \leftarrow t_2 \oplus d; \\
v_4 \leftarrow b \oplus c \oplus d;
\end{array}
\xRightarrow{\textsc{Pair}(b, c)}
\begin{array}{l}
t_1 \leftarrow a \oplus b; \\
t_2 \leftarrow t_1 \oplus c; \\
t_3 \leftarrow b \oplus c; \\
\hline
v_3 \leftarrow t_2 \oplus d; \\
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$$

The commutative version of RePair accommodates

$$\text{Commutativity} : x \oplus y = y \oplus x, \ \text{Associativity} : (x \oplus y) \oplus z = x \oplus (y \oplus z).$$

In the paper, we extend it to XorRePair by accommodating

$$\text{Cancellativity: } x \oplus x \oplus y = y.$$

文法圧縮との出会い ▷ Tozawa & Minamide, FOSSACS'07.

https://link.springer.com/chapter/10.1007%2F978-3-540-71389-0_25

Complexity Results on Balanced Context-Free Languages

Akihiko Tozawa[1] and Yasuhiko Minamide[2]

[1] IBM Research,
Tokyo Research Laboratory, IBM Japan, ltd.
[2] Department of Computer Science
University of Tsukuba

Abstract. Some decision problems related to balanced context-free languages are important for their application to the static analysis of programs generating XML strings. One such problem is the balancedness problem which decides whether or not the language of a given context-free grammar (CFG) over a paired alphabet is balanced. Another important problem is the validation problem which decides whether or not the language of a CFG is contained by that of a regular hedge grammar (RHG). This paper gives two new results; (1) the balancedness problem is in PTIME; and (2) the CFG-RHG containment problem is 2EXPTIME-complete.

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論文の前半では:

- CFG $G \subseteq_? \mathbf{Dyck}$ を解く
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Uezato & Minamide DLT'16 で
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文法圧縮そのものについては次がオススメ:
The smallest grammar problem, Charikar+, 2005
`https://ieeexplore.ieee.org/document/1459058`

Memory Access Optimization: **MultiSLP**

Optimization Metric: $\#_{\mathsf{mem}}(\_)$ = the number of memory access.

Quiz: How many times will this program access memory?

$$\#_{\mathsf{mem}}\left(\quad v \leftarrow A \oplus B \oplus C \oplus D \quad \right) = \text{?}$$

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$$\#_{\mathsf{mem}}\left(\quad v \leftarrow A \oplus B \oplus C \oplus D \quad \right) = 9$$

because each $\oplus$ issues two read and one write:

$$t_1 \leftarrow A \oplus B; \quad t_2 \leftarrow t_1 \oplus C; \quad v \leftarrow t_2 \oplus D;$$

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$$t_1 \leftarrow A \oplus B; \quad t_2 \leftarrow t_1 \oplus C; \quad v \leftarrow t_2 \oplus D;$$

t_1 and t_2 are wasteful: they are released immediately after allocated.

To reduce such wastefulness,
we extend SLP to *Multi*SLP, which allows n-arity XORs.

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On MultiSLP, we can

$$v \leftarrow \oplus_4(A, B, C, D);$$

Thus, we have $\#_{\text{mem}} = 5$.

```
⊕4(A, B, C, D: [byte]) {
  var v = Array::new(A.len);
  for i in [0..A.len):
    byte r = A[i] ^ B[i]
    r = r ^ C[i];
    v[i] = r ^ D[i];
  return v;
}
```

New Metric and Memory Optimization Problem

From a given P, can we quickly ($=$ in polynomial time)
find an equivalent and most memory efficient Q w.r.t. $\#_{\mathsf{mem}}$?

$$P: \begin{array}{l} v_1 \leftarrow a \oplus b \oplus c \oplus d \oplus e; \\ v_2 \leftarrow a \oplus b \oplus c \oplus d \oplus f; \end{array}$$

$$\#_{\mathsf{mem}}(P) = 24$$

New Metric and Memory Optimization Problem

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Unfortunately, we showed the following intractability result:

Theorem (Our NEW theoretical result)

Unless **P = NP**, *for a given SLP* P, *in polynomial time,*
we cannot find Q *that* $[\![P]\!] = [\![Q]\!]$ *and minimizes* $\#_{\mathsf{mem}}(Q)$.

メモ: Deforestation

$$
\begin{array}{ccccc}
\begin{array}{r}
\underline{P \qquad\qquad\qquad\qquad\qquad\qquad} \\
\\
v_1 \leftarrow a \oplus b \oplus c \oplus d \oplus e; \\
v_2 \leftarrow a \oplus b \oplus c \oplus d \oplus f; \\
\#_{\mathsf{mem}}(P) = 24
\end{array}
& \Longrightarrow &
\begin{array}{r}
\underline{P' \qquad\qquad\qquad\qquad\quad} \\
t \leftarrow a \oplus b \oplus c \oplus d; \\
v_1 \leftarrow t \oplus e; \\
v_2 \leftarrow t \oplus f; \\
\#_{\mathsf{mem}}(P) = 15
\end{array}
& \Longrightarrow &
\begin{array}{r}
\underline{Q \qquad\qquad\qquad\qquad\quad} \\
t \leftarrow \oplus_4(a, b, c, d); \\
v_1 \leftarrow t \oplus e; \\
v_2 \leftarrow t \oplus f; \\
\#_{\mathsf{mem}}(Q) = 11
\end{array}
\end{array}
$$

$P' \Longrightarrow Q$ でやっている合成による最適化（中間データの削除）は関数プログラミングでは「Deforestation」と呼ばれる。

メモ: Deforestation

P		P'		Q
		$t \leftarrow a \oplus b \oplus c \oplus d;$		$t \leftarrow \oplus_4(a, b, c, d);$
$v_1 \leftarrow a \oplus b \oplus c \oplus d \oplus e;$	$\Longrightarrow$	$v_1 \leftarrow t \oplus e;$	$\Longrightarrow$	$v_1 \leftarrow t \oplus e;$
$v_2 \leftarrow a \oplus b \oplus c \oplus d \oplus f;$		$v_2 \leftarrow t \oplus f;$		$v_2 \leftarrow t \oplus f;$
$\#_{\mathsf{mem}}(P) = 24$		$\#_{\mathsf{mem}}(P) = 15$		$\#_{\mathsf{mem}}(Q) = 11$

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DEFORESTATION: TRANSFORMING PROGRAMS TO ELIMINATE TREES *

Philip WADLER

Department of Computer Science, University of Glasgow, Glasgow G12 8QQ, UK

Abstract. An algorithm that transforms programs to eliminate intermediate trees is presented. The algorithm applies to any term containing only functions with definitions in a given syntactic form, and is suitable for incorporation in an optimizing compiler.

メモ: Deforestation

P		P'		Q
		$t \leftarrow a \oplus b \oplus c \oplus d;$		$t \leftarrow \oplus_4(a, b, c, d);$
$v_1 \leftarrow a \oplus b \oplus c \oplus d \oplus e;$	$\Longrightarrow$	$v_1 \leftarrow t \oplus e;$	$\Longrightarrow$	$v_1 \leftarrow t \oplus e;$
$v_2 \leftarrow a \oplus b \oplus c \oplus d \oplus f;$		$v_2 \leftarrow t \oplus f;$		$v_2 \leftarrow t \oplus f;$
$\#_{\mathsf{mem}}(P) = 24$		$\#_{\mathsf{mem}}(P) = 15$		$\#_{\mathsf{mem}}(Q) = 11$

$P' \Longrightarrow Q$ でやっている合成による最適化（中間データの削除）は関数プログラミングでは「Deforestation」と呼ばれる。

```
sum (map (\x.  x * x) (upto 1 n)) ⟹
```

$h\ 0\ 1\ n$

where

$h\ a\ m\ n = \textbf{if}\ m > n$
$\quad \textbf{then}\ a$
$\quad \textbf{else}\ h(a + square\ m)(m+1)n.$

Our Heuristic: XOR Fusion

We fuse XORs when the following holds:

$$\begin{array}{l} \alpha \leftarrow \bigoplus(x_1, \ldots, x_n); \\ \quad \vdots \\ \beta \leftarrow \bigoplus(y_1, \ldots, \alpha, \ldots, y_m) \end{array} \xRightarrow{\text{fuse}} \beta \leftarrow \bigoplus(y_1, \ldots, x_1, \ldots, x_n, \ldots, y_m)$$

☆ α appears once in the program

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Example.

$$\begin{array}{r} v_1 \leftarrow a \oplus b \oplus c \oplus d \oplus e; \\ v_2 \leftarrow a \oplus b \oplus c \oplus d \oplus f; \\ \#_{\text{mem}}(24) \end{array} \xRightarrow{\text{RePair}} \begin{array}{r} t_1 \leftarrow a \oplus b; \\ t_2 \leftarrow t_1 \oplus c; \\ t_3 \leftarrow t_2 \oplus d; \\ v_1 \leftarrow t_3 \oplus e; \\ v_2 \leftarrow t_3 \oplus f; \\ \#_{\text{mem}}(15) \end{array} \xRightarrow{\text{fuse}(t_1)} \begin{array}{r} t_2 \leftarrow \oplus_3(a, b, c); \\ t_3 \leftarrow t_2 \oplus d; \\ v_1 \leftarrow t_3 \oplus e; \\ v_2 \leftarrow t_3 \oplus f; \\ \#_{\text{mem}}(13) \end{array}$$

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$$\xRightarrow{\text{fuse}(t_2)} \begin{array}{l} t_3 \leftarrow \oplus_4(a, b, c, d); \\ v_1 \leftarrow t_3 \oplus e; \\ v_2 \leftarrow t_3 \oplus f; \\ \hfill \#_{\text{mem}}(11) \end{array}$$

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Cache Optimization: SLP + LRU Cache

Metric $\#_{\mathsf{I/O}}(K, \_)$: the total number of I/O transfers between memory andf cache of K-capacity.

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We have three kinds of operations for cache:

- $\mathcal{H}(x)$: Cache Hit for an element x. $\#_{\mathsf{I/O}} = 0$.
- $\mathcal{R}(x)$: Cache miss. Evict LRU to mem. and read x from mem. $\#_{\mathsf{I/O}} = 2$.
- $\mathcal{W}(x)$: Cache miss. Evict LRU to mem. and write x to cache. $\#_{\mathsf{I/O}} = 1$.

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Example: Calculate $\#_{\mathsf{I/O}}(4, P)$ for the following example SLP P:

$v_1 \leftarrow A \oplus B;$ $\quad *_1 *_2 *_3 *_4$

$v_2 \leftarrow \oplus(E, D, A);$

$v_3 \leftarrow v_1 \oplus E;$

$v_4 \leftarrow v_1 \oplus C;$

return(v_2, v_3, v_4);

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$v_1 \leftarrow {\color{red}A} \oplus B;$ $\quad *_1 *_2 *_3 *_4 \xrightarrow[2]{\mathcal{R}({\color{red}A})} *_2 *_3 *_4 A$

$v_2 \leftarrow \oplus(E, D, A);$

$v_3 \leftarrow v_1 \oplus E;$

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$\mathsf{return}(v_2, v_3, v_4);$

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- $\mathcal{R}(x)$: Cache miss. Evict LRU to mem. and read x from mem. $\#_{\mathsf{I/O}} = 2$.
- $\mathcal{W}(x)$: Cache miss. Evict LRU to mem. and write x to cache. $\#_{\mathsf{I/O}} = 1$.

Example: Calculate $\#_{\mathsf{I/O}}(4, P)$ for the following example SLP P:

$v_1 \leftarrow A \oplus B;$ $\quad *_1 *_2 *_3 *_4 \xrightarrow[2]{\mathcal{R}(A)} *_2 *_3 *_4 A \xrightarrow[2]{\mathcal{R}(B)} *_3 *_4 AB$

$v_2 \leftarrow \oplus(E, D, A);$

$v_3 \leftarrow v_1 \oplus E;$

$v_4 \leftarrow v_1 \oplus C;$

$\text{return}(v_2, v_3, v_4);$

Cache Optimization: SLP + LRU Cache

Metric $\#_{\mathsf{I/O}}(K, \_)$: the total number of I/O transfers between memory andf cache of K-capacity.

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$v_2 \leftarrow \oplus(E, D, A);$ $\quad *_4\, ABv_1$

$v_3 \leftarrow v_1 \oplus E;$

$v_4 \leftarrow v_1 \oplus C;$

return$(v_2, v_3, v_4);$

Cache Optimization: SLP + LRU Cache

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\begin{array}{ll}
v_1 \leftarrow A \oplus B; & *_1 *_2 *_3 *_4 \xrightarrow[2]{\mathcal{R}(A)} *_2 *_3 *_4 A \xrightarrow[2]{\mathcal{R}(B)} *_3 *_4 AB \xrightarrow[1]{\mathcal{W}(v_1)} \\
v_2 \leftarrow \oplus(E, D, A); & *_4 ABv_1 \xrightarrow[2]{\mathcal{R}(E)} ABv_1E \xrightarrow[2]{\mathcal{R}(D)} Bv_1ED \xrightarrow[2]{\mathcal{R}(A)} v_1EDA \xrightarrow[1]{\mathcal{W}(v_2)} \\
v_3 \leftarrow v_1 \oplus E; & EDAv_2 \xrightarrow[2]{\mathcal{R}(v_1)} DAv_2v_1 \xrightarrow[2]{\mathcal{R}(E)} Av_2v_1E \xrightarrow[1]{\mathcal{W}(v_3)} \\
v_4 \leftarrow v_1 \oplus C; & v_2v_1Ev_3 \xrightarrow[0]{\mathcal{H}(v_1)} v_2Ev_3v_1 \xrightarrow[2]{\mathcal{R}(C)} Ev_3v_1C \xrightarrow[1]{\mathcal{W}(v_4)} \\
\text{return}(v_2, v_3, v_4); & v_3v_1Cv_4 \Longrightarrow \#_{\mathsf{I/O}}(4, P) = 20.
\end{array}
$$

First approach: Register Assignment

Idea: Reducing the number of variables can relax the pressure of cache, and thus may reduce $\#_{\text{I/O}}$.

We do Recycling variables by *Register assignment*.

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$$\begin{array}{ll} & \#_{\text{I/O}} \\ v_1 \leftarrow A \oplus B; & [5] \\ v_2 \leftarrow \oplus(E, D, A); & [7] \\ v_3 \leftarrow v_1 \oplus E; & [5] \\ v_4 \leftarrow v_1 \oplus C; & [3] \\ \text{return}(v_2, v_3, v_4); & \end{array} \xRightarrow{\text{Register assignment}} \begin{array}{ll} & \#_{\text{I/O}} \\ v_1 \leftarrow A \oplus B; & [5] \\ v_2 \leftarrow \oplus(E, D, A); & [7] \\ v_3 \leftarrow v_1 \oplus E; & [5] \\ v_1 \leftarrow v_1 \oplus C; & [2] \\ \text{return}(v_2, v_3, v_1); & \end{array}$$

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$$
\begin{array}{lc}
 & \#_{\mathsf{I/O}} \\
v_1 \leftarrow A \oplus B; & [5] \\
v_2 \leftarrow \oplus(E, D, A); & [7] \\
v_3 \leftarrow v_1 \oplus E; & [5] \\
{\color{red} v_4} \leftarrow v_1 \oplus C; & [{\color{red}3}] \\
\text{return}(v_2, v_3, {\color{red} v_4}); &
\end{array}
\quad \xRightarrow{\text{Register assignment}} \quad
\begin{array}{lc}
 & \#_{\mathsf{I/O}} \\
v_1 \leftarrow A \oplus B; & [5] \\
v_2 \leftarrow \oplus(E, D, A); & [7] \\
v_3 \leftarrow v_1 \oplus E; & [5] \\
{\color{green} v_1} \leftarrow v_1 \oplus C; & [{\color{green}2}] \\
\text{return}(v_2, v_3, {\color{green} v_1}); &
\end{array}
$$

$$
\xrightarrow[0]{\mathcal{H}(v_1)} v_2 E v_3 v_1 \xrightarrow[2]{\mathcal{R}(C)} E v_3 v_1 C \xrightarrow[{\color{red}1}]{\mathcal{W}(v_4)} v_3 v_1 C v_4
$$

$$
\Downarrow
$$

$$
\xrightarrow[0]{\mathcal{H}(v_1)} v_2 E v_3 v_1 \xrightarrow[2]{\mathcal{R}(C)} E v_3 v_1 C \xrightarrow[{\color{green}0}]{\mathcal{H}(v_1)} E v_3 C v_1
$$

It works, but the effect is so limited.

Next Approach: Reordering Statements and Arguments

No side effects on SLPs; thus, we can reorder statements and arguments.

	#I/O			#I/O
$v_1 \leftarrow A \oplus B;$	[5]		$v_2 \leftarrow \oplus(A, D, E);$	[5]
$v_2 \leftarrow \oplus(E, D, A);$	[7]	$\xRightarrow{\text{Reordering}}$	$v_1 \leftarrow A \oplus B;$	[3]
$v_3 \leftarrow v_1 \oplus E;$	[5]		$v_3 \leftarrow v_1 \oplus E;$	[3]
$v_4 \leftarrow v_1 \oplus C;$	[3]		$v_4 \leftarrow v_1 \oplus C;$	[3]
return$(v_2, v_3, v_4);$	20		return$(v_2, v_3, v_4);$	14

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$$
\begin{array}{ll}
 & \#\text{I/O} \\
v_1 \leftarrow A \oplus B; & [5] \\
v_2 \leftarrow \oplus(E, D, A); & [7] \\
v_3 \leftarrow v_1 \oplus E; & [5] \\
v_4 \leftarrow v_1 \oplus C; & [3] \\
\text{return}(v_2, v_3, v_4); & 20
\end{array}
\quad \xRightarrow{\text{Reordering}} \quad
\begin{array}{ll}
 & \#\text{I/O} \\
v_2 \leftarrow \oplus(A, D, E); & [5] \\
v_1 \leftarrow A \oplus B; & [3] \\
v_3 \leftarrow v_1 \oplus E; & [3] \\
v_4 \leftarrow v_1 \oplus C; & [3] \\
\text{return}(v_2, v_3, v_4); & 14
\end{array}
$$

Using *Pebble Game*, we can integrate $\left\{ \begin{array}{l} \text{Recycling Variables and} \\ \text{Reordering} \end{array} \right.$

⋆ R. Sethi, 1975, *Complete register allocation problems*.

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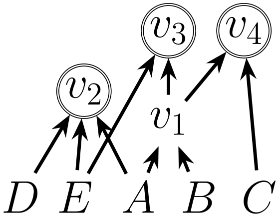
	#I/O			#I/O
$v_1 \leftarrow A \oplus B;$	[5]		$v_2 \leftarrow \oplus(A, D, E);$	[5]
$v_2 \leftarrow \oplus(E, D, A);$	[7]	$\xRightarrow{\text{Reordering}}$	$v_1 \leftarrow A \oplus B;$	[3]
$v_3 \leftarrow v_1 \oplus E;$	[5]		$v_3 \leftarrow v_1 \oplus E;$	[3]
$v_4 \leftarrow v_1 \oplus C;$	[3]		$v_4 \leftarrow v_1 \oplus C;$	[3]
return$(v_2, v_3, v_4);$	20		return$(v_2, v_3, v_4);$	14

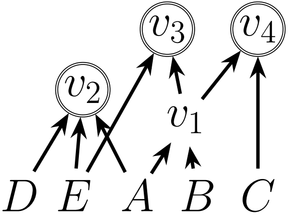
Using *Pebble Game*, we can integrate $\left\{ \begin{array}{l} \text{Recycling Variables and} \\ \text{Reordering} \end{array} \right.$

- ★ R. Sethi, 1975, *Complete register allocation problems*.
- We play the pebble game on DAGs or abstract syntax graphs.
- We aim to put pebbles in return nodes.



Pebble Game & Intractability of Optimization Problem

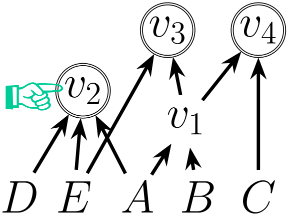
Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



Example: Evaluating strategy based on Depth-first-search

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



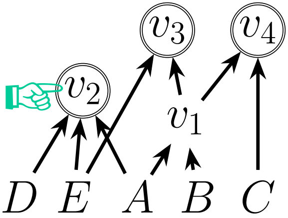
v_2 :

Example: Evaluating strategy based on Depth-first-search

1. Choose v_2 from unvisited roots: alphabetical small $v_2 \prec v_3 \prec v_4$.

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game $=$ Deciding Evaluation Order $+$ Variable Recycling



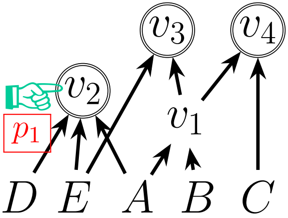
v_2 : $\leftarrow \oplus(A, D, E);$

Example: Evaluating strategy based on Depth-first-search

1. Choose v_2 from unvisited roots: alphabetical small $v_2 \prec v_3 \prec v_4$.
2. Evaluate the children of v_2 in alphabetical order.

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



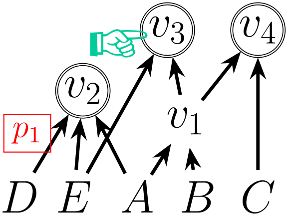
$$v_2 : \quad p_1 \leftarrow \oplus(A, D, E);$$

Example: Evaluating strategy based on Depth-first-search

1. Choose v_2 from unvisited roots: alphabetical small $v_2 \prec v_3 \prec v_4$.
2. Evaluate the children of v_2 in alphabetical order.
3. Put a pebble p_1 on v_2 to denote v_2 is visited.

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



v_2 : $p_1 \leftarrow \oplus(A, D, E);$

v_3 : $\leftarrow E \oplus$

Example: Evaluating strategy based on Depth-first-search

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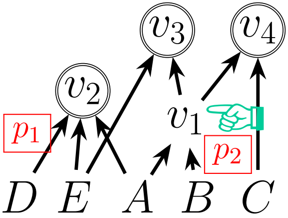
2. Evaluate the children of v_2 in alphabetical order.
3. Put a pebble p_1 on v_2 to denote v_2 is visited.
4. Choose v_3 from 2 unvisited roots, and first visit E.

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



$$
\begin{array}{ll}
v_2 : & p_1 \leftarrow \oplus(A, D, E); \\
v_1 : & p_2 \leftarrow A \oplus B; \\
v_3 : & \quad\ \leftarrow E \oplus
\end{array}
$$

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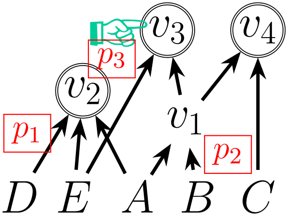
2. Evaluate the children of v_2 in alphabetical order.
3. Put a pebble p_1 on v_2 to denote v_2 is visited.
4. Choose v_3 from 2 unvisited roots, and first visit E.
5. Visit the unvisited child v_1 of v_3, evaluate, and pebble p_2

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



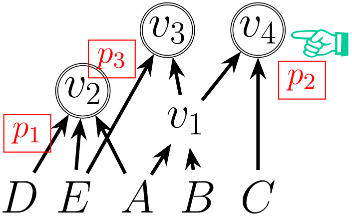
$$
\begin{aligned}
v_2 &: \quad p_1 \leftarrow \oplus(A, D, E);\\
v_1 &: \quad p_2 \leftarrow A \oplus B;\\
v_3 &: \quad p_3 \leftarrow E \oplus p_2;
\end{aligned}
$$

Example: Evaluating strategy based on Depth-first-search

1. Choose v_2 from unvisited roots: alphabetical small $v_2 \prec v_3 \prec v_4$.
2. Evaluate the children of v_2 in alphabetical order.
3. Put a pebble p_1 on v_2 to denote v_2 is visited.
4. Choose v_3 from 2 unvisited roots, and first visit E.
5. Visit the unvisited child v_1 of v_3, evaluate, and pebble p_2
6. Back to v_3 and pebble p_3

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



$$
\begin{aligned}
v_2 &: \quad p_1 \leftarrow \oplus(A, D, E);\\
v_1 &: \quad p_2 \leftarrow A \oplus B;\\
v_3 &: \quad p_3 \leftarrow E \oplus p_2;\\
v_4 &: \quad p_2 \leftarrow C \oplus p_2;
\end{aligned}
$$

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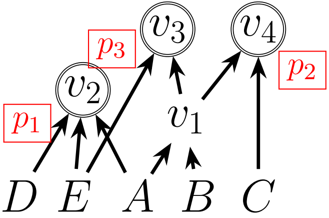
1. Choose v_2 from unvisited roots: alphabetical small $v_2 \prec v_3 \prec v_4$.
2. Evaluate the children of v_2 in alphabetical order.
3. Put a pebble p_1 on v_2 to denote v_2 is visited.
4. Choose v_3 from 2 unvisited roots, and first visit E.
5. Visit the unvisited child v_1 of v_3, evaluate, and pebble p_2
6. Back to v_3 and pebble p_3
7. Finally, we compute v_4 with *moving/recycling* pebble p_2.

Pebble Game & Intractability of Optimization Problem

Playing Pebble Game = Deciding Evaluation Order + Variable Recycling



		$\#_{\text{I/O}}$
$v_2:$	$p_1 \leftarrow \oplus(A, D, E);$	$[7]$
$v_1:$	$p_2 \leftarrow A \oplus B;$	$[3]$
$v_3:$	$p_3 \leftarrow E \oplus p_2;$	$[3]$
$v_4:$	$p_2 \leftarrow C \oplus p_2;$	$[2]$
	return$(p_1, p_3, p_2);$	15

Example: Evaluating strategy based on Depth-first-search

Can we find the best reordering and pebbling in polynomial time?

Theorem (Sethi 1975, Papp & Wattenhofer 2020)

Unless **P = NP**, *for a given* P, *in polynomial time,*
we cannot find a Q *that* $[\![P]\!] = [\![Q]\!]$ *and minimizes* $\#_{\text{I/O}}(Q)$.

We use DFS-based strategy as above in our evaluation.

Pebble Game は何から勉強すれば良い?

1975 年に出版された
Complete Register Allocation Problems, Ravi Sethi
がオススメです。https://epubs.siam.org/doi/abs/10.1137/0204020

こんな感じに図を書いて、3-SAT を pebble game 化します:

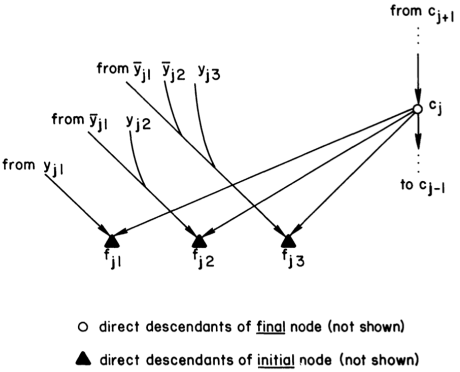


FIG. 6. *The subdag that checks if clause j is true*

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これを I/O を考えるために拡張したのが
I/O complexity: The red-blue pebble game, Jia-Wei & Kung, STOC'81
`https://dl.acm.org/doi/10.1145/800076.802486`

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よりモダンな Red-blue pebble game の話は

- *On the Hardness of Red-Blue Pebble Games*
  Papp & Wattenhofer, SPAA'20
- *Red-blue pebbling revisited: near optimal parallel matrix-matrix multip.*
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教科書なら *Models Of Computation*, J. E. Savage
`http://cs.brown.edu/people/jsavage/book/`

Evaluation

Data Set & Evaluation Environment

We consider RS(10, 4) as an example data set.

- We have 1-encoding SLP P_{enc}.
- We have $\binom{14}{4} = 1001$ decoding SLPs.

We used two environments in my paper:

name	CPU	Clock	Core	RAM
intel	i7-7567U	4.0GHz	2	DDR3-2133 16GB
amd	Ryzen 2600	3.9GHz	6	DDR4-2666 48GB

In a distributed computation,
our test environments correspond to single nodes.

L1 cache specification:

Size	Associativity	Line Size
32KB/core	8-way	64 bytes

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
$\#_{\oplus}$	755			
$\#_{\mathsf{mem}}$	2265			

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Metric		Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755			
	$\#_{\text{mem}}$	2265			
$\mathcal{B} = 512$:	$\#_{\text{I/O}}(K = 64)$	570			
$\mathcal{B} = 1\text{K}$:	$\#_{\text{I/O}}(K = 32)$	1262			
$\mathcal{B} = 2\text{K}$:	$\#_{\text{I/O}}(K = 16)$	1598			

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

$\mathcal{B}$-Byte Blocking for Cache Efficiency

$$
\begin{aligned}
&v_1 = \mathtt{xor}(A, B);\\
&v_2 = \mathtt{xor}(v_1, C, D);\\
&\mathtt{return}(v_1, v_2);
\end{aligned}
\quad\Longrightarrow\quad
\begin{aligned}
&\mathtt{for}\ i \leftarrow 0 \mathrel{..} (A\mathtt{.len} \,/\, \mathcal{B})\ \{\\
&\quad v_1^{[i]} = \mathtt{xor}(A^{[i]}, B^{[i]});\\
&\quad v_2^{[i]} = \mathtt{xor}(v_1^{[i]}, C^{[i]}, D^{[i]});\\
&\}\\
&\mathtt{return}(v_1, v_2);
\end{aligned}
$$

where $A^{[i]}$ is the i-th $\mathcal{B}$-bytes block.

		Base	
$\mathcal{B} = $			
$\mathcal{B} = $			
$\mathcal{B} = 2\text{K}$:	$\#_{\text{I/O}}(K = 16)$	1598	

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	$\#_{\mathsf{mem}}$	2265			
$\mathcal{B} = 512$:	$\#_{\mathsf{I/O}}(K = 64)$	570			
	Throughput (GB/s)	3.10			
$\mathcal{B} = 1\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 32)$	1262			
	Throughput (GB/s)	4.03			
$\mathcal{B} = 2\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598			
	Throughput (GB/s)	4.45			

Improvements by heuristics for the encoding SLP on Intel PC

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$\mathcal{B} = 2\mathrm{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598			
	Throughput (GB/s)	4.45			

Why smaller blocks are slower than the large one?

Pros: Smaller blocks,

- More cache-able blocks $\frac{32K}{\mathcal{B}}$.

Cons: Smaller blocks,

- Due to cache conflicts, using cache identically is more difficult.
- Latency penalty becomes totally large.

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385		
	$\#_{\text{mem}}$	2265	1155		
$\mathcal{B} = 512$:	$\#_{\text{I/O}}(K = 64)$	570			
	Throughput (GB/s)	3.10			
$\mathcal{B} = 1\text{K}$:	$\#_{\text{I/O}}(K = 32)$	1262			
	Throughput (GB/s)	4.03			
$\mathcal{B} = 2\text{K}$:	$\#_{\text{I/O}}(K = 16)$	1598			
	Throughput (GB/s)	4.45			

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385		
	$\#_{\text{mem}}$	2265	1155		
$\mathcal{B} = 512$:	$\#_{\text{I/O}}(K = 64)$	570	1231		
	Throughput (GB/s)	3.10			
$\mathcal{B} = 1\text{K}$:	$\#_{\text{I/O}}(K = 32)$	1262	1465		
	Throughput (GB/s)	4.03			
$\mathcal{B} = 2\text{K}$:	$\#_{\text{I/O}}(K = 16)$	1598	1599		
	Throughput (GB/s)	4.45			

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$\mathcal{B} = 512$:	$\#_{\mathsf{I/O}}(K = 64)$	570	1231		
	Throughput (GB/s)	3.10	4.18		
$\mathcal{B} = 1\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 32)$	1262	1465		
	Throughput (GB/s)	4.03	4.36		
$\mathcal{B} = 2\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598	1599		
	Throughput (GB/s)	4.45	4.86		

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385	N/A	
	$\#_{\mathsf{mem}}$	2265	1155	677	
$\mathcal{B} = 512$:	$\#_{\mathsf{I/O}}(K = 64)$	570	1231		
	Throughput (GB/s)	3.10	4.18		
$\mathcal{B} = 1\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 32)$	1262	1465		
	Throughput (GB/s)	4.03	4.36		
$\mathcal{B} = 2\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598	1599		
	Throughput (GB/s)	4.45	4.86		

Improvements by heuristics for the encoding SLP on Intel PC

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	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385	N/A	
	$\#_{\mathsf{mem}}$	2265	1155	677	
$\mathcal{B} = 512$:	$\#_{\mathsf{I/O}}(K = 64)$	570	1231	936	
	Throughput (GB/s)	3.10	4.18		
$\mathcal{B} = 1\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 32)$	1262	1465	1086	
	Throughput (GB/s)	4.03	4.36		
$\mathcal{B} = 2\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598	1599	1144	
	Throughput (GB/s)	4.45	4.86		

Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385	N/A	
	$\#_{\mathsf{mem}}$	2265	1155	677	
$\mathcal{B} = 512$:	$\#_{\mathsf{I/O}}(K = 64)$	570	1231	936	
	Throughput (GB/s)	3.10	4.18	6.98	
$\mathcal{B} = 1\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 32)$	1262	1465	1086	
	Throughput (GB/s)	4.03	4.36	7.50	
$\mathcal{B} = 2\mathsf{K}$:	$\#_{\mathsf{I/O}}(K = 16)$	1598	1599	1144	
	Throughput (GB/s)	4.45	4.86	7.12	

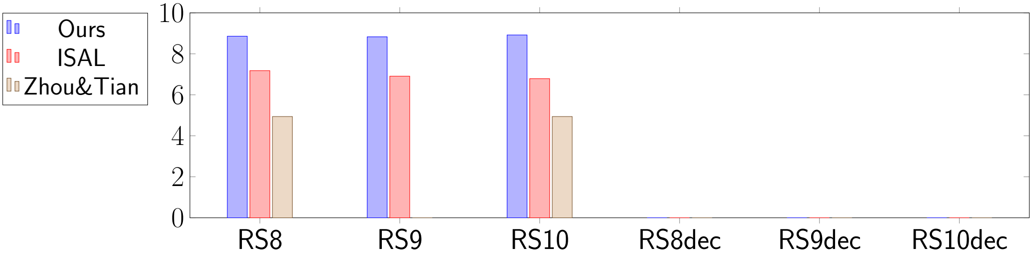
Improvements by heuristics for the encoding SLP on Intel PC

Throughput is Avg. of 1000-runs for 10MB randomly generated data

	Metric	Base P_{enc}	RePair	RePair + Fuse	RePair + Fuse + Pebbling
	$\#_{\oplus}$	755	385	N/A	
	$\#_{\text{mem}}$	2265	1155	677	
$\mathcal{B} = 512$:	$\#_{\text{I/O}}(K = 64)$	570	1231	936	636
	Throughput (GB/s)	3.10	4.18	6.98	7.24
$\mathcal{B} = 1\text{K}$:	$\#_{\text{I/O}}(K = 32)$	1262	1465	1086	779
	Throughput (GB/s)	4.03	4.36	7.50	8.92
$\mathcal{B} = 2\text{K}$:	$\#_{\text{I/O}}(K = 16)$	1598	1599	1144	845
	Throughput (GB/s)	4.45	4.86	7.12	8.55

Throughput Comparison (Intel + 1K-Blocking)

Enc	$\#_{\mathsf{mem}}$	$\#_{\mathsf{I/O}}$	Ours	ISA-L v2.30	Zhou & Tian
RS$(8,4)$	543	585	8.86 GB/s	7.18 GB/s	4.94 GB/s
RS$(9,4)$	611	671	8.83	6.91	N/A in their paper
RS$(10,4)$	677	779	8.92	6.79	4.94



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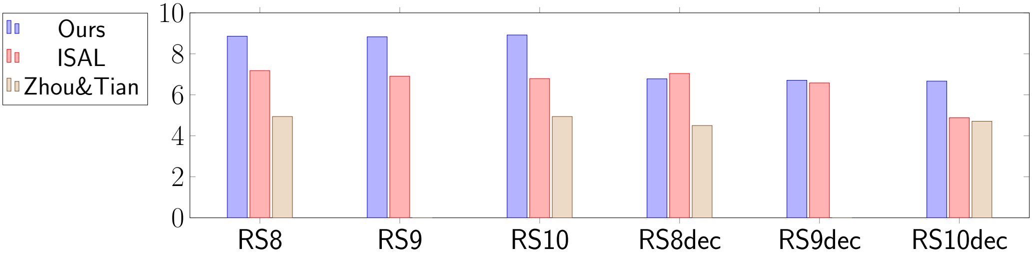
RS$(10,4)$	677	779	8.92	6.79	4.94

Dec	$\#_{\text{mem}}$	$\#_{\text{I/O}}$	Ours	ISA-L v2.30	Zhou & Tian
RS$(8,4)$	747	811	6.78 GB/s	7.04 GB/s	4.50 GB/s
RS$(9,4)$	829	968	6.71	6.58	N/A
RS$(10,4)$	923	1077	6.67	4.88	4.71



Conclusion (+ Other Throughput Scores)

intel 1K (GB/sec)	Ours Enc	Ours Dec	ISA-L v 2.30 Enc	ISA-L v 2.30 Dec	Zhou & Tian Enc	Zhou & Tian Dec
RS$(8, 3)$	12.32	8.82	9.09	9.25	6.08	5.57
RS$(9, 3)$	11.97	8.27	7.31	7.92	6.17	5.66
RS$(10, 3)$	11.78	8.89	6.78	7.93	6.15_S	5.90
RS$(8, 2)$	18.79	14.59	12.99	13.34	8.13_E	8.07_E
RS$(9, 2)$	18.93	14.27	11.85	12.03	8.34_E	8.04
RS$(10, 2)$	18.98	14.66	12.12	12.61	8.40_E	8.22_E

Conclusion

- We identified bitmatrix multiplication as straight line programs (SLP).
- We optimized XOR-based EC by optimizing SLPs using various program optimization techniques.
- Each of our techniques is not difficult; however, it suffices to match Intel's high performance library ISAL.
- As future work on cache optimization, I plan to accommodate multi-layer cache L1, L2, and L3 cache.